Definition: Degenerate Basic Feasible Solutions

Consider a Linear Program in Standard form and let B be a feasible basis with basic feasible solution x^*.

We say x^* is <u>degenerate</u> when x_B^* has at least one entry that is zero. Otherwise, we say that x^* is <u>non-degenerate</u>.

Example 1

Determine the basic feasible solution to the basis B_1 and B_2 given below, and state if the BFS is degenerate:

B_1 = {1,2,3}

B_2 = {3,4,5}

$$\begin{aligned} \min \quad & c^T x \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 & 3 \\ 1 & -1 & -1 & -1 & 1 \\ 1 & 0 & 0 & 1 & 1 \end{bmatrix}, b = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

<u>Solution:</u>

Solving $[A_{B_1} | \, b] \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 1 & 6 \\ 1 & -1 & -1 & -2 \\ 1 & 0 & 0 & 1 \end{array}\right]$ → RREF → $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{array}\right]$. Since all entries are positive in $x_{B_1}^*$, this is non-degenerate.

Solving $[A_{B_2} | \, b] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 6 \\ -1 & -1 & 1 & -2 \\ 0 & 1 & 1 & 1 \end{array}\right]$ → RREF → $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right]$. Since we have an entry = 0 in $x_{B_2}^*$, this is degenerate.

Finding Better BFS

Let us consider a LP in standard form (Rank(A) = # rows) given below and its dual.

$$(P)\quad \begin{array}{ll} \min & c^T x \\ st & Ax = b \\ & x \geq 0 \end{array} \qquad\qquad (D)\quad \begin{array}{ll} \max & b^T y \\ st & A^T y \leq c \\ & y \text{ is free} \end{array}$$

Say we found a non-degenerate basic feasible solution x' with a corresponding basis B.

Let us look at the complementary slackness conditions for (D) (note we know they are already satisfied for (P) since they are all equations!). Since x' is non-degenerate, we get the following:

$$\begin{array}{lll} {x_1}' = 0 & \text{or} & {A^T}_1 y' = c_1 \\ {x_2}' = 0 & \text{or} & {A^T}_2 y' = c_2 \\ \dots & & \\ {x_n}' = 0 & \text{or} & {A^T}_n y' = c_m \end{array}$$

Since x_B' has no zeros, we cannot remove those corresponding conditions.

This means we get ${A_B}^T y' = c_B$ must hold if x' will be optimal.

Finding Better BFS Con'td

Remember, B is a basis which means ${A_B}^T$ is invertible (since A_B is invertible) and so y'can only have one solution: $y' = {A_B}^{-T} c_B$

We would check to make sure that this is feasible in the dual (i.e. satisfies all the other dual constraints, of course we can ignore the constraints that correspond to basis B as we have already forced equality for these ones).

Let us presume that it is not optimal. What do we do?

If it is not optimal, then there is a constraint k that is not satisfied in the dual. This means that ${A_k}^T y' > c_k$ for at least one (possibly many) different k values not from our basis B (those constraints hold at equality!).

To see where we should look for the next basic feasible solution, we will look at $z = c^T x$ (our original objective function).

We can split up the objective function by : $z = c_B^T x_B + c_N^T x_N$ where N is the set of the indices not in our basis (i.e. the rest of the indices).

Finding Better BFS Con'td

Splitting out constraints in (P) (used in basis B) in a similar way will change our constraints to look like:

$$z = c_B^T x_B + c_N^T x_N$$
$$A_B x_B + A_N x_N = b$$

This is a linear system, so we can multiply both sides of the second equation by ${A_B}^{-1}$:

$$z = c_B^T x_B + c_N^T x_N$$
$$x_B + A_B^{-1} A_N x_N = A_B^{-1} b$$

Then multiply the second equation by c_B^T and rearranging the gives:

$$z = c_B^T x_B + c_N^T x_N$$
$$c_B^T x_B = c_B^T A_B^{-1} b - c_B^T A_B^{-1} A_N x_N$$

Subbing 2 into 1 gives: $z = c_B^T {A_B}^{-1} b - c_B^T {A_B}^{-1} A_N x_N + c_N^T x_N$

Recall that $y' = {A_B}^{-T} c_B$, and so taking the transpose on both sides gives ${y'}^T = c_B^T {A_B}^{-1}$, also ${x_B}' = {A_B}^{-1} b$, this gives the new system as:

$$\begin{aligned} z &= {y'}^T b - {y'}^T A_N x_N + c_N^T x_N \\ x_B + {A_B}^{-1} A_N x_N &= {x_B}' \end{aligned} \longrightarrow \begin{aligned} z &= (c_N^T - {y'}^T A_N) x_N + {y'}^T b \\ x_B + {A_B}^{-1} A_N x_N &= x_B' \end{aligned}$$

