Computation Tree Logic (CTL) & Basic Model Checking Algorithms

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What you'll learn

1. **Rationale behind declarative specifications:**
   - Why operational style is insufficient
2. **Computation Tree Logic CTL:**
   - Syntax
   - Semantics: Kripke models
3. **Explicit-state model checking of CTL:**
   - Recursive coloring

Operational models

Nowadays, a lot of ES design is based on executable behavioral models of the system under design, e.g. using

- Statecharts (a syntactically sugared variant of Moore automata)
- VHDL.

Such operational models are good at

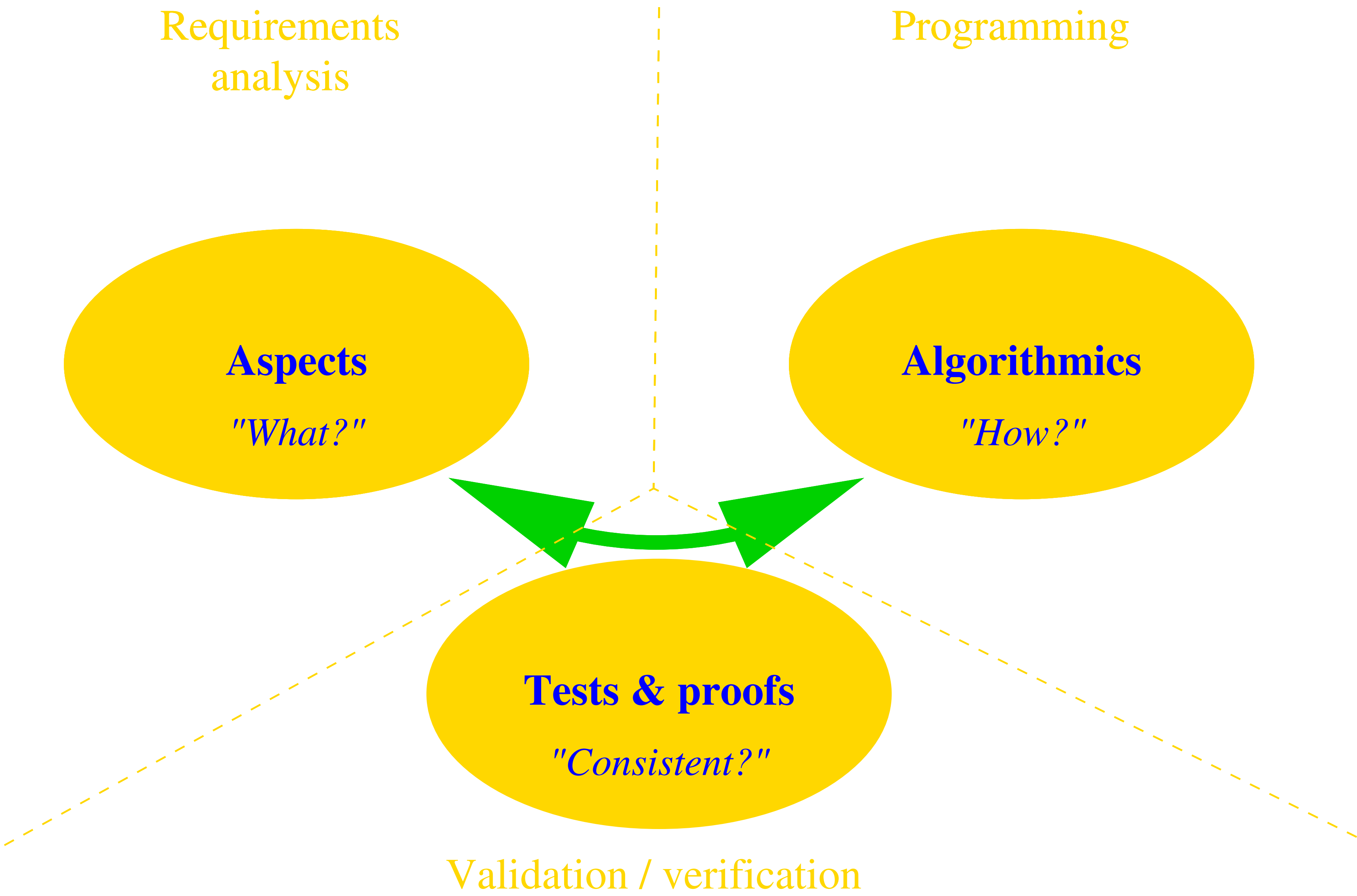
- supporting system analysis
  - simulation / virtual prototyping
- supporting incremental design
  - executable models
- supporting system deployment
  - executable model as “golden device”
  - code generation for rapid prototyping or final product
  - hardware synthesis

