Graph Algorithms:
Minimum Spanning Tree (Ch 23)

Minimum spanning tree

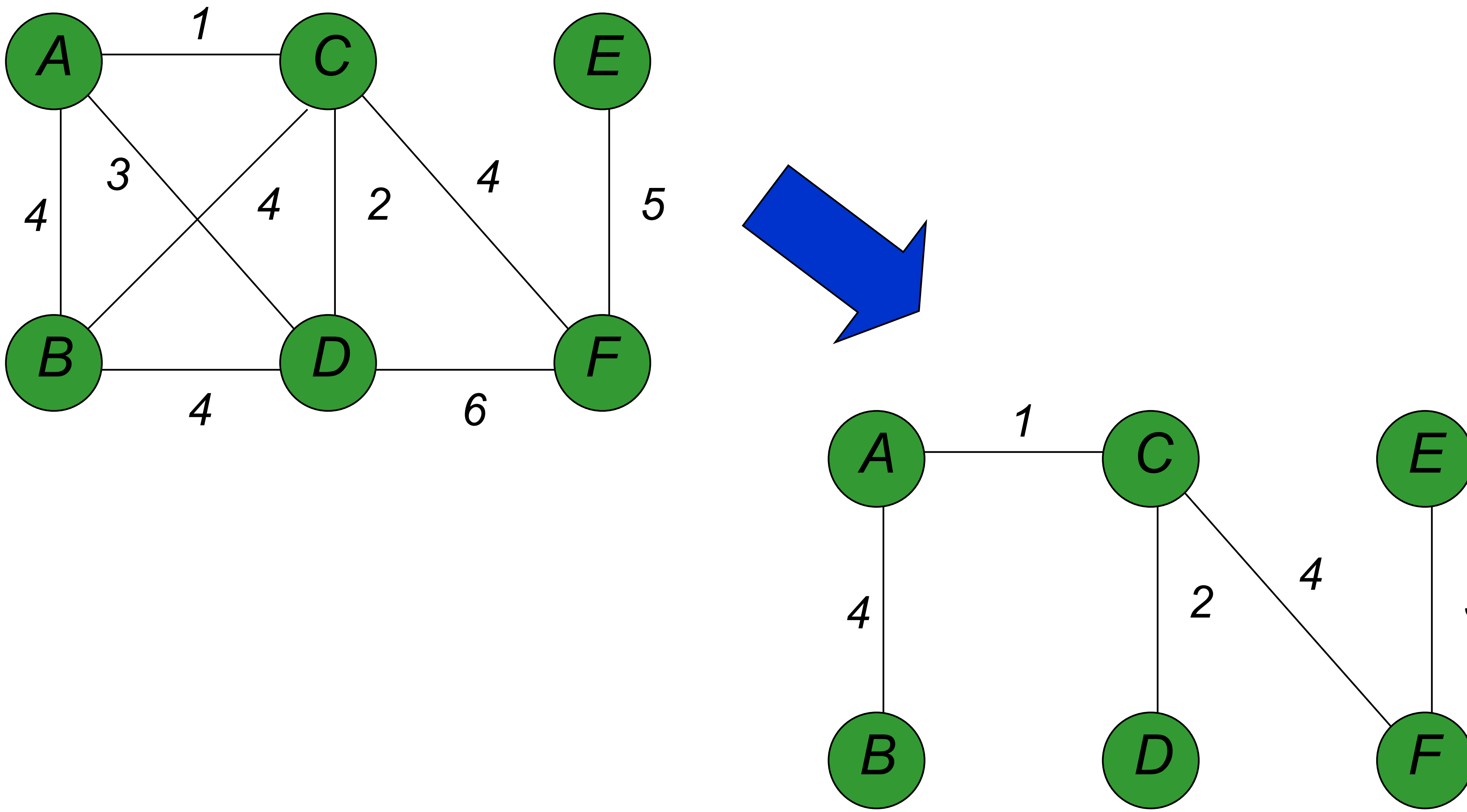
What is the lowest weight set of edges that connects all vertices of an undirected graph with positive weights

Input: An undirected, positive weight graph, G=(V,E)

Output: A tree T=(V,E') where E' $\subseteq$ E that minimizes

$$weight(T) = \sum_{e \in E'} w_e$$

MST example



Applications?

Connectivity

- Networks (e.g. communications)
- Circuit design/wiring

hub/spoke models (e.g. flights, transportation)

Traveling salesman problem?

