Maximization of Network throughput

August 1, 2021

Braess's paradox is the observation that adding one or more roads to a road network can slow down overall traffic flow through it. The paradox was postulated in 1968 by German mathematician Dietrich Braess. The paradox may have analogies in electrical power grids and biological systems. It has been suggested that in theory, the improvement of a malfunctioning network could be accomplished by removing certain parts of it. The paradox has been used to explain instances of improved traffic flow when existing major roads are closed.

Dietrich Braess, a mathematician at Ruhr University, Germany, noticed the flow in a road network could be impeded by adding a new road, when he was working on traffic modelling. His idea was that if each driver is making the optimal self-interested decision as to which route is quickest, a shortcut could be chosen too often for drivers to have the shortest travel times possible.

"For each point of a road network, let there be given the number of cars starting from it and the destination of the cars. Under these conditions, one wishes to estimate the distribution of traffic flow. Whether one street is preferable to another depends not only on the quality of the road, but also on the density of the flow. If every driver takes the path that looks most favourable to them, the resultant running times need not be minimal. Furthermore, it is indicated by an example that an extension of the road network may cause a redistribution of the traffic that results in longer individual running times."

1 Mathematical approach

Consider a road network as shown in the adjacent diagram on which 4000 drivers wish to travel from point Start to End. The travel time in minutes on the Start-A road is the number of travelers (T) divided by 100, and on Start-B is a constant 45 minutes (likewise with the roads across from them). If the dashed road does not exist (so the traffic network has 4 roads in total), the time needed to drive Start-A-End route with a drivers would be $\frac{a}{100} + 45$. The time needed to drive the Start-B-End route with b drivers would be $\frac{b}{100} + 45$. As there are 4000 drivers, the fact that $a + b = 4000$ can be used to derive the fact that $a = b = 2000$ when the system is at equilibrium. Therefore, each route takes

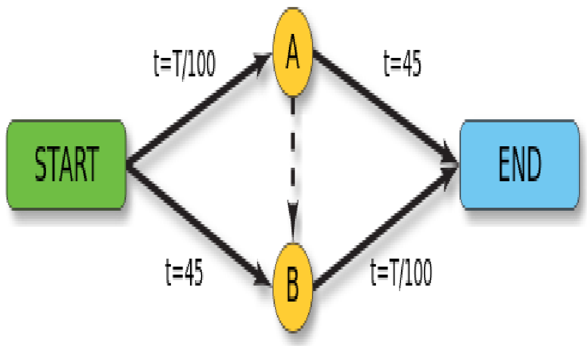

Figure 1: Braess's paradox Block Diagram

$\frac{2000}{100} + 45 = 65$ minutes. If either route took less time, it would not be a Nash equilibrium: a rational driver would switch from the longer route to the shorter route.

Now suppose the dashed line A-B is a road with an extremely short travel time of approximately 0 minutes. Suppose that the road is opened and one driver tries Start-A-B-End. To his surprise he finds that his time is $\frac{2000}{100} + \frac{2001}{100} = 40.01$ minutes, a saving of almost 25 minutes. Soon, more of the 4000 drivers are trying this new route. The time taken rises from 40.01 and keeps climbing. When the number of drivers trying the new route reaches 2500, with 1500 still in the Start-B-End route, their time will be $\frac{2500}{100} + \frac{4000}{100} = 65$ minutes, which is no improvement over the original route. Meanwhile, those 1500 drivers have been slowed to $45 + \frac{4000}{100} = 85$ minutes, a 20-minute increase. They are obliged to switch to the new route via A too, so it now takes $\frac{4000}{100} + \frac{4000}{100} = 80$ minutes. Nobody has any incentive to travel A-End or Start-B because any driver trying them will take 85 minutes. Thus, the opening of the cross route triggers an irreversible change to it by everyone, costing everyone 80 minutes instead of the original 65. If every driver were to agree not to use the A-B path, or if that route were closed, every driver would benefit by a 15-minute reduction in travel time.

Finding an equilibrium

The above proof outlines a procedure known as best response dynamics, which finds an equilibrium for a linear traffic graph and terminates in a finite number of steps. The algorithm is termed "best response" because at each step of the algorithm, if the graph is not at equilibrium then some driver has a best response to the strategies of all other drivers and switches to that response.

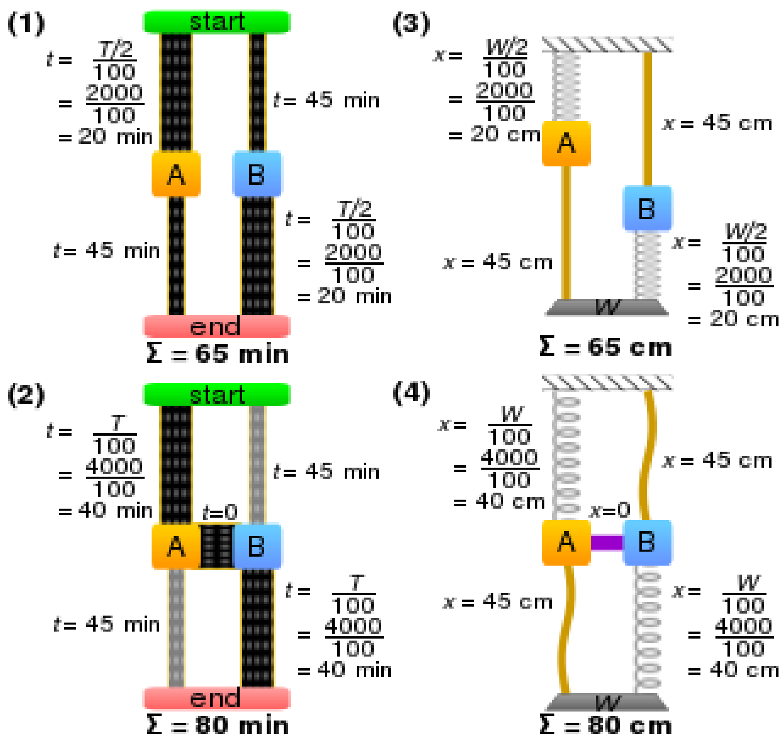

Figure 2: Braess's paradox Mathematical Interpretation

Pseudocode for Best Response Dynamics:

Let P be some traffic pattern.
while P is not at equilibrium:
{ compute the potential traffic T of P
for each node n in P:
{ for each alternate path p available to n:
{compute the potential traffic N of the pattern when n takes path p
if $N \leq T$:
{modify P so that n takes path p}}}}
continue the topmost while

At each step, if some particular node could do better by taking an alternate path (a "best response"), doing so strictly increases the efficiency of the network. If no any node has a best response, the network is at equilibrium. Since the efficiency of the network strictly increases with each step, the best response dynamics algorithm must eventually halt.