Course Name: HW

Name
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Question 1

Let $a_n = \dfrac{18n^5 + 2n - 5}{6n^5 - n^3 - 3}$. We want to show that $(\forall\epsilon > 0)(\exists K \in \mathbb{N})[n \geq K \Rightarrow |a_n - 3| < \epsilon]$.

We have:

$$\begin{aligned}
\left|\frac{18n^5+2n-5}{6n^5-n^3-3} - 3\right| &= \left|\frac{18n^5+2n-5-3(6n^5-n^3-3)}{6n^5-n^3-3}\right| \\
&= \left|\frac{3n^3+2n+4}{6n^5-n^3-3}\right| \\
&< \left|\frac{3n^3+3n^3+4n^3}{6n^5-n^3-3}\right| && \text{(as for } n \in \mathbb{N},\ 2n < 3n \leq 3n^3 \text{ and } 4 \leq 4n \leq 4n^3\text{)} \\
&\leq \left|\frac{10n^3}{6n^5-n^5-3n^5}\right| && \text{(as for } n \in \mathbb{N},\ n^5 \geq n^3 \text{ and } 3n^5 \geq 3\text{)} \\
&= \left|\frac{10n^3}{2n^5}\right| \\
&= \left|\frac{5}{n^2}\right| \\
&= \frac{5}{n^2} && (*)
\end{aligned}$$

Note that for $\epsilon > 0$ and $n > 0$, $\dfrac{5}{n^2} < \epsilon \iff n > \sqrt{\dfrac{5}{\epsilon}}$.

Proof. Let $\epsilon > 0$ be given. By the Archimedean property of $\mathbb{R}$, $\exists K \in \mathbb{N}$ such that $K > \sqrt{\dfrac{5}{\epsilon}}$. Then for all $n \geq K$, we have:

$$\begin{aligned}
\left|\frac{18n^5+2n-5}{6n^5-n^3-3} - 3\right| &< \frac{5}{n^2} && \text{(by } (*)\text{)} \\
&\leq \frac{5}{K^2} && \text{(since } n \geq K\text{)} \\
&< \frac{5}{\left(\frac{5}{\epsilon}\right)} && \text{(since } K > \sqrt{\tfrac{5}{\epsilon}}\text{)} \\
&= \epsilon
\end{aligned}$$

Hence, we have $\displaystyle\lim_{n\to\infty} \frac{18n^5+2n-5}{6n^5-n^3-3} = 3$, as required. □

Question 2

Let a be the common limit of the subsequences. We want to show that $(\forall\epsilon > 0)(\exists K \in \mathbb{N})[n \geq K \Rightarrow |a_n - a| < \epsilon]$.

We know that:

$$(\forall\epsilon > 0)(\exists K_1 \in \mathbb{N})[k \geq K_1 \Rightarrow |a_{2k} - a| < \epsilon]$$

$$(\forall\epsilon > 0)(\exists K_2 \in \mathbb{N})[k \geq K_2 \Rightarrow |a_{4k-1} - a| < \epsilon]$$

$$(\forall \epsilon > 0)(\exists K_3 \in \mathbb{N})[k \geq K_3 \Rightarrow |a_{4k-3} - a| < \epsilon]$$

We also note that these 3 subsequences partition (a_n). This is because the integers can be partitioned into the equivalences classes of $\mathbb{Z}_4$. When n is even, we have the union of equivalence classes $[0] \cup [2]$. In other words, if we only look at the positive, even values of n, we get the sequence (a_{2k}). If n is odd, we have the union of equivalence classes $[1] \cup [3]$. In other words, if we only look at the positive, odd values of n, we get either the sequence (a_{4k-1}) or (a_{4k-3}).

