Filter Design

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1 Introduction

We're assigned the creation of FIR and IIR filter designs confined to a particular filter type. Our focus is on creating a bandpass filter, and we've been provided with the specifications for this filter.

2 Filter Specifications

2.1 The Digital Filter

1. Passband: The passband is from {4 + 0.6(j)}kHz to {4 + 0.6(j+2)}kHz. where

$$j = (r - 11000) \mod \sigma \tag{1}$$

where σ is sum of digits of roll number and r is roll number.

$$r = 11034 \tag{2}$$
$$\sigma = 9 \tag{3}$$
$$j = 7 \tag{4}$$

By substituting $j = 7$, we determine the passband range for our bandpass filter as 8.2 kHz to 9.4 kHz.
Therefore, the un-normalized discrete-time filter passband frequencies are $F_{p1} =$ 8.2 kHz and $F_{p2} = 9.4$ kHz.
The corresponding normalized digital filter passband frequencies are:

$$\omega_{p1} = 2\pi \frac{F_{p1}}{F_s} = 0.34\pi \tag{5}$$
$$\omega_{p2} = 2\pi \frac{F_{p2}}{F_s} = 0.39\pi \tag{6}$$

2. Tolerances: The passband (δ_1) and stopband (δ_2) tolerances are given to be equal, so we let $\delta_1 = \delta_2 = \delta = 0.15$.

3. Stopband: The transition band for bandpass filters is $\Delta F = 0.3$ kHz on either side of the passband.

$$F_{s1} = 8.2 - 0.3 = 7.9\text{KHz} \tag{7}$$
$$F_{s2} = 9.4 + 0.3 = 9.7\text{KHz} \tag{8}$$

$$\omega_{s1} = 2\pi \frac{F_{s1}}{F_s} = 0.329\pi \tag{9}$$
$$\omega_{s2} = 2\pi \frac{F_{s2}}{F_s} = 0.404\pi \tag{10}$$
$$\tag{11}$$

2.2 The Analog filter

In the bilinear transform, the analog filter frequency (Ω) is related to the corresponding digital filter frequency (ω) by the equation:

$$\Omega = \tan \frac{\omega}{2} \tag{12}$$

Using this relation, we obtain the analog passband and stopband frequencies as: $\Omega_{p1} = 0.5913$, $\Omega_{p2} = 0.702$ and $\Omega_{s1} = 0.5662$, $\Omega_{s2} = 0.7361$ respectively.

3 The IIR Filter Design

To design filters with a monotonic stopband and equiripple passband, we utilize the Chebyshev approximation method to design our bandpass IIR filter.

3.1 The Analog Filter

1. Low Pass Filter Specifications: Let $H_{a,BP}(j\Omega)$ represent the desired analog bandpass filter, incorporating the specifications outlined in Section 2.2. Additionally, let $H_{a,LP}(j\Omega_L)$ denote the equivalent low pass filter.

$$\Omega_L = \frac{\Omega^2 - \Omega_0^2}{B\Omega} \tag{13}$$

where $\Omega_0 = \sqrt{\Omega_{p1}\Omega_{p2}} = 0.644$ and $B = \Omega_{p2} - \Omega_{p1} = 0.1107$.

Substituting Ω_{s1} and Ω_{s2} in (13) we obtain the stopband edges of lowpass filter

$$\Omega_{Ls1} = \frac{\Omega_{s1}^2 - \Omega_0^2}{B\Omega_{s1}} = -1.502 \tag{14}$$
$$\Omega_{Ls2} = \frac{\Omega_{s2}^2 - \Omega_0^2}{B\Omega_{s2}} = 1.559 \tag{15}$$

And we choose the minimum of these two stopband edges

$$\Omega_{Ls} = \min(|\Omega_{Ls_1}|, |\Omega_{Ls_2}|) = 1.502. \tag{16}$$

2. The Low Pass Chebyschev Filter Paramters: The magnitude of frequency response of the low pass filter is given by

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L/\Omega_{Lp})} \tag{17}$$

The passband edge of the low pass filter is chosen as $\Omega_{Lp} = 1$. Therfore ,

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\Omega_L)} \tag{18}$$

Here c_N denote the chebyshev polynomials for a particular order N of the filter.

