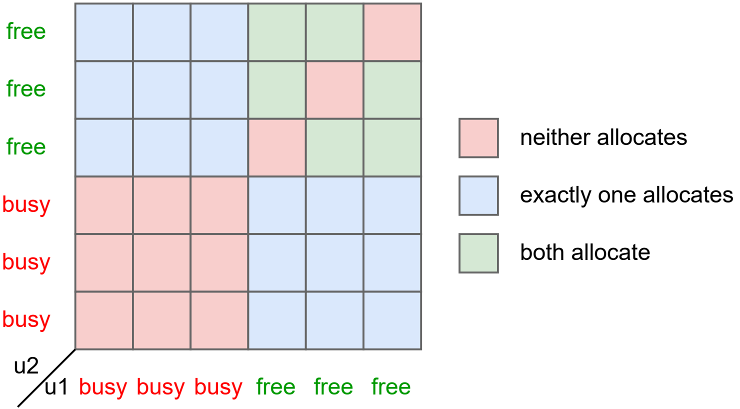
MSP 2024 - Tutorial 1

Exercise 1. A service provider manages a frequency bandwidth divided into 6 channels that can be used for communication. A user that attempts to establish a new connection randomly and uniformly picks the channel; if the channel is free, it is allocated to the user, otherwise the user repeats the attempt. Assume that one attempt takes roughly 1ms.
Assume that currently 3 out of 6 channels are busy and that two users simultaneously attempt to allocate free channels. If both users randomly pick the same free channel, they both restart the procedure.

1. Construct the Markov chain that models such a system.
2. Compute the probability that it takes up to 3ms to find free channels for both users.
3. Compute the probability that it takes exactly 3ms to find free channels for both users.
4. Compute the expected time required to find free channels for both users.

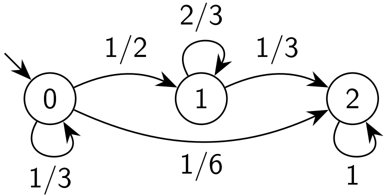
Solution

1. Decision diagram for state 0:



Solution

1. State of the DTMC encodes how many users have the channel allocated.



Solution

2. $P(2 \text{ channels are allocated up to time } 3) = P(X(3) = 2) = \boldsymbol{t}_3(2)$:

$$
\begin{aligned}
\boldsymbol{t}_0(0, 1, 2) &= (1, 0, 0) \\
\boldsymbol{t}_1 = \boldsymbol{t}_0 \cdot \boldsymbol{P} &= \left(\frac{1}{3}, \frac{1}{2}, \frac{1}{6}\right) \\
\boldsymbol{t}_2 = \boldsymbol{t}_1 \cdot \boldsymbol{P} &= \left(\frac{1}{9}, \frac{1}{2}, \frac{7}{18}\right) \\
\boldsymbol{t}_3 = \boldsymbol{t}_2 \cdot \boldsymbol{P} &= \left(\frac{1}{27}, \frac{7}{18}, \frac{31}{54}\right)
\end{aligned}
$$

$\Rightarrow \boldsymbol{t}_3(2) = \frac{31}{54}$

Solution

3

$$
\begin{aligned}
&P(\text{two channels are allocated exactly at time 3}) = P(X(2) \neq 2, X(3) = 2) \\
&\quad = P(X(2) = 0, X(3) = 2) + P(X(2) = 1, X(3) = 2) \\
&\quad = P(X(2) = 0) \cdot P(X(3) = 2 \mid X(2) = 0) \\
&\qquad + P(X(2) = 1) \cdot P(X(3) = 2 \mid X(2) = 1) \\
&\quad = \boldsymbol{t}_2(0) \cdot \boldsymbol{P}(0,2) + \boldsymbol{t}_2(1) \cdot \boldsymbol{P}(1,2) = \frac{1}{9} \cdot \frac{1}{6} + \frac{1}{2} \cdot \frac{1}{3} = \frac{5}{27}
\end{aligned}
$$

Coincidentally, since state 2 is absorbing, $P(X(2) \neq 2, X(3) = 2)$ is also equal to $\boldsymbol{t}_3(2) - \boldsymbol{t}_2(2)$.

