Hausdorff Measure and Dimension

Name
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Introduction

The **Hausdorff measure** is a measure on metric spaces which generalizes the Lebesgue measure on $\mathbb{R}^n$. As such, one can imagine how useful this measure can be in areas such as differential geometry, where it can be used to define "volumes" of submanifolds (such as the unit sphere in $\mathbb{R}^3$). In **Assignment 3**, we've seen how to construct the Hausdorff measure on a metric space. We shall now look at some further properties of the measure (such as its *measurable* sets) as well as introduce **Hausdorff dimension**, which generalizes the notion of the usual (topological) dimension.

1 Measurable sets of the Hausdorff measure

Let (X, ρ) be a metric space. Recall (from **Assignment 3**) that if $d > 0$ and $\delta > 0$, we define $H^d_\delta : \mathcal{P}(X) \to [0, \infty]$ by:

$$H^d_\delta(E) = \inf \left\{ \sum_{i \geqslant 1} \operatorname{diam}(U_i)^d : E \subseteq \bigcup_{i \geqslant 1} U_i,\ \operatorname{diam}(U_i) < \delta \right\}$$

for all $E \subseteq X$, where $\operatorname{diam}(U) = \sup\{\rho(x, y) : x, y \in U\}$. Then the d-dimensional Hausdorff *outer* measure H^d on (X, ρ) is defined by:

$$H^d(E) = \lim_{\delta \to 0^+} H^d_\delta(E)$$

for all $E \subseteq X$. That is, $H^n(E)$ is approximated by the sum of d^{th} powers of the diameters of the sets of countable covers of E where each diameter is less than δ, and a smaller value of δ means a better approximation. We wish to find the *measurable* sets of H^d.

We know that a set $E \subseteq X$ is H^d-measurable when $H^d(A) = H^d(E \cap A) + H^d(A \setminus E)$ for all $A \subseteq E$. We therefore have:

$$\mathcal{B} = \{E \subseteq X : H^d(A) = H^d(E \cap A) + H^d(A \setminus E)\ \forall A \subseteq X\}$$

is the set of all H^d-measurable sets. In particular, the Carathéodory theorem tells us that $\mathcal{B}$ is a σ-algebra and that H^d restricted to $\mathcal{B}$ is a *complete* measure. Of course, this definition doesn't tell us much about exactly which sets are H^d-measurable. If we wish for the Hausdorff measure to extend the notion of the Lebesgue measure, it's reasonable to ask that the Borel sets in X be H^d-measurable. Recall (again from

Assignment 3) that two sets $A, B \subseteq X$ in a metric space (X, ρ) are said to be **positively separated** when $\inf\{\rho(x, y) : x \in A,\ y \in B\} > 0$. For brevity, we will define $\rho(A, B) = \inf\{\rho(x, y) : x \in A,\ y \in B\}$. We have seen that if A and B are positively separated, then $H^d(A \cup B) = H^d(A) + H^d(B)$.

Definition 1.1 In general, if μ^* is an outer measure on a metric space (X, ρ) such that $\mu^*(A \cup B) = \mu^*(A) + \mu^*(B)$ whenever $A, B \subseteq X$ are positively separated, then μ^* is called a **metric outer measure** (or a **Carathéodory outer measure**).

Hence, the Hausdorff outer measure H^d is a metric outer measure. We have the following general result about metric outer measures that states that the Borel sets must be measurable.

Proposition 1.2 Let (X, ρ) be a metric space and μ^* a metric outer measure on (X, ρ). Then every Borel set in X is μ^*-measurable.

Proof. (Adapted from [**?**]).

Let $\mathrm{Bor}(X)$ denote the set of Borel sets, which we know is a σ-algebra. In particular, we have that $\mathrm{Bor}(X) = \sigma(\{F \subseteq X : F \text{ is closed}\})$. Let $\mathcal{B}$ be the σ-algebra of μ^*-measurable sets in X. Then to show that $\mathrm{Bor}(X) \subseteq \mathcal{B}$, it suffices to prove that every closed set is μ^*-measurable.

