The Only Way is Up

Prabhav Kumar

The Question

Let's play a game, where you pick random real numbers between 0 and 1. The rule is that if your current pick is greater than your previous pick, then you continue playing the game or else the game stops. What is the expected number of picks you make?

Important Assumptions & Deductions

- **pick random numbers between 0 and 1**: The underlying probability distribution is uniform.
- **real**: Thankfully, we are not dealing with complex numbers :)
- **current pick is greater than your previous pick**: Since we are dealing with a continuous distribution, the probability of two picks being equal to each other is 0.
- **expected number of picks**: If the game ends on the 3^{rd} pick, then we say the length of the game is 3.

Important Identities

- If X is a random variable, then

$$P(X > x) = 1 - P(X \leq x)$$

- If X is a discrete random variable that only takes on positive integers, then

$$E(X) = \sum_{x=1}^{\infty} xP(X = x)$$

Solution

Calculating Probabilities

Let's denote the picks as $r_1, r_2, ...$ and the number of picks or game length as L. Note that L has to be a positive integer because the first pick is always made.

- $L = 1$: The game never stops after 1 pick, so

$$P(L = 1) = 0$$

- $L = 2$: Here, we need $r_1 < r_2$. This is where we must think of the numbers as ordered sequences. Suppose we are given r_1 and r_2 and are asked to put it in ascending order. Then, the ordered sequence is either (r_1, r_2) or (r_2, r_1)*. As each sequence is equally likely (by common sense), for the game to continue, we want (r_1, r_2). Thus:

  1. $P(L > 2) = \frac{1}{2} \Rightarrow P(L \leq 2) = 1 - \frac{1}{2} = \frac{1}{2}$

2. $P(L=2) = P(L \leq 2) - P(L \leq 1) = P(L \leq 2) - P(L=1) = \dfrac{1}{2} - 0 = \dfrac{1}{2}$

- $L = 3$: Here, we need $r_1 < r_2 < r_3$. There are $3! = 6$ ways to order r_1, r_2, and r_3 and only (r_1, r_2, r_3) allows to continue the game. So $P(L > 3) = \dfrac{1}{6}$. Thus:

  1. $P(L > 3) = \dfrac{1}{6} \Rightarrow P(L \leq 3) = 1 - \dfrac{1}{6} = \dfrac{5}{6}$
  2. $P(L = 3) = P(L \leq 3) - P(L \leq 2) = \dfrac{5}{6} - \dfrac{1}{2} = \dfrac{1}{3}$

- $L = 4$: Here, we need $r_1 < r_2 < r_3 < r_4$. There are $4! = 24$ ways to order r_1, r_2, r_3, and r_4 and only (r_1, r_2, r_3, r_4) allows to continue the game. So $P(L > 4) = \dfrac{1}{24}$. Thus:

  1. $P(L > 4) = \dfrac{1}{24} \Rightarrow P(L \leq 4) = 1 - \dfrac{1}{24} = \dfrac{23}{24}$
  2. $P(L = 4) = P(L \leq 4) - P(L \leq 3) = \dfrac{23}{24} - \dfrac{5}{6} = \dfrac{1}{8}$

- ...

- $L = n$: Here, we need $r_1 < r_2 < ... < r_n$. There are $n!$ ways to order $r_1, r_2, ...,$ and r_n and only $(r_1, r_2, ..., r_n)$ allows to continue the game. So $P(L > n) = \dfrac{1}{n!}$. Thus:

  1. $P(L > n) = \dfrac{1}{n!} \Rightarrow P(L \leq n) = 1 - \dfrac{1}{n!}$
  2. $P(L = n) = P(L \leq n) - P(L \leq n-1) = 1 - \dfrac{1}{n!} - \left(1 - \dfrac{1}{(n-1)!}\right) = \dfrac{1}{(n-1)!} - \dfrac{1}{n!}$

$(*)$ Let $(a_1, a_2, ..., a_n)$ denote the ordered sequence such that $a_1 < a_2 < ... < a_n$.

Calculating Expected Value

As we have found a generalised formula for the probabilities, we simply obtain the expected value as follows:

$$\begin{aligned}
E(L) &= 1 \times 0 + 2 \times \frac{1}{2} + 3 \times \frac{1}{3} + 4 \times \frac{1}{8} + ... \\
&= 2 \times \frac{1}{2} + 3 \times \frac{1}{3} + 4 \times \frac{1}{8} + ... \\
&= \sum_{n=2}^{\infty} n \times \left(\frac{1}{(n-1)!} - \frac{1}{n!}\right) \\
&= \sum_{n=2}^{\infty} \left(\frac{n}{(n-1)!} - \frac{n}{n!}\right) \\
&= \sum_{n=2}^{\infty} \left(\frac{n}{(n-1)!} - \frac{1}{(n-1)!}\right) \\
&= \sum_{n=2}^{\infty} \frac{n-1}{(n-1)!} \\
&= \sum_{n=2}^{\infty} \frac{1}{(n-2)!} \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \\
&= e
\end{aligned}$$

Euler's number is back at it again. A neat solution to a neat problem.

Checking Answer with Python:

```python
# Importing Packages
import random
import math

# Variables
trials = 10000000

# The Game
# List to store results
game_length = []
for _ in range(trials):
    # List to store picks per game
    picks = []

    while True:
        # Our Pick
        pick = random.uniform(0, 1)
        picks.append(pick)

        # Following Game Rules
        if len(picks) >= 2 and picks[-1] <= picks[-2]:
            length = len(picks)
            game_length.append(length)
            break

# Calculating Expected Value
ev = sum(game_length) / trials

# Theoretical Answer
answer = math.e

# Outputting Results
print(f'Answer: {round(answer, 4)} | Simulated Expected Value: {round(ev, 4)}')
```

Output

```
Answer: 2.7183 | Simulated Expected Value: 2.7183
```