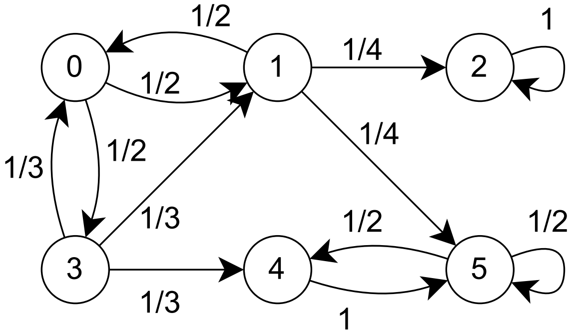
MSP 2024 - Tutorial 2

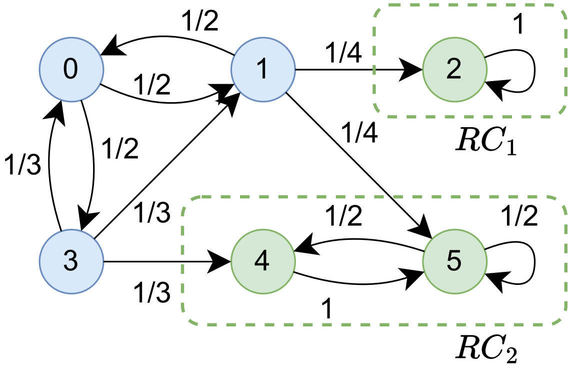
Exercise 1. Assume DTMC below.



1. Identify recurrent classes and transient states of this DTMC.
2. Compute the limiting distribution of this DTMC given that initially the chain starts in state:

   a) 2, b) 4, c) 5, d) 0, e) 1

Exercise 1. Assume DTMC below.



1. Identify recurrent classes and transient states of this DTMC.
2. Compute the limiting distribution of this DTMC given that initially the chain starts in state:

   a) 2, b) 4, c) 5, d) 0, e) 1

Solution

1. recurrent classes: $RC_1 = \{2\}$, $RC_2 = \{4, 5\}$; transient states: $\{0, 1, 3\}$.

Let $\boldsymbol{t}_\infty^s$ denote the limiting distribution when starting in state s.

2. a) $\boldsymbol{t}_\infty^2 = (0, 0, 1, 0, 0, 0) =: \boldsymbol{\pi}^{RC_1}$

   b,c) Computing steady-state for RC_2:

$$\boldsymbol{\pi}(4) = \frac{1}{2} \cdot \boldsymbol{\pi}(5)$$
$$\boldsymbol{\pi}(5) = \boldsymbol{\pi}(4) + \frac{1}{2} \cdot \boldsymbol{\pi}(5)$$
$$\boldsymbol{\pi}(4) + \boldsymbol{\pi}(5) = 1$$

$$\boldsymbol{t}_\infty^4 = \boldsymbol{t}_\infty^5 = \left(0, 0, 0, 0, \tfrac{1}{3}, \tfrac{2}{3}\right) =: \boldsymbol{\pi}^{RC_2}$$

Solution

Reachability to RC_1: $T = RC_1,\ x(s) \coloneqq P(s \to T)\,.\ S_0 = \{4, 5\}$

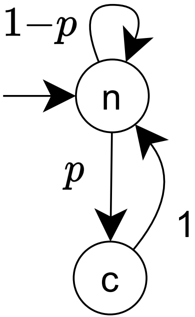
$$\begin{aligned}
\boldsymbol{x}(2) &= 1, \quad \boldsymbol{x}(4) = \boldsymbol{x}(5) = 0 \\
\boldsymbol{x}(0) &= \frac{1}{2} \cdot \boldsymbol{x}(1) + \frac{1}{2} \cdot \boldsymbol{x}(3) \\
\boldsymbol{x}(1) &= \frac{1}{2} \cdot \boldsymbol{x}(0) + \frac{1}{4} \cdot \boldsymbol{x}(2) + \frac{1}{4} \cdot \boldsymbol{x}(5) \\
\boldsymbol{x}(3) &= \frac{1}{3} \cdot \boldsymbol{x}(0) + \frac{1}{3} \cdot \boldsymbol{x}(1) + \frac{1}{3} \cdot \boldsymbol{x}(4)
\end{aligned}$$