$$c_N(x) = \cosh(N \cosh^{-1} x), x = \Omega_L \tag{19}$$
$$c_0(x) = 1 \tag{20}$$
$$c_1(x) = x \tag{21}$$

There exists a recurssive relation from which all the polynomials can be found out.

$$c_{N+2} = 2xc_{N+1} - c_N \tag{22}$$

Imposing the band restrictions on (17)

$$|H_{a,LP}(j\Omega_L)|^2 < \delta_2 \text{ for } \Omega_L = \Omega_{Ls} \tag{23}$$
$$1 - \delta_1 < |H_{a,LP}(j\Omega_L)|^2 < 1 \text{ for } \Omega_L = \Omega_{Lp} \tag{24}$$
$$\tag{25}$$

we obtain :

$$\frac{\sqrt{D_2}}{c_N(\Omega_{Ls})} \le \epsilon \le \sqrt{D_1},$$
$$N \ge \left\lceil \frac{\cosh^{-1} \sqrt{D_2/D_1}}{\cosh^{-1} \Omega_{Ls}} \right\rceil, \tag{26}$$

where $D_1 = \frac{1}{(1-\delta)^2} - 1$ and $D_2 = \frac{1}{\delta^2} - 1$ and $\lceil . \rceil$ is known as the ceiling operator .

Parameter	Value
D_1	0.384
D_2	43.44
N	4
$c_4(x)$	$8x^4 + 8x^2 + 1$

Table 1: Parameter Table

we get $N \geq 4$ and $0.278 \leq \epsilon \leq 0.61$

The below code plots (17) for different values of ϵ .

