Approximation Algorithms for Preference Aggregation Using CP-Nets

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Abstract

This paper studies the design and analysis of approximation algorithms for aggregating preferences over combinatorial domains, represented using Conditional Preference Networks (CP-nets). Its focus is on aggregating preferences over so-called swaps, for which optimal solutions in general are already known to be of exponential size. We first analyze a trivial 2-approximation algorithm that simply outputs the best of the given input preferences, and establish a structural condition under which the approximation ratio of this algorithm is improved to 4/3. We then propose a polynomial-time approximation algorithm whose outputs are provably no worse than those of the trivial algorithm, but often substantially better. A family of problem instances is presented for which our improved algorithm produces optimal solutions, while, for any ε , the trivial algorithm cannot attain a $(2-\varepsilon)$ -approximation. These results may lead to the first polynomial-time approximation algorithm that solves the CP-net aggregation problem for swaps with an approximation ratio substantially better than 2.

Introduction

The goal of preference aggregation is to find, given a set of individual rankings over objects called outcomes, either the best collective outcome or the best collective ranking over the outcomes. Preference aggregation has applications in the domain of recommender systems, multi-criteria object selection, and meta-search engines (Dwork et al. 2001). In this paper, we study preference aggregation over combinatorial domains, using so-called Conditional Preference Networks (CP-nets, (Boutilier et al. 2004)) as a compact representation model for outcome rankings. A combinatorial domain defines outcomes as vectors of attribute-value pairs. By expressing conditional dependencies between attributes, a CPnet represents the preferences over a large number of outcome pairs using compact statements. For example, given five attributes V_1 through V_5 , such a statement might be "Given value 1 in attribute V_4 , I prefer value 0 over value 1 in attribute V_5 ." This statement means that all outcomes with value assignment (1,0) for attributes (V_4,V_5) are preferred over those with value assignment (1, 1), irrespective of their values in the attributes V_1 through V_3 . This saves the

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resources needed for explicitly listing preferences between pairs of outcomes with various values in V_1 through V_3 . In this example, V_4 is called a parent of V_5 ; in general, an attribute can have more than one parent.

In this paper, we study the problem of aggregating multiple CP-nets N_1, \ldots, N_t (without cyclic dependencies between attributes) into a single CP-net N which forms the best possible consensus of N_1, \ldots, N_t in terms of the associated outcome rankings. One objective might be to minimize the total number of outcome pairs that are ordered differently by N and N_s , summed up over all $s \in \{1, ..., t\}$. However, feasible solutions to this problem are unlikely to be found, partly because even determining how a given CPnet orders a given outcome pair is NP-hard (Boutilier et al. 2004). For this and other reasons, Ali et al. (2021) propose to focus on an objective function that counts only *swaps* that are ordered differently by N and N_s . A swap is a pair of outcomes that differ only in the value of a single attribute; deciding how a CP-net orders any given swap can be done in polynomial time (Boutilier et al. 2004).

We adopt the objective function proposed by Ali et al. (2021). However, they showed that this function cannot be optimized in polynomial time; in particular, sometimes the size of the only optimal solution is exponential in the size of the input. In a preliminary result, we show that only the number of input CP-nets (not the number of attributes) contribute to the hardness of optimal aggregation. Motivated by these results, we study efficient approximation algorithms for Ali et al.'s objective function.

A first (trivial) approximation algorithm simply outputs an input that obtains the smallest value of the objective function among all inputs. It is a well-known fact that this trivial algorithm guarantees a 2-approximation, but Endriss and Grandi (2014) showed that the bound of 2 cannot be improved. Our first main result states that the trivial algorithm obtains an approximation ratio of 4/3 in case the inputs satisfy a natural (yet limiting) symmetry condition.

We then propose an improved algorithm that, given an attribute V_n , considers the parent sets for V_n in the input CP-nets $N_1, \ldots N_t$. For each such parent set P, the algorithm

¹Technically, we limit this objective function to only swaps in which the two outcomes differ in a fixed attribute V_n , since it is sufficient to reduce the CP-net aggregation problem to an attributewise aggregation problem. Technical details will follow.

first computes a provably optimal aggregate among all CP-nets that use P as a parent set for V_n . It then computes the objective value for each resulting aggregate and outputs one with the smallest such value. Our formal results on this improved algorithm entail that it is guaranteed to be no worse than the trivial algorithm. We then define a family of problem instances for which the improved algorithm is optimal (i.e., has approximation ratio 1), while the trivial algorithm has ratio at least 3/2. In particular, the ratio of the trivial algorithm cannot be bounded below 2 for this family. Whether the improved algorithm obtains a ratio of at most 4/3 in general, remains an open question.

We hope to thus initiate a line of research on approximation algorithms for CP-net aggregation, and to enrich the research on approximation algorithms in the more general context of binary aggregation (Endriss and Grandi 2014).

Related Work

A substantial body of research addresses preference aggregation using explicit preference orders, represented as permutations over the outcome space, with the goal of finding a permutation that minimizes some objective function (Sculley 2007; Dwork et al. 2001; Dinu and Manea 2006; Bachmaier et al. 2015). For outcomes defined over a combinatorial domain, one works with compact preference representations (Airiau et al. 2011), e.g., CP-nets (Boutilier et al. 2004), LP-Trees (Booth et al. 2010), utility-based models, or logical representation languages (Lang 2004).

One approach to preference aggregation using CP-nets is that of mCP-nets (Rossi, Venable, and Walsh 2004; Lukasiewicz and Malizia 2016, 2019, 2022). In this approach, the input is a set of partial CP-nets. No single aggregate model is constructed. Instead, preference reasoning tasks such as outcome ordering and optimization are performed using some voting rule on the set of input CP-nets, and all input CP-nets must be stored.

Given a set of acyclic input CP-nets, Lang (2007) proposes to elicit votes sequentially over the attributes, using the value assigned to any parent attributes, thus constructing a consensus outcome. Xia, Lang, and Ying (2007a,b) showed that sequential voting may lead to paradoxical outcomes, which can be avoided by assuming some linear ordering over the attributes. Further work on sequential voting was presented by Lang and Xia (2009); Grandi et al. (2014). A similar approach considers voting over general CP-nets using a hypercube-wise decomposition, (Xia, Conitzer, and Lang 2008; Conitzer, Lang, and Xia 2011; Li, Vo, and Kowalczyk 2011; Brandt et al. 2016). Lastly, (Cornelio et al. 2013, 2015, 2021) address aggregating CP-nets using PCP-nets, which are an extension to CP-nets that allow for probabilistic uncertainty.

In the present study, we are not interested in finding the joint best outcome of the given input CP-nets. Instead we want to create a consensus preference ordering (over all so-called swap pairs, i.e., pairs of outcomes that differ only in a single attribute) that best aggregates the given preference orders, under the constraint that this consensus ordering can be represented as a CP-net. Our approach is similar to that of Ali et al. (2021), in that we treat preference aggregation as

an optimization problem where the input profile and the optimal output are both represented using CP-nets. In contrast to the mCP-nets or PCP-nets approach, this avoids storing all input CP-nets and allows for applying existing CP-net algorithms for reasoning about preferences. However, Ali et al. (2021) showed that there is no polynomial-time algorithm solving the problem that we focus on. This motivates us to study approximation algorithms for said problem.

Hardness results for aggregation were also established outside the context of CP-nets. For rank aggregation over explicit total or partial orders over the outcome space, Dwork et al. (2001) showed that optimizing based on the cumulative pairwise distance from each input ordering, known as Kemeny optimization, is intractable. In the related field of judgement aggregation, Endriss, Grandi, and Porello (2012) proved that distance-based aggregation is intractable, which motivates a simple 2-approximation algorithm (Endriss and Grandi 2014). Ailon, Charikar, and Newman (2008); Ailon (2010) studied (expected) approximation ratios of a randomized algorithm as well as of a linear programming approach.

