Name - Swati kesarwani

Class - B.Sc. III (pm) ; sem-6

Sub- Maths

Paper - Discrete mathematics

Graph Theory

A graph is denoted by G consisting two non-Empty Set.
V and E i.e G = (V, E)
where V Represent vertex (Node) and E Represented edge (side) vertex is represented as • (Vertex, Node) and Edge is represented as ——— Edge

There are many types of graph -

Simple graph -

1). No direction
2). No Loop
3). No Multiple Edge between two vertex.
4) It is undirected graph

(simple graph Undirected graph)

Multigraph

1). No direction
2). No loop
3). Multiple Edges Consist between Vertex
i.e. Multiple edges consist between vertex

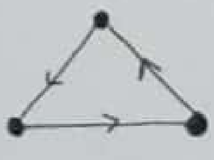
(Multigraph)

Pseudo graph

1). No direction
2). Loop consists
3). Multiple edges possible

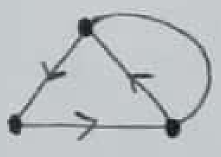
Directed graph

1). Direction contain
2). No Multiple Edge
3). No Loop contain



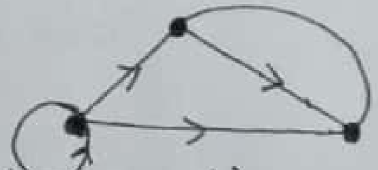
Directed multigraph -

(1) More than one edges between two Nodes
(2) Direction are available



Mixed graph -

1). Direction may be Possible
2) Multiple edges (May be or not)



Ques. In which Multiple Loop are allowed

① simple graph
② Directed graph
③ Pseudograph (✓)
④ N.O.T

Ques. In a directed graph

① graph is fixed
② graph is not fixed
③ Direction are fixed (✓)
④ N.O.T.

Ques. In which of the following graph Loops are not allowed

① Pseudograph
② Simple graph (✓)
③ Mixed graph
④ Directed Multigraph

Ques. find the total number of distinct simple graphs with up to three vertices is

ⓐ 9
ⓑ 7 ✓
ⓒ 15
ⓓ 16

Soln -

Null graph -

(No edges between Nodes)

Trival grap -

only one Node

Connected graph

If same graph are connected, is called connected graph.

Regular graph (complete graph)

A simple graph G is said to be complete. If every vertex in G has connected with every other vertex.

No. of vertex = 3 = n

No. of edge = $\frac{n(n-1)}{2}$

$= \frac{3\times2}{2}$

$= 3$

Note – A complete graph (regular graph) has n vertex. Then No. of edges are $\frac{n(n-1)}{2}$

Node = 3

No. of edge = $\frac{n(n-1)}{2}$

$= \frac{3\times(3-1)}{2}$

$= 3$

Node = 4

No. of edge = $\frac{n(n-1)}{2} = \frac{4\times(4-1)}{2}$

$= 2\times3 = 6$

(complete graph)

$n = 5$

No. of edge = $\frac{n(n-1)}{2}$

$= \frac{5\times4}{2}$

$= 10$

Ques. Let G be a complete graph of n vertices Then Total No. of edges are

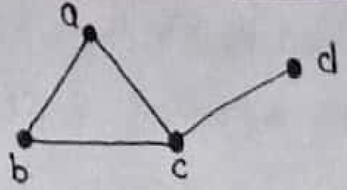
(a) n (b) $\frac{n(n-1)}{2}$ (✓) (c) n^2-1 (d) $2n$

Degree of vertex (For undirected graph)

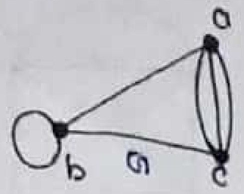
The degree of a vertex V in a graph G is written as d(V) is equal to No. of edges which are incident on V with self looped counted twice.

* loop is formed at any point (vertex) then degree of loop is 2.

deg(a) = 2
deg(b) = 2
deg(c) = 3
deg(d) = 1



In G
deg(a) = 4
deg(b) = 4
deg(c) = 4



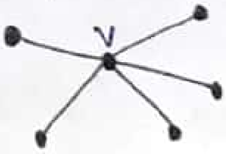
Isolated Vertex- A vertex in a graph G having No edge incident onto is an isolated Vertex. It's degree is 0.

