# Random Numbers

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*Abstract*—This manual provides a simple introduction to the generation of random numbers

## 1 UNIFORM RANDOM NUMBERS

Let $U$ be a uniform random variable between 0 and 1.

1.1 Generate $10^6$ samples of $U$ using a C program and save into a file called uni.dat .
**Solution:** Download the following files

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−probability/master/
    manual/codes/exrand.c
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−probability/master/
    manual/codes/coeffs.h
```

and compile and execute the C program using

```
$ gcc exrand.c −lm −Wall −g
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of $U$ using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \qquad (1.1)$$

**Solution:** The following code plots Fig. 1.2

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−probability/master/
    manual/codes/cdf_plot.py
```

It is executed with
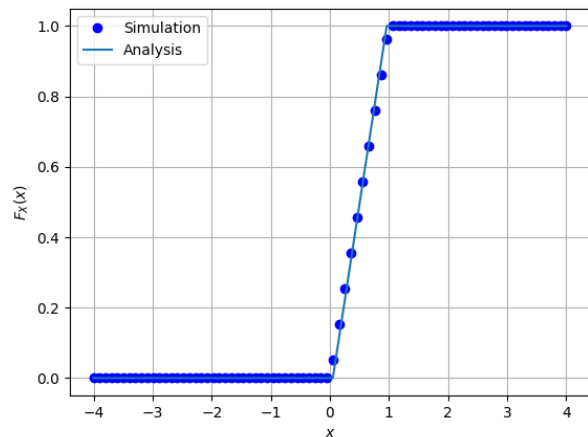
```
$ python3 cdf_plot.py
```



Fig. 1.2: The CDF of $U$

1.3 Find a theoretical expression for $F_U(x)$.
**Solution:** The CDF of $U$ is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(u)du \qquad (1.2)$$

We now have three cases:

a)  $x < 0$: $p_X(x) = 0$, and hence $F_U(x) = 0$.
b)  $0 \le x < 1$: Here,

$$F_U(x) = \int_{0}^{x} du = x \qquad (1.3)$$

c)  $x \ge 1$: Put $x = 1$ in (1.3) as $U$ is uniform in [0, 1] to get $F_U(x) = 1$.
Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} \qquad (1.4)$$

This is verified in Figure (1.2)

1.4 The mean of $U$ is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \qquad (1.5)$$

and its variance as

$$\text{var}\,[U] = E\,[U - E\,[U]]^2 \qquad (1.6)$$

Write a C program to find the mean and variance of $U$.

**Solution:** The C program can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−probability/master/
    manual/codes/mean_var_uni.c
```

and compiled and executed with

```
$ gcc mean_var_uni.c −lm −Wall −g
$ ./a.out
```

The calculated mean is 0.500007 and the calculated variance is 0.083301.

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)dx \qquad (1.7)$$

**Solution:** We write

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \qquad (1.8)$$

$$= \int_{-\infty}^{\infty} x^2 p_U(x)dx \qquad (1.9)$$

$$= \int_0^1 x^2 dx = \frac{1}{3} \qquad (1.10)$$

and

$$E\,[U] = \int_{-\infty}^{\infty} x dF_U(x) \qquad (1.11)$$

$$= \int_{-\infty}^{\infty} x p_U(x)dx \qquad (1.12)$$

$$= \int_0^1 x dx = \frac{1}{2} \qquad (1.13)$$

which checks out with the empirical mean on 0.500007. Now, using linearity of expectation,

$$\text{var}\,[U] = E\,[U - E\,[U]]^2 \qquad (1.14)$$

$$= E\left[U^2 - 2UE\,[U] + (E\,[U])^2\right] \qquad (1.15)$$

$$= E\left[U^2\right] - 2\,(E\,[U])^2 + (E\,[U])^2 \qquad (1.16)$$

$$= E\left[U^2\right] - (E\,[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \qquad (1.17)$$

$$\qquad (1.18)$$

and this checks out with the empirical variance 0.083301 of the sample data.
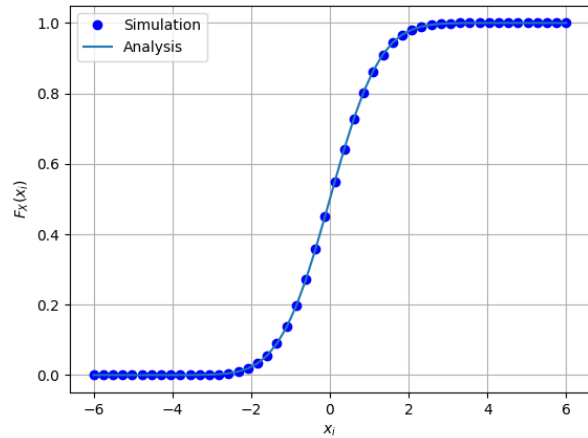


Fig. 2.2: The CDF of $X$

## 2  Central Limit Theorem

2.1 Generate $10^6$ samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \qquad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

**Solution:** The sample data is generated by the C file in Question 1.1

2.2 Load gau.dat in python and plot the empirical CDF of $X$ using the samples in gau.dat. What properties does a CDF have?