Operational models

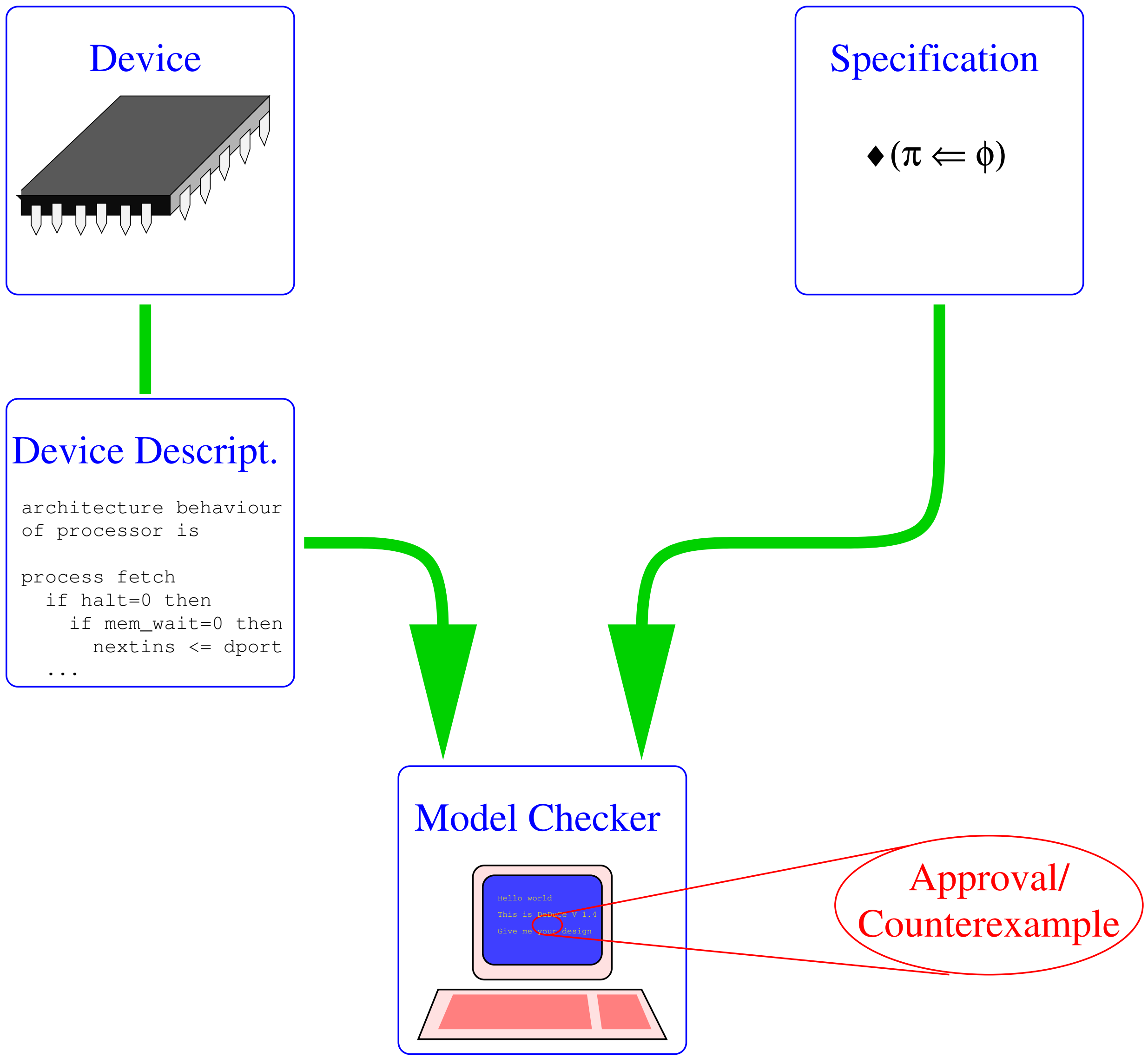
...are bad at

- supporting non-operational descriptions:
  - *What* instead of *how*.
  - E.g.: Every request is eventually answered.
- supporting negative requirements:
  - “Thou shalt not...”
  - E.g.: The train ought not move, unless it is manned.
- providing a structural match for requirement *lists*:
  - System has to satisfy R_1 *and* R_2 *and* ...
  - If system fails to satisfy R_1 then R_2 should be satisfied.

