Cooperative Control of Dynamical Systems With Application to Autonomous Vehicles

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Abstract

In this paper, a new framework based on matrix theory is proposed to analyze and design cooperative controls for a group of individual dynamical systems whose outputs are sensed by or communicated to others in an intermittent, dynamically changing, and local manner. In the framework, sensing/communication is described mathematically by a time-varying matrix whose dimension is equal to the number of dynamical systems in the group and whose elements assume piecewise-constant and binary values, and dynamical systems are generally heterogeneous and can be transformed into a canonical form of different, arbitrary but finite relative degrees. Utilizing a set of new results on augmentation of irreducible matrices and on lower triangulation of reducible matrices, the framework allows a designer to study how a general local-and-output-feedback cooperative control can determine group behaviors of the dynamical systems and to see how changes of sensing/communication would impact on the group behaviors over time. A necessary and sufficient condition on convergence of a multiplicative sequence of reducible row-stochastic matrices is explicitly derived and, through simple choices of a gain matrix in the cooperative control law, the overall closed loop system is shown to exhibit cooperative behaviors (such as single group behavior, multiple group behaviors, adaptive cooperative behavior for the group, and cooperative formation including individual behaviors). Examples, including formation control of nonholonomic systems in the chained form, are used to illustrate the proposed framework.

I. Introduction

This paper proposes a matrix-theory-based framework of analysis and cooperative control designs for a group of individual but heterogeneous dynamical systems and seeks for the least restrictive requirement on sensing and communication among the systems. As an example, a group of unmanned autonomous vehicles are commanded to perform a set of tasks as a group, and individual robots of different capabilities are to exhibit not only certain group behavior but also their individual behaviors. In the general case, the dynamical systems operate in a dynamically changing and uncertain environment. As such, sensing and communication among the systems are intermittent and local, and their changes are not known apriori or predictable by either deterministic or probabilistic models. The fundamental questions are what is the least restrictive condition on sensing/communication and how to design cooperative control to achieve a guaranteed performance.

There have been many earlier results on distributed robotics, and these results are obtained using heuristic approaches. For example, artificial intelligence methods [1] have been extensively used to explore the architecture, task allocation, mapping building, coordination and control algorithms in multi-robot motion systems [2], [3], [4], [5]. Multi-robot localization and exploration are studied in [6] using a probabilistic approach. Path planning and formation control are investigated in [7] using behavior based control paradigm [8], where the rule based formation behaviors have been defined and evaluated through simulations. A simple heuristic distributed algorithm is proposed in [9] for identical mobile robots to form a circle of a given radius, where each robot updates its position according to a set of rules. In many cases, cooperative rules are chosen to mimic animal behaviors [10]. The basic cohesion, separation and alignment rules are extracted by observing the animal flocking and simulated through computer animation [11]. The alignment problem is recently studied in [12], and the so-called nearest neighbor rule is derived experimentally. That is, all the particles of point mass move in the plane with the same speed, and their headings are updated individually by the same discrete and local rule of averaging its own heading and the headings of its neighbors. The group flocking behaviors such as avoidance, aggregation and dispersion have also been explored [13].

While heuristic and bio-inspired approaches have produced many interesting and very useful results, there was a lack of theoretical frameworks for both analysis and control design. Under the assumption that sensing/communication is time invariant, analysis and control of multi-vehicle systems can be done using various standard approaches in control theory, for instance, [14], [15], [16], [17], [18], [19], [20]. However, cooperative control of dynamical systems often involve intermittent, local, and dynamically changing communication/sensing. Thus, the central and difficult question is twofold: What is the least restrictive condition on sensing/communication to ensure cooperative controllability, and how to design cooperative controls for a group of general dynamical systems in the network?

A truly distributed cooperative law based on the nearest neighbor rule [12] is proposed in [21] for mobile autonomous agents, dynamic changes of communication topology are represented by an undirected graph, performance of the cooperative law is analyzed by graph theory, and convergence is obtained under the assumption that the undirected graph is connected. It is significant and ground-breaking that the result in [21] provides a graph-theory-based framework to analyze a group of networked agents. More recently, the result in [21] is extended in [22] to a multi-agent system in which communication is represented by a directed graph, and convergence of consensus is ensured under the less restrictive condition that there exists a spanning tree (see the definition in [22]). In parallel, local control strategies for groups of mobile autonomous agents are proposed and analyzed in [23], and formation control is shown in [24] to be convergent for unicycles under the condition that the graph has a globally reachable node (see the definition in [24]). Cooperative controls have also been analyzed by other researchers using the combination of graph theory and Nyquist stability criterion [25], [26], using proximity graphs [27], and using the combination of graph theory, convexity, and discrete set-valued Lyapunov functions [28]. All of these results use graph theory as the main approach to derive sensor graph conditions and to conclude convergence, and most of them deal with identical agents or linear systems which, using the terminology of control theory, are of relative degree one. Extension of [21] to second-order dynamics have been pursued in [29], [30].

Different from and complementary to graph theory, matrix theory and control theory are used as the means in this paper to develop a new framework for analyzing a group of dynamical systems and for designing cooperative controls (and so are the preliminary results in [31], [32], [33], [34], [35]). Specifically, by using a piecewise constant and binary-valued matrix to capture the changes of sensing/communication, two sets of new results on matrix reducibility and irreducibility have been developed. The first set is on augmentation of irreducible and reducible matrices, which enables us to analyze not only identical agents but also heterogeneous dynamical systems of arbitrary but finite relative degree in the presence of dynamically changing communication/sensing among them. The second set is on reducible matrices and, by utilizing the canonical form of lower triangulation of reducible matrices, a necessary and sufficient condition is obtained under which any multiplicative sequence of row stochastic matrices is convergent, and the condition is stated and easily explained in terms of changing communication/sensing patterns. Specifically, a multiplicative sequence of solution matrices (obtained from reducible system matrices) is convergent to a matrix of identically rows if and only if it consists of a non-vanishing, lower-triangularly complete subsequence (that is, a subsequence whose lower triangulations have at least one non-vanishing element in each lower triangular matrix block row). Through development and adoption of lower triangulation as the canonical form, all the existing graph theory results (such as a strongly connected graph, a spanning tree, and a globally reachable node) have their counterparts in algebraic matrix theory. Besides admitting high-order dynamical systems, the algebraic matrix approach also has the advantage that it allows us to explore different convergence, to obtain explicit expression of convergence rate for the matrix sequence, and to provide a new more-intuitive concept and also a new simpler convergence test. Specifically, it is shown that systems can be cooperatively controlled if they have over time only one sensing/communication group which can be tested by simply calculating the binary product of communication/sensing matrices over an interval.

In the new framework, a canonical form of vehicle-level dynamics is proposed for separating the designs of vehicle-level control and cooperative control. Using this canonical form, a designer can embed different control objectives and handle different vehicle dynamics. As such, designs of single-objective cooperative control, multi-objective cooperative control, formation control, and adaptive cooperative control are unified in this paper, and their differences boil down to simple choices of a constant gain matrix in the design. These new results and their matrix-theory-based framework of analysis and design show how existing control theories can be enriched to handle networked dynamical systems, and they also enable us to solve more complicated problems such as cooperative control of nonlinear systems such as nonholonomic systems in the chained form.

This paper is organized as follows. In section II, the cooperative control problem is formulated using control theory and matrix theory. In section III, new results on augmentation of irreducible and reducible matrices, on lower triangulation of reducible matrices, and on convergence of their multiplicative sequence are presented. In section IV, different cooperative control designs are presented by applying the convergence results obtained in section III. Illustrative examples are included in section V. Appendices I, II and III provide a summary of existing and useful results on reducible and irreducible matrices and on sequence convergence of matrix product, while appendices IV and V contain key lemmas used to prove the results in section III.

II. Problem Formulation

Consider a group of q vehicles and suppose that dynamics of the ith vehicle are described by

$$\dot{\phi}_i = f_i(\phi_i, v_i), \quad \psi_i = h_i(\phi_i), \tag{1}$$

where $i \in \{1, \cdots, q\}$, $\phi_i \in \Re^{n_i}$ is the original state, and $\psi_i(t) \in \Re^m$ is the output, and $v_i(t) \in \Re^m$ is the control input. The proposed cooperative control design consists of the following two-level control hierarchy:

Local cooperative strategy: A local vehicle-level command $u_i = u_i(t, s_{i1}(t)\psi_1, \cdots, s_{iq}(t)\psi_q)$ is synthesized by taking into account all the information available to the ith vehicle about outputs of other vehicles, where $s_{ij}(t)$ are binary time functions, $s_{ii} \equiv 1$; $s_{ij}(t) = 1$ if $\psi_j(t)$ (or its equivalence) is known to the ith vehicle at time t, and $s_{ij} = 0$ if otherwise.

Vehicle-level control: Vehicle control $v_i = v_i(t, \phi_i, u_i)$ implements the local cooperative strategy of u_i at the ith vehicle and, for the ease of designing u_i, it transforms vehicle dynamics into a canonical form (which is introduced in section II-A).

As the focus of this paper, cooperative control $u = [\ u_1^T \ \cdots \ u_q^T\ ]^T$ will be synthesized in sections II-B and IV, and its objective is to ensure that all the state variables of the systems are uniformly bounded and that, for any $i \in \{1, \cdots, q\}$, the steady state error $e^{ss} \stackrel{\triangle}{=} \lim_{t\to\infty}[\psi_i(t) - \psi_i^d(t) - y^d]$ exists and is independent of i, where constant vector y^d describes the desired cooperative behaviors for the whole group of vehicles while vector $\psi_i^d(t)$ describes the desired individual behavior(s) in addition to the group behavior and is uniformly bounded and smooth. It will be shown in section IV that the value of vector y^d can be used to characterize either single cooperative behavior or multiple cooperative behaviors.

The proposed cooperative control reacts to but requires little information about sensing/communications among the vehicles. In this paper, we consider the general cases that vehicles operate by themselves most of the time and that exchange of output information among the vehicles occurs only intermittently and locally. To capture this nature of information flow, let us define the following sensing/communication matrix and its corresponding time sequence $\{t_k^s : k = 0, 1, \cdots\}$ as:

$$S(t) = \begin{bmatrix} S_1(t) \\ S_2(t) \\ \vdots \\ S_q(t) \end{bmatrix} = \begin{bmatrix} s_{11} & s_{12}(t) & \cdots & s_{1q}(t) \\ s_{21}(t) & s_{22} & \cdots & s_{2q}(t) \\ \vdots & \vdots & \vdots & \vdots \\ s_{q1}(t) & s_{q2}(t) & \cdots & s_{qq} \end{bmatrix} \in \{0,1\}^{q\times q}, \quad \begin{cases} S(t) = S(t_k^s), & \forall t \in [t_k^s, t_{k+1}^s) \\ \\ S(k) \stackrel{\triangle}{=} S(t_k^s), \end{cases} \tag{2}$$

where elements $s_{ij}(t)$ are those aforementioned; and $t_0^s \stackrel{\triangle}{=} t_0$. Time sequence $\{t_k^s\}$ and the corresponding changes in the row $S_i(t)$ of matrix $S(t)$ are detectable instantaneously by and locally at the ith vehicle, but they are *not* predictable or prescribed or known apriori or modelled in any way. Nonetheless, it can be assumed without loss of any generality that $0 < \underline{c}_t \leq t_{k+1}^s - t_k^s \leq \overline{c}_t < \infty$, where $\underline{c}_t$ and $\overline{c}_t$ are constant bounds.*

The goal of this paper is twofold: (i) determine cooperative controllability (i.e., the least restrictive condition on the inherent changes of $S(t)$ under which cooperative behavior(s) can be achieved), and (ii) develop a systematic way of synthesizing cooperative controls. In what follows, the mathematical problem of achieving a cooperative behavior is formulated in section II-C, and a necessary and sufficient condition on its solvability is explicitly found in section III.

A. Vehicle-Level Canonical Form for Designing Cooperative Control

In order to focus upon cooperative control design, the following assumption is introduced. In essence, the assumption says that, although vehicle systems are heterogeneous, vehicle level controls can be designed to make their input-output (I/O) relationship canonical. That is, for the purpose of achieving cooperative behavior in certain m-dimensional subspace, their I/O relationship is represented by triplet $\{A_i, B_i, C_i\}$.

Assumption 1: There exist a diffeomorphic state transformation $[x_i^T, \varphi_i^T]^T = \mathcal{X}_i(t, \phi_i)$ and a decentralized control law $v_i = \mathcal{V}_i(t, \phi_i, u_i)$ such that vehicle dynamics in (1) are transformed into

$$\dot{x}_i = A_i x_i + B_i u_i, \quad y_i = C_i x_i, \quad \dot{\varphi}_i = g_i(t, \varphi_i, x_i), \tag{3}$$

where $x_i \in \Re^{l_i m}$ for some integer $l_i \geq 1$, variables in x_i are output-related, $\varphi_i \in \Re^{n_i - l_i m}$ is the vector containing internal state variables, $y_i = \psi_i - \psi_i^d$, $I_{m\times m}$ is the m-dimensional identity matrix, $\otimes$ denotes the Kronecker product (defined as $D \otimes E = [d_{ij}E]$), J_k is the kth order Jordan canonical form given by

$$J_k = \begin{bmatrix} -1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 & 1 & 0 \\ 0 & 0 & 0 & \cdots & -1 & 1 \\ 0 & 0 & 0 & \cdots & 0 & -1 \end{bmatrix} \in \Re^{k\times k}, \qquad \begin{aligned} A_i &= J_{l_i} \otimes I_{m\times m} \in \Re^{(l_i m)\times(l_i m)}, \\ B_i &= \begin{bmatrix} 0 \\ I_{m\times m} \end{bmatrix} \in \Re^{(l_i m)\times m}, \\ C_i &= \begin{bmatrix} I_{m\times m} & 0 \end{bmatrix} \in \Re^{m\times(l_i m)}, \end{aligned}$$

u_i is the cooperative control to be designed, and $\dot{\varphi}_i = g_i(t, \varphi_i, x_i)$ is input-to-state stable [36]. ◇

Canonical form (3) has the structure that, if u_i converges to any given constant vector, so does output y_i. In other word, through the transformation into (3), the resulting controls $v_i(\cdot)$ are decentralized at individual vehicles and capable of following the cooperative strategy of u_i. It is straightforward to verify that canonical form (3) holds if tracking dynamics of the vehicles are input-output feedback linearizable with stable internal dynamics and that l_i is the relative degree for the ith vehicle. Hence, technical conditions equivalent to assumption 1 can be found in standard texts such as [36], [37] and are omitted here for briefness. Instead, examples are included in section V-B to illustrate assumption 1.