MST construction: Kruskal's method

- create a forest F (a set of trees), where each node in the graph is a separate (trivial) tree.
- create a set S containing all the links in the network.
- while S is nonempty and F is not yet spanned
  - remove a link with minimum cost from S
  - if that link connects two different trees, then add it to the forest, combining two trees into a single tree

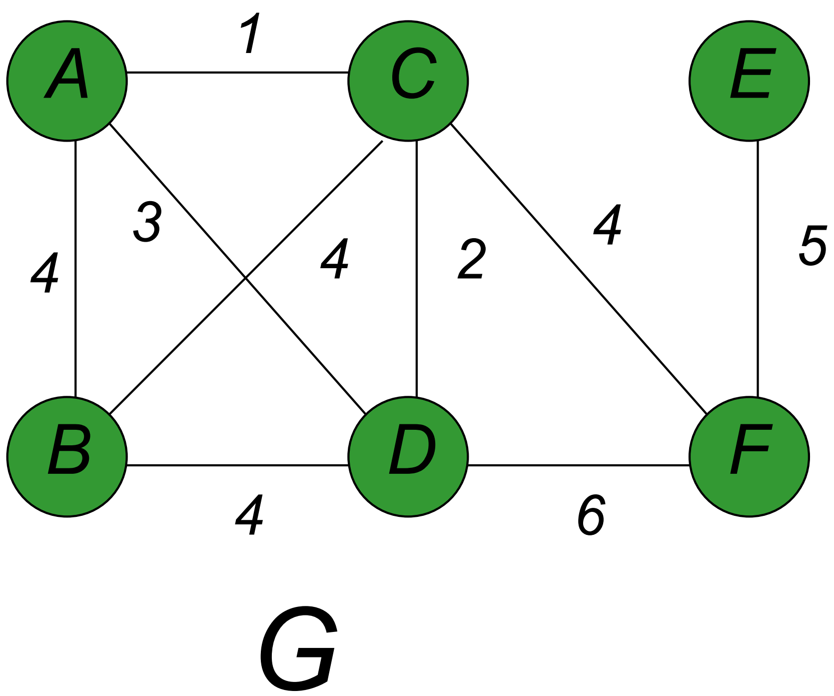
Greedy algorithm

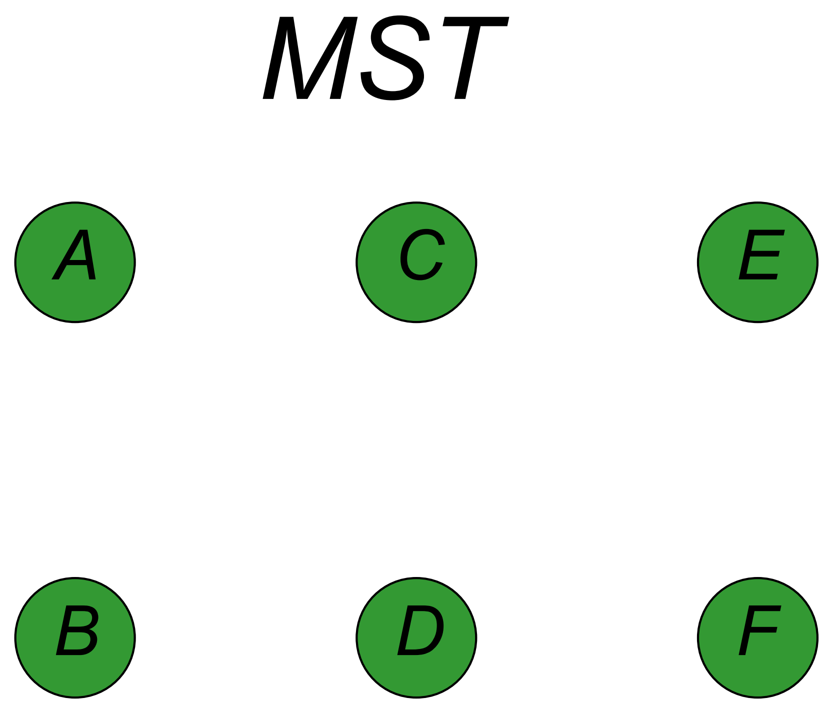
MST construction: Kruskal's method

```
MST-KRUSKAL(G, w)
A = Ø
for each vertex v ∈ G.V
    MAKE-SET(v)
sort the edges of G.E into nondecreasing order by weight w
for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
    if FIND-SET(u) ≠ FIND-SET(v)   // Test if u and v belong to same tree
        A = A ∪ {(u, v)}
        UNION(u, v)    // Combine two trees
return A
```

