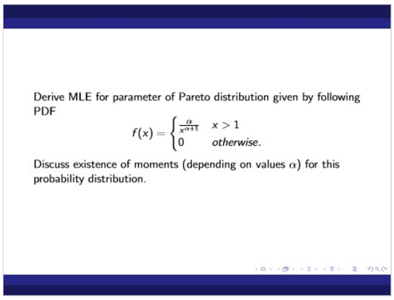
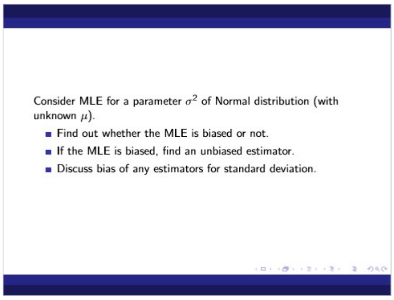



1) $L(\alpha) = \prod_{i=1}^{n} \frac{\alpha}{x^{\alpha+1}} \to \ell(\alpha) = \sum_{i=1}^{n} \ln\left(\frac{\alpha}{x_i^{\alpha+1}}\right) = n \cdot \ln(\alpha) - \sum_{i=1}^{n} (\alpha+1) \cdot \ln(x_i)$

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} \ln(x_i) \to \hat{\alpha} = \frac{n}{\sum_{i=1}^{n} \ln(x_i)}$$

2) $m^k \to \int_1^\infty x^k \cdot \frac{\alpha}{x^{\alpha+1}}\,dx \to$ diskutuj pro jaké $\alpha$ $\exists$ jaké momenty




$$\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{n}\sum_{i=1}^{n} x_i^2 - \bar{X}^2 = \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2 - 2\sum x_i \mu + n\mu^2\right) - \left(\bar{X}^2 - 2\bar{X}\mu + \mu^2\right) =$$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}(x_i - \mu)^2\right) - (\bar{X} - \mu)^2 \Rightarrow E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right) = E\left[\frac{1}{n}\left(\sum_{i=1}^{n}(x_i - \mu)^2\right) - (\bar{X} - \mu)^2\right]$$

$$E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2\right) \to x_i - \mu \sim N(0; \sigma^2) \Rightarrow E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2\right) = \frac{1}{n} \cdot n\sigma^2 = \sigma^2$$

$E\left((\bar{X} - \mu)^2\right) \to$ Rozptyl $\bar{X}$ ; $\bar{X} \sim N\left(\mu; \frac{\sigma^2}{n}\right) \Rightarrow E\left((\bar{X} - \mu)^2\right) = \frac{\sigma^2}{n}$

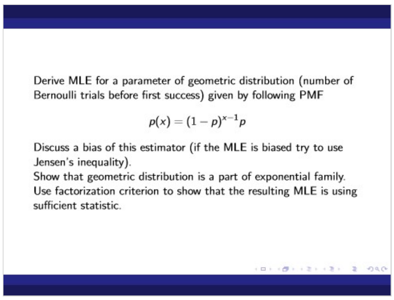
dohromady: $E\left(\frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2\right) = \sigma^2 - \frac{\sigma^2}{n} = \frac{n\sigma^2 - \sigma^2}{n} = \frac{n-1}{n} \cdot \sigma^2 \neq \sigma^2$

Nestranný odhad: $\frac{1}{n} \to \frac{1}{n-1} \Rightarrow \frac{n-1}{n-1}\sigma^2 = \sigma^2$

Směrodatná odchylka: $\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2} \to \sqrt{E(X)} \neq E(\sqrt{X})$

$\sqrt{x}$ konkávní $\Rightarrow E\left(\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}\right) < \sqrt{E\left(\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2\right)} = \sigma$

$\Rightarrow$ odhad je vychýlený a podhodnocuje




$$L(p) = \prod_{i=1}^{n}(1-p)^{x_i-1} \cdot p$$

$$\ell(p) = \sum_{i=1}^{n} \ln\left[(1-p)^{x_i-1} \cdot p\right] = \sum_{i=1}^{n}\left[(x_i - 1) \cdot \ln(1-p) + \ln p\right] =$$

$$= \sum_{i=1}^{n} x_i \cdot \ln(1-p) - n\ln(1-p) + n\ln(p)$$

$$\frac{\partial \ell}{\partial p} = -\frac{\sum_{i=1}^{n} x_i}{1-p} + \frac{n}{1-p} + \frac{n}{p} = 0$$

$$\frac{\sum_{i=1}^{n} x_i}{n} = \frac{1-p}{1-p} + \frac{1-p}{p} = \frac{1}{p} \to \hat{p} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{X}}$$