*If $S(t)$ becomes a constant matrix after some finite time, an infinite time sequence $\{t_k^s\}$ can always be chosen to yields a finite $\overline{c}_t$ except that $S(t_k^s)$ remains constant. On the other hand, requirement of $\underline{c}_t$ not being too small is needed for implementation.

B. A General Class of Cooperative Controls

Cooperative controls proposed in this paper are the class of linear, piecewise-constant, locally-feedback controls with feedback gain matrices $G_i(t) \stackrel{\triangle}{=} \left[\begin{array}{ccc} G_{i1}(t) & \cdots & G_{iq}(t) \end{array} \right]$, where $i = 1, \cdots, q$,

$$G_{ij}(t) = G_{ij}(t_k^s), \;\; \forall t \in [t_k^s, t_{k+1}^s); \quad G_{ij}(k) \stackrel{\triangle}{=} G_{ij}(t_k^s) \stackrel{\triangle}{=} \frac{s_{ij}(t_k^s)}{\sum_{\eta=1}^q s_{i\eta}(t_k^s)} K_c, \quad j = 1, \cdots, q; \tag{4}$$

$s_{ij}(t)$ are piecewise-constant as defined in (2), and $K_c \in \Re^{m \times m}$ is a constant, nonnegative, and row stochastic matrix (see the definitions in appendix I) to be selected in section IV. That is, cooperative controls are of form

$$u_i \stackrel{\triangle}{=} \sum_{j=1}^q G_{ij}(t)[s_{ij}(t) y_j] = \sum_{j=1}^q G_{ij}(t) y_j = G_i(t) y, \tag{5}$$

where $y = [y_1^T \;\; \cdots \;\; y_q^T]^T$. Although $S(t)$ is not known apriori nor can it be modelled, $S(t)$ is piecewise constant, diagonally positive and binary, and the value of row $S_i(t)$ is known at time t to the ith vehicle. The above choice of feedback gain matrix block $G_{ij}(t)$ in terms of $s_{ij}(t)$ ensures that matrices $G_i(t)$ are row stochastic and that control (5) is always local and implementable with all and only available information.

Remark 1: It is apparent that choice (4) can be generalized to $G_{ij}(t_k^s) = s_{ij}(t_k^s) w_{ij} K_{ij} / [\sum_{\eta=1}^q w_{i\eta} s_{i\eta}(t_k^s)]$, where $w_{ij} > 0$ are weighting coefficients, K_{ij} are constant and nonnegative, K_{ii} are either irreducible or diagonally positive, and the row-block sum $\sum_{j=1}^q G_{ij}(t_k^s)$ is a row stochastic matrix. Then, all the subsequent results can be similarly developed. As will be used in subsection V-C.2, choosing $K_{ij} = K_c$ and then adjusting the weighting $w_{ij} > 0$ according to the current status of sensing/communication can improve convergence performance of the cooperative control. $\diamond$

Remark 2: It follows from definition (4) that $G_i(t)$ is row stochastic and that control (5) has the following alternative expression:

$$u_i = G_i(t) y = K_c y_i + G_i(t)[y - \mathbf{1}_q \otimes y_i],$$

where vector $\mathbf{1}$ is that defined in appendix I. The above expression shows that implementation of control (5) only requires measurements of other vehicles' outputs relative to the subject vehicle. In general, for vehicle i of relative degree l_i higher than one, absolute measurement of its own output y_i may be needed for controlling its motion. In the case that $l_i = 1$, the ith subsystem becomes $\dot{y}_i = -y_i + u_i \stackrel{\triangle}{=} u_i'$, and it follows from the above alternative expression of u_i that $u_i' = (K_c - I_{m\times m}) y_i + G_i(t)[y - \mathbf{1}_q \otimes y_i]$. Hence, upon setting $K_c = I_{m \times m}$, vehicle i is cooperatively controlled without any absolute measurement. $\diamond$

C. Mathematical Problem of Cooperative Control Design and Stability/Convergence Analysis

By simply combining equations (3) and (5) for all i, we obtain the overall closed-loop system $\dot{x} = [A + D(t)]x$, where $A = \text{diag}\{A_1, \;\cdots,\; A_q\} \in \Re^{N_q \times N_q}$ for $N_q = mL_q$ and $L_q = \sum_{i=1}^q l_i$, $C = \text{diag}\{C_1, \;\cdots,\; C_q\} \in \Re^{(mq)\times N_q}$, $B = \text{diag}\{B_1, \;\cdots,\; B_q\} \in \Re^{N_q \times (mq)}$, $G(t) = \left[\begin{array}{ccc} G_1^T(t) & \cdots & G_q^T(t) \end{array} \right]^T \in$

$\Re^{(mq)\times(mq)}$, and $D(t) = BG(t)C$. The overall system equation can be simplified by merging matrices A and $D(t)$ and using their special structures; that is, the overall dynamics can be expressed as

$$\dot{x} = [-I_{N_q\times N_q} + \overline{G}(t)]x, \tag{6}$$

where $\overline{G}(t) \in \Re^{N_q\times N_q}$, $\overline{G}_{ii} \in \Re^{(l_i m)\times(l_i m)}$, and $\overline{G}_{ij}(t) \in \Re^{(l_i m)\times(l_j m)}$ are defined as[†]

$$\overline{G}(t) = \begin{bmatrix} \overline{G}_{11}(t) & \overline{G}_{12}(t) & \cdots & \overline{G}_{1q}(t) \\ \overline{G}_{21}(t) & \overline{G}_{22}(t) & \cdots & \overline{G}_{2q}(t) \\ \vdots & \vdots & \vdots & \vdots \\ \overline{G}_{q1}(t) & \overline{G}_{q2}(t) & \cdots & \overline{G}_{qq}(t) \end{bmatrix}, \quad \begin{array}{l} \overline{G}_{ii}(t) = \begin{bmatrix} 0 & I_{(l_i-1)\times(l_i-1)} \otimes I_{m\times m} \\ G_{ii}(t) & 0 \end{bmatrix}, \\ \\ \overline{G}_{ij}(t) = \begin{bmatrix} 0 & 0 \\ G_{ij}(t) & 0 \end{bmatrix}, \quad \text{if } i \neq j. \end{array} \tag{7}$$

It is obvious that matrix $\overline{G}(t)$ is also piecewise constant and row stochastic at any given time instant t.

Utilizing the property of matrix $\overline{G}(t)$ being piecewise constant, we can obtain the state solution of system (6) and the following steady state: letting $x^{ss} \triangleq \lim_{t\to\infty} x(t)$,

$$x^{ss} = \lim_{t\geq t_k^s,\ k\to\infty} \left\{ e^{[-I+\overline{G}(t_k^s)](t-t_k^s)} e^{[-I+\overline{G}(t_{k-1}^s)](t_k^s-t_{k-1}^s)} \cdots e^{[-I+\overline{G}(t_1^s)](t_2^s-t_1^s)} e^{[-I+\overline{G}(t_0)](t_1^s-t_0)} \right\} x(t_0), \tag{8}$$

provided that the above limit exists. Recalling that cooperative control design is successful if $x^{ss} = \mathbf{1}_{L_q} \otimes y^{ss}$ for some constant vector $y^{ss} \in \Re^m$, we know that the following sequence of pre-multiplying matrix products is mathematically of central importance: denoting $\aleph^+ \triangleq \{1, 2, \cdots, \infty\}$,

$$\left\{ \prod_{\eta=1}^{k} P(\eta) \triangleq P(k)P(k-1)\cdots P(2)P(1) : \quad P(k) = e^{[-I+\overline{G}(t_{k-1}^s)](t_k^s-t_{k-1}^s)} \text{ and } k \in \aleph^+ \right\}. \tag{9}$$

In section III, a necessary and sufficient condition is found to ensure convergence of $\lim_{k\to\infty} \prod_{\eta=1}^{k} P(\eta) = \mathbf{1}_{N_q} c$ for some $c \in \Re^{1\times N_q}$. In section IV, the convergence condition is used to choose K_c in order to achieve single cooperative behavior, multiple cooperative behaviors, and adaptive cooperative behaviors.

Remark 3: The proposed framework of cooperative control design also applies to discrete time systems and sampled-data systems. Specifically, the vehicle-level canonical form in the discrete time domain can be chosen to be

$$x_i(k+1) = A_{d_i} x_i(k) + B_i u_{d_i}(k), \quad y_i(k) = C_i x_i(k), \quad \varphi_i(k+1) = g_i(k, \varphi_i(k), x_i(k)),$$

where A_i, B_i, C_i are the same as those in (3), $0 < c_d < 1$ is a design constant, and $A_{d_i} \triangleq I_{(l_i m)\times(l_i m)} + (1-c_d)A_i$. Then, under cooperative control $u_{d_i}(k) = (1-c_d)u_i$ where u_i is given by (5) (except that t is replaced by k), the closed loop system consisting of all the vehicles is

$$x(k+1) = [c_d I_{N_q\times N_q} + (1-c_d)\overline{G}(k)]x(k) \triangleq P_d(k)x(k),$$

[†]Whenever $l_i - 1 = 0$, the corresponding rows and columns of $I_{(l_i-1)\times(l_i-1)} \otimes I_{m\times m}$ are empty, i.e., removed from $\overline{G}$.

where matrix $\overline{G}(k)$ is the same as $\overline{G}(t)$ in (6). In this case, stability and convergence can again be analyzed by the product sequence of $\prod_{\eta=1}^{k} P_d(\eta)$, and all the subsequent results can be similarly developed except for two differences. First, lemma III.6 is no longer needed; instead, since diagonal blocks in the lower triangulation of $P_d(k)$ are all irreducible and diagonally positive, certain-finite-length products of these diagonal blocks all become positive (according to corollary A.3 in appendix II). Second, if $\lambda(P_d(\eta)) = 0$ for some finite η, cooperative behavior(s) can be achieved in finite steps for discrete time systems (but impossible for continuous time systems as $P(\eta)$ in (9) are invertible and hence $\lambda(P(\eta)) > 0$ for all η), where matrix function $\lambda(\cdot)$ is defined by (32) in the appendix. $\diamond$

III. New Results on Row Stochastic Matrices and on Convergence of Their Products

Several results critical to the proposed cooperative controls of dynamical systems are developed below.

A. New Results on Row Stochastic Matrices

Existing results on reducible and irreducible matrices are summarized in appendix II, and lemma A.1 and corollary A.2 are instrumental to establish the following results of irreducibility and the canonical form of lower triangulation on the process of augmenting matrix $S(t)$ to closed-loop system matrix $\overline{G}(t)$.

Lemma III.1: Consider matrix $E \in \Re^{(qm)\times(qm)}$ with sub-blocks $E_{ij} \in \Re^{m\times m}$. Suppose that $E \geq 0$ and that $\overline{E} \in \Re^{[(q+1)m]\times[(q+1)m]}$ is defined by

$$\overline{E} = \begin{bmatrix} 0 & W_1 & W_2 & \cdots & W_q \\ E_{11} & F_1 & E_{12} & \cdots & E_{1q} \\ E_{21} & F_2 & E_{22} & \cdots & E_{2q} \\ \vdots & \vdots & \vdots & & \vdots \\ E_{q1} & F_q & E_{q2} & \cdots & E_{qq} \end{bmatrix},$$

where $\underline{c} > 0$ and $\overline{c} > 0$ are constants, W_1 is a diagonal matrix satisfying $\underline{c}I_{m\times m} \leq W_1 \leq \overline{c}I_{m\times m}$, $W_i \geq 0$ for $i = 2, \cdots, q$, and $F_j \geq 0$ for $j = 1, \cdots, q$. Then, if E is irreducible, so is $\overline{E}$. Furthermore, if $W_2 = \cdots = W_q = F_1 = F_2 = \cdots = F_q = 0$ and if $\overline{E}$ is irreducible, E is irreducible.

Proof: Let us first prove that $\overline{E}$ is irreducible if E is irreducible. Suppose that $z \geq 0$ and $z \neq 0$ and that, for some constant $\gamma > 0$, $\gamma z \geq \overline{E}z$ holds. Partition vector z and define vectors z' and z'' as

$$z = \begin{bmatrix} z_1^T & z_2^T & z_3^T & \cdots & z_{q+1}^T \end{bmatrix}^T, \quad z' = \begin{bmatrix} z_1^T & z_3^T & \cdots & z_{q+1}^T \end{bmatrix}^T, \quad z'' = \begin{bmatrix} z_2^T & z_3^T & \cdots & z_{q+1}^T \end{bmatrix}^T,$$

where $z_i \in \Re^m$. It follows from the definition of $\overline{E}$ that

$$\gamma z_1 \geq W_1 z_2 + \sum_{i=2}^{q} W_i z_{i+1} \geq \underline{c} z_2, \tag{10}$$

and that

$$\gamma z'' \geq E z' + F z_2 \geq E z', \tag{11}$$

where $F = \begin{bmatrix} F_1^T & F_2^T & \cdots & F_q^T \end{bmatrix}^T$. Combining (10) and (11) yields

$$\gamma \max\{\gamma/\underline{c}, 1\} z' \geq \gamma z'' \geq E z'.$$

Since E is irreducible, we know from corollary A.2 in appendix II that $z' > 0$. It follows from (11) that $z' > 0$ implies $z'' > 0$ and hence $z_2 > 0$. Thus, we have $z > 0$ and, by corollary A.2, $\overline{E}$ is irreducible.

