BRISTOL Engineering Mathematics Challenge

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Exploring the relationships between angle, distance, height and velocity when a UAV is banking.

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INTRODUCTION

Deciding from where to focus the problem presented in the competition was tough, since the topic is incredibly broad. We started by making some simple assumptions, from which to start working:

- The camera is fixed during the flightpath.
- Some parts of the field won't be photographed because we don't aim for completeness.
- The UAV will bank instantaneously.
- Wind is negligible.
- The studied field has a flat surface, is perfectly rectangular and of 400mx600m.

We found out simple relationships between the angle and height of the plane. From these relationships, as stated in the description, we deduced that the distance covered by the UAV's camera during the turning points (banking) was going to increase, thus changing the quality of the image. To us, this looked like a great field to extend on.

We started to explore some of the possible relationships between the banking angle and the quality of the image, which we intended to keep constant. From here we derived some interesting relationships between thrust force, velocity and height.

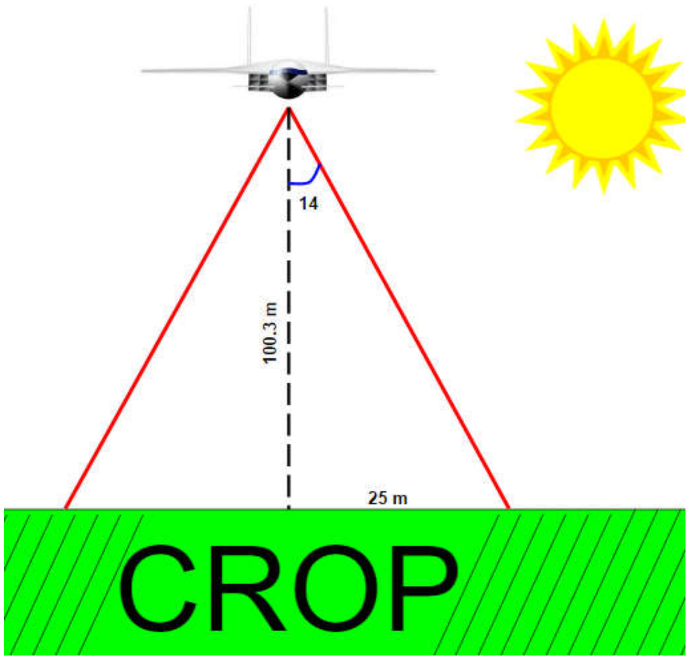
To conclude the work, we wanted to apply it to a rectangular field. This gave a real-life side to our work, despite working on an idealistic field.

HEIGHT, CAMERA ANGLE & DISTANCE

We found that the optimal height of the UAV according to the data from the description paper is 100.3 m. This is considering that the UAV is not banking and the best width for the image is 50 m:

$$\tan 14° = \frac{25}{h}; \quad h = \frac{25}{\tan 14°};$$

$$h \cong 100.3\ m$$



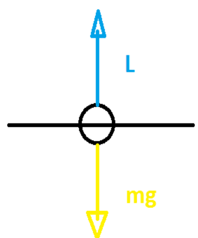
So when the camera covers 50 m width of land, the resolution of the image will have an adequate quality to find any draughts, plagues, flooding… This lead us to the main body of our project: the image resolution with respect to the banking angle.

By using $r = \frac{v^2}{g\ tan\theta}$, where r is the turning radius, in this case 25m, v is the speed of the plane, 12 m/s and θ is the banking angle. Solving for θ, the banking angle is 30.45 °.

FORCES ON THE UAV

INTRODUCTION

When a plane is flying in a straight line, there are two equal and opposite forces acting on it. Pulling down is the weight (mg) and pulling up is the lift force (L), caused by air.



So when the plane is flying straight, $L = mg$

The lift can also be calculated using the following formula

$$L = \frac{1}{2}\rho v^2 A C_L$$

Where ρ is air density, v is true airspeed, A is the wing area and C_L is the lift coefficient at the desire angle of attack, which can be simplified to $L = Kv^2$ (1) where $K = \frac{1}{2}\rho A C_L$.

