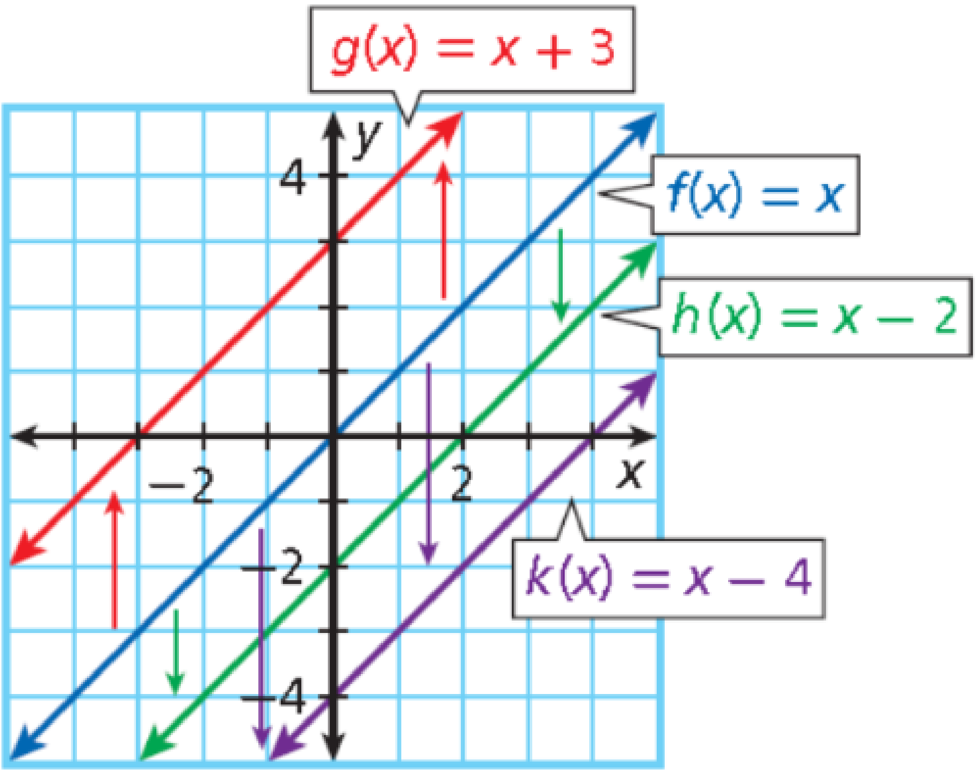
What is Linear Transformation

Linear transformations and matrices are closely related concepts in mathematics. A linear transformation is a function that maps vectors from one vector space to another in a way that preserves certain properties. Matrices, on the other hand, provide a convenient way to represent and analyze linear transformations.



Linear transformations and matrices are two important concepts in linear algebra that are used in machine learning.

A linear transformation is a function that maps an input vector to an output vector. The input and output vectors can have any number of dimensions. The transformation is said to be linear if it satisfies the following two properties:

- **Additivity:** The transformation of a sum of vectors is the sum of the transformations of the individual vectors.
- **Homogeneity:** The transformation of a scalar multiple of a vector is the scalar multiple of the transformation of the vector.

Matrices are a convenient way to represent linear transformations. A matrix is a rectangular array of numbers that can be used to represent a linear transformation between two vector spaces. The number of rows in the matrix corresponds to the dimension of the input vector, and the number of columns corresponds to the dimension of the output vector.

What are Matrices ?

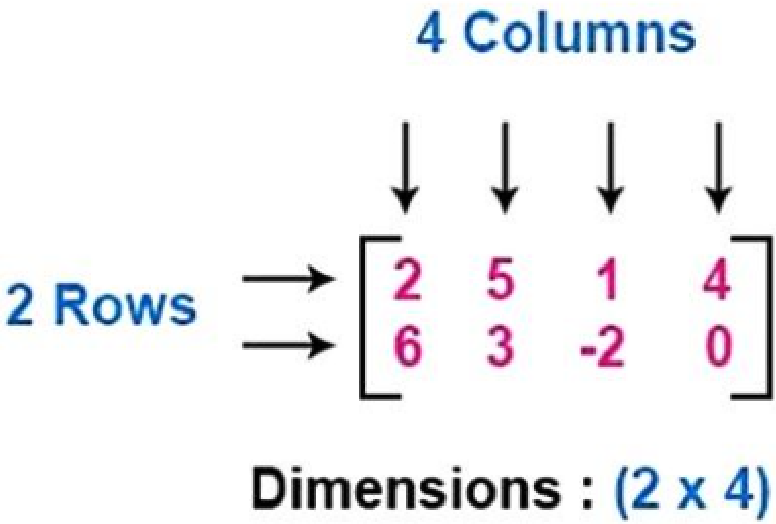
A matrix is a rectangular array of numbers, symbols, or expressions arranged in rows and columns. The numbers, symbols, or expressions are called the elements of the matrix

$$\begin{bmatrix} 6 & -3 & -1 \\ 3 & -9 & 2 \\ -1 & 1 & 3 \end{bmatrix}$$

matrix

Order of the matrix

The order of a matrix is the number of rows and columns it has. It is denoted by a pair of numbers, the first number being the number of rows and the second number being the number of columns. For example, a 3x4 matrix has 3 rows and 4 columns, and its order is 3x4.

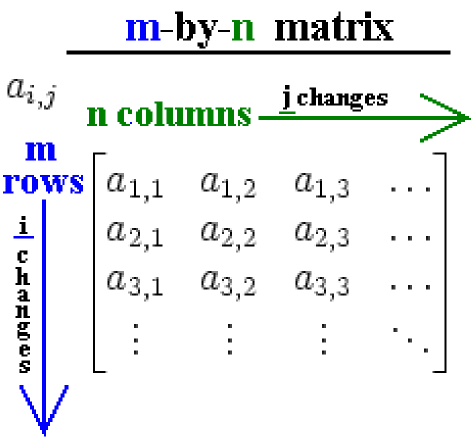


The order of a matrix is important because it determines the number of elements in the matrix. A 3x4 matrix has 3*4 = 12 elements. The order of a matrix also determines the type of operations that can be performed on it. For example, matrix multiplication can only be performed on matrices of the same order.

Here are some examples of matrices and their orders:

- A 1x1 matrix has 1 row and 1 column, and its order is 1x1.
- A 2x3 matrix has 2 rows and 3 columns, and its order is 2x3.
- A 3x4 matrix has 3 rows and 4 columns, and its order is 3x4.
- A 4x5 matrix has 4 rows and 5 columns, and its order is 4x5.

Notation



the notation for the order of a matrix.

The order of a matrix is typically denoted as "m × n," where "m" represents the number of rows and "n" represents the number of columns. For example, a matrix with 3 rows and 2 columns is written as a 3 × 2 matrix.

To represent the elements of a matrix, we often use subscripts. Each element of the matrix is identified by its position in the matrix using two subscripts. The first subscript refers to the row number, and the second subscript refers to the column number. For example, the element in the i-th row and j-th column of a matrix A is denoted by A[i, j].