Next, consider the case that $W_2 = \cdots = W_q = F_1 = F_2 = \cdots = F_q = 0$. This part of the proof is done by contradiction. To this end, suppose that $\overline{E}$ is irreducible but E is reducible. Matrix E being reducible implies that $\gamma\xi \geq E\xi$ holds for some constant $\gamma > 0$ and some vector ξ, where $\xi = \begin{bmatrix} \xi_1^T & \cdots & \xi_q^T \end{bmatrix}^T$ with $\xi_i \in \Re^m$, $\xi \geq 0$, $\xi \neq 0$, and $\xi \not> 0$. Define $\overline{\xi} = \begin{bmatrix} \xi_1^T & \xi_1^T & \xi_2^T & \cdots & \xi_q^T \end{bmatrix}^T$. Then, it follows that $\gamma \max\{\overline{c}/\gamma, 1\}\overline{\xi} \geq \overline{E}\,\overline{\xi}$ while $\overline{\xi} \geq 0$, $\overline{\xi} \neq 0$, and $\overline{\xi} \not> 0$. According to corollary A.2 one more time, this result contradicts with $\overline{E}$ being irreducible; hence, E must be irreducible. □

By applying lemma III.1 inductively and together with appropriate permutations, one can easily conclude the following corollary. Corollary III.2 provides the property needed to study the cooperative control problem of general systems whose dynamics are of different relative degrees.

Corollary III.2: Given any non-negative matrix $G(t) \in \Re^{(qm)\times(qm)}$ with sub-blocks $G_{ij}(t) \in \Re^{m\times m}$, let $\overline{G}(t) \in \Re^{(L_q m)\times(L_q m)}$ with $L_q = l_1 + \cdots + l_q$ be the augmentation of $G(t)$ as defined by (7). Then, $\overline{G}(t)$ is irreducible at time t if and only if $G(t)$ is irreducible at time t.

The following lemma shows the invariance of irreducibility under Kronecker product, and its proof becomes a straightforward application of corollary A.2 upon realizing the two facts that, for any $\overline{z} \in \Re^{mq}$, inequality $z' \otimes \mathbf{1}_m \leq \overline{z} \leq z'' \otimes \mathbf{1}_m$ always holds for some $z', z'' \in \Re^q$ and that, for any $z \in \Re^q$ with $z \geq 0$, $\gamma(z \otimes \mathbf{1}_m) \geq (S \otimes F)(z \otimes \mathbf{1}_m)$ holds if and only if $\gamma z \geq Sz$. Corollary III.4 can be concluded from lemma III.3 since irreducibility is invariant under the operation of multiplying any row of a matrix by a positive constant.

Lemma III.3: Consider a pair of nonnegative matrices $S \in \Re^{q\times q}$ and $F \in \Re^{m\times m}$, where F is irreducible and row stochastic. Then, matrix $E = S \otimes F$ is irreducible if and only if S is irreducible.

Corollary III.4: Given an irreducible and row stochastic matrix K_c, matrix $G(t)$ defined in (4) is irreducible at time t if and only if matrix $S(t)$ is irreducible at time t.

Next, let us consider the general case that matrix $S(t)$ is reducible. In this case, progression of the properties on matrix augmentations from $S(t)$ to $G(t)$ and then to $\overline{G}(t)$ is stated by the following corollary. Its proof can be completed by applying lemma A.1 in appendix II and then invoking corollaries III.4 and III.2. In fact, corollaries III.4 and III.2 can now be viewed as the special case of $p = 1$ in corollary III.5.

Corollary III.5: Consider matrices $G(t)$ and $\overline{G}(t)$ defined in (5), (4) and (7), where K_c is an irreducible and row stochastic matrix. Then, given any reducible matrix $S(k)$, there exist an integer $1 < p \leq q$ and permutation matrices $T_S(k)$, $T(k)$ and $\overline{T}(k)$ such that

$$T_S^T(k)S(k)T_S(k) = S'_{\lrcorner}(k), \quad T^T(k)G(k)T(k) = G'_{\lrcorner}(k), \quad \overline{T}^T(k)\overline{G}(k)\overline{T}(k) = \overline{G}'_{\lrcorner}(k), \tag{12}$$

where

$$S'_{\searrow}(k) \triangleq \begin{bmatrix} S'_{11}(k) & 0 & \cdots & 0 \\ S'_{21}(k) & S'_{22}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ S'_{p1}(k) & S'_{p2}(k) & \cdots & S'_{pp}(k) \end{bmatrix}, \quad G'_{\searrow}(k) \triangleq \begin{bmatrix} G'_{11}(k) & 0 & \cdots & 0 \\ G'_{21}(k) & G'_{22}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ G'_{p1}(k) & G'_{p2}(k) & \cdots & G'_{pp}(k) \end{bmatrix}, \tag{13}$$

and

$$\overline{G}'_{\searrow}(k) \triangleq \begin{bmatrix} \overline{G}'_{11}(k) & 0 & \cdots & 0 \\ \overline{G}'_{21}(k) & \overline{G}'_{22}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \overline{G}'_{p1}(k) & \overline{G}'_{p2}(k) & \cdots & \overline{G}'_{pp}(k) \end{bmatrix}, \tag{14}$$

the augmentation process from $S'_{\searrow}(k)$ to $G'_{ij}(k)$ and then to $\overline{G}'_{ij}(k) \in \Re^{r_i \times r_j}$ is the same as that from $S(k)$ to $G_{ij}(k)$ and then to $\overline{G}_{ij}(k)$ (i.e., equations (4) and (7)), and diagonal blocks in $S'_{\searrow}(k)$, $G'_{\searrow}(k)$ and $\overline{G}'_{\searrow}(k)$ are all square and irreducible.

Remark 4: In corollary III.5, $T = T_S \otimes I_{m\times m}$ and, if $l_i = l$ for all $i = 1, \cdots, q$, $\overline{T} = T \otimes I_{l\times l}$. In the general case that $l_i \neq l_j$, $\overline{T}$ is the augmentation of T by appropriately adding rows and columns with $I_{l_i\times l_i} \otimes I_{m\times m}$ and zero. Generally, index p and dimensions of diagonal blocks are functions of k. $\diamond$

B. A Less-Restrictive Condition on Sequence Convergence of Matrix Products

Development of the new and less restrictive condition on convergence requires the following definition.

Definition III.1: Consider a sequence of non-negative matrix blocks $\{E_{ij}(k) : k \in \aleph^+\}$. The sequence $\{E_{ij}(k)\}$ is said to be *uniformly non-vanishing*, denoted by $\{E_{ij}(k)\} \succ 0$, if there are a constant $\epsilon > 0$ and an infinite sub-sequence $\{l_v, v \in \aleph^+\}$ of $\aleph^+$ such that $\lim_{v\to\infty} l_v = +\infty$ and that, for every choice of $v \in \aleph^+$, at least one element of $E_{ij}(l_v)$ is greater than or equal to ϵ. $\diamond$

Lemma III.6 provides useful properties of those matrices $P(k)$ defined in (9) and it extends lemma A.4 in appendix III to the general case that $\overline{G}(t^s_k)$ is reducible. Note that, by lemma A.4, both $P(k)$ and $P'_{\searrow}(k)$ are row stochastic.

Lemma III.6: Consider matrix sequence $\{P(k) : k \in \aleph^+\}$ with $P(k) = e^{[-I+\overline{G}(t^s_{k-1})]\tau_k}$ where $\tau_k = t^s_k - t^s_{k-1}$, and matrices $S(t^s_k)$, $G(t^s_k)$ and $\overline{G}(t^s_k)$ are those in corollary III.5. Then, if K_c is irreducible and row stochastic, there exist an integer $1 < p \leq q$ and a permutation matrix $\overline{T}(k)$ such that

$$\overline{T}^T(k)P(k)\overline{T}(k) = \begin{bmatrix} P'_{11}(k) & 0 & \cdots & 0 \\ P'_{21}(k) & P'_{22}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P'_{p1}(k) & P'_{p2}(k) & \cdots & P'_{pp}(k) \end{bmatrix} \triangleq P'_{\searrow}(k), \tag{15}$$

where all diagonal blocks $P'_{ii}(k) \in \Re^{r_i\times r_i}$ are square and uniformly positive for all k. Furthermore, $\{S'_{ij}(k)\} \succ 0$ implies $\{G'_{ij}(k)\} \succ 0$ hence in turn $\{\overline{G}'_{ij}(k)\} \succ 0$ and $\{P'_{ij}(k)\} \succ 0$.

Proof: It follows from corollary III.5 and specifically from (14) that

$$\begin{aligned} P'_{\lrcorner}(k) &= \overline{T}^T(k)P(k)\overline{T}(k) \\ &= e^{-2\tau_k}\overline{T}^T(k)e^{[I+\overline{G}(t^s_{k-1})]\tau_k}\overline{T}(k) \\ &= e^{-2\tau_k}\overline{T}^T(k)\left\{\sum_{l=0}^{\infty}\frac{\tau_k^l}{l!}[I+\overline{G}(t^s_{k-1})]^l\right\}\overline{T}(k) \\ &= e^{-2\tau_k}\left\{I+\tau_k[I+\overline{G}'_{\lrcorner}(k)]+\frac{\tau_k^2}{2!}[I+\overline{G}'_{\lrcorner}(k)]^2+\cdots+\frac{\tau_k^{n-1}}{(n-1)!}[I+\overline{G}'_{\lrcorner}(k)]^{n-1}+\cdots\right\}, \end{aligned}$$

from which all the statements can be concluded by using the structural properties of $[I+\overline{G}'_{\lrcorner}(k)]^l$. Specifically, $P'_{ii}(k)=e^{-2\tau_k}e^{[I+\overline{G}'_{ii}(k)]\tau_k}$ to which lemma A.4 in appendix III can be applied; $P'_{ij}(k)\geq e^{-2\tau_k}\tau_k\overline{G}'_{ij}(k)$; by definition, $\{S'_{ij}(k)\}\succ 0$ implies $\{G'_{ij}(k)\}\succ 0$; and by augmentation, $\{G'_{ij}(k)\}\succ 0$ implies $\{\overline{G}'_{ij}(k)\}\succ 0$. □

Clearly, the lower-triangular structure in (15) should be adopted as the canonical form to study convergence of multiplicative sequence $\{\prod_{\eta=1}^k P(\eta)\}$ in (9). The existing result on sequence convergence, lemma A.5 in appendix III, is not applicable except for the simplest case that $p=1$ (that is, $P(k)$ and $P'_{\lrcorner}(k)$ are positive). Hence, more general and less restrictive conditions of sequence convergence must be developed. The proposed development is possible because lemma A.5 is only sufficient and quite restrictive. As an example, consider matrix

$$P'_{\lrcorner}=\begin{bmatrix} 1 & 0 & 0 \\ 0.1 & 0.9 & 0 \\ 0 & 0.7 & 0.3 \end{bmatrix}.$$

Since $\lambda(P'_{\lrcorner})=1$, lemma A.5 does not apply but, due to its triangular structure and to its property, its power series is convergent as $\lim_{k\to\infty}[P'_{\lrcorner}]^k=\mathbf{1}_3[1\ 0\ 0]$. The following theorem presents a relaxed convergence condition by detailing convergence cases of lower triangular matrix sequence $\{\prod_{\eta=1}^k P'_{\lrcorner}(\eta)\}$. Proof of theorem III.7 requires lemma A.6, lemma A.7, corollary A.8, and lemma A.9 which are stated and proven in appendix IV.

Definition III.2: Row stochastic matrix sequence $\{P(k):k\in\aleph^+\}$ with $P(k)$ defined in (9) is said to be *sequentially lower-triangular and of fixed block dimensions* if its associated permutation matrix $\overline{T}(k)$ in (12) and corresponding p and r_i in (15) are all independent of k. ◇

Definition III.3: Row stochastic matrix sequence $\{P(k):k\in\aleph^+\}$ with $P(k)$ defined in (9) is said to be *sequentially lower-triangular* if its associated permutation matrix $\overline{T}(k)$ in (12) is independent of k (while p and r_i in (15) may vary as $p(k)$ and $r_i(k)$). ◇

Theorem III.7: If sequence $\{P(k):k\in\aleph^+\}$ is sequentially lower-triangular and of fixed block dimensions, the corresponding multiplicative sequence in (9) is convergent, that is, there exist a squarely row stochastic matrix Q^{ss} and a permutation matrix $\overline{T}$ such that

$$\lim_{k\to\infty}\prod_{\eta=1}^k P(\eta)=\overline{T}^T\lim_{k\to\infty}Q(k)\,\overline{T}=\overline{T}^TQ^{ss}\overline{T},\quad\text{and}\quad |Q_i(k)-Q_i^{ss}|\leq\sigma_i^k\mathbf{J}_{r_i\times N_q},\tag{16}$$

where $0 < \sigma_i < 1$, $Q(k)$ is defined by $Q(k) \stackrel{\triangle}{=} \prod_{\eta=1}^{k} P'_{\lrcorner}(\eta)$, matrix operation of $|\cdot|$ is defined in appendix I, and Q_i^{ss} and $Q_i(k)$ are the ith block rows of Q^{ss} and $Q(k)$, respectively. Specifically, matrix $Q^{ss} \stackrel{\triangle}{=} [Q_{ij}^{ss}]$ has the same block structure as $P'_{\lrcorner}(k)$ (i.e., $Q_{ij}^{ss} = 0$ if $j > i$), and its block elements Q_{ij}^{ss} with $i \geq j$ can all be constructed inductively. In particular, $Q_{11}^{ss} = \mathbf{1}_{r_1} c_1$; and if, for any $i > 1$, there exists some $j < i$ such that $\{P'_{ij}(k)\} \succ 0$ holds, $Q_i^{ss} = Q_1^{ss}$ for all $i > 1$.