Preliminaries

Boutilier et al. (2004) define a CP-net N as a directed graph, in which the vertex set $\mathcal{V} = \{V_1, \dots, V_n\}$ is a set of n binary attributes, with $\{0,1\}$ as the set of possible values of each attribute V_i . A preference over V_i is now simply one of the two possible total orders over $\{0,1\}$. An edge (V_j,V_i) means that the user's preference over V_i depends on the value of V_j , in which case V_j is called a parent of V_i . We focus solely on acyclic CP-nets. By $Pa(N,V_i)$ one denotes the set of parents of V_i in a CP-net N. If $Pa(N,V_i) = \emptyset$ for all $V_i \in \mathcal{V}$, then N is called separable.

For each V_i , the user's conditional preferences over $\{0,1\}$ are listed in a Conditional Preference Table (CPT), denoted $\mathrm{CPT}(N, V_i)$. For example, suppose V_i has only one parent, namely V_i . Then the CPT entry $0:0 \succ 1$ is read "If V_i has the value 0, then 0 is preferred over 1 for V_i ." Since a CPT for V_i specifies at most one preference per assignment of values to $Pa(N, V_i)$, it lists at most 2^k preferences, called CPT rules, where $k = |Pa(N, V_i)|$. The size of a CPT is the total number of its rules. An incomplete CPT is one that is of size strictly less than 2^k . In this paper, we always assume implicitly that CP-nets are complete, i.e., any CPT contains the maximum possible number of rules.² This assumption can be limiting for algorithmic studies, but is not uncommon in the literature (see, e.g., (Alanazi, Mouhoub, and Zilles 2020; Ali et al. 2021)). Note that our study can be generalized to incomplete CP-nets with some additional effort, yet without major conceptual differences.

An instantiation of a set $\mathcal{V}' \subseteq \mathcal{V}$ is an assignment of values to each attribute in \mathcal{V}' ; then $\operatorname{Inst}(\mathcal{V}')$ denotes the set of all instantiations of \mathcal{V}' . Note that $\operatorname{Inst}(\emptyset)$ contains only the empty tuple. Assuming a fixed order over \mathcal{V} , each element $\gamma \in \operatorname{Inst}(\mathcal{V}')$ is simply a boolean vector with $|\mathcal{V}'|$

²This would allow us to represent CPTs more compactly, by only listing those rules whose preference over $\{0,1\}$ is less frequent. However, all formal results in this paper hold irrespective of whether one represents CPTs this way or by listing all rules.

components, where $\gamma[V_i]$ denotes the value of γ in V_i , if $V_i \in \mathcal{V}'$. Given $\mathcal{V}', \mathcal{V}'' \subseteq \mathcal{V}$ and $\gamma' \in \operatorname{Inst}(\mathcal{V}')$, and $\gamma'' \in \operatorname{Inst}(\mathcal{V}'')$, we say γ' is *consistent* with γ'' (and vice versa) iff $\gamma'[V] = \gamma''[V]$ for all $V \in \mathcal{V}' \cap \mathcal{V}''$. Elements of $Inst(\mathcal{V})$ are called outcomes. Thus, any outcome o corresponds to the vector $(o[V_1], \ldots, o[V_n])$. An outcome pair (o,o') is called a *swap* over V_i if o,o' differ only in their value in V_i , and $o[V_i] = 0$, $o'[V_i] = 1$.

In general, consider a CPT rule for V_i of the form γ : $b \succ b'$, where $\gamma \in \text{Inst}(Pa(N, V_i))$ and $\{b, b'\} = \{0, 1\}$. Here, γ is called the *context* of the CPT rule. The rule is interpreted using the *ceteris paribus* assumption: if (o, o')or (o', o) is a swap over V_i , $o[V_i] = b$, $o'[V_i] = b'$, and $o[V_j] = o'[V_j] = \gamma[V_j]$ for all $V_j \in Pa(N, V_i)$, then o is preferred over o', written o > o'. This way, a complete CPnet orders all swap pairs, i.e., for each swap (o, o') over any attribute V_i , the CP-net entails either $o \succ o'$ or $o' \succ o$. By identifying > with its transitive closure, one obtains a partial preference order over the space Inst(V) of all outcomes.

Problem Formulation The focus of this paper lies on forming an aggregate CP-net, given a tuple T (N_1,\ldots,N_t) of input CP-nets. This aggregate is supposed to represent a consensus among the preferences of the underlying individual input CP-nets. This raises the question of how to assess how well a CP-net N represents a consensus between several input CP-nets. One measure could be the total number of triples (s, o, o') where $1 \le s \le t$ and (o, o') is any outcome pair, such that N orders (o, o') differently than N_s . However, Boutilier et al. (2004) showed that deciding whether a CP-net entails $o \succ o'$ is NP-hard in general, which substantially hinders the design of efficient algorithms for constructing consensus CP-nets that minimize such measure. Moreover, not every outcome pair is ordered by every CP-net, which makes it non-trivial to even define when "N orders (o, o') differently than N_s " (Ali et al. 2021).

By contrast, deciding whether a CP-net entails o > o', for arbitrary swaps (o, o') can be done in polynomial time (Boutilier et al. 2004). Also, since every complete CP-net orders every swap, it is easy to test whether two CP-nets order a given swap in two different ways. We thus follow the approach by (Ali et al. 2021), namely to aggregate CP-nets with respect to a measure that counts the number of cases in which a proposed consensus CP-net disagrees with a given input CP-net. Specifically, given two CP-nets N and N_s , we define the swap disagreement $\Delta(N, N_s)$ as the number of swaps (o, o') such that N and N_s order (o, o') differently, i.e., one of them entails $o \succ o'$, while the other entails $o' \succ$ o. Our objective function for a CP-net N, given a tuple T = (N_1, \ldots, N_t) of input CP-nets, then evaluates to

$$f_T(N) = \sum_{1 \le s \le t} \Delta(N, N_s).$$

Our goal is the study of algorithms that, given T, aim at constructing an N that minimizes this objective function. Note that $f_T(N)$ can be calculated in polynomial time, given Tand N (Ali et al. 2021).

One further advantage of focusing on swaps rather than general outcome pairs is that the CPT for an attribute V_i alone determines how a CP-net orders a swap over V_i . Thus aggregating CP-nets can be done by aggregating their CPTs for each attribute separately. Therefore, we will henceforth overload the notation N and N_s (which so far only represented CP-nets) to refer to CPTs for the fixed attribute V_n , and we will focus only on aggregating CPTs for V_n .

In sum, this paper focuses on (efficient) algorithms that produce (optimal or approximate) solutions to the following problem, called the CPT aggregation problem:

- input: A tuple $T=(N_1,\ldots,N_t)$ of CPTs for an attribute V_n , over a set $\mathcal{V}=\{V_1,\ldots,V_{n-1}\}$ of n-1 potential parent attributes. (We call any such T a problem instance.)
- desired output: A CPT N for attribute V_n , over \mathcal{V} , that minimizes $f_T(N)$.

CPT aggregation is a special case of binary aggregation (Endriss and Grandi 2014). In binary aggregation, one fixes a set of issues $I = \{1, \dots, m\}$. A problem instance is a tuple of ballots, i.e., of elements of $\{0,1\}^m$, and the goal is to find the best collective ballot with a 0/1 vote for each issue. For CPT aggregation, each swap (o, o') would be an issue, and for each issue we would have one of two possible orderings.

