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MATHEMATICS II

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TOPIC OUTLINE

UNIT 3

SYSTEMS OF LINEAR EQUATIONS

STUDY GOALS

On completion of this unit, you will be able to …

- … translate text into a system of linear equations (LES).
- … express an LES using the extended matrix notation.
- … find the solutions to an LES using the Gauss algorithm.
- … use the Gauss algorithm to find the inverse of a matrix.

EXPLAIN SIMPLY

1. What is a linear equation system (LES)?
2. Explain how to find the solutions to an LES.
3. Can an LES have more than one solution? How do you find out?
4. Can an LES have no solution? How do you find out?

Example:

SmartShoes, Inc. sells sports shoes of three different models (A, B and C).

- Yesterday's revenues amounted to 1,980 EUR, where the selling prices per pair of shoes was 50 EUR for model A, 80 EUR for model B and 90 EUR for model C. $50x_1 + 80x_2 + 90x_3 = 1{,}980$
- In total, the company sold 30 pairs of shoes. $x_1 + x_2 + x_3 = 30$
- The number of model A pairs of shoes the company sold was as large as the sum of the model B and model C pairs. $x_1 - x_2 - x_3 = 0$

How many model A, model B and model C shoes did the company sell?

A **linear equation system (LES)** is a set of m equations and n variables (unknowns) $x_1, x_2, \dots, x_n$ of the following general form:

$$
\begin{aligned}
& a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
& a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
& \dots \\
& a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m
\end{aligned}
$$

where $a_{ij} \in \mathbb{R}, b_i \in \mathbb{R}$ for $i = 1, \dots, m; j = 1, \dots, n$

A linear equation system is often written in a **matrix notation**:

$A\vec{x} = \vec{b}$, where

Coefficient Matrix

Extended Coefficient Matrix

Variable Vector

Right-Hand Side Vector

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$A|b = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right)$$

An LES can have no, exactly one, or infinitely many solutions.

Case 1: One solution

Example:
$$\begin{aligned} 50x_1 &+80x_2 &+90x_3 &= 1{,}980 \\ x_1 &+x_2 &+x_3 &= 30 \\ x_1 &-x_2 &-x_3 &= 0 \end{aligned}$$

Case 2: No solution

Example:
$$\begin{aligned} x_1 &+x_2 &= 2 \\ 2x_1 &+2x_2 &= 8 \end{aligned}$$

Case 3: Infinitely many solutions

Example:
$$\begin{aligned} x_1 &+x_2 & &= 0 \\ x_1 &+x_2 &+x_3 &= 1 \end{aligned}$$

The Gauss Algorithm

- **Step 1:** Generate the extended matrix $A|b$
- **Step 2:** Choose $i = 1$.
- **Step 3:** Use **elementary row operations** to achieve that
  $a_{ii} \neq 0$ and $a_{ji} = 0$ for all $j > i$
  (i.e., all entries below a_{ii} must be 0)
- **Step 4:** Increase index i by 1.
  STOP if $i = n$
  else go to step 3.

- interchanging rows
- multiplying a row with a scalar $\neq 0$
- Adding (or subtracting) rows

Case 1

Applying the Gauss algorithm to the SmartShoes example yields:

$$\left(\begin{array}{ccc|c} 50 & 80 & 90 & 1{,}980 \\ 1 & 1 & 1 & 30 \\ 1 & -1 & -1 & 0 \end{array}\right) \xrightarrow{\text{Swap rows}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 1 & -1 & -1 & 0 \\ 50 & 80 & 90 & 1{,}980 \end{array}\right)$$

$$\xrightarrow[(3)-50\cdot(1)]{(2)-(1)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & -2 & -2 & -30 \\ 0 & 30 & 40 & 480 \end{array}\right) \xrightarrow{(2)/(-2)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & 1 & 1 & 15 \\ 0 & 30 & 40 & 480 \end{array}\right)$$

$$\xrightarrow{(3)-30\cdot(2)} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & 1 & 1 & 15 \\ 0 & 0 & 10 & 30 \end{array}\right) \xrightarrow{(3)/10} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 0 & 1 & 1 & 15 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

Hence: $x_3 = 3$

$x_2 + x_3 = x_2 + 3 = 15$, so $x_2 = 12$

$x_1 + x_2 + x_3 = x_1 + 12 + 3 = 30$, so $x_1 = 15$

Case 2

If, during the Gauss algorithm, we obtain a row containing zeros only on the left, but a non-zero value on the right-hand side, original equation system contains a **contradiction**. Hence, the equation system has **no solution**.