Solution

4. Let $T = \{2\}$ be the set of target states and let $\boldsymbol{e}(s)$ denote the expected number of steps (milliseconds) to reach T from s.
$P(0 \rightarrow T) = 1$ since the DTMC contains only one (reachable) BSCC and this BSCC contains state 2, thus, $\boldsymbol{e}(0)$ is defined and is obtained by solving the following system:

$$\begin{aligned}
\boldsymbol{e}(0) &= 1 + \frac{1}{3}\boldsymbol{e}(0) + \frac{1}{2}\boldsymbol{e}(1) + \frac{1}{6}\boldsymbol{e}(2) \\
\boldsymbol{e}(1) &= 1 + \frac{2}{3}\boldsymbol{e}(1) + \frac{1}{3}\boldsymbol{e}(2) \\
\boldsymbol{e}(2) &= 0
\end{aligned}$$

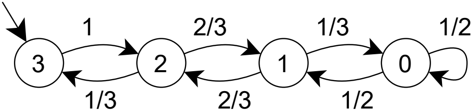
$\Rightarrow \boldsymbol{e}(0) = 3.75$ ms

Exercise 2. Assume a cluster server with three machines. Initially, all machines are on. Every hour, a technician randomly and uniformly chooses a machine to work with: a running machine is shut down for maintenance and an offline machine is turned back on. However, when all machines are off, the probability that the selected machine is restarted is reduced to 50%. The cluster as a whole is considered online if at least one machine is running.

1. Construct the Markov chain that models such a system.
2. Compute the probability that after 4 hours the server is online.
3. Assume that initially all machines are off and the technician is 2 hours late on their first day. Modify the model and recompute the probability from 2.

Solution

1. State of the DTMC encodes the number of running machines.

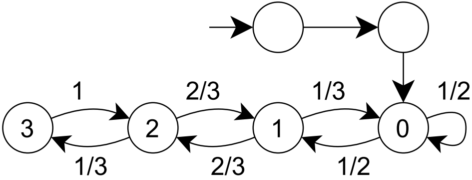


2.

$$\boldsymbol{t}_0(3,2,1,0) = (1,0,0,0)$$
$$\boldsymbol{t}_1 = \boldsymbol{t}_0 \cdot \boldsymbol{P} = (0,1,0,0)$$
$$\boldsymbol{t}_2 = \boldsymbol{t}_1 \cdot \boldsymbol{P} = \left(\frac{1}{3}, 0, \frac{2}{3}, 0\right)$$
$$\boldsymbol{t}_3 = \boldsymbol{t}_2 \cdot \boldsymbol{P} = \left(0, \frac{7}{9}, 0, \frac{2}{9}\right)$$
$$\boldsymbol{t}_4 = \boldsymbol{t}_3 \cdot \boldsymbol{P} = \left(\frac{7}{27}, 0, \frac{17}{27}, \frac{1}{9}\right)$$
$$P(\text{online after 4 hours}) = \boldsymbol{t}_4(1) + \boldsymbol{t}_4(2) + \boldsymbol{t}_4(3) = \frac{8}{9}$$

Solution

3. The Markov chain below models the late arrival



$$\boldsymbol{t}_2(3,2,1,0) = (0,0,0,1)$$

$$\boldsymbol{t}_3 = \boldsymbol{t}_2 \cdot \boldsymbol{P} = \left(0, 0, \frac{1}{2}, \frac{1}{2}\right)$$

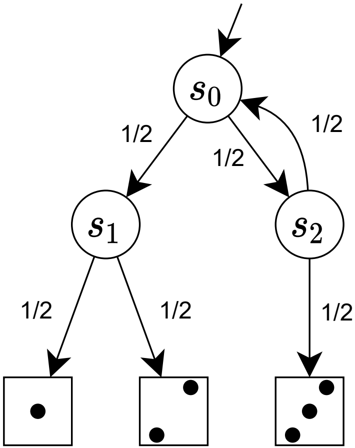
$$\boldsymbol{t}_4 = \boldsymbol{t}_3 \cdot \boldsymbol{P} = \left(0, \frac{1}{3}, \frac{1}{4}, \frac{5}{12}\right)$$

$$P(\text{online after 4 hours}) = \boldsymbol{t}_4(1) + \boldsymbol{t}_4(2) + \boldsymbol{t}_4(3) = \frac{7}{12}$$

Exercise 3. Design a protocol that uses a fair coin to simulate a toss of a 3-sided dice.

1. Verify correctness of your protocol.
2. Analyze efficiency of your protocol.
3. Compute the probability that the protocol does not terminate within 4 steps.

Solution



Solution

1. Correctness: $P(s_0 \to \{\epsdice{1}\}) = P(s_0 \to \{\epsdice{2}\}) = P(s_0 \to \{\epsdice{3}\}) = 1/3$?

   - Let $T = \{\epsdice{1}\}$, $\boldsymbol{x}(s) := P(s \to T)$. $S_0 = \{\epsdice{2}, \epsdice{3}\}$

   $$\begin{aligned}
   \boldsymbol{x}(\epsdice{1}) &= 1 \\
   \boldsymbol{x}(\epsdice{2}) &= \boldsymbol{x}(\epsdice{3}) = 0 \\
   \boldsymbol{x}(s_0) &= \frac{1}{2} \cdot \boldsymbol{x}(s_1) + \frac{1}{2} \cdot \boldsymbol{x}(s_2) \\
   \boldsymbol{x}(s_1) &= \frac{1}{2} \cdot \boldsymbol{x}(\epsdice{1}) + \frac{1}{2} \cdot \boldsymbol{x}(\epsdice{2}) \\
   \boldsymbol{x}(s_2) &= \frac{1}{2} \cdot \boldsymbol{x}(s_0) + \frac{1}{2} \cdot \boldsymbol{x}(\epsdice{3})
   \end{aligned}$$

   $\Rightarrow P(s_0 \to \{\epsdice{1}\}) = \boldsymbol{x}(s_0) = 1/3$

   - By symmetry, $P(s_0 \to \{\epsdice{2}\}) = 1/3$ as well.