Indeed, let $F \subseteq X$ be closed and let $A \subseteq X$ be any set. It's immediate that $\mu^*(A) \leqslant \mu^*(A \cap F) + \mu^*(A \setminus F)$ by subadditivity. Moreover, if $\mu^*(A) = \infty$, equality clearly holds. Thus, we may assume $\mu^*(A) < \infty$. For each $n \in \mathbb{N}$, let $A_n = \{x \in A \setminus F : \rho(\{x\}, F) = \inf\{\rho(x, y) : y \in F\} \geqslant \frac{1}{n}\}$. Observe that if $x \in A_n$, then $\rho(\{x\}, F) \geqslant \frac{1}{n} > \frac{1}{n+1}$, so $x \in A_{n+1}$. Hence, $A_1 \subseteq A_2 \subseteq \cdots \subseteq A \setminus F$.

Now for any $x \in A \setminus F$, suppose we had that $\rho(\{x\}, F) = \inf\{\rho(x, y) : y \in F\} = 0$. Then for all $n \in \mathbb{N}$, we can find some $y_n \in F$ for which $\rho(x, y_n) < \frac{1}{n}$. As ρ is a metric on X, it follows that y_n converges to x. By closure of F, we have that in fact $x \in F$, which is a contradiction. It thus follows that $\rho(\{x\}, F) > 0$ for any $x \in A \setminus F$, which means we can find a $k \in \mathbb{N}$ for which $\rho(\{x\}, F) \geqslant \frac{1}{k}$ and consequently $x \in A_k \subseteq \bigcup_{n \geqslant 1} A_n$. This proves that $\bigcup_{n \geqslant 1} A_n = A \setminus F$.

As $A = (A \cap F) \cup (A \setminus F) \supseteq (A \cap F) \cup A_n$ for all $n \in \mathbb{N}$, we have by monotonicity that:

$$\mu^*(A) \geqslant \mu^*((A \cap F) \cup A_n)$$

Moreover, we have by construction that $\rho(\{x\}, F) \geqslant \frac{1}{n}$ whenever $x \in A_n$, and thus (in particular) $\rho(A_n, A \cap F) \geqslant \frac{1}{n} > 0$, so A_n and $A \cap F$ are positively separated. Since μ^* is a *metric* outer measure:

$$\mu^*((A \cap F) \cup A_n) = \mu^*(A \cap F) + \mu^*(A_n)$$

Now consider $D_n = A_{n+1} \setminus A_n = \{x \in A \setminus F : \frac{1}{n+1} \leqslant \rho(\{x\}, F) < \frac{1}{n}\}$. By construction, we have that the D_n's are disjoint and $A \setminus F = \bigcup_{n \geqslant 1} D_n$. Let $x \in D_{n+1}$ and suppose $y \in X$ satisfies $\rho(x, y) < \frac{1}{n(n+1)}$. Then for all $z \in F$, $x \in D_{n+1} = A_{n+2} \setminus A_{n+1}$ means $\frac{1}{n+2} \leqslant \rho(x, z) < \frac{1}{n+1}$, so the triangle inequality gives:

$$\rho(y, z) \leqslant \rho(y, x) + \rho(x, z) < \frac{1}{n(n+1)} + \frac{1}{n+1} = \frac{n+1}{n(n+1)} = \frac{1}{n}$$

and thus $\rho(y, F) < \frac{1}{n} \implies y \notin A_n$. This means if $y \in A_n$, then we must have $\rho(x, y) \geqslant \frac{1}{n(n+1)}$ so that $\rho(x, A_n) \geqslant \frac{1}{n(n+1)}$. As our choice of $x \in D_{n+1}$ was arbitrary, it follows that $\rho(D_{n+1}, A_n) \geqslant \frac{1}{n(n+1)} > 0$ so that D_{n+1} and A_n are positively separated. Thus, for all $n \in \mathbb{N}$:

$$\mu^*(A_{2n+1}) = \mu^*(D_{2n} \cup A_{2n}) \geqslant \mu^*(D_{2n} \cup A_{2n-1}) = \mu^*(D_{2n}) + \mu^*(A_{2n-1})$$

$$\mu^*(A_{2n}) = \mu^*(D_{2n-1} \cup A_{2n-1}) \geqslant \mu^*(D_{2n-1} \cup A_{2n-2}) = \mu^*(D_{2n-1}) + \mu^*(A_{2n-2})$$

where we can set $A_0 = \emptyset$. It hence follows that (inductively):

$$\sum_{k=1}^{n} \mu^*(D_{2k}) \leqslant \sum_{k=1}^{n} \mu^*(D_{2k}) + \mu^*(A_1) \leqslant \mu^*(A_{2n+1}) \leqslant \mu^*(A) < \infty$$

$$\sum_{k=1}^{n} \mu^*(D_{2k-1}) = \sum_{k=1}^{n} \mu^*(D_{2k-1}) + \mu^*(A_0) \leqslant \mu^*(A_{2n}) \leqslant \mu^*(A) < \infty$$

