Signals and Systems HW Week13 Reference Solutions

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Write at the beginning:

This reference solutions are used for better understanding of these principles of signals and systems within the problems. The purpose is not only to complete the homework, but also improve your understanding.

In view of that most of you have complained about the problems in English quite confusing and incomprehensible, I tried to explain how the problem is organized and what is the internal logic of it. I hope this will help you not only understand these problems but also be used to the expression habits of English. The latter one will be of great use in your future study and research works.

I will give you a Chinese version if there is something critical. The expression of Chinese will be like [这个就是一个中文解释的案例，我会将中文解释的部分放在这样的一个中括号里]。For a Chinese paragraph, I will use this pattern

[

这是一个中文段落解释的案例，中括号会在中文段落前后的行首。

]

For better systematic comprehension, I will add some other principles and concepts that I think is relevant to this problem. You can have a look at them and if there is something not clear you can contact me or refer to your books.

5.39

Original problem is:

Explain what have done to the signals.

Twenty-four voice signals are sampled uniformly and are then time-division multiplexed, using PAM. The PAM signal is reconstructed from flat-topped pluses with $1\ \mu s$ duration. The multiplexing operation provides for synchronization by adding an extra pulse of sufficient amplitude and also $1\ \mu s$ duration. The highest frequency component of each voice signal is 3.4 kHz.

Specify the PAM. How PAM is done.

Additional expression for multiplexing.

Extra information of the voice signal.

(a) Assuming a sampling rate of 8 kHz, calculate the spacing between successive pulse of the multiplexed signal.
(b) Repeat your calculation, assuming the use of Nyquist rate sampling.

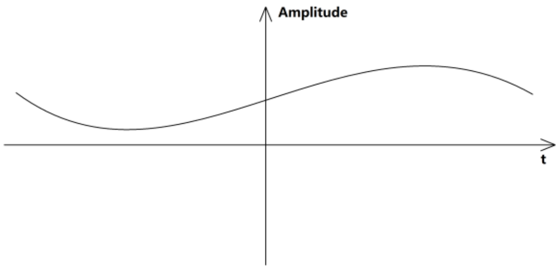
Problem description ends.

General View:

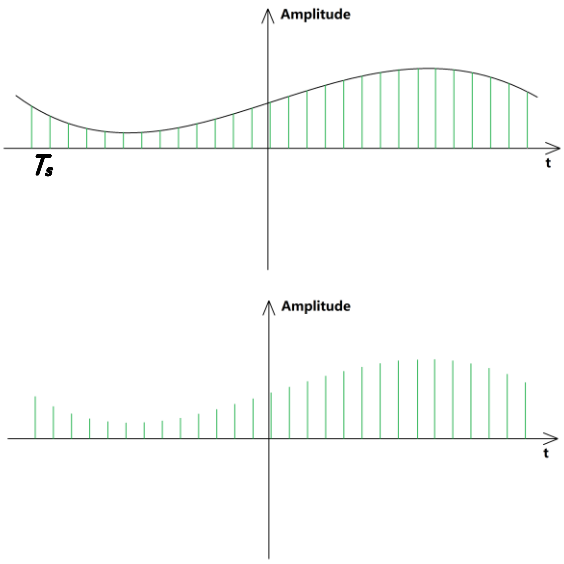
This problem is not quite hard. The purpose of the problem is to find the spacing of the nearest pulse when using time-division multiplex.

Here is how the whole process be like

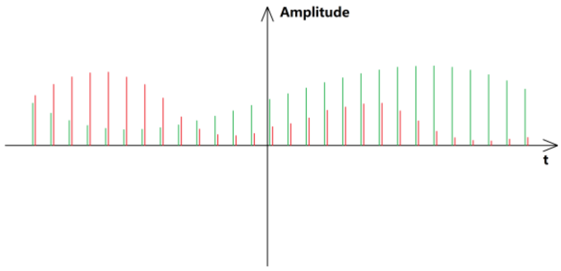
Assuming each signal of the 24 voice signals, the original continuous-time analog signal is



Then each of the signal is sampled with desired sampling rate, for (a) is 8 kHz and for (b) is two times the highest frequency the voice signal contains. The green lines train are the discrete-time analog signal, with interval $T_s = \frac{1}{f_s}$



And for the time-division multiplex by using PAM, we can directly combine the signals with a little time shift. For example, there is another sampled signal, shown in red, by shift the red signal a bit and combined with green signal, we have



Also, for the rest 22 voice signals and the extra synchronization signal, we can do the same thing.

I also write a simple MATLAB script for demonstration, and the source code is attached to the end of this document.
Here is an example if there are 4 voice signals and 1 extra synchronization signal.
The four signals are

```
x1 = cos(2*pi*n);
x2 = cos(2*pi*2*n);
x3 = cos(2*pi*3*n);
x4 = cos(2*pi*4*n);
```

and we only take $-0.5s < t < 0.5s$ interval for demonstration.
Fig 5.39.1 shows the original four continuous-time analog signals.

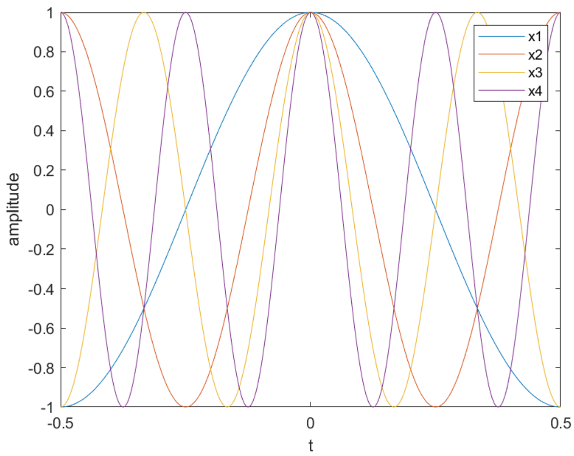


Fig 5.39.1

We sample these continuous-time analog signals and get discrete-time analog signals.
Fig 5.39.2 shows the sampled signals, but we still in continuous-time domain. Because we use a continuous-time sampling signal to do the sampling.
Fig 5.39.3 shows the sampled signals in discrete-time domain. We just take the non-zero value from above to each discrete point.
[就是说图 5.39.2 仍然是连续信号，横坐标仍然为时间，在非采样点都为零，我们提取所有的非零值并依次赋值给每一个离散点就得到了图 5.39.3。]

Note that the only difference between Fig 5.39.2 and Fig 5.39.3 is the meaning and unit of x-axis.