Kruskal's algorithm

Add smallest edge that connects two sets not already connected





Kruskal's algorithm

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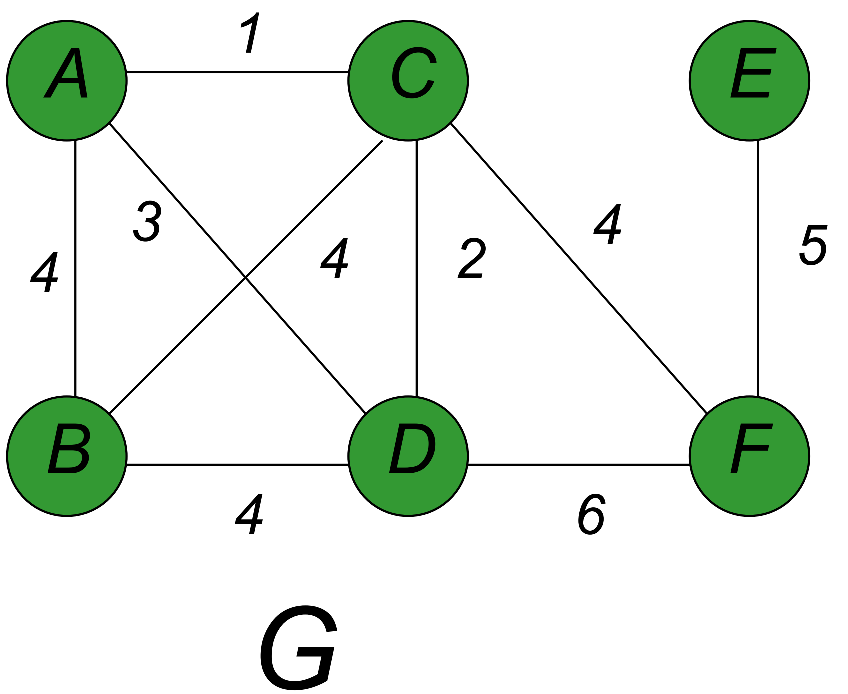