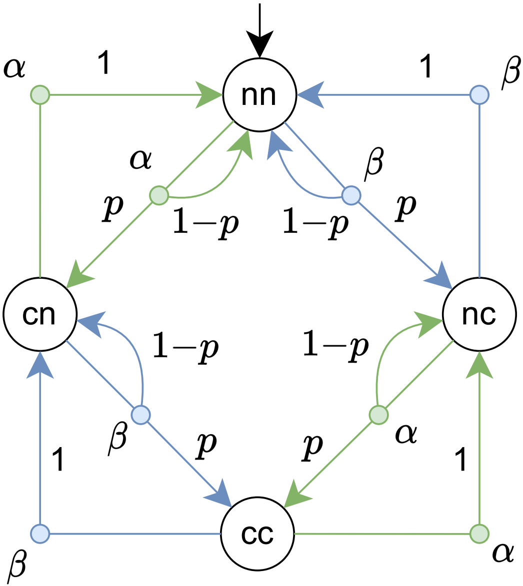
$\Rightarrow P(0 \to RC_1) = 1/3$ and $P(1 \to RC_1) = 5/12$. Then,
$P(0 \to RC_2) = 1 - P(0 \to RC_1) = 2/3$ and $P(1 \to RC_2) = 7/12$.
d)$\boldsymbol{t}^0_\infty = P(0 \to RC_1) \cdot \boldsymbol{\pi}^{RC_1} + P(0 \to RC_2) \cdot \boldsymbol{\pi}^{RC_2} = \left(0, 0, \frac{1}{3}, 0, \frac{2}{9}, \frac{4}{9}\right)$
e)$\boldsymbol{t}^1_\infty = P(1 \to RC_1) \cdot \boldsymbol{\pi}^{RC_1} + P(1 \to RC_2) \cdot \boldsymbol{\pi}^{RC_2} = \left(0, 0, \frac{5}{12}, 0, \frac{7}{36}, \frac{14}{36}\right)$

Exercise 2. During each time step, a process enters a critical section (CS) with probability p $(0 < p < 1)$. Once in the CS, the process leaves it before the next time step.

1. Model such a process as a Markov chain.
2. Model a concurrent execution of two such processes on a single-core CPU as a Markov decision process. Non-deterministic actions correspond to the choice of the process to be run on a CPU during the next time step.

Solution





Exercise 2 (cont.). We say that the scheduler guarantees

- safety: the two processes can never both be in the CS
- liveness: with probability 1, every process can enter and leave CS infinitely often

3. Find scheduler that minimizes/maximizes probability of both processes being in the CS.
4. Show that no deterministic memoryless scheduler can guarantee safety and liveness.
5. Find a randomized memoryless scheduler that guarantees safety and liveness.
6. Find a deterministic non-memoryless scheduler that guarantees safety and liveness.

Solution

❸ • minimizing scheduler: $\sigma_{\min}(nn) = \sigma_{\min}(cn) = \alpha$.

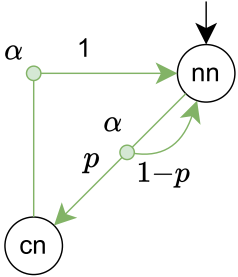


Figure: DTMC $\mathcal{M}_{\sigma_{\min}}$ induced by $\sigma_{\min}$

$P_{\sigma_{\min}}(nn \to \{cc\}) = 0$

Solution

❸ • maximizing scheduler: $\sigma_{\mathsf{max}}(nn) = \alpha,\ \sigma_{\mathsf{max}}(cn) = \beta,\ \sigma_{\mathsf{max}}(cc) = *$

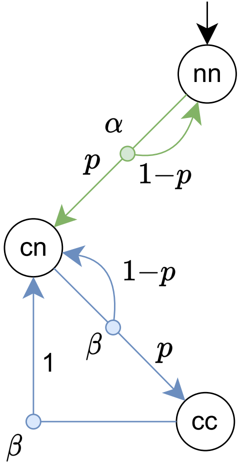
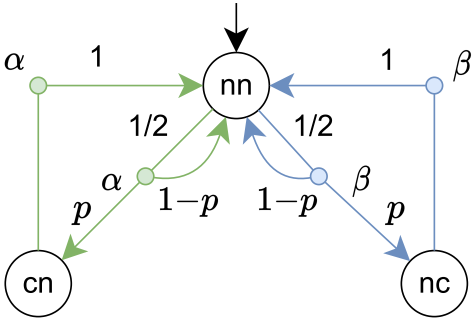


Figure: DTMC $\mathcal{M}_{\sigma_{\mathsf{max}}}$ induced by σ_{max}

Solution

4. hint: show using scheduler enumeration
5. gist: when neither of the processes is in the CS, randomize which one to schedule

$$\sigma(nn) = 1/2 : \alpha + 1/2 : \beta,\ \sigma(cn) = 1 : \alpha,\ \sigma(nc) = 1 : \beta$$