True airspeed is the difference between the groundspeed and the wind speed. As we are assuming that there is no wind present in the system, the true airspeed is equal to the groundspeed, which is the velocity of the plane.

The lift coefficient depends on the angle of the wing.

Something to take into consideration is that the plane is pulled back by another force called drag (D). The lift and drag are perpendicular, so we can calculate the resultant force, which will be equal to (1):

$$Kv^2 = \sqrt{L^2 + D^2}$$

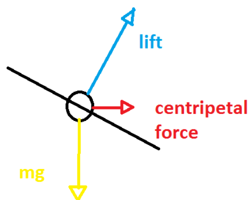
The lift/drag ratio, or L/D ratio goes from 10 (in passenger planes) to 40 (in military drones). We are assuming that the UAV has a ratio of 15. This means that $\frac{L}{D} = 15$ so $D = \frac{L}{15}$. Substituting into the previous equation,

$$(Kv^2)^2 = L^2 + \left(\frac{L}{15}\right)^2$$

Which means that the influence of the drag is minimal and can thus be disregarded.

K: CONSTANT OR VARIABLE

From the K formula, the only thing that may change is the lift coefficient. We can either assume that the wings are fixed to the body of the aircraft and thus both $K$and C_L will always be the same, or that the wings have elevators. These are flight control surfaces at the rear of the aircraft, and allow the angle of the aircraft and thus the lift coefficient and the height to change.



We have considered both possibilities. First, we assumed the plane had no elevators, so the K was constant. After that, later on in the project, the consequences of the plan having elevators will be studied.

K AS A CONSTANT

If K is a constant, the height of the plane will not change. Always at a constant height, the plane will bank and then return to a straight position. When the plan is flying normally, $L = mg$. However, when the plane banks the lift force is not anymore straight upwards as a new horizontal force, the centripetal force, is acting on the plane. The velocity will thus have to be higher to maintain height.

We can consider two different velocities, the one when the plane is going straight (v_1) and the one when the plane is banking (v_2)

We can calculate the lift when banking using Pythagoras rule: $L = \sqrt{weight^2 + centripetal\ force^2}$

So

$L = \sqrt{mg^2 + \left(\frac{{v_2}^2}{r}m\right)^2}$ which simplified is $L = m\sqrt{g^2 + \left(\frac{{v_2}^2}{r}\right)^2}$ (2)

And combining (1) and (2) we obtain that

$$Kv^2 = m\sqrt{g^2 + \left(\frac{{v_2}^2}{r}\right)^2} \tag{3}$$

As said before, when the plane is flying straight $L = mg$and $L = Kv_1^2$ so $mg = Kv_1^2$.
Solving for K,

$$K = \frac{mg}{v_1^2}$$

And replacing K into (3):

$\frac{mg}{v_1^2}v_2^2 = m\sqrt{g^2 + \left(\frac{v^2}{r}\right)^2}$ And thus $\frac{g}{v_1^2}v_2^2 = \sqrt{g^2 + \left(\frac{v^2}{r}\right)^2}$ (4)

It is said that the maximum speed of the plane is 12m/s. However, it is not mentioned if 12 is the maximum speed when flying straight and the plane is able to accelerate to keep the height during the banking or if 12 is the definite maximum speed the plane can reach and thus the banking speed is 12 and the straight speed is a smaller number.

In the equation (4), $g = 9.81m/s^2$ and $r = 25m$. If one considers that 12m/s is the speed when the plane is flying straight and thus introduces $v_1 = 12$, then $v_2 = 13.34m/s$. On the other hand, if 12m/s is the speed when banking and thus $v_2 = 12$, then $v_1 = 11.14m/s$.

We are going to assume then that the maximum speed overall is 12 m/s.