As the above holds for all $n \in \mathbb{N}$, we have in particular that $\sum_{k=1}^{\infty} \mu^*(D_{2k})$ and $\sum_{k=1}^{\infty} \mu^*(D_{2k-1})$ are convergent series. Thus, $\sum_{k=1}^{\infty} \mu^*(D_k)$ is also a convergent series. By subadditivity, we have for all $n \in \mathbb{N}$ that:

$$\mu^*(A \setminus F) = \mu^*\left(\bigcup_{k \geqslant 1} A_k\right) = \mu^*\left(A_n \cup \bigcup_{k \geqslant n+1} D_k\right) \leqslant \mu^*(A_n) + \sum_{k \geqslant n+1} \mu^*(D_k)$$

By convergence of the series:

$$\mu^*(A \setminus F) \leqslant \liminf_{n \to \infty} \left[\mu^*(A_n) + \sum_{k \geqslant n+1} \mu^*(D_k)\right] = \liminf_{n \to \infty} \mu^*(A_n) + \liminf_{n \to \infty} \sum_{k \geqslant n+1} \mu^*(D_k) = \liminf_{n \to \infty} \mu^*(A_n)$$

Moreover, $A_n \subseteq A \setminus F$ means $\mu^*(A_n) \leqslant \mu^*(A \setminus F)$ by monotonicity, so in fact:

$$\mu^*(A \setminus F) \leqslant \liminf_{n \to \infty} \mu^*(A_n) \leqslant \limsup_{n \to \infty} \mu^*(A_n) \leqslant \mu^*(A \setminus F)$$

$$\implies \lim_{n \to \infty} \mu^*(A_n) = \mu^*(A \setminus F)$$

Finally, it follows that:

$$\mu^*(A) \geqslant \mu^*(A \cap F) + \lim_{n \to \infty} \mu^*(A_n) = \mu^*(A \cap F) + \mu^*(A \setminus F)$$

and we hence get that F is μ^*-measurable. As F was an arbitrarily-chosen closed set, it follows that every Borel set is μ^*-measurable. □

The following result is therefore immediate (following the fact that H^d is a metric outer measure):

Corollary 1.3 In a metric space (X, ρ), every Borel set is H^d-measurable.

Remark 1.4 As $\mathrm{Bor}(X)$ is a σ-algebra and each $A \in \mathrm{Bor}(X)$ is H^d-measurable, it follows that $(X, \mathrm{Bor}(X), H^d)$ is a measure space (with H^d restricted to $\mathrm{Bor}(X)$).

The next step is to justify that the Hausdorff measure can indeed be considered a generalization of the Lebesgue measure (though we will not go into too much detail, as it would take quite some time).

Recall that the Lebesgue outer measure on $\mathbb{R}$ is *translation-invariant*. That is, $m^*(x + A) = m(A)$ for any $A \subseteq \mathbb{R}$ and any $x \in \mathbb{R}$. We also know that $m^*(\lambda A) = |\lambda| m^*(A)$ for $\lambda \in \mathbb{R}$. The following analogous result holds for the Hausdorff measure:

Lemma 1.5 On a metric space (X, ρ), the Hausdorff outer measure H^d is invariant under isometries of (X, ρ). Moreover, for any set Y, any $\lambda \in \mathbb{R}$, and functions $f, g : Y \to X$ satisfying $\rho(f(y_1), f(y_2)) \leqslant |\lambda|\rho(g(y_1), g(y_2))$ for all $y_1, y_2 \in Y$, we have $H^d(f(B)) = |\lambda|^d H^d(g(B))$ for all $B \subseteq Y$.

Proof. (Adapted from [**?**]).