Matrix Representation for CP-net Aggregation Consider $T = (N_1, \dots, N_t), 1 \le s \le t$, and any swap (o, o')over V_n . There is a context $\gamma \in \operatorname{Inst}(Pa(N_s, V_n))$ such that $o[V] = o'[V] = \gamma[V]$ for each $V \in Pa(N_s, V_n)$. Given γ , N_s entails $o \succ o'$ iff $\operatorname{CPT}(N_s, V_n)$ has the rule $\gamma: 0 \succ 1$. We thus encode the preference (called *vote*) of any given CPT on any given swap with a boolean value: it is 0 if N_s entails $o \succ o'$, and 1 otherwise. We also encode the 2^{n-1} swaps over V_n with their corresponding bit strings over the attributes V_1, \ldots, V_{n-1} . This encodes the votes of all input CPTs on all swaps using a $2^{n-1} \times t$ boolean matrix M(T). Any given row represents all the CPT votes for one swap, and any given column represents the votes of one CPT on all swaps. We use $M(T)_{\mu\nu}$ to denote the vote on swap μ by CPT ν . Certain sub-matrices of M(T) have useful interpretations. For $\mathcal{V}' \subseteq \mathcal{V} \setminus \{V_n\}$, $|\mathcal{V}'| = k$, and some context γ of $Inst(\mathcal{V}')$, the sub-matrix M' corresponding to only the swaps (o, o') with $o[V] = o'[V] = \gamma[V]$ for all $V \in \mathcal{V}'$ contains the 2^{n-k-1} rows with all possible instantiations of $V \setminus (V' \cup \{V_n\})$, and all t columns. For a sub-tuple τ of T, the sub-matrix M' corresponding to only the votes of input CPTs in τ contains all 2^{n-1} rows, and the $|\tau|$ columns corresponding to CPTs in τ . We now introduce some definitions based on this matrix representation.

Definition 1. $freq_M(1 > 0)$ denotes the number of votes in a matrix M encoded by 1 and $freq_M(0 > 1)$ denotes the number of votes in a matrix M encoded by 0. In particular, if M has 2^{n-1} rows and t columns,

$$\begin{split} freq_M(1\succ 0) &= \sum\nolimits_{0 \leq \mu < 2^{n-1}} \sum\nolimits_{1 \leq \nu \leq t} M_{\mu\nu} \\ freq_M(0\succ 1) &= t \cdot 2^{n-1} - freq_M(1\succ 0) \end{split}$$

$$freq_M(0 \succ 1) = t \cdot 2^{n-1} - freq_M(1 \succ 0)$$

For any given swap (o, o'), a given row of M(T) gives us the votes of the t input CPTs for (o, o'). Hence, we call the corresponding row vector of length t the voting configuration (of T for swap (o, o')). There are 2^t possible bit strings of length t, but not all of them necessarily occur as voting configurations in M(T).

Optimal Solutions

Ali et al. (2021) showed that the CPT aggregation problem cannot be solved optimally in polynomial time, simply because in some cases the size of any optimal solution is exponential in the size of the input tuple T (measured in terms of the total number of CPT rules in N_1, \ldots, N_t):

Theorem 2 (Ali et al. (2021)). There is a family $\mathcal{F}_{bad} = (T_n)_{n \in \mathbb{N}}$ of problem instances, such that any N^* minimizing $f_{T_n}(N^*)$ is of size exponential in the size of T_n .

The family \mathcal{F}_{bad} of problem instances witnessing Theorem 2 contains, for each $n \geq 4$, a tuple $T_n = (N_1^n, \ldots, N_{n-1}^n)$ of t = n-1 input CPTs, where $Pa(N_s, V_n) = \{V_s\}$. Every optimal aggregate CPT for T_n must have the full set $\{V_1, \ldots, V_{n-1}\}$ as a parent set and thus have 2^{n-1} CPT rules, while T_n contains only 2(n-1) rules (two rules per input CPT).

On the positive side, Ali et al. (2021) noted that optimal CPT aggregation is possible in polynomial time for the subset of problem instances in which the smallest input parent set has at most k attributes less than the union of all input parent sets, for some fixed k. We here offer an additional positive result, based on the observation that the design of \mathcal{F}_{bad} critically hinges on the number t of input CPTs growing linearly with the number n of attributes.

Proposition 3. Let $\mathcal{F}_{t \in O(1)}$ be a family of problem instances $T = (N_1, \ldots, N_t)$ for V_n over \mathcal{V} , where t is a constant. Then there is a linear-time algorithm that optimally solves the CPT aggregation problem for $\mathcal{F}_{t \in O(1)}$.

Proof. The size of the input is in $\Theta(2^{n'})$ where $n' = \max_{1 \leq s \leq t} |Pa(N_s, V_n)|$. Since t is a constant, we have $n' = \Theta(|\bigcup_{1 \leq s \leq t} Pa(N_s, V_n)|)$. Now consider an algorithm that constructs a CPT over $P := \bigcup_{1 \leq s \leq t} Pa(N_s, V_n)$; for each context γ over P, it checks each of the t input CPTs and determines whether $1 \succ 0$ is the majority preference over the input CPTs for context γ . If yes, $\gamma: 1 \succ 0$ is added to the output CPT; otherwise $\gamma: 0 \succ 1$ is added to the output CPT. Clearly, this algorithm optimally solves the CPT aggregation problem for $\mathcal{F}_{t \in O(1)}$ in linear time.

In particular, the difficulty of scaling with the number t of input parent sets is the only cause for the hardness result in Theorem 2—scaling with the number n of attributes does not pose any problems to efficient optimization. Given the overall hardness of optimally solving the CPT aggregation problem, the main focus of this paper is on efficiently constructing approximate solutions. First, we will look at obtaining approximate solutions by simply picking the best input CPT from the tuple T given as problem instance.

Best Input CPTs as Approximate Solutions

As shown by Endriss, Grandi, and Porello (2012), judgement aggregation modeled as a distance-based optimization problem is intractable. This motivated the work by Endriss and Grandi (2014), which still aims at a distance optimization approach, but restricts the solution space to the inputs

provided, aiming to find what they call the most representative voter. For our problem this is equivalent to using the input CPT minimizing the sum of pairwise distances from every other input CPT as an approximation to the optimal consensus CPT. The paper discusses three approaches to guide the selection of input to be used as the consensus. Two of these rules are shown to be 2-approximations of the optimal solution with a distance minimization approach. However, the paper also establishes that neither these rules, nor any other rule restricted to the input ballots submitted, can guarantee a better approximation. Their result immediately carries over to CPT aggregation.

Theorem 4. Let $T=(N_1,\ldots,N_t)$ be any problem instance and N any optimal solution for T. Then $\min\{f_T(N_s)\mid 1\leq s\leq t\}<2f_T(N)$. Moreover, for every $\varepsilon>0$, there exists a problem instance $T_\varepsilon=(N_1^\varepsilon,\ldots,N_{t_\varepsilon}^\varepsilon)$ such that $\min\{f_{T_\varepsilon}(N_s^\varepsilon)\mid 1\leq s\leq t_\varepsilon\}>(2-\varepsilon)f_{T_\varepsilon}(N_\varepsilon)$, where N_ε is any optimal solution for T_ε .