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https://github.com/dhanushnayakh03/EE1205/tree/main/Audio_%20Filter/
    codes/plot1.py
```

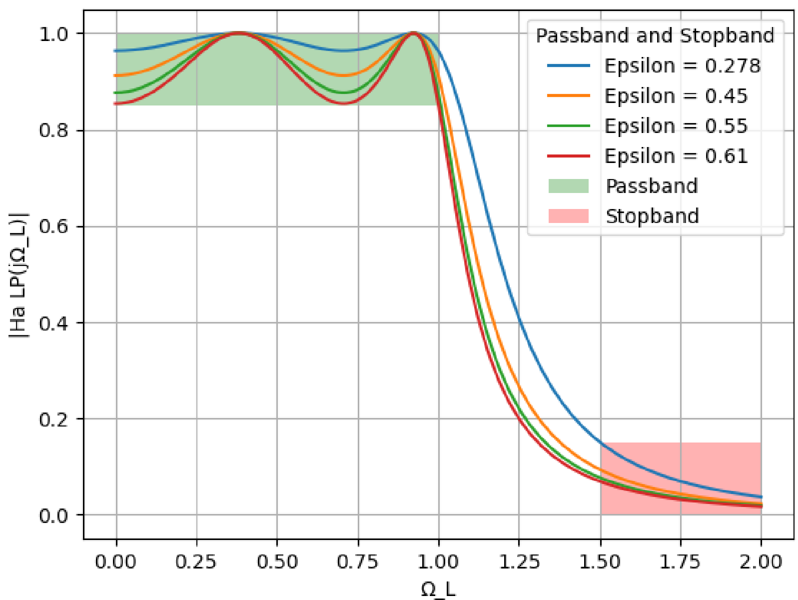

Figure 1: The Analog Low-Pass Frequency Response for $0.278 \le \epsilon \le 0.61$

In Figure 1, we can observe equiripple behavior in the passband and monotonic behavior in the stopband.
As the value of ϵ increases, the magnitude of $|H_{a,LP}(j\Omega_L)|$ decreases.

3. The Low Pass Chebyschev Filter: The next step in the design process is to derive an expression for the magnitude response in the s-domain for the low pass Chebyshev filter

   Using $s = j\Omega$ or in this case $s_L = j\Omega_L$ we obtain:

$$|H_{a,LP}(j\Omega_L)|^2 = \frac{1}{1 + \epsilon^2 c_N^2(\frac{s_L}{j})} \tag{27}$$

   To find poles equate the denominator to zero:

$$1 + \epsilon^2 c_N^2 \left(\frac{s_L}{j}\right) = 0 \quad \text{where} \quad c_N(x) = cos\left(Ncos^{-1}(x)\right) \tag{28}$$

On solving (28) we obtain poles :

$$s_k = -\Omega_{Lp} \sin(A_k) \sinh(B_k) - j\Omega_{Lp} \cos(A_k) \cosh(B_k) \tag{29}$$

where k is the index of the pole and

$$A_k = (2k+1)\frac{\pi}{2N} \tag{30}$$

$$B_k = \frac{1}{N} \sinh^{-1}\left(\frac{1}{\epsilon}\right) \tag{31}$$

The below code computes the values of s_k and stores it in a text file.

https://github.com/dhanushnayakh03/EE1205/blob/main/Filter_Design/codes/sk_gen.c

The poles obtained are formulated in the table below.

Pole	*Value*
s_1	0.4604 + j0.4276
s_2	0.4604 - j0.4276
s_3	0.1907 - j1.0322
s_4	0.1907 + j1.0322
s_5	-0.1907 - j1.0322
s_6	-0.4604 - j0.4276
s_7	-0.4604 + j0.4276
s_8	-0.1907 + j1.0322