Proof. Let $\epsilon > 0$ be given. Let $K = \max\{2K_1, 4K_2 - 1, 4K_3 - 1\}$. Then for all $n \geq K$, we have:

$$\begin{cases} |a_n - a| = |a_{2k} - a| < \epsilon & \text{if } n = 2k \text{ (since } k = \frac{n}{2} \geq \frac{K}{2} \geq K_1) \\ |a_n - a| = |a_{4k-1} - a| < \epsilon & \text{if } n = 4k-1 \text{ (since } k = \frac{n+1}{4} \geq \frac{K+1}{4} \geq K_2) \\ |a_n - a| = |a_{4k-3} - a| < \epsilon & \text{if } n = 4k-3 \text{ (since } k = \frac{n+3}{4} \geq \frac{K+3}{4} \geq K_3) \end{cases}$$

□

Question 3

Note that

$$x_1 = 6, x_2 = 5.33, x_3 = 5.12, x_4 = 5.05, x_5 = 5.02, x_6 = 5.01, ...$$

Proof. We first show that $x_n \geq 5$ for all $n \in \mathbb{N}$. Let $P(n)$ be the statement that $x_n \geq 5$. Clearly, $P(1)$ holds, as $x_1 = 6 \geq 5$. Assume $P(k)$ holds, i.e. $x_k \geq 5$. Then $x_{k+1} = \dfrac{8x_k}{3 + x_k} = 8 - \dfrac{24}{3 + x_k} \geq 8 - \dfrac{24}{3+5}$, which gives $x_{k+1} \geq 5$. So, $P(k+1)$ holds. Thus, by the principle of mathematical induction, $x_n \geq 5$ for all $n \in \mathbb{N}$, i.e., (x_n) is bounded below by 5.

Now we show that x_n is decreasing for all $n \in \mathbb{N}$. Let $D(n)$ be the statement $x_{n+1} \leq x_n$. Then, since $x_2 = \dfrac{48}{9} \leq 6$, $D(1)$ holds. Assume $D(k)$ holds, i.e. $x_{k+1} \leq x_k$. Then $x_{k+2} - x_{k+1} = \dfrac{8x_{k+1}}{3 + x_{k+1}} - \dfrac{8x_k}{3 + x_k} = \dfrac{24(x_{k+1} - x_k)}{(3 + x_{k+1})(3 + x_k)}$. Note that $(3 + x_{k+1})(3 + x_k) \geq 0$, since $x_k \geq 5$ for all $k \in \mathbb{N}$, and by our induction hypothesis, $x_{k+1} - x_k \leq 0$. Thus, $\dfrac{24(x_{k+1} - x_k)}{(3 + x_{k+1})(3 + x_k)} \leq 0$. So, $D(k+1)$ holds. Thus, by the principle of mathematical induction, $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$, i.e., x_n is decreasing.

Since (x_n) is decreasing and bounded below, by Corollary 3.3.2(ii), it follows from the Monotone Convergence Theorem that (x_n) converges. Let $\lim\limits_{n\to\infty} x_n = x \in \mathbb{R}$. By Theorem 3.4.1, as

(x_{n+1}) is a subsequence of (x_n), we know that $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n = x$. Hence,

$$\begin{aligned} x = \lim_{n\to\infty} x_{n+1} &= \lim_{n\to\infty} \frac{8x_n}{3+x_n} \\ &= \frac{\lim_{n\to\infty} 8x_n}{\lim_{n\to\infty} 3+x_n} \qquad \text{(as } x_n \text{ is bounded below by 5, so } 3+x_n \neq 0 \text{ for any } n\text{)} \\ &= \frac{8\cdot \lim_{n\to\infty} x_n}{3+\lim_{n\to\infty} x_n} \\ &= \frac{8x}{3+x} \end{aligned}$$

Thus, $x = \dfrac{8x}{3+x}$, which gives $x^2 - 5x = 0$. So either $x = 5$ or $x = 0$. Let $y_n = 5$ be a constant sequence. So, by Theorem 3.2.9(b), as $x_n \geq 5$ (from above), $x = \lim_{n\to\infty} x_n \geq \lim_{n\to\infty} y_n = 5$. Hence, $x = 5$. $\square$

