Definition: Linear Programs in Standard Form

We say a LP is in standard form when it looks like the following:

$$\begin{array}{ll} \min & c^T x \\ st & Ax = b \\ & x \geq 0 \end{array}$$

Where A has <u>full row Rank</u>.

It turns out we can get any LP into this form as we will see in our next strategy.

Strategy: Converting LPs into Standard Form

How To Use It	When To Use It	Why It Works
To convert a linear program into standard form, we can: 1) Change max to a min by taking the negative of the objective function (and recall that the optimum value will need to be negated once solved). 2) To change single variable constraints that look like $x_i \leq b_i$, we replace $x_i = -\overline{x}_i + b_i$ for all of our variables and this inequality will change direction and becomes $\overline{x_i} \geq 0$. 3) To change single variable constraints that look like $x_i \geq b_i$, we replace $x_i = \overline{x}_i + b_i$ for all of our variables and this inequality will become $\overline{x_i} \geq 0$. 4) To change a constraint of form $\mathrm{a}_\mathrm{i}^\mathrm{T} x \geq b_i$ into equations, we can introduce a new variable u_i and change the constraint to $\mathrm{a}_\mathrm{i}^\mathrm{T} x - u_i = b_i$ where $u_i \geq 0$. 5) To change a constraint of form $\mathrm{a}_\mathrm{i}^\mathrm{T} x \leq b_i$ into equations, we can introduce a new variable u_i and change the constraint to $\mathrm{a}_\mathrm{i}^\mathrm{T} x + u_i = b_i$ where $u_i \geq 0$. 6) To change free variables, we can let $x_i = u_i - v_i$ where $u_i \geq 0$ and $v_i \geq 0$. 7) Finally, check that all of the constraints are linearly independent (full row rank) by row reducing the matrix and keeping only the rows that are linearly independent. (also, if it happens that you have [0 0 … 0 | #] then the system is infeasible) Note that it is usually better to rename all variables to the form of z_i (or another different letter name) to make easier to number the variables. It just means when finding the solution to the original problem, you will need to know what the original x_i happened to be in terms of z_j.	When we want to convert a linear program into standard form.	This is using our properties of inequalities in some creative ways. The reason part 4/5 works is because we know we can remove/add from a scale to get the scale to become equal. The reason 6 works is because we can always get $x_i > 0$ by selecting a value for u_i and $x_i < 0$ by selecting a value for v_i.

Example 1

Change the following into standard form:

$$\begin{aligned} \max \quad & x_1 - x_2 + 2x_3 \\ s.t \quad & x_1 + x_2 + x_3 \geq 1 \\ & x_1 + 2x_2 + 3x_3 \leq 2 \\ & x_1 \geq 0 \\ & x_2 \leq -2 \\ & x_3 \ free \end{aligned}$$

Solution:

Change Max to Min

$$\begin{aligned} \min \quad & -x_1 + x_2 - 2x_3 \\ s.t \quad & x_1 + x_2 + x_3 \geq 1 \\ & x_1 + 2x_2 + 3x_3 \leq 2 \\ & x_1 \geq 0 \\ & x_2 \leq -2 \\ & x_3 \ free \end{aligned}$$

$\longrightarrow$

Change RHS to be 0

$$\begin{aligned} \min \quad & -x_1 - \overline{x_2} - 2x_3 - 2 \\ s.t \quad & x_1 - \overline{x_2} + x_3 \geq 3 \\ & x_1 - 2\overline{x_2} + 3x_3 \leq 6 \\ & x_1 \geq 0 \\ & \overline{x_2} \geq 0 \\ & x_3 \ free \end{aligned}$$

$\longrightarrow$

Change constraints to $=$

$$\begin{aligned} \min \quad & -x_1 - \overline{x_2} - 2x_3 - 2 \\ s.t \quad & x_1 - \overline{x_2} + x_3 - u_1 = 3 \\ & x_1 - 2\overline{x_2} + 3x_3 + u_2 = 6 \\ & x_1 \geq 0 \\ & \overline{x_2} \geq 0 \\ & x_3 \ free \\ & u_1, u_2 \geq 0 \end{aligned}$$