Multiple viewpoints

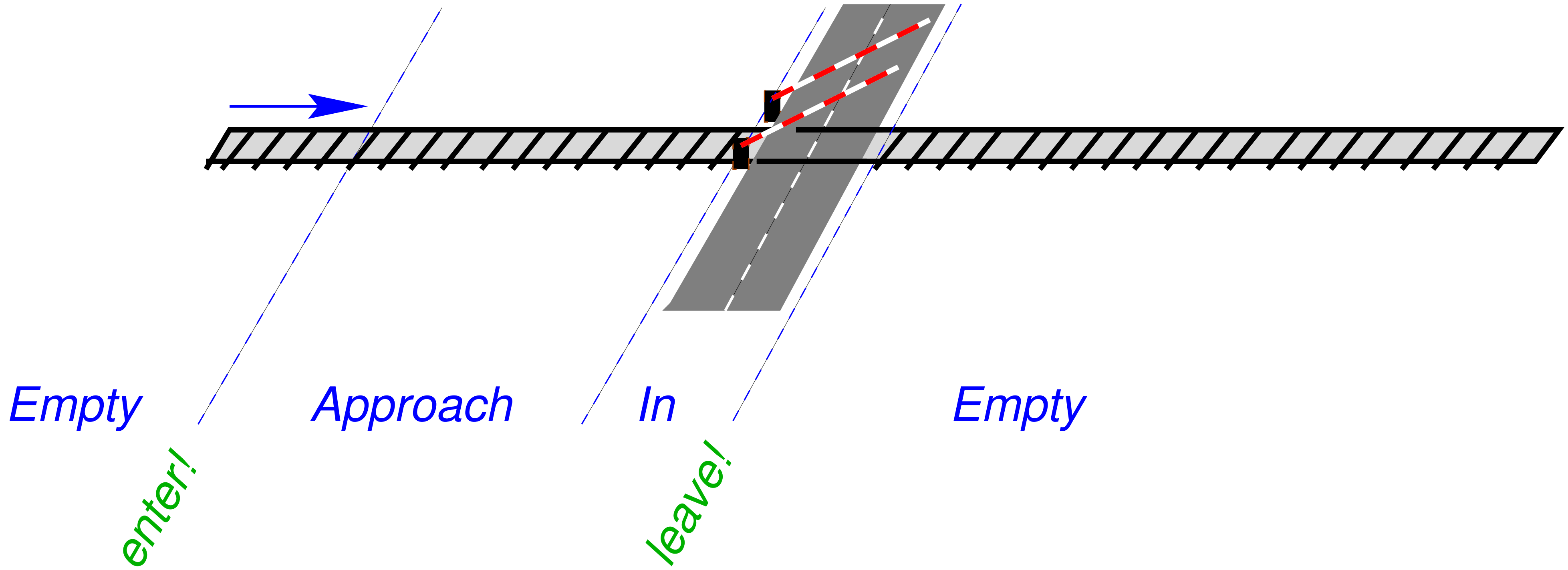


Model checking



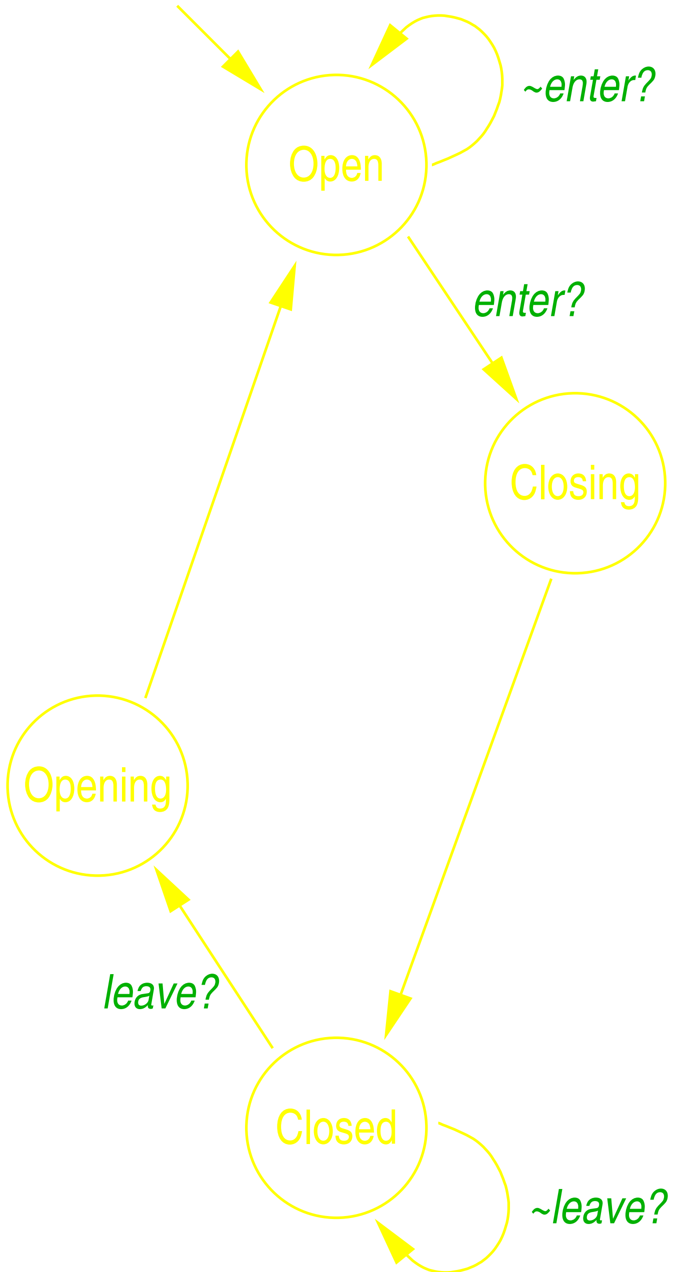
Exhaustive state-space search

Automatic verification/falsification of invariants



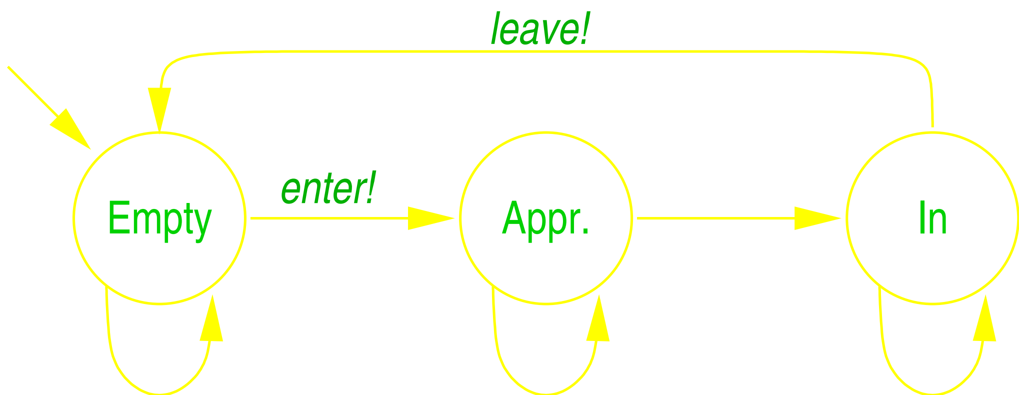
Safety requirement: Gate has to be closed whenever a train is in “In”.

