Stochastic dynamical modeling in biology

- Homework 1 solutions -

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Exercise 1

For any numerical n-dimensional DTSP with autocovariance function $\mathbb{K}$ we can show that if $k, m \in \mathbb{N}_0$ then $\mathbb{K}(k,m) = \mathbb{K}(m,k)^T$. We can prove this equivalence by:

1. Defining our equations ...

$$\mathbb{K}(k,m) := Cov(X_k, X_m) := \mathbb{E}\{(X_k - \mathbb{E}X_k)(X_m - \mathbb{E}X_m)^T\} \tag{1}$$

and

$$\mathbb{K}(m,k)^T := Cov(X_m, X_k)^T := \mathbb{E}\{(X_m - \mathbb{E}X_m)(X_k - \mathbb{E}X_k)^T\}^T \tag{2}$$

2. Write out their matrices ...

$$\mathbb{K}(k,m) = \begin{bmatrix} Cov(X_k^1, X_m^1) & \cdots & Cov(X_k^1, X_m^n) \\ \vdots & \ddots & \vdots \\ Cov(X_k^n, X_m^1) & \cdots & Cov(X_k^n, X_m^n) \end{bmatrix} \tag{3}$$

and

$$\mathbb{K}(m,k)^T = \begin{bmatrix} Cov(X_m^1, X_k^1) & \cdots & Cov(X_m^n, X_k^1) \\ \vdots & \ddots & \vdots \\ Cov(X_m^1, X_k^n) & \cdots & Cov(X_m^n, X_k^n) \end{bmatrix} \tag{4}$$

3. Now that we have defined our functions and their subsequent matrices we can see that they are equivalent to each other i.e. ...

$$\mathbb{K}(k,m) = \mathbb{K}(m,k)^T \tag{5}$$

Exercise 2

We have some n-dimensional covariance-stationary DTSP $(X_k)_{k\in\mathbb{Z}}$ with autocovariance function $\mathbb{K}$, and we let $\mu := \mathbb{E}\{X_0\}$ be the mean of the process. In stationary DTSP we have some time lag that is defined $l := k - m$. We want to show that ...

$$(X_k)_{k\in\mathbb{Z}} = (X_k - \mu)_{k\in\mathbb{Z}} \tag{6}$$

We can do that as follows ...

1. Define the covariance-stationary function ...

$$\mathbb{K}(l) := Cov(X_k, X_{k-l}) := \mathbb{E}\{(X_k - \mathbb{E}X_k)(X_{k-l} - \mathbb{E}X_{k-l})^T\} \tag{7}$$

2. Given by the definition of a covariance-stationary function we have some time lag l, the matrix $\mathbb{K}(l)$, specifically the right portion of of each factor in equation (7) is not dependent on k. This means that for any covariance-stationary DTSP the μ is the same at any offset l, which means that we can replace the $\mathbb{E}\{X_0\}$ variable with μ. Since the value is constant ...

$$\mathbb{K}(l) := \mathbb{E}\{(X_k - \mu)(X_{k-l} - \mu)^T\} \tag{8}$$

3. This shows by definition that (6) is equivalent.

Exercise 3

We analyzed electroencephelogram (EEG) results. The samples were taken from a patient's scalp over a timeframe of 2.5 minutes. We are interested in the results for neurons F5, C1, and C3. However, we are interested in the last 80% of the data for these three samples.

1. First we slice the data and drop the first 20% of time series data. After slicing the data it is useful to visualize it by graphing. We need to make sure to transform the Time portion of the data so that it is in seconds and then we can view a Hertz vs Time graph ...

