Assignment 11

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Question : We have shown that if X is a random variable with distribution $F(x)$, then the random variable $y = F(x)$ is uniform in the interval $(0, 1)$. The following is a generalization.
Given n arbitrary random variables x_i; we form the random variables $y_1 = F(x_1), y_2 = F(x_2|x_1), ..., y_n = F(x_n|x_{n-1}, ..., x_1)$. We shall show that these random variables are independent and each is uniform in the interval $(0, 1)$.

Solution :
The random variables y_1 are functions of the random variables x_i as assumed in initial stage of the problem ,For $0 \leq y \leq 1$, the system
$y_1 = F(x_1), y_2 = F(x_2|x_1), ..., y_n = F(x_n|x_{n-1}, ..., x_1)$
has a unique solution $x_1, x_2, x_3, ..., x_n$ and its jacobian equals

$$J = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & 0 & 0..... & 0 \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & 0..... & 0 \\ . & . & .... & . \\ \frac{\partial y_n}{\partial x_n} & . & .... & \frac{\partial y_1}{\partial x_1} \end{vmatrix}$$

This determinant is triangular; hence it equals the product of its diagonal elements

$$\frac{\partial y_k}{\partial x_k} = f(x_k|x_{k-1}, ..., x_1)$$

But we know by definition of conditional probability,

$$f(x_1, ..., x_k) = f(x_k|x_{k-1}, ..., x_1)......f(x_2|x_1).f(x_1) \tag{1}$$

So we have

$$f(y_1, ..., y_n) = \frac{f(y_1, ..., y_n)}{f(x_n|x_{n-1}, ..., x_1)......f(x_2|x_1).f(x_1)} = 1 \tag{2}$$

Hence it is proved that in the n-dimensional cube $0 \leq y_i \leq 1$, and 0 otherwise.