1. **(a)** Find the distance between points $P = (5, -3)$ and $Q = (-4, 7)$.

$$\text{dist}(P,Q) = \sqrt{(5-(-4))^2 + (-3-7)^2} = \sqrt{81+100} = \boxed{\sqrt{181}}$$

(b) Find the midpoint of the line segment joining the points P and Q.

$$M = \left(\frac{5+(-4)}{2}, \frac{(-3)+7}{2} \right) = \boxed{(0.5,\ 2)}$$

7. Let $P = (-2, 3)$, $Q = (2, -4)$ and $\vec{v} = \begin{bmatrix} 5 \\ -4 \end{bmatrix}$. Compute the following quantities:

(a) $\text{Dist}(P, Q) = \sqrt{(-2-2)^2 + (3-(-4))^2} = \sqrt{16+49} = \boxed{\sqrt{65}}$

(b) $\overrightarrow{PQ} = Q - P = \begin{bmatrix} 2-(-2) \\ -4-3 \end{bmatrix} = \boxed{\begin{bmatrix} 4 \\ -7 \end{bmatrix}}$ **(c)** $P + \vec{v} = \boxed{(3,\ -1)}$

(d) $Q - \vec{v} = \boxed{(-3,\ 0)}$ **(e)** $P - Q = \boxed{\begin{bmatrix} -4 \\ 7 \end{bmatrix}}$

11. Let $P = (-2, 3)$, $\vec{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -4 \\ 6 \end{bmatrix}$. Compute the following quantities:

(a) $P + \vec{v} = \boxed{(1,\ 1)}$ **(b)** $P + \vec{w} = \boxed{(-6,\ 9)}$ **(c)** $\vec{v} + \vec{w} = \boxed{\begin{bmatrix} -1 \\ 4 \end{bmatrix}}$

(d) $P + \vec{v} + \vec{w} = \boxed{(-3,\ 7)}$ **(e)** $\vec{w} - \vec{v} = \boxed{\begin{bmatrix} -7 \\ 8 \end{bmatrix}}$ **(f)** $\text{proj}_{\vec{v}}(\vec{w}) = \boxed{\frac{-24}{13}\begin{bmatrix} 3 \\ -2 \end{bmatrix}}$

15. Let $\vec{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$. Compute the following quantities:

(a) $\vec{v} + \vec{w} = \boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$ **(b)** $\vec{w} - \vec{v} = \boxed{\begin{bmatrix} -3 \\ 8 \end{bmatrix}}$ **(c)** $3\vec{v} - 2\vec{w} = \boxed{\begin{bmatrix} 8 \\ -19 \end{bmatrix}}$

(d) $\frac{1}{2}(\vec{v} + \vec{w}) = \boxed{\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}}$ **(e)** $\|\vec{w}\| = \boxed{\sqrt{26}}$ **(f)** $\text{proj}_{\vec{v}}(\vec{w}) = \boxed{\frac{-17}{13}\begin{bmatrix} 2 \\ -3 \end{bmatrix}}$

19. Let $\vec{u} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -12 \\ 5 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Find:

(a) $2\vec{v} - 3(\vec{u} + \vec{w}) = 2\begin{bmatrix} -12 \\ 5 \end{bmatrix} - 3\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix} - \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \boxed{\begin{bmatrix} -24 \\ 25 \end{bmatrix}}$

(b) $\|\vec{v}\| = \sqrt{144+25} = \boxed{13}$

(c) $\hat{v} = \boxed{\frac{1}{13}\begin{bmatrix} -12 \\ 5 \end{bmatrix}}$

(d) $\hat{v}^{\perp} = \boxed{\frac{1}{13}\begin{bmatrix} 5 \\ 12 \end{bmatrix}}$ or $= \boxed{\frac{1}{13}\begin{bmatrix} -5 \\ -12 \end{bmatrix}}$

23. (a) Find a unit vector in the direction of $\vec{u} = \begin{bmatrix} -8 \\ 15 \end{bmatrix}$.

Answer: $\|\vec{u}\| = 17 \Rightarrow \hat{u} = \boxed{\frac{1}{17}\begin{bmatrix} -8 \\ 15 \end{bmatrix}}$

(b) Find a unit vector in the direction of $\vec{v} = \begin{bmatrix} 20 \\ -21 \end{bmatrix}$.