Matrices Uses and Application Areas

1. **Linear Systems**: Matrices can be used to represent and solve systems of linear equations. A system of linear equations can be written in matrix form as Ax = b, where A is the matrix of coefficients, x is the column vector of unknowns, and b is the column vector of constants. Methods such as Gaussian elimination, LU decomposition and matrix inversion can be employed to find the solutions to the system
2. **Linear Transformations**: Matrices are used to represent linear transformations between vector spaces. A matrix can define a linear transformation that maps vectors from one space to another while preserving the operations of vector addition and scalar multiplication. For example, rotation, scaling, and reflection transformations in geometry can be represented using matrices.
3. **Eigenvalues and Eigenvectors**: Matrices are used in the study of eigenvalues and eigenvectors, which are essential in various applications such as differential equations, stability analysis, and diagonalization of matrices. An eigenvalue- eigenvector pair (λ, v) of a square matrix A satisfies the equation Av = λv.

4. **Graph Theory**: In graph theory, matrices can be used to represent graphs through adjacency matrices, incidence matrices, and Laplacian matrices. These matrix representations provide a convenient way to analyze the properties of graphs and perform operations on them.

5. **Markov Chains**: Matrices are used in the study of Markov chains, which are stochastic processes that undergo transitions from one state to another according to certain probabilistic rules. Transition matrices describe the probabilities of transitioning between different states in a Markov chain and can be used to analyze the long-term behavior of the system.

6. **Computer Graphics**: Matrices are used extensively in computer graphics to represent transformations such as translation, rotation, scaling, and projection. These transformations are applied to 2D or 3D models to manipulate their position, orientation, and size in a virtual environment

What are Types of Matrices?

1. **Row Matrix**: A row matrix is a matrix with a single row and multiple columns. It can be represented as a 1 × n matrix.

   For example: A = [1 2 3] , This is a row matrix with 1 × 3 dimensions.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots\cdots & a_{1n} \end{bmatrix}_{1 \times n}$$

2. **Column Matrix**: A column matrix is a matrix with a single column and multiple rows. It can be represented as an m × 1 matrix.

   For example:

```
B = [ 1 ]
    [ 2 ]
    [ 3 ]

    This is a column matrix with 3 × 1 dimensions.
```

$$A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ a_{41} \\ \vdots \\ a_{n1} \end{bmatrix}_{n \times 1}$$

3. **Square Matrix**: A square matrix is a matrix where the number of rows is equal to the number of columns. In other words, it has an equal number of rows and columns.

   For example:

```
    C = [ 1  0  0 ]
        [ 0  2  0 ]
        [ 0  0  3 ]

        This is a 3 × 3 square matrix.
```

square #rows = #cols

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2} \qquad \boxed{i=j} \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times 3}$$

a_{11} a_{22} a_{11} a_{22} a_{33}

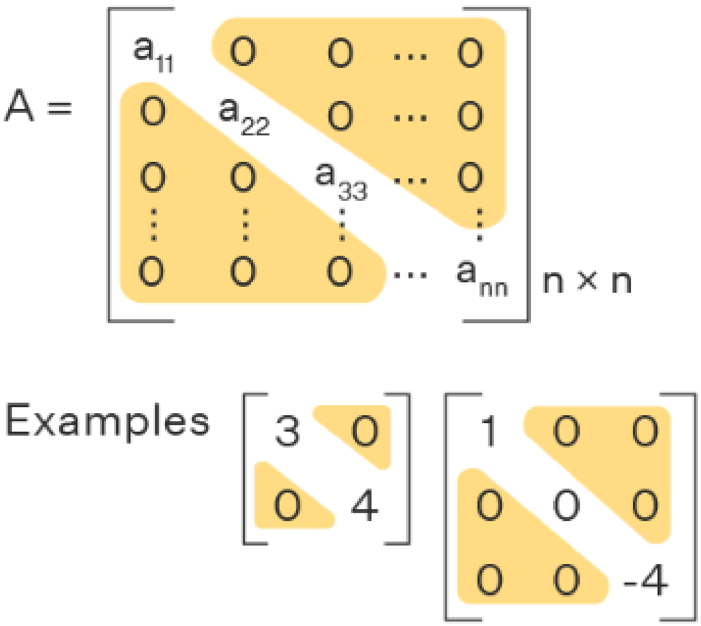
i = row

j = cols

4 .**Diagonal Matrix**: A diagonal matrix is a square matrix in which all the non-diagonal elements are zero. The diagonal elements can be zero or nonzero.

```
For example:
    E = [ 2  0  0 ]
        [ 0  5  0 ]
        [ 0  0 -1 ]

        In this matrix, all the elements outside the diagonal (the main diagon
al from the top left to the bottom right) are zero.
```



5. **Scalar Matrix**: A scalar matrix is a diagonal matrix where all the diagonal elements are the same. It is a special case of a diagonal matrix with all the diagonal elements equal to a scalar value.

   For example:

   ```
   F = [ 3  0  0 ]
       [ 0  3  0 ]
       [ 0  0  3 ]

       In this matrix, all the diagonal elements are 3
   ```

6. **Identity Matrix**: An identity matrix, denoted as I, is a square matrix in which all the diagonal elements are 1, and all the non-diagonal elements are zero. It has ones on the main diagonal and zeros elsewhere.

   For example:

   ```
   I = [ 1  0  0 ]
       [ 0  1  0 ]
       [ 0  0  1 ]
   ```

   This is a 3 × 3 identity matrix.

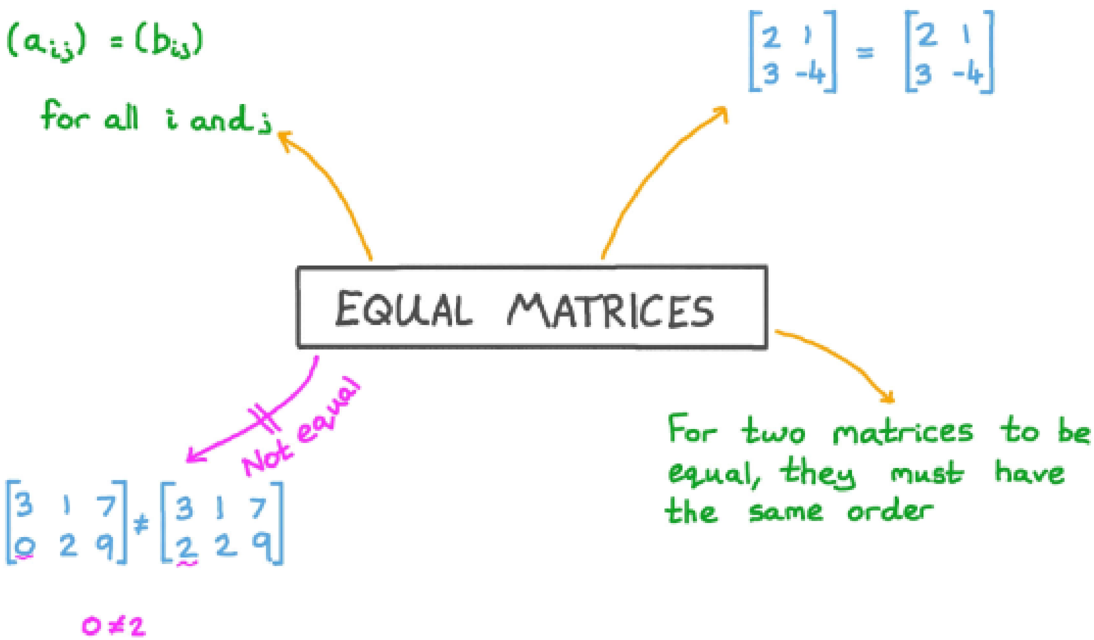
7. **Zero Matrix**: A zero matrix, denoted as O or 0, is a matrix where all the elements are zero. It can have any number of rows and columns.