Let $h : X \to X$ be an isometry, so that $\rho(x, y) = \rho(h(x), h(y))$ for all $x, y \in X$. In particular, for any $U \subseteq X$, the fact that h is isometric and bijective implies:

$$\operatorname{diam}(U) = \sup\{\rho(x, y) : x, y \in U\} = \sup\{\rho(h(x), h(y)) : x, y \in U\}$$

$$= \sup\{\rho(h(x), h(y)) : h(x), h(y) \in h(U)\} = \operatorname{diam}(h(U))$$

and likewise, $\operatorname{diam}(h^{-1}(U)) = \operatorname{diam}(U)$ (in other words, diam is invariant under isometries). Let $A \subseteq X$. Then for any $\delta > 0$:

$$H^d_\delta(h(A)) = \inf\left\{\sum_{i\geqslant 1} \operatorname{diam}(U_i)^d : h(A) \subseteq \bigcup_{i\geqslant 1} U_i,\ \operatorname{diam}(U_i) < \delta\right\}$$

$$= \inf\left\{\sum_{i\geqslant 1} \operatorname{diam}(h^{-1}(U_i))^d : A \subseteq \bigcup_{i\geqslant 1} h^{-1}(U_i),\ \operatorname{diam}(h^{-1}(U_i)) < \delta\right\} = H^d_\delta(A)$$

and it follows by letting $\delta \to 0$ that $H^d(h(A)) = H^d(A)$, so that H^d is invariant under isometries.

For the second statement, we may assume WLOG that $\lambda \neq 0$ (the case where $\lambda = 0$ is clear). Let $B \subseteq Y$ and let $\delta, \varepsilon > 0$ be arbitrary. Then we can find a cover $V_1, V_2, ... \subseteq X$ of $g(B)$ such that $\operatorname{diam}(V_i) < \frac{\delta}{|\lambda|}$ and:

$$\sum_{i=1}^{\infty} \operatorname{diam}(V_i)^d \leqslant H^d_\delta(g(B)) + \frac{\varepsilon}{|\lambda|^d}$$

Let $U_i = f(g^{-1}(V_i))$. Then we have by construction that $U_1, U_2, ... \subseteq X$ is a cover for $f(A)$, and by the assumption we have:

$$\operatorname{diam}(U_i) = \sup\{\rho(x_1, x_2) : x_1, x_2 \in U_i\} = \sup\{\rho(f(y_1), f(y_2)) : y_1, y_2 \in g^{-1}(V_i)\}$$

$$\leqslant |\lambda| \sup\{\rho(g(y_1), g(y_2)) : y_1, y_2 \in g^{-1}(V_i)\} = |\lambda| \sup\{\rho(x_1, x_2) : x_1, x_2 \in V_i\} = |\lambda| \operatorname{diam}(V_i)$$

Hence, it follows that:

$$H^d_\delta(f(B)) \leqslant \sum_{i=1}^{\infty} \operatorname{diam}(U_i)^d \leqslant |\lambda|^d \sum_{i=1}^{\infty} \operatorname{diam}(V_i)^d \leqslant |\lambda|^d H^d_\delta(g(B)) + \varepsilon$$

Now since δ and ε were arbitrary, letting $\delta, \varepsilon \to 0$ gives:

$$H^d(f(B)) \leqslant |\lambda|^d H^d(g(B))$$

which completes the proof. $\square$

The following result justifies that the Hausdorff measure can be considered a generalization of the Lebesgue measure:

Proposition 1.6 For $\mathbb{R}^n$ equipped with the usual Euclidean metric and m_n the Lebesgue measure on $\mathbb{R}^n$, there exists a constant $\gamma_n > 0$ for which $H^n = \gamma_n m_n$.

Proof. We will *sketch* the general idea of the proof. Consider the unit cube $Q = [0,1]^n$ and define $\gamma_n = H^n(Q)$. We can show that $\gamma_n < \infty$ by choosing $\delta > 0$, writing Q as a union of cubes of length $< \frac{\delta}{\sqrt{n}}$, and verifying that the sum of diameters of the cubes is $n^{n/2}$. It will follow that $\gamma_n = H^n(Q) < \infty$ (as δ doesn't depend on n).