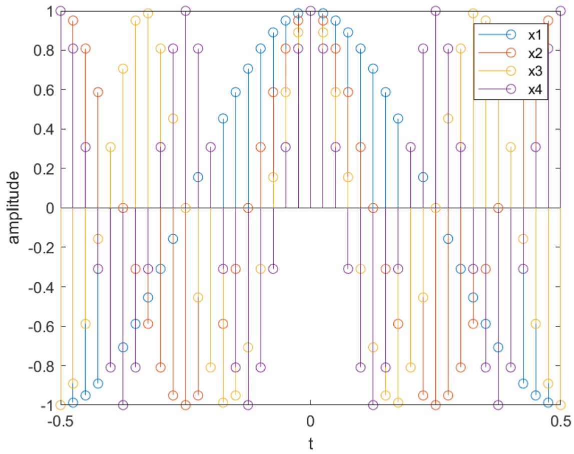


Fig 5.39.2

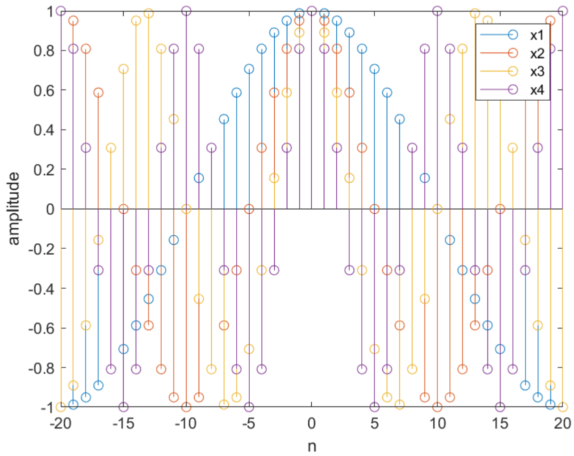


Fig 5.39.3

Now for time-division multiplex, we take each discrete-time analog signal at a time and put it on a discrete point. Assume the synchronization signal is $x[n] = 2$.

[

我们依次从原来的四个信号和同步信号中抽取一个值，并依次放在新信号的离散点上，可以得到如下的图像

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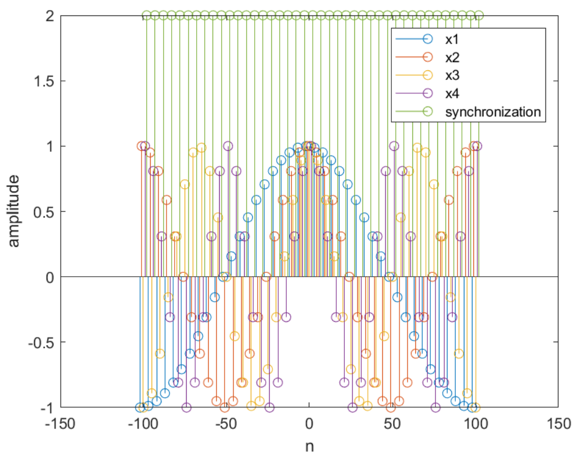


Fig 5.39.4 (if you do not see this clear you can run the code below)

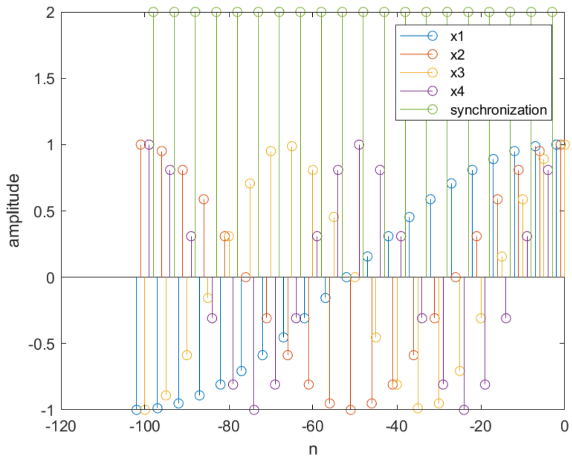


Fig 5.39.5 Enlarged one of Fig 5.39.4

If we do not use color to distinguish the five signals (include synchronization signal), we have

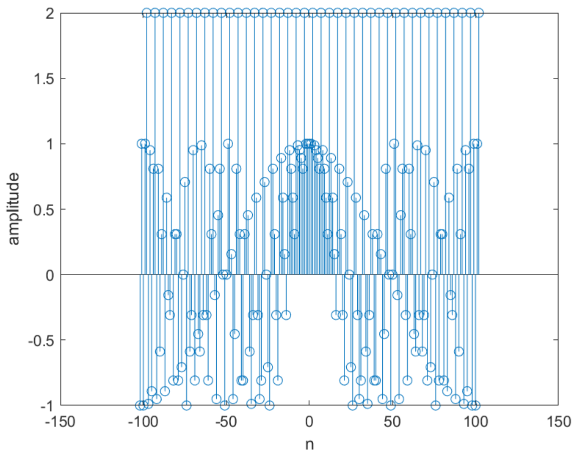


Fig 5.39.6

Now it is time to get the modulated signal by PAM (Pulse Amplitude Modulation), we use a flat-topped pulse with duration $1\ \mu s$ to represent one point in the discrete-time analog signal, and scale the pulse train to time interval $-0.5s < t < 0.5s$ (The $1\ \mu s$ flat-topped pulse is hard to draw, I still use ideal pulse to demonstrate). This is the final modulated signal.

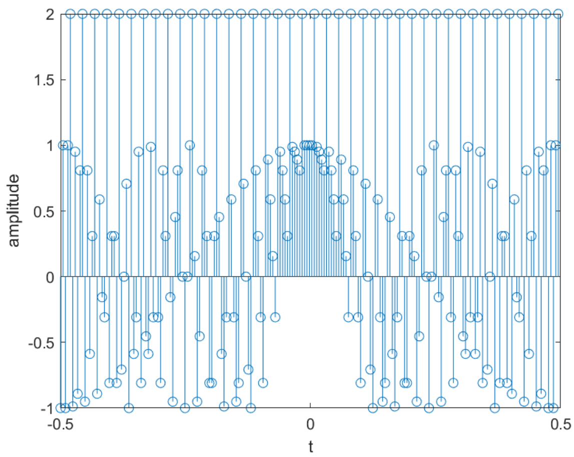


Fig 5.39.7

Note that for real-time transmission, the transmission speed should at least be equal to the speed the voice signal being generated. This is why in the last step in modulation, we need to scale the flat-topped pulse train to time-interval $-0.5s < t < 0.5s$. In other words. If the discrete-time voice signal is generated each t_s s (That is the sampling time interval, each time we get a discrete-time value the time elapses t_s, where $t_s = \frac{1}{f_s}$ and f_s is the sampling rate), and we have 25 signals, then the modulated signal should have 25 flat-topped pulses in the time interval t_s. This is the core of this problem. When we know the pulse number and total time interval, the time spacing between the two successive pulses can be easily got by just divide them.