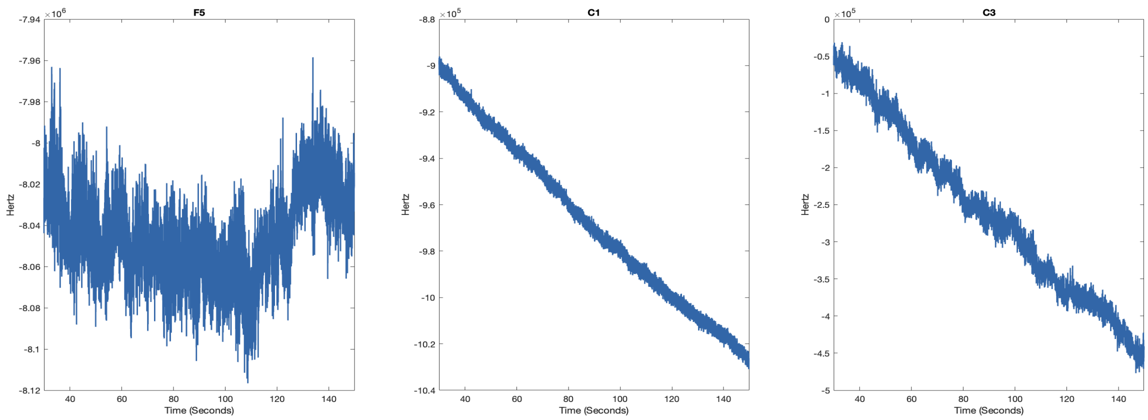

Figure 1: Hertz Vs. Time

2. We can see that there are strong trends within C1 and C3. We should detrend those samples, so we can have a completely unbiased sample. This detrending will convert the samples to stationary. Making our math and analysis easier. We can put this now detrended data into the autocovariance-stationary function ...

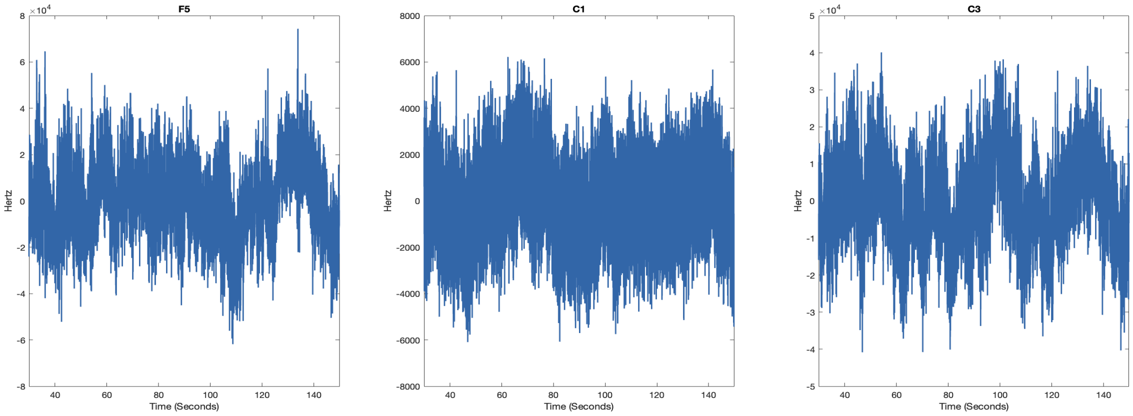

Figure 2: Hertz Vs. Time Detrended

3. Now that our data has no strong trends and is stationary we can put our data through the autocovariance-stationary function. Then we can graph the covariance of a sample j and it's effects on sample i at some time lag $l := k - m$. Which is written out ...

$$\mathbb{K}(l) := Cov(X_k, X_{k-l}) := \mathbb{E}\{(X_k - \mathbb{E}X_k)(X_{k-l} - \mathbb{E}X_{k-l})\} \tag{9}$$

   We can see the autocovariance-stationary function in action looking at the graph below (Fig.3). The x-axis has been constrained from a range of [0, 60] seconds.

4. Finally we can look at the autocorrelation function and check to see if any of the samples X_k are correlated with each other. We can do this by taking the autocovariance function and dividing by some normalization factors. These normlization factors are when the autocovariance function has a time lag equal to 0 ...

$$\mathbb{K}^o := \frac{\mathbb{K}_{i,j}(k,m)}{\sqrt{\mathbb{K}_{i,i}(k,k)\mathbb{K}_{j,j}(m,m)}} \tag{10}$$

   This normalization allows us to view a correlation between the element i and j. The values range from [-1,1]. Looking below (Fig.4) we can see $\mathbb{K}^o(l)$ zoomed in between [0,60] seconds.

