Assignment
11.9.2 - 11

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Question

Sum of the first p, q and r terms of an A.P. are a, b and c, respectively.
Prove that $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$

Solution

Symbol	Value	Description
$x(n)$	$(x_0+nd)\times u(n)$	n^{th} term of an A.P
x_0	x_0	1^{st} term of the A.P
d	d	Common difference
$u(n)$	unit step function	$u(n) = 0 \ (n<0)$ $u(n)=1 \ (n \geq 0)$
$y(n)$	$\sum_{k=0}^{n} x(k)$	Sum of n terms of an AP
a	$y(p-1)$	Sum of first p terms of the AP
b	$y(q-1)$	Sum of first q terms of the AP
c	$y(r-1)$	Sum of first r terms of the AP

TABLE 0
Variable description

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{1}$$
$$X(z)=\sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \tag{2}$$
$$X(z)=\sum_{n=-\infty}^{\infty} (x_0+nd)u(n)z^{-n} \tag{3}$$
$$u(n) \overset{\mathcal{Z}}{\longleftrightarrow} U(z)=\frac{1}{1-z^{-1}}, |z|>1 \tag{4}$$
$$X(z)=\frac{x_0}{1-z^{-1}}+\frac{dz^{-1}}{(1-z^{-1})^2} \tag{5}$$
$$y(n) \overset{\mathcal{Z}}{\longleftrightarrow} Y(z) \tag{6}$$
$$Y(z)=\sum_{n=-\infty}^{\infty} y(n)z^{-n} \tag{7}$$
$$y(n)=x(n)*u(n) \tag{8}$$
$$Y(z)=X(z)\,U(z) \tag{9}$$
$$Y(z)=\left(\frac{x_0}{1-z^{-1}}+\frac{dz^{-1}}{(1-z^{-1})^2}\right)\left(\frac{1}{1-z^{-1}}\right) \tag{10}$$

By performing Z transform on Y(z) using contour integration

$$y(n)=\frac{1}{2\pi j}\oint_C Y(z)\, z^{n-1}\, dz \tag{11}$$
$$y(n)=\frac{1}{2\pi j}\oint_C \left(\frac{x_0 z^{n-1}}{\left(1-z^{-1}\right)^2}+\frac{dz^{n-2}}{\left(1-z^{-1}\right)^3}\right) dz \tag{12}$$
$$R=\frac{1}{(m-1)!}\lim_{z\to a}\frac{d^{m-1}}{dz^{m-1}}\left((z-a)^m f(z)\right) \tag{13}$$
$$R=\sum_{i=1}^{n} R_i \tag{14}$$

For R_1 we can observe that the pole has been repeated twice.

$$R_1 = \frac{1}{(1)!}\lim_{z\to1}\frac{d}{dz}\left((z-1)^2\frac{x_0 z^{n+1}}{(z-1)^2}\right) \tag{15}$$
$$R_1 = x_0\,(n+1)\lim_{z\to1}(z^n) \tag{16}$$
$$R_1 = x_0\,(n+1) \tag{17}$$
$$\tag{18}$$

For R_2 we can observe that the pole has been repeated thrice.

$$R_2 = \frac{1}{(2)!}\lim_{z\to1}\frac{d^2}{dz^2}\left((z-1)^3\frac{dz^{n+1}}{(z-1)^3}\right) \tag{19}$$
$$R_2 = \frac{d\,(n+1)}{2}\lim_{z\to1}\frac{d}{dz}(z^n) \tag{20}$$
$$R_2 = \frac{d\,(n+1)\,(n)}{2}\lim_{z\to1}\left(z^{n-1}\right) \tag{21}$$
$$R_2 = \frac{d\,(n)\,(n+1)}{2} \tag{22}$$
$$R = x_0\,(n+1) + \frac{d\,(n)\,(n+1)}{2} \tag{23}$$

Finally,

$$y\,(n) = x\,(0)\,(n+1)\,u\,(n) + d\left(\frac{n\,(n+1)}{2}\right)u\,(n) \tag{24}$$
$$y\,(n) = \frac{n+1}{2}\,(2x\,(0) + nd)\,u\,(n) \tag{25}$$
$$a = \frac{p}{2}(2x_0 + (p-1)d) \tag{26}$$
$$b = \frac{q}{2}(2x_0 + (q-1)d) \tag{27}$$
$$c = \frac{r}{2}(2x_0 + (r-1)d) \tag{28}$$

Back substituting values into the term $\frac{a}{p}(q-r)$ it can be rewritten as $\frac{p}{2}\times\frac{1}{p}(q-r)(2x_0+(p-1)d)$ On further simplification it can be rewritten as

$$\frac{(q-r)}{2}(2x_0 - d + pd) \tag{29}$$

Assuming $2x_0 - d$ as a constant k

$$\frac{a}{p}(q-r) = \frac{(q-r)}{2}(k+pd) \tag{30}$$
$$\frac{(q-r)}{2}(k+pd) = \frac{kq+pqd-kr-prd}{2} \tag{31}$$
$$\frac{(r-p)}{2}(k+qd) = \frac{kr+qrd-kp-pqd}{2} \tag{32}$$
$$\frac{(p-q)}{2}(k+rd) = \frac{kp+prd-kq-qrd}{2} \tag{33}$$

Upon on addition of (31), (32) and (33) the total sum adds up to 0.