FUEL

Other factor than can be calculated is the fuel usage of the plane.

$$efficiency\ coefficient = \frac{useful\ output}{fuel\ input}$$

So

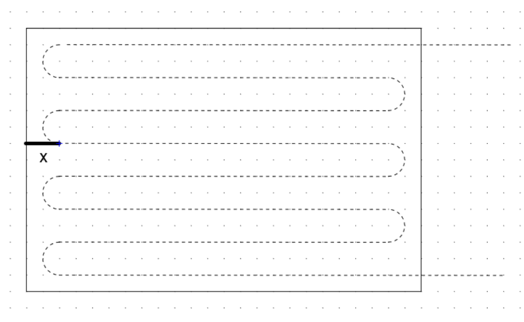
$$useful\ output = efficiency\ coefficient \cdot fuel\ input$$

The useful output can be calculated using the work done. Work (*W)* equals force (*F)* multiplied distance (*d)*, but the plane has two different types of movement: straight line and banking. Thus, the total work done will be:

$$W = F_{straight} \cdot d_{straight} + F_{banking} \cdot d_{banking}$$

On the studied field, a perfect rectangle, we assumed that the plane is going to come in the field through one of the shorter sides, 25m away from the edge.

The lenght of the field is denominated *l* and *x* is equal to the distance between the edge of the field and the beginning of the banking.



The formula developes as follows:

$$useful\,fuel = \ mg \cdot [(l - 2x)(n - 1) + 2(l - x)] + \left[m\sqrt{g^2 + \left(\frac{v^2}{r}\right)^2} \cdot \pi rn\right]$$

Where n equals the number of turns. (for example, n=7 in the diagram above)
A ratio between the fuel use when the plane is banking in respect to when it is flying straight can be calculated. The travelled distance will be assumed as one for both just to calculate the ratio more easily:

$$\frac{m\sqrt{g^2+\left(\frac{v^2}{r}\right)^2}}{mg} = \frac{\sqrt{g^2+\left(\frac{v^2}{r}\right)^2}}{g}$$

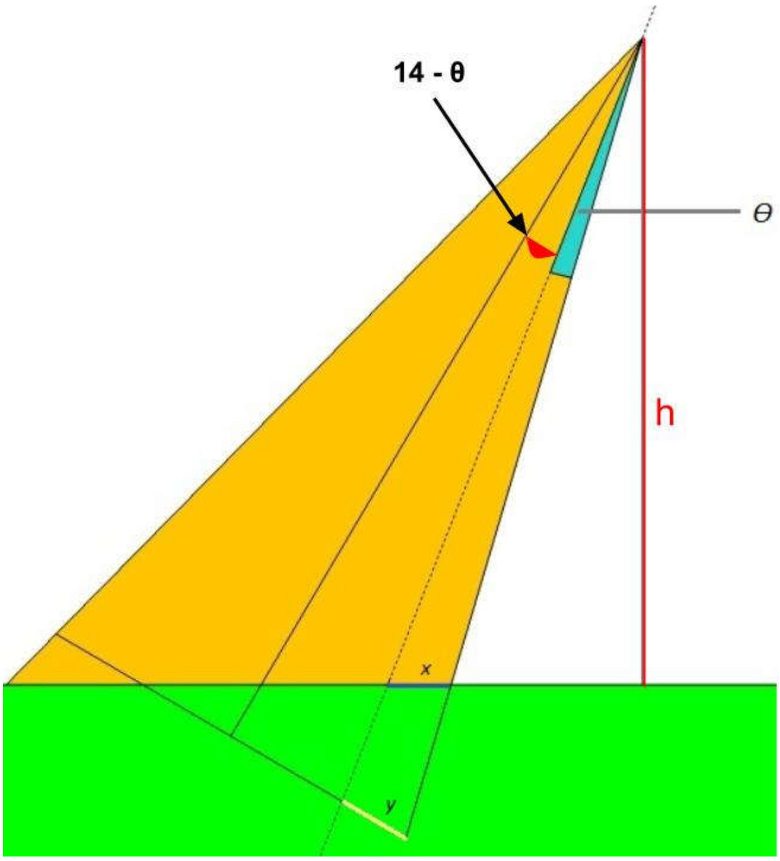
And when all the numbers are substituted in,

$$\frac{\sqrt{9.81^2 + \left(\frac{12^2}{25}\right)^2}}{9.81} = 1.16$$

Thus, when the plane is banking it uses 1.16 more fuel than flying straight.

