Kathmandu University

Department of Computer Science and Engineering

Dhulikhel, Kavre

A Project Report

on

“Linear/Bilinear spline interpolation in image processing”

MCSC 202

Submitted by:

Ankit Bhattarai (9)

Prabesh Guragain (22)

Abhiyan Dhakal (13)

Pranish Kafle (26)

Sugat Sujakhu (55)

Submitted to:

Dr. Gokul K.C.

Department of Mathematics

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Abstract

This report explores the significance of interpolation techniques, particularly linear and bilinear splines, in image processing applications, emphasizing their role in resizing and resampling images. The advantages of simplicity and computational efficiency, coupled with the absence of Runge's phenomenon, make linear and bilinear interpolation methods widely adopted. However, their limitations, such as the lack of smoothness in functions and the tendency to produce blocky artifacts in resized images, highlight the need for higher-order splines in certain applications. The report introduces spline functions, providing a mathematical definition and emphasizing their preference over polynomial interpolation due to their ability to mitigate Runge's phenomenon. It explores linear and bilinear spline interpolation methods, outlining their mathematical definitions and construction. The application of these methods in image processing is illustrated, particularly in resizing images using bilinear spline interpolation for 2D matrices. Overall, the report explains the principles, advantages, limitations, and practical applications of linear and bilinear spline interpolation in the context of image processing.

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Chapter 1 Introduction

1.1 Background

Image processing often involves the need to resize or resample images, and interpolation is a crucial technique used to estimate pixel values at non-integer coordinates. Spline interpolation methods, particularly linear and bilinear splines, play a significant role in achieving smooth and visually pleasing results during such operations.

1.2 Definition

Interpolation is a mathematical technique used to estimate values between known or given values. In other words, it is the process of estimating data points within the range of discrete, known data points. The goal of interpolation is to provide a reasonable guess of the values that lie between the given data points, allowing for the analysis, visualization, or computation of values at non-integer positions

1.2.1 Advantages:

- Simplicity and computational efficiency.
- Linear/Bilinear interpolation is easy to implement.
- Absence of Runge's phenomenon.

1.2.2 Limitations:

- The functions will not be smooth like higher order spline.
- Tends to produce blocky artifacts in resized images.
- Not suitable for applications where higher accuracy is required.

Chapter 2 Spline

Spline is a piecewise polynomial function defined over intervals. Spline is Used for interpolating or approximating values between given data points.

Spline interpolation is often preferred to polynomial interpolation because it yields similar results, even when using low degree polynomials, while avoiding Runge's phenomenon for higher degrees.

2.1 Mathematical Definition of Splines:

A function s(x) is called as a spline function if it satisfies the following properties:

1. s(xi) = f (x_i) for i = 1, 2..., n.
2. On each subinterval [x_i, x_i +1] where i = 1, 2…, n, s(x) is a polynomial of degree n-1.
3. s(x) and its first (n − 2) derivatives are continuous on (a, b).