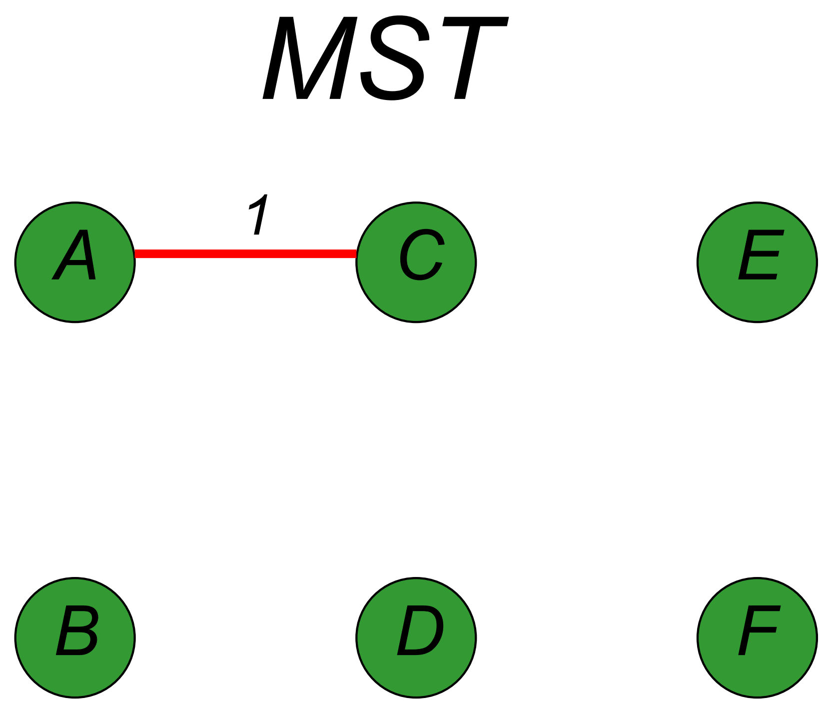

Kruskal's algorithm

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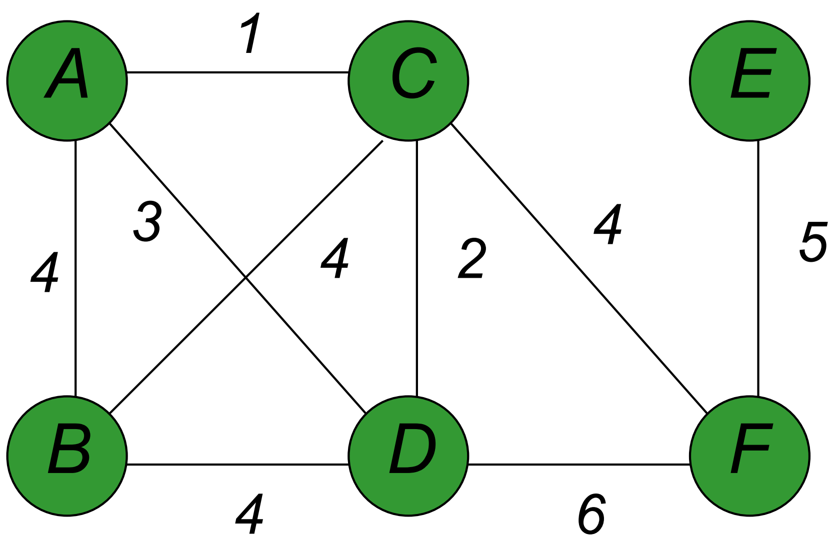
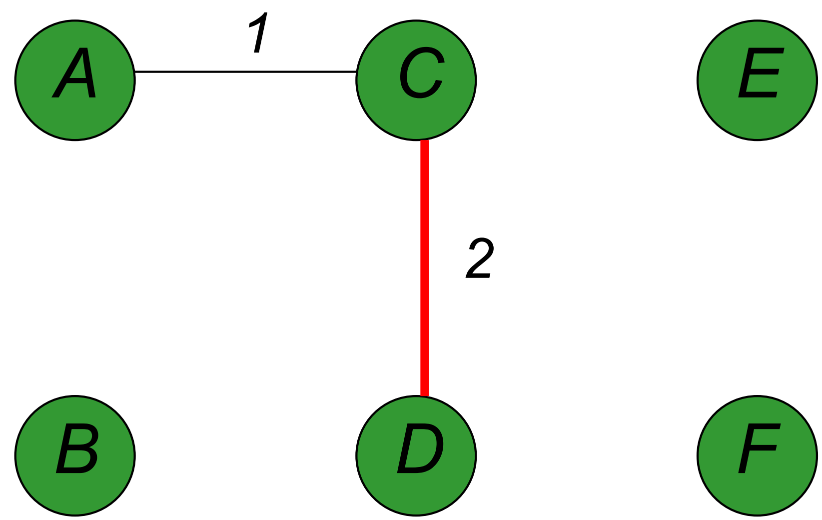