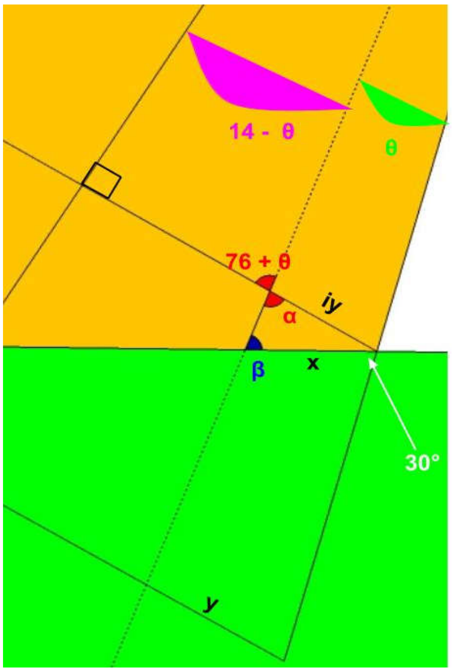
RESOLUTION

Originally, we wanted to find the resolution of the image at a height of 100.237 meters, but later developments in our strategy have brought us to generalise the formula including the height as a variable. For the sake of non-repetition, we will demonstrate the formula as generalised.
The diagrams shows the grasp of the camera when the drone is banking and situated at a distance h (red) from the ground.
The aim is to get a generalised formula for the average resolution of a segment of the ground of length x, (blue), beginning with the point of the ground closest to the normal of the drone to the terrain.
Our clearest indication of the camera's resolution is very simple: at a distance of 100.237 meters, a 50 meters line has a resolution of 1. To find the average resolution of the segment, the simplest way seems to compare its length to the length of the R=1 line (which is imaginary in the example) described by the same angle θ that x describes with the camera.



There is no clear proportion between the two segments. However, if we draw a line parallel to the base with its vertex together with segment x, we can explore the trigonometric relationships between the angles.

By drawing the parallel line, because there is a right-angle triangle, we find two angles: $14-\theta$ and $76+\theta$. Now, the small triangle has two known angles ($30°$ - obtained through the banking angle - and α). So we are left with a triangle such that:



$$\alpha = (76+\theta)°,\ \ \beta = (74-\theta)°,\ \ \gamma = 30°$$

Thus applying the sine rule with the sides iy and x:

$$\frac{\mathbf{sin(74-\boldsymbol{\theta})}}{\mathbf{sin(76+\boldsymbol{\theta})}} = \frac{\boldsymbol{iy}}{\boldsymbol{x}}$$

To find an expression of x, y and h, we had to find an expression of iy in terms of h. This we achieved through similar triangle theorem (Euclid's theorem):

The ratio between iy and y is proportional to ratio of the line up to the ground and the line full line:

$$\frac{iy}{y} = \frac{(\frac{h}{\cos 16{,}419})}{(\frac{100.237}{\cos 14})};\qquad \frac{iy}{y} = \frac{h}{\cos 16{,}419} \times \frac{\cos 14}{100.237};\qquad iy = y\frac{h\cos 14}{100.237\cos 16.419}$$

We plug this into our sine rule expression finding that:

$$\frac{y}{x} = \frac{100.237\cos 16{,}419}{h\cos 14} \times \frac{\sin(74-\theta)}{\sin(76+\theta)}$$

Therefore

$$RESOLUTION = \frac{100.237\cos 16{,}419}{h\cos 14} \times \frac{\sin(74-\theta)}{\sin(76+\theta)}$$

The resolution is expressed in θ and h. For later use, we need a function expressed in terms of x. Therefore, we must find θ in terms of x and h.

