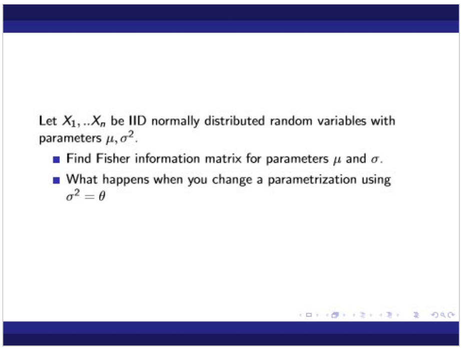
MSP_6_cviko

úterý 22. října 2024 8:07



$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \to l(\mu,\sigma) = -\ln(\sqrt{2\pi}) - \ln(\sigma) - \frac{(x-\mu)^2}{2\sigma^2}$$

$$\frac{\partial l}{\partial \mu} = \frac{x-\mu}{\sigma^2} \qquad \frac{\partial l}{\partial \sigma} = -\frac{1}{\sigma} + \frac{(x-\mu)^2}{\sigma^3}$$

$$\frac{\partial^2 l}{\partial \mu^2} = -\frac{1}{\sigma^2} \qquad \frac{\partial^2 l}{\partial \mu \partial \sigma} = -\frac{2}{\sigma^3}(x-\mu)$$

$$\frac{\partial^2 l}{\partial \sigma^2} = \frac{1}{\sigma^2} - \frac{3(x-\mu)^2}{\sigma^4} \qquad \boxed{E(X) = \mu}$$

$$-E\left[-\frac{1}{\sigma^2}\right] = \frac{1}{\sigma^2} \;\Big|\; -E\left(-\frac{2}{\sigma^3}(x-\mu)\right) = \frac{2}{\sigma^3} \cdot \int (x-\mu) f(x)\, dx = 0$$

$$-E\left[\frac{1}{\sigma^2} - \frac{3(x-\mu)^2}{\sigma^4}\right] = -\frac{1}{\sigma^2} + \frac{3}{\sigma^4} \cdot E\left((x-\mu)^2\right) = -\frac{1}{\sigma^2} + \frac{3\sigma^2}{\sigma^4} = \frac{2}{\sigma^2}$$

$$E\left[(X - E(X))^2\right] = \sigma^2$$

$$J(\mu,\sigma) = \begin{pmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{2}{\sigma^2} \end{pmatrix}$$

$$f(x) = \frac{1}{\sqrt{2\pi\theta}} \cdot e^{-\frac{(x-\mu)^2}{2\theta}} \to l(\mu;\theta) = -\frac{1}{2}\ln(\theta) - \frac{1}{2}\ln(2\pi) - \frac{(x-\mu)^2}{2\theta}$$

$$\frac{\partial l}{\partial \mu} = \frac{x-\mu}{\theta} \qquad \frac{\partial l}{\partial \theta} = -\frac{1}{2\theta} + \frac{(x-\mu)^2}{2\theta^2}$$

$$\frac{\partial^2 l}{\partial \mu^2} = -\frac{1}{\theta} \qquad \frac{\partial^2 l}{\partial \mu \partial \theta} = -\frac{x-\mu}{\theta^2} \qquad -E\left[-\frac{x-\mu}{\theta^2}\right] = 0$$

$$\frac{\partial^2 l}{\partial^2 \theta} = \frac{1}{2\theta^2} - \frac{(x-\mu)^2}{\theta^3} \qquad -E\left[+\frac{1}{2\theta^2} - \frac{(x-\mu)^2}{\theta^3}\right] = -\frac{1}{2\theta^2} + \frac{\theta}{\theta^3} = \frac{1}{2\theta^2}$$

$$J(\mu;\theta) = \begin{pmatrix} \frac{1}{\theta} & 0 \\ 0 & \frac{1}{2\theta^2} \end{pmatrix}$$

Let $X_1, \dots X_n$ be IID distributed random variables with PMF:

$$P(x) = \binom{x-1}{r-1} \pi^r (1-\pi)^{x-r}; x \in \{r, r+1, r+2, \dots\}.$$

Negative binomial distribution - number of observed Bernoulli trials with probability of success π to get r successes. Assume that r is known.