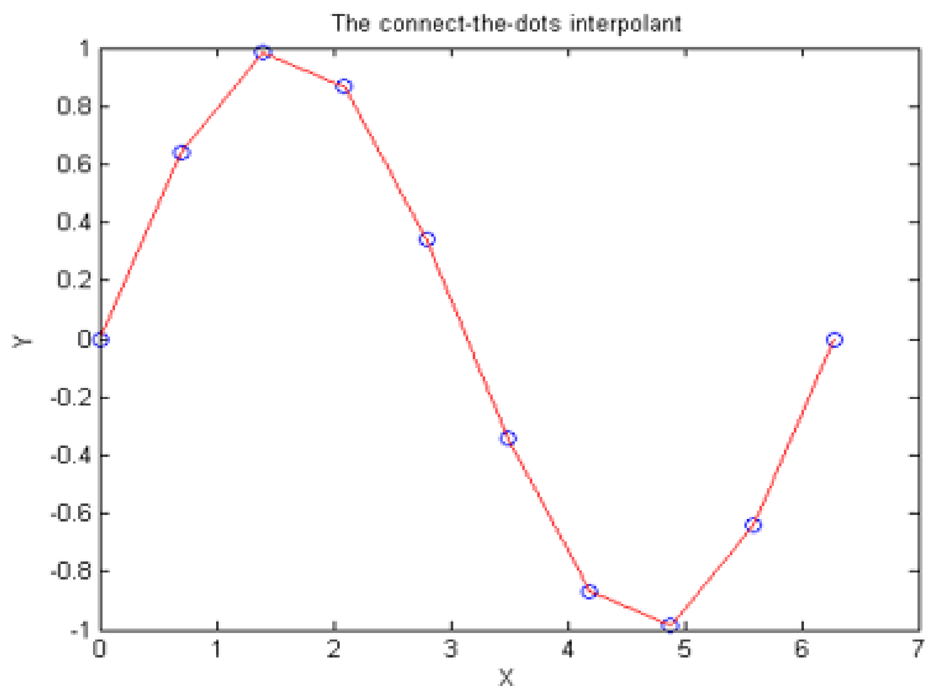


Figure 1Graph of spline function

Chapter 3 Spline Interpolation

Spline interpolation is a method to create a smooth curve that passes through a set of given points. Spline interpolation breaks the interval between two adjacent data points into smaller subintervals and fits a polynomial function to each subinterval.

3.1 Types of Spline Interpolations

3.1.1 Linear Spline Interpolation

Linear spline interpolation is a method used to estimate values between two known data points by creating a series of straight-line segments, or "splines," between those points.

Mathematical Definition:

A function s(x) is called as a linear spline if it satisfies the following properties:

1. $s(x_i) = f(x_i)$ for $i = 1, 2\ldots, n$.
2. s(x) is a polynomial of degree 1, denoted by $s_i(x)$, on each subinterval $[x_i, x_{i+1}]$ where $i = 1, 2, \ldots, n - 1$,
3. s(x) is a continuous function on (a, b).

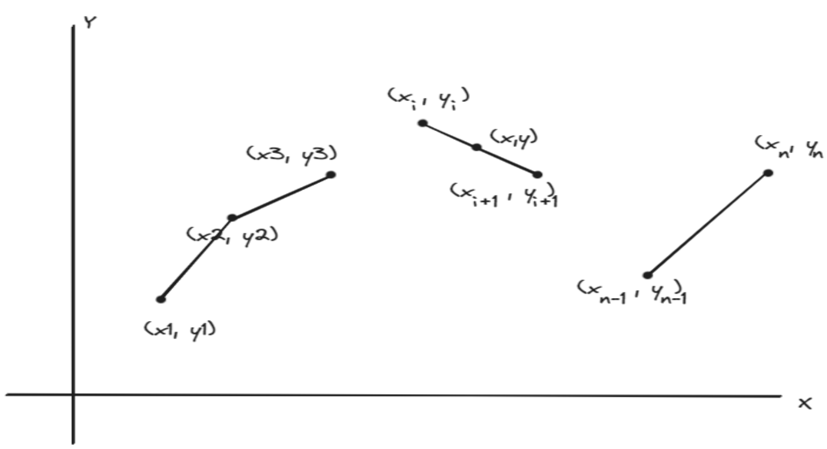


Figure 2 Demonstration of linear spline interpolation

Construction of Linear Splines:

In a linear spline, each function is merely a straight line connecting two data points, (x_i , $f(x_i)$) and (x_{i+1}, $f(x_{i+1})$) in the subinterval [x_i , x_{i+1}]. The line joining these two points is given by:

$$\frac{s_i(x) - f(x_i)}{x - x_i} = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}$$

$$\Longrightarrow \quad \boxed{s_i(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i)}$$

This can also be written as

$$\boxed{s_i(x) = a_i + b_i(x - x_i)}$$

where

$$a_i = f(x_i) \quad \text{and} \quad b_i = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}.$$

3.1.2 Bilinear Spline Interpolation

In mathematics, bilinear interpolation is a method for interpolating functions of two variables (e.g., x and y) using repeated linear interpolation. This is achieved by first interpolating in the x-direction then in the y-direction or vice-versa.