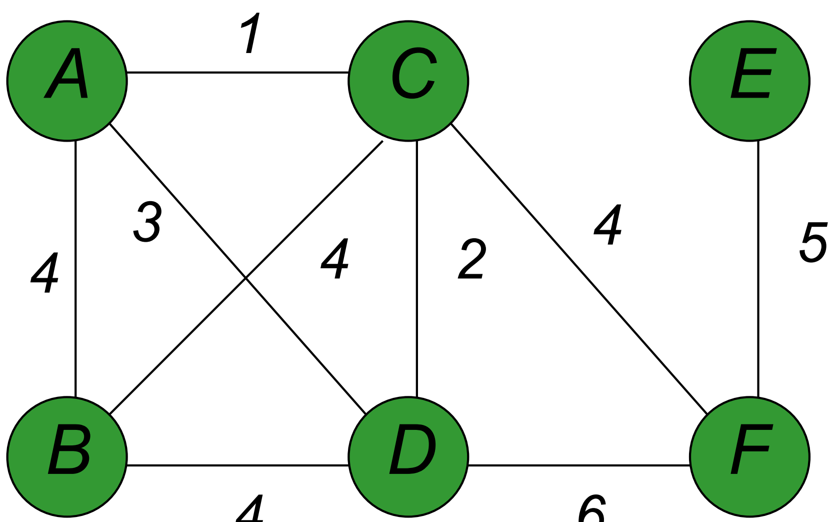
G

MST



Kruskal's algorithm

Add smallest edge that connects two sets not already connected



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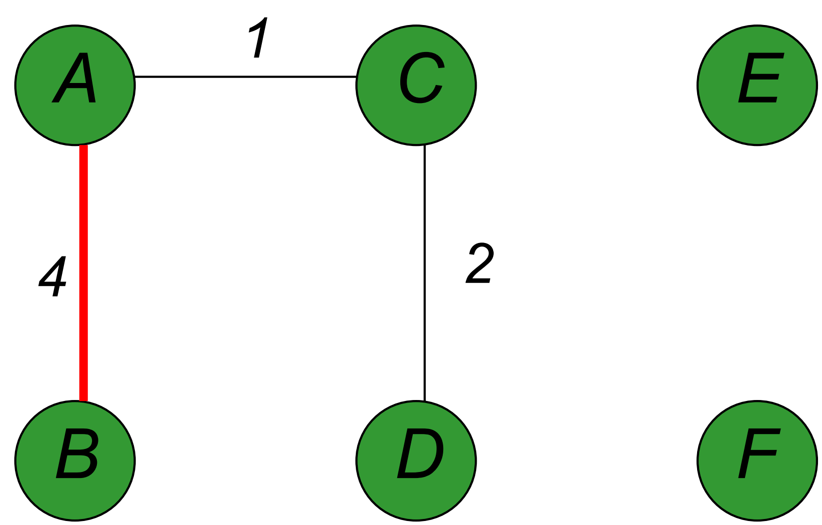
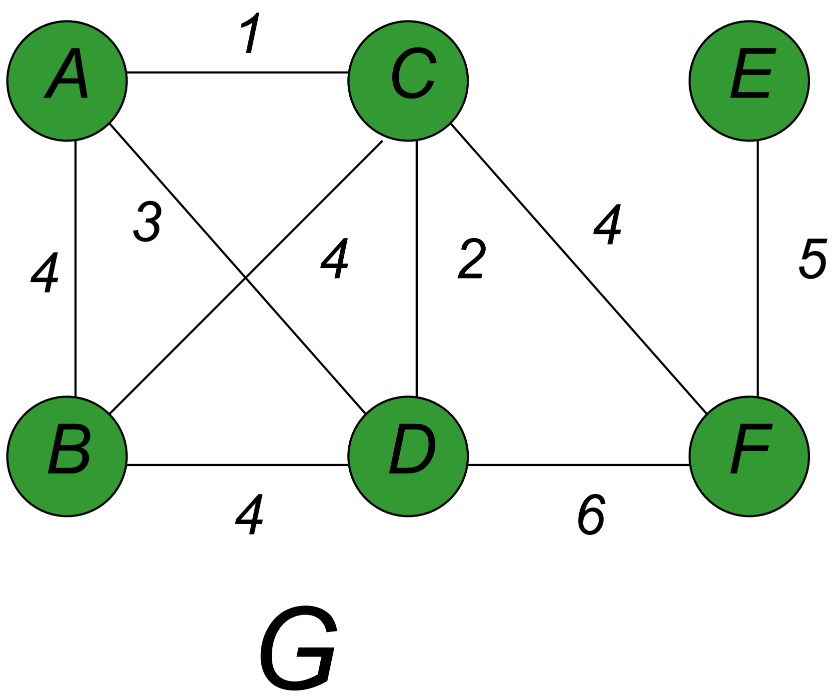
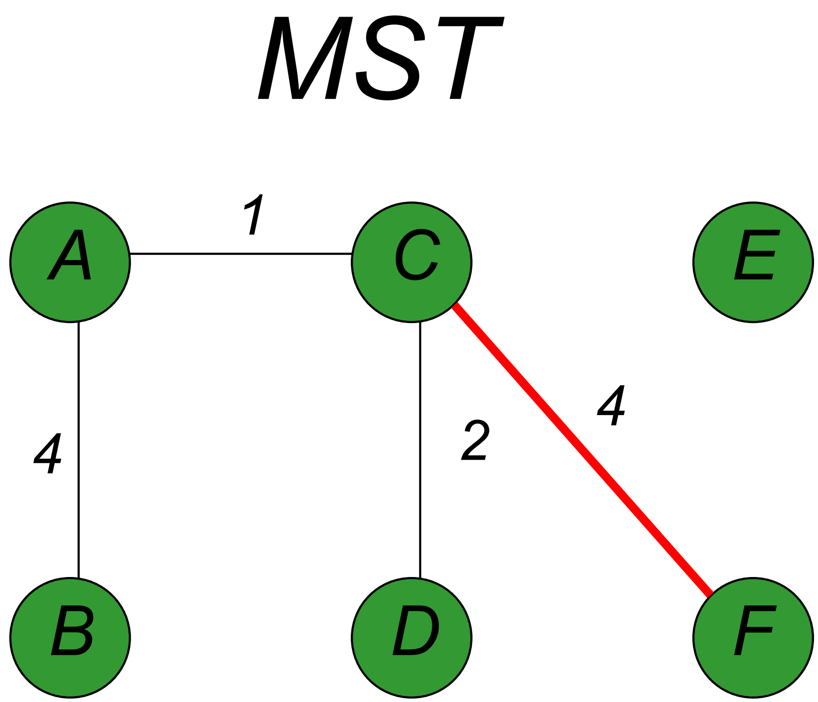


