Lecture 6 : Vector space & subspace

Consider $\mathbb{R}^3 = \{(x_1, x_2, x_3) ; x_1, x_2, x_3 \in \mathbb{R}\}$

two vectors (x_1, x_2, x_3) and (y_1, y_2, y_3) in $\mathbb{R}^3$ can be added in the following way :

$$(x_1, x_2, x_3) + (y_1, y_2, y_3) = (x_1 + y_1, x_2 + y_2, x_3 + y_3)$$

For $\lambda \in \mathbb{R}$, the scalar multiplication is

$$\lambda (x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3)$$

This scalar multiplication satisfies the following distributive law :

$$\lambda \{(x_1, x_2, x_3) + (y_1, y_2, y_3)\} = \lambda (x_1, x_2, x_3) + \lambda (y_1, y_2, y_3)$$

& $(\lambda + \mu) (x_1, x_2, x_3) = \lambda (x_1, x_2, x_3) + \mu (x_1, x_2, x_3)$.

We will generalize these algebraic structures on $\mathbb{R}^3$ to define the notion of 'vector space'. Each element of $\mathbb{R}^3$ can be written as linear combination of 3 vectors $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$. These will be extended to basis vectors and dimension of vector space.

Vector space : Let V be a non-empty set & K be set of real (or complex) numbers. Suppose V is equipped with two operations as follows:

(i) Addition: $+ : V \times V \to V$
$(x, y) \mapsto x + y$

(ii) Scalar Multiplication $\cdot : K \times V \to V$
$(\lambda, x) \mapsto \lambda \cdot x$

V with these two operations is said to vector space if it satisfies the following (I & II) properties:

I. $(V, +)$ is an abelian group i.e.

(a) $(x+y)+z = x+(y+z) \ \forall x, y, z \in V$

(b) there exists an element $0 \in V$ (called identity or zero element) in V such that
$x + 0 = 0 + x = x \ \forall \ x \in V$

(c) $\forall \ x \in V$ there exists $-x \in V$ such that $x + (-x) = (-x) + x = 0$.

(d) Abelian: $x + y = y + x \ \forall x, y \in V$

(first three (a), (b), (c) are called group properties)

II. Scalar multiplication satisfies following properties:

(a) $\lambda \cdot (x+y) = \lambda \cdot x + \lambda \cdot y$

(b) $(\lambda + \mu) \cdot x = \lambda \cdot x + \mu \cdot x$

(d) $\lambda \cdot (\mu \cdot x) = (\lambda \mu) \cdot x$

(e) $1 \cdot x = x$

$\forall \; x, y \in V \; \& \; \forall \; \lambda, \mu \in K.$

The elements of V are called vectors and elements of K are called scalars

V is called real or complex vector space accordingly K is $\mathbb{R}$ or $\mathbb{C}$.
($\mathbb{C}$ is set of all complex numbers)

Example: (i) $\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_i \in \mathbb{R}, \; i = 1, \ldots, n\}$

is real vector space with addition and scalar multiplication defined componentwise as:

$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$

$\& \; \lambda \cdot (x_1, x_2, \ldots, x_n) = (\lambda x_1, \lambda x_2, \ldots, \lambda x_n)$

similarly, $\mathbb{C}^n = \{(z_1, z_2, \dots, z_n) : z_i \in \mathbb{C}, 1 \le i \le n\}$ is a complex vector space (here $K = \mathbb{C}$)

(ii) Let $K = \mathbb{R}$ or $\mathbb{C}$ & suppose $M_n(K)$ be the set of all $n \times n$ matrices with entries from K.

Let $A = (a_{ij})$ & $B = (b_{ij})$ be two matrices in $M_n(K)$

$$A + B := (a_{ij} + b_{ij})$$

& for $\lambda \in K$

$$\lambda \cdot A := (\lambda a_{ij})$$

$M_n(K)$ with these operations becomes a vector space with zero element being the zero matrix.