Answer: $\|\vec{v}\| = 29 \Rightarrow \hat{v} = \boxed{\frac{1}{29}\begin{bmatrix} 20 \\ -21 \end{bmatrix}}$

(c) Find a unit vector in the direction of $\vec{w} = \begin{bmatrix} 12 \\ -21 \\ 28 \end{bmatrix}$.

Answer: $\|\vec{w}\| = 37 \Rightarrow \hat{w} = \boxed{\frac{1}{37}\begin{bmatrix} 12 \\ -21 \\ 28 \end{bmatrix}}$

(d) Find two unit vectors perpendicular to $\vec{a} = \begin{bmatrix} 21 \\ -12 \\ 16 \end{bmatrix}$.

Answer: e.g. $\boxed{\frac{1}{\sqrt{65}}\begin{bmatrix} 7 \\ 4 \\ 0 \end{bmatrix}}$, $\boxed{\frac{1}{5}\begin{bmatrix} 0 \\ 4 \\ 3 \end{bmatrix}}$, $\boxed{\frac{1}{\sqrt{697}}\begin{bmatrix} 16 \\ 0 \\ -21 \end{bmatrix}}$, $\boxed{\frac{1}{\sqrt{34}}\begin{bmatrix} 4 \\ 3 \\ -3 \end{bmatrix}}$ etc.

27. Let $P = (2, 5)$, $Q = (4, -1)$ and $R = (5, 2)$.

(a) Find the midpoints M_{PQ} and M_{PR} of segments $\overline{PQ}$ and $\overline{PR}$.

(b) Find the midpoint M of the segment $\overline{M_{PR}Q}$.

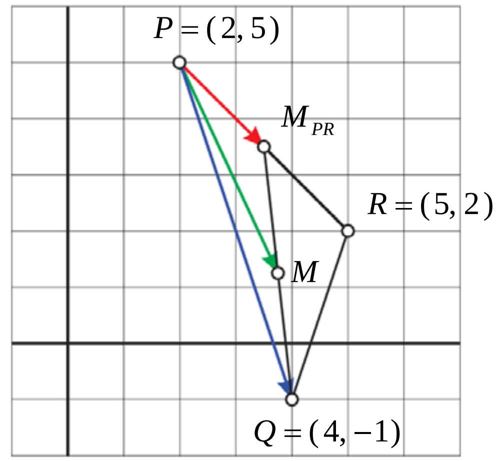
(c) Express $\overrightarrow{PM}$ in terms of the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$: $a \cdot \overrightarrow{PQ} + b \cdot \overrightarrow{PR} = \overrightarrow{PM}$

Answers: (a) $M_{PQ} = \left(\dfrac{2+4}{2}, \dfrac{5+(-1)}{2}\right) = \boxed{(3,2)}$

$M_{PR} = \left(\dfrac{2+5}{2}, \dfrac{5+2}{2}\right) = \boxed{(3.5, 3.5)}$

(b) $M = \left(\dfrac{3.5+4}{2}, \dfrac{3.5+(-1)}{2}\right) = \boxed{(3.75, 1.25)}$

(c) $\overrightarrow{PM} = \dfrac{1}{2}\left(\overrightarrow{PQ} + \dfrac{1}{2}\overrightarrow{PR}\right) = \dfrac{1}{2}\overrightarrow{PQ} + \dfrac{1}{4}\overrightarrow{PR}$



29. Let $\vec{u} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$ $\vec{v} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$. Compute the following quantities:

(a) $2\vec{u} + 3\vec{v} = 2\begin{bmatrix} 7 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ -3 \end{bmatrix} = \boxed{\begin{bmatrix} 20 \\ -11 \end{bmatrix}}$

(b) $3\vec{u} - 2\vec{v} + \vec{w} = 3\begin{bmatrix} 7 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ -3 \end{bmatrix} + \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \boxed{\begin{bmatrix} 23 \\ 7 \end{bmatrix}}$

