ECE 7995: Detection and Estimation
Midterm Exam, Winter 2018

Instructions: Answer the following questions, showing your work. Open book/notes, open calculator. Partial credit will be granted where appropriate, and no points will be granted for answers without justification. You have four hours to complete the exam. These four hours can be any contiguous four-hour block between when the exam is handed out and the start of class on Monday, March 19th. Write your start and end time (and date) in the space below as you begin and finish the exam.

Name:
Start Time:
Finish Time:

1. (20 points) Let $X \in \{1, 2, 3\}$ and $Y \in \{1, 2\}$ be discrete random variables, and let their joint probability mass function be given by the matrix

$$p(x, y) = \begin{array}{cc} & Y \\ & \begin{bmatrix} 0.1 & 0.2 \\ 0 & 0.1 \\ 0.2 & 0.3 \end{bmatrix} \end{array}. \; X$$

Find the marginal distributions $p(x)$ and $p(y)$ and the conditional distributions $p(x|y)$ and $p(y|x)$.

(mistake in problem: doesn't sum to one)

$$p(x) = \sum_y p(x,y) = \begin{bmatrix} 0.3 \\ 0.1 \\ 0.5 \end{bmatrix}$$

$$p(y) = \sum_x p(x,y) = [0.3, 0.6]$$

$$p(x|y) = \frac{p(x,y)}{p(y)} = \begin{bmatrix} 0.1/0.3 & 0.2/0.6 \\ 0 & 0.1/0.6 \\ 0.2/0.3 & 0.3/0.6 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 \\ 0 & 1/6 \\ 2/3 & 1/2 \end{bmatrix}$$

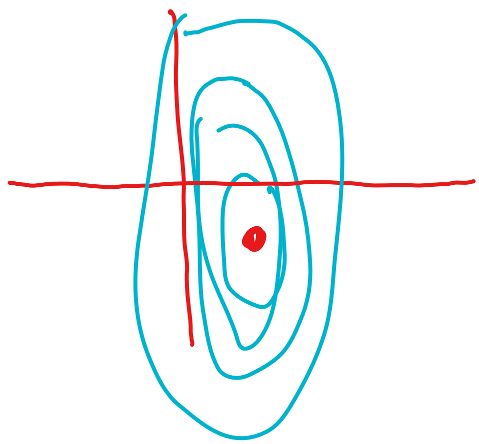
$$p(y|x) = \frac{p(x,y)}{p(x)} = \begin{bmatrix} 0.1/0.3 & 0.2/0.3 \\ 0 & 0.1/0.1 \\ 0.2/0.5 & 0.3/0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 2/3 \\ 0 & 1 \\ 2/5 & 3/5 \end{bmatrix}$$

2. (20 points) Let $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be a Gaussian random vector with mean and covariance

$$\mu = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \Sigma = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

(a) (5 points) Sketch the contour plot of the distribution $p(\mathbf{x})$.



(b) (15 points) Consider the random vector formed by the matrix multiplication $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}.$$

What is the distribution of $\mathbf{y}$?

y is Gaussian, with mean:

$$\mu_y = A \cdot \mu = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

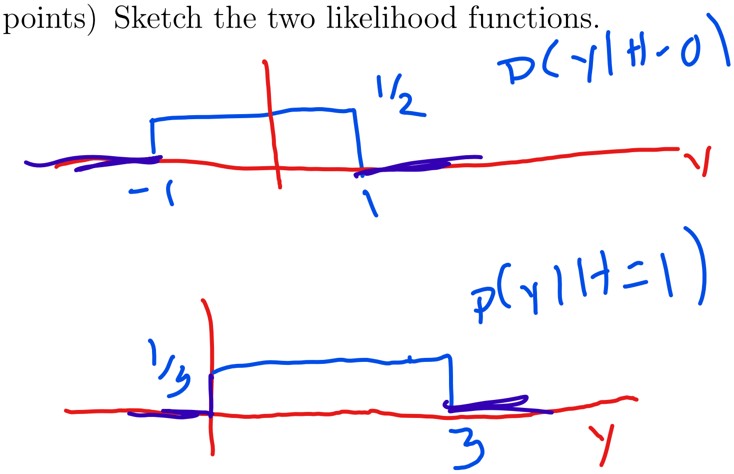
covariance:

$$\Sigma_y = A \Sigma A^T$$

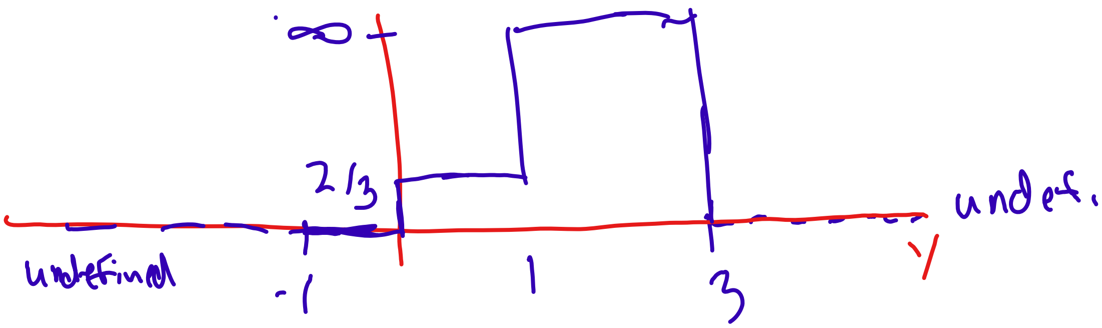
$$= \begin{bmatrix} 2 & -2 \\ 2 & 6 \end{bmatrix}$$

