Assignment

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Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files

```
$ wget https://github.com/Prabhav11004/
    AI1110-ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/exrand.c
$ wget https://github.com/Prabhav11004/
    AI1110-ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/coeffs.h
```

and compile and execute the C program using

```
$ gcc exrand.c -lm
$ ./a.out
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
$ wget https://github.com/Prabhav11004/
    AI1110-ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/cdf_plot.py
```

It is executed with

```
$ python3 cdf_plot.py
```

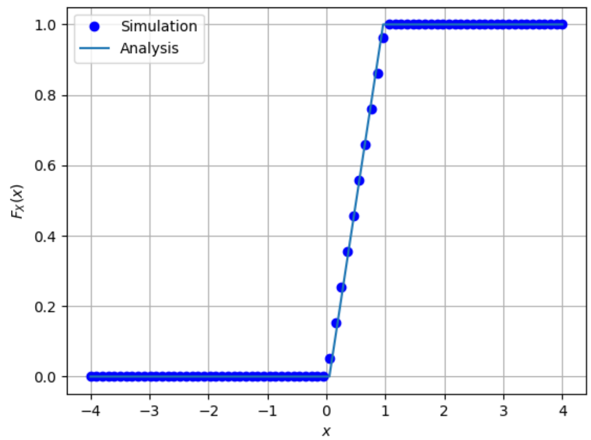

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The CDF of U is given by

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^{x} p_U(u)du \tag{1.2}$$

We now have three cases:

a) $x < 0$: $p_X(x) = 0$, and hence $F_U(x) = 0$.
b) $0 \leq x < 1$: Here,

$$F_U(x) = \int_0^x du = x \tag{1.3}$$

c) $x \geq 1$: Put $x = 1$ in (1.3) as U is uniform in $[0, 1]$ to get $F_U(x) = 1$.

Therefore,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases} \tag{1.4}$$

This is verified in Figure (1.2)

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.5}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.6}$$

Write a C program to find the mean and variance of U.

Solution: The C program can be downloaded using

```
$ wget https://github.com/Prabhav11004/
    AI1110-ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/mean_var.c
$ wget https://github.com/Prabhav11004/
    AI1110-ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/mean_var.h
```

and compiled and executed with

```
$ gcc mean_var.c -lm
$ ./a.out
```

The calculated mean is 0.500007 and the calculated variance is 0.083301.

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) dx \tag{1.7}$$

Solution: We write

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dF_U(x) \tag{1.8}$$

$$= \int_{-\infty}^{\infty} x^2 p_U(x) dx \tag{1.9}$$

$$= \int_0^1 x^2 dx = \frac{1}{3} \tag{1.10}$$

and

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.11}$$

$$= \int_{-\infty}^{\infty} x p_U(x) dx \tag{1.12}$$

$$= \int_0^1 x dx = \frac{1}{2} \tag{1.13}$$

which checks out with the empirical mean on 0.500007. Now, using linearity of expectation,

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.14}$$

$$= E\left[U^2 - 2UE[U] + (E[U])^2\right] \tag{1.15}$$

$$= E\left[U^2\right] - 2(E[U])^2 + (E[U])^2 \tag{1.16}$$

$$= E\left[U^2\right] - (E[U])^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12} \tag{1.17}$$

$$\tag{1.18}$$

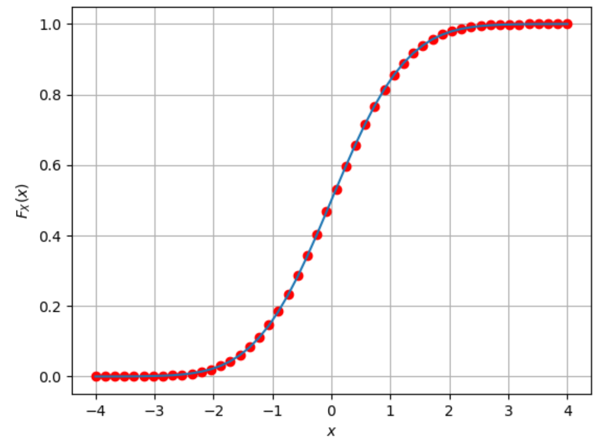

Fig. 2.2: The CDF of X

and this checks out with the empirical variance 0.083301 of the sample data.