Table 2: Values of s_k

The below code plots the pole-zero plot.

https://github.com/dhanushnayakh03/EE1205/blob/main/Filter_Design/codes/plot1.py

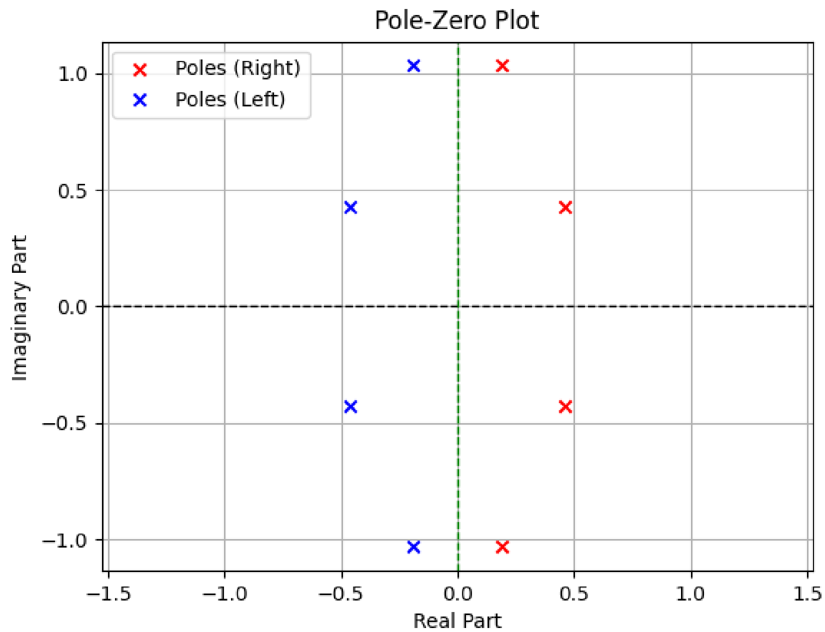

Figure 2: The Pole zero plot and all the poles lie on an ellipse. The left and right poles have been identified as shown.

The poles in the left half of the plane are considered in the design as we intend to design a stable system.

Therefore, the magnitude response is expressed as:

$$H_{a,LP}(s_L) = \frac{G_{LP}}{(s_L - s_5)(s_L - s_6)(s_L - s_7)(s_L - s_8)} \tag{32}$$

where G_{LP} is the gain of the Low pass filter. Refer to Table 2 for s_k values.

We know that from (17):-

$$\left|H_{a,LP}(s_L)\right| = \frac{1}{\sqrt{1+\epsilon^2}} \text{at } \Omega_L = 1 \implies s_L = j \tag{33}$$

Substituting respective values in (33) we get $G_{LP} = 0.4166$

$$H_{a,LP}(s_L) = \frac{0.4166}{(s_L - s_5)(s_L - s_6)(s_L - s_7)(s_L - s_8)} \tag{34}$$

$$= \frac{0.4166}{s_L^4 + 1.3022 s_L^3 + 1.84781 s_L^2 + 1.16512 s_L + 0.435003} \tag{35}$$

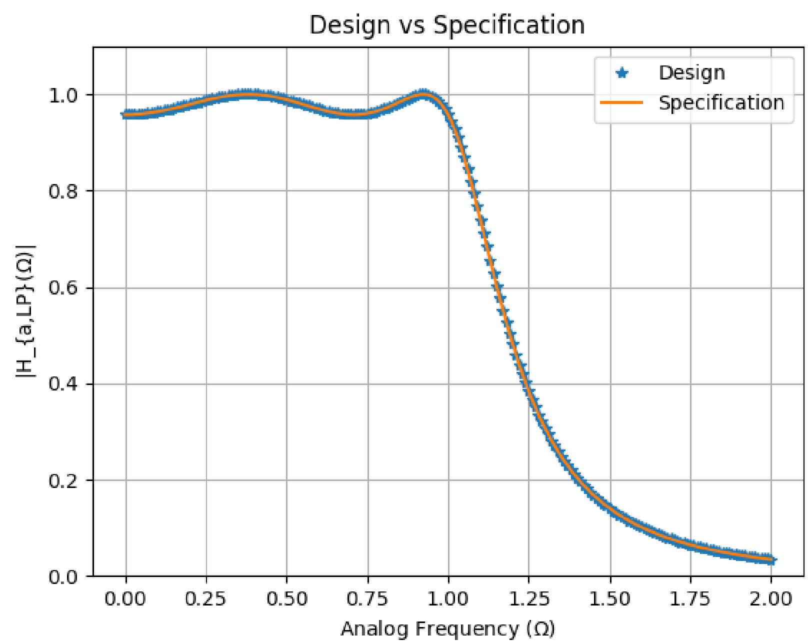

Figure 3: Design vs Specification corresponding to (35) and (18)

4. The Band Pass Chebyschev Filter:
5. The Band Pass Chebyschev Filter: After verifying the design with the required specifications, the next step is to transition to the desired type of filter using frequency transformation.

$$s_L = \frac{s^2 + \Omega_0^2}{Bs} \tag{36}$$

$$H_{a,BP}(s) = G_{BP} H_{a,LP}(s_L)\Big|_{s_L = \frac{s^2+\Omega_0^2}{Bs}}, \tag{37}$$

As there is one to one correspondence between the filters so $\Omega = \Omega_{p1}$ should correspond to Ω_{Lp}

$$s = j\Omega_{p1} \tag{38}$$

