Simplex Tableau

In this part of the lesson, we will explore a visualization of how to run the simplex method using what is called a Simplex Tableau.

Consider a LP in our standard form:

$$\begin{aligned} \min \quad & c^T x \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

We can construct a Simplex Tableau as follows:

a_{11}	a_{12}	…	a_{1m}	b_1
a_{21}	a_{22}	…	a_{2m}	b_2
…	…	…	…	…
a_{n1}	a_{n2}	…	a_{nm}	b_n
c_1	c_2	…	c_m	z

Basic Feasible Solution in the Tableau

When we have a basic feasible solution, (that is when we solve $A_B x_B = b$), we note that we could reduce the entire matrix A and this will result in a matrix where only the columns of B form an identity matrix. We could also reduce the OF matrix as well and our Tableau will look like the following.

Say our matrix is 4 x 8 and our B = {1, 2, 4, 8}

1	0	r_{13}	0	r_{15}	r_{16}	r_{17}	0	RS_1
0	1	r_{23}	0	r_{25}	r_{26}	r_{27}	0	RS_2
0	0	r_{33}	1	r_{35}	r_{36}	r_{37}	0	RS_3
0	0	r_{43}	0	r_{45}	r_{46}	r_{47}	1	RS_4
0	0	OF_3	0	OF_5	OF_6	OF_7	0	$z - z^*$

If we assign our basic variables (x_1, x_2, x_4, x_8) to be the right hand side and all other variables 0, we could get a basic feasible solution (provided that the RS ≥ 0 at this stage, which it should be since it is a BFS). If we consider the dual to find y^*this will solve to $y^* = (0,0,0,0)$ since we also reduced the OF. Since reducing a matrix system does not change the LP (they both have the same solution set and OF) this means solving the optimal solution in the tableau will also solve the original LP, but this one is easier to play with (the dual is 0 and the basis is clear).

When is it optimal?

1	0	r_{13}	0	r_{15}	r_{16}	r_{17}	0	RS_1
0	1	r_{23}	0	r_{25}	r_{26}	r_{27}	0	RS_2
0	0	r_{33}	1	r_{35}	r_{36}	r_{37}	0	RS_3
0	0	r_{43}	0	r_{45}	r_{46}	r_{47}	1	RS_4
0	0	$\boxed{OF_3}$	0	$\boxed{OF_5}$	$\boxed{OF_6}$	$\boxed{OF_7}$	0	$z - z^*$

Recall we know a solution is optimal if the dual solution y^* satisfies all of the dual constraints. This will clearly happen if and only if the last row (other than $z - z^*$) is non-negative (as our dual will be of the form $A^T y \leq c^T$ and $A_B^T y^* = 0$ forces $y^* = (0,0,0,0)$ so we get that we need to check $A^T y^* = 0 \leq c^T$).

What is the optimal solution? Well since we had $z = OF$ we know that $z - z^* = OF_3 x_3 + OF_5 x_5 + OF_6 x_6 + OF_7 x_7$ but since these are all of the non-Basis variables (and are thus = 0) we get: $z - z^* = 0$ or simply $z = z^*$

How do we know if it is unbounded?

1	0	r_{13}	0	r_{15}	r_{16}	r_{17}	0	RS_1
0	1	r_{23}	0	r_{25}	r_{26}	r_{27}	0	RS_2
0	0	r_{33}	1	r_{35}	r_{36}	r_{37}	0	RS_3
0	0	r_{43}	0	r_{45}	r_{46}	r_{47}	1	RS_4
0	0	$OF_3 - a$	0	OF_5	OF_6	OF_7	0	$z - z^*$

If one of the entries in the last Row is negative, then it will produce $0 \leq -a$ in the dual problem meaning that the current BFS is not an optimal solution. Say in this case it is OF_3. Our Simplex Method says "Solve $A_B d_B = -A_k$". Since our tableau has $A_B = I$, this means we get $d_B = -A_k$. It then asks us to check if all d_B is non-negative (≥ 0). In this case we simply check if the column A_k is non-positive (≤ 0) as $-d_B = A_k$. Thus, if the entire column is non-positive (i.e. ≤ 0), this means the problem is unbounded (we can select $x_3 = t$ and solve for the remaining basic variables to derive a certificate).

How do we get to the next BFS?