[

对于实时通信系统来说，调制之后传输的信号的传输速度要至少与语音信号生成的速度一致。这也就是为什么我们在调制的最后一步需要把时间尺度调整至$-0.5s < t < 0.5s$。换句话说，如果离散语音信号生成的速度是t_s（这个就是采样间隔），如果我们有 25 个信号，并且这 25 个信号要实时发送出去，那么在一个采样间隔t_s内我们最终调制的信号需要包含 25 个平顶冲激。这也就是这道题的核心。当我们知道了调制后的时分复用的时间间隔以及在间隔内的冲激串的冲激的数量，那我们可以通过简单的相除来得到两个连续相邻冲激的时间间隔。

]

Note that the synchronization signal is distinguishing enough for us to distinguish each sampling moment. This is why we need such a synchronization signal.

[

注意到这个同步信号可以明显区分开每一个采样时刻。我们可以认为这 24 个语音信号是在同一时刻采样，只是发送的时候被调制到不同的时分复用的时间地址上，但本质上还是同一时刻采样的不同语音信号。而每一个同步信号前后都是这 24 个语音信号两个相邻时刻的采样结果。

]

Solution:

(a)

For the sampling rate is 8 kHz we have the sampling interval

$$T_s = \frac{1}{8 \times 10^3} = 125\mu s$$

For there are 24 voice signals and an extra pulse, thus there are totally 25 signals. The time interval between two successive ideal pulses is

$$T_i = \frac{T_s}{25} = 5\ \mu s$$

Note that each pulse is flat-topped pulse with 1 μs duration, thus the time spacing between two successive modulated flat-topped pulses is

$$T_{spacing} = T_i - T_d = 5 - 1 = 4\ \mu s$$

(b)

If the sampling rate is Nyquist rate, the sampling rate should be twice the highest frequency component, which is 3.4 kHz here. That is $f_{s2} = 2 \times 3.4 = 6.8$ kHz, thus the sampling interval is

$$T_{s2} = \frac{1}{6.8 \times 10^3} = 147\mu s$$

Hence

$$T_{i2} = \frac{T_{s2}}{25} = 5.88\ \mu s$$

$$T_{spacing2} = T_{i2} - T_d = 5.88 - 1 = 4.88\ \mu s$$

Addition questions:

In (a) and (b) respectively, how many different voice signals at most can we transmit at the same time? Assume that to distinguish two pulses we need the spacing between two successive pulses at least $1\ \mu s$.

For (a), the time interval for each sampling moment is $T_s = 125\mu s$, for each pulse has a duration of $1\ \mu s$ and $1\ \mu s$ time spacing to the next one, thus the maximum pulse number we can get is

$$N_m = floor\left(\frac{T_s}{2}\right) = floor\left(\frac{125}{2}\right) = 62$$

Besides the synchronization signal, there are 61 signals at most we can transmit at the same time.

For (b), this is similar

$$N_{m2} = floor\left(\frac{T_{s2}}{2}\right) = floor\left(\frac{147}{2}\right) = 73$$

Besides the synchronization signal, there are 72 signals at most we can transmit at the same time.

5.48

Original problem is:

The first step of what we should do

Consider a multiplex system in which four input signals $m_1(t)$, $m_2(t)$, $m_3(t)$, and $m_4(t)$ are respectively multiplied by the carrier waves

$$[\cos(\omega_a t) + \cos(\omega_b t)],$$
$$[\cos(\omega_a t + \alpha_1) + \cos(\omega_b t + \beta_1)],$$
$$[\cos(\omega_a t + \alpha_2) + \cos(\omega_b t + \beta_2)],$$

and

$$[\cos(\omega_a t + \alpha_3) + \cos(\omega_b t + \beta_3)],$$

The second step of what we should do

and the resulting DSB-SC signals are summed and then transmitted over a common channel. In the receiver, demodulation is achieved by multiplying the sum of the DSB-SC signals by the four carrier waves separately and then using filtering to remove the unwanted components. Determine the conditions that the phase angles α_1, α_2, α_1 and β_1, β_2, β_3 must satisfy in order that the output of the kth demodulator be $m_k(t)$, where $k = 1, 2, 3, 4$.

Our purpose in this problem

Problem description ends.

General view:

This problem is not very hard if you have a sense of phase and be aware that not only the frequency can contain information, the phase can also carry information. The modulation and demodulation procedures are given to you and the purpose of the problem is to designate correct phase angles to realize the demodulation.

Here is how the whole procedure be like

We have 4 message signals, they are $m_1(t)$, $m_2(t)$, $m_3(t)$, and $m_4(t)$. And they are respectively multiplied by the carrier waves descripted in the problem. Finally we get 4 DSB-SC signals, they are

$$x_1(t) = m_1(t)[\cos(\omega_a t) + \cos(\omega_b t)]$$
$$x_2(t) = m_2(t)[\cos(\omega_a t + \alpha_1) + \cos(\omega_b t + \beta_1)]$$
$$x_3(t) = m_3(t)[\cos(\omega_a t + \alpha_2) + \cos(\omega_b t + \beta_2)]$$
$$x_4(t) = m_4(t)[\cos(\omega_a t + \alpha_3) + \cos(\omega_b t + \beta_3)]$$

And then they are all summed for they will be all transmitted over a common channel at the same time. The transmission signal is

$$s(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t) = \sum_{i=1}^{4} m_i(t)\,[\cos(\omega_a t + \alpha_{i-1}) + \cos(\omega_b t + \beta_{i-1})]$$

Where $\alpha_0 = \beta_0 = 0.$

Additional explanation for channel. What is channel?

When we need to transmit a signal or signals from transmitter to receiver, there is media the signal passing through. For wireless communication, the media is the space. But because there are blocks (walls or floors), there might be several rays from transmitter to receive, resulting several transmission paths with different path length and attenuation (This will be explained in detail if you further learn communication principle and wireless communication). Abstractly we can consider the channel as system. When transmitter transmits signals out, the signals pass through the channel and finally be received by the receiver. This process can be equivalent to a signal is sent to a system and then be given at the output of the system. The input of the system is the signal transmitter transmits, and the output of the system is the signal the receiver receives. In this problem, the channel is ideal and this ideal channel is equivalent to the system with the unit impulse response $h(t) = \delta(t)$. If we have four independent ideal channels, which means we have four independent ideal systems, the four message signals can be transmitted separately and independently and we do not need to use such a modulation procedure. But because all the four messages have to pass through the same channel at the same time, which means they all pass through the same system at the same time, we need some methods to distinguish them otherwise the signal will be aliased. The modulation procedure helps us to take more use of the channel, or the system frequency or phase resources.

[

补充一些关于信道的信息，什么是信道?