Change free variable

$$\begin{aligned} \min \quad & -x_1 - \overline{x_2} - 2(u_3 - v_3) - 2 \\ s.t \quad & x_1 - \overline{x_2} + (u_3 - v_3) - u_1 = 3 \\ & x_1 - 2\overline{x_2} + 3(u_3 - v_3) + u_2 = 6 \\ & x_1 \geq 0 \\ & \overline{x_2} \geq 0 \\ & u_1, u_2, u_3, v_3 \geq 0 \end{aligned}$$

$\longrightarrow$

Simplify

$$\begin{aligned} \min \quad & -x_1 - \overline{x_2} - 2u_3 + 2v_3 - 2 \\ s.t \quad & x_1 - \overline{x_2} + u_3 - v_3 - u_1 = 3 \\ & x_1 - 2\overline{x_2} + 3u_3 - 3v_3 + u_2 = 6 \\ & x_1 \geq 0 \\ & \overline{x_2} \geq 0 \\ & u_1, u_2, u_3, v_3 \geq 0 \end{aligned}$$

$\longrightarrow$

Rename Variables

$$\begin{aligned} \min \quad & -z_1 - z_2 - 2z_3 + 2z_4 - 2 \\ s.t \quad & z_1 - z_2 + z_3 - z_4 - z_5 = 3 \\ & z_1 - 2z_2 + 3z_3 - 3z_4 + z_6 = 6 \\ & z_1, z_2, z_3, z_4, z_5, z_6 \geq 0 \end{aligned}$$

Note that the two equations are Linearly Independent, so it is full row rank.

Example 2

Explain why the following is not in Standard form, then express the Linear Program in Standard form

$$\begin{array}{ll} \min & -x_1 - x_2 - 2x_3 - 2x_4 \\ & x_1 - x_2 + x_3 - x_4 - x_5 = 3 \\ s.t & x_1 - 2x_2 + 3x_3 - 3x_4 + x_6 = 6 \\ & 2x_1 - 3x_2 + 4x_3 - 4x_4 - x_5 + x_6 = 9 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

Solution:

We note that it is pretty much in standard form, except we must check that A has full row rank:

$$[A|b] \to \left[\begin{array}{cccccc|c} 1 & -1 & 1 & -1 & -1 & 0 & 3 \\ 1 & -2 & 3 & -3 & 0 & 1 & 6 \\ 2 & -3 & 4 & -4 & -1 & 1 & 9 \end{array}\right] \longrightarrow RREF = \left[\begin{array}{cccccc|c} 1 & 0 & -1 & 1 & -2 & -1 & 0 \\ 0 & 1 & -2 & 2 & -1 & -1 & -3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

This means the last row is redundant (as does not have a pivot) and so we keep only the equations that are useful

$$\begin{array}{ll} \min & -x_1 - x_2 - 2x_3 - 2x_4 \\ & x_1 - x_3 + x_4 - 2x_5 - x_6 = 0 \\ s.t & x_2 - 2x_3 + 2x_4 - x_5 - x_6 = -3 \\ & x_1, x_2, x_3, x_4, x_5, x_6 \geq 0 \end{array}$$

Note that if the system happens to be inconsistent, then that means the linear program is infeasible.

Definition: Feasible Basis and Basic Feasible Solution

Consider a Linear Program in Standard form where $A \in M_{mn}$:

$$\begin{aligned} \min \quad & c^T x \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

Let B be a basis for R^m that use some set of columns of A. Let A_B represent the square matrix that stores these columns and let x_B store only the $x's$ that correspond to these columns.

If when we solve $A_B x_B = b$ and $x_B \geq 0$ (i.e. it is feasible). Then we call B a feasible basis.

We call the corresponding x a basic feasible solution where $x_i = 0$ for $i \notin B$ and x_B is the solution from solving $A_B x_B = b$.