(c) $\vec{u} \cdot \vec{v} = \boxed{17}$

(d) $\vec{v} \cdot \vec{w} = \boxed{0}$

(e) $(2\vec{u} + 3\vec{v}) \cdot \vec{w} = \begin{bmatrix} 20 \\ -11 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 4 \end{bmatrix} = \boxed{76}$

35. Find the angles between the following pairs of vectors:

(a) $\begin{bmatrix} 8 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ 2 \end{bmatrix}$ **(b)** $\begin{bmatrix} -4 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 12 \\ -6 \end{bmatrix}$ **(c)** $\begin{bmatrix} -3 \\ 11 \end{bmatrix}$ and $\begin{bmatrix} 4 \\ -3 \end{bmatrix}$

(d) $\begin{bmatrix} 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 1 \end{bmatrix}$ **(e)** $\begin{bmatrix} 6 \\ -10 \end{bmatrix}$ and $\begin{bmatrix} 15 \\ 9 \end{bmatrix}$

Answers:

(a) $\theta = \cos^{-1}\left(\dfrac{52}{\sqrt{68}\sqrt{53}}\right) = \boxed{29.9816°}$

(b) $\theta = \cos^{-1}\left(\dfrac{-60}{\sqrt{20}\sqrt{180}}\right) = \cos^{-1}(-1) = \boxed{180°}$. Note: $\begin{bmatrix} 12 \\ -6 \end{bmatrix} = -3\cdot\begin{bmatrix} -4 \\ 2 \end{bmatrix}$

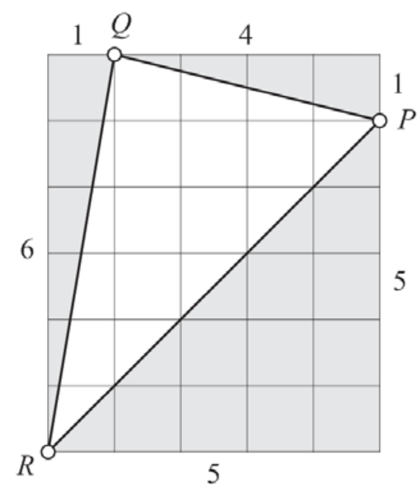
(c) $\theta = \cos^{-1}\left(\dfrac{-45}{5\sqrt{130}}\right) = \boxed{142.1250°}$

(d) $\theta = \cos^{-1}\left(\dfrac{-13}{\sqrt{13}\sqrt{26}}\right) = \cos^{-1}\left(\dfrac{-1}{\sqrt{2}}\right) = \boxed{135°}$

(e) $\theta = \cos^{-1}(0) = \boxed{90°}$.

37. Let $P = (2, 3)$, $Q = (-2, 4)$ and $R = (-3, -2)$.

(a) Find the area of ΔPQR :



$$6\cdot 5 - \frac{1}{2}\cdot 5\cdot 5 - \frac{1}{2}\cdot 6\cdot 1 - \frac{1}{2}\cdot 4\cdot 1 = \boxed{12.5}$$

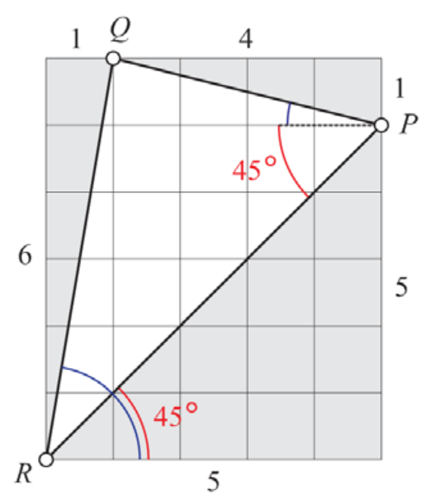
or, $\dfrac{1}{2}bc\sin(\alpha) = \dfrac{1}{2}\sqrt{50}\sqrt{37}\sin(\alpha) = 12.5$

[with b and c from part (c) and α from (b)]

or, using Heron's formula $\sqrt{s(s-a)(s-b)(s-c)} = 12.5$

[with a, b and c from part (c)]