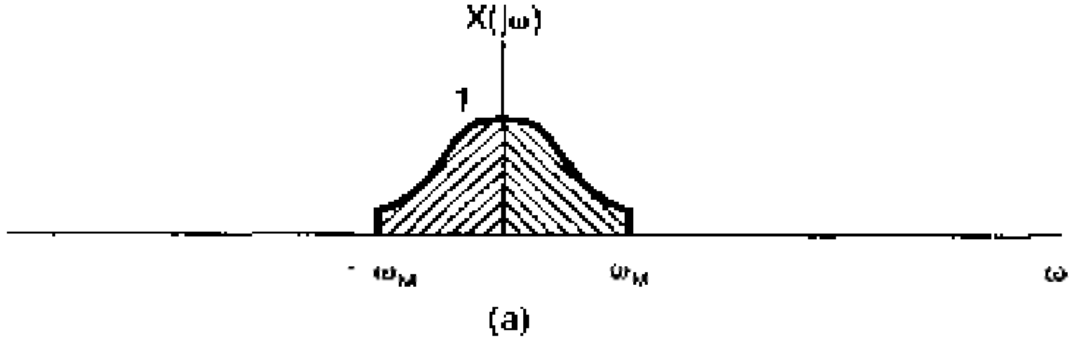
当我们需要将一个信号或一些信号从发射机传输到接收机时，这些信号会通过一些媒介。对于无线通信来说，媒体就是空间。但由于空间中有障碍物（比如墙壁或者底板等），从发射机到接收机可能会有多条射线，每一条射线可以代表一种传输路径，这也就意味着多条传输路径，并且路径长度和衰减大概率不同(如果你进一步学习通信原理和无线通信的话，你会发现这些内容会在那里被详细解释)。如果我们抽象信道，我们可以把信道看作一个系统。当发射机发出信号时，信号经过信道，最后被接收机接收。这个过程可以等价于一个信号被发送进一个系统，然后在系统的输出端输出。系统的输入是发射机发射的信号，系统的输出是接收机接收的信号，系统就是信道。在这个问题中，信道是理想的，这个理想信道等价于单位脉冲响应为$h(t) = \delta(t)$的理想系统。如果我们有四个独立的理想信道，这意味着我们有四个独立的理想系统，这四个消息信号可以分别独立地传输，我们不需要使用这道题中所说的调制步骤。但是因为所有四个消息必须同时通过同一个通道，这意味着它们都同时通过同一个系统，所以我们需要一些方法来区分它们，否则信号将会被混淆，因此这里的调制方式可以帮助我们充分利用这个信道，或者说系统的频率、相位资源。

]

Additional explanation for DSB-SC (Double Side Band, Suspend Carrier).

There are some other signals named DSB-WC, SSB-SC, SSB-WC. The detailed information can be found in section 8.4 on page 430 for Chinese version and page 597 for English version.

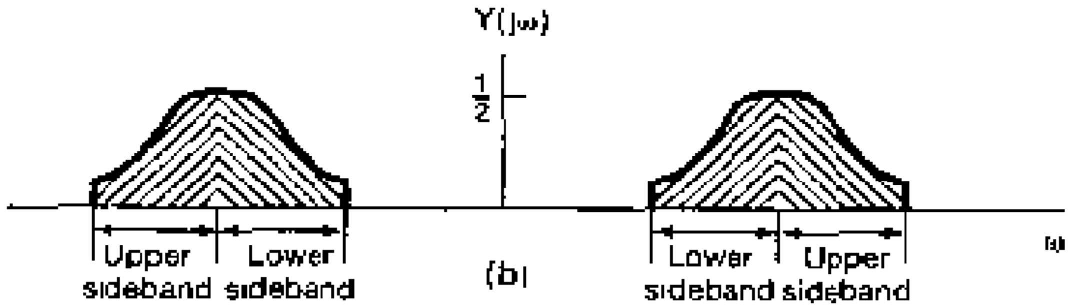
Assume a signal $x(t)$, its spectrum is



Then if we multiple it with another signal $\cos(\omega_c t)$. We get

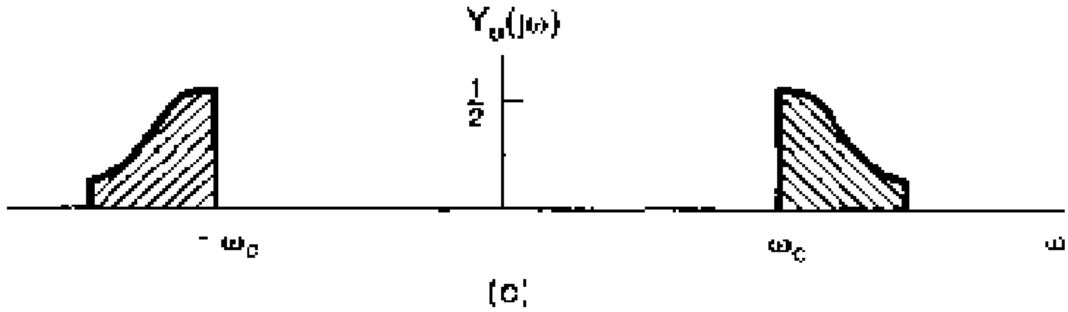
$$y(t) = x(t)\cos(\omega_c t)$$

With its spectrum

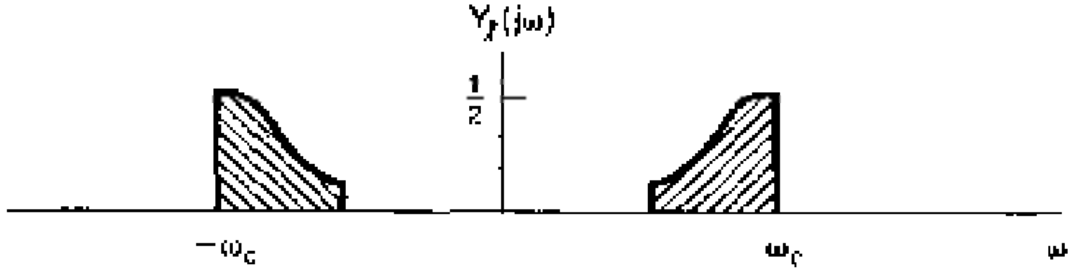


This is what we call DSB-SC, the two copies of the spectrum have the whole information, both lower sideband and upper sideband, thus we call this Double Side-Band. SC will be discussed later.

And if we use an ideal high pass filter with cutoff frequency $f_{cutoff} = \omega_c$, we have



In this case the signal only contains the upper sideband.
Similarly, if we use a low pass filter with cutoff frequency $f_{cutoff} = \omega_c$, we have



(d)

In this case the signal only contains the lower sideband.
Notice that the lower and upper means the frequency.
For these two cases we call them Single Side-Band.

