Assignment

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Abstract—**This manual provides a simple introduction to the generation of random numbers**

1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files

```
$ wget https://github.com/Prabhav11004/
    AI1110−ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/exrand.c
$ wget https://github.com/Prabhav11004/
    AI1110−ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/coeffs.h
```

and compile and execute the C program using

```
$ gcc exrand.c −lm
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \qquad (1.1)$$

Solution: The following code plots Fig. 1.2

```
$ wget https://github.com/Prabhav11004/
    AI1110−ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/cdf_plot.py
```

It is executed with
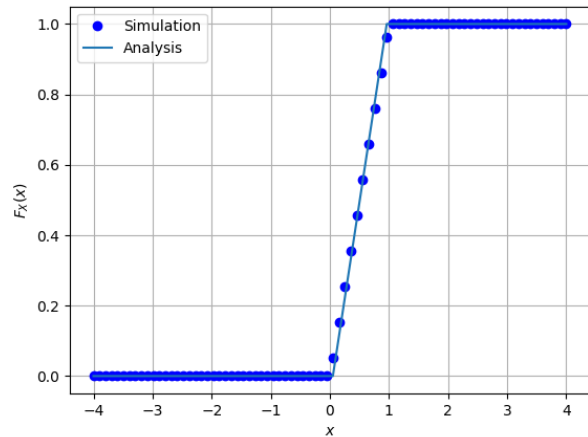
```
$ python3 cdf_plot.py
```


Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \le x) = \int_{-\infty}^{x} p_U(u)du \qquad (1.2)$$

We now have three cases:

a) $x < 0$: $p_X(x) = 0$, and hence $F_U(x) = 0$.
b) $0 \le x < 1$: Here,

$$F_U(x) = \int_0^x du = x \qquad (1.3)$$

c) $x \ge 1$: Put $x = 1$ in (1.3) as U is uniform in [0, 1] to get $F_U(x) = 1$.
Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases} \qquad (1.4)$$

This is verified in Figure (1.2)

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \qquad (1.5)$$

and its variance as

$$\text{var}\,[U] = E\,[U - E\,[U]]^2 \qquad (1.6)$$

Write a C program to find the mean and variance of U.

Solution: The C program can be downloaded using

```
$ wget https://github.com/Prabhav11004/
    AI1110−ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/mean_var.c
$ wget https://github.com/Prabhav11004/
    AI1110−ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/mean_var.h
```

and compiled and executed with

```
$ gcc mean_var.c −lm
$ ./a.out
```

The calculated mean is 0.500007 and the calculated variance is 0.083301.

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)dx \qquad (1.7)$$

Solution: We write

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \qquad (1.8)$$

$$= \int_{-\infty}^{\infty} x^2 p_U(x)dx \qquad (1.9)$$

$$= \int_{0}^{1} x^2 dx = \frac{1}{3} \qquad (1.10)$$

and

$$E\,[U] = \int_{-\infty}^{\infty} x dF_U(x) \qquad (1.11)$$

$$= \int_{-\infty}^{\infty} x p_U(x)dx \qquad (1.12)$$

$$= \int_{0}^{1} x dx = \frac{1}{2} \qquad (1.13)$$

which checks out with the empirical mean on 0.500007. Now, using linearity of expectation,

$$\text{var}\,[U] = E\,[U - E\,[U]]^2 \qquad (1.14)$$

$$= E\left[U^2 - 2UE\,[U] + (E\,[U])^2\right] \qquad (1.15)$$

$$= E\left[U^2\right] - 2\,(E\,[U])^2 + (E\,[U])^2 \qquad (1.16)$$

$$= E\left[U^2\right] - (E\,[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \qquad (1.17)$$
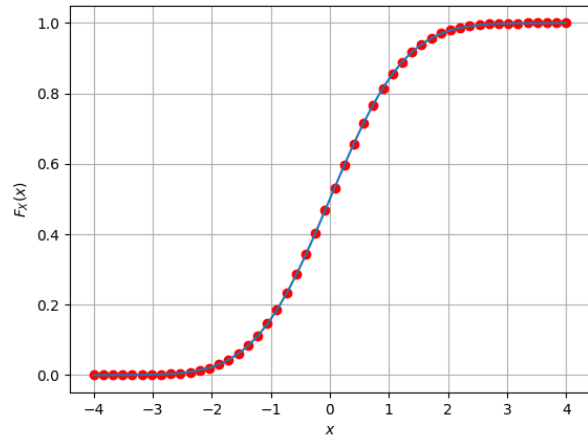
$$(1.18)$$

Fig. 2.2: The CDF of X

and this checks out with the empirical variance 0.083301 of the sample data.

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \qquad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
$ wget https://github.com/Prabhav11004/
    AI1110−ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/exrand.c
$ wget https://github.com/Prabhav11004/
    AI1110−ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/coeffs.h
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2 The required python file can be downloaded using