Finding Better BFS Con’td

So, we will look at two of these developed equations:

$$z = (c_N^T - y'^T A_N)x_N + y'^T b$$
$$x_B + {A_B}^{-1} A_N x_N = x'_B$$

Now when we have that y’ is not a solution (ie not optimal), and this was because there was (at least one) k such that ${A_k}^T y' > c_k$ or in other words: $y'^T A_k > c_k^T$ which gives $0 > c_k^T - y'^T A_k$. This means that in our first constraint we have (when only considering k):

$$\underbrace{z}_{\text{OF value}} = \underbrace{(c_k^T - y'^T A_k)}_{<0}\underbrace{x_k}_{\geq 0} + \underbrace{y'^T b}_{\text{Our current objective value (not optimal) as } x_N = 0}$$

This means if we can adjust x_k, we will end up with a lower value of z (OF value). Keep in mind though, we still require to satisfy the second constraint and so we must be careful when changing x_k (remember that all $x_i \geq 0$ needs to happen):

$$x_B + {A_B}^{-1} A_k x_k = x'_B \longrightarrow x_B = x'_B - {A_B}^{-1} A_k x_k \geq 0$$

Finding Better BFS Con’td

Let us choose x_k = t. Then we need to find a new x’’ such that:

$$x''_B + {A_B}^{-1} A_k x_k = x'_B$$
$$x''_k = t$$

Thus, we just force it to be true by solving for x''_B: $x''_B = x'_B - {A_B}^{-1} A_k t$

Clearly, this will satisfy the constraint given before: (we are only changing one value of x_N (the kth one), and so subbing in our new x''_B will allow for equality to hold. It will also make a smaller z, so we have a better optimal value.

$$z = (c_N^T - y'^T A_N) x_N + y'^T b$$
$$x_B + {A_B}^{-1} A_N x_N = x'_B$$

Keep in mind, we cannot simply choose any t we want (usually), as we require $x''_B = x'_B - {A_B}^{-1} A_k t \geq 0$. However, we will want to choose t as large as possible as we know it lowers our z value. So, to find the next candidate, we are going to choose a t that makes one of the components of x'_B = 0 (and this will essentially give us a new BFS as we are going to take a positive value for x_k (where k is from N) and force a positive value from x_B to be 0 (call it r).

Our new BFS will be $B \cup \{k\} - \{r\}$ (i.e. start with B, add k, and remove r).

Unboundedness

This idea also allows to see if the LP is unbounded as well (so no need to constantly check the dual or complementary slackness conditions!) From $x''_B = x'_B - {A_B}^{-1}A_k t \geq 0$, what happens if $-{A_B}^{-1}A_k \geq 0$?

This means $x''_B = x'_B - {A_B}^{-1}A_k t \geq 0$ will hold for every value of t as it is the sum of two positives.

This means that we can keep choosing larger and larger values of t, which will make our z smaller and smaller and still satisfies all of our constraints (including $x_B \geq 0$)

Thus, if we ever have $-{A_B}^{-1}A_k \geq 0$ in our algorithm, we would stop and conclude unboundedness.

Strategy: Simplex Method for finding a Better BFS

How To Use It	When To Use It	Why It Works
Start with a non-degenerate BFS x*: (with $x^*_B > 0$) **Step 1:** Find the potential dual solution by solving ${A_B}^T y^* = c_B$ using RREF **Step 2:** See if it is feasible by comparing all constraints not in B in the dual by checking to see if ${A_N}^T y^* \leq c_N$ holds. If it satisfies all of the constraints then x* is optimal, if it doesn't, we will have (at least one) inequality of the form: ${A_k}^T y^* > c_k$. Select one of these contradicting inequalities. **Step 3:** Set x'_k = t and solve $x'_B = x^*_B - {A_B}^{-1} A_k t \geq 0$ so that t is as large as possible. To solve this without inverses we should do the following: a) let $d_B = -{A_B}^{-1} A_k$ and solve for d_B by solving: $A_B d_B = -A_k$ b) if $d_B \geq 0$, we can choose t→∞. Stop as the problem is unbounded. c) otherwise, of any $d_B < 0$ choose t = $\min\{x^*_i / -d_i\}$ and let r represent the index that produces the minimum t. **Step 4:** Our new BFS is x' where we change $x'_B = x^*_B + t d_B$ and change x'_k = t, the rest is unchanged (ie is still 0). Our new basis B' is made by taking B adding k and taking out r. Go to step 1 to see if this new BFS is optimal.	When we want to determine the optimal solution of a linear program or if a linear program is unbounded. Note we need to start with a Linear Program in standard form and an initial basic feasible solution. We will see how we can determine an initial Basic Feasible Solution In later lessons.	We worked to justify this in the past few slides. Essentially through some algebraic manipulations, we take advantage of increasing one variable to decrease our objective value.