$$s_L = \frac{(j\Omega_{p1})^2 + \Omega_0^2}{B(j\Omega_{p1})} \tag{39}$$

$$\left|H_{a,BP}(j\Omega_{p1})\right| = 1 \tag{40}$$

$$G_{BP}\left|H_{a,LP}(s_L)\right| = 1 \tag{41}$$

Substituting (39) in (41) we obtain Gain of required bass pass filter:

$$G_{BP} = 1.0370 \tag{42}$$

Thus the response in s domain

$$H_{a,BP}(s) = \frac{6.49 \times 10^{-5} s^4}{s^8 + 0.144s^7 + 0.1682s^6 + 0.1810s^5 + 1.05s^4 + 0.750s^3 + 0.289s^2 + 0.0102s + 0.029} \tag{43}$$

The expressions in the s-domain and gain factors are computed using a Python code.
In Figure 3, we plot $|H_{a,BP}(j\Omega)|$ as a function of Ω for both positive and negative frequencies.
We observe that the passband and stopband frequencies in the figure closely match those obtained analytically through the bilinear transformation.

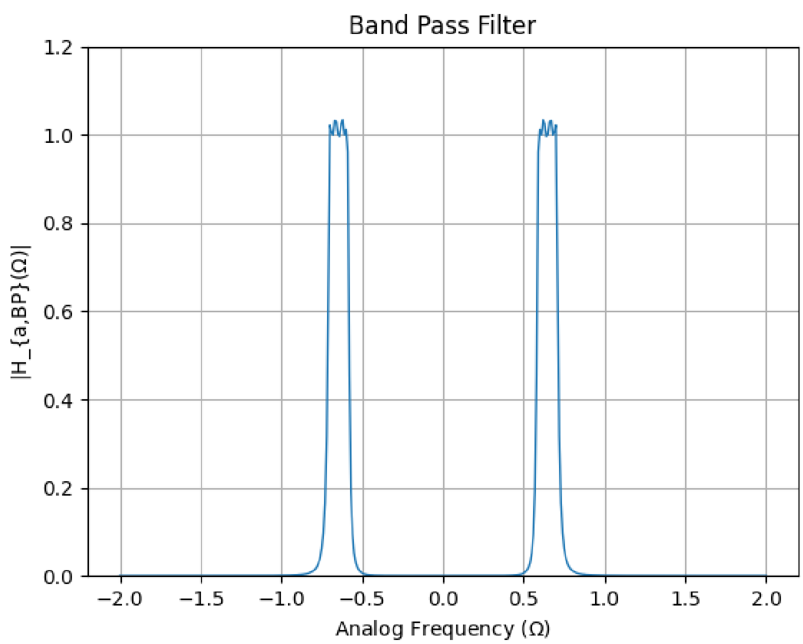

Figure 4: The Analog Bandpass Magnitude Response from (43).The filter design specifications are satisfied

3.2 The Digital Filter

From the bilinear transformation, we obtain the digital bandpass filter from the corresponding analog filter as

$$H_{d,BP}(z) = GH_{a,BP}(s)\big|_{s=\frac{1-z^{-1}}{1+z^{-1}}} \tag{44}$$

Substituting $s = \frac{1-z^{-1}}{1+z^{-1}}$ in (43) and calculating expression using a python code we get :

$$H_{d,BP}(z) = \frac{G\left(1 - 4z^{-2} + 6z^{-4} - 4z^{-6} + z^{-8}\right)}{3.6214 - 6.9496z^{-1} + 26.7376z^{-2} - 60.1464z^{-3} + 73.758z^{-4} - 49.572z^{-5} + 26.144z^{-6} - 7.62z^{-7} + 1.451z^{-8}} \tag{45}$$

where $G = 6.49 \times 10^{-5}$

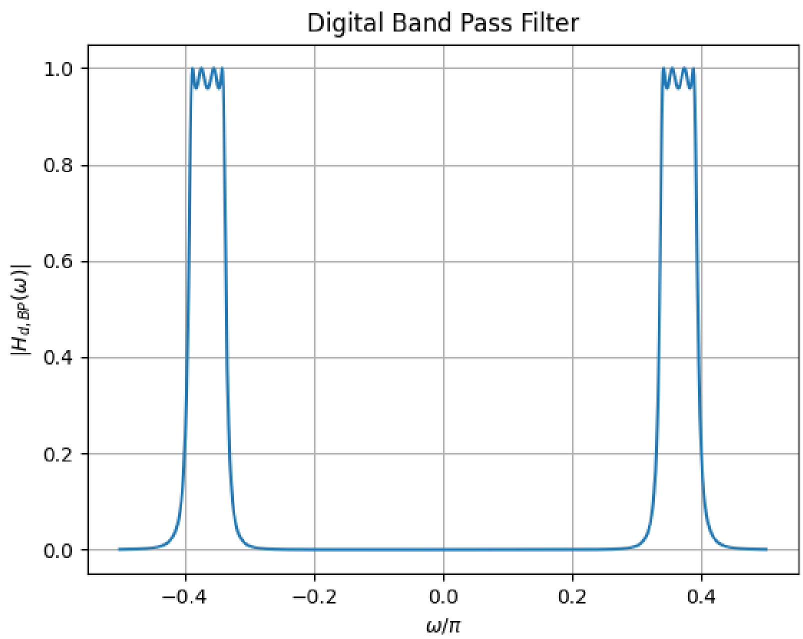

Figure 5: Digital Specifications are met. Passband and stopband frequencies are same