   - Since $\epsdice{1}$, $\epsdice{2}$ and $\epsdice{3}$ are the only BSCCs, then $P(s_0 \to \{\epsdice{3}\}) = 1 - P(s_0 \to \{\epsdice{1}\}) - P(s_0 \to \{\epsdice{2}\}) = 1/3$

Solution

2. Efficiency = expected number of tosses to execute the protocol. Let $T = \{\text{⚀}, \text{⚁}, \text{⚂}\}$ and let $\boldsymbol{e}(s)$ denote the expected number of steps to reach T from s. Then:

$$
\begin{aligned}
\boldsymbol{e}(\text{⚀}) = \boldsymbol{e}(\text{⚁}) = \boldsymbol{e}(\text{⚂}) &= 0 \\
\boldsymbol{e}(s_0) &= 1 + \frac{1}{2} \cdot \boldsymbol{e}(s_1) + \frac{1}{2} \cdot \boldsymbol{e}(s_2) \\
\boldsymbol{e}(s_1) &= 1 + \frac{1}{2} \cdot \boldsymbol{e}(\text{⚀}) + \frac{1}{2} \cdot \boldsymbol{e}(\text{⚁}) \\
\boldsymbol{e}(s_2) &= 1 + \frac{1}{2} \cdot \boldsymbol{e}(s_0) + \frac{1}{2} \cdot \boldsymbol{e}(\text{⚂})
\end{aligned}
$$

$\Rightarrow \boldsymbol{e}(s_0) = 8/3$

Solution

3.

$$
\begin{aligned}
\boldsymbol{t}_0(s_0, s_1, s_2, \text{⚀}, \text{⚁}, \text{⚂}) &= (1, 0, 0, 0, 0, 0) \\
\boldsymbol{t}_1 = \boldsymbol{t}_0 \cdot \boldsymbol{P} &= \left(0, \frac{1}{2}, \frac{1}{2}, 0, 0, 0\right) \\
\boldsymbol{t}_2 = \boldsymbol{t}_1 \cdot \boldsymbol{P} &= \left(\frac{1}{4}, 0, 0, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \\
\boldsymbol{t}_3 = \boldsymbol{t}_2 \cdot \boldsymbol{P} &= \left(0, \frac{1}{8}, \frac{1}{8}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \\
\boldsymbol{t}_4 = \boldsymbol{t}_3 \cdot \boldsymbol{P} &= \left(\frac{1}{16}, 0, 0, \frac{5}{16}, \frac{5}{16}, \frac{5}{16}\right)
\end{aligned}
$$

$$P(\text{protocol is running after 4 tosses}) = \boldsymbol{t}_4(s_0) + \boldsymbol{t}_4(s_1) + \boldsymbol{t}_4(s_2) = 1/16$$

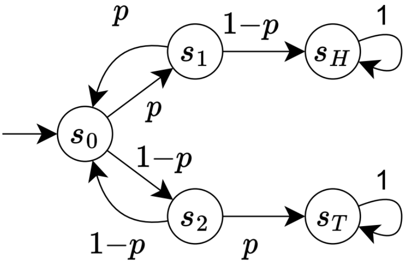
Exercise 4 (homework). Design a protocol that uses a fair coin to simulate a toss of a 5-sided dice. Ensure that your Markov chain does not have unnecessary states/transitions.

1. Verify correctness of your protocol.
2. Analyze efficiency of your protocol.
3. Compute the probability that the protocol produces result 1 or 5 within first 5 steps.

Exercise 5. Design a protocol that uses an unfair coin with bias p to simulate a toss of a fair coin.

1. Verify correctness of your protocol.
2. Compute expected number $t(p)$ of tosses until protocol termination. Plot $t(p)$ and interpret the result.