Proof: For the first block row, it is obvious from the lower triangular structure that $Q_{11}(k) = \prod_{\eta=1}^{k} P'_{11}(\eta)$. Since $P'_{11}(k)$ is uniformly positive, it follows from lemma A.5 in appendix III that $Q_{11}^{ss} = \mathbf{1}_{r_1} c_1$ for some $c_1 \in \Re^{1\times r_1}$. Given $P'_{11}(k) \geq \epsilon_1 \mathbf{J}_{r_1\times r_1}$, we have that $0 \leq \lambda(P'_{11}(k)) \leq (1 - r_1\epsilon_1) \stackrel{\triangle}{=} \sigma_1 < 1$. It follows from lemma A.6 and then from inequality (33) in appendix III that $|Q_{11}(k) - Q_{11}^{ss}| \leq \delta\left(Q_{11}(k)\right) \mathbf{J}_{r_1\times r_1} \leq \sigma_1^k \mathbf{J}_{r_1\times r_1}$.

In what follows, the ith block row of Q^{ss} is constructed recursively by assuming that rows Q_j^{ss} with $1 \leq j < i$ have been obtained. The value of Q_i^{ss} is determined by the three cases of $\{P'_{ij}(k)\}$.

The first case is that, for some integer $\overline{k}_i > 0$, $P'_{ij}(k) = 0$ for all $k > \overline{k}_i$ and for all $j < i$. Consider the infinite product $R_i^{ss} \stackrel{\triangle}{=} \prod_{k=\overline{k}_i+1}^{\infty} P'_{\lrcorner}(k)$. Since $P'_{ij}(k) = 0$ for all $k > \overline{k}_i$ and for all j except for $P'_{ii}(k) > 0$, we can repeat the analysis of Q_{11}^{ss} and show that $R_{ij}^{ss} = 0$ for all j except that $R_{ii}^{ss} = \mathbf{1}_{r_i} c_i$ for some $c_i \in \Re^{1\times r_i}$. Post-multiplying matrices $P'_{\lrcorner}(\overline{k}_i)\cdots P'_{\lrcorner}(1)$ yields $Q_i^{ss} = R_i^{ss} P'_{\lrcorner}(\overline{k}_i)\cdots P'_{\lrcorner}(1)$.

Other than the first case, uniform non-vanishing property of $\{P'_{ij}(k)\} \succ 0$ holds for $j \in \Omega_i$ where $\Omega_i \stackrel{\triangle}{=} \{j_1,\ \cdots,\ j_l\}$ is nonempty set of integers and of maximum $(i-1)$ elements. For every $j_\mu \in \Omega_i$, check if $\{[P'_{ij_\mu}(k)P'_{j_\mu j'}(k)]\} \succ 0$ holds for some $j' \notin \Omega_i$; if so, add j' into set Ω_i and reset $P'_{ij'}(k) = P'_{ij_\mu}(k)P'_{j_\mu j'}(k)$; and continue the search to exhaust all the possibilities of Ω_i being expanded. Once set Ω_i is finalized, Q_i^{ss} can be found under the following two situations.

As the second case, row blocks $Q_{j_\mu}^{ss}$ are assumed to have the same individual rows as $Q_{j_\mu}^{ss} = \mathbf{1}_{r_{j_\mu}} c'$ for some $c' \in \Re^{1\times N_q}$ and for all $j_\mu \in \Omega_i$. It follows from corollary A.8 and its preceding discussion that $\{k_l\}$ exists such that $Q(k_l - 2) = \prod_{\eta=1}^{k_l-2} P'_{\lrcorner}(\eta) = \prod_{k=1}^{l} E_{\lrcorner}(k) \stackrel{\triangle}{=} Q'(k)$, where $E_{ii}(k) > 0$, and lower-triangular blocks $E_{ij}(k)$ have the property that $E_{ij} > 0$ for $j \in \Omega_i$ and $E_{ij} = 0$ for any $j < i$ and $j \notin \Omega_i$. Therefore, the ith block row of matrix equation $Q'(k+1) = E_{\lrcorner}(k+1)Q'(k)$ is

$$Q'_i(k+1) = E_{ii}(k+1)Q'_i(k) + \sum_{j\in\Omega_i} E_{ij}(k+1)Q'_j, \tag{17}$$

where $Q'_l(k) \in \Re^{r_i\times N_q}$ is the lth block row of $Q'(k)$. Applying lemma A.9 in appendix IV to equation (17) yields $Q_i^{ss} = \mathbf{1}_{r_i} c'$.

The third case deals with the situation that row blocks $Q_{j_\mu}^{ss}$ exist but their individual rows are not same for all $j_\mu \in \Omega_i$. In this case, Q_i^{ss} should be found by searching for an infinite subsequence $\{k'_v : v \in \aleph^+\}$ of $\aleph^+$ such that $k'_v - k'_{v-1} \geq \underline{k}$ for some large integer $\underline{k} > 0$, that $P'_{ij_\mu}(k) \neq 0$ for some k in every interval

(k'_{v-1}, k'_v) and for all $j_\mu \in \Omega_i$, and that

$$\lim_{v\to\infty} \left[R_{ii}(k'_v, k'_{v-1}) Q_i^{ss} + \sum_{j_\mu \in \Omega_i} R_{ij_\mu}(k'_v, k'_{v-1}) Q_{j_\mu}^{ss} \right] = Q_i^{ss}, \tag{18}$$

where $R(k'_v, k'_{v-1}) = \prod_{\eta=k'_{v-1}}^{k'_v} P'_{\searrow}(\eta)$. Then, it follows from the proof of lemma A.7 that, given equation (18), sequence $\{E_{\searrow}(k)\}$ can be chosen (or adjusted) such that

$$Q_i^{ss} = \lim_{k\to\infty} \left[E_{ii}(k+1) Q_i^{ss} + \sum_{j\in\Omega_i} E_{ij}(k+1) Q_j^{ss} \right].$$

Subtracting both sides of the above equation from those of (17) yields equation (37) with $\tilde{Q}_i(k) = Q'_i(k) - Q_i^{ss}$, $\tilde{R}_{j_v}(k) = Q'_{j_v}(k) - Q_{j_v}^{ss}$, and $j_v \in \Omega_i$. Thus, the convergence rate of σ_i can be concluded by repeating the steps following equation (37) and in the proof of lemma A.9 in appendix IV. □

The following corollary is stated in the simple form conducive to the subsequent designs of cooperative control. For a sequence of fixed block dimensions, its proof involves repeated applications of the second case in the proof of theorem III.7. For a sequence whose $P'_{ii}(k)$ are of different dimensions with respect to k, products of $\prod_{\eta=k_1}^{k_2} P'_{\searrow}(\eta)$ contain diagonal blocks that are irreducible, diagonally positive, and no smaller in size than those of $P'_{\searrow}(\eta)$. Hence, by corollary A.3 in appendix II, product sequence of $\prod_{\eta=1}^{\infty} P'_{\searrow}(\eta)$ can be regrouped into a new sequence of $\prod_{\eta'=1}^{\infty} P''_{\searrow}(\eta')$ such that diagonal blocks of $P''_{\searrow}(\eta')$ are all positive and of their largest dimensions, which implies that new sequence $\{P''_{\searrow}(\eta')\}$ must again have fixed block dimensions. Therefore, corollary III.8 can be concluded again from mimicking the second case in the proof of theorem III.7 but without the requirement of fixed block dimensions.

Definition III.4: Row stochastic matrix sequence $\{P(k) : k \in \aleph^+\}$ with $P(k)$ defined in (9) is said to be *sequentially lower-triangularly complete* if it is sequentially lower-triangular and if, in every row i of its canonical triangulation $P'_{\searrow}(k)$ in (15) (if sizes of block rows change, the larger one or their union of overlapping block rows should be considered; see example III.1, corollary III.5 and lemma III.6), there is at least one $j < i$ such that the corresponding block is uniformly non-vanishing as $\{P'_{ij}(k)\} \succ 0$. ◇

Corollary III.8: If sequence $\{P(k) : k \in \aleph^+\}$ with $P(k)$ defined in (9) is sequentially lower-triangularly complete, $\lim_{k\to\infty} \prod_{\eta=1}^{k} P(\eta) = \mathbf{1}_{N_q} c$ where $c \in \Re^{1\times N_q}$ may contain up to $\max_k r_1(k)$ non-zero elements.

As will be shown later in section IV, having a limit of $\mathbf{1}_{N_q} c$ can be viewed as convergence to a single cooperative behavior, while $Q_i^{ss} = \mathbf{1}_{r_i} c'_i$ implies multiple cooperative behaviors. Multiple behaviors arise naturally from the first case in the proof of theorem III.7 but, through cooperative control designs in section IV, they can be intentionally generated by imposing a diagonal structure to which corollary III.8 can also be applied.

The following two remarks further elaborate the concept of a matrix sequence being sequentially lower-triangularly complete.

Remark 5: Suppose that sequence $\{P(k) : k \in \aleph^+\}$ with $P(k)$ defined in (9) is sequentially lower-triangularly complete. If there exists j for every i such that $\{P'_{ij}(k)\} \neq 0$ at a given k, all those nodes of $S(t)$ being associated in the augmentation $P'_{11}(k)$ are globally reachable (according to the definition in [24]), and the directed graph corresponding to $S(t)$ has a spanning tree beginning at any one of those nodes (according to the definition in [22]). Thus, the proposed matrix-theory-based framework is not only very general (in terms of system dynamics and of convergence rate) but also complementary (as it admits the best results obtained using the graph theory). $\diamond$

Remark 6: It is clear from theorem III.7 and corollary III.8 that the convergence rate is determined by the lower-triangularly non-vanishing subsequence contained in a sequentially lower-triangularly complete sequence. Whenever the frequency of this subsequence is known, convergence over time can be concluded using the result in theorem III.7. This explicit result of convergence rate also applies to the general case, theorem III.9, in the next subsection. $\diamond$

C. A Necessary and Sufficient Condition on Sequence Convergence of Matrix Products

Although corollary III.8 no longer requires irreducibility of matrix $S(t)$ or $G(t)$, it is applicable only to the case that permutation matrix $\overline{T}(t)$ be time independent. In general, $\overline{T}(t)$ would be time varying, and $\prod_{\eta=1}^{k} P(\eta) = \prod_{\eta=1}^{k} \overline{T}^T(t_\eta^s) P'_{\triangle}(\eta) \overline{T}(t_\eta^s)$. Nonetheless, a necessary and sufficient condition on the convergence of $\prod_{\eta=1}^{k} P(\eta)$ can be explicitly found, and is stated in the following theorem. As such, the theorem provides the least restrictive condition for cooperative control design. Its proof brings together corollary III.8 and those in appendix V, especially lemma A.10 which provides a necessary and sufficient condition on convergence of a subsequence of lower-triangular matrices.

Definition III.5: Row stochastic matrix sequence $\{P(k) : k \in \aleph^+\}$ with $P(k)$ defined in (9) is said to be *sequentially complete* if an infinite multiplicative subsequence extracted from $\prod_{k=1}^{\infty} P(k)$ (or its re-grouped version) is sequentially lower-triangularly complete. $\diamond$

Theorem III.9: Consider the row stochastic matrix sequence $\{P(k) : k \in \aleph^+\}$ with $P(k)$ defined in (9). Then, its multiplicative sequence is convergent as $\lim_{k\to\infty} \prod_{\eta=1}^{k} P(\eta) = \mathbf{1}_{N_q} c$ where $c \in \Re^{1\times N_q}$ if and only if $\{P(k) : k \in \aleph^+\}$ is sequentially complete.

Proof: Necessity can easily be seen by contradiction and by invoking theorem III.7. In other words, if a sequentially lower-triangularly complete subsequence cannot be found in $\prod_{k=1}^{\infty} P(k)$, matrix $P(k)$ becomes a diagonal-block matrix for all $k > \overline{k}$, and the multiplicative sequence does not converge to a matrix of identical rows.

To show sufficiency, let $\{P(k_v) : v \in \aleph^+\}$ denote the sequentially lower-triangularly complete subsequence contained in $\{P(k) : k \in \aleph^+\}$. It follows that the infinite product $\prod_{k=1}^{\infty} P(k)$ can be grouped as

$$\prod_{k=1}^{\infty} P(k) = \prod_{v=1}^{\infty} \left\langle \left[\overline{T}_c P'_{\triangle}(k_v) \overline{T}_c^T\right] F(k_v) \right\rangle, \tag{19}$$

where $\overline{T}_c$ is a fixed and appropriate permutation matrix, $F(k_v)$ is the product of $P(k_v - 1) \cdots P(k_{v-1} + 1)$. Since subsequence $\{P(k_v) : v \in \aleph^+\}$ is sequentially lower-triangularly complete, so is subsequence

$\{P'_{\lrcorner}(k_v)\}$. It follows from corollary III.8 that $\prod_{v=1}^{\infty} P'_{\lrcorner}(k_v) = \mathbf{1}_{N_q} c$ with $c = [c_1 \ 0 \cdots ; 0] \in \Re^{1 \times N_q}$ and $c_1 \in \Re^{1 \times r_1}$. On the other hand, the multiplicative sequence (19) can be re-grouped as

$$\overline{T}_c^T \left[\prod_{k=1}^{\infty} P(k) \right] \overline{T}_c = \prod_{v=1}^{\infty} \left\langle P'_{\lrcorner}(k_v) \left[\overline{T}_c^T F(k_v) \overline{T}_c \right] \right\rangle,$$

in which diagonal elements of $F(k_v)$ are uniformly positive and so are all those of matrices $\left[\overline{T}_c^T F(k_v) \overline{T}_c \right]$ for all $v \in \aleph^+$. Thus, the convergence of $\overline{T}_c^T \left[\prod_{k=1}^{\infty} P(k) \right] \overline{T}_c$ and hence $\prod_{k=1}^{\infty} P(k)$ can be concluded by invoking lemmas A.10, A.11, and A.5. □

D. Explanation, Examples, and Comparisons

The concept of matrix sequence $\{P(k) : k \in \aleph^+\}$ being sequentially complete and its corresponding convergence result, theorem III.9, have explicit physical meanings in terms of the status of sensing/communication network. To this end, the following definitions are introduced.