- Find a MLE of π for known r.
- Decide whether this Negative binomial distribution belongs to exponential family.
- Try to find a sufficient statistic for a parameter π
- Compute Fisher information for a parameter π
- You observed n NB trials with $r = 3$. $\sum_{i=1}^{100} x_i = 2999$. Find the 95% asymptotic CI for π

$$l(\pi) = \sum \ln \binom{x_i - 1}{r-1} + nr \cdot \ln(\pi) + \sum_{i=1}^{n} (x_i - r) \ln(1-\pi) =$$

$$= \sum \ln \binom{x_i - 1}{r-1} + \sum x_i \ln(1-\pi) + n \cdot r \cdot \ln(\pi) - nr \ln(1-\pi)$$

$$\frac{\partial l}{\partial \pi} = -\frac{\sum_{i=1}^{n} x_i}{1-\pi} + \frac{n \cdot r}{\pi} + \frac{n \cdot r}{1-\pi} = 0 \Rightarrow \frac{\sum_{i=1}^{n} x_i}{n \cdot r} = \frac{1}{\pi}$$

$$\hat{\pi} = \frac{n \cdot r}{\sum_{i=1}^{n} x_i}$$

$$\binom{x-1}{r-1} \cdot e^{r \ln(\pi) + (x-r) \cdot \ln(1-\pi)} = \binom{x-1}{r-1} \cdot e^{x \cdot \ln(1-\pi) + r \ln\left(\frac{\pi}{1-\pi}\right)}$$

$$\binom{x_i - 1}{r-1}^n \cdot e^{\sum x_i \ln(1-\pi) + rn \ln\left(\frac{\pi}{1-\pi}\right)} \Rightarrow T(x) = \sum_{i=1}^{n} x_i$$

pro $n=1$

$$\frac{\partial^2 l}{\partial \pi^2} = -\frac{x}{(1-\pi)^2} - \frac{r}{\pi^2} + \frac{r}{(1-\pi)^2}$$

$$-E\left(\frac{-x}{(1-\pi)^2}\right) = \frac{1}{(1-\pi)^2} \cdot E(X) = \frac{r}{\pi(1-\pi)^2} + \frac{r}{\pi^2} - \frac{r}{(1-\pi)^2} = \frac{r(\pi + (1-\pi)^2 - \pi^2)}{\pi^2(1-\pi)^2} = \frac{r(\pi + 1 - 2\pi + \pi^2 - \pi^2)}{\pi^2(1-\pi)^2} =$$

$$= \frac{r}{\pi^2(1-\pi)} = J(\pi)$$

$$CI: \left\langle \hat{\pi} - u_{1-\alpha/2} \sqrt{\frac{1}{J_n(\pi)}} ; \hat{\pi} + u_{1-\alpha/2} \sqrt{\frac{1}{J_n(\pi)}} \right\rangle =$$

$$\hat{\pi} \doteq 0{,}1 \Rightarrow \left\langle 0{,}1 - 1{,}96 \cdot \sqrt{\frac{1}{\frac{100 \cdot 3}{0{,}1^2(1-0{,}1)}}} ; 0{,}1 + 0{,}01074 \right\rangle = \langle 0{,}0893 ; 0{,}1107 \rangle$$

Let $X_1, \dots X_n$ be IID random variables following a Poisson distribution with a parameter λ.