For Suspend Carrier and With Carrier, let's look back at the modulation procedure.

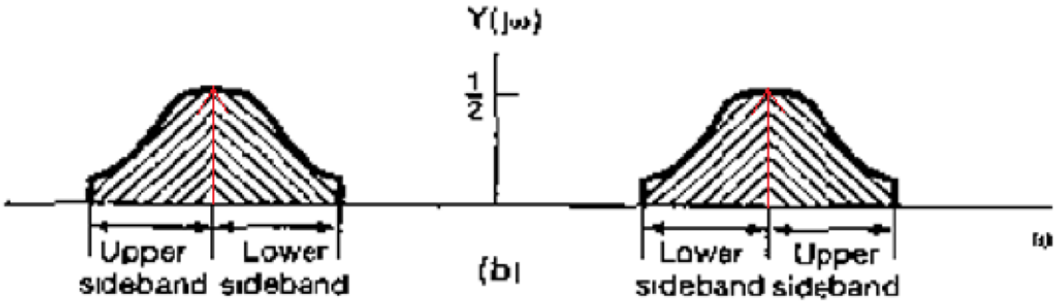
$$y(t) = x(t)\cos(\omega_c t)$$

Where $y(t)$ is the modulated signal and $x(t)$ is a message signal with maximum amplitude 1. You can find that the signal envelope value can be positive and negative and sometimes it is not good for transmission, thus we plus one to the modulation to raise the signal up, which is

$$y_2(t) = [x(t) + 1]\cos(\omega_c t)$$

Then there is no negative signal envelope value for $y_2(t)$.

But there is one problem in the spectrum. Shown in red, the spectrum of the carrier will also be added in one of the final modulated signal.



(b)

But actually, it is not a very big problem sometimes because we still have methods to deal with it.
You will know the methods if you further study in these fields.
Back to our main problem, we now know that the receiver receives exactly what the transmitter transmits. The main problem for us is how to separate these four messages apart (demodulation).

Before we start to deal with the received signals, we have to know two properties and one formula group.

The first property is orthogonality of the cos and sin functions, the other one is the phase can cancel each other in some case (actually the two properties are somehow describing the same thing).
The formula group is popular, which turns the multiplication of sinusoidal functions to summation of them. There are four formulas in total, which are

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha+\beta) + \cos(\alpha-\beta)]$$

$$\sin\alpha\sin\beta = -\frac{1}{2}[\cos(\alpha+\beta) - \cos(\alpha-\beta)]$$

You might be familiar to these formulas in your high school. The proof is simple and you can review by yourself. [就是积化和差公式]

Let's take a look at the case if there are only two message signals to be transmitted. We can still use the same carrier wave frequency to transmit the two message signals simultaneously. One message signal is modulated (multiplied) by cos signal while the other one is by sin signal. We have

$$y_1(t) = m_1(t)\cos(\omega_c t)$$
$$y_2(t) = m_2(t)\sin(\omega_c t)$$

Then the signal to be transmitted is the sum of the two above, we have (s is source)

$$s(t) = y_1(t) + y_2(t)$$

And receiver also receives this, to separate the two message signals apart, we need to multiple the received signal with the two carrier waves with the same phase. First, we deal with the cos one (r is reconstruct).

$$\begin{aligned} r_1(t) = s(t)\cos(\omega_c t) &= [y_1(t) + y_2(t)]\cos(\omega_c t) = y_1(t)\cos(\omega_c t) + y_2(t)\cos(\omega_c t) \\ &= m_1(t)\cos(\omega_c t)\cos(\omega_c t) + m_2(t)\sin(\omega_c t)\cos(\omega_c t) \\ &= m_1(t)\cos^2(\omega_c t) + m_2(t)\sin(\omega_c t)\cos(\omega_c t) \\ &= \frac{1}{2}m_1(t)[1+\cos(2\omega_c t)] + \frac{1}{2}m_2(t)\sin(2\omega_c t) \\ &= \frac{1}{2}m_1(t) + \left[\frac{1}{2}m_1(t)\cos(2\omega_c t) + \frac{1}{2}m_2(t)\sin(2\omega_c t)\right] \end{aligned}$$

Notice that the part in the square brackets [] is the high frequency part, though there is cos and sin and they are different, their frequency is the same, $2\omega_c$. They just have a 90 degrees phase difference.

[注意式子方括号中的部分，虽然有 cos 和 sin 两个函数，但是它们的频率实际上都是$2\omega_c$，它们仅仅只有 90 度的相位差而已]

Pass $r_1(t)$ to a low pass filter we will remove the high frequency part, which is the part in the square brackets in this case. Then we obtain $\frac{1}{2}m_1(t)$ if the filter has a unit amplitude. And we successfully recover the message the transmitter transmits.

It's quite similar for the other message signal, you can do it by yourself.

Now you have a sense that the phase can also help us to transmit the message signals in the same channel simultaneously. We can now start to solve the problem.

Solution:

We have 4 message signals, they are $m_1(t)$, $m_2(t)$, $m_3(t)$, and $m_4(t)$. And they are respectively multiplied by the carrier waves descripted in the problem. Finally we get 4 DSB-SC signals, they are

$$x_1(t) = m_1(t)[\cos(\omega_a t) + \cos(\omega_b t)]$$
$$x_2(t) = m_2(t)[\cos(\omega_a t + \alpha_1) + \cos(\omega_b t + \beta_1)]$$
$$x_3(t) = m_3(t)[\cos(\omega_a t + \alpha_2) + \cos(\omega_b t + \beta_2)]$$
$$x_4(t) = m_4(t)[\cos(\omega_a t + \alpha_3) + \cos(\omega_b t + \beta_3)]$$

And then they are all summed for they will be all transmitted over a common channel at the same time. The transmission signal is

$$s(t) = x_1(t) + x_2(t) + x_3(t) + x_4(t) = \sum_{i=1}^{4} \{m_i(t)[\cos(\omega_a t + \alpha_{i-1}) + \cos(\omega_b t + \beta_{i-1})]\}$$

Where $\alpha_0 = \beta_0 = 0$.