Kruskal’s algorithm

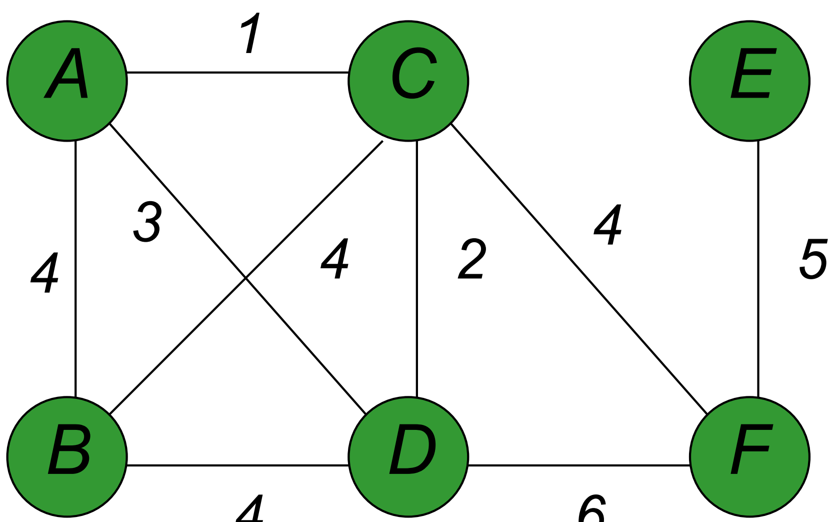
Add smallest edge that connects two sets not already connected





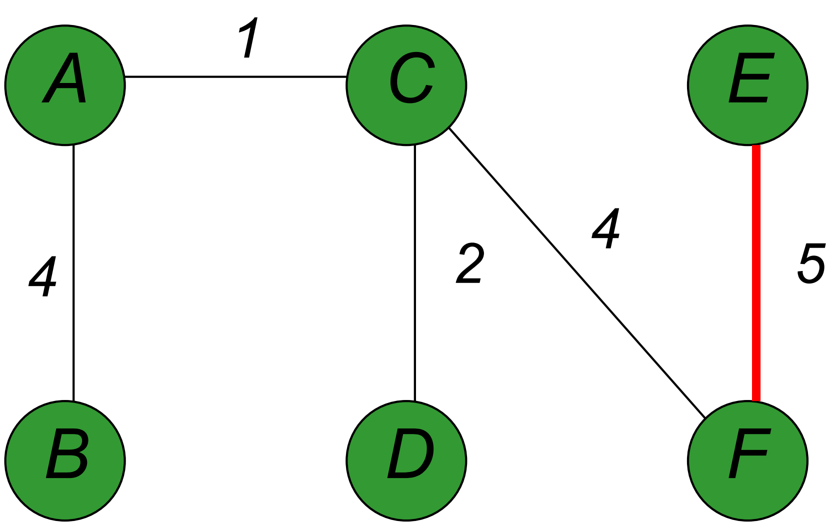
Kruskal's algorithm

Add smallest edge that connects two sets not already connected



G

MST



Kruskal Alg's Complexity

```
MST-KRUSKAL(G, w)
A = ∅
for each vertex v ∈ G.V
    MAKE-SET(v)
sort the edges of G.E into nondecreasing order by weight w
for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
    if FIND-SET(u) ≠ FIND-SET(v)      O(1)
        A = A ∪ {(u, v)}
        UNION(u, v)                   O(lg |V|) per iteration (amortized)
return A
```

Sorting: $O(|E| \lg |E|)$

For loop: $O(|E| \lg |E|)$

Thus, total time= $O(|E| \lg |E|)$

$= O(|E| \lg |V|)$ since $(\lg |E|) = O(\lg |V|)$ as $|E| < |V|^2$

MST construction: Prim's method

- Initialize a tree with a single node, chosen arbitrarily from the network.
- Grow the tree by one edge: of the edges that connect the tree to nodes not yet in the tree, find the minimum-cost link, and transfer it to the tree.
- Repeat step 2 (until all nodes are in the tree).

