Proof of Complementary Slackness
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We will need Strong Duality to assert that if we have optimal solutions $\mathbf{x}^*$ to the primal and $\mathbf{y}^*$ to the dual then $\mathbf{c}\cdot\mathbf{x}^* = \mathbf{b}\cdot\mathbf{y}^*$.

Theorem (Complementary Slackness) Let $\mathbf{x}$ be a feasible solution to the primal and $\mathbf{y}$ be a feasible solution to the dual where

$$\begin{array}{lccc} & \max & \mathbf{c}\cdot\mathbf{x} & & \min & \mathbf{b}\cdot\mathbf{y} \\ \text{primal} & & A\mathbf{x}\le\mathbf{b} & \text{dual} & & A^T\mathbf{y}\ge\mathbf{c}\,. \\ & & \mathbf{x}\ge\mathbf{0} & & & \mathbf{y}\ge\mathbf{0}\end{array}$$

Then $\mathbf{x}$ is optimal to the primal and $\mathbf{y}$ is optimal to the dual if and only if the conditions of Complementary Slackness hold:

$$(b_i - \sum_{j=1}^{n} a_{ij}x_j)y_i = x_{n+i}y_i = 0 \text{ for } i = 1, 2, \ldots, m$$

and

$$(\sum_{i=1}^{m} a_{ji}y_i - c_j)x_j = y_{m+j}x_j = 0 \text{ for } j = 1, 2, \ldots, n.$$

Proof: Given that $\mathbf{x}$ and $\mathbf{y}$ are feasible implies $A\mathbf{x}\le\mathbf{b}$, $\mathbf{x}\ge\mathbf{0}$, $A^T\mathbf{y}\ge\mathbf{c}$, and $\mathbf{y}\ge\mathbf{0}$. We now use the ideas in our proof of Weak Duality. Now $\mathbf{x}\ge\mathbf{0}$, $A^T\mathbf{y}\ge\mathbf{c}$ implies $\mathbf{x}^TA^T\mathbf{y}\ge\mathbf{x}^T\mathbf{c}$ where $\mathbf{x}^T\mathbf{c} = \mathbf{c}\cdot\mathbf{x}$. Also $A\mathbf{x}\le\mathbf{b}$, $\mathbf{y}\ge\mathbf{0}$ implies $\mathbf{y}^TA\mathbf{x}\le y^T\mathbf{b}$ where $\mathbf{y}^T\mathbf{b} = \mathbf{b}\cdot\mathbf{y}$. We have

$$\mathbf{c}\cdot\mathbf{x} = \mathbf{x}^T\mathbf{c} \le \mathbf{x}^TA^T\mathbf{y} = \mathbf{y}^TA\mathbf{x} \le \mathbf{y}^T\mathbf{b} = \mathbf{b}\cdot\mathbf{y}$$

using the fact that $\mathbf{x}^TA^T\mathbf{y}$ is a 1×1 matrix and so $\mathbf{x}^TA^T\mathbf{y} = (\mathbf{x}^TA^T\mathbf{y})^T = \mathbf{y}^TA\mathbf{x}$. Now by Strong Duality, $\mathbf{x}$ and $\mathbf{y}$ are both optimal to their respective Linear Programs if and only if $\mathbf{c}\cdot\mathbf{x} = \mathbf{b}\cdot\mathbf{y}$ and so if and only if $\mathbf{x}^T\mathbf{c} = \mathbf{x}^TA^T\mathbf{y}$ and $\mathbf{y}^TA\mathbf{x} = y^T\mathbf{b}$ and hence if and only if $\mathbf{x}\cdot(A^T\mathbf{y}-\mathbf{c}) = 0$ and $\mathbf{y}\cdot(\mathbf{b}-A\mathbf{x}) = 0$. We note that $0 = \mathbf{x}\cdot(A^T\mathbf{y}-\mathbf{c}) = \sum_{j=1}^{n}\Big(x_j(\sum_{i=1}^{m} a_{ji}y_i - c_j)\Big)$ and since $x_j\ge 0$ and $\sum_{i=1}^{m} a_{ji}y_i - c_j \ge 0$ for each $j = 1, 2, \ldots, n$ we deduce that $x_j(\sum_{i=1}^{m} a_{ji}y_i - c_j)\ge 0$ for each $j = 1, 2, \ldots, n$. But then each of the n terms in the sum $\sum_{j=1}^{n}\Big(x_j(\sum_{i=1}^{m} a_{ji}y_i - c_j)\Big)$ are positive and yet the sum is 0 and so we deduce that each term is 0, namely $x_j(\sum_{i=1}^{m} a_{ji}y_i - c_j) = 0$ for each $j = 1, 2, \ldots, n$ which is the second condition for Complementary Slackness. Similarly we deduce that $\mathbf{y}\cdot(\mathbf{b}-A\mathbf{x}) = 0$ if and only if

$$y_i(b_i - \sum_{j=1}^{n} a_{ij}x_j) = y_ix_{n+i} = 0 \text{ for } i = 1, 2, \ldots, m.$$

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