The Full Two-Phase Simplex Method

The Simplex Method typically has two phases:

Phase 1: Find an Initial BFS that is feasible. If you cannot, terminate as the problem is infeasible.

Phase 2: Using your BFS from Phase 1, traverse BFS to BFS by making sure that the objective value function gets better (or at least not worse).

Note: We have seen Phase 2 in our previous lesson (using the simplex method). Our goal in this lesson will be to find an efficient way to do Phase 1.

Definition: Auxiliary Problem

Given a Linear Program in Standard form:

$$P \quad \begin{array}{ll} \min & c^T x \\ st & \begin{bmatrix} a_{1,1} & a_{1,2} & \dots & a_{1,n} \\ a_{2,1} & a_{2,2} & \dots & a_{2,n} \\ \dots & \dots & \dots & \dots \\ a_{m,1} & a_{m,2} & \dots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \dots \\ b_m \end{bmatrix} \\ & x \geq 0 \end{array}$$

We call the corresponding <u>Auxiliary Problem</u>:

$$P' \quad \begin{array}{ll} \min & s_1 + s_2 + \dots + s_m \\ st & \begin{bmatrix} a'_{1,1} & a'_{1,2} & \dots & a'_{1,n} & 1 & 0 & \dots & 0 \\ a'_{2,1} & a'_{2,2} & \dots & a'_{2,n} & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a'_{m,1} & a'_{m,2} & \dots & a'_{m,n} & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \dots \\ x_n \\ s_1 \\ \dots \\ s_m \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ \dots \\ b'_m \end{bmatrix} \\ & x \geq 0 \\ & s \geq 0 \end{array}$$

Where $a'_{i,j}$ and b'_i are created in such a way that right hand side (the new $b's$) are ≥ 0.

To construct such equations, we simply, we simply multiply all constraints from P that have $b_i < 0$ by -1.

Strategy: Constructing an Auxiliary Problem

How To Use It	When To Use It	Why It Works
To find the Auxiliary problem for a Linear Program in Standard Form, you do the following: **Step 1:** Multiply all equations that have –b values by -1. **Step 2:** For each equation in the constraints, add a different auxiliary variable + s_i. **Step3:** State that all $s_i \geq 0$. **Step 4:** Change the objective function as the sum of the auxiliary variables: $s_1 + s_2 + \ldots + s_m$ where m is the number of rows in the constraints.	When we want to construct a Linear Program that will allow us to find an auxiliary problem. Note that this problem will help us find an initial basic feasible solution.	This is simply using the definition in a systematic way.

Example 1

Find the Auxiliary Problem for the following LP:

$$P \quad \begin{array}{ll} \min & c^T x \\ st & Ax = b \\ & x \geq 0 \end{array} \qquad A = \begin{bmatrix} 1 & 0 & -1 & 4 & 0 \\ -1 & 2 & 0 & 2 & 3 \\ 3 & -1 & -1 & -1 & 6 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, c = \begin{bmatrix} 3 \\ -2 \\ 1 \\ -1 \\ 2 \end{bmatrix}$$

Solution:

We first fix our equations so the right hand side is ≥ 0:

$$\begin{bmatrix} 1 & 0 & -1 & 4 & 0 \\ -1 & 2 & 0 & 2 & 3 \\ 3 & -1 & -1 & -1 & 6 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} \longrightarrow \begin{bmatrix} -1 & 0 & 1 & -4 & 0 \\ -1 & 2 & 0 & 2 & 3 \\ -3 & 1 & 1 & 1 & -6 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

We can then construct the auxiliary problem:

$$P' \quad \begin{array}{ll} \min & s_1 + s_2 + s_3 \\ st & A'x = b' \\ & x \geq 0 \\ & s \geq 0 \end{array} \qquad A' = \begin{bmatrix} -1 & 0 & 1 & -4 & 0 & 1 & 0 & 0 \\ -1 & 2 & 0 & 2 & 3 & 0 & 1 & 0 \\ -3 & 1 & 1 & 1 & -6 & 0 & 0 & 1 \end{bmatrix}, b' = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Theorem: Auxiliary Problem Properties

Given a linear program in standard form and the corresponding auxiliary problem:

$$P \quad \begin{array}{ll} \min & c^T x \\ st & \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix} \\ & x \geq 0 \end{array}$$

$$P' \quad \begin{array}{ll} \min & s_1 + s_2 + \ldots + s_m \\ st & \begin{bmatrix} a'_{1,1} & a'_{1,2} & \cdots & a'_{1,n} & 1 & 0 & \ldots & 0 \\ a'_{2,1} & a'_{2,2} & \cdots & a'_{2,n} & 0 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \ldots & \ldots & \ldots & \cdots \\ a'_{m,1} & a'_{m,2} & \cdots & a'_{m,n} & 0 & 0 & \ldots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ \cdots \\ x_n \\ s_1 \\ \cdots \\ s_m \end{bmatrix} = \begin{bmatrix} b'_1 \\ b'_2 \\ \cdots \\ b'_m \end{bmatrix} \\ & x \geq 0 \\ & s \geq 0 \end{array}$$

We have the following properties about the auxiliary problem:

1) **It is in standard form (thus we can run the simplex method).**
2) **It has a clear initial basic feasible solution of $(x, s) = (0, b')$ as all of the b' are ≥ 0. With basis $B = \{n+1, \ldots, n+m\}$.**
3) **Solving the Auxiliary Problem using the Simplex Method will have two outcomes:**
   a) **The optimal solution will be 0 (thus all of the s variables are 0) and the resulting BFS (not including the s-variables) will be a Basic Feasible Solution in the original problem.**
   b) **The optimal solution will be larger than 0, (thus at least one of the s variables is larger than 0) and there is no BFS in the original problem and thus the problem is infeasible.**

Why this works:
1) and 2) should be clear, and the proof of 3 can be found in the proofs section.