**Solution:** The CDF of $X$ is plotted in Fig. 2.2 The required python file can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−probability/master/
    manual/codes/cdf_gauss_plot.py
```

and executed using

```
$ python3 cdf_gauss_plot.py
```

a) The CDF is non-decreasing
b) It is right-continuous.
c) $\lim_{x \to -\infty} F_X(x) = 0$
d) $\lim_{x \to \infty} F_X(x) = 1$

2.3 Load gau.dat in python and plot the empirical PDF of $X$ using the samples in gau.dat. The
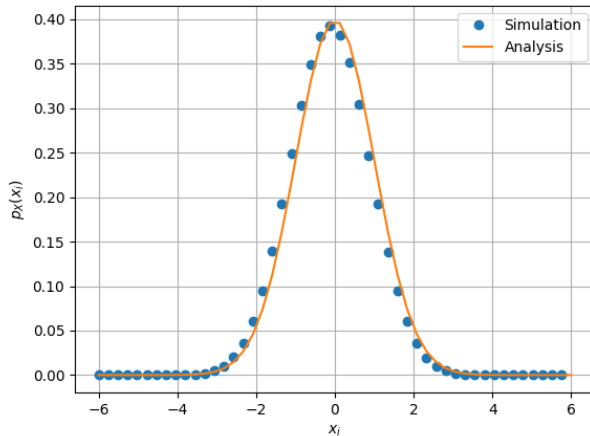
Fig. 2.3: The PDF of $X$

PDF of $X$ is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \qquad (2.2)$$

What properties does the PDF have?
**Solution:** The PDF of $X$ is plotted in Fig. 2.3 using the code below

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/pdf_plot.py
```

The figure is generated using

```
$ python pdf_plot.py
```

The properties of a PDF $p_X(x)$ are as follows:
a) $\forall x \in \mathbb{R},\ p_X(x) \geq 0$
b) $\int_{-\infty}^{\infty} p_X(x)dx = 1$
c) For $a < b,\ a,b \in \mathbb{R}$

$$\Pr(a < X < b) = \Pr(a \leq X \leq b) \qquad (2.3)$$

$$= \int_{a}^{b} p_X(x)dx \qquad (2.4)$$

If we take $a = b$, then we get $\Pr(X = a) = 0$.

2.4 Find the mean and variance of $X$ by writing a C program.
**Solution:** The mean and variance have been calculated using (1.5) and (1.6) respectively. The C program can be downloaded using

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−assignments/master/
    manual/codes/mean_var_gau.c
```

and compiled and executed with the following commands

```
$ gcc mean_var_gau.c −lm −Wall −g
$ ./a.out
```

The calculated mean is 0.000326 and the calculated variance is 1.000906.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.
**Solution:** The mean is given by

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) = 0 \qquad (2.6)$$

as the integrand is odd. This checks out with the empirical mean of 0.000326. The variance is given by

$$\text{var}[X] = E\left[X^2\right] - (E[X])^2 \qquad (2.7)$$

$$= \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)dx \qquad (2.8)$$

$$= \int_{0}^{\infty} \frac{2}{\sqrt{2\pi}} \sqrt{2}te^{-t}dt \qquad (2.9)$$

$$= \frac{2}{\sqrt{\pi}}\Gamma\left(\frac{3}{2}\right) \qquad (2.10)$$

$$= \frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) = 1 \qquad (2.11)$$

where we have used $t = \frac{x^2}{2}$ and so $dt = xdx$. We have also used the gamma function given as

$$\Gamma(n) = \int_{-\infty}^{\infty} x^{n-1}e^{-x}dx \qquad (2.12)$$

$$\Gamma(n) = (n-1)\Gamma(n-1) \text{ for } n > 1 \qquad (2.13)$$

and the fact that $\Gamma(1/2) = \sqrt{\pi}$. This agrees with the empirical variance of 1.000906.

## 3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \qquad (3.1)$$

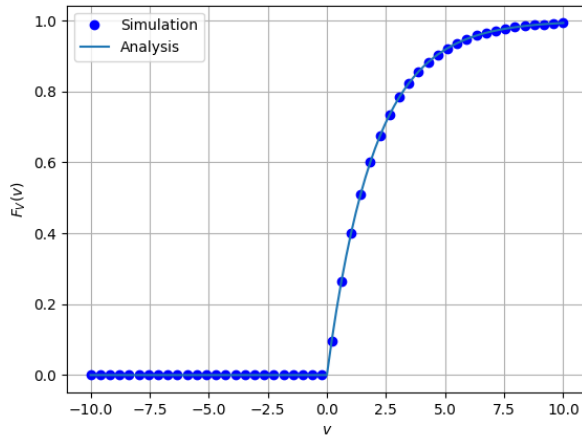and plot its CDF.
**Solution:** The relevant python code is at

Fig. 3.1: The CDF of $V$

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−probability/master/
    manual/codes/cdf_exp_plot.py
```

and can be executed with

```
$ python3 cdf_exp_plot.py
```

and the CDF is plotted in Figure (3.1).

3.2 Find a theoretical expression for $F_V(x)$.

**Solution:** Note that the function

$$v = f(u) = -2\ln(1-u) \qquad (3.2)$$

is monotonically increasing in $[0, 1]$ and $v \in \mathbb{R}^+$. Hence, it is invertible and the inverse function is given by

$$u = f^{-1}(v) = 1 - \exp\left(-\frac{v}{2}\right) \qquad (3.3)$$

Therefore, from the monotonicity of $v$, and using (1.4),

$$F_V(v) = F_U\left(1 - \exp\left(-\frac{v}{2}\right)\right) \qquad (3.4)$$

$$\implies F_V(v) = \begin{cases} 0 & v < 0 \\ 1 - \exp\left(-\frac{v}{2}\right) & v \geq 0 \end{cases} \qquad (3.5)$$