(b) Find the angles of ΔPQR



$$\alpha = \tan^{-1}(6) - 45° = 35.5377°$$
$$\beta = \tan^{-1}(1/4) + 45° = 59.0362$$
$$\gamma = 180° - \beta - \alpha = 85.4261°$$

or, using a, b and c from part (c)

$$\cos(\alpha) = \frac{b^2+c^2-a^2}{2bc} = \frac{50+37-17}{2\sqrt{50}\sqrt{37}} \Rightarrow \boxed{\alpha = 35.5377^\circ}$$

$$\cos(\beta) = \frac{a^2+c^2-b^2}{2ac} = \frac{17+50-37}{2\sqrt{17}\sqrt{50}} \Rightarrow \boxed{\beta = 59.0362^\circ}$$

$$\cos(\gamma) = \frac{a^2+b^2-c^2}{2ab} = \frac{17+37-50}{2\sqrt{17}\sqrt{37}} \Rightarrow \boxed{\gamma = 85.4261^\circ}$$

(c) Find the lengths of the sides of ΔPQR

$\text{dist}(P,Q) = \boxed{\sqrt{17}}$, $\text{dist}(Q,R) = \boxed{\sqrt{37}}$ and $\text{dist}(R,P) = \boxed{\sqrt{50}}$

(d) Find the base of the altitude from Q, using a projection vector.

$$B = R + \text{proj}_{\overrightarrow{RP}}(\overrightarrow{RQ}) = (-3,-2) + \frac{35}{50}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \boxed{(\,0.5,\ 1.5\,)}$$

43. Use the law of cosines to find the angle between the lines $2x+y = 5$ and $3x-5y = 1$ [Hint: First find the point of intersection of the two lines, call it Q. Then check that the points $P = (\,-1, 7\,)$ and $R = (\,7, 4\,)$ are each on one of the lines.

Use these three points, i.e. ΔPQR, and the law of cosines to compute the angle.]

The intersection of the two lines: $Q = (2,1)$. The lengths of the sides of ΔPQR are

$|PQ| = \sqrt{9+36} = \sqrt{45}$, $|QR| = \sqrt{25+9} = \sqrt{34}$ and $|PR| = \sqrt{64+9} = \sqrt{73}$,

So that by the rule of cosines: $\angle PQR = \cos^{-1}\left(\dfrac{45+34-73}{2\sqrt{45}\sqrt{34}}\right) = \boxed{85.6013^\circ}$

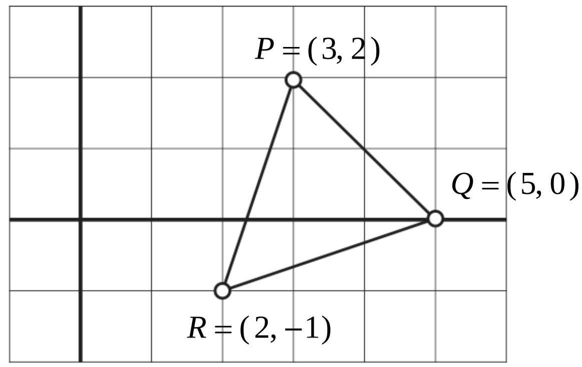
47. (a) Graph in one picture the points $P = (3, 2)$, $Q = (5, 0)$ and $R = (2, -1)$.

(b) Compute the distances: dist(P, Q), dist(P, R) and dist(Q, R).

(c) Compute the angles of the triangle $\triangle PQR$, using the law of cosines and the lengths you computed in part (b).

Answers:

(a)



(b) $\text{dist}(P,Q) = \boxed{\sqrt{8}}$

$\text{dist}(P,R) = \boxed{\sqrt{10}}$

$\text{dist}(Q,R) = \boxed{\sqrt{10}}$

(c) $$\angle RPQ = \cos^{-1}\left(\frac{a^2+b^2-c^2}{2ab}\right) = \cos^{-1}\left(\frac{8+10-10}{2\sqrt{8}\sqrt{10}}\right) = \boxed{63.43495^\circ}$$

$$\angle PQR = \cos^{-1}\left(\frac{a^2+c^2-b^2}{2ac}\right) = \cos^{-1}\left(\frac{8+10-10}{2\sqrt{8}\sqrt{10}}\right) = \boxed{63.43495^\circ} = \angle RPQ$$

$$\angle PRQ = 180^\circ - 2\angle RPQ = 180^\circ - 126.8699^\circ = \boxed{53.1301^\circ}$$

51. Let $\vec{u} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$ and $\vec{w} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$.