Strategy: Finding an Initial Basic Feasible Solution

How To Use It	When To Use It	Why It Works
To find an initial basic feasible solution to a linear programming problem in standard form, you can: **Step 1:** Create the Auxiliary Problem P' using the initial problem P. **Step 2:** Start with the clear BFS in P' and run the simplex method on P'. $BFS = \begin{bmatrix} x_1 \\ \ldots \\ x_n \\ s_1 \\ \ldots \\ s_m \end{bmatrix} = \begin{bmatrix} 0 \\ \ldots \\ 0 \\ b_1 \\ \ldots \\ b_m \end{bmatrix}$ **Step 3:** One of the following will happen: a) If the optimum value ever becomes 0 during the simplex method, the Basic Feasible Solution you end up with in step 2 is a Basic Feasible Solution for the original when you delete the auxiliary variables $s_1, s_2, \ldots, s_m$. b) If the optimum value is larger than 0 when the simplex method terminates with P', then P is infeasible.	When you want to find an initial basic feasible solution to a linear program in standard form. (Note that if the problem is not in standard form, you can always construct one in standard form).	This is simply using the previous theorem in a systematic way.

Example 2

Consider the LP problem given:

$$\begin{aligned} \min \quad & c^T x \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad A = \begin{bmatrix} 1 & 0 & 3 & 1 \\ -1 & 2 & 1 & 0 \end{bmatrix}, b = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, c = \begin{bmatrix} 1 \\ -1 \\ 2 \\ 4 \end{bmatrix}$$

Determine the Auxiliary Problem P’, then use P’ to find the initial BFS for P.

Solution:

The auxiliary Problem P’ will be: $A' = \begin{bmatrix} 1 & 0 & 3 & 1 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}, b' = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, c' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

The corresponding basic feasible solution is $B = \{5,6\}$: $x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 6 \end{bmatrix}$

Example 2 Continued

$$\begin{array}{ll} \min & c'^T x \\ st & A'x = b' \\ & x \geq 0 \end{array} \quad A' = \begin{bmatrix} 1 & 0 & 3 & 1 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}, b' = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, c' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad B = \{5,6\} \quad x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 6 \end{bmatrix}$$

Step 1: Solve: $A_B{}^T y^* = c_B$ $\begin{array}{c} 5 \\ 6 \end{array}\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{array}{c} 5 \\ 6 \end{array}\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$

Step 2: Check to see if the solution is feasible in the dual by checking: $A_N{}^T y^* \leq c_N$ $\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 1 \end{bmatrix} \cancel{\leq} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Here we see the inequality is not satisfied so we do not have an optimal solution. Thus, we will choose k = 2.

Example 2 Continued

$$A' = \begin{bmatrix} 1 & 0 & 3 & 1 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}, b' = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, c' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad B = \{5,6\} \quad x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 3 \\ 6 \end{bmatrix} \qquad k = 2$$

Step 3 **a):** Solve: $A_B d_B = -A_k$

$$\begin{array}{c} 5 \\ 6 \end{array}\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -2 \end{array}\right] \longrightarrow \begin{array}{c} 5 \\ 6 \end{array}\left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & -2 \end{array}\right] \longrightarrow \begin{bmatrix} d_5 \\ d_6 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

b) We have some d's that are < 0 (not all are positive, thus it is not unbounded as of this point)

c) Find t and r that is the minimum: $t = \min\left\{\frac{x_6^*}{-d_6}\right\} = \min\left\{\frac{6}{2}\right\} = 3$

Thus, t = 3 and we can choose r = 6

Step 4: Find our new BFS: $x'_B = x^*_B + td_B$ and change $x'_k = t$, also change our basis to B'

$$x'_B = \begin{bmatrix} x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} + (3)\begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$x'_B = \begin{bmatrix} 3 \\ 0 \end{bmatrix} \longrightarrow x' = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \qquad B' = \{2,5\}$$

Example 2 Continued

$$\min \quad c'^T x \qquad st \quad A'x = b', \quad x \geq 0$$

$$A' = \begin{bmatrix} 1 & 0 & 3 & 1 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}, b' = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, c' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad B = \{2,5\} \qquad x^* = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix}$$

Step 1: Solve: $A_B{}^T y^* = c_B$

$$\begin{array}{c} 2 \\ 5 \end{array}\left[\begin{array}{cc|c} 0 & 2 & 0 \\ 1 & 0 & 1 \end{array}\right] \longrightarrow \begin{array}{c} 2 \\ 5 \end{array}\left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 0 \end{array}\right]$$

Step 2: Check to see if the solution is feasible in the dual by checking: $A_N{}^T y^* \leq c_N$

$$\begin{array}{c} 1 \\ 3 \\ 4 \\ 6 \end{array}\begin{bmatrix} 1 & -1 \\ 3 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 1 \\ 0 \end{bmatrix} \cancel{\leq} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Here we see the inequality is not satisfied so we do not have an optimal solution. Thus, we will choose k = 1.