Definition III.6: Systems of (1) or their sensor/communication matrix sequence $\{S(k), k \in \aleph^+\}$ with $S(k)$ defined in (2) are said to have a *sequentially lower-triangularly complete sensing/communication* if its permutation matrix $T_S^T(k)$ in (12) is independent of k (while p and dimensions of $S'_{ii}(k)$ may vary) and if, in every row i of its canonical triangulation $S'_{\lrcorner}(k)$ in (13) (if sizes of block rows change, the larger one or their union of overlapping block rows should be considered; see example III.1), there is at least one $j < i$ such that the corresponding element is uniformly non-vanishing as $\{S'_{ij}(k)\} \succ 0$. ◇

Definition III.7: Systems of (1) or their sensor/communication matrix sequence $\{S(k), k \in \aleph^+\}$ are said to have a *sequentially complete sensing/communication* if an infinite multiplicative subsequence extracted from sequence $\bigwedge_{k=1}^{\infty} S(k) \triangleq \lim_{k \to \infty} S(k) \bigwedge S(k-1) \bigwedge \cdots \bigwedge S(1)$ (or its regrouped version) forms a sequentially lower-triangularly complete sensing/communication, where $\bigwedge$ denotes a binary product of binary matrices (that is, binary multiplication and binary addition of the elements are used). ◇

It follows from corollary III.5 and lemma III.6 that definitions III.6 and III.7 are equivalent to definitions III.3, III.4 and III.5. The following example is to illustrate the concepts of sequentially lower-triangularly complete sensing/communication and sequentially complete sensing/communication.

Example III.1: Consider the sensing/communication sequence $\{S(k), k \in \aleph^+\}$ defined by $S(k) = S_1$ if k is odd and $S(k) = S_2$ if k is even, where

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix} \quad \Longrightarrow \quad S_2 \bigwedge S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Obviously, matrices S_1 and S_2 are reducible (with $p = 4, 3$, respectively). It follows from definition III.6 (applied to either the pair S_1, S_2 directly or their product $S_2 \bigwedge S_1$) that the sequence is sequentially lower-triangularly complete. $\diamond$

Example III.2: Suppose that a sensing/communication sequence $\{S(k), k \in \aleph^+\}$ is defined by $S(k) = S_1$ if $k = 3\eta + 1$, $S(k) = S_2$ if $k = 3\eta + 2$, and $S(k) = S_3$ if $k = 3\eta$, where $\eta \in \aleph$,

$$S_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad S_2 = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad S_3 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

It is apparent that matrices S_1, S_2 and S_3 are all reducible and that $S'_{\triangle}(3\eta+1) = S_1$ with $T_s(3\eta+1) = I$ and $p = 5$, $S'_{\triangle}(3\eta) = S_3$ with $T_s(3\eta) = I$ and $p = 5$, and

$$S'_{\triangle}(3\eta+2) = T_s^T(3\eta+2) S_2 T_s(3\eta+2) = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{with} \quad T_s(3\eta+2) = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and $p = 3$. Neither of three subsequences $\{S(3\eta+1)\}$, $\{S(3\eta+2)\}$, $\{S(3\eta)\}$ is sequentially lower-triangularly complete by itself. Nonetheless, sequence $\{S(k), k \in \aleph^+\}$ is sequentially complete because an infinite lower-triangularly complete subsequence $\{S''(k), k \in \aleph^+\}$ can be found by grouping the original sequence and then extracting subsequence from the (regrouped) original sequence. Specifically, the sequence $\bigwedge_{\eta'=1}^{\infty} S''(\eta')$ is lower-triangularly complete, where $S''(\eta') = S(3\eta'+1) \bigwedge S(3\eta'+2) = S_1 \bigwedge S_2$ if η' is odd and $S''(\eta') = S(3\eta'/2) = S_3$ if η' is even. This is because the sequence of repeating $\{S_1 \bigwedge S_2,\ S_3\}$ is sequentially lower triangularly complete according to definition III.6 and their canonical triangulations: setting $T_s \triangleq T_s(\eta') \triangleq T_s(3\eta+2)$ for all η',

$$T_s^T \left(S_1 \bigwedge S_2\right) T_s = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad T_s^T S_3 T_s = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

$\diamond$

Definition III.7 on sequentially complete sensing/communciation is defined and derived mathematically. Nonetheless, it is straightforward to see that the concept of sequentially complete sensing/communciation is equivalent to the systems having only one sensing/communication group which is defined below in layman's language.

Definition III.8: Systems of (1) are said to form *two (or more) sensing/communication groups* over finite time interval $[t-\Delta, t]$ for some $\Delta > 0$ (or over infinite interval $[t, \infty)$) if over the interval the set of vehicles' indices, $\mathcal{I} \stackrel{\triangle}{=} \{1, 2, \cdots, q\}$, can be divided into two (or more) disjoint, nonempty and complementary sets $\mathcal{S} \stackrel{\triangle}{=} \{i_1, i_2, \cdots, i_\mu\}$ and $\mathcal{S}^c \stackrel{\triangle}{=} \mathcal{I}/\mathcal{S} = \{j_1, j_2, \cdots, j_\nu\}$ (with $\mu + \nu = q$) such that there is no sensing/communication between any pair of vehicles i and j where $i \in \mathcal{S}$ and $j \in \mathcal{S}^c$ (that is, the lower triangular form of binary product $\left[\bigwedge_{t_k^s \in [t-\Delta, t]} S(t_k^s)\right]$ (or $\left[\bigwedge_{t_k^s \in [t, \infty)} S(t_k^s)\right]$) is block diagonal of some sizes). Alternatively, systems of (1) are said to have *only one sensing/communication group* if they do not form two or more sensing/communication groups. $\diamond$

The above discussion together with theorems III.7 and III.9 shows that, for cooperativeness and for its exponential-rate convergence, binary product of sensing/communication matrix $S(t)$ over certain interval can be reducible but every diagonal block in its lower triangular form must be coupled to other row blocks, as prescribed by the following theorem. Its proof is obvious by summarizing all the above results.

Theorem III.10: Binary matrix sequence $\{S(k)\}$ is sequentially complete and the corresponding sequence $\{P(k)\}$ in (9) is convergent if and only if systems of (1) have only one sensing/communication group over $[t, \infty)$ (for any $t \geq t_0$). Furthermore, sequence $\{P(k)\}$ is exponentially convergent with respect to a given interval length $\Delta > 0$ if, for all $\eta \in \aleph^+$, lower triangular forms of binary products $\left[\bigwedge_{t_k^s \in [t_0+(\eta-1)\Delta, t_0+\eta\Delta]} S(t_k^s)\right]$ are not block diagonal.

As discussed in remarks 5 and 6, theorem III.7 (except for its convergence rate analysis) can be viewed as the matrix-theoretical counter-part of known graph-theoretical results. Theorem III.9 is more general than theorem III.7 since all time-varying permutation matrices are admissible. Although there is no need to account for any specific vehicle (or node in terms of graph theory) in applying theorem III.9, one could still argue that theorem III.9 nonetheless mirrors known graph-theoretical results. Theorem III.10 is new because its concept of only one sensing/communication group is physically motivated (and more intuitive to non-experts while being necessary and sufficient mathematically) and, more importantly, real-time calculation of binary product of the changes of $S(k)$ over any interval is trivial, its corresponding lower triangular form can be easily found and monitored periodically, and both cooperative controllability and performance measure (of exponential convergence rate) can be determined by simply checking whether the resulting product's canonical form is block-diagonal.

IV. Designs of Cooperative Control

New convergence conditions revealed in corollary III.8, theorems III.9 and III.10 can directly be applied to cooperative control designs. As the main result of this paper, the following theorem can be concluded by invoking theorem III.10 and by noting that the internal dynamics are input-to-state stable.

Theorem IV.1: Consider dynamical systems in (1), under assumption 1, and under cooperative control (5) and (4). Then, systems of (1) exhibit a single cooperative behavior as,

$$x^{ss} = \mathbf{1}_{N_q} c x(t_0) = c_0 \mathbf{1}_{N_q}, \quad \text{and} \quad y^{ss} = c_0 \mathbf{1}_m, \quad c \in \Re^{1 \times N_q},\ c_0 \in \Re, \tag{20}$$

provided that: (i) Gain matrix K_c is chosen to be irreducible and row stochastic; (ii) Systems in (1) have only one sensing/communication group (or a sequentially complete sensing/communication). Furthermore, convergence to the cooperative behavior is exponential with respect to certain length Δ if systems in (1) have only one sensing/communication group over consecutive time intervals of period Δ.

In the above theorem, the first condition can easily and always be met in the control design, and the second condition defines cooperative controllability (i.e., the minimum requirement on operational environment). Several results can be derived from the general design result, theorem IV.1, and they are presented in the next three subsections.

A. Two Special Designs of Cooperative Control

The first result, implied by theorem IV.1 and given below as corollary IV.2, generalizes the existing result of cooperative control design in [21], [23] to dynamical systems of high-relative degree. Its proof is obvious as $S(t)$ being irreducible is equivalent to triangulation with $p = 1$ and, as shown in [38], is also equivalent to the corresponding directed sensor graph being strongly connected.

Corollary IV.2: Under assumption 1 and under cooperative control (5) and (4) with irreducible and row stochastic matrix K_c, systems of (1) exhibit a single cooperative behavior as described in (20) if their sensor/communication sequence $\{S(k)\}$ contains an infinite subsequence of irreducible matrices.

The second result, implied by theorem IV.1 and given below as corollary IV.3, deals with the case that communication pattern in the system has a lower triangular structure. In this case, the structure of $S'(k)$ with $T_s(k)$ being constant means that, in terms of sensing and communication, the system observers a leader-follower structure. Thus, corollary IV.3 (or simply corollary III.8) can be viewed as the result on leader-follower cooperative control, while theorem IV.1 deals with the leaderless cooperative control. As pointed out in remark 5, the corresponding graph has a spanning tree or a globally reachable node. Thus, theorem IV.1 and the following corollary extend the result in [22] to dynamical systems of high-relative degree as well as the results of [23], [24] to formation control of nonholonomic systems in a high-order chained form (see example V.3 in section V-B) and within a dynamically changing environment.

Corollary IV.3: Under assumption 1 and under cooperative control (5) and (4) with irreducible and row stochastic matrix K_c, systems of (1) exhibit a single cooperative behavior as described in (20) if their sensor/communication sequence $\{S(k)\}$ is sequentially lower-triangularly complete.

B. Multiple-Objective Cooperative Control

Utilizing the design flexibility embedded in gain matrix K_c, theorem IV.1 can be extended to the case that multiple cooperative behaviors are desired. Corollary IV.4, given below, specifies the cooperative control design for the case that each channel of vehicle output has a distinct behavior. Its proof is straightforward as K_c being diagonal yields completely decoupled dynamics among the m channels of all the vehicles and their associated state variables, and theorem IV.1 can be applied to each of the channels.

By analogy, one can work out a design to generate any given combination of behaviors (greater than one but less than m) among vehicles' output.

Corollary IV.4: Under assumption 1 and under cooperative control (5) and (4) with $K_c = I_{m\times m}$, systems of (1) exhibit m distinct cooperative behaviors described by

$$x^{ss} = \mathbf{1}_{L_q} \otimes c_0, \quad \text{and} \quad y_i^{ss} = c_0, \quad c_0 \in \Re^{m\times 1},$$

if their sensor/communication sequence $\{S(k)\}$ is sequentially complete.

C. Adaptive Cooperative Control

The single cooperative behavior described in (20) does not necessarily mean that, if $y^d = c_0^d \mathbf{1}_m$, the desired behavior represented by constant c_0^d is achieved. In order to ensure $c_0 = c_0^d$ in (20), we must employ an adaptive version of cooperative control (5). To this end, an virtual vehicle representing a hand-off operator is introduced as

$$\dot{x}_0 = -x_0 + u_0, \quad y_0(t) = x_0(t), \quad u_0 = K_c x_0(t),$$

where $x_0 \in \Re^m$ with $x_0(t_0) = c_0^d \mathbf{1}_m$. Communication from the virtual vehicle to the physical vehicles is also intermittent and local, thus we can introduce the following augmented sensor/communication matrix and its associated time sequence $\{\overline{t}_k^s : k = 0, 1, \cdots\}$ as:

$$\overline{S}(t) = \begin{bmatrix} 1 & 0 \ \cdots \ 0 \\ s_{10} & \\ \vdots & S(t) \\ s_{q0} & \end{bmatrix} \in \Re^{(q+1)\times(q+1)}, \quad \begin{cases} \overline{S}(t) = \overline{S}(\overline{t}_k^s), & \forall t \in [\overline{t}_k^s, \overline{t}_{k+1}^s) \\ \overline{S}(k) \triangleq \overline{S}(\overline{t}_k^s), & \end{cases} \tag{21}$$

Accordingly, cooperative control is modified from (5) and (4) to the following adaptive version:

$$u_i(t) = \sum_{j=0}^{q} \frac{s_{ij}(t)}{\sum_{\eta=0}^{q} s_{i\eta}(t)} K_c[s_{ij}(t) y_j], \quad i = 1, \cdots, q, \tag{22}$$

where $s_{ij}(t)$ are piecewise-constant entries (21). Applying theorem IV.1 to the resulting augmented closed loop system renders the following corollary.

Corollary IV.5: Under assumption 1 and under cooperative control (22) with irreducible and row stochastic matrix K_c, systems of (1) exhibit the desired cooperative behavior y^d, i.e.,

$$x^{ss} = \mathbf{1}_{L_q+1} \otimes y^d, \quad \text{and} \quad y_i^{ss} = y^d,$$

if $y^d = c_0^d \mathbf{1}_m$ for $c_0^d \in \Re$ and if their augmented sensor/communication sequence $\{\overline{S}(k)\}$ defined by (21) is sequentially complete.

If $y^d \neq c_0^d \mathbf{1}_m$, a multi-objective adaptive cooperative control can be designed by combining corollaries IV.4 and IV.5. The results corresponding to corollaries IV.2 and IV.3 can also be stated here.

V. Illustrative Examples

In subsection V-A, a few cooperative control objectives are explicitly formulated. Then, in subsection V-B, several examples are used to illustrate the decentralized vehicle-level control design (which, together with its associated state transformation, renders the canonical form of (3)). Finally, the proposed cooperative controls are simulated in subsection V-C to illustrate their performance.