Calculation of Bilinear Spline interpolation

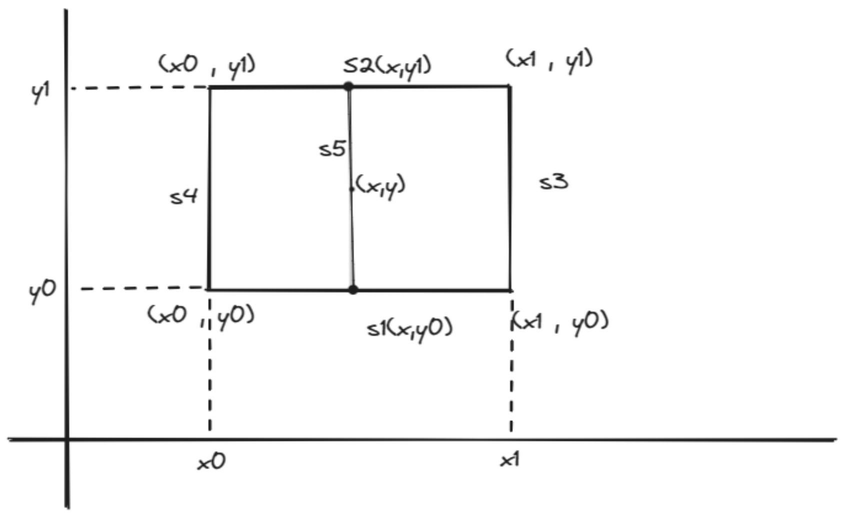


Figure 3 Bi-linear spline interpolation

Suppose that we want to find the value of the unknown function *f* at the point (*x*, *y*). We can calculate using repeated linear Interpolation

We first do linear interpolation in the *x*-direction. This gives

$$s_1(x,\, y_0) \;=\; f(x_0,\, y_0) \;+\; \frac{f(x_1,\, y_0) \;-\; f(x_0, y_0)}{x_1 - x_0}(x \;-\; x_0)\,.\,.\,.\,.\,.\,.\,.\;(i)$$

$$s_2(x,\, y_1) \;=\; f(x_0,\, y_1) \;+\; \frac{f(x_1,\, y_1) \;-\; f(x_0, y_1)}{x_1 - x_0}(x \;-\; x_0)\,.\,.\,.\,.\,.\,.\,.\;(ii)$$

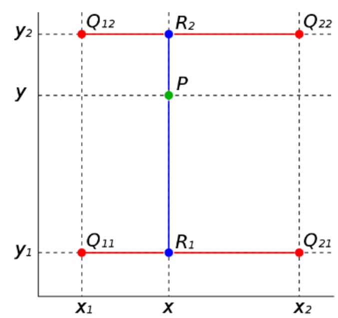
We proceed by linear interpolating in the *y*-direction to obtain the desired estimate:

$$s_5(x,\, y) \;=\; s_1(x,\, y_0) \;+\; \frac{s_2(x,\, y_1) \;-\; s_1(x, y_0)}{y_1 - y_0}(y \;-\; y_0)$$

where value of s1 and s2 are given equation (i) and (ii) respectively.

Chapter 4 Image Processing Using Spline Interpolation

Image interpolation is performed using bilinear spline interpolation because it involves a 2D matrix. First, we calculate the interpolated value in the horizontal direction using the spline interpolation formula. Then, we proceed to vertical interpolation using the linear spline interpolation formula, which can also be done in reverse. To find R2 and R1, we utilize equations (i) and (ii) from the bilinear interpolation. Finally, we determine the value of p using equation (iii) in the vertical direction.



For example, in matrix form, the original image (considered as a 3x3 matrix) is resized to a 7x7 matrix using the Bilinear interpolation method.