3. (20 points) Consider a binary hypothesis testing problem in which the null hypothesis has likelihood $p(y|H=0) = \mathcal{U}(-1,1)$ and the alternative hypothesis has likelihood $p(y|H=1) = \mathcal{U}(0,3)$.

   (a) (10 points) Sketch the two likelihood functions.



   (b) (5 points) Sketch likelihood *ratio* $l(y) = \frac{p(y|H=1)}{p(y|H=0)}$.



   (c) (5 points) Derive the Neyman-Pearson dectector $\delta(y)$ if the threshold of the likelihood ratio test is set at $\nu = 1$.

chose $H=1$ if $l(y) \geq \nu$:

$$\delta(y) = \begin{cases} 0, & -1 \leq y \leq 1 \\ 1, & 1 < y \leq 3, \end{cases}$$

undefined otherwise.

4. (20 points) Consider the binary hypothesis testing problem associated with a detector that measures the time interval y_i between the arrival of photons. The null hypothesis is that the time between photon arrivals is exponentially-distributed with a rate $\lambda = 1$ photons/sec, and the alternative hypothesis is that the arrivals are exponentially-distributed with rate $\lambda = 2$ photons/sec; i.e. each arrival time y_i has the distribution $p(y_i|H) = \lambda \exp(-\lambda y_i)$.

(a) (5 points) Suppose you observe n i.i.d. measurements y_i, $1 \le i \le n$. Write the log-likehihood ratio for this hypothesis test.

$$p(y|H_1) = \prod_{i=1}^{n} 2 \cdot \exp(-2y_i) = 2^n \exp(-2\sum_i y_i)$$

$$p(y|H_0) = \exp(-\sum y_i)$$

$$L(y) = \log\left(\frac{p(y|H_1)}{p(y|H_0)}\right) = \log(2^n \cdot \exp(-2\sum_i y_i)) - \log(\exp(-\sum y_i))$$

$$= n\log(2) - \sum_{i=1}^{n} y_i$$

(b) (10 points) Supposing equal costs and prior distribution $\pi_0 = 3/4$, find the Bayes-optimimum detector for this problem.

$$\log(\pi_0/\pi_1) = \log\left(\frac{3/4}{1/4}\right) = \log(3)$$

so,
$$\delta(y) = \begin{cases} 0, & n\log(2) - \sum_i y_i < \log(3) \\ 1, & o\backslash w \end{cases}$$

$$= \begin{cases} 0, & \frac{1}{n}\sum_i y_i > \log(2) - \frac{1}{n}\log(3) \\ 1, & \frac{1}{n}\sum_i y_i < \log(2) - 1/n \log(3) \end{cases}$$

(c) (5 points) For $n = 5$, consider the measurement $y = (5.0, 1.24, 0.24, 2.78, 5.2)$. What is the output of the detector in this case?

For $n=5$, the threshold is $\log(2) - \frac{1}{5}\log(3) = 0.4734$

$\frac{1}{n}\sum_i y_i = 2.89$,

$$\boxed{\text{so } \delta(y) = 0}$$

5. (20 points) Consider a binary Gaussian hypothesis test where $\mathbf{y}$ is a multivariate Gaussian distribution. Under the null hypothesis, the mean is $\mu_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and under the alternative hypothesis the mean is $\mu_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$. Under both hypothesis, the common covariance matrix is $\Sigma = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$.

(a) (5 points) Write the log-likelihood ratio for this scenario.

(from Ex 3.4)

$$L(y) = [5\ 2]\begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix} y \quad \frac{1}{2}\left([5\ 2]\begin{bmatrix} 1 & 0 \\ 0 & 1/5 \end{bmatrix}\begin{bmatrix} 5 \\ 2 \end{bmatrix}\right)$$

$$= [5\ \ 2/5]\, y - 12.9$$

(b) (10 points) For equal priors and costs, find the Bayes-optimum detector for this problem.

For equal costs/priors, threshold is 0:

$$\delta(y) = \begin{cases} 0, & [5\ \ 2/5]\, y < 12.9 \\ 1, & [5\ \ 2/5]\, y > 12.9 \end{cases}$$

(c) (5 points) Consider the measurement $\mathbf{y} = \begin{bmatrix} 3.1 \\ -0.1 \end{bmatrix}$. What is the output of the detector in this case?

$$[5\ \ 2/5]\begin{bmatrix} 3.1 \\ -0.1 \end{bmatrix} = 15.46, > 12.9$$

$$\boxed{\delta(y) = 1}$$