Proof. Since every problem instance of CPT aggregation is also a problem instance of binary aggregation, the first statement follows directly from the corresponding result by Endriss and Grandi (2014). The second statement likewise follows from (Endriss and Grandi 2014), since the proof therein of the corresponding binary aggregation statement uses a problem instance for which both the instance itself and the optimal solution can be cast as binary CPTs.³

Since one can calculate $f_T(N_s)$ for all $s \in \{1, \ldots, t\}$ in polynomial time, Theorem 4 trivially yields a polynomial-time 2-approximation algorithm. By exploiting the special structure of CPTs (as opposed to general ballots in binary aggregation), we can present an improved approximation ratio for the special case of so-called *symmetric* CPTs.

Definition 5. A CPT N_s for V_n , with $|Pa(N_s, V_n)| = k$, is called symmetric iff its corresponding column vector (in matrix representation) has an equal number of zeros and ones in any sub-matrix corresponding to some fixed context of a proper subset of $Pa(N_s, V_n)$.

In particular, when all input CPTs are symmetric and have pairwise disjoint parent sets, the trivial algorithm witnessing Theorem 4 is guaranteed to yield a 4/3-approximation. Moreover, *any* input CPT yields an equally good approximation in this case:

Theorem 6. Let $t \geq 3$ and $T = (N_1, \ldots, N_t)$ be a problem instance in which every N_s is symmetric, such that $Pa(N_s, V_n) \cap Pa(N_{s'}, V_n) = \emptyset$ for $s \neq s'$. Let N be any optimal solution for T. Then, for all $s \in \{1, \ldots, t\}$, we have $f_T(N_s) = \min\{f_T(N_{s'}) \mid 1 \leq s' \leq t\} \leq \frac{4}{3}f_T(N)$.

The proof of Theorem 6 is sketched via Lemmas 7-12, with details given in the appendix. Note that Lemmas 8-12 assume the same premises as Theorem 6.

Lemma 7. Let $T' = (N'_1, \dots, N'_t)$ be any problem instance of t CPTs. Assuming each of the 2^t voting configurations

³We will get back to this problem instance in Theorem 17.

occurs exactly once in M(T'), and N is an optimal solution for T', we have

$$f_{T'}(N) = \begin{cases} 2 \cdot \sum_{\kappa=0}^{c-1} \kappa \binom{2c-1}{\kappa} & \text{if } t = 2c - 1\\ 2 \cdot \sum_{\kappa=0}^{c-1} \kappa \binom{2c}{\kappa} + c \binom{2c}{c} & \text{if } t = 2c \end{cases}$$

Proof. Each possible voting configuration can be represented by some bit string of length t. Consider a voting configuration with κ CPTs voting $0 \succ 1$ and $t - \kappa$ CPTs voting $1 \succ 0$, $\kappa \le t - \kappa$. Clearly, N makes κ errors on this, $0 \le \kappa \le \lfloor \frac{t}{2} \rfloor$. The total error of N on all voting configurations where $0 \succ 1$ is in the minority is $\sum_{\kappa=0}^{\lfloor \frac{t}{2} \rfloor} \kappa \cdot \binom{t}{\kappa}$. Accounting for those where $1 \succ 0$ is in the minority, and (in case t is even) those where $0 \succ 1$ and $1 \succ 0$ are equally frequent, we obtain the desired expression for $f_{T'}(N)$. \square

Lemma 8. Each of the 2^t voting configurations of T is the row vector for exactly 2^{n-t-1} swaps.

Proof. Let $\mathcal{U}=(\mathcal{V}\setminus\{V_n\})\setminus\bigcup_{1\leq s\leq t}Pa(N_s,V_n)$. Consider a bit string of length t where the s-th bit represents the ordering entailed by N_s . Assume the s-th bit is 0. Given symmetry, we know that exactly 2^{p_s-1} contexts of $Pa(N_s,V_n)$ entail $0\succ 1$, where $p_s=|Pa(N_s,V_n)|$. Thus, 2^{p_s-1} of the voting configurations have 0 in the s-th bit. Extending this argument, a given voting configuration can be generated by $\prod_{s=1}^t 2^{p_s-1}$ contexts. Each such bit string also occurs once for each context of \mathcal{U} . Thus, $2^{|\mathcal{U}|}\prod_{s=1}^t 2^{p_s-1}$ swaps of V_n have this given voting configuration. Since all input parent sets are pairwise disjoint, this simplifies to 2^{n-t-1} .

The next four lemmas are stated, with proof details given in the appendix.

Lemma 9. Suppose each of the 2^t voting configurations occurs for the same number of swaps. Let N be an optimal solution for T. Then

$$f_T(N) = \begin{cases} t \cdot 2^{n-2} - 2^{n-t-1} \cdot c \binom{2c-1}{c} & \text{if } t = 2c - 1\\ t \cdot 2^{n-2} - 2^{n-t-1} \cdot c \binom{2c}{c} & \text{if } t = 2c \end{cases}$$

For a problem instance T under our premises, this yields combinatorial expressions to compute the error made by an optimal CPT. Next, we give an expression for the error made by any of the input CPTs N_s , which we claim to be at most 4/3 times the objective value of an optimal solution.

Lemma 10.
$$f_T(N_s) = (t-1) \cdot 2^{n-2}$$
 for all $s \in \{1, ..., t\}$.

Lemma 11. If
$$1 \le s \le t = 2c - 1$$
, $c > 1$ then $\frac{f_T(N_s)}{f_T(N)} \le \frac{4}{3}$.

Lemma 12. If
$$1 \le s \le t = 2c$$
, $c > 1$ then $\frac{f_T(N_s)}{f_T(N)} \le \frac{4}{3}$.

This finally completes the proof of Theorem 6.

Optimal Solution for a Fixed Parent Set

Theorem 4 states that the *best input CPT* is never worse than the optimal solution by more than a factor of 2. This raises the question whether the *best input parent set* yields better guarantees on the approximation ratio. To this end, instead of taking the best input CPT unmodified as an aggregate output, we propose Algorithm 1 as an approximation algorithm.

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Algorithm 1: Build \operatorname{CPT}(N^a, V_n) that minimizes f_T subject to Pa(N^a, V_n) \subseteq Pa(N_s, V_n) for some s \in \{1, \dots, t\} Input: A tuple T = (N_1, \dots, N_t) of CPTs for V_n Output: \operatorname{CPT}(N^a, V_n) with Pa(N^a, V_n) \subseteq Pa(N_s, V_n)
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1 for each $s \in \{1, \dots, t\}$ do

compute a CPT N_s^* that minimizes f_T among all CPTs with parent set contained in $Pa(N_s, V_n)$

for some $s \in \{1, \ldots, t\}$

3 end

4 $N^a = N_{s^*}^*$ where $f_T(N_{s^*}^*) = \min\{f_T(N_s^*) \mid 1 \le s \le t\}$.

Clearly, Algorithm 1 cannot produce worse outputs than the trivial algorithm that simply selects the best input CPT. What still needs to be addressed is (i) can this algorithm be designed to run in polynomial time?, and (ii) what approximation ratio can this algorithm achieve (under which circumstances)? To address (i), all that is needed is a polynomial-time algorithm that, given $T = (N_1, \dots, N_t)$ and a parent set $P \in \{Pa(N_s, V_n) \mid 1 \leq s \leq t\}$, produces a CPT N_s^* that minimizes f_T among all CPTs with parent set $Pa(N_s, V_n)$. With Algorithm 2, we provide an algorithm that solves a more general problem: given $T=% \frac{1}{2}\left(\frac{1}{2}\right) \left(\frac{$ (N_1, \ldots, N_t) and any parent set $P \subseteq \bigcup_{1 \le s \le n} Pa(N_s, V_n)$, it produces a CPT N_s^* that minimizes f_T^- among all CPTs with parent set contained in P. We will see below, that this algorithm runs in time polynomial in the size of its input, when the cardinality of P is bounded by the size of the largest input parent set.