We can similarly show that $\gamma_n > 0$ by choosing an arbitrary countable cover $U_1, U_2, ...$ of Q and covering each U_i by a cube Q_i of side length $\operatorname{diam}(U_i)$. Monotonicity will then give that $m_n(Q) = 1 \leqslant \sum_{i=1}^{\infty} m_n(Q_i) = \sum_{i=1}^{\infty} (\operatorname{diam}(U_i))^n$, and taking the infimum over all such covers leads to $1 \leqslant H^n(Q)$ (so that in particular, $\gamma_n = H^n(Q) > 0$).

Finally, we can make use of the uniqueness of the Carathéodory theorem in the construction of m_n to show that $H^n(B) = \gamma_n m_n(B)$ for all open *boxes* $B \subseteq \mathbb{R}^n$. This is done by approximating the boxes by "almost disjoint" cubes (as done in **A1 Q5 (2)**) and using **Lemma ??** to deal with the scaling and translating of each cube to $[0,1]^n$. It will then follow that in fact $H^n(A) = \gamma_n m_n(A)$ for all Lebesgue-measurable $A \subseteq \mathbb{R}^n$. $\square$

Remark 1.7 The proof doesn't require the computation of the value of γ_n, but only that it's a (strictly) positive finite value. It turns out that $\gamma_n = \frac{\operatorname{vol}(B_1^n)}{2^n}$ where $\operatorname{vol}(B_1^n)$ is the volume of the n-ball $B_1^n \subseteq \mathbb{R}^n$ of radius 1 (diameter 2). In particular, $\gamma_1 = 1$ so that H^1 is exactly the 1-dimensional Lebesgue measure m_1. We will prove this latter fact formally (as it will help later).

Lemma 1.8 $\gamma_1 = 1$. Hence, $H^1 = m_1$ (that is, the 1-dimensional Hausdorff measure on $\mathbb{R}$ is exactly the Lebesgue measure on $\mathbb{R}$).

Proof. Since $H^1([0,1]) = \gamma_1 m_1([0,1]) = \gamma_1$, we must show that $H^1([0,1]) = 1$. Indeed, let $\delta > 0$ and choose $N \in \mathbb{N}$ for which $\frac{1}{N} < \delta$. Then for each $1 \leqslant n \leqslant N$, set $U_n = [\frac{n-1}{N}, \frac{n}{N}]$ so that $U_1, ..., U_N$ is a cover for $[0,1]$ with $\operatorname{diam}(U_n) = \frac{1}{N} < \delta$. Then:

$$H_\delta^1([0,1]) \leqslant \sum_{n=1}^{N} \operatorname{diam}(U_n) \leqslant \sum_{n=1}^{N} \frac{1}{N} = 1$$

and so letting $\delta \to 0$, we have $H^1([0,1]) \leqslant 1$. On the other hand, let $\delta > 0$ and choose any cover $U_1, U_2, ... \subseteq \mathbb{R}$ for $[0,1]$ satisfying $\operatorname{diam}(U_i) < \delta$. We can see that $\operatorname{diam}(U_i) \geqslant m_1^*(U_i)$ (where m_1^* is the Lebesgue *outer* measure) because $\operatorname{diam}(U_i) < \delta$ implies we can find $\alpha_i = \inf(U_i)$ and $\beta_i = \sup(U_i)$ so that $\operatorname{diam}(U_i) = \beta_i - \alpha_i$ and $J_i = [\alpha, \beta] \supseteq U_i$ so that $\operatorname{diam}(U_i) = \beta_i - \alpha_i = m_1(J_i) \geqslant m_1^*(U_i)$ by monotonicity. It then follows (by subadditivity and monotonicity) that:

$$\sum_{i=1}^{\infty} \operatorname{diam}(U_i) \geqslant \sum_{i=1}^{\infty} m_1^*(U_i) \geqslant m_1^* \left(\bigcup_{i=1}^{\infty} U_i \right) \geqslant m_1([0,1]) = 1$$

so that (since our choice of cover was arbitrary) $H_\delta^1([0,1]) \geqslant 1$. Letting $\delta \to 0$ then gives $H^1([0,1]) \geqslant 1$, and thus $H^1([0,1]) = \gamma_1 = 1$. $\square$

2 Hausdorff dimension

Before defining the Hausdorff dimension of a set, we first need the following lemma:

Lemma 2.1 Let (X, ρ) be a metric space, let $E \subseteq X$, and let $\alpha, \beta \in \mathbb{R}$ with $0 < \alpha < \beta$. If $H^\alpha(E) < \infty$, then $H^\beta(E) = 0$.