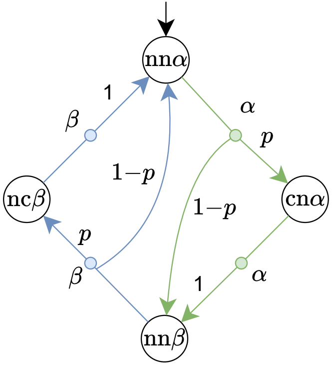


Solution

❻ gist: alternate the scheduling of the two processes

$$\sigma(nn) = \sigma\left([S\ Act]^* cn\right) = \sigma\left(S\ [Act\ S]^* \beta nn\right) = \alpha$$
$$\sigma\left([S\ Act]^* nc\right) = \sigma\left(S\ [Act\ S]^* \alpha nn\right) = \beta$$

to construct the induced DTMC, introduce additional variable (memory) encoding "*which process has the CS priority*"

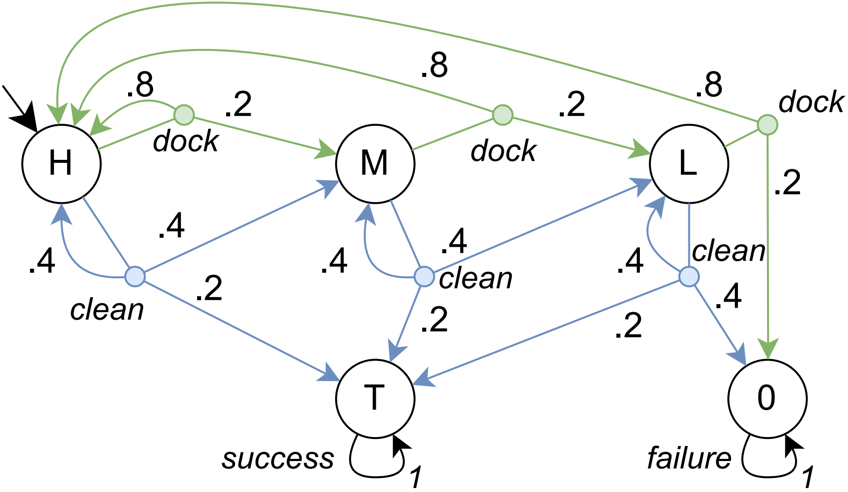


Exercise 3. Assume a cleaning robot has four distinct battery levels: high, medium, low, and zero. During each step, the robot can decide whether to resume cleaning or return to the charging dock.

- When the robot decides to resume cleaning, it will successfully finish the task with probability $p_{clean} = 0.2$. With each cleaning attempt, regardless of the success, the battery will be either drained by one level (with probability $p_{drain} = 0.5$) or remain the same. When the battery level drops to zero, the robot is cannot continue.
- If the robot decides to charge, it will succeed in finding the dock with probability $p_{dock} = 0.8$, fully charging its battery. Otherwise, the robot gets lost and loses one battery level.

1. Find a strategy that maximizes the robot's chances of cleaning the room before running out of battery.
   - Use all three techniques presented in the lecture (scheduler enumeration, LP, value iteration)

Solution


.8
.2
dock
.8
.2
dock
.8
dock
.2
H
M
L
.4
.4
.4
.4
clean
.4
clean
.4
.4
clean
.2
.2
.2
clean
T
0
success
1
failure
1

Solution

- scheduler enumeration = investigate all 8 possible schedulers:

$$\sigma_1 = \{H \mapsto \text{dock},\ M \mapsto \text{dock},\ L \mapsto \text{dock},\}, P_{\sigma_1}(H \to \{T\}) = 0$$
$$\sigma_2 = \{H \mapsto \text{dock},\ M \mapsto \text{dock},\ L \mapsto \text{clean},\}, P_{\sigma_2}(H \to \{T\}) \approx .333$$
$$\sigma_3 = \{H \mapsto \text{dock},\ M \mapsto \text{clean},\ L \mapsto \text{dock},\}, P_{\sigma_3}(H \to \{T\}) \approx .714$$
$$\sigma_4 = \{H \mapsto \text{dock},\ M \mapsto \text{clean},\ L \mapsto \text{clean},\}, P_{\sigma_4}(H \to \{T\}) \approx .555$$
$$\sigma_5 = \{H \mapsto \text{clean},\ M \mapsto \text{dock},\ L \mapsto \text{dock},\}, P_{\sigma_5}(H \to \{T\}) \approx .925$$
$$\sigma_6 = \{H \mapsto \text{clean},\ M \mapsto \text{dock},\ L \mapsto \text{clean},\}, P_{\sigma_6}(H \to \{T\}) \approx .809$$
$$\sigma_7 = \{H \mapsto \text{clean},\ M \mapsto \text{clean},\ L \mapsto \text{dock},\}, P_{\sigma_7}(H \to \{T\}) \approx .862$$
$$\sigma_8 = \{H \mapsto \text{clean},\ M \mapsto \text{clean},\ L \mapsto \text{clean},\}, P_{\sigma_8}(H \to \{T\}) \approx .703$$