   For example:

```
O = [ 0  0  0 ]
    [ 0  0  0 ]
```

Equality of Matrices

Matrix equality refers to the condition where two matrices have the same dimensions and their corresponding elements are equal. In other words, two matrices A and B are considered equal if and only if they have the same number of rows and columns, and the elements at each corresponding position in the matrices are identical.



Formally, let A and B be matrices of the same size (m × n). The matrices A and B are equal if A[i, j] = B[i, j] for all i and j, where A[i, j] denotes the element at the i-th row and j-th column of matrix A, and B[i, j] denotes the element at the i-th row and j-th column of matrix B.

Scalar Operations in Matrices

A scalar operation is an operation that is performed on each element of a matrix. The most common scalar operations are addition, subtraction, multiplication, and division.

- **Scalar addition:** Scalar addition is the operation of adding a scalar to each element of a matrix. For example, if we have a matrix A and a scalar k, then the scalar addition of k to A is the matrix B, where each element of B is equal to the corresponding element of A plus k.

  ```
  B = A + k
  ```

- **Scalar subtraction:** Scalar subtraction is the operation of subtracting a scalar from each element of a matrix. For example, if we have a matrix A and a scalar k, then the scalar subtraction of k from A is the matrix B, where each element of B is equal to the corresponding element of A minus k.

  ```
  B = A - k
  ```

- **Scalar multiplication:** Scalar multiplication is the operation of multiplying each element of a matrix by a scalar. For example, if we have a matrix A and a scalar k, then the scalar multiplication of k by A is the matrix B, where each element of B is equal to the corresponding element of A multiplied by k.

  ```
  B = k * A
  ```

- **Scalar division:** Scalar division is the operation of dividing each element of a matrix by a scalar. For example, if we have a matrix A and a scalar k, then the scalar division of A by k is the matrix B, where each element of B is equal to the corresponding element of A divided by k.

  ```
  B = A / k
  ```

Scalar operations are a powerful tool for manipulating matrices. They can be used to simplify matrices, to perform transformations on matrices, and to solve systems of linear equations.

single 2 , 3 , −5 add / multiply

$\boxed{K+A} = \quad K = 2 \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

$\rightarrow K+A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$

$\rightarrow KA = 2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$

negative

$A = -A$

$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \quad K = -1$

$KA = -1 \times \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 4 \end{bmatrix}$

$\boxed{-4}$

$\boxed{A, B} \quad K$

$K(A+B) = \underset{mat}{KA} + \underset{mat}{KB}$

K(A + B) = KA + KB

The equation K(A + B) = KA + KB represents the distributive property of scalar multiplication over matrix addition.

In this equation, A and B are matrices of the same size, and K is a scalar (a single number). The distributive property states that multiplying a scalar by the sum of two matrices is the same as multiplying the scalar by each matrix separately and then adding the results.

Let's break down the equation step by step:

1. K(A + B): Here, we add matrices A and B element-wise first. The resulting matrix is then multiplied by the scalar K.
2. KA + KB: In this case, we multiply matrix A by the scalar K, and we also multiply matrix B by the scalar K. Finally, we add the two resulting matrices element-wise.

The equation states that these two expressions are equal: K(A + B) is the same as KA + KB. It is important to note that this property holds only when matrices A and B have the same dimensions.

The distributive property is a fundamental property in linear algebra and is commonly used in various matrix calculations and transformations. It allows for simplification and efficient computation by breaking down operations into smaller steps.

Matrices Addition and substraction

Matrix addition and subtraction are basic operations performed on matrices of the same size. Let's explore these operations in more detail:

- **Matrix Addition:** To add two matrices A and B of the same size (with the same number of rows and columns), you simply add the corresponding elements of the matrices to create a new matrix. The resulting matrix will have the same dimensions as the original matrices.

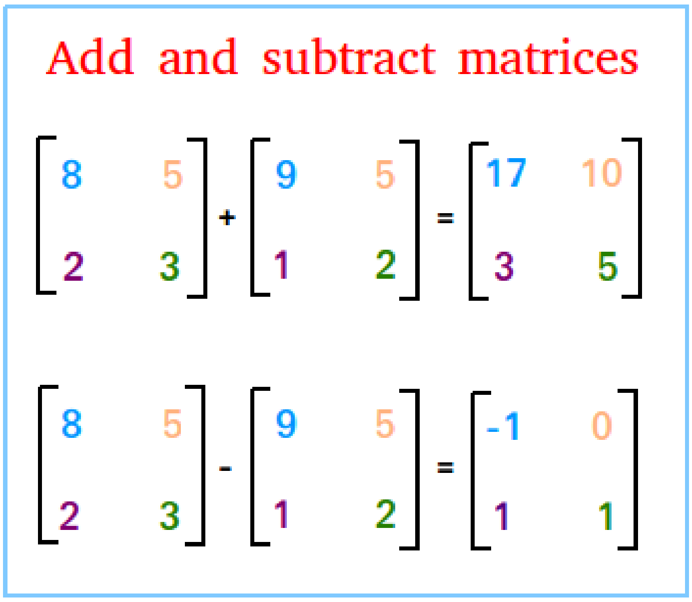
Formally, if A and B are matrices of size m × n, then the sum of A and B, denoted as A + B, is a matrix C of the same size, where each element C[i, j] is obtained by adding the corresponding elements from A and B: C[i, j] = A[i, j] + B[i, j] for all i and j.

For example, let's consider the matrices A and B: A = | 1 2 | | 3 4 |

B = | 5 6 | | 7 8 |

The sum of A and B, A + B, will be: A + B = | 1 + 5 2 + 6 | | 3 + 7 4 + 8 |

```
| 6   8 |
| 10  12 |
```



$$\begin{bmatrix} 8 & 5 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 9 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 17 & 10 \\ 3 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 8 & 5 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 9 & 5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}$$

- **Matrix Subtraction:** To subtract matrix B from matrix A, you subtract the corresponding elements of B from A to obtain a new matrix. Similar to matrix addition, the resulting matrix will have the same dimensions as the original matrices.

Formally, if A and B are matrices of size m × n, then the difference between A and B, denoted as A - B, is a matrix C of the same size, where each element C[i, j] is obtained by subtracting the corresponding elements from A and B: C[i, j] = A[i, j] - B[i, j] for all i and j.

Using the previous example, let's find the difference between A and B, A - B: A - B = | 1 - 5 2 - 6 | | 3 - 7 4 - 8 |

```
| -4    -4 |
| -4    -4 |
```

Matrix addition and subtraction are essential operations in linear algebra. They are used in various mathematical computations, transformations, and algorithms. These operations allow for combining or comparing matrices element-wise, which is often necessary in solving systems of linear equations, performing linear transformations, and analyzing data in various fields.

The additive inverse and additive identity in the context of matrix operations:

- **Additive Inverse**: The additive inverse of a matrix A, denoted as -A, is a matrix that, when added to A, results in the zero matrix. In other words, -A is the matrix that, when added to A, cancels out its elements and produces a matrix where all elements are zero.

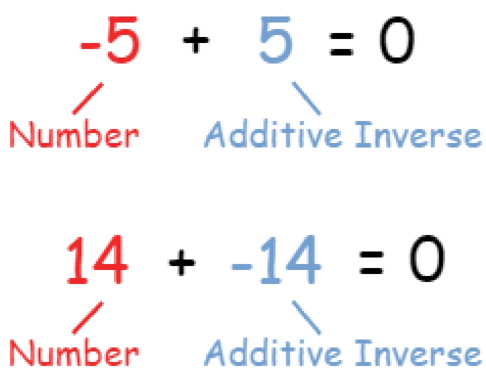
Mathematically, for any matrix A, the additive inverse -A is given by: -A = (-1) * A, where (-1) is the scalar -1 multiplied by every element of the matrix A.