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following files and execute the C program.

```
$ wget https://github.com/Prabhav11004/
    AI1110-ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/exrand.c
$ wget https://github.com/Prabhav11004/
    AI1110-ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/coeffs.h
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2 The required python file can be downloaded using

```
$ wget https://github.com/Prabhav11004/
    AI1110-ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/q2.2
    _cdf_gau_plot.py
```

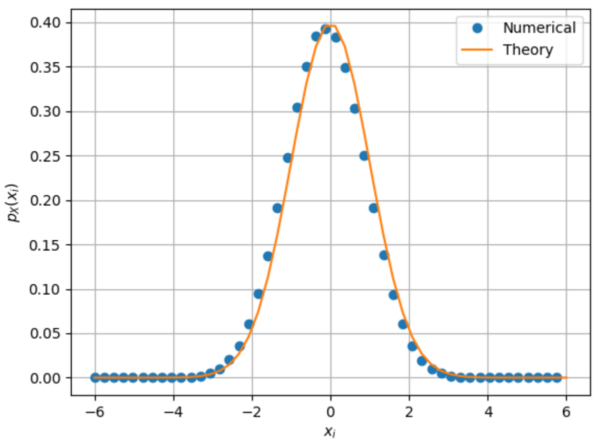

Fig. 2.3: The PDF of X

and executed using

```
$ python3 q2.2_cdf_gauss_plot.py
```

a) The CDF is non-decreasing
b) It is right-continuous.
c) $\lim_{x\to-\infty} F_X(x) = 0$
d) $\lim_{x\to\infty} F_X(x) = 1$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
$ wget https://github.com/Prabhav11004/
    AI1110-ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/pdf_plot.py
```

The figure is generated using

```
$ python pdf_plot.py
```

The properties of a PDF $p_X(x)$ are as follows:
a) $\forall x \in \mathbb{R},\ p_X(x) \geq 0$
b) $\int_{-\infty}^{\infty} p_X(x)dx = 1$
c) For $a < b,\ a, b \in \mathbb{R}$

$$\Pr(a < X < b) = \Pr(a \leq X \leq b) \tag{2.3}$$

$$= \int_a^b p_X(x)dx \tag{2.4}$$

If we take $a = b$, then we get $\Pr(X = a) = 0$.

2.4 Find the mean and variance of X by writing a C program.
Solution: The mean and variance have been calculated using (1.5) and (1.6) respectively. The C program can be downloaded using

```
$ wget https://github.com/Prabhav11004/
    AI1110-ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/q2.4
    _mean_var.c
$ wget https://github.com/Prabhav11004/
    AI1110-ASSIGNMENTS/blob/main/
    R_N_Assignment/Codes/mean_var.h
```

and compiled and executed with the following commands

```
$ gcc q2.4_mean_var.c -lm
$ ./a.out
```

The calculated mean is 0.000326 and the calculated variance is 1.000906.

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.5}$$

repeat the above exercise theoretically.
Solution: The mean of X is given by

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) \mathrm{d}x \tag{2.6}$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.7}$$

Now, let

$$g(x) = \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.8}$$

$$\implies g(-x) = \frac{-x}{\sqrt{2\pi}} \exp\left(-\frac{(-x)^2}{2}\right) \tag{2.9}$$

$$= -\frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.10}$$

$$= -g(x) \tag{2.11}$$

Thus, $g(x)$ is an odd function

$$\therefore E[X] = \int_{-\infty}^{\infty} g(x) \mathrm{d}x = 0 \tag{2.12}$$

Now,

$$E\left[X^2\right] = \int_{-\infty}^{\infty} x^2 p_X(x)\mathrm{d}x \tag{2.13}$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \tag{2.14}$$

$$= 2\int_{0}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \tag{2.15}$$

since $\frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)$ is an even function
Using integration by parts,

$$E\left[X^2\right] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \tag{2.16}$$

$$= \sqrt{\frac{2}{\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x\right)\Bigg|_0^{\infty} - \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \tag{2.17}$$