$$\Rightarrow \sigma_{\max} = \sigma_5$$

Solution

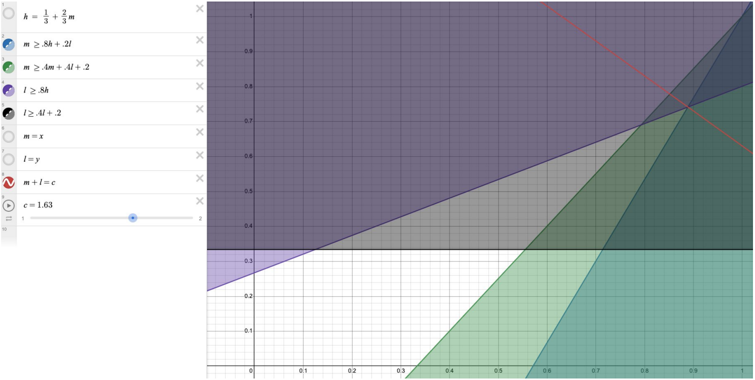
- using linear programming:
  $S_1 = \{T\} \;\Rightarrow\; \boldsymbol{x}(T) = 1 \quad S_0 = \{0\} \;\Rightarrow\; \boldsymbol{x}(0) = 0$
  - solve the following LP: minimize $\boldsymbol{x}(H) + \boldsymbol{x}(M) + \boldsymbol{x}(L)$ subject to

$$
\begin{aligned}
\boldsymbol{x}(H) &\geq 0.8 \cdot \boldsymbol{x}(H) + 0.2 \cdot \boldsymbol{x}(M) && \textit{h\_dock}\\
\boldsymbol{x}(H) &\geq 0.4 \cdot \boldsymbol{x}(H) + 0.4 \cdot \boldsymbol{x}(M) + 0.2 \cdot \boldsymbol{x}(T) && \textit{h\_clean}\\
\boldsymbol{x}(M) &\geq 0.8 \cdot \boldsymbol{x}(H) + 0.2 \cdot \boldsymbol{x}(L) && \textit{m\_dock}\\
\boldsymbol{x}(M) &\geq 0.4 \cdot \boldsymbol{x}(M) + 0.4 \cdot \boldsymbol{x}(L) + 0.2 \cdot \boldsymbol{x}(T) && \textit{m\_clean}\\
\boldsymbol{x}(L) &\geq 0.8 \cdot \boldsymbol{x}(H) + 0.2 \cdot \boldsymbol{x}(0) && \textit{l\_dock}\\
\boldsymbol{x}(L) &\geq 0.4 \cdot \boldsymbol{x}(L) + 0.4 \cdot \boldsymbol{x}(0) + 0.2 \cdot \boldsymbol{x}(T) && \textit{l\_clean}\\
0 \leq \boldsymbol{x}(H)&, \boldsymbol{x}(M), \boldsymbol{x}(L) \leq 1
\end{aligned}
$$

  $\Rightarrow \boldsymbol{x}(H) \approx 0.925,\; \boldsymbol{x}(M) \approx 0.888,\; \boldsymbol{x}(L) \approx 0.740$
  - to deduce the optimizing scheduler, plug $\boldsymbol{x}$ into constraints above and see that constraints *h_clean*, *m_dock*, *l_dock* yield equality (at lower bound, slack=0): $\Rightarrow \sigma_{\text{max}} = \{H \mapsto \text{clean},\; M \mapsto \text{dock},\; L \mapsto \text{dock}\}$
  - try out this linear program in LP solver

Solution

- graphical solution using graphing calculator:
  assume $\sigma(H) = \mathit{clean}$ (why?), then
  $\boldsymbol{x}(H) = 0.4 \cdot \boldsymbol{x}(H) + 0.4 \cdot \boldsymbol{x}(M) + 0.2 \Rightarrow \boldsymbol{x}(H) = \frac{1}{3} + \frac{2}{3} \cdot \boldsymbol{x}(M)$
  let $h := \boldsymbol{x}(H),\ x = m := \boldsymbol{x}(M),\ y = l := \boldsymbol{x}(L)$



- value iteration: see matlab/python script

Exercise 4 (homework). Engineers designed a budget version of the cleaning robot with a weaker navigation system, which allows the robot to find the charging dock with probability $p_{dock} = 0.5$ during each attempt.

1. Find a strategy that maximizes the robot's chances of cleaning the room before running out of battery.

Solution: $\sigma_{\max}(H) = \sigma_{\max}(M) = \mathrm{clean}$, $\sigma_{\max}(L) = \mathrm{dock}$, $P_{\max} \approx 0.714$