(a) Find the coordinates of the vectors $\vec{u}+\vec{v}$, $\vec{v}-\vec{u}$, $-3\vec{w}$ and $\vec{u}+\vec{v}-3\vec{w}$.

(b) Graph in **one** picture the vectors $\vec{u}+\vec{v}$, $\vec{v}-\vec{u}$, $2\vec{w}$ and $\vec{v}-\vec{u}+2\vec{w}$.

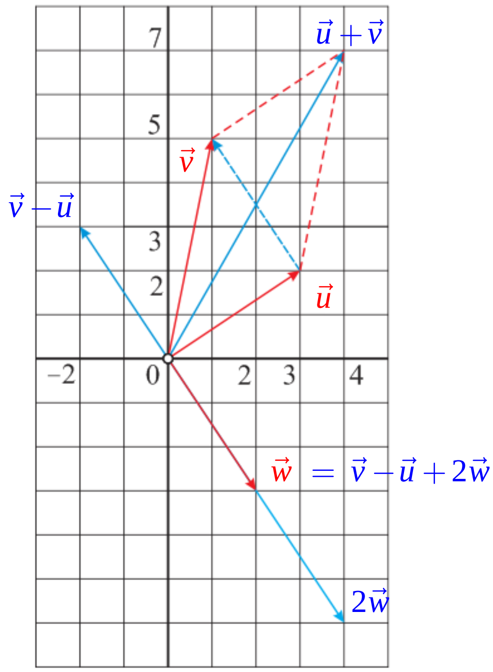
(c) Compute the dot products $\vec{u}\cdot\vec{w}$, $(\vec{u}+\vec{v})\cdot(-3\vec{w})$ and $(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}-3\vec{w})$.

(d) Compute the angle between the vectors $\vec{u}$ and $\vec{v}$.

(e) Using the dot product find the angle between the vectors $\vec{v}-\vec{u}$ and $\vec{w}-\vec{u}$. How does this compare to the angle $\angle QPR$ in $\triangle PQR$ from problem 48. Explain.

(a) $\vec{u}+\vec{v} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$, $\vec{v}-\vec{u} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$, $-3\vec{w} = \begin{bmatrix} -6 \\ 9 \end{bmatrix}$ and $\vec{u}+\vec{v}-3\vec{w} = \begin{bmatrix} -2 \\ 16 \end{bmatrix}$.

(b) Note that $2\vec{w} = \begin{bmatrix} 4 \\ -6 \end{bmatrix}$ and $\vec{v}-\vec{u}+2\vec{w} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} = \vec{w}$ so that the all these vectors in one picture look like:



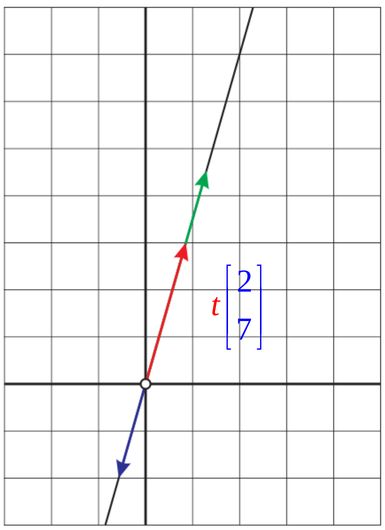
(c) $\vec{u}\cdot\vec{w} = \boxed{0}$, $(\vec{u}+\vec{v})\cdot(-3\vec{w}) = \boxed{39}$ and $(\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v}-3\vec{w}) = \boxed{-26}$

(d) $\angle QPR = \cos^{-1}\left(\dfrac{(\vec{v}-\vec{u})\cdot(\vec{w}-\vec{u})}{\|\vec{v}-\vec{u}\|\|\vec{w}-\vec{u}\|}\right) = \cos^{-1}\left(\dfrac{26}{\sqrt{13}\sqrt{65}}\right) = \boxed{26.5651°}$

55. (a) Find two vectors that are perpendicular to $\vec{u} = \begin{bmatrix}-7\\2\end{bmatrix}$: e.g. $\begin{bmatrix}2\\7\end{bmatrix}$, $\begin{bmatrix}0\\0\end{bmatrix}$, $\begin{bmatrix}-4\\-14\end{bmatrix}$ etc.