Given T and P as input, Algorithm 2 computes, for each context γ of P, the frequency of both possible preference orderings, and assigns the ordering 0 > 1 if it is in the majority, $1 \succ 0$ otherwise. Lines 7-11 iterate over each N_s in T, and count the number of rules in $CPT(N_s, V_n)$ that would apply to a swap consistent with context γ and entail 0 > 1. This count is multiplied by the number of swaps ordered by each rule in $CPT(N_s, V_n)$ and then divided by the number of possible contexts for $P \setminus Pa(N_s, V_n)$, since P is fixed to γ . At each iteration, zerovotes has the number of swaps for which N_s entails 0 > 1 given γ . In line 12, zerovotes has the total number of swaps for which 0 > 1 is entailed given γ , summed over all input CPTs. The number of votes for $1 \succ 0$ is then found by subtracting this from the total number of votes given γ . Lines 13-17 then assign the ordering 0 > 1 if it is in the majority, 1 > 0 otherwise. Repeating this for each possible context of P, we obtain $CPT(N^a, V_n)$ with $2^{|P|}$ rules. Some of the attributes in P might be irrelevant parents for this CPT in the sense that their value does not affect the preference order of the CPT, see (Koriche and Zanuttini 2010; Allen 2015). Removing these attributes yields a more compact representation semantically equivalent to N^a . This can be done in time linear in the size of N^a and quadratic in n (Ali et al. 2021).

First, we show that Algorithm 2 is correct, i.e., it produces the optimal aggregate CPT with the given parent set.

Theorem 13. Algorithm 2 constructs $CPT(N^a, V_n)$ such that $f_T(N^a) \leq f_T(N)$ for all N with $Pa(N, V_n) \subseteq P$.

Algorithm 2: Build $CPT(N^a, V_n)$ that minimizes f_T with $Pa(N^a, V_n) \subseteq P$ for given T and P

```
Input : T = (N_1, ..., N_t), P \subseteq \{V_1, ..., V_{n-1}\}
  Output: An optimal CPT(N^a, V_n) wrt f_T
            Pa(N^a, V_n) \subseteq P
5 for each \gamma \in \text{Inst}(P) do
       zerovotes = 0
6
        for each s \in \{1, \ldots, t\} do
          numRules = the number of rules in CPT(N_s, V_n)
            voting 0 \succ 1 with contexts consistent with \gamma
           numSwaps = numRules \cdot 2^{n-|Pa(N_s,V_n)|-1}
8
          numSwaps = numSwaps/2^{|P-Pa(N_s,V_n)|}
          zerovotes = zerovotes + numSwaps
10
11
       onevotes = t \cdot 2^{n-|P|-1} - zerovotes
12
        if zerovotes > onevotes then
          add \gamma: 0 \succ 1 to CPT(N^a, V_n)
13
       end
14
15
       | add \gamma: 1 \succ 0 to CPT(N^a, V_n)
16
17
      remove irrelevant parents from CPT(N^a, V_n)
18
19 end
```

Proof. Assume there is some N, $Pa(N,V_n) = P_N \subseteq P$, such that $f_T(N^a) > f_T(N)$. Then there exists $\gamma \in Inst(P_N)$ for which N and N^a entail different orders, and N disagrees with T on fewer swaps consistent with γ than N^a does. Now γ corresponds to some sub-matrix M'. The corresponding sub-matrices for N and N^a contain a constant value each—one of them 0, the other 1. Wlog, assume N^a has an all-ones sub-matrix in place of M'. By Algorithm 2, this implies $freq_{M'}(0 \succ 1) \leq freq_{M'}(1 \succ 0)$. By our assumption, N' now has the all-zeros sub-matrix in place of M', and this all-zeros matrix has fewer inconsistencies with M' than N^a 's all-ones matrix. From this, we have $freq_{M'}(0 \succ 1) > freq_{M'}(1 \succ 0)$ —a contradiction.

Second, Algorithm 2 runs in polynomial time, when |P| is bounded by the size of the largest input parent set.

Theorem 14. Algorithm 2 runs in time $O(2^{|P|} \cdot \sum_{N_s \in T} |\operatorname{CPT}(N_s, V_n)|)$. In particular, if $|P| \leq \max\{|Pa(N_s, V_n) \mid 1 \leq i \leq t\}$, it runs in time polynomial in $\sum_{1 \leq s \leq t} |\operatorname{CPT}(N_s, V_n)|$. Moreover, Algorithm 1 runs in polynomial time.

Proof. The total runtime of the inner loop (lines 7-11 iterated) is in $O(\sum_{1 \leq s \leq t} |CPT(N_s, V_n)|)$, which is linear in input size. The inner loop runs once per $\gamma \in \operatorname{Inst}(P)$, i.e., $2^{|P|}$ times. Since removing irrelevant parents can be done in time linear in the size of N^a and quadratic in n, this yields a runtime in $O(2^{|P|} \cdot \sum_{1 \leq s \leq t} |CPT(N_s, V_n)|)$. The remaining statements of the theorem follow immediately. \square

Thus Algorithm 1 is an efficient method providing the optimal CPT wrt any of the input parent sets. It clearly cannot

produce an output worse than the trivial algorithm, which simply outputs the best input CPT. To demonstrate that it can do substantially better, we define a family of input instances that provides useful insights into CPT aggregation.

Definition 15. For $n \geq 3$ and $k \in \{2, \ldots, n-1\}$ define $T^{k,n} = (N_1^{k,n}, \ldots, N_t^{k,n})$ as follows. Let $t = \binom{n-1}{k} 2^k$. Then each number $s \in \{1, \ldots, t\}$ corresponds to a unique pair (P, γ) , where P is a k-element subset of $\mathcal{V} \setminus V_n$ and γ is a context over P. Now let $N_s^{k,n}$ be the CPT with rules $\gamma: 1 \succ 0$, and $\gamma': 0 \succ 1$ for all contexts $\gamma' \in \operatorname{Inst}(P) \setminus \{\gamma\}$.

For the case k=n-1, $(T^{k,n})_{k,n}$ was already mentioned by Endriss and Grandi (2014). It will turn out (Theorem 16) that Algorithm 1 produces an optimal solution for $T^{k,n}$ when $n\geq 3$ and $2\leq k\leq n-1$. By contrast, for k=n-1, Endriss and Grandi (2014) proved that the trivial algorithm, which outputs the best input CPT, cannot obtain an approximation ratio better than 2 for the family $(T^{n-1,n})_{n\geq 3}$ (Theorem 17). Moreover, we will argue that the trivial algorithm provides solutions whose objective value is at least a factor of 3/2 above the optimum (Theorem 18).

Theorem 16. Let $n \geq 3$ and $k \in \{2, ..., n-1\}$. Then Algorithm 1 outputs an optimal solution for $T^{k,n}$.

Proof. Each k-element subset $\mathcal{V}' \subseteq \mathcal{V} \setminus V_n$ is the parent set of 2^k input CPTs. By definition of $T^{k,n}$, for any context over any \mathcal{V}' , all but one of the CPTs with \mathcal{V}' as parent set entail $0 \succ 1$. Thus, for all swaps of V_n , $0 \succ 1$ is the majority ordering and the optimal solution is the separable $0 \succ 1$. Now assume $Pa(N_s^{k,n}, V_n)$ is the input to Algorithm 2, for some $s \in \{1, \ldots, \binom{n-1}{k} 2^k$. Consider the 2^k sub-matrices corresponding to contexts of $Pa(N_s^{k,n}, V_n)$, each with 2^{n-k-1} rows. On each row, $0 \succ 1$ occurs more often than $1 \succ 0$. So, over each sub-matrix M, $freq_M(0 \succ 1) > freq_M(1 \succ 0)$. Thus Algorithm 2 outputs the optimal separable $0 \succ 1$, and Algorithm 1 outputs an optimal solution. □

Theorem 17 (cf. (Endriss and Grandi 2014)). Let $\varepsilon > 0$ be any positive real number. Then there is some $n \geq 3$ such that

$$f_{T^{n-1,n}}(N_s^{n-1,n}) > (2-\varepsilon)f_{T^{n-1,n}}(N)$$

for all $1 \leq s \leq 2^{n-1}$ (= $\binom{n-1}{n-1}2^{n-1}$), where N is any optimal solution for $T^{n-1,n}$.