For example, if we have matrix A: A = | 2 3 | | 4 5 |

The additive inverse of A, denoted as -A, will be: -A = (-1) * A = | -2 -3 | | -4 -5 |

When you add A to its additive inverse, you get the zero matrix: A + (-A) = | 2 + (-2) 3 + (-3) | | 4 + (-4) 5 + (-5) |

```
| 0   0 |
| 0   0 |
```



- **Additive Identity**: The additive identity is a special matrix called the zero matrix, denoted as O or 0. It is a matrix where all elements are zero.

For any matrix A, the zero matrix O has the property that A + O = O + A = A. Adding the zero matrix to any matrix leaves the original matrix unchanged.

For example, if we have matrix A: A = | 1 2 | | 3 4 |

Adding the zero matrix O of the same size as A to A will give: A + O = | 1 + 0 2 + 0 | | 3 + 0 4 + 0 |

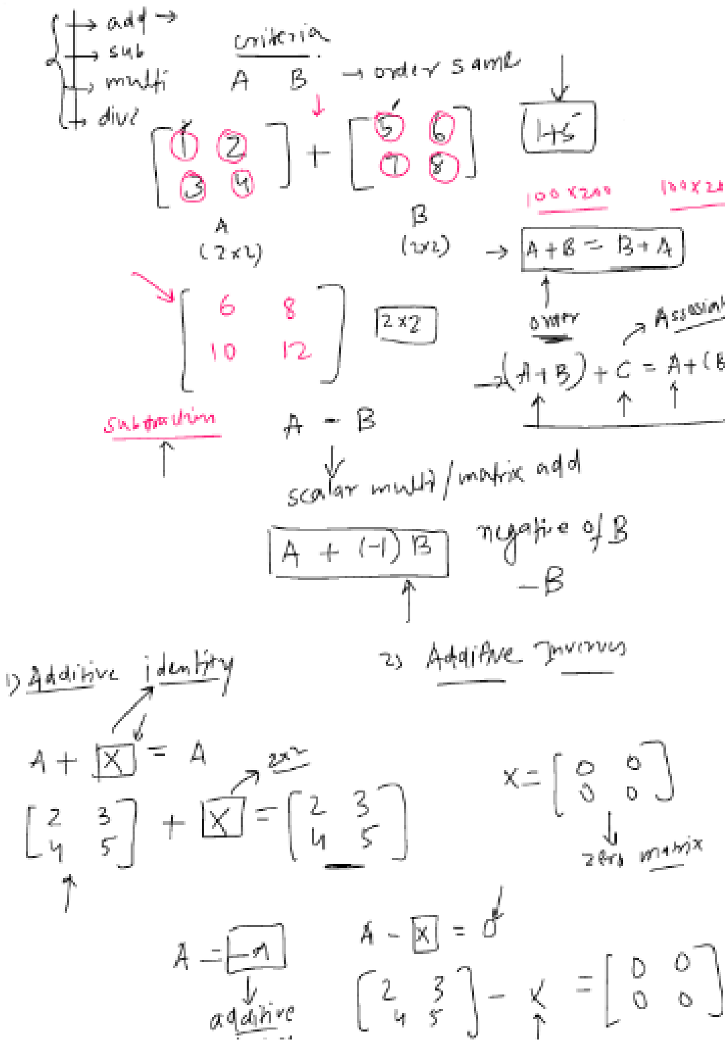
```
| 1   2 |
| 3   4 |
```

Similarly, adding A to the zero matrix O will also give the same result: O + A = | 0 + 1 0 + 2 | | 0 + 3 0 + 4 |

```
| 1   2 |
| 3   4 |
```

The zero matrix serves as the identity element for matrix addition.

Both the additive inverse and additive identity play important roles in matrix operations, providing properties and elements that allow for simplification, cancellation, and maintaining consistency in computations involving matrices.



Rules matrix addition and subtraction

some important rules and properties related to matrix addition and subtraction:

1. **Same Order**: Matrix addition and subtraction are defined only for matrices of the same size. That is, you can add or subtract two matrices only if they have the same number of rows and columns.

2. **Commutative Property**: For matrices A and B of the same size, **A + B = B + A**. The order of addition does not affect the result.

3. **Associative Property**: For matrices A, B, and C of the same size, **(A + B) + C = A + (B + C)**. The grouping of addition operations does not affect the result.

4. **Additive Identity**: There exists a special matrix called the zero matrix, denoted as O or 0, with the same size as any given matrix A, such that **A + O = O + A = A**. In other words, adding the zero matrix to any matrix leaves the matrix unchanged.

5. **Additive Inverse**: For any matrix A, there exists a matrix called the additive inverse or negative of A, denoted as -A, such that **A + (-A) = (-A) + A = O**, where O is the zero matrix. In other words, adding a matrix to its additive inverse results in the zero matrix.

These rules and properties are fundamental to matrix addition and subtraction and are analogous to the properties of addition in arithmetic. They ensure that these operations are well-defined and behave consistently.

These properties allow for simplification and efficient computation, as well as providing a foundation for solving systems of linear equations, analyzing linear transformations, and performing matrix manipulations in various applications.

Matrices Multiplication

Matrix multiplication is an operation that combines two matrices to create a new matrix. It is a fundamental operation in linear algebra and is used extensively in various mathematical and computational applications.

To multiply two matrices, the number of columns in the first matrix must be equal to the number of rows in the second matrix. Let's consider two matrices, A and B, where A has dimensions m × n and B has dimensions n × p. The resulting matrix C, denoted as C = A * B, will have dimensions m × p.

To compute the element C[i, j] of the resulting matrix C, you take the dot product of the i-th row of matrix A and the j-th column of matrix B. This involves multiplying corresponding elements of the row and column and summing them. Mathematically, for

C = A * B:

C[i, j] = A[i, 1] * B[1, j] + A[i, 2] * B[2, j] + ... + A[i, n] * B[n, j]

Here's an example to illustrate matrix multiplication:

Let matrix A be a 2 × 3 matrix: A = | 1 2 3 | | 4 5 6 |

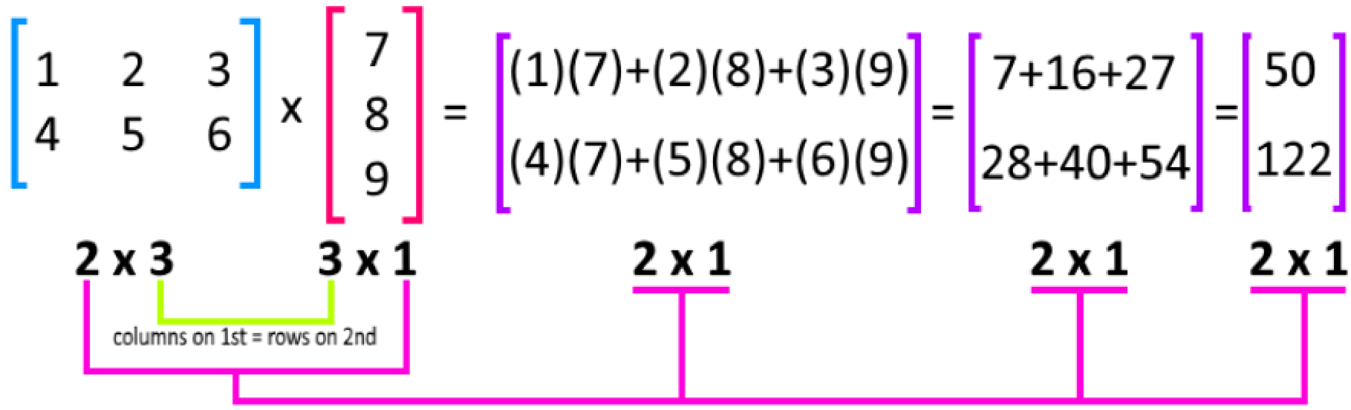
And let matrix B be a 3 × 2 matrix: B = | 7 8 | | 9 10 | | 11 12 |