And the output of the signals after multiplying the carrier wave is

$$r_j(t) = s(t)\left[\cos\left(\omega_a t + \alpha_{j-1}\right) + \cos\left(\omega_b t + \beta_{j-1}\right)\right]$$

Where $j = 1, 2, 3, 4$

It can also be converted to

$$r_j(t) = \sum_{i=1}^{4} \{m_i(t)[\cos(\omega_a t + \alpha_{i-1}) + \cos(\omega_b t + \beta_{i-1})]\} \left[\cos\left(\omega_a t + \alpha_{j-1}\right) + \cos\left(\omega_b t + \beta_{j-1}\right)\right]$$

$$= \sum_{i=1}^{4} \{m_i(t)[\cos(\omega_a t + \alpha_{i-1}) + \cos(\omega_b t + \beta_{i-1})]\left[\cos\left(\omega_a t + \alpha_{j-1}\right) + \cos\left(\omega_b t + \beta_{j-1}\right)\right]\}$$

And we just use association law to the cos part, we have

$$r_j(t) = \sum_{i=1}^{4} \{m_i(t)[\cos(\omega_a t + \alpha_{i-1}) \cos\left(\omega_a t + \alpha_{j-1}\right) + \cos(\omega_a t + \alpha_{i-1}) \cos\left(\omega_b t + \beta_{j-1}\right) + \cos(\omega_b t + \beta_{i-1}) \cos\left(\omega_a t + \alpha_{j-1}\right) + \cos(\omega_b t + \beta_{i-1}) \cos\left(\omega_b t + \beta_{j-1}\right)]\}$$

Use the product to sum formula, we have

$$r_j(t) = \frac{1}{2} \sum_{i=1}^{4} \Big\{m_i(t) \Big[\cos(2\omega_a t + \alpha_{i-1} + \alpha_{j-1}) + \cos\left(\alpha_{i-1} - \alpha_{j-1}\right) + \cos\left((\omega_a + \omega_b)t + \alpha_{i-1} + \beta_{j-1}\right) + \cos\left((\omega_a - \omega_b)t + \alpha_{i-1} - \beta_{j-1}\right) + \cos\left((\omega_a + \omega_b)t + \alpha_{j-1} + \beta_{i-1}\right) + \cos\left((\omega_a - \omega_b)t + \alpha_{j-1} - \beta_{i-1}\right) + \cos(2\omega_b t + \beta_{i-1} + \beta_{j-1}) + \cos\left(\beta_{i-1} - \beta_{j-1}\right)\Big]\Big\}$$

That is

$$r_j(t) = \frac{1}{2}\sum_{i=1}^{4}\Big\{m_i(t)\Big[\cos(\alpha_{i-1}-\alpha_{j-1}) + \cos(\beta_{i-1}-\beta_{j-1})$$

$$+\cos(2\omega_a t + \alpha_{i-1} + \alpha_{j-1}) + \cos(2\omega_b t + \beta_{i-1} + \beta_{j-1})$$

$$+\cos\Big((\omega_a+\omega_b)t + \alpha_{i-1} + \beta_{j-1}\Big) + \cos\Big((\omega_a-\omega_b)t + \alpha_{i-1} - \beta_{j-1}\Big)$$

$$+\cos\Big((\omega_a+\omega_b)t + \alpha_{j-1} + \beta_{i-1}\Big) + \cos\Big((\omega_a-\omega_b)t + \alpha_{j-1} - \beta_{i-1}\Big)\Big]\Big\}$$

Then you can find that there are totally 5 different frequency regions, which are 0, $2\omega_a$, $2\omega_b$, $(\omega_a+\omega_b)$, $(\omega_a-\omega_b)$. We can apply a low pass filter to remove the six high frequency components of $r_i(t)$, thus the output is (d is demodulation)

$$d_j(t) = \frac{1}{2}\sum_{i=1}^{4} m_i(t)\big(\cos(\alpha_{i-1}-\alpha_{j-1}) + \cos(\beta_{i-1}-\beta_{j-1})\big)$$

If we want to recover the first message, then $j = 1$, we have

$$d_1 = \frac{1}{2}\sum_{i=1}^{4} m_i(t)(\cos\alpha_{i-1} + \cos\beta_{i-1})$$

It should be

$$d_1 = \frac{1}{2}\sum_{i=1}^{4} m_i(t)(\cos\alpha_{i-1} + \cos\beta_{i-1})$$

$$= \frac{1}{2}[m_1(t)(\cos\alpha_0 + \cos\beta_0) + m_2(t)(\cos\alpha_1 + \cos\beta_1)$$

$$+ m_3(t)(\cos\alpha_2 + \cos\beta_2) + m_4(t)(\cos\alpha_3 + \cos\beta_3)] = m_1(t)$$

Then it is obvious that

$$\cos\alpha_0 + \cos\beta_0 = \cos 0 + \cos 0 = 2$$

And

$$\cos\alpha_1 + \cos\beta_1 = \cos\alpha_2 + \cos\beta_2 = \cos\alpha_3 + \cos\beta_3 = 0$$

For a general requirement, let's look back at the output we get after the low pass filter

$$d_j(t) = \frac{1}{2}\sum_{i=1}^{4} m_i(t)\big(\cos(\alpha_{i-1}-\alpha_{j-1}) + \cos(\beta_{i-1}-\beta_{j-1})\big)$$

For kth message, we have $j = k$, and

$$d_k(t) = \frac{1}{2}\sum_{i=1}^{4} m_i(t)(\cos(\alpha_{i-1}-\alpha_{k-1}) + \cos(\beta_{i-1}-\beta_{k-1}))$$

In the summation, when $i = k$, the message is what we want now, and the sum of the two cos functions should be two to totally recover the message signal because there is a coefficient $\frac{1}{2}$ here.

$$\cos(\alpha_{i-1}-\alpha_{k-1})+\cos(\beta_{i-1}-\beta_{k-1})=2,\ \ if\ i=j$$

And for any other message signals, that is $i \neq j$, the sum of the two cos functions should be cancelled (equal to 0) because we do not want any other message signal influence the kth message signal. That is

$$\cos(\alpha_{i-1}-\alpha_{k-1})+\cos(\beta_{i-1}-\beta_{k-1})=0,\ \ if\ i\neq j$$

Combine the two conclusions above, we have the final requirement for the phase chosen to demodulate the message signals.

$$\cos(\alpha_{i-1}-\alpha_{k-1})+\cos(\beta_{i-1}-\beta_{k-1})=\begin{cases}2, & i=k\\ 0, & i\neq k\end{cases}$$

Where $i,k=1,2,3,4$

So what does this equation mean?