4 The FIR Filter

We design the FIR filter by first obtaining the (non-causal) lowpass equivalent using the Kaiser window and then converting it to a causal bandpass filter.

4.1 The Equivalent Lowpass Filter

The lowpass filter has a passband frequency ω_l and transition band $\Delta\omega = 2\pi\frac{\Delta F}{F_s} = 0.0125\pi$. The stopband tolerance is $\delta = 0.15$.The cutoff-frequency is given by :

$$\omega_l = \frac{B}{2} \tag{46}$$

$$= 0.025\pi \tag{47}$$

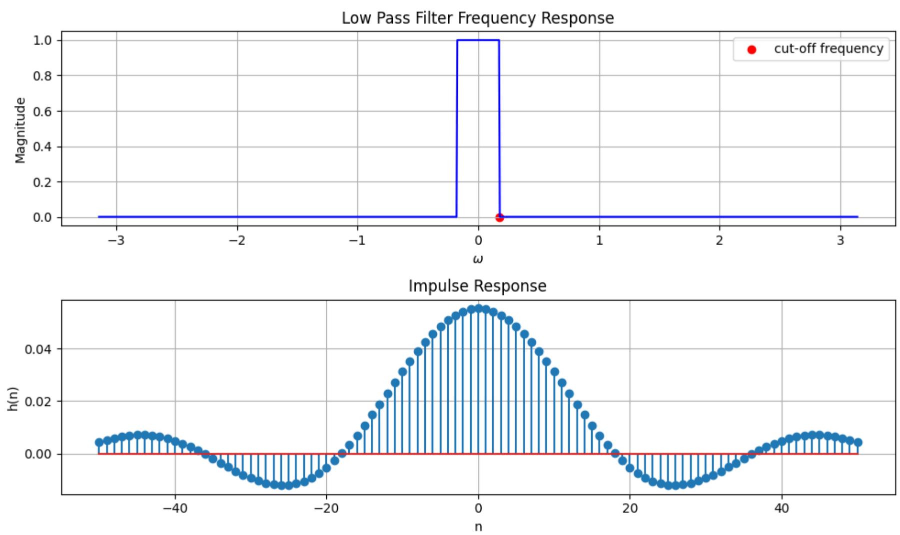

Figure 6: Frequency response and impulse response of an ideal Low Pass Filter

The impulse response of ideal Low Pass Filter is given by :

$$h(n) = \begin{cases} \frac{w_l}{\pi}, & \text{if } n = 0 \\ \frac{\sin(w_l n)}{n\pi}, & \text{if } n \neq 0 \end{cases} \tag{48}$$

From equation (48), we can infer that $h(n)$ for an ideal Low Pass Filter is not causal, and it cannot be made causal by introducing a finite delay.
Additionally, $h(n)$ does not converge, resulting in an unstable system.
To address this issue, we employ windowing on the impulse response. A window function is selected and multiplied. The Kaiser window is chosen for this purpose.

$$w(n) = \begin{cases} \frac{I_0\left[\beta N\sqrt{1-\left(\frac{n}{N}\right)^2}\right]}{I_0(\beta N)}, & -N \leq n \leq N, \quad \beta > 0 \\ 0 & \text{otherwise,} \end{cases}$$

1. N is chosen according to

$$N \geq \frac{A-8}{4.57\Delta\omega}, \tag{49}$$

   where $A = -20\log_{10}\delta$. Substituting the appropriate values from the design specifications, we obtain $A = 16.4782$ and $N \geq 48$.

2. β is chosen according to