Theorem: Simplex Theorem

Consider the simplex method for finding a better BFS. Then the following are true:

1. **Our new B' = B remove r add k will also be a basis.**
2. **When our BFS is non-degenerate, our new z-value is guaranteed to get smaller in the next iteration.**

Why this works?
Check proofs section for a formal proof, but we have also justified this in a semi-"hand wavey" way through previous slides.

Corollary: Simplex Theorem

Consider the simplex method for finding a better BFS. Then the following are true:

1. **When our BFS is always non-degenerate, the algorithm will terminate.**
2. **If we have degenerate BFS's, we could cycle forever and never terminate.**

This means:
1) We will need to find a rule that allows us to avoid cycling in the algorithm.
2) We will need a way to find an initial basic feasible solution.

Example 2

Consider the LP problem given:

$$\begin{aligned} \min \quad & c^T x \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 2 & 3 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, c = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

Given that x* = [3,1,0,0] is a BFS with B={1,2}, show that it is not optimal, then choose a new better basis and check if it is optimal.

<u>Solution:</u>

Step 1: Solve: $A_B{}^T y^* = c_B$

$$\begin{array}{c} 1 \\ 2 \end{array}\left[\begin{array}{cc|c} 1 & 1 & 4 \\ -1 & 0 & 3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 7 \end{array}\right]$$

Step 2: Check to see if the solution is feasible in the dual by checking: $A_N{}^T y^* \leq c_N$ $\longrightarrow$ $\begin{array}{c} 3 \\ 4 \end{array}\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \end{bmatrix} \not\leq \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Here we see the inequality is not satisfied (for either) so we do not have an optimal solution. Thus we will choose k = 4 (we could choose 3 if you like).

Example 2 Continued

Step 3 **a):** Solve: $A_B d_B = -A_k$

$$\begin{array}{c} 1 \\ 2 \end{array}\left[\begin{array}{cc|c} 1 & -1 & -2 \\ 1 & 0 & -3 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & -1 \end{array}\right] \longrightarrow \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

b) We have some d's that are < 0 (not all are positive, thus it is not unbounded as of this point)

c) Find t and r that is the minimum:

$$t = \min\left\{\frac{x_1^*}{-d_1}, \frac{x_2^*}{-d_2}\right\} = \min\left\{\frac{3}{3}, \frac{1}{1}\right\} = 1$$

thus t = 1 and we can choose r = 2 (or r = 1 if you like, as both produce 1)

Step 4: Find our new BFS: $x'_B = x^*_B + td_B$ and change $x'_k = t$, also change our basis to B'

$$x_B^{'} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} + (1)\begin{bmatrix} -3 \\ -1 \end{bmatrix}$$

$$x_B^{'} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow x' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

and our B' = {1,4}

Example 2 Continued

Lastly, we need to check if this is optimal. To do so we go back to the CS conditions one last time with Basis B = {1,4} (Step1) and make sure that the solution satisfies the other constraints in N = {2,3} (Step 2).

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 2 & 3 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, c = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} \qquad x' = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Step 1: Solve: $A_B{}^T y^* = c_B$

$$\begin{array}{c} 1 \\ 4 \end{array} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & 3 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 0 & 11 \\ 0 & 1 & -7 \end{array}\right]$$

Step 2: Check to see if the solution is feasible in the dual by checking: $A_N{}^T y^* \leq c_N \longrightarrow \begin{array}{c} 2 \\ 3 \end{array} \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 11 \\ -7 \end{bmatrix} = \begin{bmatrix} -11 \\ -3 \end{bmatrix} \leq \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Since it is feasible, we have that x' is an optimal solution (you can also check that they both obtain the same OV in their respective problems!)

Example 3

Consider the LP problem given:

$$\begin{aligned} \min \quad & c^T x \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad A = \begin{bmatrix} 1 & 1 & -1 & -1 & 2 \\ 0 & -3 & 1 & 3 & 2 \\ 1 & 2 & 0 & -2 & 0 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

Given that x* = [1,0,1,0,1] is a BFS with B={1,3,5}, show that it is not optimal. Then find a better solution or show that the problem is unbounded. If you find a better solution, use the method again to find an even better solution (or show it is unbounded)

Solution:

Step 1: Solve: $A_B{}^T y^* = c_B$

$$\begin{array}{c} 1 \\ 3 \\ 5 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0.5 \end{array}\right]$$

Step 2: Check to see if the solution is feasible in the dual by checking: $A_N{}^T y^* \leq c_N$

$$\longrightarrow \begin{array}{c} 2 \\ 4 \end{array}\begin{bmatrix} 1 & -3 & 2 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \not\leq \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

Here we see the inequality is not satisfied (for either) so we do not have an optimal solution. Thus, we will choose k = 2 (could choose 4 if you like).