Ques. The No. of Vertices in odd degree of a graph is
ⓐ always even ⓑ always odd ⓒ even as well as odd ⓓ N.O.T

deg = 3

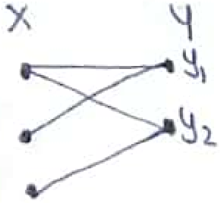
deg V = 3
No. of Nodes (Vertex) = 4 (even)

No. of vertices even



degree V = 5

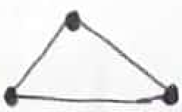
BIPARTITE Graph-



It arrange the set and the edge between Node same set is not allowed and the edge between Node of different set can be allowed.

Plannar graph- In Plannar graph edges does not intersect.

Plannar graph



Plannar graph

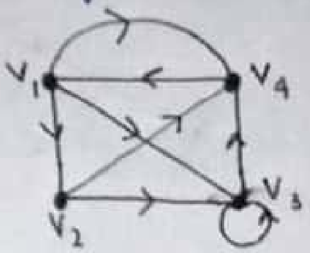
These are Plannar graph because edges does not intersect in these graph.

degree for directed graph - In directed graph there are two types of degree

1). Indegree - No. of edges incident into vertex(v_i) $\{d^-(v_i)\}$

2). Outdegree - No of edges incident out of a vertex (v_i) $\{d^+(v_i)\}$

Ques. Find indegree and outdegree of the following graph.



Indegree - $deg^-(v_1) = 1$
$deg^-(v_2) = 1$
$deg^-(v_3) = 3$
$deg^-(v_4) = 3$

Outdegree - $deg^+(v_1) = 3$
$deg^+(v_2) = 2$
$deg^+(v_3) = 1 + 1$ (due to loop)
$= 2$
$deg^+(v_4) = 1$

Note - degree of loop in directed graph is divided in one is indegree and other one in outdegree

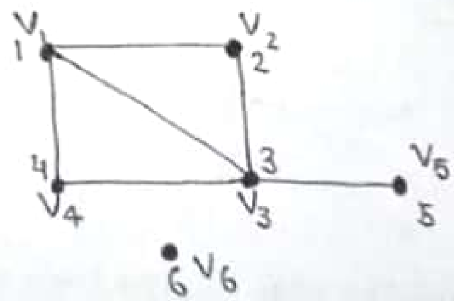
Hand shaking ~~graph~~ Theorem/ sum of degree Theorem -

Let $G = (V, E)$ be a undirected graph with e edges.

Then $\sum_{i=1}^{n} deg(v_i) = 2e$

$[\ d(v_1) + d(v_2) + d(v_3) + \cdots + d(v_n) = 2e\ ]$

Example - $deg(v_1) = 3$
$deg(v_2) = 2$
$deg(v_3) = 4$
$deg(v_4) = 2$
$deg(v_5) = 1$
$deg(v_6) = 0$



$d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + d(v_6) = 12$

$\sum_{i=1}^{6} d(v_i) = 12$

$e = 6$

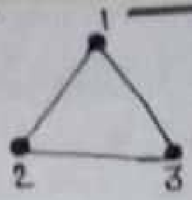
$2e = 12 \Rightarrow \left[\sum_{i=1}^{6} d(v_i) = 2e \right]$

Example -

$\deg(1) = 2$

$\deg(2) = 2$

$\deg(3) = 2$



$\deg(1) + \deg(2) + \deg(3) = 6$

$\sum_{i=1}^{3} d(v) = 6$

$e = 3$

$2e = 6$

$$\left[\sum_{i=1}^{3} d(v_i) = 2e \right]$$

Ques. An isolated vertex of a graph has

① one degree ② zero degree (✓) ③ two degree ④ NOT

Ques. find the no. of edges in a graph with 10 vertices each of degree 4 is

ⓐ 10 ⓑ 20 (✓) ⓒ 40 ⓓ 80

$\sum^{10} d(v_i) = 2e$

$d(v_1) + d(v_2) + \text{----} d(v_{10}) = 2e$

$4 \times 10 = 2e$

$e = 20$

Euler's formula Let G be a connected Planar simple graph with edge e and vertices V. Let r be the No. of Region (bounded and unbounded) of G, then