$$\beta N = \begin{cases} 0.1102(A-8.7) & A > 50 \\ 0.5849(A-21)^{0.4} + 0.07886(A-21) & 21 \leq A \leq 50 \\ 0 & A < 21 \end{cases} \tag{50}$$

The window function is defined as :

$$w(n) = \begin{cases} 1, & \text{for } -48 \le n \le 48 \\ 0, & \text{otherwise} \end{cases} \tag{51}$$

Therefore the desired impulse response is :

$$h_{lp} = h_n w_n \tag{52}$$

$$h(n) = \begin{cases} \frac{\sin(w_l n)}{n\pi}, & \text{for } -48 \le n \le 48 \\ 0 & \text{otherwise} \end{cases} \tag{53}$$

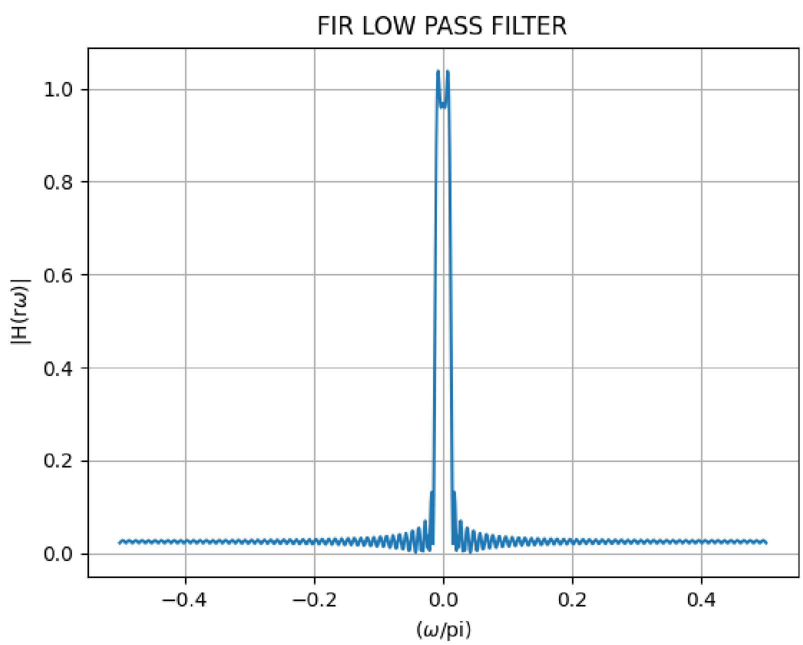

Figure 7: Magnitude Response of Low Pass Filter after using Kaiser Window

4.2 The Equivalent Band Pass Filter

A Band-Pass Filter (BPF) is derived by subtracting the magnitude response of a Low-Pass Filter (LPF) with a cutoff frequency of ω_{p1} from another LPF magnitude response with a cutoff frequency of ω_{p2}..

$$h_{BP}(n) = \begin{cases} \frac{\sin(w_{p2} n)}{n\pi} - \frac{\sin(\omega_{p1} n)}{n\pi}, & \text{for } n \neq 0 \\ \frac{\omega_{p2} - \omega_{p1}}{\pi} & \text{for} n = 0 \end{cases} \tag{54}$$

$$\frac{\sin(\omega_{p2}n)}{n\pi} - \frac{\sin\left(\omega_{p1}n\right)}{n\pi} = 2\cos\left(\frac{\omega_{p2}n + \omega_{p1}n}{2}\right)\sin\left(\frac{\omega_{p2}n - \omega_{p1}n}{2}\right) \tag{55}$$

$$= \frac{2\cos\left(0.365n\pi\right)\sin\left(0.025n\pi\right)}{n\pi} \tag{56}$$

Multipying by window function we get :

$$h_{BP}\left(n\right) = \begin{cases} \frac{2\cos\left(0.365n\pi\right)\sin\left(0.025n\pi\right)}{n\pi}, & \text{for } -48 \leq n \leq 48 \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

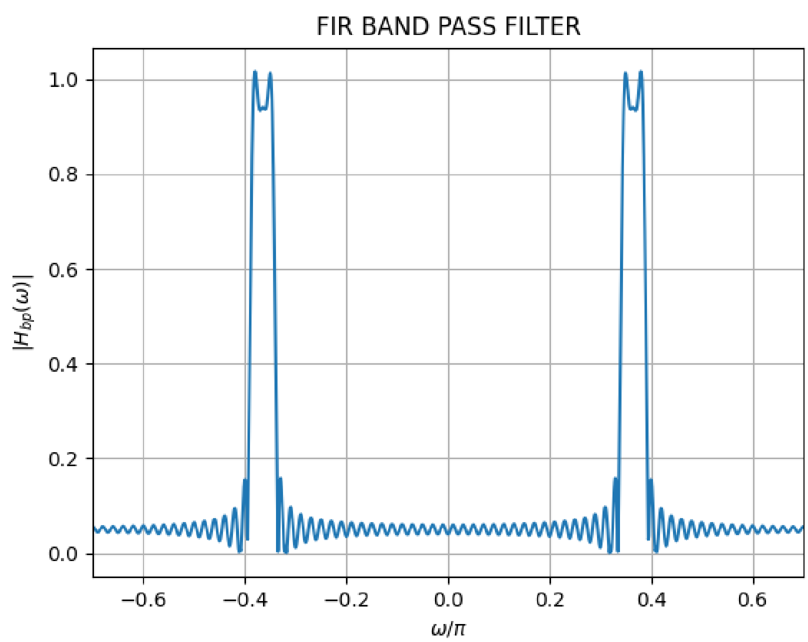

Figure 8: Magnitude Response of Band Pass Filter after using Kaiser Window