(b) Give a geometric interpretation of the set of all vectors perpendicular to $\vec{u} = \begin{bmatrix}-7\\2\end{bmatrix}$:

All vectors on the line through the origin with $\vec{u}$ as normal, i.e. all multiples of $\begin{bmatrix}2\\7\end{bmatrix}$.

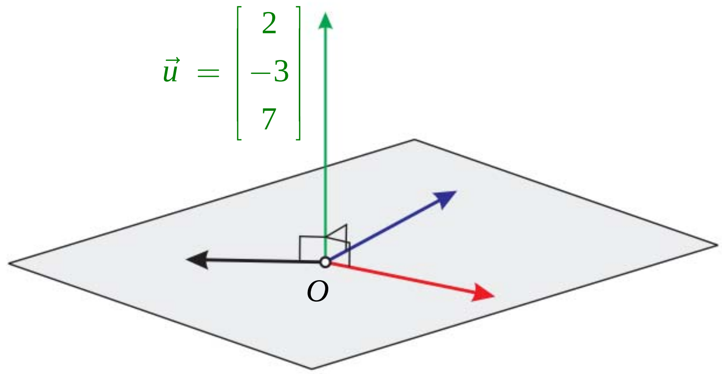


(c) Find four vectors that are perpendicular to $\vec{v} = \begin{bmatrix}2\\-3\\7\end{bmatrix}$.

e.g. $\begin{bmatrix}0\\7\\3\end{bmatrix}$, $\begin{bmatrix}7\\0\\-2\end{bmatrix}$, $\begin{bmatrix}3\\2\\0\end{bmatrix}$, $\begin{bmatrix}0\\0\\0\end{bmatrix}$, $\begin{bmatrix}-2\\1\\1\end{bmatrix}$ etc.

(d) Give a geometric interpretation of the set of all vectors perpendicular to $\vec{u} = \begin{bmatrix} 2 \\ -3 \\ 7 \end{bmatrix}$:

All vectors in the plane through the origin, with $\vec{u}$ as normal.



59. Find the orthogonal projection of $\vec{u} = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$ onto $\vec{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$.

Answer: $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{\begin{bmatrix} -1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}}{\begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}} \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \boxed{-\dfrac{5}{10} \begin{bmatrix} 3 \\ 1 \end{bmatrix}} = \boxed{\begin{bmatrix} -1.5 \\ -0.5 \end{bmatrix}}$

63. Find the orthogonal projection of $\vec{u} = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ onto $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$.

Answer: $\text{proj}_{\vec{v}}(\vec{u}) = \dfrac{\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}}{\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \boxed{\dfrac{3}{6} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}} = \boxed{\begin{bmatrix} 1 \\ 0.5 \\ 0.5 \end{bmatrix}}$

67. Show that the vectors $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$.

a) $\mathrm{proj}_{\vec{v}}(\vec{w}) \parallel \vec{v}$

Answer: $\mathrm{proj}_{\vec{v}}(\vec{w}) = \dfrac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\cdot\vec{v} = t\cdot\vec{v}$, i.e. a multiple of $\vec{v}$, i.e. $\mathrm{proj}_{\vec{v}}(\vec{w}) \parallel \vec{v}$.

b) $\left(\vec{w} - \mathrm{proj}_{\vec{v}}(\vec{w})\right) \perp \vec{v}$

Answer:
$$\begin{aligned}\left(\vec{w} - \mathrm{proj}_{\vec{v}}(\vec{w})\right)\cdot\vec{v} &= \left(\vec{w} - \frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\cdot\vec{v}\right)\cdot\vec{v} \\ &= \vec{w}\cdot\vec{v} - \frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\cdot(\vec{v}\cdot\vec{v}) \\ &= \vec{w}\cdot\vec{v} - \vec{w}\cdot\vec{v} = 0\end{aligned}$$

Hence: $\left(\vec{w} - \mathrm{proj}_{\vec{v}}(\vec{w})\right) \perp \vec{v}$