$[r = e - v + 2]$

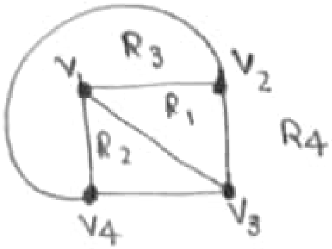
or $[v - e + r = 2]$

Example -

Here $V = 4$

$e = 6$

$r = 4$



$v - e + r = 4 - 6 + 4$

$v - e + r = 2$

$r = e - v + 2$

$= 6 - 4 + 2$

$= 8 - 4$

$r = 4$

Ques. If There are 20 vertex each of degree 3. Then How many Region of a Plannar graph

Soln -

$r = e - v + 2$, $V = 20$,

$2e = 20 \times 3 \Rightarrow e = 30$

$r = 30-20+2 = 32-20$

$r = 12$

Ques. In theory of graph, The Euler formula is

ⓐ $r+e-v=2$ ⓑ $v-e+r=3$ ⓒ $e-r+v=2$ ⓓ $v-e+r=2$ (✓)

Graph Representation

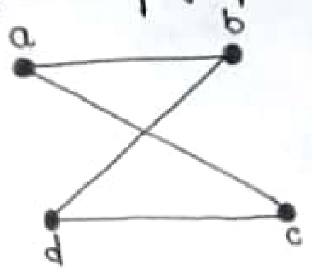
1). <u>The Adjacency Matrix (for Undirected graph)</u>. Let G is a graph. Then Adjacency matrix is Represented as

$$A = [a_{ij}] = \begin{cases} 1 & \text{if } (i,j) \text{ is an edge of } G \\ 0 & \text{if There is no edge b/w Them} \end{cases}$$

Ques. Find the adjacency Matrix of following graphs

(i)

Vertex	Adjacency
a	b, c
b	d, a
c	d, a
d	b, c

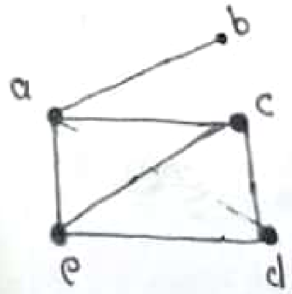


Matrix =

$$\begin{array}{c|c} & \begin{matrix} a & b & c & d \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

(ii)

Vertex	Adjacency
a	b, c, e
b	a
c	d, e, a
d	e, c
e	c, d, a



matrix =

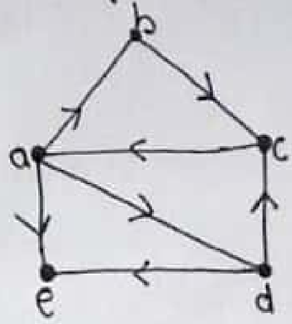
$$\begin{array}{c|c} & \begin{matrix} a & b & c & d & e \end{matrix} \\ \begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Adjacency Matrix of directed graph The adjacency Matrix of a directed graph is represented as

$$A = [a_{ij}] = \begin{cases} 1, & \text{if an edge directed from } i^{th} \text{ vertex to } j^{th} \text{ vertex} \\ 0, & \text{otherwise} \end{cases}$$

Ques. Find the Adjacency matrix of following graph

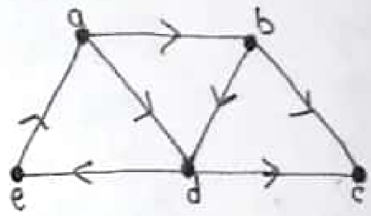
Vertex	Adjacency
a	b, d, e
b	c
c	a
d	e, c
e	—



$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Ques. Find the adjacency matrix of following graph

Vertex	Adjacency
a	b, d
b	c, d
c	—
d	e, c
e	a



$$A = [a_{ij}] \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Incidence Matrix of undirected graph - Let $G = (V, E)$ be an undirected graph. Let $1, 2, 3 \ldots n$ are n vertices and $e_1, e_2, e_3 \ldots e_m$ are the edges of G.