Theorem 18. Let $n \geq 3$, $k \in \{2, ..., n-1\}$, and $1 \leq s \leq {n-1 \choose k} 2^k$. Let N be an optimal solution for $T^{k,n}$. Then $f_{T^{k,n}}(N_s^{k,n}) \geq (3/2) f_{T^{k,n}}(N)$.

In order to prove this theorem, we will need to establish two helpful lemmas.

Lemma 19. Let N be an optimal solution for $T^{k,n}$. Then $f_{T^{k,n}}(N)=2^{n-1}\binom{n-1}{k}$.

Proof. Each $N_2^{k,n}$ in $T^{k,n}$ entails $1\succ 0$ for one rule applying to 2^{n-k-1} swaps. By the proof of Theorem 16, N is the separable $0\succ 1$, implying $\Delta(N,N_s^{k,n})=2^{n-k-1}$. Summing up over all t values of s proves the claim. \square

Lemma 20. Let
$$1 \le s \le \binom{n-1}{k} 2^k$$
. Then $f_{T^{k,n}}(N_s^{k,n}) = (2^n - 2^{n-k}) \sum_{k'=0}^k \binom{k}{k'} \binom{n-k-1}{k-k'}$.

Proof. Note that $f_{T^{k,n}}(N_s^{k,n}) = \sum_{s' \neq s} \Delta(N_s^{k,n}, N_{s'}^{k,n})$. The value $\Delta(N_s^{k,n}, N_{s'}^{k,n})$ depends on $|Pa(N_s^{k,n}, V_n) \cap Pa(N_{s'}^{k,n}, V_n)| =: k'$, as well as the contexts γ_1 and γ_2 for which $N_s^{k,n}$ and $N_{s'}^{k,n}$, resp., entail $1 \succ 0$. Let P denote $Pa(N_{s'}^{k,n}, V_n)$ for $some \ s' \neq s, 1 \leq s' \leq \binom{n-1}{2} 2^k$.

 $Pa(N_{s'}^{k,n},V_n)$ for some $s'\neq s,$ $1\leq s'\leq \binom{n-1}{k}2^k$. Given $0\leq k'\leq k$, there are $\binom{k}{k'}\binom{n-k-1}{k-k'}$ parent sets P of size k such that $Pa(N_s^{k,n},V_n)\cap P=k'$. For each such P, if k'=k, the tuple T has 2^k-1 CPTs other than $N_s^{k,n}$ that have parent set P. For k'< k, T has 2^k CPTs with parent set P. While it is not necessary to treat the cases k'=k and k'=0 separately, we still do so, as it may help the reader better understand our argument for general values of k'.

For k'=k there are 2^k contexts of $Pa(N_s^{k,n},V_n)\cup P$ (= $Pa(N_s^{k,n},V_n)$). By construction, $N_s^{k,n}$ and $N_{s'}^{k,n}$ disagree on 2 contexts and thus on 2^{n-k} swaps, i.e., $\Delta(N_s^{k,n},N_{s'}^{k,n})=2^{n-k}$ for each of the 2^k-1 CPTs $N_{s'}^{k,n}$ other than $N_s^{k,n}$ that have parent set P.

other than $N_s^{k,n}$ that have parent set P.

For k'=0 there are 2^{2k} contexts of $Pa(N_s^{k,n},V_n)\cup P$. $N_s^{k,n}$ entails $0\succ 1$ for 2^k-1 contexts and $N_s^{k,n}$ entails $1\succ 0$ for one context. These contexts are independent of each other since k'=0. This yields $(2^k-1)\cdot 1=2^k-1$ contexts of $Pa(N_s^{k,n},V_n)\cup P$ on which the two CPTs disagree. Accounting also for the symmetric case with the roles of $N_s^{k,n}$ and $N_{s'}^{k,n}$ exchanged, we obtain a disagreement on $2^{k+1}-2$ contexts of $Pa(N_s^{k,n},V_n)\cup P$, each corresponding to 2^{n-2k-1} swaps. This yields $\Delta(N_s^{k,n},N_{s'}^{k,n})=2^{n-k}-2^{n-2k}$ for each CPT $N_{s'}^{k,n}$ that has a parent set P disjoint from $Pa(N_s^{k,n},V_n)$, i.e., a parent set P yielding k'=0. There are $2^k\binom{n-k-1}{k}$ such CPTs, namely 2^k CPTs for each choice of k-element set P disjoint from $Pa(N_s^{k,n},V_n)$.

Lastly, for $1 \leq k' \leq k-1$, there are $2^{2k-k'}$ contexts of $Pa(N_s^{k,n},V_n) \cup P$. Note that, for $\gamma_1 \in \operatorname{Inst}(Pa(N_s^{k,n},V_n))$ and $\gamma_2 \in \operatorname{Inst}(P)$, we have: $(N_s^{k,n} \text{ has } \gamma_1 : 1 \succ 0 \text{ and } N_s^{k,n} \text{ has } \gamma_2 : 1 \succ 0)$ iff γ_1 and γ_2 are consistent, i.e., they have the same values on all attributes in $Pa(N_s^{k,n},V_n) \cap P$. Now suppose $N_s^{k,n}$ entails $1 \succ 0$ for γ_1 . There are $2^{k-k'}$ CPTs with parent set P that entail $1 \succ 0$ for some γ_2 consistent with γ_1 , and $2^k - 2^{k-k'}$ CPTs with parent set P that do not.

Consider CPTs of the first type. There are 2^k contexts of $Pa(N_s^{k,n}, V_n)$; for exactly one such context, namely γ_1 , the CPT $N_s^{k,n}$ entails $1 \succ 0$. Each of the $2^{k-k'}$ contexts of $P \setminus Pa(N_s^{k,n}, V_n)$ can be appended to γ_1 . For $2^{k-k'}-1$ of these, $N_{s'}^{k,n}$ entails $0 \succ 1$; so $N_s^{k,n}$ and $N_{s'}^{k,n}$ disagree on all such contexts appended to γ_1 . Also counting the symmetric case where $N_{s'}^{k,n}$ entails $1 \succ 0$, $N_s^{k,n}$ and $N_{s'}^{k,n}$ disagree on $2^{k-k'+1}-2$ contexts of $Pa(N_s^{k,n}, V_n) \cup P$, each of which orders $2^{n-2k+k'-1}$ swaps. Thus $\Delta(N_s^{k,n}, N_{s'}^{k,n}) = 2^{n-k}-2^{n-2k+k'}$ in this case.

