Introduction to Mathematical Programming

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Lecture 19

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Definition

We will discuss an important (special case) example of constrained optimization: Linear Programming.
Consider the following problem,

$$
\begin{aligned}
\text{Minimize :} \quad & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\
\text{subject to :} \quad & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n && = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n && = b_2 \\
& \vdots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n && = b_m \\
x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0
\end{aligned}
$$

It is a constrained optimization problem with linear objective function and linear constrains.

Linear Programming, Matrix Form

It can be re-written as,

$$\text{Minimize :} \quad \vec{c}^{\top}\vec{x}$$
$$\text{subject to :} \quad \mathbf{A}\vec{x} = \vec{b} \quad \text{and} \quad \vec{x} \geq \vec{\mathbf{0}}$$

where $\vec{x} \geq \vec{\mathbf{0}}$ is defined component wise, i.e., $x_i \geq 0$.

Goal

Thus given $\mathbf{A}$, $\vec{b}$ and $\vec{c}$, the problem is asking to find a $\vec{x}$ such that the objective function is at its minimal

Before we discuss the fundamental theorem regarding Linear Programming, we have to establish a few terms and lemmas.

Hyperplanes and Half-Spaces

Definition

A hyperplane H in $\mathbb{R}^n$ is a collection of points of the form, $\{\vec{x} \in \mathbb{R}^n : \vec{p}^\top \vec{x} = \alpha\}$, where $\vec{p}$ is a non-zero vector in $\mathbb{R}^n$ and α is a scalar.

Moreover,

- $\vec{p}$ is called the normal vector of the hyperplane H.
- A hyperplane defines two closed half-spaces $H^+ = \{\vec{x} \in \mathbb{R}^n : \vec{p}^\top \vec{x} \geq \alpha\}$ and $H^- = \{\vec{x} \in \mathbb{R}^n : \vec{p}^\top \vec{x} \leq \alpha\}$.
- And two open half-spaces $\{\vec{x} \in \mathbb{R}^n : \vec{p}^\top \vec{x} > \alpha\}$ and $\{\vec{x} \in \mathbb{R}^n : \vec{p}^\top \vec{x} < \alpha\}$.
- Any point in $\mathbb{R}^n$ lies in H^+, or in H^-, or in both.
- Let $\bar{\vec{x}} \in H$, then $H = \{\vec{x} \in \mathbb{R}^n : \vec{p}^\top (\vec{x} - \bar{\vec{x}}) = 0\}$.
- Example: $H = \{(x_1, x_2) : x_1 + x_2 = 4\}$, then $\vec{p} = (1, 1)^\top$.

Separating Hyperplanes

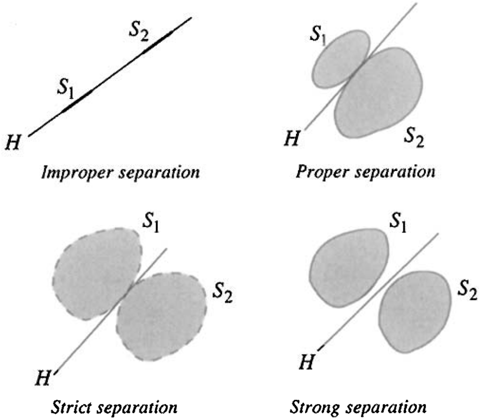
Definition

Let S_1 and S_2 be two non-empty sets in $\mathbb{R}^n$. A hyperplane $H = \{\vec{x} \in \mathbb{R}^n : \vec{p}^\top \vec{x} = \alpha\}$ is said to separate S_1 and S_2 if $\vec{p}^\top \vec{x} \geq \alpha$ for each $\vec{x} \in S_1$ and $\vec{p}^\top \vec{x} \leq \alpha$ for each $\vec{x} \in S_2$.