Example 3 Continued

Step 3 **a):** Solve: $A_B d_B = -A_k$

$$\begin{array}{c} 1 \\ 3 \\ 5 \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 2 & -1 \\ 0 & 1 & 2 & 3 \\ 1 & 0 & 0 & -2 \end{array}\right] \longrightarrow \begin{array}{c} 1 \\ 3 \\ 5 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \longrightarrow \begin{bmatrix} d_1 \\ d_3 \\ d_5 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

b) We have some d's that are < 0 (not all are positive, thus it is not unbounded as of this point)

c) Find t and r that is the minimum (there is only one choice in this case):

$$t = \min\left\{\frac{x_1^*}{-d_1}\right\} = \min\left\{\frac{1}{2}\right\} = 1/2$$

thus, t = 1/2 and we can choose r = 1

Step 4: Find our new BFS: $x'_B = x^*_B + td_B$ and change $x'_k = t$, also change our basis to B'

$$x_B^{'} = \begin{bmatrix} x_1 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + (0.5)\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1.5 \\ 1.5 \end{bmatrix} \longrightarrow x' = \begin{bmatrix} 0 \\ 0.5 \\ 1.5 \\ 0 \\ 1.5 \end{bmatrix}$$

and our B' = {2,3,5}

Example 3 Continued

Consider the LP problem given:

$$\begin{aligned} \min \quad & c^T x \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad A = \begin{bmatrix} 1 & 1 & -1 & -1 & 2 \\ 0 & -3 & 1 & 3 & 2 \\ 1 & 2 & 0 & -2 & 0 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

Given that x* = [0,0.5,1.5,0,1.5] is a BFS with B={2,3,5}. We repeat Step 1 to find a better solution

Step 1: Solve: $A_B^T y^* = c_B$

$$\begin{array}{c} 2 \\ 3 \\ 5 \end{array} \left[\begin{array}{ccc|c} 1 & -3 & 2 & -1 \\ -1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

Step 2: Check to see if the solution is feasible in the dual by checking: $A_N^T y^* \leq c_N$

$$\longrightarrow \begin{array}{c} 1 \\ 4 \end{array} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \not\leq \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Here we see the inequality is not satisfied (for the second one) so we do not have an optimal solution. Thus, we will choose k = 4 (only choice, as the first one is satisfied)

Example 3 Con'td

Step 3 **a):** Solve: $A_B d_B = -A_k$

$$\begin{array}{c} 2 \\ 3 \\ 5 \end{array}\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ -3 & 1 & 2 & -3 \\ 2 & 0 & 0 & 2 \end{array}\right] \longrightarrow \begin{array}{c} 2 \\ 3 \\ 5 \end{array}\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \longrightarrow \begin{bmatrix} d_2 \\ d_3 \\ d_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

b) Here we have that the problem is unbounded as all of the d ≥ 0

We have thus arrived at a conclusion (unboundedness).

Note this also can give us a certificate of unboundedness as well: x_4 = t will give:

$x'_B = x^*_B + td_B$:

$$x'_B = \begin{bmatrix} x_2 \\ x_3 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1.5 \\ 1.5 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \longrightarrow x' = \begin{bmatrix} 0 \\ 0.5+t \\ 1.5 \\ t \\ 1.5 \end{bmatrix}$$

Example 3 Continued

We can test that it is indeed a certificate of unboundedness as per the usual method:

$$x' = \begin{bmatrix} 0 \\ 0.5+t \\ 1.5 \\ t \\ 1.5 \end{bmatrix} \qquad \begin{array}{ll} \min & c^T x \\ st & Ax = b \\ & x \geq 0 \end{array} \qquad A = \begin{bmatrix} 1 & 1 & -1 & -1 & 2 \\ 0 & -3 & 1 & 3 & 2 \\ 1 & 2 & 0 & -2 & 0 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

Subbing this into the LP gives:

$$\begin{array}{ll} \min & c^T x \\ st & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{l} \min \quad -0.5 - t - t + 3 \quad = \quad 2.5 - 2t \\ Ax = b \rightarrow \begin{bmatrix} 0.5 + t - 1.5 - t + 3 \\ -1.5 - 3t + 1.5 + 3t + 3 \\ 1 + 2t - 2t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \end{array}$$

As t goes to infinity, we will have an unbounded OF and the bound constraints are satisfied as t is going to infinity will allow for all entries be ≥ 0.