Greedy Algorithm

Prim's Algorithm

MST-PRIM(G, w, r)

for each $u \in G.V$
    $u.key = \infty$
    $u.\pi = \text{NIL}$ *Parent*
$r.key = 0$
$Q = G.V$ *Priority queue*
while $Q \neq \emptyset$
    $u = \text{EXTRACT-MIN}(Q)$
    for each $v \in G.Adj[u]$
        if $v \in Q$ and $w(u, v) < v.key$
            $v.\pi = u$
            $v.key = w(u, v)$

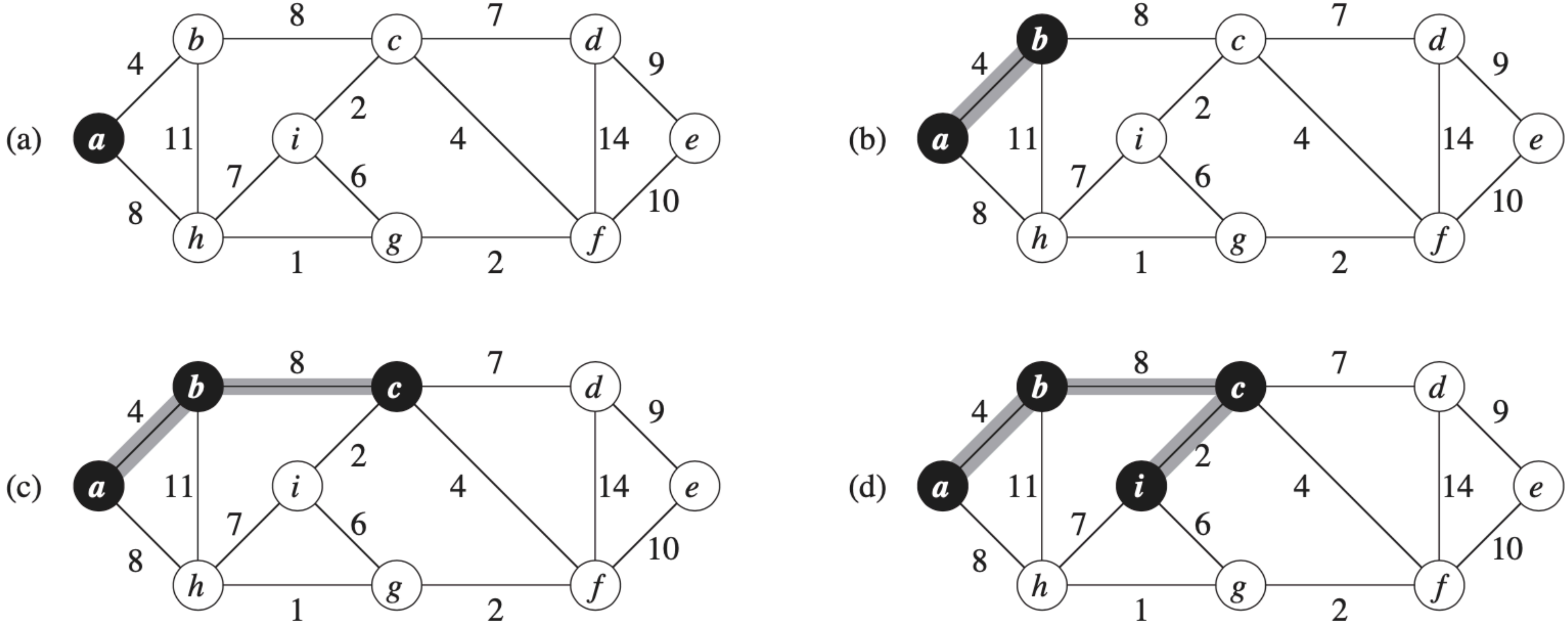
Set of vertices is partitioned into two subsets: V-Q and Q.

Set V-Q indicates current MST.

Extract u in Q incident on least weight edge between V-Q and Q. And add u in V-Q.