Proof. (Adapted from [**?**]).

Let $\delta > 0$. As $H^\alpha(E) < \infty$, we can choose $U_1, U_2, ... \subseteq X$ for which $E \subseteq \bigcup_{i \geqslant 1} U_i$, each $\text{diam}(U_i) < \delta$, and:

$$\sum_{i \geqslant 1} \text{diam}(U_i)^\alpha < H^\alpha_\delta(E) + 1$$

We then see that as $\alpha < \beta$ and $\text{diam}(U_i) < \delta$:

$$H^\beta_\delta(E) \leqslant \sum_{i \geqslant 1} \text{diam}(U_i)^\beta = \sum_{i \geqslant 1} \text{diam}(U_i)^{\beta - \alpha + \alpha} \leqslant \delta^{\beta - \alpha} \sum_{i \geqslant 1} \text{diam}(U_i)^\alpha \leqslant \delta^{\beta - \alpha} \left(H^\alpha_\delta(E) + 1\right)$$

Moreover, since $H^\alpha_\delta(E)$ is increasing as $\delta \to 0$ (as proved in **A3**):

$$H^\beta_\delta(E) \leqslant \delta^{\beta - \alpha} \left(H^\alpha_\delta(E) + 1\right) \leqslant \delta^{\beta - \alpha} \left(H^\alpha(E) + 1\right)$$

Finally, we have:

$$H^\beta(E) = \lim_{\delta \to 0} H^\beta_\delta(E) \leqslant \lim_{\delta \to 0} \delta^{\beta - \alpha} \left(H^\alpha(E) + 1\right) = 0$$

which forces $H^\beta(E) = 0$. □

The lemma makes the following definition well-defined:

Definition 2.2 Let (X, ρ) be a metric space and let $E \subseteq X$. The **Hausdorff dimension** $\dim_H(E)$ of E is defined to be:

$$\dim_H(E) = \inf\{\alpha \geqslant 0 : H^\alpha(E) = 0\}$$

Using **Lemma ??** and **Proposition ??**, we can compute the Hausdorff dimensions of certain shapes (subsets of $\mathbb{R}^n$). The following are examples in $\mathbb{R}^2$.

Example 2.3 Consider the closed unit disk $\mathbb{D} = \{x \in \mathbb{R}^2 : \|x\|_2 \leqslant 1\}$. We claim that its Hausdorff dimension $\dim_H(\mathbb{D}) = 2$. By using **Proposition ??**, we have that:

$$H^2(\mathbb{D}) = \gamma_2 m_2(\mathbb{D}) = \gamma_2 \cdot \pi > 0$$

where $m_2(\mathbb{D}) = \pi$ is just the $\mathbb{R}^2$-Lebesgue measure (area) of $\mathbb{D}$. Then **Lemma ??** tells us that $H^\alpha(\mathbb{D}) = 0$ for all $\alpha > 2$, and so:

$$\dim_H(\mathbb{D}) = \inf\{\alpha \geqslant 0 : H^\alpha(\mathbb{D}) = 0\} = 2$$

We used **Proposition ??** to prove that $H^2(\mathbb{D}) > 0$, but we could also have done an infimum argument to do it. The former method saves us time, however.

It can be more difficult to compute Hausdorff dimensions of more general submanifolds of $\mathbb{R}^n$, as **Proposition ??** would not guarantee that H^n gives a non-zero value. In the next example, we compute the Hausdorff dimension of the submanifold S^1 (the unit circle in $\mathbb{R}^2$), and the following lemma proves useful to do so:

Lemma 2.4 Let (X, ρ) be a metric space and let $A \subseteq X$ be a *connected* set. Then $H^1(A) \geqslant \operatorname{diam}(A)$.