To multiply A and B, we can compute each element of the resulting matrix C: C = A * B = | 1*7 + 2*9 + 3*11* 1*8 + 2*10 + 3*12 | | 4*7 + 5*9 + 6*11 4*8 + 5*10 + 6*12 |

```
        = |  58   64  |
          |  139  154  |
```

So, the resulting matrix C is a 2 × 2 matrix: C = | 58 64 | | 139 154 |

Matrix multiplication is not commutative, which means that the order in which matrices are multiplied matters. In general, AB ≠ BA, unless both matrices are square and represent special cases.



The number of rows in the 1st matrix and the number of columns in the 2nd matrix, make the dimensions of the final matrix

Multiplicative identity

The multiplicative identity is a concept applicable to matrix multiplication. It refers to a special matrix that, when multiplied by any other matrix, leaves the other matrix unchanged. In other words, it acts as the identity element for matrix multiplication.

The multiplicative identity matrix is typically denoted as I or sometimes as 1. It is always a square matrix, meaning it has an equal number of rows and columns. The dimensions of the identity matrix are usually indicated by a subscript, such as I_n for an n × n identity matrix.

The defining property of the identity matrix is that, for any matrix A of appropriate size, the product of A and the identity matrix I will result in A itself.

Mathematically, if A is an m × n matrix, I is an n × n identity matrix, and A_{mn} is the resulting matrix, then:

$A \cdot I = A_{mn}$

Similarly, if B is an n × p matrix, I is an n × n identity matrix, and B_{np} is the resulting matrix, then:

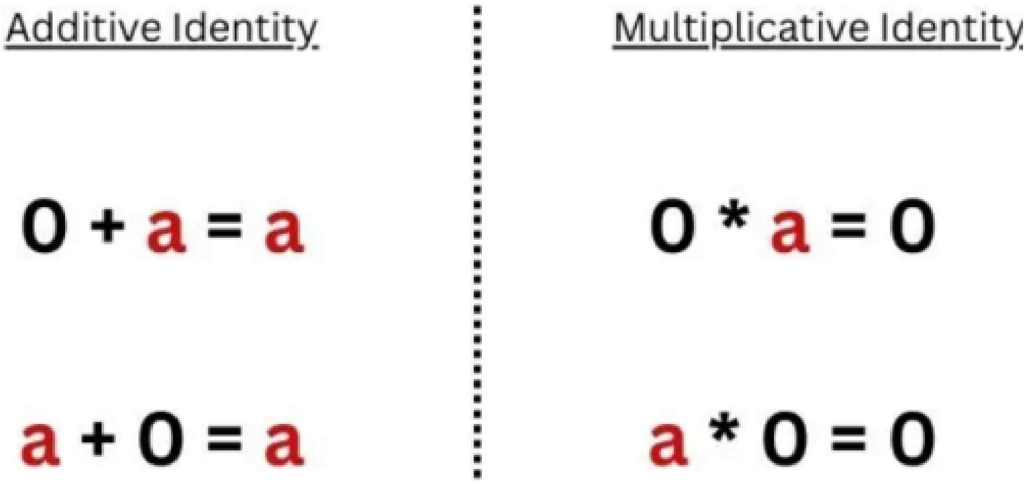
$I \cdot B = B_{np}$

The identity matrix is characterized by having ones (1) on its main diagonal (from the top left to the bottom right) and zeros (0) everywhere else.

For example, here is a 3 × 3 identity matrix, denoted as I_3:

I_3 = | 1 0 0 | | 0 1 0 | | 0 0 1 |

When multiplied by any 3 × 3 matrix, the identity matrix will leave the matrix unchanged.



Properties of Matix Multiplication

Matrix multiplication has several properties, including:

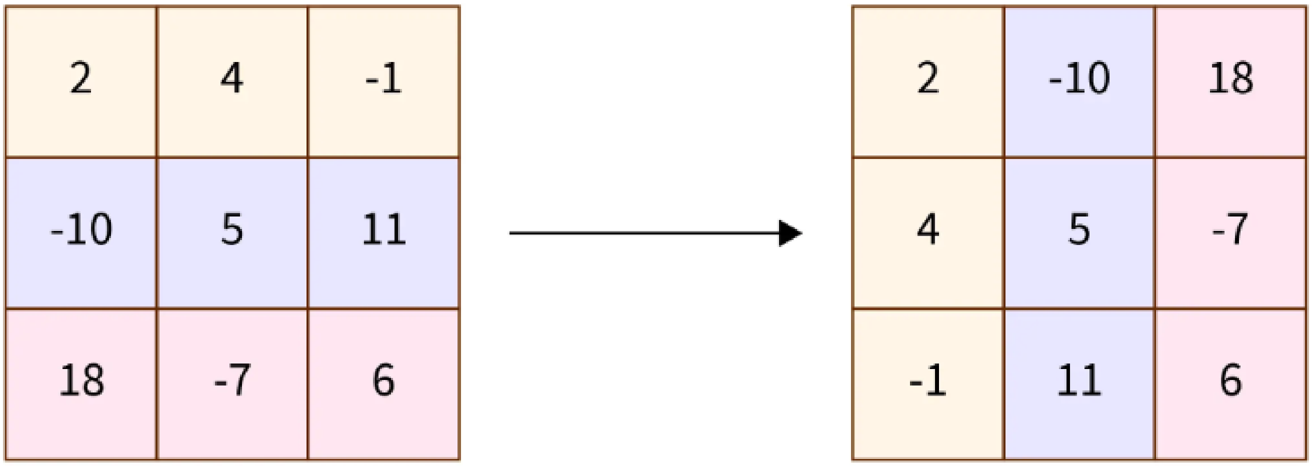
- Associativity: **(AB)C = A(BC)**
- Distributivity: **A(B + C) = AB + AC and (A + B)C = AC + BC**
- Distributivity over scalar multiplication: **k(AB) = (kA)B = A(kB)**
- Identity element: The identity matrix I is the multiplicative identity for matrices. This means that **AI = IA = A** for any matrix A.
- Inverses: If A is an invertible matrix, then there exists a matrix B such that **AB = BA = I.** The matrix B is called the inverse of A and is denoted by A^-1.

Transpose Matrix

The transpose of a matrix is an operation that swaps its rows and columns, creating a new matrix. In other words, the transpose of a matrix reflects it across its main diagonal (from the top left to the bottom right) to obtain a new matrix.

The transpose of a matrix A is denoted as **A^T or A**. If A has dimensions m × n, then the transpose of **A, A^T**, will have dimensions n × m.