When $i=k$, it is easy to get that

$$\cos(\alpha_{i-1}-\alpha_{k-1})+\cos(\beta_{i-1}-\beta_{k-1})=\cos 0+\cos 0=2$$

And when $i\neq k$,

$$\cos(\alpha_{i-1}-\alpha_{k-1})+\cos(\beta_{i-1}-\beta_{k-1})=0$$

Assume $\theta=\alpha_{i-1}-\alpha_{k-1}$ and $\gamma=\beta_{i-1}-\beta_{k-1}$. Then

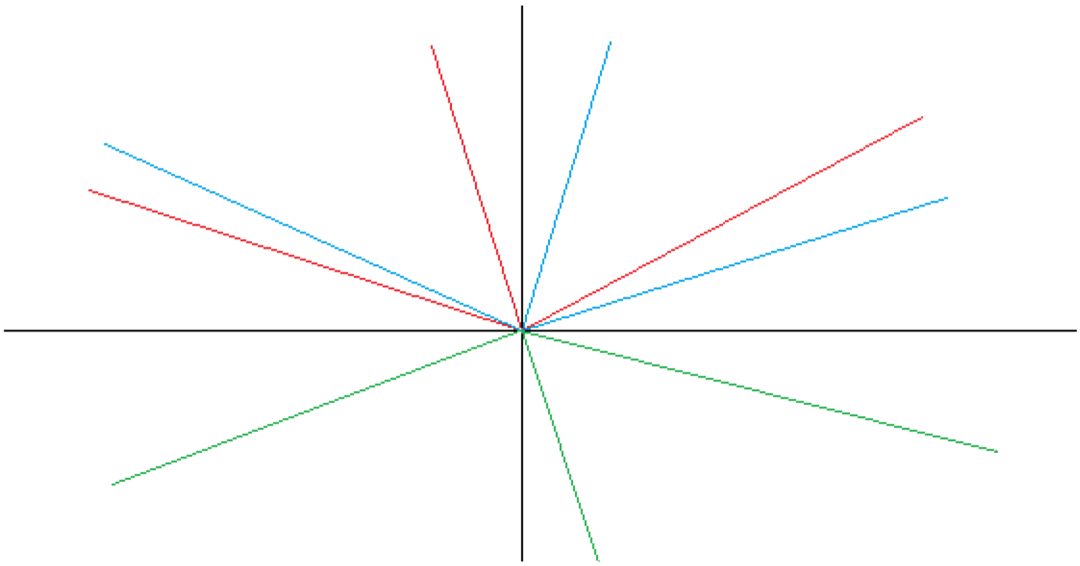
$$\cos\theta+\cos\gamma=0$$

Use sum to product formula, we have

$$\cos\frac{\theta+\gamma}{2}\cos\frac{\theta-\gamma}{2}=0$$

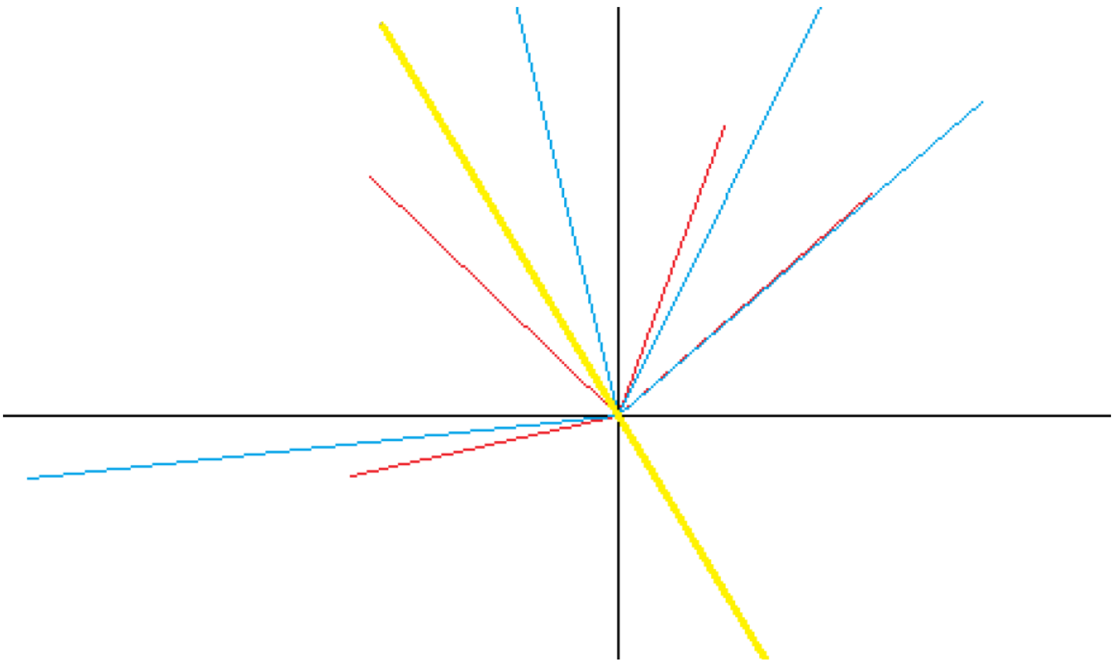
That is to say either $\theta+\gamma=\pi+2\pi n$ or $\theta-\gamma=\pi+2\pi n$, where n is an integer.
$\theta+\gamma=\pi+2\pi n$ tells us they are symmetric to the y-axis, and $\theta-\gamma=\pi+2\pi n$ tells us they point to the opposite direction.

Then it is easy to explain the original phase angle set α and β. The two sets can have no relation, but the difference between any two phase angles in one set to the one in the other set with the corresponding phase angle should be same.
[上面我们得到了θ和γ只需要是关于 y 轴对称或者关于原点对称就可以了，因为θ和γ实际上是两α的差和两β的差，也就是说我们并不需要约束每一个α和β的值，我们约束的是α集合和β集合的在自己集合中的差就可以了。]
Here is an example of the phase angles, the red one indicates α and the red and green one indicates β (with red satisfying $\theta+\gamma=\pi+2\pi n$, green satisfying $\theta-\gamma=\pi+2\pi n$). Note that $\alpha_0=\beta_0=0$ and we do not draw it.



Note that the blue and red are symmetrical related to the y axis, and the green and red are symmetrical related to the origin, blue and green are symmetrical related to the x axis.

Further, we can have none-zero α_0 and β_0. Like the example below ($\alpha_0 = \beta_0$).