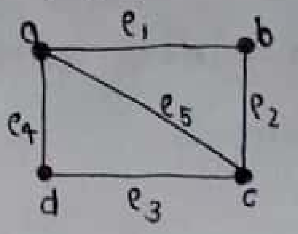
Then incidence Matrix

$B = \{b_{ij}\}$ is $n \times m$ matrix

$B = [b_{ij}] = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } i \\ 0 & \text{otherwise i.e. } e_j \text{ is not incident with vertex } i \end{cases}$

Ques. find the incidence Matrix of following graph

Vertex	Adjacency
a	e_1, e_4, e_5
b	e_2, e_1
c	e_2, e_3, e_5
d	e_4, e_3

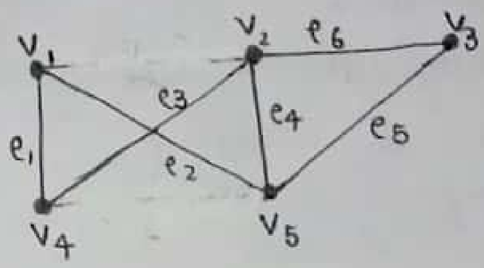


incidence matrix

$$\begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{ccccc} e_1 & e_2 & e_3 & e_4 & e_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Ques. find the incidence Matrix.

Vertex	Adjacency
V_1	e_1, e_2
V_2	e_3, e_4, e_6
V_3	e_5, e_6
V_4	e_1, e_3
V_5	e_2, e_5, e_4

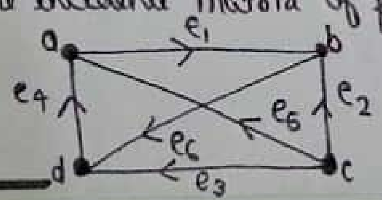


$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

incidence matrix of directed graph – The incidence matrix of directed graph is $B = \{b_{ij}\}$.

where $b_{ij} = \begin{cases} 1 & \text{if edge } j \text{ is directed away from vertex} \\ -1 & \text{" " " toward " "} \\ 0 & \text{otherwise} \end{cases}$

Ques. find the incidence matrix of following graph

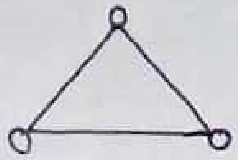


Vertex	adjacency
a	e_1, e_5, e_4
b	e_2, e_1, e_6
c	e_5, e_2, e_3
d	e_4, e_3, e_6

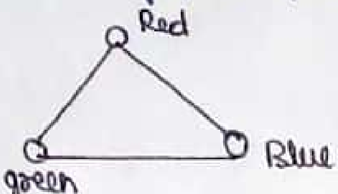
matrix

$$
\begin{array}{c|cccccc}
 & e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \\
\hline
a & 1 & 0 & 0 & -1 & -1 & 0 \\
b & -1 & -1 & 0 & 0 & 0 & 1 \\
c & 0 & 1 & 1 & 0 & 1 & 0 \\
d & 0 & 0 & -1 & 1 & 0 & -1
\end{array}
$$

Graph coloring



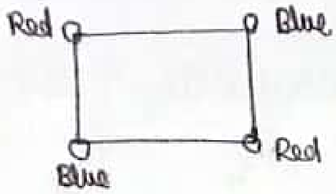
coloring of graph constitute vertex (Node) of graph. In coloring of graph it is necessary the two adjacent vertex are not in the same color. The least no. of color needed for coloring the graph is called chromatic Number. It is also known as K-chromatic graph, Represented as $\chi(G)$ or $X(G)$
chai