To obtain the element at the i-th row and j-th column in the transpose of matrix A, denoted as (A^T)[i, j], you take the element at the j-th row and i-th column in matrix A, denoted as A[j, i].



Properties

Properties of the transpose operation include:

- **(A^T)^T = A**: Taking the transpose of a transpose results in the original matrix.
- **(A + B)^T = A^T + B^T**: The transpose of the sum of two matrices is equal to the sum of their transposes.
- **(kA)^T = k(A^T)**: The transpose of a scalar multiple of a matrix is equal to the scalar multiple of the transpose.

solving eq(n) Rules

1) $(A^T)^T = A$

2) $(A+B)^T = A^T + B^T$

3) $(AB)^T = B^T \cdot A^T$

Transpose

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

A^T A'

$A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

$(A^T)^T = A$

$(m \times n)^T$

$(n \times m)^T = \boxed{m \times n}$

$\boxed{B^T} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

$(A+B)^T = A^T + B^T$

$C = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 2 & 4 \end{bmatrix}_{2\times 3}$

$C^T = \begin{bmatrix} 2 & 1 \\ 3 & 2 \\ 4 & 4 \end{bmatrix}_{3\times 2}$

$m \times n = n \times m$

$\boxed{(AB)^T = B^T \cdot A^T}$

$\boxed{\text{geometrical}}$

$\boxed{\text{Symmetric}}$ matrix

$\boxed{A = A^T}$

$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$

Skew symmetric

$\boxed{A^T = -A}$

What is Determinant?

The determinant is a scalar value computed from a square matrix (a matrix with the same number of rows and columns) that carries important information about the matrix. It has several uses in linear algebra, including determining the invertibility of a matrix, finding the solution to systems of linear equations, and calculating the volume scaling factor for linear transformations.

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = a \bullet \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} - b \bullet \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + c \bullet \det\begin{bmatrix} d & e \\ g & h \end{bmatrix}$$

The determinant is a scalar value associated with a square matrix. It is a fundamental concept in linear algebra and carries important information about the matrix.

The determinant of a square matrix is denoted as det(A), |A|, or sometimes simply A. For a matrix A of size n × n, the determinant is a single value.

To calculate the determinant of a 2 × 2 matrix, such as A: A = | a b | | c d |

The determinant is given by: det(A) = ad - bc

For example, if A is: A = | 2 3 | | 4 5 |

The determinant of A is: det(A) = (2 * 5) - (3 * 4) = 10 - 12 = -2

For larger matrices, the calculation of the determinant becomes more complex. Different methods, such as cofactor expansion, Gaussian elimination, or using properties of determinants, can be used to find the determinant of matrices of higher dimensions.

$$\begin{aligned}\det\begin{bmatrix} 2 & -3 & 1 \\ 2 & 0 & -1 \\ 1 & 4 & 5 \end{bmatrix} &= 2 \bullet \det\begin{bmatrix} 0 & -1 \\ 4 & 5 \end{bmatrix} - (-3) \bullet \det\begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} + 1 \bullet \det\begin{bmatrix} 2 & 0 \\ 1 & 4 \end{bmatrix} \\ &= 2[0-(-4)] + 3[10-(-1)] + 1[8-0] \\ &= 2(0+4) + 3(10+1) + 1(8) \\ &= 2(4) + 3(11) + 8 \\ &= 8 + 33 + 8 \\ &= 49 \checkmark \end{aligned}$$

Properties of Determinants

Properties and characteristics of determinants include:

1. **Determinant of the Identity Matrix**: The determinant of the identity matrix I_n, where n is the number of rows (equal to the number of columns), is equal to 1. **$\det(I_n) = 1$**.

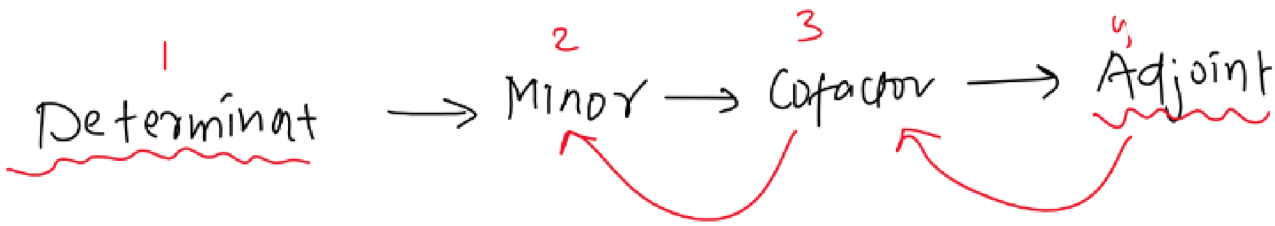
2. **Determinant of a Scalar Multiple**: If a matrix A is multiplied by a scalar k, the determinant of the scaled matrix is given by: **$\det(kA) = k^n * \det(A)$**, where n is the dimension of the matrix.

3. **Determinant of Transpose**: The determinant of the transpose of a matrix is equal to the determinant of the original matrix: **det(A^T) = det(A)**.

4. **Determinant of Inverse**: The determinant of a matrix and its inverse are related. If A is an invertible matrix (non-singular), then det(A) ≠ 0, and the determinant of the inverse matrix A^{-1} is given by: **det(A^{-1}) = 1 / det(A)**.

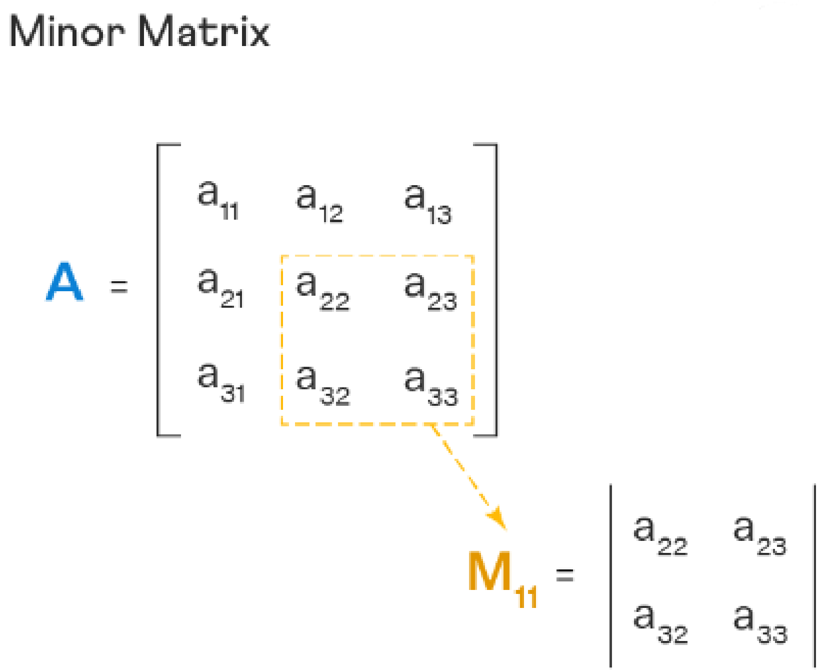
5. **Determinant and Matrix Operations**: The determinant can provide information about the properties of matrices, such as whether they are invertible or singular, linear dependence of vectors represented by the matrix, and whether a system of linear equations has a unique solution.

The determinant plays a crucial role in various areas, including solving systems of linear equations, finding eigenvalues and eigenvectors, diagonalizing matrices, and determining the volume or area scale factor of linear transformations.



what is Minor?