A. Cooperate Control Objectives

For the vehicles described by (1), there are two choices in cooperative control design: group behavior described by vector y^d, and individual behavior described by vector $\psi_i^d(t)$. As has been shown in corollaries IV.4 and IV.5 that y^d can be chosen to be anything between the two extremes of $y^d = c_0^d \mathbf{1}_m$ for $c_0^d \in \Re$ and $y^d \in \Re^m$. In other words, through the choice of K_c, one or a given number of cooperative behaviors can be achieved for the group of vehicles.

On the other hand, some or all of the vehicles can also exhibit their individual behavior(s) by choosing among the following list (and more can be added):

- Consensus problem: $\psi_i^d(t) = 0$ for all $i = 1, \cdots, q$.
- Formation control problem: $\psi_i^d(t) = \int_{t_0}^t w^d(\tau)d\tau + \psi_i^d(t_0)$, where $w^d(t)$ is the desired velocity of the whole formation, and $\psi_i^d(t_0)$ is the relative position of the vehicle in the formation.

B. Vehicle Platforms

In general, the vehicles described by (1) are heterogeneous, i.e., the vehicles can be any combination of the following vehicle platforms or their equivalence/extensions. To simplify the notation, the subscript $_i$ denoting the ith vehicle is omitted below if the discussion is limited to just one vehicle.

Example V.1: A point-mass agent whose equation of motion is

$$\dot{\phi}_j = \phi_{j+1},\ j = 1, \cdots, l-1; \quad \dot{\phi}_l = v, \quad \psi = \phi_1, \tag{23}$$

where $\phi_j \in \Re^m$ are the state sub-vectors, ψ is the output, and $v \in \Re^m$ is the control. Then, under the state and input transformations of

$$x_j = \sum_{k=0}^{j-1} \frac{(j-1)!}{(j-1-k)!k!}\left(\phi_{k+1} - \frac{d^k}{dt^k}\psi_i^d(t)\right), \quad v = -\sum_{k=0}^{l-1} \frac{l!}{(l-k)!k!}\left(\phi_{k+1} - \frac{d^k}{dt^k}\psi_i^d(t)\right) + \frac{d^l}{dt^l}\psi_i^d(t) + u,$$

dynamical model of (23) is transformed into (3), and both are mathematically equivalent. $\diamond$

Example V.2: A simple model of *unmanned aerial vehicle* is [39]:

$$\begin{array}{lll} \dot{P}_x = P_V \cos(P_\gamma)\cos(P_\phi), & \dot{P}_y = P_V \cos(P_\gamma)\sin(P_\phi), & \dot{P}_h = P_V \sin(P_\gamma), \\ \dot{P}_V = \frac{T-D}{M} - g\sin(P_\gamma), & \dot{P}_\gamma = \frac{g}{P_V}\left(n\cos\delta - \cos(P_\gamma)\right), & \dot{P}_\phi = \frac{L\sin(P_\delta)}{mP_V\cos(P_\gamma)}, \end{array} \tag{24}$$

where P_x is the down-range displacement, P_y is the cross-range displacement, P_h is the altitude, P_V is the ground speed and is assumed to be equal to the airspeed, P_γ is the flight path angle, P_ϕ is the heading angle, T is the aircraft engine thrust, D is the drag, M is the aircraft mass, g is the gravity acceleration, L is the aerodynamic lift, and δ is the banking angle. Control variables are δ, engine thrust T, and load factor $n = L/gm$. Then, by defining the output $\psi = [P_x \; P_y \; P_h]^T \in \Re^3$ and under the input transformation of $v = [v_1, v_2, v_3]^T \in \Re^3$ where

$$\begin{aligned}
\delta &= \tan^{-1}\left[\frac{v_2\cos(P_\phi) - v_1\sin(P_\phi)}{\cos(P_\gamma)(v_3+g) - \sin(P_\gamma)(v_1\cos(P_\phi) + v_2\sin(P_\phi))}\right], \\
n &= \frac{\cos(P_\gamma)(v_3+g) - \sin(P_\gamma)(v_1\cos(P_\phi) + v_2\sin(P_\phi))}{g\cos\delta}, \\
T &= [\sin(P_\gamma)(v_3+g) + \cos(P_\gamma)(v_1\cos(P_\phi) + v_2\sin(P_\phi))]m + D,
\end{aligned}$$

system (24) is transformed into (23) with $m = 3$, $l = 2$, and no internal dynamics. $\diamond$

Example V.3: It is well known [40] that many nonholonomic systems (such as differential-driven wheeled mobile robots, car-like mobile robots, etc.) can be transformed into the chained form by state and control transformations. Although the chained form is not feedback linearizable, the following discussion shows that the proposed framework applies to cooperative formation control of a groups of vehicles, some or all of which involve those nonholonomic dynamics.

Without loss of any generality, let us assume that the 2-dimensional formation control objective be given by $\psi_i^d(t) = \int_{t_0}^t w^d(\tau)d\tau + \psi_i^d(t_0)$ where $w^d(t) \triangleq [w_1^d(t) \; w_2^d(t)]^T \in \Re^2$ and $\inf_{t \geq t_0} |w_j^d(t)| \geq \underline{w}^d > 0$ for $j = 1, 2$ and that the ith vehicle be described by the following $(4, 2)$ chained form (while other vehicles are described by either chained forms of same or different orders or by system (3)):

$$\begin{cases} \dot{\phi}_{i,1} = v_{i,1}, \\ \dot{\phi}_{i,2} = \phi_{i,3}v_{i,1}, \;\; \dot{\phi}_{i,3} = \phi_{i,4}v_{i,1}, \;\; \dot{\phi}_{i,4} = v_{i,2}, \end{cases} \qquad \psi_i = \begin{bmatrix} \phi_{i,1} \\ \phi_{i,2} \end{bmatrix}. \tag{25}$$

Then, for any given $w^d(t)$ and $\psi_i^d(t_0)$ and for a given $\Delta t > 0$, there exist individual open-loop steering controls [40], [41] and smooth closed-loop exponentially-stabilizing controls [42] such that, for the ith vehicle in chained form (25) as well as for other vehicles,

$$\|\psi_{j,1}(t_0 + \Delta t) - \psi_{j,1}^d(t_0 + \Delta t)\| < \underline{w}^d/(2q), \quad j \in \{i, \cdots, q\}. \tag{26}$$

Now, for $t \geq t_0 + \Delta t$, let us define the decentralized state transformation

$$x_i(t) = \begin{bmatrix} \phi_{i,1} - \psi_{i,1}^d \\ \phi_{i,2} - \psi_{i,2}^d \\ \phi_{i,1} - \psi_{i,1}^d + v_{i,1} - w_1^d \\ \phi_{i,3}v_{i,1} - w_2^d + \phi_{i,2} - \psi_{i,2}^d, \\ \phi_{i,1} + 2v_{i,1} + \dot{v}_{i,1} - \psi_{i,1}^d - 2w_1^d - \dot{w}_1^d \\ \phi_{i,2} + 2\phi_{i,3}v_{i,1} + \phi_{i,4}v_{i,1}^2 + \phi_{i,3}\dot{v}_{i,1} - \psi_{i,2}^d - 2w_2^d - \dot{w}_2^d \end{bmatrix} \triangleq \begin{bmatrix} x_{i,1} \\ x_{i,2} \\ x_{i,3} \\ x_{i,4} \\ x_{i,5} \\ x_{i,6} \end{bmatrix}, \tag{27}$$

and decentralized control mapping

$$\begin{aligned}
\ddot{v}_{i,1}(t) &= \psi^d_{i,1} + 3w^d_1 + 3\dot{w}^d_1 + \ddot{w}^d_1 - \phi_{i,1} - 3v_{i,1} - 3\dot{v}_{i,1} + u_{i,1}, \\
v_{i,2}(t) &= \frac{1}{v^2_{i,1}} \left\{ -[\phi_{i,2} + 2\phi_{i,3}v_{i,1} + \phi_{i,4}v^2_{i,1} + \phi_{i,3}\dot{v}_{i,1} - \psi^d_{i,2} - 2w^d_2 - \dot{w}^d_2] \right. \\
& \left. -[2\phi_{i,4}v_{i,1} + 2\phi_{i,3}\dot{v}_{i,1} - 2\dot{w}^d_2 + \phi_{i,3}v_{i,1} - w^d_2 + 3\phi_{i,4}v_{i,1}\dot{v}_{i,1} + \phi_{i,3}\ddot{v}_{i,1} - \ddot{w}^d_2] + u_{i,2} \right\}, \quad (28)
\end{aligned}$$

where initial conditions of $v_{i,1}(t_0 + \Delta t)$ and $\dot{v}_{i,1}(t_0 + \Delta t)$ are set to be

$$\left\{ \begin{aligned}
x_{i,3}(t_0 + \Delta t) &= \phi_{i,1}(t_0 + \Delta t) - \psi^d_{i,1}(t_0 + \Delta t) + v_{i,1}(t_0 + \Delta t) - w^d_1(t_0 + \Delta t) = 0, \\
x_{i,5}(t_0 + \Delta t) &= \phi_{i,1}(t_0 + \Delta t) + 2v_{i,1}(t_0 + \Delta t) + \dot{v}_{i,1}(t_0 + \Delta t) \\
& \quad -\psi^d_{i,1}(t_0 + \Delta t) - 2w^d_1(t_0 + \Delta t) - \dot{w}^d_1(t_0 + \Delta t) = 0.
\end{aligned} \right. \quad (29)$$

It is straightforward to verify that, under transformations (27) and (28), chained form (25) is mapped into canonical form (3) with $m = 2$ and $l_i = 3$. Applying the multiple-objective cooperative control design in section IV-B and applying the initial conditions in (26) and (29), we know that transformations (27) and (28) are globally well defined because

$$\begin{aligned}
|v_{i,1}(t)| &= |x_{i,3}(t) - x_{i,1}(t) + w^d_1| \\
&\geq \underline{w}^d - |x_{i,3}(t) - x_{i,1}(t)| \\
&= \underline{w}^d - \left| [Q_{i,3}(t) - Q_{i,1}(t)] \begin{bmatrix} x_{i,1}(t_0 + \Delta T) & x_{i,3}(t_0 + \Delta T) & x_{i,5}(t_0 + \Delta T) \end{bmatrix}^T \right| \\
&\geq \underline{w}^d - 2\sum_{j=1}^{q} |x_{j,1}(t_0 + \Delta T)| > 0, \quad (30)
\end{aligned}$$

where $Q(t) = \prod_{t \geq t^s_k} P(k)$ is the row-stochastic matrix solution, and $Q_{i,3}(t)$ and $Q_{i,1}(t)$ are the rows corresponding to $x_{i,3}(t)$ and $x_{i,1}(t)$, respectively. Further research is needed to remove the condition of $\inf_{t \geq t_0} |w^d_j(t)| \geq \underline{w}^d > 0$. One possibility is to use the smooth time varying state and control transformations recently introduced in [42] for global and smooth regulation of chained systems. $\diamond$

C. Simulation Results

In this subsection, the proposed cooperative control is simulated for a group of 3 vehicles described by (3). For the ease of presentation, two dimensional output (i.e., vehicles moving in a plane) is considered.

In the simulations, the sensing/communication matrix $S(t)$ is randomly switched among the following four topologies:

$$S_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; S_2 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; S_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \; S_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}. \quad (31)$$

The union of infinite subsequences of S_1 and S_3 is sequentially lower-triangularly complete. Thus, sensing/communication consisting of randomly switching S_1, S_2, S_3 and S_4 is sequentially complete.

To achieve a single cooperative behavior, K_c is set to be

$$K_c = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which is irreducible and row stochastic. To have multiple cooperative behaviors for the two dimensional outputs of vehicles, $K_c = I_{2\times 2}$ is chosen according to corollary IV.4.

C.1 Consensus of velocity/motion: $m = 2$ and $l_i = 1$ in (3).

Figure 1 shows the single cooperative behavior for (velocity) states of the vehicles, and figure 2 shows two different cooperative behaviors by output channel. In both simulations, the initial velocities are set to be $[0.5\ \ 0.2]^T$, $[0.2\ \ 0.5]^T$, and $[0.3\ \ 0.1]^T$, respectively.