(iii) Let $P(x)$ be the set of all one variable polynomials with coeffecients from K ($= \mathbb{R}$ or $\mathbb{C}$) i.e

$$P(x) = \{a_0 + a_1 x + \dots + a_n x^n : a_0, \dots, a_n \in K, \; n \in \mathbb{N} \cup \{0\}\}$$

For the polynomial $a_0 + a_1 x + \dots + a_n x^n$

if $a_n \neq 0$, n is called degree of polynomial.
Let $f(x)$, $g(x)$ be polynomials of degree m, n respectively.
Let $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ &
$g(x) = b_0 + b_1 x + \cdots + b_n x^n$
& suppose $m \leq n$

$$f(x) + g(x) := (a_0 + b_0) + (a_1 + b_1)x + \cdots + (a_m + b_m)x^m + b_{m+1} x^{m+1} + \cdots + b_n x^n.$$

For $\lambda \in K$

$$\lambda \cdot f(x) := \lambda a_0 + (\lambda a_1)x + \cdots + (\lambda a_m)x^m$$

With these operations $P(X)$ becomes a real or complex vector space accordingly $K = \mathbb{R}$ or $\mathbb{C}$.

(iv) The set of all solutions of the system of linear equations $Ax = 0$ is a vector space. If x & y are two solutions then
$A(x+y) = Ax + Ay = 0 \Rightarrow x + y$ is a solution.

Let $\lambda \in \mathbb{R}$ and x be a solution of $AX = 0$

$A(\lambda x) = \lambda A x = 0$

$\Rightarrow \lambda x$ is also a solution of $AX = 0$.

☆ There exists a unique zero element in a vector space: If 0 & $0'$ are two zero elements then $0 = 0 + 0' = 0'$.

☆ $\alpha \cdot 0 = 0 \quad \forall \alpha \in \mathbb{R}$:

$\alpha \cdot 0 = \alpha \cdot (0 + 0) = \alpha \cdot 0 + \alpha \cdot 0 \Rightarrow \alpha \cdot 0 = 0$

☆ $0 \cdot x = 0 \quad \forall x \in V$

☆ $(-1) \cdot x = -x \quad \forall x \in V$:

$0 = 0 \cdot x = (1-1)x = x + (-1) \cdot x$

$\Rightarrow (-1) \cdot x = -x$.

Vector Subspace:

Consider the vector space $\mathbb{R}^3$.

Let $(a, b, c) \neq (0, 0, 0)$.

Let $W = \{(x, y, z) \in \mathbb{R}^3 : ax + by + cz = 0\}$

Then W represents the plane passing through origin and orthogonal to the vector (a, b, c).

Observe that $\alpha(x_1, x_2, x_3) + \beta(y_1, y_2, y_3) \in W$ for all $(x_1, x_2, x_3), (y_1, y_2, y_3) \in W$ and for all scalars $\alpha, \beta \in \mathbb{R}$.

This property will help us to define the notion of subspace of a vector space. Note that if W is a plane not passing through origin then the above property is not satisfied.

Subspace: Let V be a real or complex vector space. A non-empty subset W of V is called a subspace of V if

$$\alpha \cdot x + \beta \cdot y \in W \quad \text{for all } x, y \in W$$

& for all scalars α, β.

☆ Let $x \in W$ then $0 = 0 \cdot x + 0 \cdot x \in W$.

☆ Note that if W is a subspace of V then W with '$+$' & '$\cdot$' operations of V is itself a vector space.

Example: (i) Any plane passing through origin is a subspace of $\mathbb{R}^3$.

In general, a hyperplane given by the equation

$$a_1 x_1 + a_2 x_2 + \dots + a_n x_n = 0$$

is a subspace of $\mathbb{R}^n$.

(ii) For any vector space, $\{0\}$ is a subspace.

(iii) Let $W = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y \in \mathbb{R} \right\}$ then

W is a subspace of $M_2(\mathbb{R})$.

Proposition: Let W_1, W_2 be subspaces of a vector space V. $W_1 \cup W_2$ is a subspace of V if and only if $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$

Proof: Suppose $W_1 \cup W_2$ is a subspace.

Let $W_1 \nsubseteq W_2$ then $\exists\ x \in W_1$ such that $x \notin W_2$.

Claim: $W_2 \subseteq W_1$

Let $y \in W_2$, then $x + y \in W_1 \cup W_2$

if $x + y \in W_2 \Rightarrow x = (x+y) - y \in W_2$, contradiction

Therefore, $x + y \in W_1$.

Hence, $y = (x+y) - x \in W_1$

So, $W_2 \subseteq W_1$.

Converse is obvious.

Non-example: The previous proposition shows that union of any two distinct straight lines passing through origin is not a subspace of $\mathbb{R}^3$.

Proposition: Intersection of any two subspaces of a vector space is a subspace.

Proof: Exercise.