Consider CPTs of the second type. There are 2^k contexts of $Pa(N_s^{k,n}, V_n)$; for exactly one such context, namely γ_1 , the CPT $N_s^{k,n}$ entails $1 \succ 0$. Each of the $2^{k-k'}$ contexts of $P \setminus Pa(N_s^{k,n}, V_n)$ can be appended to γ_1 . Since CPTs of the second type entail $0 \succ 1$ for every $\gamma_2 \in \operatorname{Inst}(P)$ that

is consistent with γ_1 , we have that $N^{k,n}_{s'}$ entails $0 \succ 1$ for all $2^{k-k'}$ completions of γ_1 in $\operatorname{Inst}(P)$. Counting also the symmetric case where $N^{k,n}_{s'}$ entails $1 \succ 0$, $N^{k,n}_{s}$ and $N^{k,n}_{s'}$ disagree on $2^{k-k'+1}$ contexts of $Pa(N^{k,n}_{s}, V_n) \cup P$, each of which orders $2^{n-2k+k'-1}$ swaps. Thus $\Delta(N^{k,n}_{s}, N^{k,n}_{s'} = 2^{n-k}$ in this case.

Combining these pieces, the value of $f_{T^{k,n}}(N_s^{k,n})$ equals

$$\begin{aligned} &(2^k-1)\cdot 2^{n-k} + 2^k \binom{n-k-1}{k} (2^{n-k}-2^{n-2k}) \\ + & \sum_{k'=1}^{k-1} \left[2^{k-k'} \binom{k}{k'} \binom{n-k-1}{k'} (2^{n-k}-2^{n-2k+k'}) \right. \\ & \left. + 2^k - 2^{k-k'} \binom{k}{k'} \binom{n-k-1}{k'} 2^{n-k} \right] \end{aligned}$$

Simplifying this with straightforward calculations yields $f_{T^{k,n}}(N_s^{k,n}) = (2^n - 2^{n-k}) \sum_{k'=0}^k {k \choose k'} {n-k-1 \choose k'}.$

Proof of Theorem 18. Follows from Lemmas 19 and 20, using Vandermonde's identit; see appendix for details. \Box

Conclusions

Since CP-net aggregation (wrt swap preferences) is known to be intractable, the design and analysis of approximation algorithms for preference aggregation is one of few viable approaches to efficient preference aggregation in this context. Proposition 3 implies that optimal CP-net aggregation is intractable not due to any difficulties in scaling with the number of attributes, but just due to the difficulty of scaling with the number of input CPTs. In particular, the cause of intractability lies solely in the parent set size of optimal solutions, which can be asymptotically larger than the size of the largest input parent set.

Therefore, we focused on approximation algorithms that keep the size of the output parent set linear in the size of the largest input parent set. A trivial such algorithm is one that simply outputs the best input CPT, which yields a 2-approximation in general. When imposing a symmetry constraint on the input CPT, the approximation ratio of this algorithm improves from 2 to 4/3, but in general, the ratio can be arbitrarily close to 2 (see Theorem 17).

Algorithm 1 instead considers each input parent set and calculates a provably optimal CPT for that parent set. Finally it outputs the best thus attained CPT. This polynomial-time method is never worse than the trivial algorithm, yet substantially better for some families of input instances. At the time of writing this paper, we are not aware of any problem instance on which Algorithm 1 attains an approximation ratio greater than 4/3. One open problem is to either prove that 4/3 is indeed an upper bound on this algorithm's approximation ratio, or else to provide a problem instance for which Algorithm 1 has a ratio exceeding 4/3.

Due to the relations between binary aggregation and CPnet aggregation, we hope that our work provides insights that are useful beyond the aggregation of CP-nets. Acknowledgements. Boting Yang was supported in part by an NSERC Discovery Research Grant, Application No.: RGPIN-2018-06800. Sandra Zilles was supported in part by an NSERC Discovery Research Grant, Application No.: RGPIN-2017-05336, by an NSERC Canada Research Chair, and by a Canada CIFAR AI Chair held at the Alberta Machine Intelligence Institute (Amii).

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Appendix An example CP-net

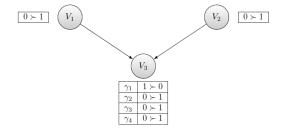


Figure 1: N_1

In Figure 1, N_1 is defined over $V = \{V_1, V_2, V_3\}$. For attributes V_1 and V_2 , the individual with N_1 as their preference model unconditionally prefers $0 \succ 1$. For V_3 , their preference is conditioned on the values assigned to V_1 and V_2 . Each vertex, denoting an attribute, is annotated with a Conditional Preference Table (CPT), with a total preference ordering given for each possible instantiation of the corresponding parent set. Any outcome pair that differs on the value assigned to exactly one attribute is called a swap, e.g. 000,001 is a swap of V_3 . Assuming $\gamma_1=00$, the best outcome for N_1 is 001, and the worst outcome is 000. The outcome pair 001,101 cannot be ordered by N_1 , because CP-net semantics does not tell us how to trade-off between the best value for V_1 and that for V_2 .

Matrix Representation

Notice that N_1 in Figure 1 entails $1 \succ 0$ for only one context of $Pa(N_1,V_3)$, namely $\gamma_1=00$. We now define a tuple T of four CP-nets. Each $N_i \in T$ has identical dependency graph, and identical CPTs for V_1 and V_2 . The only difference is that for each N_i , $CPT(N_i,V_3)$ entails $1 \succ 0$ for γ_i , and $0 \succ 1$ for all other contexts. We now give the matrix representation for T. Each row represents an instantiation of $\{V_1,V_2\}$, corresponding to a different swap of V_3 . Each column represents $N_i \in T$.

$o[\{V_1, V_2\}]$	N_1	N_2	N_3	N_4
00	1	0	0	0
01	0	1	0	0
10	0	0	1	0
11	0	0	0	1

In fact, T is an instance of the special family of input instances mentioned in Definition 15. In particular, T is $T^{2,3}$. It is easy to see that $f_T(N)=4$ for an optimal solution N. We can also verify that none of the inputs attain a ratio strictly less than $\frac{3}{2}$, while Algorithm 1 produces an optimal solution.

Proofs of Lemmas For Theorem 6

Lemma 7 and 8 were proven in the main body of the paper: **Lemma 7.** Let $T' = (N'_1, \dots, N'_t)$ be any problem instance of t CPTs. Assuming each of the 2^t voting configurations occurs exactly once in M(T'), and N is an optimal solution for T', we have

$$f_{T'}(N) = \begin{cases} 2 \cdot \sum_{\kappa=0}^{c-1} \kappa \binom{2c-1}{\kappa} & \text{if } t = 2c - 1\\ 2 \cdot \sum_{\kappa=0}^{c-1} \kappa \binom{2c}{\kappa} + c \binom{2c}{c} & \text{if } t = 2c \end{cases}$$

Proof. See main body of the paper.

Lemma 8. Each of the 2^t voting configurations of T is the row vector for exactly 2^{n-t-1} swaps.

These lemmas are used in the proofs of the following four lemmas.

Lemma 9. Suppose each of the 2^t voting configurations occurs for the same number of swaps. Let N be an optimal solution for T. Then

$$f_T(N) = \begin{cases} t \cdot 2^{n-2} - 2^{n-t-1} \cdot c \binom{2c-1}{c} & \text{if } t = 2c - 1\\ t \cdot 2^{n-2} - 2^{n-t-1} \cdot c \binom{2c}{c} & \text{if } t = 2c \end{cases}$$

Proof. Lemma 7 provides the total error of N when each voting configuration occurs for exactly one swap. If each voting configuration occurs for the same number of swaps, we can simply multiply the error made on each, by the number of swaps for which that configuration occurs. For the tuples satisfying the conditions of Lemma 8, we obtain $f_T(N) =$

$$\begin{cases} 2^{n-t} \cdot \sum_{\kappa=0}^{c-1} \kappa \binom{2c-1}{\kappa} & \text{if } t = 2c - 1\\ 2^{n-t} \cdot \sum_{\kappa=0}^{c-1} \kappa \binom{2c}{\kappa} + 2^{n-t-1} \cdot c \binom{2c}{c} & \text{if } t = 2c \end{cases}$$

which simplifies to the desired expression. \Box

For a problem instance T under our premises, this gives us combinatorial expressions to compute the error made by an optimal CPT. Next, we give an expression to compute the error made by any of the input CPTs N_s , which we claim to be at most 4/3 times the objective value of an optimal solution.