1 Exercise 4

1. First, let us define what the PSD function is ...

$$\mathbb{D}(w) = \frac{\tau}{2\pi} \sum_{l=-\infty}^{\infty} \mathbb{K}(l) e^{-il\tau\omega} \tag{11}$$

2. We also know that ...

$$\mathbb{D}(w) = \lim_{N \to \infty} \mathbb{E}\{\mathcal{F}_N(\omega) \cdot \mathcal{F}_N(\omega)^H\} \tag{12}$$

3. In regards to the values begin real numbers. We know that a complex square matrix that is equal to its own conjugate transpose (H), which we can see above (12). Which means we can set them equal to each other.

$$\frac{\tau}{2\pi} \sum_{l=-\infty}^{\infty} \mathbb{K}(l) e^{-il\tau\omega} = \lim_{N \to \infty} \mathbb{E}\{\mathcal{F}_N(\omega) \cdot \mathcal{F}_N(\omega)^H\} \tag{13}$$

   We also know that Hermitian matrices are complex extensions of real numbers. Leading to the diagonal entries of our matrix being real.

4. In regards to our values always being non-negative values. Within our PSD diagonal entries they represent the power of the signal at some specific frequency. By definition power is non-negative as it is the magnitude squared of the signal. Therefore, our diagonal entries cannot be negative.