In addition,

- If $S_1 \cup S_2 \not\subset H$, H is said to properly separate S_1 and S_2.
- Strictly separate if $\vec{p}^\top \vec{x} > \alpha$ for each $\vec{x} \in S_1$ and $\vec{p}^\top \vec{x} < \alpha$ for each $\vec{x} \in S_2$.
- Strongly separate if $\vec{p}^\top \vec{x} \geq \alpha + \epsilon$ for each $\vec{x} \in S_1$ and $\epsilon > 0$, and $\vec{p}^\top \vec{x} \leq \alpha$ for each $\vec{x} \in S_2$.

Example


S_2
S_1
H
Improper separation
S_1
H
S_2
Proper separation
S_1
S_2
H
Strict separation
S_1
S_2
H
Strong separation

Separation Theorem

Theorem

Let S be a non-empty close convex set in $\mathbb{R}^n$ and $\vec{y} \notin S$. Then there exists a non-zero vector $\vec{p}$ and a scalar α such that $\vec{p}^\top \vec{y} > \alpha$ and $\vec{p}^\top \vec{x} \leq \alpha$ for each $\vec{x} \in S$.

We will not prove that theorem, however we will discuss some important consequences induced from the theorem,

- S is a closed convex set in $\mathbb{R}^n$, then S is the intersection of all half-spaces containing S.
- If $\vec{y}$ is not in the closure of the convex hull of S, then there is an H strongly separating S and $\vec{y}$.

Polyhedral Sets

Polyhedral sets are important special cases of convex sets,

- Any closed convex set is the intersection all closed half-spaces containing it.
- In the case of polyhedral sets, only a finite number of half-spaces are needed.

Definition

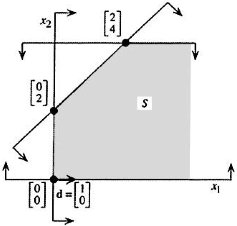
A set S in $\mathbb{R}^n$ is called a polyhedral set if it is the intersection of a finite number of closed half-spaces; that is,
$S = \{\vec{x} \in \mathbb{R}^n : \vec{p}_i^\top \vec{x} = \alpha_i, \text{ for } i = 1, \cdots, m\}$, where $\vec{p}_i$ is a non-zero vector and α_i is a scalar for $i = 1, \cdots, m$.

Note that polyhedral sets are closed and convex.

Some Examples

The following are some typical examples of polyhedral sets,

$$S = \{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} \leq \vec{b}\}$$
$$S = \{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \vec{b}, \vec{x} \geq \vec{\mathbf{0}}\}$$
$$S = \{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} \geq \vec{b}, \vec{x} \geq \vec{\mathbf{0}}\}$$
$$S = \{(x_1, x_2) : -x_1 + x_2 \leq 2, x_2 \leq 4, x_1 \geq 0, x_2 \geq 0\}$$



Extreme Points

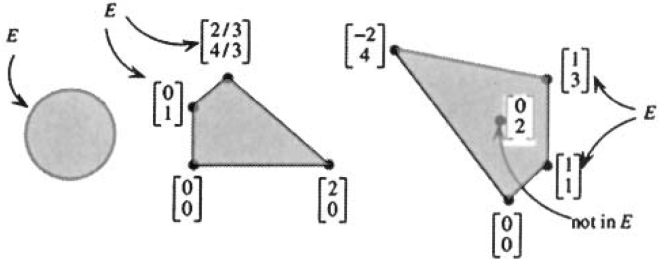
Definition

Let S be a non-empty convex set in $\mathbb{R}^n$. A vector $\vec{x} \in S$ is called an extreme point of S if $\vec{x} = \lambda\vec{x}_1 + (1-\lambda)\vec{x}_2$ with $\vec{x}_1, \vec{x}_2 \in S$, and $\lambda \in (0,1)$ implies that $\vec{x} = \vec{x}_1 = \vec{x}_2$.