Lemma 10.
$$f_T(N_s) = (t-1) \cdot 2^{n-2}$$
 for all $s \in \{1, ..., t\}$.

Proof. Consider any $s' \neq s$. Since the CPTs in the tuple T are symmetric and have pairwise disjoint parent sets, N_s and $N_{s'}$ disagree on half of all swaps, i.e., on 2^{n-2} swaps. Thus $f_T(N_s) = (t-1) \cdot 2^{n-2}$.

Lemma 11. If $1 \le s \le t = 2c - 1$, c > 1 then $\frac{f_T(N_s)}{f_T(N)} \le \frac{4}{3}$.

Proof. This is equivalent to proving

$$(2c-1) \cdot 2^{n-2} - 2^{n-2c} \cdot c \binom{2c-1}{c} \ge \frac{3}{4} (2c-2) \cdot 2^{n-2}$$

$$\Leftarrow (2c-1) \cdot 2^{n-2} - 2^{n-2c} \cdot c \binom{2c-1}{c} \geq (6c-6) \cdot 2^{n-4}$$

$$\Leftarrow (8c - 4) \cdot 2^{n - 4} - 2^{n - 2c} \cdot c \binom{2c - 1}{c} \ge (6c - 6) \cdot 2^{n - 4}$$

$$\Leftarrow 2^{n-2c} \cdot c \binom{2c-1}{c} \le (2c+2) \cdot 2^{n-4}$$

$$\Leftarrow c \binom{2c-1}{c} \le (2c+2) \cdot 2^{2c-4}$$

$$\Leftrightarrow c \binom{2c-1}{c} \le (c+1) \cdot 2^{2c-3}$$

We prove the latter for all c > 1 using induction.

When c = 2, the inequality obviously holds.

Assume the inequality holds for some fixed c and we need to prove that

$$(c+1)$$
 $\binom{2c+1}{c+1} \le (c+2) \cdot 2^{2c-1}$

We know

$$\binom{2c+1}{c+1} = \binom{2c+1}{c} = \frac{2c+1}{c+1} \binom{2c}{c} = \frac{4c+2}{c+1} \binom{2c-1}{c}$$

and

$$(c+1)\binom{2c+1}{c+1} = (4c+2)\binom{2c-1}{c}$$

By inductive hypothesis,

$$c\binom{2c-1}{c} \le (c+1) \cdot 2^{2c-3},$$

so, multiplying both sides by 4

$$4c \binom{2c-1}{c} \le (c+1) \cdot 2^{2c-1}.$$

Adding $2 \cdot {2c-1 \choose c}$ on both sides yields

$$4c\binom{2c-1}{c} + 2 \cdot \binom{2c-1}{c} \le (c+1) \cdot 2^{2c-1} + 2 \cdot \binom{2c-1}{c}$$

$$(4c+2)\binom{2c-1}{c} \le (c+1) \cdot 2^{2c-1} + \binom{2c}{c}$$

$$(c+1)\binom{2c+1}{c+1} \le (c+1) \cdot 2^{2c-1} + \binom{2c}{c}$$

We complete the induction by proving

$$\binom{2d}{d} \le 2^{2d-1}$$

for all $d \ge 1$, also using induction.

For d = 1, this is obviously true.

Assume $\binom{2d}{d} \leq 2^{2d-1}$ for a fixed d. We have to prove $\binom{2d+2}{d+1} \leq 2^{2d+1}$. We know $\binom{2d+2}{d+1} = \frac{2\cdot(2d+1)}{d+1}\binom{2d}{d}$. By inductive hypothesis $\binom{2d}{d} \leq 2^{2d-1}$. Multiplying both sides by $\frac{2\cdot(2d+1)}{d+1}$ gives us

$$\frac{2 \cdot (2d+1)}{d+1} \binom{2d}{d} \le \frac{2 \cdot (2d+1)}{d+1} \cdot 2^{2d-1}$$

$$\binom{2d+2}{d+1} \le \frac{2 \cdot (2d+1)}{d+1} \cdot 2^{2d-1} \le 4 \cdot 2^{2d-1}$$

The above holds because $\frac{2\cdot(2d+1)}{d+1}<4$. This proves ${2d\choose d}\leq 2^{2d-1}$ for all values of $d\geq 1$, which in turn completes our first induction.

Lemma 12. If $1 \le s \le t = 2c$, c > 1 then $\frac{f_T(N_s)}{f_T(N)} \le \frac{4}{3}$.

Proof. This is equivalent to proving

$$\begin{aligned} 2c \cdot 2^{n-2} - 2^{n-2c-1} \cdot c \binom{2c}{c} &\geq \frac{3}{4} (2c-2) \cdot 2^{n-2} \\ &\Leftarrow 2c \cdot 2^{n-2} - 2^{n-2c-1} \cdot c \binom{2c}{c} \geq (6c-3) \cdot 2^{n-4} \\ &\Leftarrow 8c \cdot 2^{n-4} - 2^{n-2c-1} \cdot c \binom{2c}{c} \geq (6c-3) \cdot 2^{n-4} \\ &\Leftarrow 2^{n-2c-1} \cdot c \binom{2c}{c} \leq (2c+3) \cdot 2^{n-4} \\ &\iff c \binom{2c}{c} \leq (2c+3) \cdot 2^{2c-3} \\ &\iff c \binom{2c-1}{c} \leq (2c+3) \cdot 2^{2c-4} \end{aligned}$$

In the proof of Lemma 11, we established $c\binom{2c-1}{c} \leq (c+1)\cdot 2^{2c-3} = (2c+2)\cdot 2^{2c-4}$. Since c>1, the above immediately follows.

Proof of Theorem 18

Lemmas 19 and 20 were proven in the main body of the paper:

Lemma 19. Let N be an optimal solution for $T^{k,n}$. Then $f_{T^{k,n}}(N) = 2^{n-1} \binom{n-1}{k}$.

Lemma 20. Let $1 \le s \le \binom{n-1}{k} 2^k$. Then $f_{T^{k,n}}(N_s^{k,n}) = (2^n - 2^{n-k}) \sum_{k'=0}^k \binom{k}{k'} \binom{n-k-1}{k-k'}$.

Theorem 18. Let $n \geq 3$, $k \in \{2, ..., n-1\}$, and $1 \leq s \leq {n-1 \choose k} 2^k$. Let N be an optimal solution for $T^{k,n}$. Then $f_{T^{k,n}}(N_s^{k,n}) \geq (3/2) f_{T^{k,n}}(N)$.

Proof. By Lemmas 19 and 20, and since $(3/2)2^{n-1}=3\cdot 2^{n-2}$, we need to show

$$(2^{n}-2^{n-k})\sum_{k'=0}^{k} \binom{k}{k'} \binom{n-k-1}{k-k'} \ge 3 \cdot 2^{n-2} \binom{n-1}{k}.$$

Since $k \ge 2$, we have $2^n - 2^{n-k} \ge 2^n - 2^{n-2} = 3 \cdot 2^{n-2}$. Thus it suffices to show

$$\sum_{k'=0}^{k} {k \choose k'} {n-k-1 \choose k-k'} \ge {n-1 \choose k},$$

which holds with equality by Vandermonde's Identity.