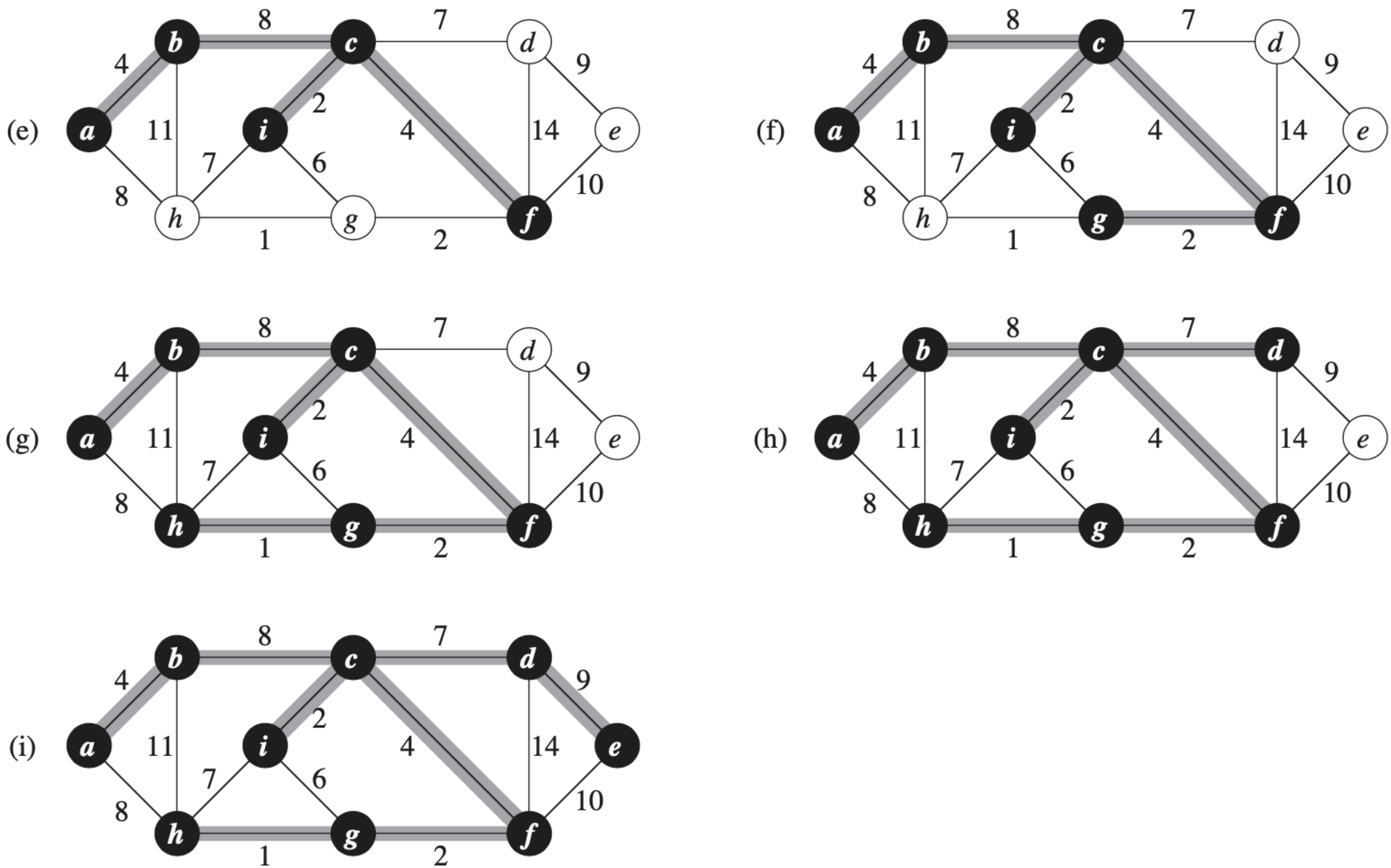
update the key and parent of each node v adjacent to u but not in the tree.

Prim's Algorithm



V-Q indicates current MST and is shaded in black.

Prim's Algorithm



Prim's Algorithm: Time Complexity

```
MST-PRIM(G, w, r)
1   for each u ∈ G.V
2       u.key = ∞
3       u.π = NIL
4   r.key = 0
5   Q = G.V
6   while Q ≠ ∅
7       u = EXTRACT-MIN(Q)
8       for each v ∈ G.Adj[u]
9           if v ∈ Q and w(u, v) < v.key
10              v.π = u
11              v.key = w(u, v)
```

Depends on how we implement priority queue

Buildheap:O(|V|)

|V| calls to Extract-Min, each of O(lg|V|)

lines 8–11 executes O(|E|) times altogether, since the sum of the lengths of all adjacency lists is 2|E|

Line 11 involves Decrease-Key: O(lg|V| for binary heap and O(1) for fibonacci heap.

Using binary heap, *time=* $O(|V| \lg |V|) + O(|E| \lg |V|) = O(|E| \lg |V|)$

Using fibonacci heap, *time=* $O(|V| \lg |V| + |E|)$

Prim's Algorithm: Correctness

- Straightforward using induction