Examples,

- $S = \{(x_1, x_2) : x_1^2 + x_2^2 \leq 1\}$ and $E = \{(x_1, x_2) : x_1^2 + x_2^2 = 1\}$.
- $S = \{(x_1, x_2) : x_1 + x_2 \leq 2, -x_1 + 2x_2 \leq 2, x_1 \geq 0, x_2 \geq 0\}$ and $E = \{(0,0), (0,1), (2/3, 4/3), (2,0)\}$.
- S is the polytope generated by $(0,0), (1,1), (1,3), (-2,4), (0,2)$ and $E = \{(0,0), (1,1), (1,3), (-2,4)\}$.
- Note that for these S's, any point of S can be represented as a convex combination of the extreme points; not true for unbounded sets.

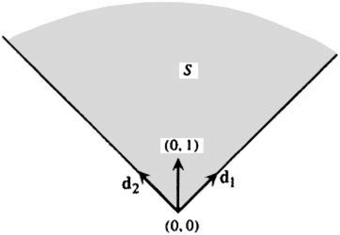
Example



Extreme Directions

Let S be a non-empty, closed convex set in *real*^{n},

- A non-zero vector $\vec{d} \in \mathbb{R}^n$ is called a direction, or a recession direction, of S if for each $\vec{x} \in S$, $\vec{x} + \lambda\vec{d} \in S$ for all $\lambda > 0$.
- Two directions $\vec{d}_1$ and $\vec{d}_2$ are called distinct if $\vec{d}_1 \neq \alpha\vec{d}_2$ for any $\alpha > 0$.
- A direction $\vec{d}$ of S is called an extreme direction if it cannot be written as a positive linear combination of two distinct directions, i.e., if $\vec{d} = \lambda_1\vec{d}_1 + \lambda_2\vec{d}_2$ for $\lambda_1, \lambda_2 > 0$, then $\vec{d}_1 = \alpha\vec{d}_2$ for some $\alpha > 0$.



Extreme Points and Extreme Directions of Polyhedral Sets

Consider the polyhedral set: $S = \{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \vec{b}, \vec{x} \geq 0\}$,

- $\mathbf{A}$ is an $m \times n$ matrix and $\vec{b}$ is a m-vector.
- Assume $\mathbf{A}$ is of rank m, if not assuming $\mathbf{A}\vec{x} = \vec{b}$ is consistent.
- The number of extreme points of S is finite; S has at least one extreme point.
- The number of extreme directions of S is finite.
- Any point in S can be represented as a convex combination of its extreme points plus a non-negative linear combination of its extreme directions, that is, $\vec{x} \in S$ if and only if $\vec{x}$ can be written as

$$\vec{x} = \sum_{j=1}^{k} \lambda_j \vec{x}_j + \sum_{j=1}^{\ell} \mu_j \vec{d}_j,$$

  where $\sum_{j=1}^{k} \lambda_j = 1$ and $\lambda_j \geq 0$ for $j = 1, \cdots, k$; and $\mu_j \geq 0$ for $j = 1, \cdots, \ell$.

Existence of Extreme Directions

Theorem

Let S be a non-empty polyhedral set of the form, $\{\vec{x} \in \mathbb{R}^n : \mathbf{A}\vec{x} = \vec{b}, \vec{x} \geq 0\}$ *where* $\mathbf{A}$ *is an* $m \times n$ *matrix with rank* m*. Then* S *has at least one extreme direction if and only if it is unbounded.*

Proof.

If S has an extreme direction, it is obviously unbounded. Now Suppose S is unbounded and by contradiction, supposed that S has no extreme directions. For then for $\vec{x} \in S$,

$$||\vec{x}|| = ||\sum_{j=1}^{k} \lambda_j \vec{x}_j|| \leq \sum_{j=1}^{k} \lambda_j ||\vec{x}_j|| \leq \sum_{j=1}^{k} ||\vec{x}_j||$$

Contradiction. □