Substitute $t = \frac{x^2}{2} \implies \mathrm{d}t = x\mathrm{d}x$

$$\int x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x = \int \exp(-t)\mathrm{d}t \tag{2.18}$$

$$= -\exp(-t) \tag{2.19}$$

$$= -\exp\left(-\frac{x^2}{2}\right) \tag{2.20}$$

Now,

$$-x \exp\left(-\frac{x^2}{2}\right)\Bigg|_0^{\infty} = 0 - 0 = 0 \tag{2.21}$$

$$\because \lim_{x\to\infty} x \exp\left(-\frac{x^2}{2}\right) = \lim_{x\to\infty} \frac{x}{\exp\left(\frac{x^2}{2}\right)} = 0 \tag{2.22}$$

as exponential function grows much faster than a polynomial function

Also,

$$\int_{0}^{\infty} -\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \tag{2.23}$$

$$\xleftrightarrow{x=t\sqrt{2}} \int_{0}^{\infty} -\exp(-t^2)\mathrm{d}t\,\sqrt{2} \tag{2.24}$$

$$= -\sqrt{2}\int_{0}^{\infty} \exp(-t^2)\mathrm{d}t \tag{2.25}$$

$$= -\sqrt{2}\frac{\sqrt{\pi}}{2} \tag{2.26}$$

$$= -\sqrt{\frac{\pi}{2}} \tag{2.27}$$

Therefore,

$$E\left[X^2\right] = 0 - \sqrt{\frac{2}{\pi}}\left(-\sqrt{\frac{\pi}{2}}\right) \tag{2.28}$$

$$= 1 \tag{2.29}$$

$$\therefore \mathrm{Var}\left[X\right] = E\left[X^2\right] - \left(E\left[X\right]\right)^2 \tag{2.30}$$

$$= 1 - 0 \tag{2.31}$$

$$= 1 \tag{2.32}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln\left(1 - U\right) \tag{3.1}$$

and plot its CDF.
Solution: The relevant python and c codes are at at

```
$ wget https://raw.githubusercontent.com/
    goats-9/ai1110-probability/master/
    manual/codes/cdf_exp_plot.py
```

and can be executed with

```
$ python3 cdf_exp_plot.py
```

and the CDF is plotted in Figure (3.1).

3.2 Find a theoretical expression for $F_V(x)$.
Solution: We have:

$$F_V(x) = \Pr\left(V \le x\right) \tag{3.2}$$

$$= \Pr\left(-2\ln(1-U) \le x\right) \tag{3.3}$$

$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$

$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.6}$$

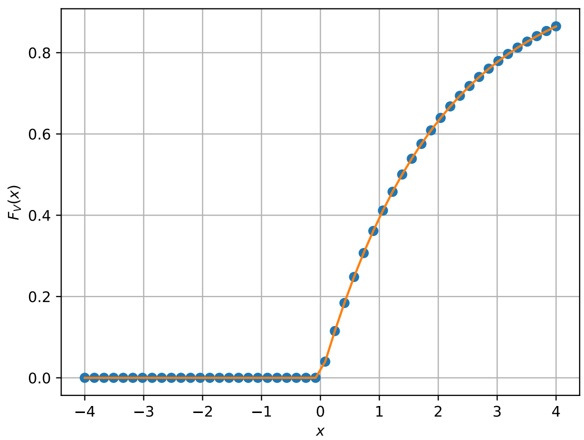

Fig. 3.1: The CDF of V

Therefore,

$$F_V(x) = \begin{cases} 0, & 1 - \exp\left(-\frac{x}{2}\right) \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & 1 - \exp\left(-\frac{x}{2}\right) \in (0, 1) \\ 1, & 1 - \exp\left(-\frac{x}{2}\right) \in (1, \infty) \end{cases} \tag{3.7}$$

From this we get:

$$F_V(x) = \begin{cases} 0, & x \in (-\infty, 0) \\ 1 - \exp\left(-\frac{x}{2}\right), & x \in (0, \infty) \end{cases} \tag{3.8}$$