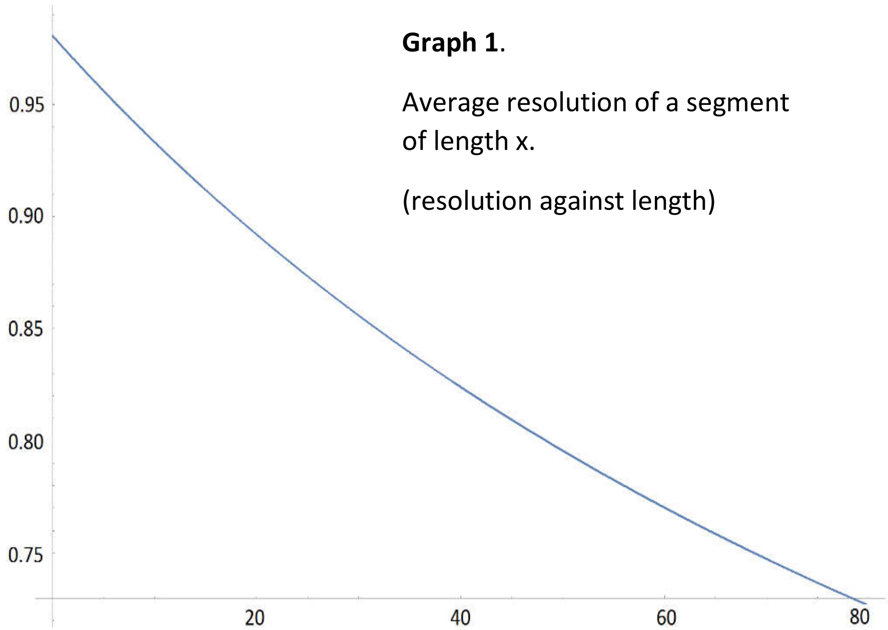
$$x = \tan(16.419+\theta) *\ h - \tan(16.419) * h$$

Rearranging for θ:

$$\theta =\ Arctan\left(\frac{x+\tan(16.419) * h}{h}\right) - 16.42$$

Our final expression for the average resolution of a segment of length x will therefore be:

$$R = \frac{100.237\cos 16.419}{h\cos 14} \times \frac{\sin(74-(ArcTan\left(\frac{x+tan(16.419)h}{h}\right) -\ 16.42))}{\sin(76+(ArcTan\left(\frac{x+tan(16.419)h}{h}\right) -\ 16.42))}$$



Graph 1.

Average resolution of a segment of length x.

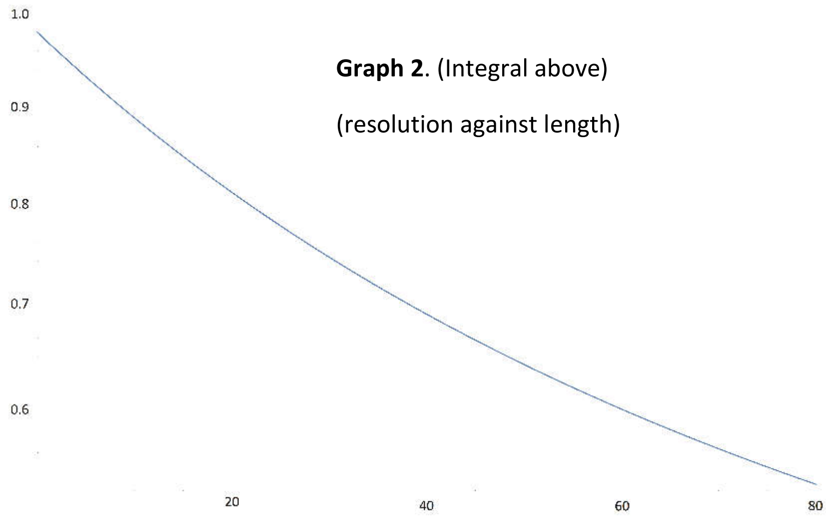
(resolution against length)

This function, though, expresses the average resolution of the segment. We'd like to know exactly where the camera take will become of a lower resolution that we would accept. To find that formula, we thought: if we had the graph of the resolution of a point plotted against its distance from the beginning of the image, we would need to integrate it and then divide it by x to find the average resolution of a segment of length x.