Original image:

$$\begin{bmatrix} 0 & 255 & 0 \\ 255 & 0 & 255 \\ 0 & 255 & 0 \end{bmatrix}$$

Resized image:

$$\begin{bmatrix} 0 & 85 & 170 & 255 & 170 & 85 & 0 \\ 85 & 113 & 141 & 170 & 141 & 113 & 85 \\ 170 & 141 & 113 & 85 & 113 & 141 & 170 \\ 255 & 170 & 85 & 0 & 85 & 170 & 255 \\ 170 & 141 & 113 & 85 & 113 & 141 & 170 \\ 850 & 113 & 141 & 170 & 141 & 113 & 85 \\ 0 & 85 & 170 & 255 & 170 & 85 & 0 \end{bmatrix}$$

The interpolated values in the horizontal and vertical axes are found using the linear spline interpolation formula with the help of the first two neighboring values. These interpolated values are then used to estimate the remaining values in the matrix, resulting in a 7x7 matrix.

$$\boxed{s_i(x) = f(x_i) + \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i}(x - x_i)}$$

Chapter 5 Application of Bilinear Spline Interpolation

The major application of linear/ bilinear spline interpolation are:

- In computer vision and image processing, bilinear interpolation is used to resample images and textures.
- Linear or bilinear interpolation can be used to estimate missing values in a signal, helping in the reconstruction of the original signal.
- Linear interpolation is used for simple curve fitting, and bilinear interpolation may be used for surfaces in multivariate data.
- Bilinear interpolation is commonly used when resizing images to estimate pixel values between existing pixels.

Chapter 6 Code

Please refer to the attached files for the code.

Major library used:

Matplotlib.pyplot: is a state-based interface to matplotlib. It provides an implicit, MATLAB-like, way of plotting. It also opens figures on your screen, and acts as the figure GUI manager.

NumPy: is a library for the Python programming language, adding support for large, multi-dimensional arrays and matrices, along with a large collection of high-level mathematical functions to operate on these arrays.

Scipy.interpolate: The scipy.interpolate module in SciPy is used for interpolation and smoothing of data in 1, 2 and higher dimensions. The choice of a specific interpolation routine depends on the data: whether it is one-dimensional, is given on a structured grid or is unstructured.

Outputs:

