HW #2: Solutions

Tuesday, March 13, 2018 8:41 AM

11.2-11

a) $H_0: Y \sim N\left(\overset{s_0}{[1, 1, \cdots, 1]^T}, \sigma^2 I\right)$

$H_1: Y \sim N\left(\underset{s_1}{[1, \cos(\tfrac{2\pi}{n}), \cos(\tfrac{4\pi}{n}), \cdots, \cos(2\pi \tfrac{n-1}{n})]^T}, \sigma^2 I\right)$

log-likelihood ratio:

$$l(y) = \frac{-1}{2\sigma^2}(y - s_1)^T(y - s_1) + \frac{1}{2\sigma^2}(y - s_0)^T(y - s_0)$$

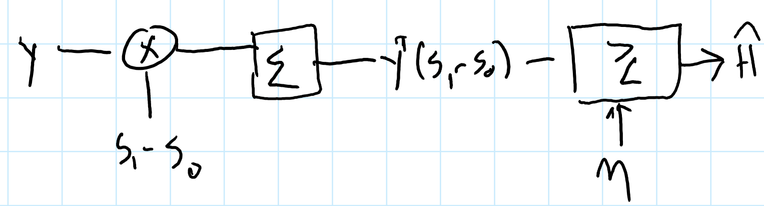
$$= \frac{1}{2\sigma^2}\left(-y^Ty + 2s_1^Ty - s_1^Ts_1 + y^Ty - 2s_0^Ty + s_0^Ts_0\right)$$

$$= \frac{1}{2\sigma^2}\left(2(s_1 - s_0)^Ty + \|s_0\|^2 - \|s_1\|^2\right)$$

So, the N-P detector is

$$\boxed{\delta(y) = \begin{cases} 0, & (s_1 - s_0)^T y < \eta \\ 1, & (s_1 - s_0)^T y \geq \eta \end{cases}}$$

b) Block diagram:



11.2-12

$$\Gamma(x) = \int_0^\infty \underset{(u)}{t^{x-1}} \, \underset{(dv)}{e^{-t}dt} \qquad v = -e^{-t}$$

$$du = (x-1)t^{x-2}dt$$

$$= -t^{x-1}e^{-t}\Big|_0^\infty + \int e^{-t}(x-1)t^{x-2}dt$$

$$= (x-1)\int_0^\infty t^{x-2}e^{-t}dt = \boxed{(x-1)\Gamma(x-1)}$$

$$= (x-1)\int_0^\infty t^{x-2} e^{-t} dt = \boxed{(x-1)\Gamma(x-1)}$$

11.2-15: a) $L(y) = \begin{cases} \frac{1}{2} e^{y}, & 0 \le y \le 2 \\ \infty, & y \ge 2 \end{cases}$

$$\ell(y) = \begin{cases} y - \log(2), & 0 \le y \le 2 \\ \infty, & y \ge 2 \end{cases}.$$

So:

$$\delta(y) = \begin{cases} 0, & y \le \eta + \log(2), \quad y \ge 2 \\ 1, & \eta + \log(2) \le y \le 2 \end{cases}.$$

b) $\pi_0 = 1/2$, $\eta = \log(\pi_0/\pi_1) = 0$

$$P_e = \tfrac{1}{2} P_r(\delta(y)=1 \mid H=0) + \tfrac{1}{2} P_r(\delta(y)=0 \mid H=1)$$

$$= \tfrac{1}{2}\left[\int_0^{\log(2)} e^{-y} dy + \int_2^\infty e^{-y} dy\right] + \frac{1}{2}\left(\frac{1}{2}(2-\log(2))\right)$$

$$= \frac{1}{2}\left(1/2 + 1/e^2\right) + \frac{1}{4}(2-\log(2)) = \boxed{0.644}$$

11.4-16:

a) $H_0: Y \sim \frac{1}{4}\exp(-|Y_1+1| - |Y_2+1|)$ $\quad H_1: Y \sim \frac{1}{4}\exp(-|Y_1-1| - |Y_2-1|)$

LLR: $\ell(y) = |Y_1+1| - |Y_2+1| - |Y_1-1| + |Y_2-1|.$

For equal priors, threshold is zero.

If $Y_1 \ge 1$, $Y_2 \le -1$, then:

$\ell(y) = -(Y_1+1) + (Y_2+1) + (Y_1-1) - (Y_2-1) = 0$ always, <u>either choice</u>.

If $Y_1 \le -1$, $Y_2 \ge 1$, then $\ell(y) = 0$ always, <u>either choice</u>

If $Y_1, Y_2 \ge 1$, $\ell(y) = (Y_1+1) - (Y_2+1) - (Y_1-1) + (Y_2-1)$
$= 4$, <u>choose H_1</u>

If $Y_1, Y_2 \le -1$, $\ell(y) = -4$, <u>choose H_0</u>.

If $y_1, y_2 \le -1$, $l(y) = -4$, choose H_0.

If $|y_1|, |y_2| \le 1$, $\boxed{\text{Need } y_2 \ge -y_1 \text{ for } l(y) \ge 0.}$

b) $$P_e = \frac{1}{2}\int_{y_2 \ge -y_1} p(y|H_0)\,dy + \frac{1}{2}\int_{y_2 \le -y_1} p(y|H_1)\,dy$$

$$= \frac{1}{8}\int_{-\infty}^{\infty}\int_{-y_1}^{\infty} \exp(-|y_1+1| - |y_2+1|)\,dy_2\,dy_1$$

$$+ \frac{1}{8}\int_{-\infty}^{\infty}\int_{-\infty}^{-y_1} \exp(-|y_1-1| - |y_2-1|)\,dy_2\,dy_1$$

$$= \frac{1}{8}\cdot\frac{4}{e^2} + \frac{1}{8}\frac{4}{e^2} = \boxed{\frac{1}{e^2} \approx 0.13}$$