Therefore, starting from this assumption:

$$\int \frac{resolution(x)}{x} dx = AvR(x)$$

$$resolution\ (x) = \frac{dy}{dx}\ (AvR(x) * x)$$



Graph 2. (Integral above)

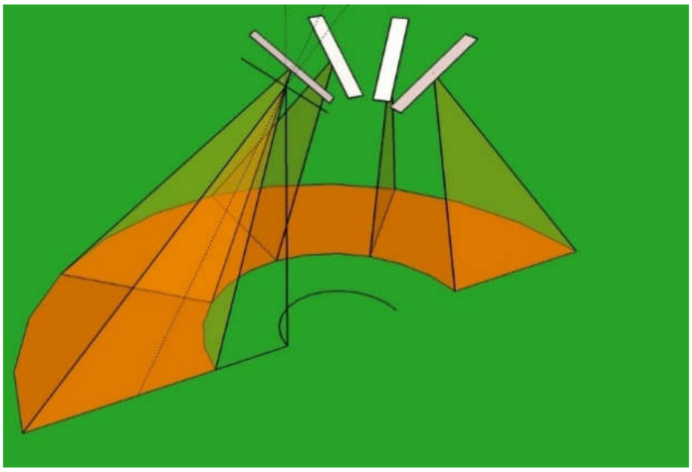
(resolution against length)

To derivate such a complex function we have used Wolfram Mathematica. The result is a too long and not helpful formula that we have decided to omit, although we have plotted the graph. From now on, we will refer the inverse of this function in later work simply referring to it as FRes(R)

BANKING

As we have previously stated, we decided to explore how to use the drone's camera take when it is banking. With previous considerations about the banking angle and the resolution of the image we now can investigate mathematically in the rectangular field. We do not aim for completeness, but maximum $\frac{Covered\ area}{Fuel}$ coefficient.

Background

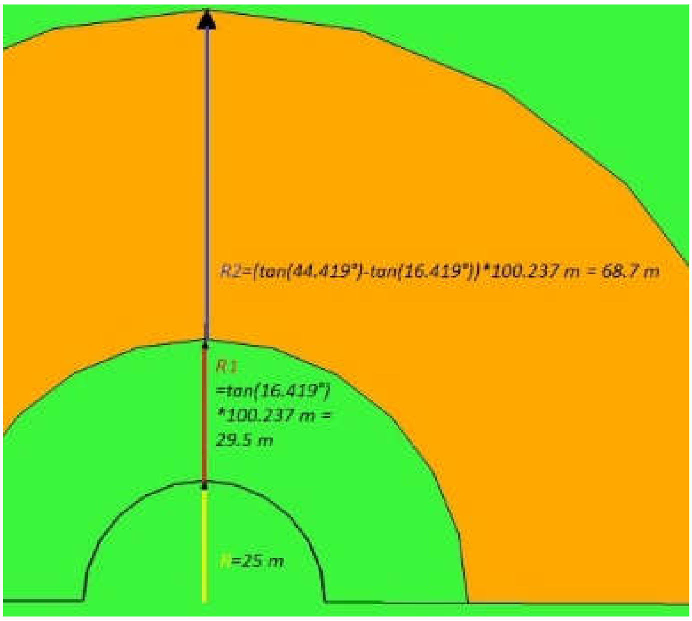


when a drone banks like showed on the information sheet it describes a semi-circular section

R2 is quite big, but its resolution is not uniform. Thanks to the function Res[x] that we found, though, it is quite easy to determine the resolution of the ground at a distance x from the beginning of the image.

As we mentioned earlier, we are trying to use as much of the drone's banking camera take as we can. It appeared to us that to achieve that, the drone should start banking (25+R1+R2) meters before the end of the field. Also, for reasons of completeness and efficiency, it is important that we maximize the "straight" trajectory. Therefore, we should turn exactly (25+R1+R2) meters before the end of the field.