Question 4

Proof. Let $m_1 = \liminf x_n$, $m_2 = \liminf y_n$ and $m = \min\{m_1, m_2\}$. Note that as $(x_n), (y_n)$ are bounded, $\exists a, b, c, d \in \mathbb{R}$ such that $a \leq x_n \leq b$ and $c \leq y_n \leq d$ for all $n \in \mathbb{N}$. As $z_n = \min\{x_n, y_n\}$, each term in (z_n) is either in (x_n) or (y_n). So, $\min\{a, c\} \leq z_n \leq \max\{b, d\}$, i.e., z_n is bounded. We know that $(\forall n \in \mathbb{N})[(z_n \leq x_n) \wedge (z_n \leq y_n)]$, so by Theorem 3.5.4, $\liminf z_n \leq \liminf x_n = m_1$ and $\liminf z_n \leq \liminf y_n = m_2$. Hence, $\liminf z_n \leq \min\{m_1, m_2\} = m$.

(Note that since x_n, y_n, z_n are bounded, by the Bolzano-Weierstrass Theorem, each of the sequences have at least one convergent subsequence, which means $S(x_n), S(y_n)$, and $S(z_n)$ are non-empty and thus $\liminf x_n$, $\liminf y_n$, and $\liminf z_n$ exist.)

Now let $z \in S(z_n)$. Then, there exists a subsequence (z_{n_k}) of (z_n) (with each $n_k \geq k$) such that $\lim_{n\to\infty} z_n = z$. Now as $(x_n), (y_n)$ are bounded sequences, by Theorem 3.5.2:

$$(\forall \epsilon > 0)(\exists K_1 \in \mathbb{N})[n \geq K_1 \Rightarrow x_n > m_1 - \epsilon]$$

and

$$(\forall \epsilon > 0)(\exists K_2 \in \mathbb{N})[n \geq K_2 \Rightarrow y_n > m_2 - \epsilon]$$

So, let $\epsilon > 0$ be given and $K = \max\{K_1, K_2\}$. Then $(\forall n \geq K)[(x_n > m_1 - \epsilon) \wedge (y_n > m_2 - \epsilon)]$. So, $(x_n > \min\{m_1 - \epsilon, m_2 - \epsilon\}) \wedge (y_n > \min\{m_1 - \epsilon, m_2 - \epsilon\})$. Hence, $z_n > \min\{m_1 - \epsilon, m_2 - \epsilon\}$. As ϵ is a constant, we get $z_n > \min\{m_1, m_2\} - \epsilon$, which is equivalent to $z_n > m - \epsilon$. Now, as

(z_{n_k}) is a subsequence of (z_n), $n_k \geq k$, so:

$$\begin{aligned} k \geq K \Rightarrow n_k &\geq k \geq K \\ \Rightarrow z_{n_k} &> m - \epsilon \\ \Rightarrow z_{n_k} &\geq m - \epsilon \end{aligned}$$

Thus, we have shown that $(\forall \epsilon > 0)(\exists K \in \mathbb{N})[k \geq K \Rightarrow z_{n_k} \geq m - \epsilon]$. Now, if we let $k \to \infty$, by Theorem 3.2.9, $z = \lim_{n\to\infty} z_{n_k} \geq m - \epsilon$. As $\epsilon > 0$ is arbitrary, $z \geq m$.

Hence, as z is arbitrary, we have shown that $(\forall z \in S(z_n))[z \geq m]$. So, m is a lower bound of $S(z_n)$. By definition, $\inf S(z_n)$ is greatest lower bound of $S(z_n)$, so $\liminf z_n \geq m$.

Consequently, as $(\liminf z_n \geq m) \wedge (\liminf z_n \leq m)$, $\liminf z_n = m = \min\{\liminf x_n, \liminf y_n\}$, as required. $\square$

Question 5

(a)

False

(b)

False

(i)

False

(ii)

False