1	0	r_{13}	0	r_{15}	r_{16}	r_{17}	0	RS_1	$\div r_{13}$
0	1	r_{23}	0	r_{25}	r_{26}	r_{27}	0	RS_2	$\div r_{23}$
0	0	r_{33}	1	r_{35}	r_{36}	r_{37}	0	RS_3	$\div r_{33}$
0	0	r_{43}	0	r_{45}	r_{46}	r_{47}	1	RS_4	$\div r_{43}$
0	0	$OF_3 - a$	0	OF_5	OF_6	OF_7	0	$z - z^*$	

If at least one of the column values is positive, then it asks us to calculate:

$t = \min\left\{\frac{x_i}{-d_i}\right\}$ for all $i \in B$ that had positive values. In this case it simply means divide $\frac{RS_i}{r_{i3}}$ as $d_i = -r_{i3}$ (but **only for the positive values** that appear in that column!!)

This gives us what the next BFS would be (calculating $x_B' = x_B^* + td_B, x_k' = t$, etc…), however, we wish to keep using the tableau, so it turns out all that matters is which row gives the minimum t, then we simply reduce the column by forcing the pivot to be the row that corresponds to the minimum t.

When is it infeasible?

This would actually come out of solving Phase 1 (ie, setting up the Auxiliary Problem). If we solve the Auxiliary LP problem and we cannot get an OVF = 0 (for the $s_1 + s_2 + \cdots + s_j$). However, since this problem is also a LP, it can also be solved using the Simplex Tableau!

We should also note that we still should follow Bland's anticycling rule of selecting the smallest value at any choice. Going forward, you should only use Bland's rule (when selecting k or r, always select the lowest such variable in the table - i.e. the leftmost column (k) when looking at a negative in the last row, or the topmost row (r) when there are more than one r that give the same t value).

1	0	r_{13}	0	r_{15}	r_{16}	r_{17}	0	RS_1	$\div r_{15}$	$\cancel{7}$
0	1	r_{23}	0	r_{25}	r_{26}	r_{27}	0	RS_2	$\div r_{25}$	$\cancel{-7}$
0	0	r_{33}	1	r_{35}	r_{36}	r_{37}	0	RS_3	$\div r_{35}$	2
0	0	r_{43}	0	r_{45}	r_{46}	r_{47}	1	RS_4	$\div r_{45}$	$\cancel{2}$
0	0	$\cancel{OF_3 \geq 0}$	0	$OF_5 < 0$	$\cancel{OF_6 \geq 0}$	$\cancel{OF_7 < 0}$	0	$z - z^*$		

When selecting r, choose the smallest value in the **topmost** row that is ≥ 0. This will follow Bland's Rule

When selecting k, choose the **leftmost** column that is < 0
This will follow Bland's Rule

Strategy: Using the Simplex Tableau to solve LPs

How To Use It

To use the simplex tableau to solve Linear Programs starting with a feasible basis B (either given or found using the Auxiliary LP) and a LP in standard form, you can do the following:

Step 1: Set up the Tableau given here:

A	b
c	z

Step 2: Row reduce the system so that the columns in B form an identity matrix (including the last row which stores the OF and z – z*).

Step 3: Search the entries in the OF row (last row without the $z - z^*$ box):

1) If all of the entries are ≥ 0 then stop, as your BFS is optimal (the optimum value will be the negative of the coefficient stored in the z box (as a negative), and the BFS values will appear on the RHS (all other BFS values are zero)).
2) Select the **smallest column (leftmost column)** k that has negative in the OF Row (last row):
   i) If this column has only non-positive (≤ 0) entries, then stop the problem is unbounded. You can select $x_k = t$ and solve for the certificate of unboundedness.
   ii) For all positive entries in this column (except OF row) calculate $\frac{RHS_i}{A_{ik}}$. Select the **minimum** of such values as your new pivot r. If you have a choice of minimum values (i.e. 2 or more identical divisions producing minimum t values), select the **smallest r (topmost row).**
3) Reduce the matrix again using A_{rk} as the pivot and repeat **Step 3** until termination.

When To Use It

When we want to solve a LP using an organized simplex tableau.

Why It Works

This was simply using our understanding of the Simplex Method in an organized way.