The distances R1 and R2 we found thanks to the tangent relationships of the triangle of the diagram in the resolution section. During banking, the drone moves along the 25 meters' radius line, with a height of 100,237. So looking back at the triangle:



$$R1 = \tan 16° \times 100{,}237m = \mathbf{29.5\ m}$$

From this distance onwards, the drone does photograph the field (R2). The area photographed is derived similarly, such that:

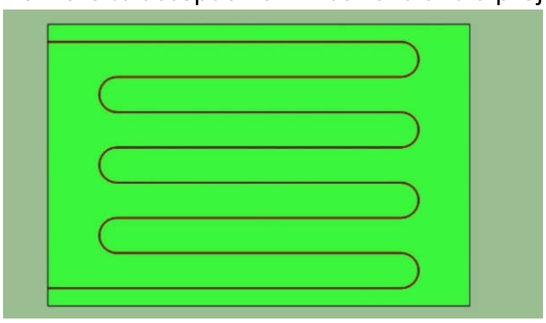
$$R2 = (\tan(16+28) - R1) \times 100{,}237m;$$

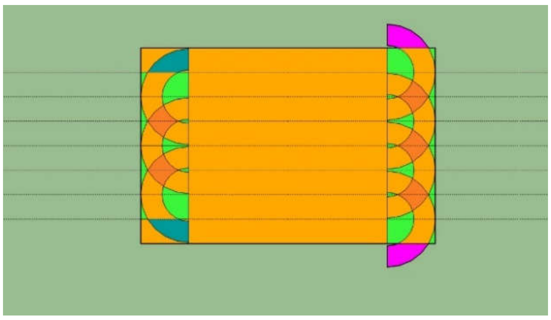
$$R2 = (\tan 44° - \tan 16°) \times 100{,}237m = \mathbf{68.7m}$$

Summarizing, the drone turns $\mathrm{R1} + \mathrm{R2} = \mathbf{98.2\ meters}$ before the end of the field.

DETERMINING THE AREA COVERED WITH BANKED TURN ON THE FIELD 600X400 MATHEMATICALY

To help us with our reasoning, we have drawn what would be the optimal trajectory for the drone if we were to accept all of R2 as valid and a projection of its shot.





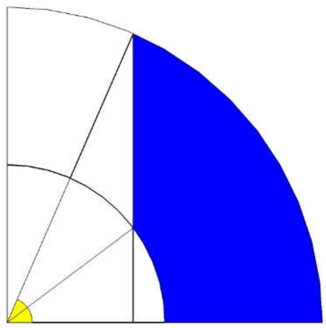
Calculating the total area covered by the drone is quite problematic.
We have to take into account three anomalies:
- Banking shot projections superpose (In light red)
- Some of the banking take is superfluous in the first and last sections (in light blue)
- Some of the banking take is "wasted" in the first and last sections, when the camera take is out of the field. (in light magenta).

Therefore:
A (Area covered by the drone) = Area of the central rectangle + (Area covered by the drone when banking)*7– [Area of superposition]*5 – [Area of camera take out of the field]*2 + Area of the two rectangular bits - [Area of "superposing blue bit"]*2

We are going to try to express this equation in terms of the inverse of Res(x), to effectively express the banking efficiency in terms of the resolution desired.

The area of the rectangle looks quite easy, but if we want to take in consideration Res it becomes a little bit more complicated. It is equal to

$$400 * \left(300 - (25 + FresR)\right) * 2$$



Total area of semicircles is also quite easy:

$$\frac{1}{2} * \pi * (\mathrm{Fres(R)} + 25 + 29.5374761)^2 - (25 + 29.5374761^2)$$

After looking at Area of superposition for quite a long time, we have decided that for the sake of brevity we will be treating its area as that of a square of side R2, and therefore we will express their area as $(\mathrm{Fres(R)})^2$
The area of the camera take out of the field and the area of the two "superposing blue bit" are equal. They should both be equal to a 50 meters segment of two concentric quarters of a circle of radius (25+29.5374761) and (25+29.5374761+ Fres(R)).
The general formula for the area of a segment of a circumference when angles are expressed in degrees is