The gate model

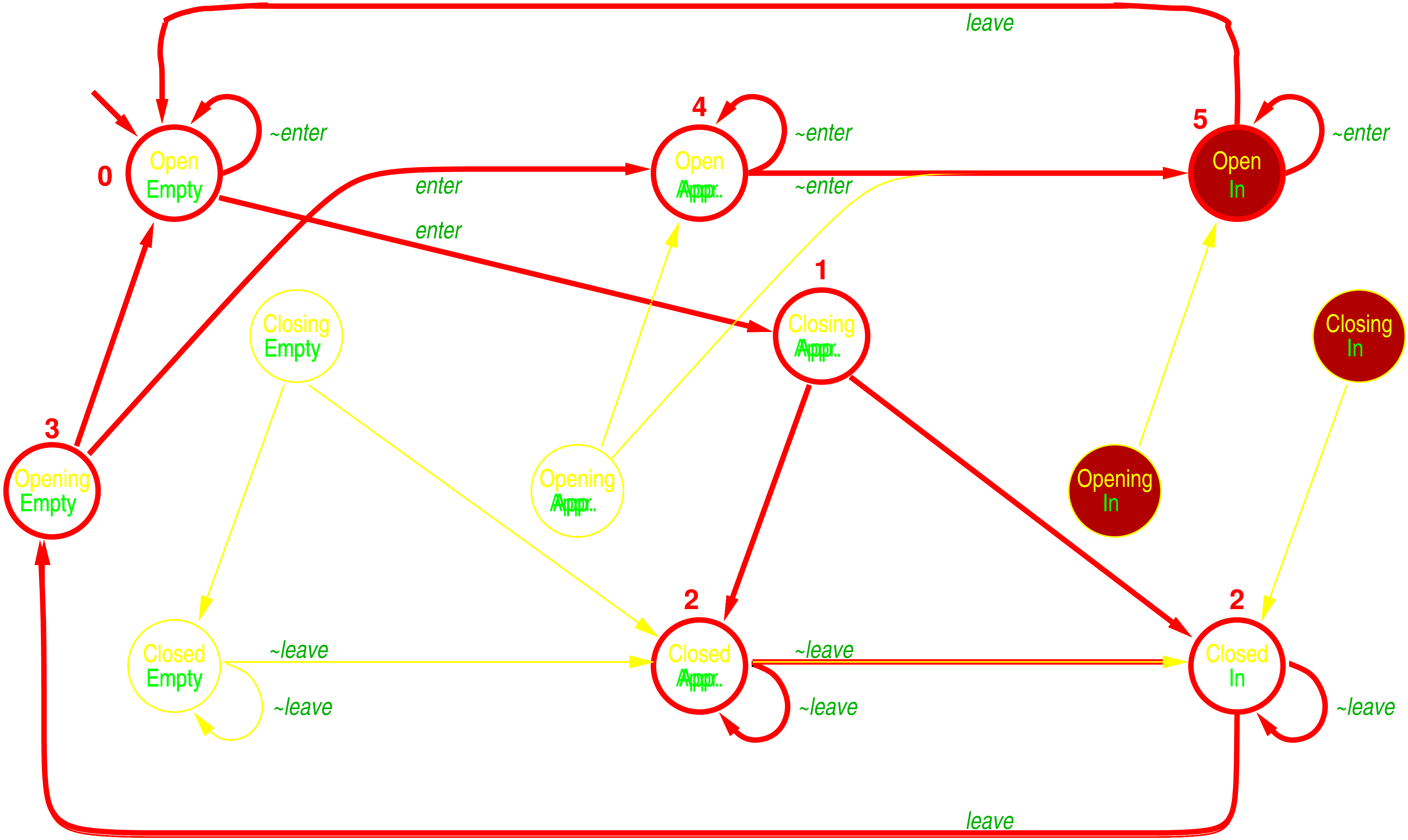


Track model

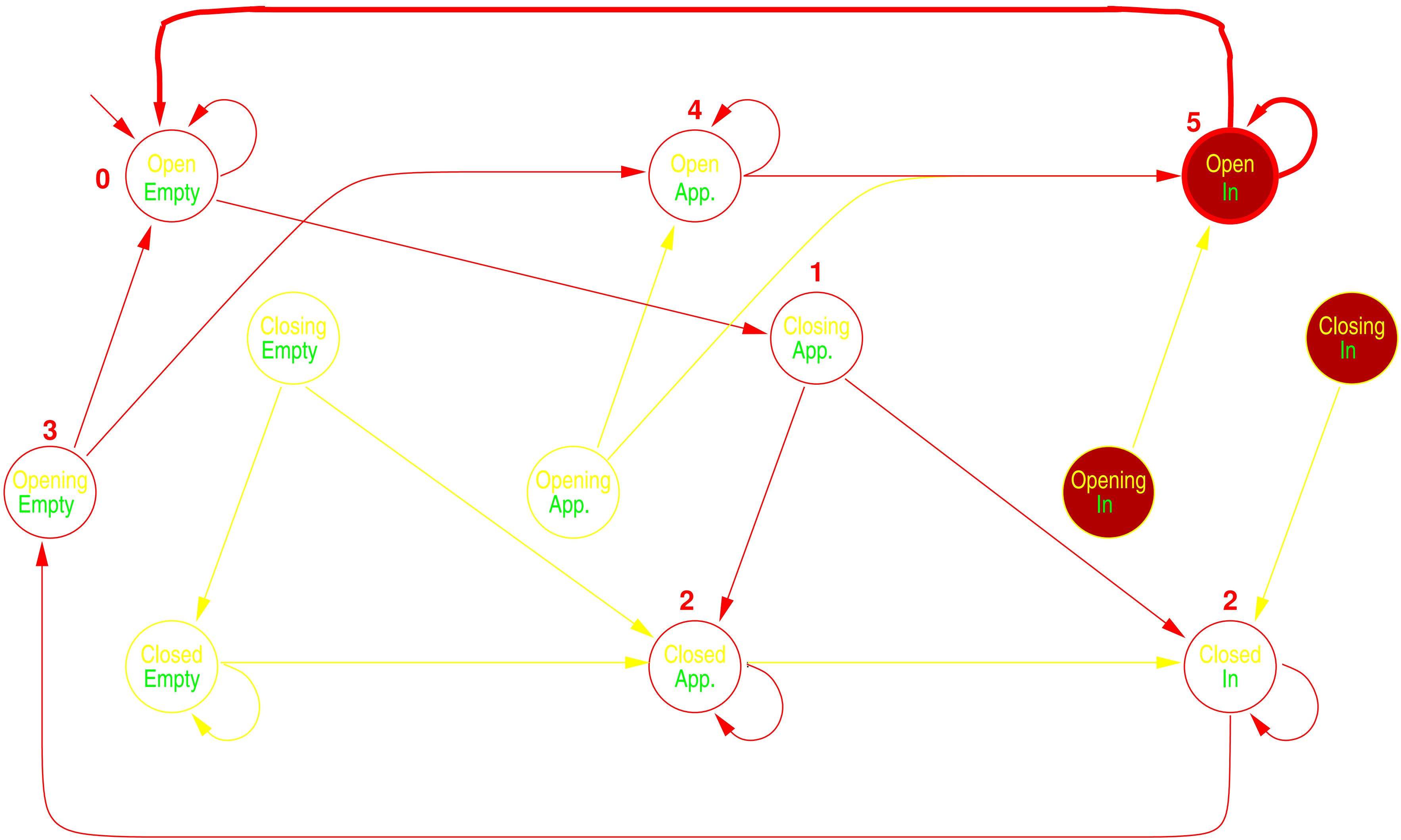
— safe abstraction —



Automatic check



Verification result



Stimuli: Empty, Approach, In , Empty , Approach, In.

Gate reaction: Open , Closing , Closed, Opening, Open , Open.

Computation Tree Logic

Syntax of CTL

We start from a countable set AP of atomic propositions.
The CTL formulae are then defined inductively:

- Any proposition $p \in AP$ is a CTL formula.
- The symbols $\perp$ and $\top$ are CTL formulae.
- If ϕ and ψ are CTL formulae, so are

  $\neg\phi$, $\phi \land \psi$, $\phi \lor \psi$, $\phi \to \psi$

  $\mathrm{EX}\,\phi$, $\mathrm{AX}\,\phi$

  $\mathrm{EF}\,\phi$, $\mathrm{AF}\,\phi$

  $\mathrm{EG}\,\phi$, $\mathrm{AG}\,\phi$

  $\phi\,\mathrm{EU}\,\psi$, $\phi\,\mathrm{AU}\,\psi$

Semantics (informal)