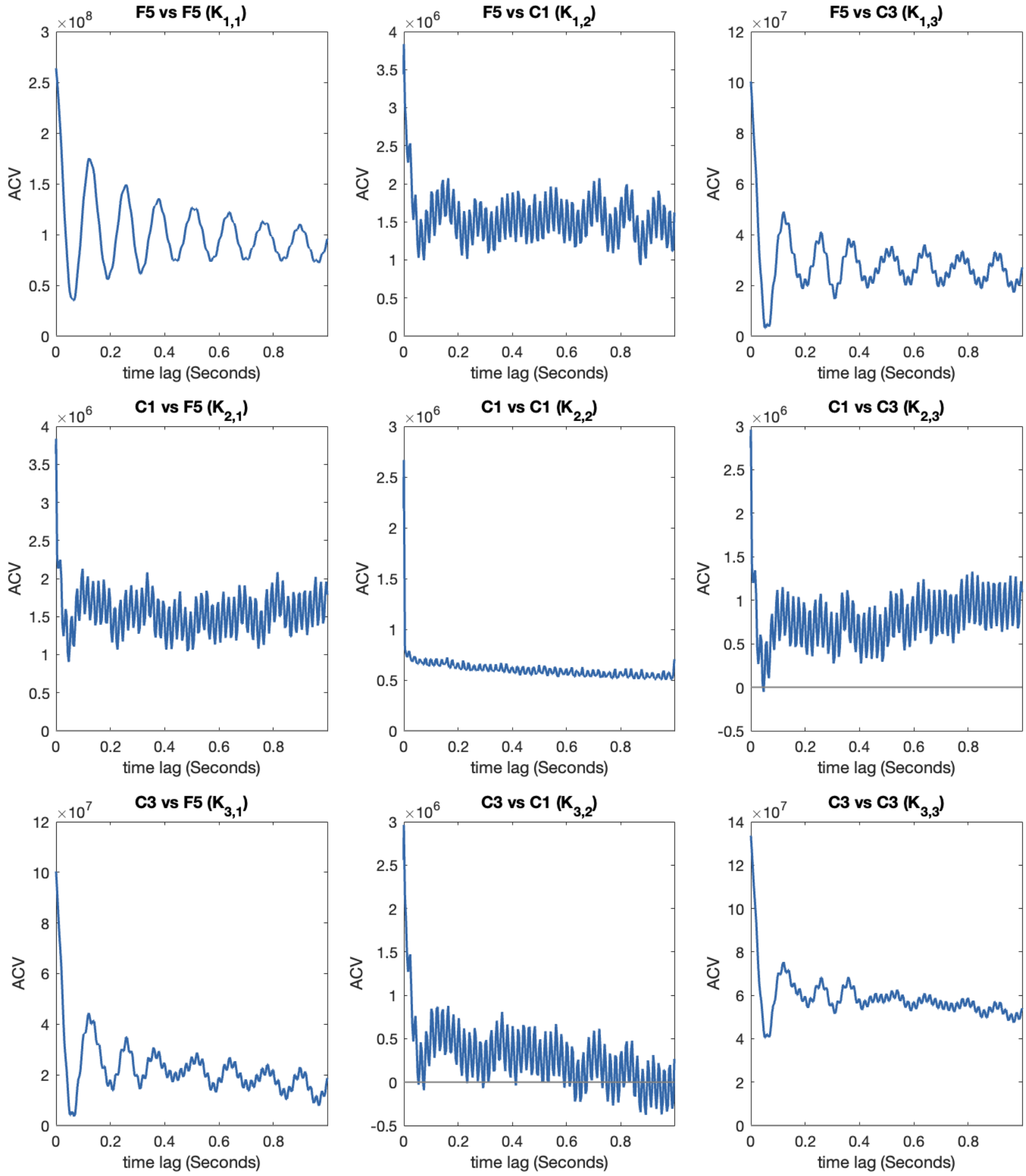

Figure 3: Autocovariance of $X_{k_{(i,j)}}$

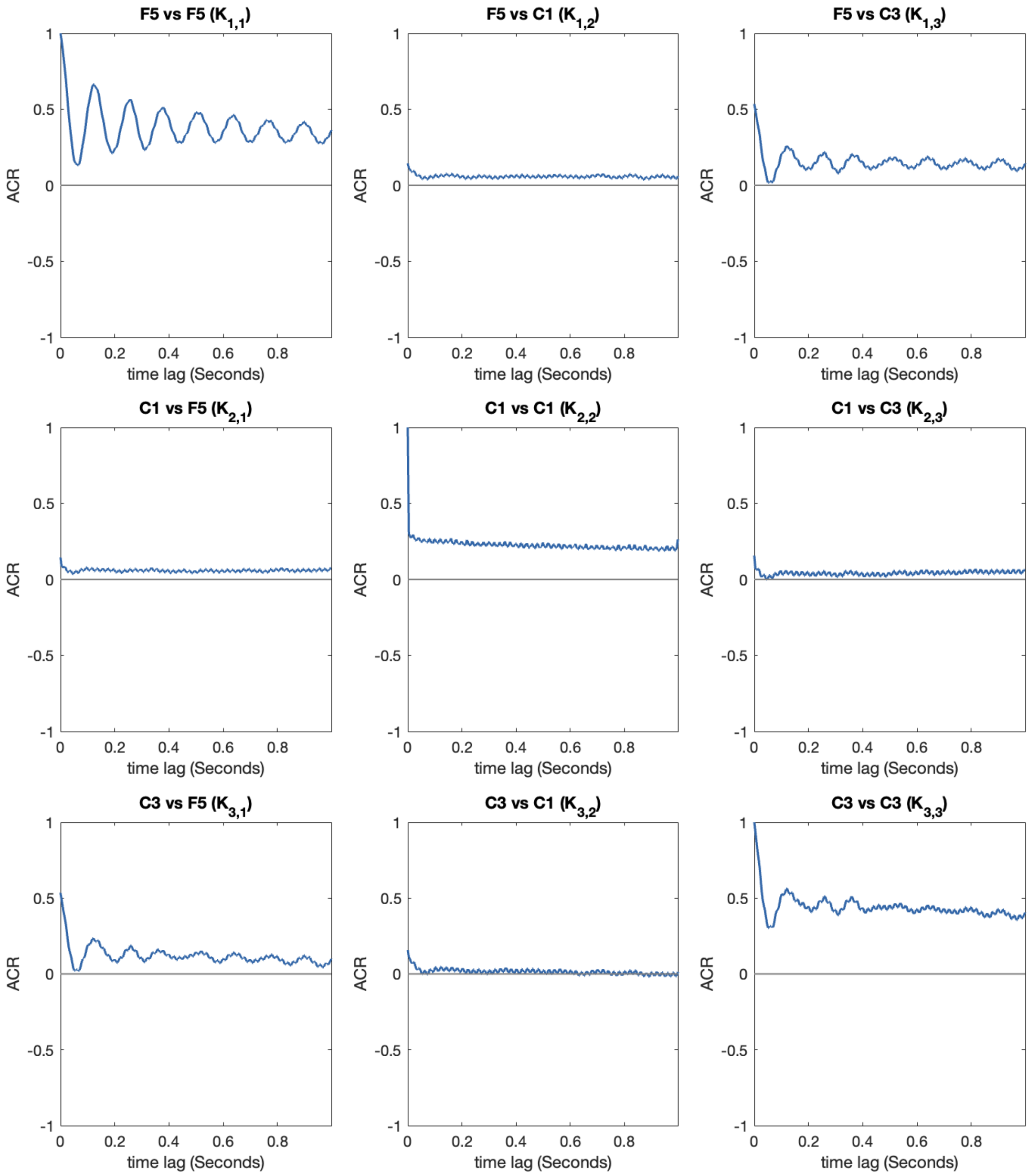

Figure 4: Autocorrelation of $X_{k_{(i,j)}}$