Proof. (Adapted from [**?**])

For each $a \in X$, consider the map $d_a : X \to \mathbb{R}$ defined by $d_a(x) = \rho(a, x)$. We can see that for $x, y \in X$ (assuming WLOG $\rho(a, x) \geqslant \rho(a, y)$):

$$|d_a(x) - d_a(y)| = |\rho(a, x) - \rho(a, y)| = \rho(a, x) - \rho(a, y) \leqslant \rho(a, y) + \rho(x, y) - \rho(a, y) = \rho(x, y)$$

where the inequality comes from the triangle inequality on ρ. This shows that d_a is 1-Lipschitz, thus (in particular) continuous. We hence get that:

$$\operatorname{diam}(d_a(U)) = \sup\{|d_a(x) - d_a(y)| : x, y \in U\} \leqslant \sup\{\rho(x, y) : x, y \in d_a(U)\} = \operatorname{diam}(U)$$

for any $U \subseteq X$, which implies for any $\delta > 0$:

$$H^1_\delta(d_a(A)) = \inf\left\{\sum_{i\geqslant 1} \operatorname{diam}(J_i) : d_a(A) \subseteq \bigcup_{i\geqslant 1} J_i,\ \operatorname{diam}(J_i) < \delta\right\}$$

$$\leqslant \inf\left\{\sum_{i\geqslant 1} \operatorname{diam}(d_a(U_i)) : A \subseteq \bigcup_{i\geqslant 1} U_i,\ \operatorname{diam}(d_a(U_i)) \leqslant \operatorname{diam}(U_i) < \delta\right\}$$

$$\leqslant \inf\left\{\sum_{i\geqslant 1} \operatorname{diam}(U_i) : A \subseteq \bigcup_{i\geqslant 1} U_i,\ \operatorname{diam}(U_i) < \delta\right\} = H^1_\delta(A)$$

so that $H^1(d_a(A)) \leqslant H^1(A)$ for all $a \in X$. Finally since d_a is continuous and A is connected, it follows that $d_a(A) \subseteq \mathbb{R}$ is connected (i.e. an interval), which means $H^1(d_a(A)) = \gamma_1 m_1(d_a(A)) = \gamma_1 \operatorname{diam}(d_a(A))$. By **Lemma ??**, $\gamma_1 = 1$ and so $H^1(d_a(A)) = \operatorname{diam}(d_a(A)) \leqslant H^1(A)$ for all $a \in X$. Finally:

$$\sup_{a\in A} \operatorname{diam}(d_a(A)) = \sup_{a\in X} \sup\{|d_a(x) - d_a(y)| : x, y \in A\} = \sup_{a\in X} \sup\{|\rho(x, a) - \rho(y, a)| : x, y \in A\}$$

$$= \sup_{a\in X} \sup\{\rho(x, a) - \rho(y, a) : x, y \in A\} \geqslant \sup\{\rho(x, y) - \rho(y, y) : x, y \in A\}$$

$$= \sup\{\rho(x, y) : x, y \in A\} = \operatorname{diam}(A)$$

so that $\operatorname{diam}(A) \leqslant \sup_{a\in A} \operatorname{diam}(d_a(A)) \leqslant H^1(A)$, which completes the proof. ☐

Example 2.5 Consider the unit circle $S^1 = \{x \in \mathbb{R}^2 : \|x\|_2 = 1\}$. We claim that its Hausdorff dimension $\dim_H(S^1) = 1$. We know that S^1 has $\mathbb{R}^2$-Lebesgue measure zero, so that:

$$H^2(S^1) = \gamma_2 m_2(S^2) = \gamma_2 \cdot 0 = 0$$

But this only tells us that $\dim_H(S^1) \leqslant 2$. To show that $\dim_H(S^1) = 1$, we observe that $S^1 \subseteq \mathbb{R}^2$ is *connected*, and so by **Lemma ??**:

$$H^1(S^1) \geqslant \operatorname{diam}(S^1) = 2 > 0$$

so that $\dim_H(S^1) = 1$ (by **Lemma ??**).

References

[Folland] Folland, G. B. (1999). *Real analysis: Modern techniques and their applications*. New York: Wiley.

[Royden] Royden, H. L. & Fitzpatrick, P. (2010). *Real analysis*. Boston: Prentice Hall.

[Semmes] Semmes, S. (2010). *Some elementary aspects of Hausdorff measure and dimension*. arXiv:1008.2637v1.