Minor of an element aij of a Determinant is the determinant obtained by deleting its ith row and jth col. It is denoted by Mij



To find the minor M_{ij} of a matrix A for the element a_{ij}, you follow these steps:

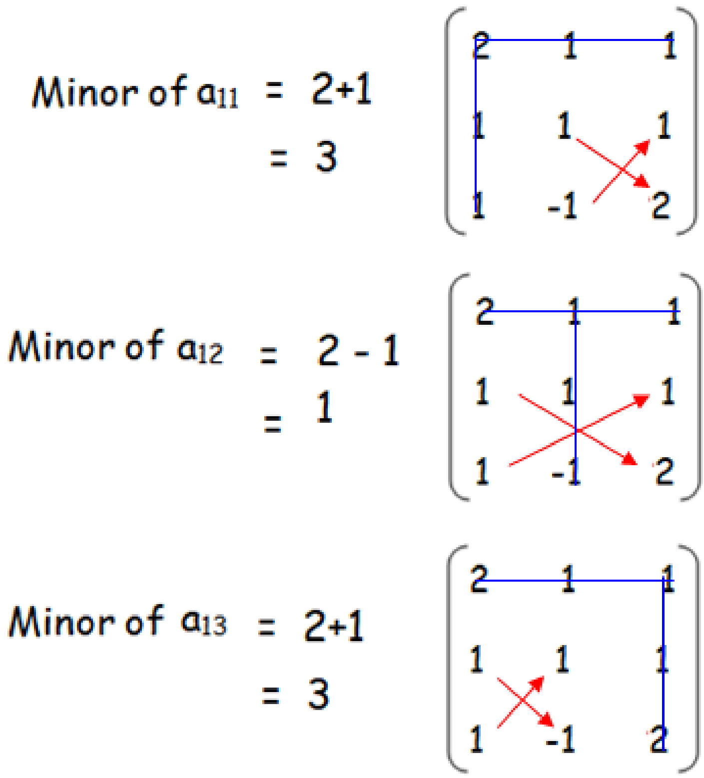
1. Start with a square matrix A.
2. Delete the i-th row and j-th column of A to obtain a submatrix.
3. Calculate the determinant of the submatrix. This determinant is denoted as M_{ij} and represents the minor of the element a_{ij}.

For example, let's consider the matrix A: A = | 2 3 4 | | 1 0 -1 | | 5 2 3 |

To find the minor M_{12} associated with the element a_{12}, we remove the first row and second column to form the submatrix B: B = | 1 -1 | | 5 3 |

The minor M_{12} of A associated with the element a_{12} is: M_{12} = det(B) = (1 * 3) - (-1 * 5) = 8

The minors of a matrix play a significant role in the computation of the cofactor matrix, which is used to find the inverse of a matrix and to evaluate determinants.



What is cofactor?

The cofactor of an element a_{ij} in a matrix is the signed minor associated with that element. It is obtained by multiplying the minor M_{ij} by a sign factor (-1)^(i+j), where i is the row index and j is the column index of the element.

Cofactor of an element of aij of a determinant is defined by Aij = (-1)^i+jMij where Mij is the minor of aij

$$A_{11} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \qquad C_{11} = +\det\begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \qquad C_{12} = -\det\begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad C_{13} = +\det\begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

The determinant of a square matrix can be calculated as the sum of the product of elements from any row (or column) with their corresponding cofactors.

Let's assume we have a square matrix A of size n × n. The determinant of A, denoted as det(A) or |A|, can be found using the cofactor expansion method along a row or column. According to this method:

1. Choose a row (or column) of the matrix A.
2. For each element a_{ij} in the chosen row (or column), calculate the product of a_{ij} with its corresponding cofactor C_{ij}.
3. Take the sum of all these products to obtain the determinant of A.

Mathematically, if we choose the i-th row (or column) for expansion, the determinant is given by:

$\det(A) = a_{i1} * C_{i1} + a_{i2} * C_{i2} + \ldots + a_{in} * C_{in}$

Alternatively, if we choose the j-th column (or row) for expansion, the determinant is given by:

$\det(A) = a_{1j} * C_{1j} + a_{2j} * C_{2j} + \ldots + a_{nj} * C_{nj}$

In both cases, the cofactors C_{ij} are calculated using the previously explained method, where each C_{ij} is the cofactor of the corresponding element a_{ij}.

This cofactor expansion method allows us to express the determinant of a matrix as a sum of products involving the elements and their cofactors along a chosen row or column.

This property of the determinant is useful for calculating determinants and evaluating matrix properties. It can be employed when the matrix has a particular row or column structure that simplifies the computation or when using recursive formulas for determinants.

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad M$$

$$A_{21} = (-1)^{2+1} M_{21} = -M_{21}$$

$$A_{11} = (-1)^{2} M_{11}$$

$i=1 \quad j=1$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & \boxed{a_{32}} & a_{33} \end{bmatrix}$$

$$\to A_{32} = (-1)^{3+2} M_{32}$$

$$\det = \boxed{a_{11} A_{11} + a_{12} A_{12} + a_{13} A_{13}}$$

What is Adjoint Matrix?

The adjugate of a matrix, also known as the classical adjoint, is a matrix formed by replacing each element in the original matrix with its corresponding cofactor and then taking the transpose of the resulting matrix. The adjugate of matrix A is denoted as adj(A).

$$\boxed{A \cdot A^{-1} = I} \qquad \boxed{A^{-1}}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}^{T}$$

$$\frac{5}{7} \to \boxed{5 \times 7^{-1}} \longrightarrow adj(A) = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

The adjoint of a square matrix is also known as the adjugate or classical adjoint. It is denoted as adj(A) or adjugate(A).

The adjoint of a matrix A is obtained by taking the transpose of the matrix of cofactors of A. In other words, each element of the adjoint matrix is the cofactor of the corresponding element in the original matrix, transposed across the main diagonal.

To find the adjoint of a matrix A, follow these steps:

1. Start with a square matrix A.
2. Calculate the cofactors C_{ij} of each element a_{ij} in A.
3. Construct the matrix of cofactors C, where C_{ij} is the cofactor of a_{ij}.
4. Take the transpose of matrix C to obtain the adjoint matrix adj(A).

Mathematically, if A is a square matrix, the adjoint matrix adj(A) is given by:

adj(A) = (C^T)

where C^T denotes the transpose of the matrix of cofactors C.

The adjoint of a matrix is related to the calculation of the inverse of a matrix. Specifically, if A is an invertible matrix (non-singular), then the inverse of A, denoted as A^(-1), can be calculated using the adjoint as follows:

A^(-1) = (1 / det(A)) * adj(A)

Here, det(A) represents the determinant of A.

The adjoint matrix is used in various applications, including finding inverses, solving systems of linear equations, evaluating determinants, and performing transformations. It allows us to obtain important information about the original matrix and provides a means to solve linear systems and perform matrix computations.