Solution



Solution

2. Correctness: $P(s_0 \to \{s_H\}) = P(s_0 \to \{s_T\}) = 1/2$.

   Let $T = \{s_H\}$, $\boldsymbol{x}(s) := P(s \to T)$. $S_0 = \{s_T\}$.

$$
\begin{aligned}
\boldsymbol{x}(s_H) &= 1 \\
\boldsymbol{x}(s_T) &= 0 \\
\boldsymbol{x}(s_0) &= p \cdot \boldsymbol{x}(s_1) + (1-p) \cdot \boldsymbol{x}(s_2) \\
\boldsymbol{x}(s_1) &= p \cdot \boldsymbol{x}(s_0) + (1-p) \cdot \boldsymbol{x}(s_H) \\
\boldsymbol{x}(s_2) &= p \cdot \boldsymbol{x}(s_T) + (1-p) \cdot \boldsymbol{x}(s_0)
\end{aligned}
$$

   $\Rightarrow P(s_0 \to \{s_H\}) = \boldsymbol{x}(s_0) = 1/2$. Since s_H and s_T are the only BSCCs, it must hold that $P(s_0 \to \{s_T\}) = 1 - P(s_0 \to \{s_H\}) = 1/2$.

Solution

2. Let $T = \{s_H, s_T\}$ and let $\boldsymbol{e}(s)$ denote the expected number of steps to reach T from s. Then:

$$\begin{aligned}
\boldsymbol{e}(s_H) &= \boldsymbol{e}(s_T) = 0 \\
\boldsymbol{e}(s_0) &= 1 + p \cdot \boldsymbol{e}(s_1) + (1-p) \cdot \boldsymbol{e}(s_2) \\
\boldsymbol{e}(s_1) &= 1 + p \cdot \boldsymbol{e}(s_0) + (1-p) \cdot \boldsymbol{e}(s_H) \\
\boldsymbol{e}(s_2) &= 1 + p \cdot \boldsymbol{e}(s_T) + (1-p) \cdot \boldsymbol{e}(s_0)
\end{aligned}$$

$\Rightarrow t(p) = \boldsymbol{e}(s_0) = \frac{1}{p(1-p)}$

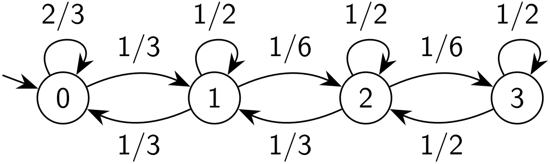
Exercise 6 (2022 midterm test). Consider a router that processes and forwards incoming packets. Every millisecond, the following *independent* events may occur:

- With probability $1/3$, there is an incoming packet that is stored in the buffer. The buffer can store up to 3 packets and is initially empty. If the buffer is full at the beginning of the millisecond, the incoming packet is lost.
- If the buffer is not empty at the beginning of the millisecond, with probability $1/2$ the router processes and forwards the first packet in the buffer.

1. Construct the Markov chain that models such a system.
2. Compute the probability that after 3ms the buffer contains at least 2 packets.
3. Compute the expected time until the buffer becomes full.

Solution

1. State of the DTMC encodes the number of packets in the buffer.



Solution

2

$$\boldsymbol{t}_0(0,1,2,3) = (1,0,0,0)$$
$$\boldsymbol{t}_1 = \left(\frac{2}{3}, \frac{1}{3}, 0, 0\right)$$
$$\boldsymbol{t}_2 = \left(\frac{5}{9}, \frac{7}{18}, \frac{1}{18}, 0\right)$$
$$\boldsymbol{t}_3 = \left(\frac{1}{2}, \frac{43}{108}, \frac{10}{108}, \frac{1}{108}\right)$$

$$P(2+ \text{ packets after 3ms}) = \boldsymbol{t}_3(2) + \boldsymbol{t}_3(3) = \frac{11}{108}$$

Solution

3. Expected time until the buffer is full = expected number of milliseconds (steps) to reach state 3. Let $T = \{3\}$ be the set of target states and let $\boldsymbol{e}(s)$ denote the expected number of steps to reach T from s.
$P(0 \to T) = 1$ since the DTMC contains no transient states, thus, $\boldsymbol{e}(0)$ is defined and is obtained by solving the following system:

$$\begin{aligned}
\boldsymbol{e}(0) &= 1 + \frac{2}{3} \cdot \boldsymbol{e}(0) + \frac{1}{3} \cdot \boldsymbol{e}(1) \\
\boldsymbol{e}(1) &= 1 + \frac{1}{3} \cdot \boldsymbol{e}(0) + \frac{1}{2} \cdot \boldsymbol{e}(1) + \frac{1}{6} \cdot \boldsymbol{e}(2) \\
\boldsymbol{e}(2) &= 1 + \frac{1}{3} \cdot \boldsymbol{e}(1) + \frac{1}{2} \cdot \boldsymbol{e}(2) + \frac{1}{6} \cdot \boldsymbol{e}(3) \\
\boldsymbol{e}(3) &= 0
\end{aligned}$$

$\Rightarrow \boldsymbol{e}(0) = 45$ ms