Things to note:

1) **A_B must be invertible as it is square and B is a basis (and thus linearly independent). This means x_B has a unique solution.**
2) **We know there must be at least one basis B as the A has full row rank. However, there may not be any feasible basis (if all bases require x_B to be less than 0 when solving $A_B x_B = b$).**

Example 3

Which of the following bases are feasible bases for the LP given below: $B_1 = \{1,2,3\}$, $B_2 = \{1,2,4\}$, $B_3 = \{1,3,5\}$

$$\begin{array}{ll} \min & c^T x \\ st & Ax = b \\ & x \geq 0 \end{array} \qquad A = \begin{bmatrix} 1 & 2 & 1 & -5 & 3 \\ 1 & -1 & 1 & 1 & 3 \\ 1 & 0 & 0 & 0 & 2 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 0 \\ 1 \end{bmatrix}$$

<u>Solution:</u>

For B_1: $A_{B_1} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and we solve $[A_{B_1} | \, b]$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \rightarrow RREF \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

Thus B_1 is a feasible basis as it has a unique solution, and all $x_{B_1} \geq 0$.

Example 3 continued

For B_2: $A_{B_2} = \begin{bmatrix} 1 & 2 & -5 \\ 1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and we solve $\left[A_{B_2} \mid b\right]$:

$$\left[\begin{array}{ccc|c} 1 & 2 & -5 & 6 \\ 1 & -1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right] \rightarrow RREF \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right]$$

Thus B_2 is a basis as it has a unique solution, but it is not a feasible basis as $x_{B_2} < 0$ (specifically x_4).

For B_3: $A_{B_3} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 1 & 0 & 2 \end{bmatrix}$ and we solve $\left[A_{B_3} \mid b\right]$:

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 6 \\ 1 & 1 & 3 & 0 \\ 1 & 0 & 2 & 1 \end{array}\right] \rightarrow RREF \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Thus B_3 is not even a basis (the columns of A_{B_3} do not row reduce to the identity). Thus this is not a feasible basis.

Note that B_3 was not consistent, but even if it was consistent, we would still not call it a feasible basis, as we need it to be a basis!

Theorem: Basic Feasible Solutions and Extreme Points

All Extreme Points are Basic Feasible Solutions.

Why this works:
Check proofs section for a formal proof, but intuitively we knew that an extreme point satisfied $Rank(A^{=}) = n$.

Since we have m rows that are holding at equality for certain (the m equations) and we are forcing all other variables $(n - m)$ to be zero, they are holding their constraint $x_i \geq 0$ at equality as well. Thus, we have at least n constraints holding at equality (the equations and the variables of the form $x_i \geq 0$) which means we should have a total rank of n.

Corollary: Basic Feasible Solutions Are Optimal Solutions

If a Linear Program has a Basic Feasible Solution and is not unbounded, then one of these Basic Feasible Solutions must be an optimal solution.

Why this works:
We had a theorem that said that one of the extreme points must be the optimal solution (when not unbounded). Since Extreme points must be a Basic Feasible Solution, we must have one of the Basic Feasible Solutions is an optimal solution (when not unbounded).

Theorem: Feasible iff Basic Feasible Exists

A Linear Program in Standard form is feasible iff it has a basic feasible solution.

Why this works:
Check Proofs section.

Note:
This means if we have no basic feasible solutions, then there is nothing that is feasible in our system. This will allow for our next strategy to give us a complete way to find Optimal Solutions.

Strategy: Using Basic Feasible Solutions to Solve LPs

How To Use It	When To Use It	Why It Works
To solve a Linear Program Using Basic Feasible Solutions, you can: 1) Get the Linear Program to Standard Form. 2) Consider all possible n choose m possible columns as Basis candidates. 3) Solve $A_B x_B = b$ for each basis, and only keep solutions that have all $x_B \geq 0$ and their corresponding basic feasible solution (by setting all other variables $x_i = 0$ when $i \notin B$) 4) Determine which basic feasible solution has the best optimum value. If there are no basic feasible solutions, then the linear program is infeasible. 5) Check the dual to see if the dual is feasible. If it is feasible, then the optimal solution found in part 4 is indeed optimal. If the dual is infeasible, then the problem is unbounded. Once again, instead of checking for a feasible point in the dual (as that can be tricky), one could check the complementary slackness conditions instead and ensure that the solution to the complementary slackness conditions is feasible in the dual.	When we want to use Basic Feasible Solutions to solve a linear program.	This is using our previous theorems. However, it is still a long one depending on how far n and m differ.