Example 2 Continued

$$A' = \begin{bmatrix} 1 & 0 & 3 & 1 & 1 & 0 \\ -1 & 2 & 1 & 0 & 0 & 1 \end{bmatrix}, b' = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, c' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \qquad B = \{2,5\} \quad x^* = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} \qquad k = 1$$

Step 3 **a):** Solve: $A_B d_B = -A_k$

$$\begin{array}{c} 2 \\ 5 \end{array}\left[\begin{array}{cc|c} 0 & 1 & -1 \\ 2 & 0 & 1 \end{array}\right] \longrightarrow \begin{array}{c} 2 \\ 5 \end{array}\left[\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & -1 \end{array}\right] \longrightarrow \begin{bmatrix} d_2 \\ d_5 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$$

b) We have some d's that are < 0 (not all are positive, thus it is not unbounded as of this point)

c) Find t and r that is the minimum: $t = \min\left\{\frac{x_5^*}{-d_5}\right\} = \min\left\{\frac{3}{1}\right\} = 3$

Thus, t = 3 and we can choose r = 5

Step 4: Find our new BFS: $x'_B = x^*_B + td_B$ and change $x'_k = t$, also change our basis to B'

$$x'_B = \begin{bmatrix} x_2 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} + (3)\begin{bmatrix} 1/2 \\ -1 \end{bmatrix}$$

$$x'_B = \begin{bmatrix} 9/2 \\ 0 \end{bmatrix} \longrightarrow x' = \begin{bmatrix} 3 \\ 9/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad B' = \{1,2\}$$

Here we stop as this is a BFS that has optimum value of 0 (i.e. all of our s-variables are 0).

$$x^* = \begin{bmatrix} 3 \\ 9/2 \\ 0 \\ 0 \end{bmatrix}$$

Example 3

Consider the LP problem given:

$$\begin{aligned} \min \quad & c^T x \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad A = \begin{bmatrix} -1 & 0 & 1 & -2 & -4 \\ 1 & -1 & -2 & 1 & 3 \\ 1 & 0 & -1 & 0 & 2 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 12 \\ -4 \end{bmatrix}, c = \begin{bmatrix} -1 \\ -2 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Determine the Auxiliary Problem P’, then use P’ to find the initial BFS for P.

<u>Solution:</u>

Negating our third equation and introducing our s-variables gives us:

$$A' = \begin{bmatrix} -1 & 0 & 1 & -2 & -4 & 1 & 0 & 0 \\ 1 & -1 & -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & -2 & 0 & 0 & 1 \end{bmatrix}, b' = \begin{bmatrix} 6 \\ 12 \\ 4 \end{bmatrix}, c' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

Thus our initial BSF becomes $B = \{6,7,8\}$ with $x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 12 \\ 4 \end{bmatrix}$

Example 3 Continued

$$\begin{aligned} \min \quad & c'^T x \\ st \quad & A' x = b' \\ & x \ge 0 \end{aligned}$$

$$A' = \begin{bmatrix} -1 & 0 & 1 & -2 & -4 & 1 & 0 & 0 \\ 1 & -1 & -2 & 1 & 3 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & -2 & 0 & 0 & 1 \end{bmatrix}, b' = \begin{bmatrix} 6 \\ 12 \\ 4 \end{bmatrix}, c' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$B = \{6,7,8\} \quad x^* = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 6 \\ 12 \\ 4 \end{bmatrix}$$

Step 1: Solve: $A_B{}^T y^* = c_B$

$$\begin{array}{c} 6 \\ 7 \\ 8 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{array}{c} 6 \\ 7 \\ 8 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Step 2: Check to see if the solution is feasible in the dual by checking $A_N{}^T y^* \le c_N$

$$\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{bmatrix} -1 & 1 & -1 \\ 0 & -1 & 0 \\ 1 & -2 & 1 \\ -2 & 1 & 0 \\ -4 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ -3 \end{bmatrix} \le \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Here we have the inequality holding true, so our original BFS is optimal in P’.

Example 3 Continued

Since our original BFS in P’ is optimal, the optimal solution is 6 + 12 + 4 = 22 (and not 0). This means that the original problem P has no feasible BFS and thus P is infeasible.

Note that this was tremendously faster (we only had to create the Auxiliary problem and perform one step of the Simplex Method) than simply checking all potential BFS candidates (5 Choose 3 = 10 candidates to check). Although this example was lucky that it stopped after the first step, it could go through a few iterations of the simplex method (but in practice it is no where near the number of checking all potential BFS candidates in the original P)