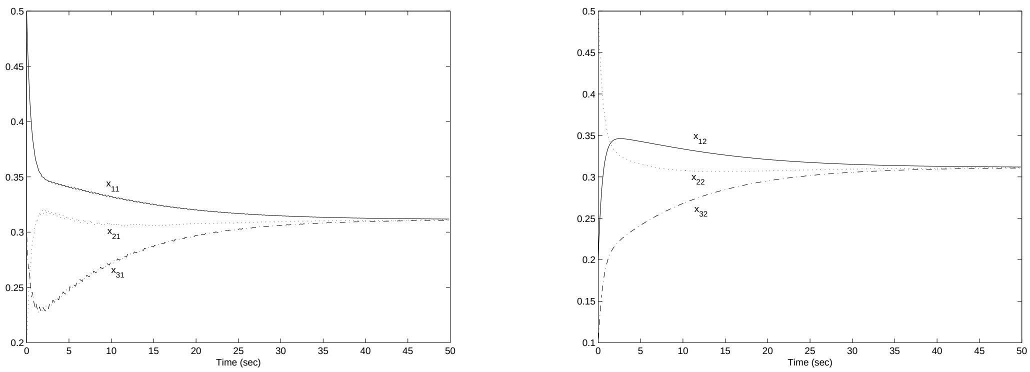


Fig. 1. Single cooperative behavior: Velocity convergence under cooperative control

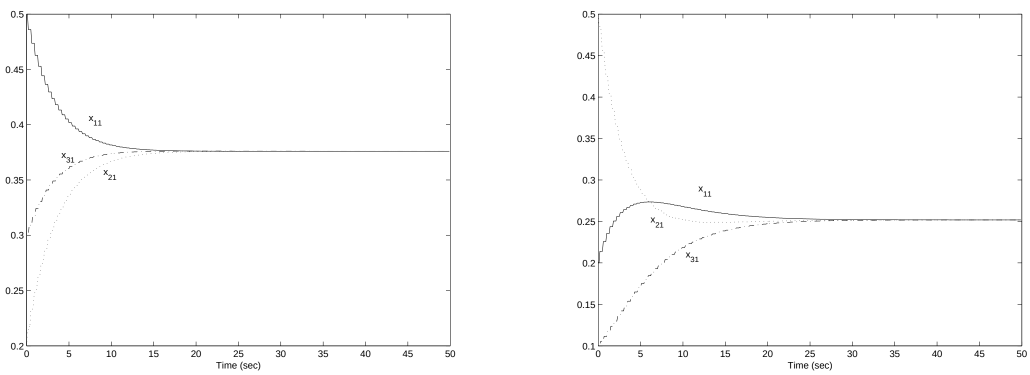


Fig. 2. Cooperative behaviors by output channel: Velocity convergence under cooperative control

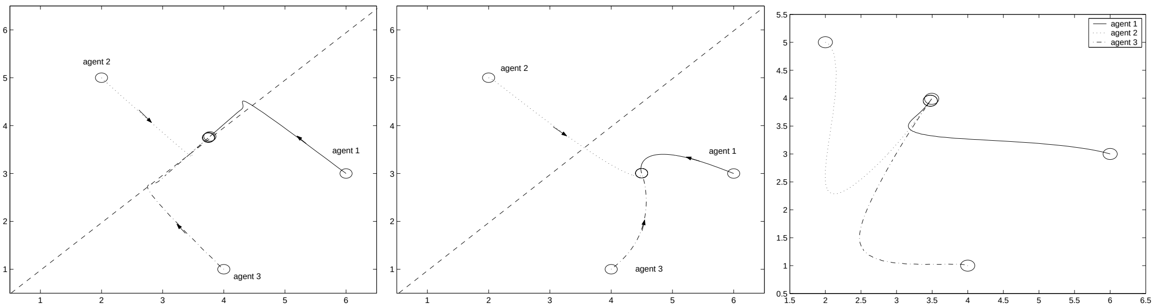
C.2 Rendezvous (consensus of positions): $m = 2$ and $l_i = 2$ in (3).

Suppose that initial positions are at $[6\ \ 3]^T$, $[2\ \ 5]^T$, and $[4\ \ 1]^T$, respectively. Figure 3a shows a convergent consensus of both channels, while figure 3b shows convergence of separate consensus for the two channels.

If the target position is specified to be $[3.5 \quad 4]^T$, the adaptive cooperative control design in section IV-C can be used. In the simulation, it is assumed that vehicle 1 receives information from the virtual vehicle about the target position. Then, the augmented sensor/communiction matrix $\overline{S}(t)$ is given by (21) together with (31), where s_{10} assumes a binary value randomly assigned, and $s_{20} = s_{30} = 0$. The control gains $G_{ij}(t)$ can be chosen according to remark 1 and given by

$$G_{ij} = \frac{w_{ij}s_{ij}}{\sum_{\eta=0}^{3} w_{i\eta}s_{i\eta}} K_c, \quad j = 0, 1, 2, 3,$$

with $w_{ij} = 0.9$ for $i \neq j$, and $w_{ii} = 0.1$. Convergence to the given target position is shown by figure 3c.

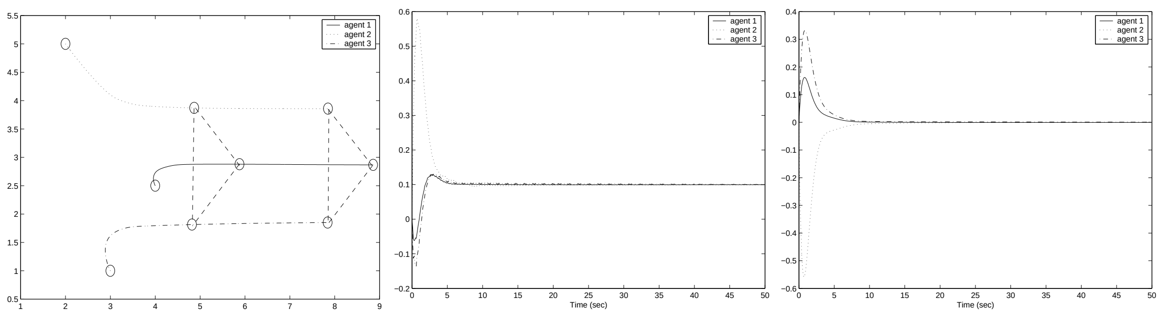


(3a) (3b) (3c)

Fig. 3. Rendezvous under cooperative controls

C.3 Formation Control: $m = 2$ and $l_i = 2$ in (3).

The desired formation trajectory is chosen to be one with velocity $\omega^d(t) = [0.1t \quad 0]^T$ and with a triangular formation of vertices at $\psi_1^d = [4 \quad 3]^T$, $\psi_2^d = [3 \quad 4]^T$, and $\psi_3^d = [3 \quad 2]^T$. Suppose that initial positions are $[4 \;\; 2.5]^T$, $[5 \;\; 2]^T$ and $[3 \;\; 1]^T$, respectively. Under the proposed cooperative control and the randomly-switched sensing/communication sequence, cooperative performance is shown by the simulation results in figure 4.



(4a) Phase portrait of formation convergence (4b) Horizontal velocities (4c) Vertical velocities

Fig. 4. Formation velocities under cooperative control

VI. Conclusion

In this paper, a matrix-theory-based framework is presented to design cooperative controls for a group of dynamical systems networked by dynamically changing communication/sensing. The framework contains a set of new results on augmentation of reducible and irreducible matrices so that, beginning with a simple sensing/communication matrix, dynamics of arbitrarily finite orders can be admitted into analysis and designs. For stability and convergence analysis, the framework develops a set of new results in terms of lower triangulation of reducible matrices for the overall closed loop system, including a necessary and sufficient condition on convergence on a sequence of time-varying multiplicative matrix sequence.

As enrichments to control theory, lower triangulation of reducible matrices and the corresponding convergence condition are fundamental to the understanding of the properties of networked dynamical systems, and their roles are analogous to that of Jordan decomposition for one linear dynamical system itself. Using the lower triangulation method, dynamical systems of any finite dimension can be studied, convergence rate can be explicitly obtained, and nonlinear and nonholonomic systems such as those in the chained form become admissible. By introducing a canonical form for cooperative controls, different objectives of cooperation such as individual behaviors, single group behavior, multiple group behaviors, and adaptive cooperation can also be embedded as a part of the proposed design framework so that the corresponding cooperative controls can be designed systematically in the same way.

The proposed framework is also complementary to the existing results obtained using graph theory. Since it is rooted in matrix theory and conducive to further incorporation of advanced control theory such as nonlinear systems and control, we believe that the proposed framework provides the means for solving more complicated problems, especially, analysis and control of networked systems whose dynamics are nonlinear and uncertain and whose sensing and/or communication are both time-varying and uncertain.

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Appendix

I. Non-Negative Matrices and Row Stochastic Matrices

Consider two matrices/vectors $E, F \in \Re^{r_1 \times r_2}$. The notations of $E = F$, $E \geq F$, and $E > F$ are defined with respect to all their elements as, for all i and j, $e_{ij} = f_{ij}$, $e_{ij} \geq f_{ij}$, and $e_{ij} > f_{ij}$, respectively. Operation $E = |F|$ of any matrix F is defined element-by-element as $e_{ij} = |f_{ij}|$. Matrix/vector E is said to be *non-negative* if $E \geq 0$ and *positive* if $E > 0$. Matrix $\mathbf{J}_{r\times r} \in \Re^{r\times r}$ and vector $\mathbf{1}_r \in \Re^r$ are the special positive matrix and vector, respectively, whose elements are all 1. Matrix E is said to be *binary* if its elements are either 0 or 1. Matrix $E \in \Re^{r\times r}$ is said to be *diagonally positive* if $e_{ii} > 0$ for all $i = 1, \cdots, r$.

A nonnegative square matrix $E \in \Re^{r\times r}$ is said to be *squarely row stochastic* or simply *row stochastic* if all the sums of its rows equal 1, that is, $E\mathbf{J}_{r\times r} = \mathbf{J}_{r\times r}$ or $E\mathbf{1}_r = \mathbf{1}_r$. Similarly, a nonnegative rectangular matrix $E \in \Re^{r_1 \times r_2}$ is said to be *rectangularly row stochastic* if $E\mathbf{1}_{r_2} = \mathbf{1}_{r_1}$.

Given a squarely or rectangularly row stochastic matrix $E \in \Re^{r_1 \times r_2}$, one can define the following two measures:

$$\delta(E) = \max_{1 \le j \le r_2} \max_{1 \le i_1, i_2 \le r_1} |e_{i_1 j} - e_{i_2 j}|, \quad \text{and} \quad \lambda(E) = 1 - \min_{1 \le i_1, i_2 \le r_1} \sum_{j=1}^{r_2} \min(e_{i_1 j}, e_{i_2 j}). \tag{32}$$

It is obvious that $0 \le \delta(E), \lambda(E) \le 1$ and that $\lambda(E) = 0$ if and only if $\delta(E) = 0$. Both quantities measure how different the rows of E are: $\delta(E) = 0$ if all the rows of E are identical, and $\lambda(E) < 1$ implies that, for every pair of rows i_1 and i_2, there exists a column j (which may depend on i_1 and i_2) such that both $e_{i_1 j}$ and $e_{i_2 j}$ are positive.

II. Summary of Existing Results on Reducible and Irreducible Matrices

A nonnegative matrix $E \in \Re^{r \times r}$ with $r \ge 2$ is said to be *reducible* if the set of its indices, $\mathcal{I} \triangleq \{1, 2, \cdots, r\}$, can be divided into two disjoint nonempty sets $\mathcal{S} \triangleq \{i_1, i_2, \cdots, i_\mu\}$ and $\mathcal{S}^c \triangleq \mathcal{I}/\mathcal{S} = \{j_1, j_2, \cdots, j_\nu\}$ (with $\mu + \nu = r$) such that $e_{i_\alpha j_\beta} = 0$, where $\alpha = 1, \cdots, \mu$ and $\beta = 1, \cdots, \nu$. Matrix E is said to be *irreducible* if it is not reducible. The following theorem provides the most basic property of a reducible matrix and that of an irreducible matrix, and its proof can be done by definition as shown in standard texts [43], [44]. Hence, the lower triangular structure of matrix $F_\searrow$ is the canonical form for reducible matrices.

Lemma A.1: Consider matrix $E \ge 0$ where $E \in \Re^{r \times r}$ and $r \ge 2$. If E is reducible, there exist an integer $p > 1$ and a permutation matrix T such that

$$T^T E T = \begin{bmatrix} F_{11} & 0 & \cdots & 0 \\ F_{21} & F_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{p1} & F_{p2} & \cdots & F_{pp} \end{bmatrix} \triangleq F_\searrow,$$

where $F_{ii} \in \Re^{r_i \times r_i}$ are square irreducible sub-matrices, and $\sum_{i=1}^{p} r_i = r$. If E is irreducible, vector $z' = (I_{r \times r} + E)z$ has more than η positive entries for any vector $z \ge 0$ containing exactly η positive entries, where $1 \le \eta < r$ and $I_{r \times r} \in \Re^{r \times r}$ is the identity matrix.

The following corollaries can directly be concluded from the above theorem.

Corollary A.2: Consider matrix $E \ge 0$ where $E \in \Re^{r \times r}$. Then, if and only if E is irreducible, inequality $\gamma z \ge Ez$ with constant $\gamma > 0$ and vector $z \ge 0$ implies either $z = 0$ or $z > 0$.

Corollary A.3: Consider matrix $E \ge 0$ where $E \in \Re^{r \times r}$. Then, E is irreducible if and only if $(cI_{r \times r} + E)^{r-1} > 0$ for any scalar $c > 0$. If all the matrices in sequence $\{E(k)\}$ are irreducible and diagonally positive, $E(k+\eta) \cdots E(k+1)E(k) > 0$ for some $1 \le \eta \le r-1$ and for all k.

III. Summary of Existing Results on Sequence Convergence of Matrix Products

Lemma A.4 given below summarizes the relevant results in [45], [23] and it links positiveness of the state transient matrix of a continuous-time system to irreducibility of its corresponding design matrix. It follows from (32) that $P > 0$ implies that both $\lambda(P) < 1$ and $\delta(P) < 1$, which provides the condition necessary for applying lemma A.5 in convergence analysis.

Lemma A.4: Consider $P = e^{(-I+E)\tau}$ where $E \in \Re^{r\times r}$ is a row stochastic matrix. Then, for every finite $\tau > 0$, matrix P is also row stochastic, and it is positive if and only if E is irreducible.

The following theorem provides the convergence result on a sequence of products of row stochastic matrices. It was first reported in [46] and then restated in [47], [23], and its proof is based upon the simple yet powerful inequality of

$$\delta\left(\prod_{\eta=1}^{k} P(\eta)\right) \leq \prod_{\eta=1}^{k} \lambda(P(\eta)) \tag{33}$$

for any $k > 0$ and for any sequence of (squarely) row stochastic matrices $\{P(k)\}$. Most of the existing results on cooperative control use this result for analysis of stability and convergence.

Lemma A.5: Given a sequence of (squarely) row stochastic matrices $\{P(k) \in \Re^{r\times r} : \ k = 1, \cdots\}$, consider the product $\prod_{\eta=1}^{k} P(\eta) = P(k)P(k-1)\cdots P(2)P(1)$. If inequality $0 \leq \lambda(P(k)) \leq c_p < 1$ holds for all k and for matrix function $\lambda(\cdot)$ defined in (32), there exists a row vector $c \in \Re^{1\times r}$ such that $\lim_{k\to\infty} \prod_{\eta=1}^{k} P(\eta) = \mathbf{1}_r c$. That is, the multiplicative sequence converges to a matrix of identical rows.