```
$ wget https://github.com/Prabhav11004/
    AI1110−ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/q2.2
    _cdf_gau_plot.py
```
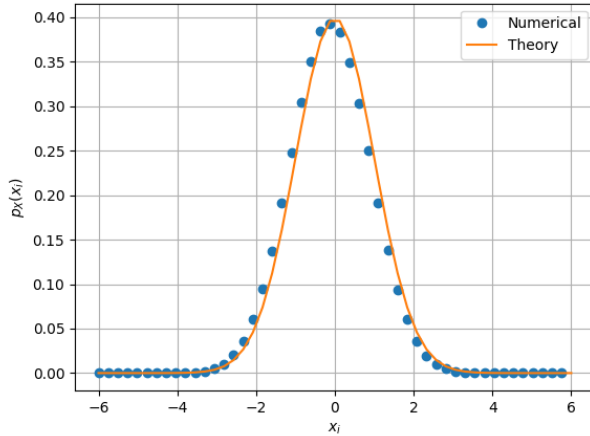
Fig. 2.3: The PDF of X

and executed using

```
$ python3 q2.2_cdf_gauss_plot.py
```

a) The CDF is non-decreasing
b) It is right-continuous.
c) $\lim_{x \to -\infty} F_X(x) = 0$
d) $\lim_{x \to \infty} F_X(x) = 1$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \qquad (2.2)$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
$ wget https://github.com/Prabhav11004/
    AI1110−ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/pdf_plot.py
```

The figure is generated using

```
$ python pdf_plot.py
```

The properties of a PDF $p_X(x)$ are as follows:
a) $\forall x \in \mathbb{R}, \ p_X(x) \geq 0$
b) $\int_{-\infty}^{\infty} p_X(x) dx = 1$
c) For $a < b, \ a, b \in \mathbb{R}$

$$\Pr(a < X < b) = \Pr(a \leq X \leq b) \qquad (2.3)$$

$$= \int_a^b p_X(x) dx \qquad (2.4)$$

If we take $a = b$, then we get $\Pr(X = a) = 0$.

2.4 Find the mean and variance of X by writing a C program.
Solution: The mean and variance have been calculated using (1.5) and (1.6) respectively. The C program can be downloaded using

```
$ wget https://github.com/Prabhav11004/
    AI1110−ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/q2.4
    _mean_var.c
$ wget https://github.com/Prabhav11004/
    AI1110−ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/mean_var.h
```

and compiled and executed with the following commands

```
$ gcc q2.4_mean_var.c −lm
$ ./a.out
```

The calculated mean is 0.000326 and the calculated variance is 1.000906.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.5)$$

repeat the above exercise theoretically.
Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \qquad (2.6)$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \qquad (2.7)$$

Now, let

$$g(x) = \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \qquad (2.8)$$

$$\implies g(-x) = \frac{-x}{\sqrt{2\pi}} \exp\left(-\frac{(-x)^2}{2}\right) \qquad (2.9)$$

$$= -\frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \qquad (2.10)$$

$$= -g(x) \qquad (2.11)$$

Thus, $g(x)$ is an odd function

$$\therefore E[X] = \int_{-\infty}^{\infty} g(x) dx = 0 \qquad (2.12)$$

Now,

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x) \mathrm{d}x \qquad (2.13)$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \qquad (2.14)$$

$$= 2 \int_{0}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \qquad (2.15)$$