ques find the chromatic no. of the following graphs

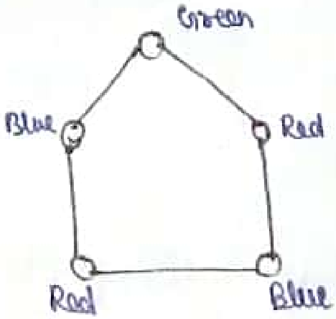
(i) The Minimum color is 2

$[\ \chi(G) = 2\ ]$



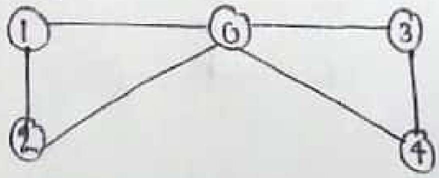
(ii) Minimum color is 3

$[\ X(G) = 3\ ]$



Trail – A walk in which no edge is Repeated (only vertex can be repeated).

Eular circuit – A Trail which start and ends at same vertex. The circuit is called Eular circuit.
A graph which has Eular circuit is called Eular graph.



$2 \to 1 \to 0 \to 3 \to 4 \to 0 \to 2$

This is a Eular circuit

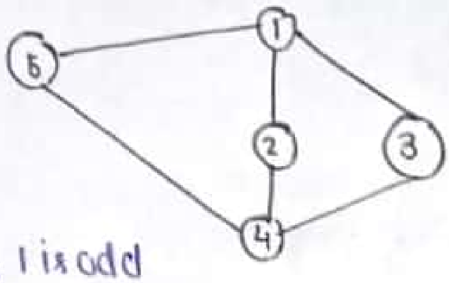
* For Eular graph Every Vertex (Node) has even degree.

deg (1) = 2
deg (0) = 4
deg (2) = 2
deg (3) = 2
deg (4) = 2

Each node has Even degree. So graph is Eular graph.

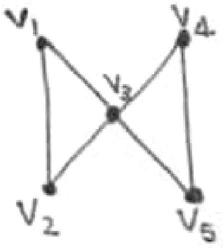
Note – Any Node which has 0 degree is Eular circuit.

Ques.



deg (5) = 2
deg (1) = 3
Since degree of 1 is odd
so it is not a

Ques. Identify the graph for Eularian circuit and Examine the graph is Eular graph or not.



Eular circuit
$V_1 \to V_3 \to V_4 \to V_5 \to V_3 \to V_2 \to V_1$

deg (V_1) = 2   deg (V_5) = 2
deg (V_2) = 2   deg (V4) = 2
deg (V_3) = 4

deg. of all vertex are even so graph is Eular graph.

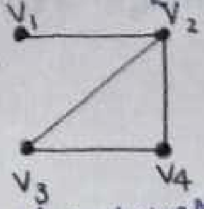
Hamiltonian graph – (Node can't be repeated) A graph G is said to be Hamiltonian graph. If it containing Hamiltonian circuit. In Hamiltonian circuit graph Passes through every vertex exactly once.

(I) In Hamiltonean graph $n \geq 3$ (No. of vertex)

(II) Now we find $\frac{n}{2}$

(III) Then graph is called Hamiltonian graph [when degree of vertex $\geq n/2$]

Ques. Examine the following
(i) graphs is Hamiltonian or not.



$n = 4 > 3$

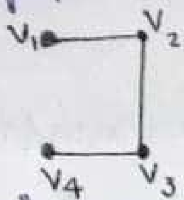
Now we find $\frac{n}{2} = \frac{4}{2} = 2$

i.e for Hamiltonian graph degree of each vertex ≥ 2.

Degree $V_1 = 1 < 2$

So it is not Hamiltonian graph.

(ii)



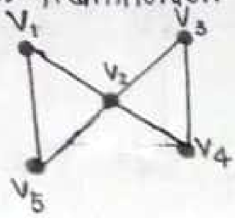
$n = 4$

$\frac{n}{2} = \frac{4}{2} = 2$

$deg(V_1) = 1 < 2$

So graph is not Hamiltonian group.

Ques. Examine the graph is Hamiltonian or not.