Example 1

Consider the LP problem given:

$$\begin{aligned} \min \quad & c^T x \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad A = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & -1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, c = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

BFS with B={1,3,5}. Perform the simplex method with Bland's Anticycling to find the optimal solution or show unboundedness using a Simplex Tableau.

Create Tableau and Reduce

1	1	−1	0	0	0
0	1	1	0	1	1
−1	0	1	1	−1	0
1	0	1	0	1	z

(Use x_1 as a pivot)

1	1	-1	0	0	0
0	1	1	0	1	1
0	1	0	1	-1	0
0	-1	2	0	1	$z-0$

(Use x_3 as a pivot)

1	2	0	0	1	1
0	1	1	0	1	1
0	1	0	1	-1	0
0	-3	0	0	-1	$z-2$

(Use x_5 as a pivot)

1	3	0	1	0	1
0	2	1	1	0	1
0	-1	0	-1	1	0
0	-4	0	-1	0	$z-2$

Example 1 continued

1	$\boxed{3}$	0	1	0	1	$\div 3$
0	2	1	1	0	1	$\div 2$
0	-1	0	-1	1	0	
0	$\boxed{-4}$	0	-1	0	$z-2$	

$1/3$	1	0	$1/3$	0	$1/3$
$-2/3$	0	1	$1/3$	0	$1/3$
$1/3$	0	0	$-2/3$	1	$1/3$
$4/3$	0	0	$1/3$	0	$z-2/3$

We have a negative value, select smallest index (k = 2).

We look for positive values (in this column) and ÷ the RHS by these values.

We find the smallest value of this division (in this case it is 1/3, so there is no choice) and force this to be the new pivot.

Here we see that our last row only has non-negative (≥ 0) values, thus we are on an optimal solution! Our x^* corresponds to the pivots, and everything else is zero, and it hold our z^* in the bottom right corner (keep in mind this is of the form $z - z^*$).
Thus $z^* = 2/3$ and x^* =[0, 1/3, 1/3, 0, 1/3]

Example 2

Consider the LP problem given:

$$\begin{aligned} \min \quad & c^T x \\ st \quad & Ax = b \\ & x \geq 0 \end{aligned} \qquad A = \begin{bmatrix} 1 & 1 & -1 & -1 & 2 \\ 0 & -3 & 1 & 3 & 2 \\ 1 & 2 & 0 & -2 & 0 \end{bmatrix}, b = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, c = \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

BFS with B={1,3,5}. Perform the simplex method with Bland's Anticycling to find the optimal solution or show unboundedness using a Simplex Tableau.

Create Tableau and Reduce

1	1	-1	-1	2	2
0	-3	1	3	2	3
1	2	0	-2	0	1
1	-1	0	-1	2	z

(Use x_1 as a pivot)

1	1	-1	-1	2	2
0	-3	1	3	2	3
0	1	1	-1	-2	-1
0	-2	1	0	0	$z-2$

(Use x_3 as a pivot)

1	-2	0	2	4	5
0	-3	1	3	2	3
0	-1	0	1	1	1
0	1	0	-3	-2	z-5

(Use x_5 as a pivot)

1	2	0	-2	0	1
0	-1	1	1	0	1
0	-1	0	1	1	1
0	-1	0	-1	0	z-3

Example 2 continued

1	$\boxed{2}$	0	-2	0	$\boxed{1}$ $\div 2$
0	-1	1	1	0	1
0	-1	0	1	1	1
0	$\boxed{-1}$	0	-1	0	z-3

We have a negative value, select smallest index (k = 2).

We look for positive values (in this column) and ÷ the RHS by these values.

We find the smallest value of this division and force this to be the new pivot.

0.5	1	0	-1	0	0.5
0.5	0	1	0	0	1.5
0.5	0	0	0	1	1.5
0.5	0	0	$\boxed{-2}$	0	z-2.5

We have a negative value, select smallest index (k = 4).

We look for positive values (in this column) and ÷ the RHS by these values.

We do not have any positive values! Thus, the problem is unbounded.

We get our certificate for unboundedness as:

$x_1 = 0$

$x_2 = 0.5 + t$

$x_3 = 1.5$

$x_4 = t$

$x_5 = 1.5$

$z - 2.5 = -2t$ (which gives $z = 2.5 - 2t$ which clearly has $z \to -\infty$ when $t \to \infty$)