since $\frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is an even function
Using integration by parts,

$$E\left[X^2\right] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \qquad (2.16)$$

$$= \sqrt{\frac{2}{\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \right)\Big|_{0}^{\infty}$$

$$- \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \qquad (2.17)$$

Substitute $t = \frac{x^2}{2} \implies \mathrm{d}t = x \mathrm{d}x$

$$\int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x = \int \exp(-t) \mathrm{d}t \qquad (2.18)$$

$$= -\exp(-t) \qquad (2.19)$$

$$= -\exp\left(-\frac{x^2}{2}\right) \qquad (2.20)$$

Now,

$$-x \exp\left(-\frac{x^2}{2}\right)\Big|_{0}^{\infty} = 0 - 0 = 0 \qquad (2.21)$$

$$\because \lim_{x\to\infty} x \exp\left(-\frac{x^2}{2}\right) = \lim_{x\to\infty} \frac{x}{\exp\left(\frac{x^2}{2}\right)} = 0 \qquad (2.22)$$

as exponential function grows much faster than a polynomial function

Also,

$$\int_{0}^{\infty} -\exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \qquad (2.23)$$

$$\xleftrightarrow{x=t\sqrt{2}} \int_{0}^{\infty} -\exp(-t^2) \mathrm{d}t \sqrt{2} \qquad (2.24)$$

$$= -\sqrt{2} \int_{0}^{\infty} \exp(-t^2) \mathrm{d}t \qquad (2.25)$$

$$= -\sqrt{2}\frac{\sqrt{\pi}}{2} \qquad (2.26)$$

$$= -\sqrt{\frac{\pi}{2}} \qquad (2.27)$$

Therefore,

$$E\left[X^2\right] = 0 - \sqrt{\frac{2}{\pi}}\left(-\sqrt{\frac{\pi}{2}}\right) \qquad (2.28)$$

$$= 1 \qquad (2.29)$$

$$\therefore \operatorname{Var}[X] = E\left[X^2\right] - (E[X])^2 \qquad (2.30)$$

$$= 1 - 0 \qquad (2.31)$$

$$= 1 \qquad (2.32)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2\ln(1 - U) \qquad (3.1)$$

and plot its CDF.
Solution: The relevant python code is at

```
$ wget https://raw.githubusercontent.com/
    goats−9/ai1110−probability/master/
    manual/codes/cdf_exp_plot.py
```

and can be executed with

```
$ python3 cdf_exp_plot.py
```

and the CDF is plotted in Figure (3.1).
3.2 Find a theoretical expression for $F_V(x)$.
Solution: We have:

$$F_V(x) = \Pr(V \le x) \qquad (3.2)$$

$$= \Pr(-2\ln(1 - U) \le x) \qquad (3.3)$$

$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \qquad (3.4)$$

$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \qquad (3.5)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \qquad (3.6)$$
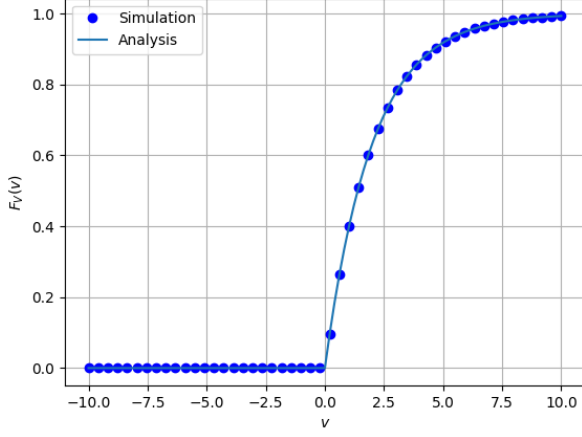
Fig. 3.1: The CDF of V

Therefore,

$$F_V(x) = \begin{cases} 0, & 1 - \exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & 1 - \exp\left(-\frac{x}{2}\right) \in (0, 1) \\ 1, & 1 - \exp\left(-\frac{x}{2}\right) \in (1, \infty) \end{cases}$$

$$(3.7)$$

From this we get:

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \quad (3.8)$$