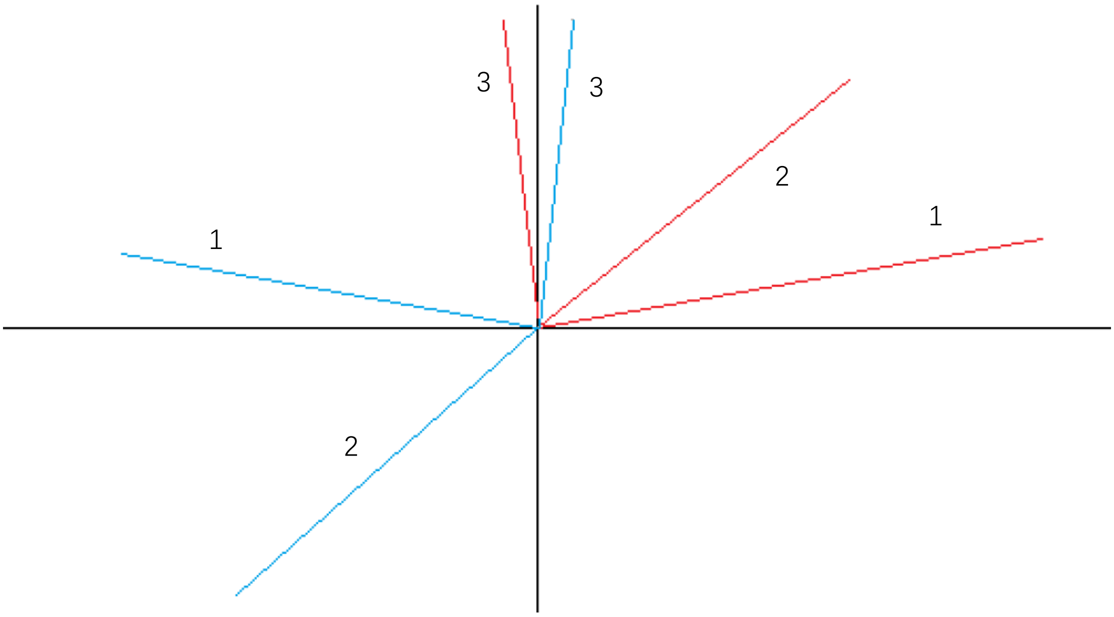


The thick yellow line is the symmetrical axis for red and blue, and is tangent to α_0 and β_0. Also you can imagine the green lines, they are still symmetrical with red related to the origin, or symmetrical with blue related to β_0.

And even more, α_0 can be not equal to β_0. Just rotate all the blue lines with any angles, but do not change the difference between the blue lines. Green lines can also be rotate, just make sure blue and green are symmetrical to the β_0.

Note that the drawings above can only recover the first message, because when it's not to α_0 and β_0, the difference does not satisfy the requirement. This is because not all α and β be symmetrical related to any α and β, and they are not monotonous.

If we want a general phase angle set and all message should be recovered, all α and β be symmetrical related to any α and β, and they are monotonous. You might say that β should be monotonous and change the symmetrical reference each time. That is α and β must be symmetrical related to y-axis and origin one by one. Like the drawing below



α_1, β_1, and α_3, β_3 are symmetrical related to the y-axis and the rest is symmetrical related to the origin.

However, it still has some issues, that is the non-symmetrical problem is not totally solved.

To solve this problem, we must use the symmetrical property and the period property of the phase angle to help use designate the final correct phase angles.

One wise idea for phase angle choosing is use special phase angle and use a arithmetic progression phase angle. Then it will be easy to obtain solutions.

For example, you can choose

$$\alpha_0 = 0, \alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{\pi}{2}, \alpha_3 = \frac{3\pi}{4}$$

$$\beta_0 = 0, \beta_1 = \frac{3\pi}{4}, \beta_2 = \frac{3\pi}{2}, \beta_3 = \frac{9\pi}{4}$$

There are other choices of the phase angles, and you can try for others.

Additional questions for phase choosing:

1. If the carrier wave is sin, the problem could be easier to solve?

No, even if we change to sin, we still have to deal with cos when choose the phase, we will meet the same problems mentioned above.

2. If α_0 and β_0 is not 0, could the problem be easier to solve?

Yes, if it is, then we do not take β_0 as a special one because this is the only phase angle for β set to be same with α, while the rest phases are not equal. You can try it yourself to find the mystery.

3. Could we use sin in demodulation?

Absolutely no, if you do so, you will get sin after low pass filter, then $\sin 0 = 0$, and you will lose your desired message signal.

Attachment

Code for 5.39

```
% We take 4 signals rather than 24 for demonstration
time_start = -0.5;
time_end = 0.5;
time_precision = 0.001;


n = time_start:time_precision:time_end;
x1 = cos(2*pi*n);
x2 = cos(2*pi*2*n);
x3 = cos(2*pi*3*n);
x4 = cos(2*pi*4*n);


figure(1);
plot(n,x1);hold on;
plot(n,x2);hold on;
plot(n,x3);hold on;
plot(n,x4);hold on;
legend('x1','x2','x3','x4');
xlabel('t');ylabel('amplitude');


% sampling
ts_interval = 25;
step = time_precision*ts_interval;
n_sampled = (time_start/step):(time_end/step);
n_sampled_time = time_start:step:time_end;
x1_sampled = x1(1:ts_interval:end);
x2_sampled = x2(1:ts_interval:end);
x3_sampled = x3(1:ts_interval:end);
x4_sampled = x4(1:ts_interval:end);


figure(2);
stem(n_sampled_time, x1_sampled);hold on;
stem(n_sampled_time, x2_sampled);hold on;
stem(n_sampled_time, x3_sampled);hold on;
stem(n_sampled_time, x4_sampled);hold on;
legend('x1','x2','x3','x4');
xlabel('t');ylabel('amplitude');
```

```
figure(3);
stem(n_sampled, x1_sampled);hold on;
stem(n_sampled, x2_sampled);hold on;
stem(n_sampled, x3_sampled);hold on;
stem(n_sampled, x4_sampled);hold on;
legend('x1','x2','x3','x4');
xlabel('n');ylabel('amplitude');




% time-division multiplex, using PAM
sig_num = 5; % the number of signals, note that there is one for
synchronization
modulated_samples = size(n_sampled,2) * sig_num;
n_modulated = (-round(modulated_samples/2-
1)):(round(modulated_samples/2-1));
modulated_x = zeros(1,modulated_samples);
modulated_x(1:sig_num:end) = x1_sampled;
modulated_x(2:sig_num:end) = x2_sampled;
modulated_x(3:sig_num:end) = x3_sampled;
modulated_x(4:sig_num:end) = x4_sampled;
modulated_x(5:sig_num:end) = 2;


figure(4);
stem(n_modulated(1:sig_num:end),x1_sampled);hold on;
stem(n_modulated(2:sig_num:end),x2_sampled);hold on;
stem(n_modulated(3:sig_num:end),x3_sampled);hold on;
stem(n_modulated(4:sig_num:end),x4_sampled);hold on;
stem(n_modulated(5:sig_num:end),2*ones(1,size(x1_sampled,2)));hold on;
legend('x1','x2','x3','x4','synchronization');
xlabel('n');ylabel('amplitude');


% uncomment this to have a enlarged one
% xlim([-120 0])
% ylim([-1.00 2.00])


figure(5);
stem(n_modulated,modulated_x);
xlabel('n');ylabel('amplitude');
```

```
n_modulated_time = time_start:(1/modulated_samples):(time_end-(1/modulated_samples));
figure(6);
stem(n_modulated_time,modulated_x);
xlabel('t');ylabel('amplitude');
```