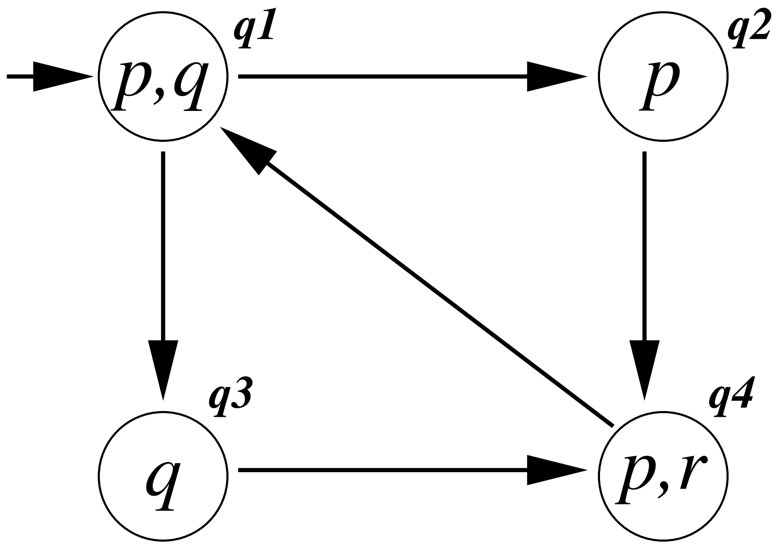
- $\mathtt{E}$ and $\mathtt{A}$ are path quantifiers:
  - $\mathtt{A}$: for all paths in the computation tree ...
  - $\mathtt{E}$: for some path in the computation tree ...
- $\mathtt{X}$, $\mathtt{F}$, $\mathtt{G}$ und $\mathtt{U}$ are temporal operators which refer to the path under investigation, as known from LTL:
  - $\mathtt{X}\phi$ (Next) : evaluate ϕ in the next state on the path
  - $\mathtt{F}\phi$ (Finally) : ϕ holds for some state on the path
  - $\mathtt{G}\phi$ (Globally) : ϕ holds for all states on the path
  - $\phi\mathtt{U}\psi$ (Until) : ϕ holds on the path at least until ψ holds

N.B. Path quantifiers and temporal operators are compound in CTL: there never is an isolated path quantifier or an isolated temporal operator. There is a lot of things you can't express in CTL because of this...

Semantics (formal)

CTL formulae are interpreted over Kripke structures.:
A Kripke structure K is a quadruple $K = (V, E, L, I)$ with

- V a set of vertices (interpreted as system states),
- $E \subseteq V \times V$ a set of edges (interpreted as possible transitions),
- $L \in V \to \mathcal{P}(AP)$ labels the vertices with atomic propositions that apply in the individual vertices,
- $I \subseteq V$ is a set of initial states.



Paths in Kripke structures

A path π in a Kripke structure $K = (V, E, L, I)$ is an edge-consistent infinite sequence of vertices:

- $\pi \in V^\omega$,
- $(\pi_i, \pi_{i+1}) \in E$ for each $i \in \mathbb{N}$.

Note that a path need not start in an initial state!

The labelling L assigns to each path π a propositional trace

$$tr_\pi = L(\pi) \stackrel{\text{def}}{=} \langle L(\pi_0), L(\pi_1), L(\pi_2), \ldots \rangle$$

that *path formulae* ($\mathtt{X}\phi, \mathtt{F}\phi, \mathtt{G}\phi, \phi\mathtt{U}\psi$) can be interpreted on.

Semantics (formal)

Let $K = (V, E, L, I)$ be a Kripke structure and $v \in V$ a vertex of K.

- $v, K \models \top$
- $v, K \not\models \bot$
- $v, K \models p$ for $p \in AP$ iff $p \in L(v)$
- $v, K \models \neg\phi$ iff $v, K \not\models \phi$,
- $v, K \models \phi \wedge \psi$ iff $v, K \models \phi$ and $v, K \models \psi$,
- $v, K \models \phi \vee \psi$ iff $v, K \models \phi$ or $v, K \models \psi$,
- $v, K \models \phi \Rightarrow \psi$ iff $v, K \not\models \phi$ or $v, K \models \psi$.

Semantics (contd.)

- $\nu, K \models \mathtt{EX}\,\phi$ iff there is *a path* π in K s.t. $\nu = \pi_1$ and $\pi_2, K \models \phi$,
- $\nu, K \models \mathtt{AX}\,\phi$ iff *all paths* π in K with $\nu = \pi_1$ satisfy $\pi_2, K \models \phi$,
- $\nu, K \models \mathtt{EF}\,\phi$ iff there is *a path* π in K s.t. $\nu = \pi_1$ and $\pi_i, K \models \phi$ for some i,
- $\nu, K \models \mathtt{AF}\,\phi$ iff *all paths* π in K with $\nu = \pi_1$ satisfy $\pi_i, K \models \phi$ for some i (that may depend on the path),
- $\nu, K \models \mathtt{EG}\,\phi$ iff there is *a path* π in K s.t. $\nu = \pi_1$ and $\pi_i, K \models \phi$ for all i,
- $\nu, K \models \mathtt{AG}\,\phi$ iff *all paths* π in K with $\nu = \pi_1$ satisfy $\pi_i, K \models \phi$ for all i,
- $\nu, K \models \phi\,\mathtt{EU}\,\psi$, iff there is *a path* π in K s.t. $\nu = \pi_1$ and some $k \in \mathbb{N}$ s.t. $\pi_i, K \models \phi$ for *each* $i < k$ and $\pi_k, K \models \psi$,
- $\nu, K \models \phi\,\mathtt{AU}\,\psi$, iff *all paths* π in K with $\nu = \pi_1$ have some $k \in \mathbb{N}$ s.t. $\pi_i, K \models \phi$ for *each* $i < k$ and $\pi_k, K \models \psi$.

A Kripke structure $K = (V, E, L, I)$ satisfies ϕ iff all its initial states satisfy ϕ,

i.e. iff $is, K \models \phi$ for all $is \in I$.

CTL Model Checking

Explicit-state algorithm

Rationale

We will extend the idea of verification/falsification by exhaustive state-space exploration to full CTL.