$n = 5$

$\frac{n}{2} = \frac{5}{2} = 2.5$

i.e 2

for Hamiltonian graph

degree of each vertex ≥ 2

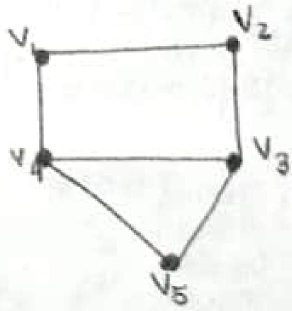
$d(V_1) = 2$ $d(V_4) = 2$

$d(V_2) = 4$ $d(V_5) = 2$

$d(V_3) = 2$

degree of each vertex is ≥ 2. So graph is Hamiltonian graph.

Ques. Examine the following graph is Hamiltonian or not.



(i) $n = 5 \geq 3$

$\frac{n}{2} = 2.5$ i.e 2

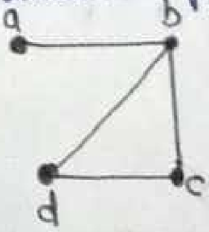
i.e. for Hamiltonian graph degree of each vertex ≥ 2

$deg(V_1) = 2$ $deg(V_2) = 3$

$deg(V_3) = 3$ $deg(V_4) = 4$

$deg(V_5) = 2$

$\Rightarrow$ degree of each vertex is ≥ 2. So graph is Hamiltonian graph.



(ii) $n = 4 \geq 3$

$\frac{n}{2} = 2$

for Hamiltonian graph degree of each vertex ≥ 2

$deg(a) = 1 < 2$

so it is not a Hamiltonian graph.

Isomorphic graph Two graph $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are said to be isomorphic graph. when it contain

(a) Same No. of vertices.
(b) Same No. of edges.
(c) Equal No. of vertices of same degree.

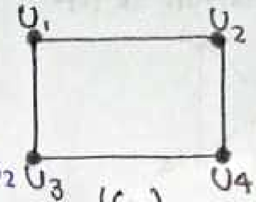
If these properties are satisfied then graph are isomorphic graph.

Ques. Examine the following graphs is isomorphic or not.

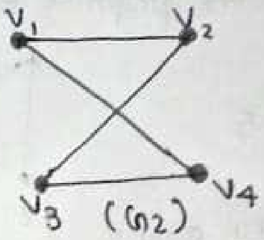
(i) No. of vertex in $G_1 = 4$

↳ No. of vertex in $G_2 = 4$

No. of vertex in G_1 = No. of vertex in G_2



(G_1)



(G_2)

↳ No. of edge in $G_1 = 4$

" " " in $G_2 = 4$

No. of edge in G_1 = No. of edge in G_2

↳ $deg\ U_1 = 2$ $\quad deg\ V_1 = 2$

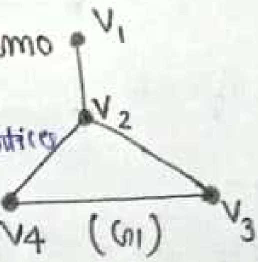
$deg\ U_2 = 2$ $\quad deg\ V_2 = 2$

$deg\ U_3 = 2$ $\quad deg\ V_3 = 2$

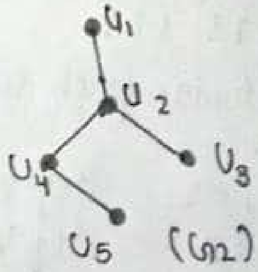
$deg\ U_4 = 2$ $\quad deg\ V_4 = 2$

degree are same in each graph of. Covers all the properties of isomorphic graph so it is called isomorphic graph.

(ii) Examine the following graph is isomorphic or not.



(G_1)



(G_2)

↳ No. of vertices in $G_1 = 5$
No. of vertices in $G_2 = 5$ } ⇒ No. of vertices in G_1 and G_2 are same

No. of edges in $G_1 = 4$

↳ No. of edges in $G_2 = 4$

No. of edges in G_1 = No. of edges in G_2

↳ $deg(V_1) = 1 = deg(U_1)$

$deg(V_2) = 2 = deg(U_2)$

$deg(V_3) \neq deg(U_3)$

So the given graph is not a Isomorphic graph.