Example:

$$\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 6 & 12 \end{array}\right) \xrightarrow{(3)/3} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 4 \end{array}\right) \xrightarrow{(3)-(1)} \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 1 \end{array}\right)$$

contradiction
→ no solution

Case 3

If the resulting upper triangular matrix of the Gauss algorithm contains less equations r than variables n, the equation system has **infinitely many solutions**: We can choose variables $x_{r+1}, \dots, x_n$ freely and calculate the values of the remaining variables by solving the inequalities bottom-up.

Example:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 6 & 7 & 8 \end{array}\right)$$

set $x_3 = s$ and $x_4 = t$ for any $s, t \in \mathbb{R}$

$x_2 + 6s + 7t = 8$ (from equation 2), hence

$\boldsymbol{x_2 = 8 - 6s - 7t}$

$x_1 + 2x_2 + 3s + 4t = 5$ (from equation 1), hence

$\boldsymbol{x_1} = 5 - 2x_2 - 3s - 4t$

$= 5 - 2(8 - 6s - 7t) - 3s - 4t = \boldsymbol{-11 + 9s + 10t}$

The Extended Gauss Algorithm (to find an inverse matrix)

- **Step 1:** Generate the extended matrix $(A|E)$, where E is the unit matrix.
- **Step 2:** Choose $i = 1$.
- **Step 3:** Use **elementary row operations** to achieve that

  $a_{ii} \neq 0$ and $a_{ji} = 0$ **for all** $\boldsymbol{j \neq i}$

  (i.e., all entries below **and above** a_{ii} must be 0)

  Difference to standard Gauss algorithm
- **Step 4:** If step 3 yields a null row in A: STOP (A does not have an inverse)
- **Step 5:** Increase index i by 1.

  STOP if $i = n$; else go to step 3.

Example: Find the inverse of $A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{(2)-(1)} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{(2)\leftrightarrow(3)} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow{(2)/2} \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right) \xrightarrow{(1)-(2)} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & -0.5 \\ 0 & 1 & 1 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$$\xrightarrow[(2)-(3)]{(1)+(3)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 1 & -0.5 \\ 0 & 1 & 0 & 1 & -1 & 0.5 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

$=A^{-1}$, i.e. inverse of A

You are now able to …

- … translate text into a system of linear equations (LES).
- … express an LES using the extended matrix notation.
- … find the solutions to an LES using the Gauss algorithm.
- … use the Gauss algorithm to find the inverse of a matrix.

SESSION 2

TRANSFER TASK 1

At a student club's summer party, soda, beer and cola were sold. All beverages were sold in bottles. At their retailer, the student club purchased soda at a price of 1 EUR per bottle, and both beer and cola at a price of 2 EUR per bottle. At the party, soda was then sold at 2 EUR per bottle, beer at 4 EUR per bottle and cola at 3 EUR per bottle.

The party's revenues with beverages amounted to 1,670 EUR. The retailer's bill was 945 EUR, and 550 bottles were consumed in total. How many bottles of soda, beer and cola have been sold each?

Please present your results.
The results will be discussed in plenary.

SESSION 2

TRANSFER TASK 2

Use the extended Gauss algorithm to find the inverse of the following matrices (or show that the matrix does not have an inverse):

$$A = \begin{pmatrix} 3 & 6 & 4 \\ 1 & 4 & 2 \\ 2 & 2 & 2 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & 5 & 0 \end{pmatrix}$$

TRANSFER TASK 2
PRESENTATION OF THE RESULTS

Please present your results.
The results will be discussed in plenary.

1. Every linear system of equations …

   a) … has one solution.
   b) … may have a solution.
   c) … has an infinite number of solutions.
   d) … has no solution.

2. Which one of the following row operations may change the set of solutions of a linear system of equations?

a) the swapping of two rows
b) replacing one row with the sum of said row and another row
c) replacing one row with the difference between said row and another row.
d) multiplication of all row elements by any real number

3. Which one of the following scenarios does **not** occur after applying the Gaussian algorithm?

a) The final expanded matrix has more rows than columns.
b) The final expanded matrix has an equal number of rows and columns.
c) The final expanded matrix has more columns than rows.
d) The final expanded matrix is an upper triangular matrix.

LIST OF SOURCES

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Robbiano L. (2011). *Linear algebra for everyone.* Springer.