- Main technique will again be breadth-first search, i.e. graph coloring.
- Need to extend this to other modalities than `AG`..
- Need to deal with nested modalities.

Model-checking CTL: General layout

Given: a Kripke structure $K = (V, E, L, I)$ and a CTL formula ϕ

Core algorithm: find the set $V_\phi \subseteq V$ of vertices in K satisfying ϕ by

1. for each atomic subformula p of ϕ, mark the set $V_p \subseteq V$ of vertices in K satisfying ϕ
2. for increasingly larger subformulae ψ of ϕ, synthesize the marking $V_\psi \subseteq V$ of nodes satisfying ψ from the markings for ψ's immediate subformulae

until all subformulae of ϕ have been processed (including ϕ itself)

Result: report "$K \models \phi$" iff $V_\phi \supseteq I$

Reduction

The tautologies

$$
\begin{aligned}
\phi \vee \psi \quad &\Leftrightarrow \quad \neg(\neg\phi \wedge \neg\psi) \\
\mathtt{AX}\,\phi \quad &\Leftrightarrow \quad \neg\mathtt{EX}\,\neg\phi \\
\mathtt{AG}\,\phi \quad &\Leftrightarrow \quad \neg\mathtt{EF}\,\neg\phi \\
\mathtt{EF}\,\phi \quad &\Leftrightarrow \quad \top\,\mathtt{EU}\,\phi \\
\mathtt{EG}\,\phi \quad &\Leftrightarrow \quad \neg\mathtt{AF}\,\neg\phi \\
\phi\,\mathtt{AU}\,\psi \quad &\Leftrightarrow \quad \neg((\neg\psi)\,\mathtt{EU}\,\neg(\phi \vee \psi)) \wedge \mathtt{AF}\,\psi
\end{aligned}
$$

indicate that we can rewrite each formula to one only containing atomic propositions, $\neg$, $\wedge$, $\mathtt{EX}$, $\mathtt{EU}$, $\mathtt{AF}$.

After preprocessing, our algorithm need only tackle these!

Model-checking CTL: Atomic propositions

Given: A finite Kripke structure with vertices V and edges E and a labelling function L assigning atomic propositions to vertices.

Furthermore an atomic proposition p to be checked.

Algorithm: Mark all vertices that have p as a label.

Complexity: $O(|V|)$

Model-checking CTL: $\neg\phi$

Given: A set V_ϕ of vertices satisfying formula ϕ.

Algorithm: Mark all vertices not belonging to V_ϕ.

Complexity: $O(|V|)$

Model-checking CTL: $\phi \wedge \psi$

Given: Sets V_ϕ and V_ψ of vertices satisfying formulae ϕ or ψ, resp.

Algorithm: Mark all vertices belonging to $V_\phi \cap V_\psi$.

Complexity: $O(|V|)$

Model-checking CTL: $\mathrm{EX}\,\phi$

Given: Set V_ϕ of vertices satisfying formulae ϕ.

Algorithm: Mark all vertices that have a successor state in V_ϕ.

Complexity: $O(|V| + |E|)$

Model-checking CTL: $\phi EU \psi$

Given: Sets V_ϕ and V_ψ of vertices satisfying formulae ϕ or ψ, resp.

Algorithm: Incremental marking by

1. Mark all vertices belonging to V_ψ.
2. Repeat
   if there is a state in V_ϕ that has some successor state
   marked then mark it also
   until no new state is found.

Termination: Guaranteed due to finiteness of $V_\phi \subset V$.

Complexity: $O(|V| + |E|)$ if breadth-first search is used.

Model-checking CTL: $AF\phi$

Given: Set V_ϕ of vertices satisfying formula ϕ.

Algorithm: Incremental marking by

1. Mark all vertices belonging to V_ϕ.
2. Repeat

   if there is a state in V that has *all* successor states marked
   then mark it also

   until no new state is found.

Termination: Guaranteed due to finiteness of V.

Complexity: $O(|V| \cdot (|V| + |E|))$.

Model-checking CTL: $\mathsf{EG}\phi$, for efficiency

Given: Set V_ϕ of vertices satisfying formula ϕ.

Algorithm: Incremental marking by

1. Strip Kripke structure to V_ϕ-states:
   $(V, E) \rightsquigarrow (V_\phi, E \cap (V_\phi \times V_\phi))$.

$\rightsquigarrow$ Complexity: $O(|V| + |E|)$

2. Mark all states belonging to loops in the reduced graph.

$\rightsquigarrow$ Complexity: $O(|V_\phi| + |E_\phi|)$ by identifying *strongly connected components*.

3. Repeat
   if there is a state in V_ϕ that has *some* successor states marked then mark it also
   until no new state is found.

$\rightsquigarrow$ Complexity: $O(|V_\phi| + |E_\phi|)$

Complexity: $O(|V| + |E|)$.

Model-checking CTL: Main result

Theorem: It is decidable whether a finite Kripke structure (V, E, L, I) satisfies a CTL formula ϕ.
The complexity of the decision procedure is $O(|\phi| \cdot (|V| + |E|))$, i.e.

- linear in the size of the formula, given a fixed Kripke structure,
- linear in the size of the Kripke structure, given a fixed formula.