$E\left(\frac{1}{\bar{X}}\right) > \frac{1}{E(\bar{X})}$ protože $\frac{1}{x}$ je konvexní $\Rightarrow$

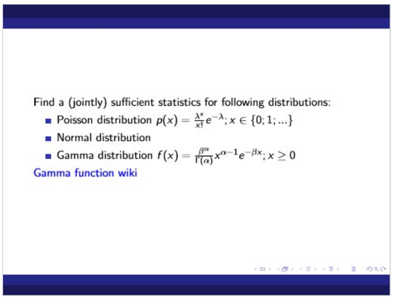
$\Rightarrow \hat{p}$ nadhodnocuje

$$p(x) = (1-p)^{x-1}p \to e^{\ln\left((1-p)^{x-1}p\right)} = e^{(x-1) \cdot \ln(1-p) + \ln p}$$

$\Rightarrow h(x) = 1$ ; $T(x) = x - 1$ $\quad \eta(\theta) = \ln(1-p)$ $\quad A(\theta) = \ln p$

$$\prod_{i=1}^{n}\left[(1-p)^{x_i-1}\cdot p\right] = (1-p)^{\sum_{i=1}^{n}(x_i-1)}\cdot p^{n} = (1-p)^{\sum_{i=1}^{n}x_i}\cdot\left(\frac{p}{1-p}\right)^{n} \Rightarrow$$

$T(x) = \sum_{i=1}^{n} x_i$ is sufficient




1) $$p(x,\lambda) = \prod_{i=1}^{n}\frac{\lambda^{x_i}}{x_i!}e^{-\lambda} = \frac{\lambda^{\sum_{i=1}^{n}x_i}}{\prod_{i=1}^{n}x_i!}\cdot e^{-n\lambda} = \prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\cdot\left[\lambda^{\sum_{i=1}^{n}x_i}\cdot e^{-n\lambda}\right] \to$$

$$u(x) = \prod_{i=1}^{n}\left(\frac{1}{x_i!}\right) ; \quad v(x,\lambda) = \lambda^{\sum_{i=1}^{n}x_i}\cdot e^{-n\lambda} \to T_1(x) = \sum_{i=1}^{n}x_i$$

2) $$f(x,\mu,\sigma^2) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\,\sigma}\cdot e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} = \left(\frac{1}{\sqrt{2\pi}\cdot\sigma}\right)^{n}\cdot e^{-\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}} =$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma}\right)^{n}\cdot e^{-\frac{1}{2\sigma^2}\cdot\sum_{i=1}^{n}x_i^2 + \frac{\mu}{\sigma^2}\cdot\sum_{i=1}^{n}x_i - \frac{\mu^2}{2\sigma^2}} \Rightarrow \begin{array}{l} T_1 = \sum_{i=1}^{n}x_i \\ T_2 = \sum_{i=1}^{n}x_i^2 \end{array}$$

3) $$f(x,\alpha,\beta) = \prod_{i=1}^{n}\frac{\beta^{\alpha}}{\Gamma(\alpha)}x_i^{\alpha-1}e^{-\beta x_i} = \frac{\beta^{n\alpha}}{\Gamma(\alpha)^{n}}\cdot e^{n(\alpha-1)\sum_{i=1}^{n}\ln(x_i)}\cdot e^{-\beta\sum_{i=1}^{n}x_i}$$

$T_1(x) = \sum_{i=1}^{n}x_i \quad T_2(x) = \sum_{i=1}^{n}\ln(x_i)$ protože $\sum_{i=1}^{n}\ln(x_i) = \ln\left(\prod_{i=1}^{n}(x_i)\right)$

nebo

$T_2^{*}(x) = \prod_{i=1}^{n}(x_i)$