[https://www.nagwa.com/en/videos/273102820428/ (https://www.nagwa.com/en/videos/273102820428/)](https://www.nagwa.com/en/videos/273102820428/)

1) Check $|A| \neq 0$
2) Find all matrix minors A_{ij}
3) Construct cofactor matrix of A
   $c_{ij} = (-1)^{i+j} |A_{ij}|$
4) Construct adjoint of A
   $adj(A) = C^T$
5) $A^{-1} = \frac{1}{|A|} adj(A)$

$$A = \begin{bmatrix} 3 & 0 & 6 & 8 \\ 2 & 1 & -1 & 0 \\ 3 & 4 & 2 & 4 \end{bmatrix}$$

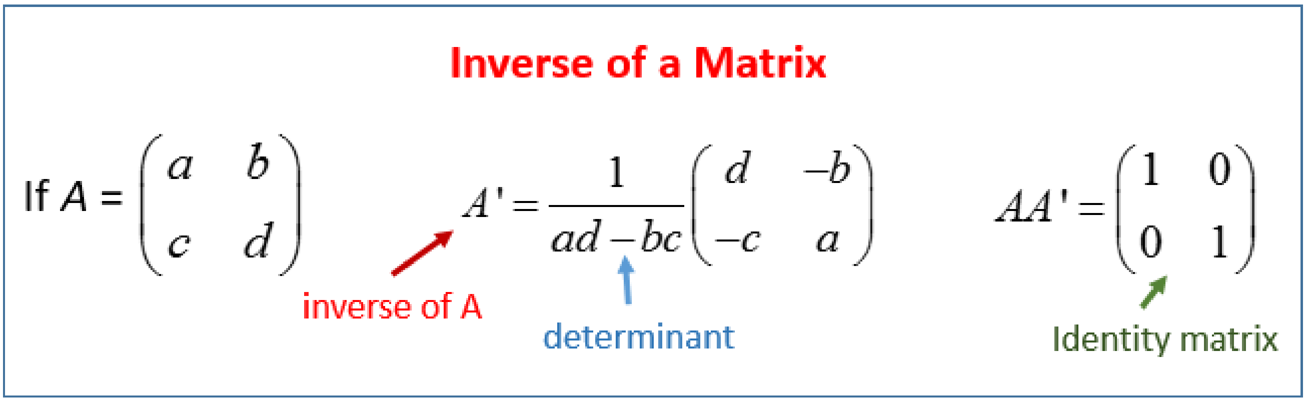
$$A_{23} = \begin{bmatrix} 3 & 0 & \cancel{6} & 8 \\ \cancel{2} & \cancel{1} & \cancel{-1} & \cancel{0} \\ 3 & 4 & \cancel{2} & 4 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 8 \\ 3 & 4 & 4 \end{bmatrix}$$

$$\boxed{\text{INVERSE OF A MATRIX: THE ADJOINT METHOD}}$$

$$C = \begin{bmatrix} (-1)^{1+1}|A_{11}| & (-1)^{1+2}|A_{12}| & \cdots & (-1)^{1+n}|A_{1n}| \\ (-1)^{2+1}|A_{21}| & (-1)^{2+2}|A_{22}| & \cdots & (-1)^{2+n}|A_{2n}| \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{n+1}|A_{n1}| & (-1)^{n+2}|A_{n2}| & \cdots & (-1)^{n+n}|A_{nn}| \end{bmatrix}$$

What is Inverse of Matrix?

An inverse matrix is a matrix that, when multiplied by the original matrix, results in the identity matrix. The inverse matrix is defined only for square matrices (matrices with the same number of rows and columns) and not all square matrices have an inverse.



- A matrix is invertible (has an inverse) if and only if it is non-singular, meaning its determinant is non-zero. If the determinant of A is zero, A is called a singular matrix, and it does not have an inverse.
- The inverse of a square matrix A, denoted as A^(-1), is a special matrix that, when multiplied by A, gives the identity matrix I. In other words, if A is invertible, the product of A and its inverse results in the identity matrix:

A * A^(-1) = A^(-1) * A = I

- The inverse of a matrix exists only for square matrices that are invertible or non-singular. A matrix is invertible if its determinant is non-zero.

To find the inverse of a matrix A, you can use various methods, such as the adjugate method or the Gaussian elimination method. One common method is to use the formula:

A^(-1) = (1 / det(A)) * adj(A)

where det(A) is the determinant of matrix A and adj(A) is the adjoint of matrix A.

Use the inverse matrix to solve $\overset{A}{\begin{bmatrix} 2 & 3 & 4 \\ -5 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}} \overset{\vec{u}}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \overset{\vec{v}}{\begin{bmatrix} 0 \\ 45 \\ -45 \end{bmatrix}}$, giving your answer as an appropriate matrix.

$\boxed{\vec{u} = A^{-1}\vec{v}}$ $\quad |A| = -45$

For nxn non-singular matrix A

$A^{-1} = \frac{1}{|A|}\,\mathrm{adj}(A)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 & 5 & -2 \\ 87 & -10 & -32 \\ -75 & 5 & 25 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} (-3\times 0) + (5\times -1) + (-2\times 1) \\ (87\times 0) + (-10\times -1) + (-32\times 1) \\ (-75\times 0) + (5\times -1) + (25\times 1) \end{bmatrix} = \begin{bmatrix} -7 \\ -22 \\ 20 \end{bmatrix}$$

$$\boxed{\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -7 \\ -22 \\ 20 \end{bmatrix}}$$

Inverse matrices play a crucial role in linear algebra and have many applications, such as solving systems of linear equations, finding the solution to a matrix equation, and performing various matrix operations. There are several methods for finding the inverse of a matrix, including Gaussian elimination, the adjugate method, and LU decomposition.

Solving a system of linear equations

Solution:

Given system of equations: **x + y = 5 ** (Equation 1) **4x + 3y = 15** (Equation 2)

- **Step 1: Write the system of equations in matrix form:**

| 1 1 | | x | | 5 | | 4 3 | * | y | = | 15 |

- **Step 2: Calculate the determinant of matrix A.**

The coefficient matrix A is: A = | 1 1 | | 4 3 |

det(A) = (1 * 3) - (1 * 4) = 3 - 4 = -1

- **Step 3: Calculate the matrix of minors.**

To find the matrix of minors, we calculate the determinant of each 2x2 submatrix formed by removing one row and one column.

The matrix of minors M is: M = | 3 1 | | 1 3 |

- **Step 4: Calculate the matrix of cofactors**.

The matrix of cofactors C is obtained by multiplying each element of the matrix of minors M by the corresponding sign factor.

The matrix of cofactors C is: C = | 3 -1 | | -1 3 |

- **Step 5: Calculate the adjoint of matrix A**.

The adjoint of matrix A, adj(A), is obtained by taking the transpose of the matrix of cofactors C.

The adjoint matrix adj(A) is: adj(A) = | 3 -1 | | -1 3 |

- **Step 6: Calculate the inverse of matrix A.**

The inverse of matrix A, A^(-1), is obtained by dividing the adjoint matrix adj(A) by the determinant det(A).

A^(-1) = (1 / det(A)) * adj(A) = (1 / -1) * | 3 -1 | | -1 3 |

```
= | -3  1 |
  | 1  -3 |
```

- **Step 7: Multiply both sides of the matrix equation AX = B by the inverse of matrix A to solve for X.**

A^(-1) * AX = A^(-1) * B

(I is the identity matrix) I * X = A^(-1) * B

X = A^(-1) * B

Substituting the values of A^(-1) and B: | x | | -3 1 | | 5 | | y | = | 1 -3 | * | 15 |

Simplifying the matrix multiplication:

| x | | (-3 * 5) + (1 * 15) | | y | = | (1 * 5) + (-3 * 15) |

```
| -30 |
| -40 |
```

Therefore, the solution to the system of linear equations is x = -30 and y = -40.

You can verify this solution by substituting x = -30 and y = -40 into the original equations. Both equations will be satisfied.

In [ ]:

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