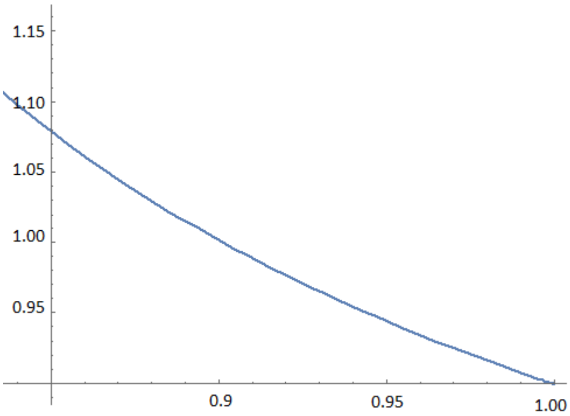
$$\frac{1}{2} * r^2 * \left(\frac{\pi}{180} * \ \theta - \sin(\theta)\right)$$

We must subtract the smaller one to the bigger, and therefore our formula will be:

$$\begin{aligned}&\left[\frac{1}{2} * \left(25 + 29.5374761 + \ (\mathrm{Fres(R)})^2\right)\right.\\ &\left.* \left(\frac{\pi}{180} * Arccos\left(25 + 29.5374761 + \frac{\mathrm{Fres(R)}}{50}\right) - \mathrm{Sin}(\mathrm{arccos}\left(25 + 29.5374761 + \frac{\mathrm{Fres(R)}}{50}\right)\right)\right]\\ &- \left[\frac{1}{2} * (25 + 29.5374761^2)\right.\\ &\left.* \left(\frac{\pi}{180} * Arccos\left(\frac{25 + 29.5374761}{50}\right) - \mathrm{Sin}(\mathrm{arccos}(25 + 29.5374761 +)\right)\right]\end{aligned}$$

This formula represents the area of a single segment. In our case, we must take 2 into account (each of the four parts is half of one). Therefore, the final formula for the area taken by a drone when banking, following the optimal trajectory for a desired resolution of R or more is:

$$Area\ covered = 400 * \left(300 - (25 + Fres(R))\right) * 2 + ((\frac{1}{2} * \pi * ((Fres(x) + 25 + 29.5374761)^2 - (25 + 29.5374761)^2 * 7 - 5 * (Fres(R)^2) + Fres(R) * 50 * 2 - \left(\frac{1}{2} * (25 + 29.5374761 + Fres(R))^2\right) * \frac{\pi}{180} * Arcos\left(25 + 29.5374761 + \frac{Fres(R)}{50}\right) - \sin\left(Arcos\left(25 + 29.5374761 + \frac{Fres(R)}{50}\right)\right) - (25 + 29.5374761)^2 * \frac{\pi}{180} * Arcos\left(\frac{25 + 29.5374761}{50}\right) - \sin(arcos(25 + 29.5374761))$$



By plotting [($A_{Banking}$/$Fuel_{Banking}$)/($A_{straight}$/$Fuel_{straight}$)] we are able to see the efficiency of our trajectory depending on the minimum resolution R accepted.
The fuel spent when banking will equal

$$constant * ((1.16 * 7 * 25 * \pi) + 8 * (600 - Fres(R))$$

The fuel spent with the normal trajectory will equal the same constant multiplied by

$$constant * ((8 * 600 + 1.16 * 7 * 25 * \pi)$$

As we expected, when we accept lower resolutions our efficiency in respect to the classic way becomes higher.
What was more unexpected is the fact that our strategy is not more efficient than the normal one when we don't accept a resolution of 0.9 or lower as valid.

By putting the maximum possible length of x inside Graph 2 we obtain that 0.83 is the minimum possible resolution. This is why in this graph the X-axis starts at this number.

CONCLUSION

We have solved the inverse of the equation for one (graph 3) and found out that the resolution at which our strategy becomes efficient in terms of area over fuel is when one accepts a resolution smaller than 0.9.

We wanted to investigate other strategies, as gliding before turning. However, due to time and space limitations we have limited our work to defining when banking is efficient in terms of area over fuel.

SOURCES

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