- Compute Fisher information for λ.
- You observed this random variable $n = 150$ times with a $\sum_{i=1}^{150} x_i = 351$. Find the 99% asymptotic CI for λ.
- Use likelihood ratio to test a hypothesis $H_0 : \lambda = 2$.

$$f(x) = \frac{\lambda^x}{x!} e^{-\lambda} \rightarrow l(\lambda) = x \ln(\lambda) - \ln(x!) - \lambda$$

$$\frac{\partial l}{\partial \lambda} = \frac{x}{\lambda} - 1 \qquad \frac{\partial^2 l}{\partial \lambda^2} = -\frac{x}{\lambda^2} \qquad -E\left(-\frac{x}{\lambda^2}\right) = \frac{1}{\lambda} = J(\lambda)$$

$$J_n(\lambda) = \frac{n}{\lambda} \qquad \bar{X} = 2{,}34 \rightarrow \lambda \in \left\langle 2{,}34 - 2{,}576 \cdot \sqrt{\frac{2{,}34}{150}} ; 2{,}34 + 0{,}322 \right\rangle$$

$$\lambda \in \langle 2{,}012 ; 2{,}662 \rangle$$

$$LR = 2[l(\hat{\lambda}) - l(\lambda_0)] =$$

$$2\left[\sum x_i \ln(\hat{\lambda}) - \ln\left(\prod_{i=1}^{n} x_i!\right) - n\hat{\lambda} - \left(\sum x_i \ln(\lambda_0) - \ln\left(\prod_{i=1}^{n} x_i!\right) - n\lambda_0\right)\right] =$$

$$= 2n\left[\bar{x} \cdot (\ln(\bar{x}) - \ln(\lambda_0)) - (\bar{x} - \lambda_0)\right] = 8{,}216$$

$$\chi^2_{0{,}95}(1) = 3{,}841 \qquad LR > 3{,}841 \Rightarrow \text{Zamítám } H_0$$

Let Y be a normally distributed random variable. Find a PDF for a random variable X created by a formula $e^Y = X$.

- Find a MLE for μ, σ^2 using observations of X.
- Prove that the Fisher information matrix will not change by the transformation.

$$f(y) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot e^{-\frac{(y-\mu)^2}{2\sigma^2}} \quad \rightarrow \quad e^y = X$$

$$y = \ln(x)$$

$$dy = \frac{dx}{X}$$

$$-\infty \rightarrow 0$$

$$\infty \rightarrow \infty$$

$$f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma\cdot X} \cdot e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}$$

$$l(\mu;\sigma) = -n\ln(\sqrt{2\pi}) - n\ln(\sigma) - n\cdot\ln\left(\prod x_i\right) - \frac{\sum_{i=1}^{n}(\ln(x_i)-\mu)^2)}{\sigma^2}$$

$$\frac{\partial l}{\partial \mu} = + \frac{\sum_{i=1}^{n}\ln(x_i) - n\mu}{\sigma^2} \quad \rightarrow \quad \hat{\mu} = \frac{\sum_{i=1}^{n}\ln(x_i)}{n}$$

$$\frac{\partial l}{\partial \sigma} = -\frac{n}{\sigma} + \frac{\sum_{i=1}^{n}(\ln(x_i)-\mu)^2}{\sigma^3} \quad \rightarrow \quad \hat{\sigma}^2 = \frac{\sum(\ln(x_i)-\hat{\mu})^2}{n}$$

$$\frac{\partial^2 l}{\partial \mu^2} = -\frac{1}{\sigma^2} \qquad \frac{\partial^2 l}{\partial \mu\,\partial \sigma} = -\frac{2}{\sigma^3}(\ln(x)-\mu)$$

$$\frac{\partial^2 l}{\partial \sigma^2} = \frac{1}{\sigma^2} + \frac{3}{\sigma^4}\cdot(\ln(x)-\mu)^2$$

$$\int_{-\infty}^{\infty}(\ln(x)-\mu)\cdot\frac{1}{x}\cdot\frac{1}{\sqrt{2\pi}\,\sigma}\cdot e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}\,dx \;\left|\begin{array}{l} y = \ln(x) \\ dy = \frac{dx}{X} \\ x\cdot dy = dx \end{array}\right| = \int_{0}^{\infty}(y-\mu)\cdot\frac{1}{\sqrt{2\pi}\,\sigma}\cdot e^{-\frac{(y-\mu)^2}{2\sigma^2}}\,dy$$

$\downarrow$

NORMÁLNÍ ROZDĚLENÍ