Example 4

You may assume that the following LP is **bounded** and Rank(A) = 3. Determine all BFS for the problem and use them to find the optimal solution.

$$\begin{aligned} \min \quad & -x_1 - 2x_2 + x_3 - x_4 \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad A = \begin{bmatrix} 1 & 0 & 1 & 2 & 4 \\ 1 & -1 & 1 & 1 & 3 \\ 1 & 0 & -1 & 0 & -2 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 12 \\ -4 \end{bmatrix}$$

<u>Solution:</u>

To solve this problem, we will iterate through all possible column Bases of A. It is ideal to use a table like the following to help keep things organized:

Potential Basis	RREF [A|b]	Is BFS?	BFS=
{1,2,3}			
{1,2,4}			
{1,2,5}			
{1,3,4}			
{1,3,5}			
{1,4,5}			
{2,3,4}			
{2,3,5}			
{2,4,5}			
{3,4,5}			

Example 4 Continued

$$\begin{aligned} \min \quad & -x_1 - 2x_2 + x_3 - x_4 \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 4 \\ 1 & -1 & 1 & 1 & 3 \\ 1 & 0 & -1 & 0 & -2 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 12 \\ -4 \end{bmatrix}$$

Potential Basis	RREF [A|b]	Is BFS?	BFS=
{1,2,3}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 5 \end{array}\right]$	No (Negatives)	
{1,2,4}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -11 \\ 0 & 0 & 1 & 5 \end{array}\right]$	No (Negatives)	
{1,2,5}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & -23/3 \\ 0 & 0 & 1 & 5/3 \end{array}\right]$	No (Negatives)	
{1,3,4}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 11 \\ 0 & 0 & 1 & -6 \end{array}\right]$	No (Negatives)	

Example 4 Continued

$$\min \quad -x_1 - 2x_2 + x_3 - x_4$$

$$st \qquad Ax = b$$

$$x \geq 0$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 4 \\ 1 & -1 & 1 & 1 & 3 \\ 1 & 0 & -1 & 0 & -2 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 12 \\ -4 \end{bmatrix}$$

Potential Basis	RREF [A|b]	Is BFS?	BFS=
{1,3,5}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 23 \\ 0 & 0 & 1 & -6 \end{array}\right]$	No (Negatives)	
{1,4,5}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -23/2 \\ 0 & 0 & 1 & 11/2 \end{array}\right]$	No (Negatives)	
{2,3,4}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$	No (Negatives)	
{2,3,5}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$	No (Negatives)	

Example 4 Continued

$$\begin{aligned} \min \quad & -x_1 - 2x_2 + x_3 - x_4 \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 0 & 1 & 2 & 4 \\ 1 & -1 & 1 & 1 & 3 \\ 1 & 0 & -1 & 0 & -2 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 12 \\ -4 \end{bmatrix}$$

Potential Basis	RREF [A|b]	Is BFS?	BFS=
{2,4,5}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$	No (Negatives)	
{3,4,5}	$\left[\begin{array}{ccc\|c} 1 & 2 & 4 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$	No (Not a basis)	

Thus, we have no BFS.

This means the problem must be infeasible.

Example 5

You may assume that the following LP is **bounded** and Rank(A) = 3. Determine all BFS for the problem and use them to find the optimal solution.

$$\begin{aligned} \min \quad & x_1 + 2x_2 + 4x_3 \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$A = \begin{bmatrix} 2 & -1 & 0 & 3 \\ 4 & -2 & -2 & 4 \\ -1 & 3 & -1 & -5 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Potential Basis	RREF [A|b]	Is BFS?	BFS=
{1,2,3}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$	Yes (All Positive)	(1,1,1,0) With Value 7
{1,2,4}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$	Yes (All Positive)	(0,2,0,1) With Value 4
{1,3,4}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -1 \end{array}\right]$	No (Negatives)	
{2,3,4}	$\left[\begin{array}{ccc\|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$	Yes (All Positive)	(0,2,0,1) With Value 4

Since the problem is bounded, the optimal solution is (0, 2, 0, 1) with a value of 4.