1. Function for 1D linear interpolation

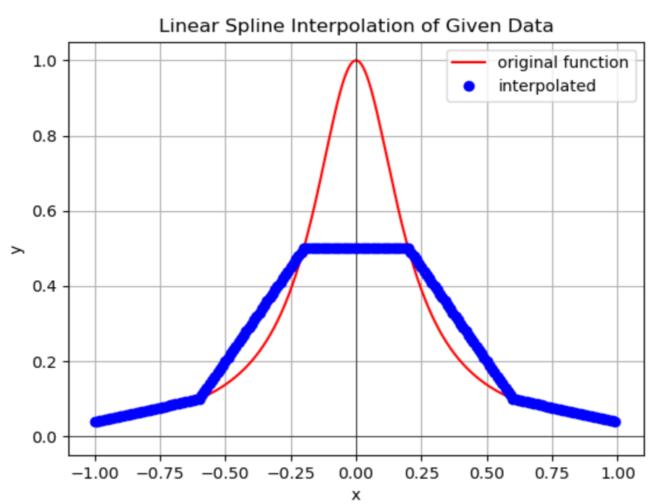


Figure 4 1D interpolation

2. Difference between Spline and Polynomial Interpolation

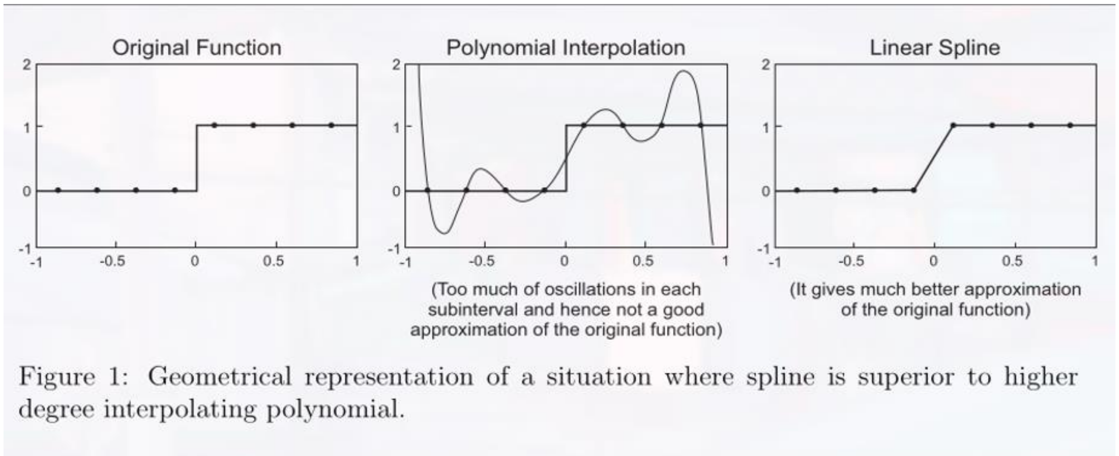


Figure 5 Polynomial vs Spline interpolation 1

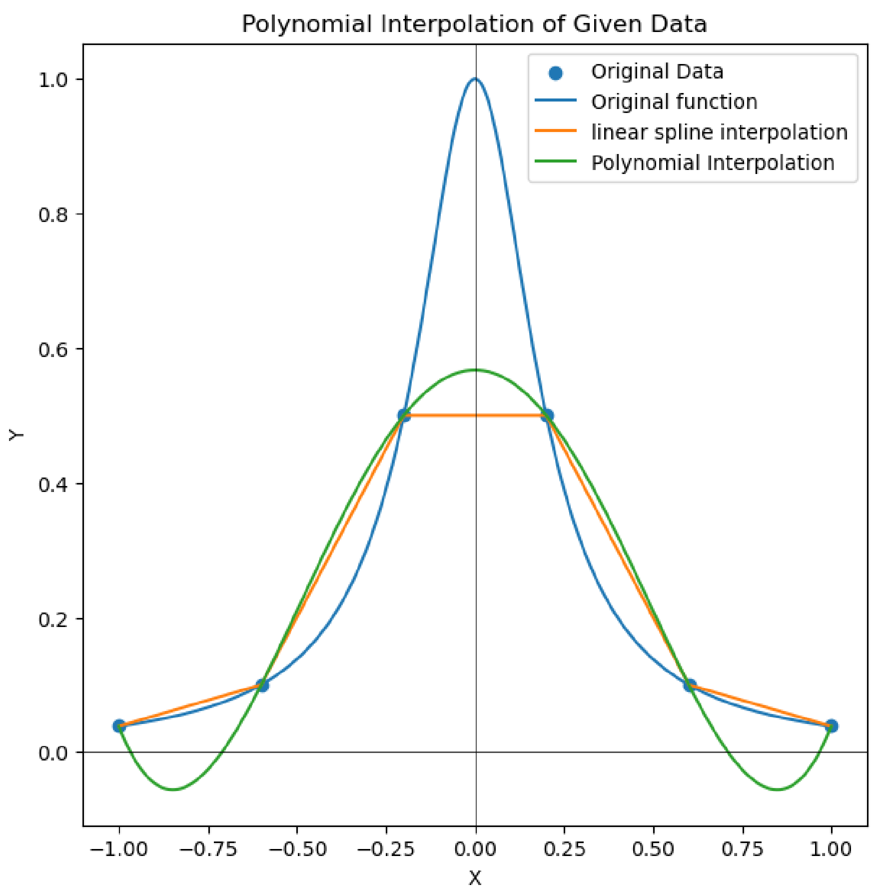


Figure 6 Polynomial vs Spline interpolation 2

3. Result Obtained after Image Processing

a. Original image:

Figure 7 Image to be scaled

b. Scaled up image:

Figure 8 Scaled up image

Chapter 7 Conclusion

We can observe that, spline interpolation has far better accuracy than ordinary polynomial interpolation. Also, the accuracy of the spline can be improved by increasing the order of the spline in general but will be best when the order of spline matches the degree of the original function. Any higher order than given function will result in Runge phenomenon and lower will compromise the accuracy, so it all about finding the most accurate one. Also, it is the matter of efficiency among the techniques, as higher order will have more accuracy but higher resource consumption where as lower order will have low resource consumption but lower accuracy in general.

So, the most important thing as a student of numerical method is to find the best technique to solve any problem under given constrains.

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