However, size of Kripke structure is exponential in number of parallel components in the system model.

Appendix

Fair Kripke Structures &

Fair CTL Model Checking

Fair Kripke Structures

A fair Kripke structure is a pair $(K, \mathcal{F})$, where

- $K = (V, E, L, I)$ is a Kripke structure
- $\mathcal{F} \subseteq \mathcal{P}(V)$ is set of vertice sets, called a fairness condition.

A fair path π in a fair Kripke structure $((V, E, L, I), \mathcal{F})$ is an edge-consistent infinite sequence of vertices which visits each set $F \in \mathcal{F}$ infinitely often:

- $\pi \in V^{\omega}$,
- $(\pi_i, \pi_{i+1}) \in E$ for each $i \in \mathbb{N}$,
- $\forall F \in \mathcal{F}. \exists^{\infty} i \in \mathbb{N}. \pi_i \in F$.

Note the similarity to (generalized) Büchi acceptance!

Fair CTL: Semantics

- $v, K, \mathcal{F} \models_F \mathtt{EX}\,\phi$ iff there is *a fair path* π in K s.t. $v = \pi_0$ and $\pi_1, K, \mathcal{F} \models_F \phi$,
- $v, K, \mathcal{F} \models_F \mathtt{AX}\,\phi$ iff *all fair paths* π in K with $v = \pi_0$ satisfy $\pi_1, K, \mathcal{F} \models_F \phi$,
- $v, K, \mathcal{F} \models_F \mathtt{EF}\,\phi$ iff there is *a fair path* π in K s.t. $v = \pi_0$ and $\pi_i, K, \mathcal{F} \models_F \phi$ for some i,
- $v, K, \mathcal{F} \models_F \mathtt{AF}\,\phi$ iff *all fair paths* π in K with $v = \pi_0$ satisfy $\pi_i, K, \mathcal{F} \models_F \phi$ for some i (that may depend on the fair path),
- $v, K, \mathcal{F} \models_F \mathtt{EG}\,\phi$ iff there is *a fair path* π in K s.t. $v = \pi_0$ and $\pi_i, K, \mathcal{F} \models_F \phi$ for all i,
- $v, K, \mathcal{F} \models_F \mathtt{AG}\,\phi$ iff *all fair paths* π in K with $v = \pi_0$ satisfy $\pi_i, K, \mathcal{F} \models_F \phi$ for all i,
- $v, K, \mathcal{F} \models_F \phi\,\mathtt{EU}\,\psi$, iff there is *a fair path* π in K s.t. $v = \pi_0$ and some $k \in \mathbb{N}$ s.t. $\pi_i, K, \mathcal{F} \models_F \phi$ for *each* $i < k$ and $\pi_k, K, \mathcal{F} \models_F \psi$,
- $v, K, \mathcal{F} \models_F \phi\,\mathtt{AU}\,\psi$, iff *all fair paths* π in K with $v = \pi_0$ have some $k \in \mathbb{N}$ s.t. $\pi_i, K, \mathcal{F} \models_F \phi$ for *each* $i < k$ and $\pi_k, K, \mathcal{F} \models_F \psi$.

A fair Kripke structure $((V, E, L, I), \mathcal{F})$ satisfies ϕ, denoted $((V, E, L, I), \mathcal{F}) \models_F \phi$, iff all its initial states satisfy ϕ, i.e. iff $is, K, \mathcal{F} \models_F \phi$ for all $is \in I$.

Model-checking CTL: Fair states

Lemma: Given a fair Kripke structure $(((V, E, L, I), \mathcal{F})$, the set *Fair* $\subseteq V$ of states from which a fair path originates can be determined algorithmically.

Alg.: This is a problem of finding adequate SCCs:

1. Find all SCCs in K.
2. Select those SCCs that do contain at least one state from each fairness set $F \in \mathcal{F}$.
3. Find all states from which at least one of the selected SCCs is reachable.

Model-checking fair CTL: $\mathrm{EX}\,\phi$

Given: Set V_ϕ of vertices fairly satisfying formulae ϕ.

Algorithm: Mark all vertices that have a successor state in $V_\phi \cap$ *Fair*.

Note that the intersection with *Fair* is necessary even though the states in V_ϕ *fairly* satisfy ϕ:

- ϕ may be an atomic proposition, in which case fairness is irrelevant;
- ϕ may start with an A path quantifier that is trivially satisfied by non-existence of a fair path.

Model-checking fair CTL: $\phi EU\psi$

Given: Sets V_ϕ and V_ψ of vertices fairly satisfying formulae ϕ or ψ, resp.

Algorithm: Incremental marking by

1. Mark all vertices belonging to $V_\psi \cap$ *Fair*.
2. Repeat

   if there is a state in V_ϕ that has some successor state marked then mark it also

   until no new state is found.

Model-checking fair CTL: $EG\phi$

Given: Set V_ϕ of vertices fairly satisfying formula ϕ.

Algorithm: Incremental marking by

1. Strip Kripke structure to V_ϕ-states:
   $(V, E) \rightsquigarrow (V_\phi, E \cap (V_\phi \times V_\phi))$.
2. Mark all states that can reach a *fair* SCC in the *reduced* graph.
   (Same algorithm as for finding the set *Fair*, yet applied to the reduced graph.)