It should be noted that, if there are finite many distinct matrices $P(\eta)$ and if all their power sequences are known to be convergent, there are results available in [46], [47] to conclude convergence of the infinite sequence $\prod_{\eta=1}^{\infty} P(\eta)$.

IV. Lemmas Needed to Establish Theorem III.7

The following lemma provides an inequality that is useful for establishing convergence rate of matrix sequences.

Lemma A.6: Given any pair of squarely row stochastic $E \in \Re^{r_1\times r_1}$ and rectangularly row stochastic matrix $Q \in \Re^{r_1\times r_2}$, then $|Q - EQ| \leq \delta(Q)\mathbf{J}_{r_1\times r_2}$ where $\delta(\cdot)$ and $|\cdot|$ are defined in appendix I.

Proof: Letting $F = EQ$ yields that, for any $1 \leq j \leq r_2$, $f_{ij} = \sum_{l=1}^{r_1} e_{il} q_{lj}$ and hence

$$\min_{i\in\{1,\cdots,r_1\}} q_{ij} \leq \min_{i\in\{1,\cdots,r_1\}} f_{ij} \leq \max_{i\in\{1,\cdots,r_1\}} f_{ij} \leq \max_{i\in\{1,\cdots,r_1\}} q_{ij},$$

which implies $\delta(EQ) \leq \delta(Q)$. It follows from the above relationship that, for any $1 \leq j \leq r_2$,

$$\max_{i\in\{1,\cdots,r_1\}} (q_{ij} - f_{ij}) \leq \max_{i_1\in\{1,\cdots,r_1\}} q_{i_1 j} - \min_{i_2\in\{1,\cdots,r_1\}} f_{i_2 j} \leq \max_{i_1\in\{1,\cdots,r_1\}} q_{i_1 j} - \min_{i_2\in\{1,\cdots,r_1\}} q_{i_2 j},$$

and

$$\min_{i\in\{1,\cdots,r_1\}} (q_{ij} - f_{ij}) \geq \min_{i_2\in\{1,\cdots,r_1\}} q_{i_2 j} - \max_{i_1\in\{1,\cdots,r_1\}} f_{i_1 j} \geq \min_{i_2\in\{1,\cdots,r_1\}} q_{i_2 j} - \max_{i_1\in\{1,\cdots,r_1\}} q_{i_1 j}.$$

Thus, $|q_{ij} - f_{ij}| \leq \max_{i_1 \in \{1,\cdots,r_1\}} q_{i_1 j} - \min_{i_2 \in \{1,\cdots,r_1\}} q_{i_2 j}$ from which inequality $|Q - EQ| \leq \delta(Q)\mathbf{J}_{r_1 \times r_2}$ can be concluded using the definition of $\delta(\cdot)$ in (32). □

The following lemma shows an important property for a multiplicative sequence of lower triangular matrices which have non-vanishing off-diagonal blocks. In essence, lemma A.7 shows that, for the multiplicative sequence with positive diagonal blocks, $\{P_{ij}(k)\} \succ 0$ is equivalent to $\{E_{ij}(\eta)\} \succ 0$ once the product of $\prod_k P_{\lrcorner}(k)$ is grouped into a new product $\prod_\eta E_{\lrcorner}(\eta)$.

Lemma A.7: Consider a sequence of squarely row stochastic and lower triangular matrices $\{P_{\lrcorner}(k) : k \in \aleph^+\}$ of form

$$P_{\lrcorner}(k) = \begin{bmatrix} P_{11}(k) & 0 & \cdots & 0 \\ P_{21}(k) & P_{22}(k) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ P_{p1}(k) & P_{p2}(k) & \cdots & P_{pp}(k) \end{bmatrix} \in \Re^{r \times r}, \tag{34}$$

where its diagonal sub-matrices $P_{ii}(k) \in \Re^{r_i \times r_i}$ are all square, of fixed dimension, and uniformly positive with respect to k. If $\{P_{ij}(k)\} \succ 0$ for some $1 \leq j < i \leq p$, the product of $\prod_{k=1}^{\infty} P_{\lrcorner}(k)$ can be grouped into another of form $\prod_{\eta=1}^{\infty} E_{\lrcorner}(\eta)$ such that $E_{ij}(\eta) > 0$ for all $\eta \geq 2$ and $\prod_{\eta=1}^{\infty} E_{\lrcorner}(\eta) = \prod_{k=1}^{\infty} P_{\lrcorner}(k)$.

Proof: By definition, $\{P_{ij}(k)\} \succ 0$ implies that there is a sub-sequence $\{k_v, v \in \aleph^+\}$ of $\aleph^+$ such that $P_{ij}(k_v) \neq 0$ for all k_v and that $\lim_{v \to \infty} k_v = +\infty$. Now, choose a subsequence $\{k'_\eta, \eta \in \aleph^+\}$ of $\{k_v, v \in \aleph^+\}$ such that $k'_1 = \min_{k_v \geq 3} k_v$ and $k'_\eta - k'_{\eta-1} \geq 3$ and define a new matrix sequence $\{E_{\lrcorner}(\eta) :\ \eta \in \aleph^+\}$ with

$$E_{\lrcorner}(1) = P_{\lrcorner}(k'_1 - 2) \cdots P_{\lrcorner}(1), \quad \text{and} \quad E_{\lrcorner}(\eta + 1) \triangleq \prod_{k=k'_\eta - 1}^{k'_{\eta+1}-2} P_{\lrcorner}(k) = P_{\lrcorner}(k'_{\eta+1} - 2) \cdots P_{\lrcorner}(k'_\eta - 1) \quad \text{for } \eta \geq 1.$$

It is obvious that $\prod_{\eta=1}^{l} E_{\lrcorner}(\eta) = \prod_{k=1}^{k'_l - 2} P_{\lrcorner}(k)$.

Next, consider the pair of i and j (with $j < i$) at which $P_{ij}(k'_s) \neq 0$. It follows that the block of product $P_{\lrcorner}(k'_s + 1)P_{\lrcorner}(k'_s)$ corresponding to $P_{ij}(k)$ is

$$[P_{\lrcorner}(k'_s)P_{\lrcorner}(k'_s - 1)]_{ij} = \sum_{w=1}^{p} P_{iw}(k'_s)P_{wj}(k'_s - 1) = \sum_{w=j}^{i} P_{iw}(k'_s)P_{wj}(k'_s - 1) \geq P_{ij}(k'_s)P_{jj}(k'_s - 1).$$

Thus, since $P_{jj}(k) > 0$ for all k, a single positive element in any row of $P_{ij}(k'_s)$ makes all the elements of the corresponding row of $[P_{\lrcorner}(k'_s)P_{\lrcorner}(k'_s - 1)]_{ij}$ positive. Furthermore, it follows that

$$[P_{\lrcorner}(k'_s + 1)P_{\lrcorner}(k'_s)P_{\lrcorner}(k'_s - 1)]_{ij} = \sum_{w=j}^{i} P_{iw}(k'_s + 1)[P_{\lrcorner}(k'_s)P_{\lrcorner}(k'_s - 1)]_{wj} \geq P_{ii}(k'_s + 1)[P_{\lrcorner}(k'_s)P_{\lrcorner}(k'_s - 1)]_{ij},$$

which together with $P_{ii}(k) > 0$ implies that a positive row in $[P_{\lrcorner}(k'_s)P_{\lrcorner}(k'_s - 1)]_{ij}$ makes the whole block of $[P_{\lrcorner}(k'_s + 1)P_{\lrcorner}(k'_s)P_{\lrcorner}(k'_s - 1)]_{ij}$ positive. By induction, we have that, for all $s \in \aleph^+$,

$$E_{ij}(s + 1) \geq P_{ii}(k'_{s+1} - 2) \cdots P_{ii}(k'_s + 2)[P_{\lrcorner}(k'_s + 1)P_{\lrcorner}(k'_s)P_{\lrcorner}(k'_s - 1)]_{ij} > 0,$$

which completes the proof. □

Lemma A.7 shows how $\{P_{ij}(k)\} \succ 0$ in sequence $\{P_{\searrow}(k)\}$ renders $E_{ij}(s) > 0$ in the new sequence $\{E_{\searrow}(s)\}$ constructed from consecutive yet disjoint products of $P_{\searrow}(k)$ and of certain length. In other words, any nonzero element of $P_{ij}(k)$ would spread first along the row and then along the column to fill up the corresponding block in a successive matrix product of length 3 or more. Such a spread of positive entries from $\{P_{ij}(k)\} \succ 0$ happens primarily within the resulting block of the same row and column after multiplicative operations of triangular matrices. If $\{P_{i_1j_1}(k)\} \succ 0$ and $\{P_{i_2i_1}(k)\} \succ 0$ (with $j_1 < i_1 < i_2$), then their joint spreads could make $[\prod P_{\searrow}(k)]_{i_2j_1}$ nonzero. For example, consider $P_{\searrow}$ in (34) with $p = 4$ and with off-diagonal blocks $P_{ij} = 0$ except that $P_{32}(k) \neq 0$ and $P_{43}(k) \neq 0$. In this case, $[P_{\searrow}(k)P_{\searrow}(k-1)]_{42} \neq 0$. When this joint spread happens out of corresponding blocks, it could induce a chain of spreads should an appropriate group of non-vanishing blocks become nonzero at the same time instants. Even so, all spreads out of their own blocks can only happen to a finite numbers of blocks on the left and in the same row, and lemma A.7 can be applied again and again to handle those blocks with emerging positive entries. Hence, the following corollary can be proven by repeatedly applying lemma A.7.

Corollary A.8: Consider the matrix sequence defined in lemma A.7. If some of lower-triangular blocks have the property that $\{P_{ij}(k)\} \succ 0$, the product of $\prod_{k=1}^{\infty} P_{\searrow}(k)$ can be grouped into another of form $\prod_{\eta=1}^{\infty} E_{\searrow}(\eta)$ such that $\prod_{\eta=1}^{\infty} E_{\searrow}(\eta) = \prod_{k=1}^{\infty} P_{\searrow}(k)$, that all the lower-triangular blocks of $E_{\searrow}(\eta)$ are either zero or positive for $\eta \geq 2$, and that the positive and lower-triangular blocks include all of those corresponding to $\{P_{ij}(k)\} \succ 0$.

If $\{P_{ij}(k)\} \succ 0$, corollary A.8 says that $E_{ij_v} > 0$ for at least one $j_v \leq j$ including $j_v = j$ and that the rest of lower-triangular blocks on the same row are zero. This property together with $E_{ii} > 0$ is critical to establish the convergence result in the following lemma.

Lemma A.9: Given a finite number of sequences $\{R_{j_v}(k) : k \in \aleph^+\}$ (where $v \in \{1, \cdots, l\}$, $1 \leq j_v < i$, and $1 \leq l < i$) consisting of rectangularly row stochastic matrices $R_{j_v}(k) \in \Re^{r_{j_v} \times r}$, consider the matrix equation: $\forall k \in \aleph^+$,

$$Q_i(k+1) = E_{ii}(k+1)Q_i(k) + \sum_{v=1}^{l} E_{ij_v}(k+1)R_{j_v}(k), \tag{35}$$

where $Q_i(k) \in \Re^{r_i \times r}$ with $Q_i(1)$ being rectangularly row stochastic, and matrices $E_{ij_v}(k) \in \Re^{r_i \times r_{j_v}}$ and $E_{ii}(k) \in \Re^{r_i \times r_i}$ are uniformly positive for all k. Then, there exist a constant $0 \leq \sigma_i < 1$ and constant vector $c \in \Re^{1 \times r}$ such that $Q_i(k)$ is also rectangularly row stochastic,

$$|Q_i(k) - \mathbf{1}_{r_i} c| \leq \sigma_i^k \mathbf{J}_{r_i \times r}, \quad \text{and} \quad \lim_{k \to \infty} Q_i(k) = \mathbf{1}_{r_i} c, \tag{36}$$

provided that composite matrix $[E_{ij_1}(k) \ \cdots \ E_{ij_l}(k) \ E_{ii}(k)]$ is also rectangularly row stochastic for all k and that $\lim_{k\to\infty} R_{j_v}(k) = \mathbf{1}_{r_{j_v}} c$ and $|R_{j_v}(k) - \mathbf{1}_{r_{j_v}} c| \leq \sigma_{j_v}^k \mathbf{J}_{r_{j_v} \times r}$ with $0 \leq \sigma_{j_v} < 1$ and for all $v = 1, \cdots, l$.

Proof: It is straightforward to recursively verify through (35) that sequence $\{Q_i(k)\}$ is also rectangularly row stochastic. Define $\tilde{Q}_i(k) = Q_i(k) - \mathbf{1}_{r_i} c$ and $\tilde{R}_{j_v}(k) = R_{j_v}(k) - \mathbf{1}_{r_{j_v}} c$. It follows from (35) and from

$\sum_{v=1}^{l} E_{ij_v}(k)\mathbf{1}_{r_{j_v}} + E_{ii}(k)\mathbf{1}_{r_i} = \mathbf{1}_{r_i}$ that

$$\begin{aligned}
\tilde{Q}_i(k+1) &= E_{ii}(k+1)Q_i(k) + \sum_{v=1}^{l} E_{ij_v}(k+1)R_{j_v}(k) - \mathbf{1}_{r_i}c, \\
&= E_{ii}(k+1)\tilde{Q}_i(k) + \sum_{v=1}^{l} E_{ij_v}(k+1)\tilde{R}_{j_v}(k) \qquad (37) \\
&= \Phi(k+1,2)\tilde{Q}_i(1) + \sum_{v=1}^{l}\left[E_{ij_v}(k+1)\tilde{R}_{j_v}(k) + \sum_{\eta=1}^{k-1}\Phi(k+1,\eta+2)E_{ij_v}(\eta+1)\tilde{R}_{j_v}(\eta)\right],
\end{aligned}$$

where $\Phi(k_2,k_1) \triangleq \prod_{\eta=k_1}^{k_2} E_{ii}(\eta) = E_{ii}(k_2)\cdots E_{ii}(k_1)$. It follows from $|\tilde{R}_{j_v}(k)| \le \sigma_{j_v}^k \mathbf{J}_{r_{j_v}\times r}$ that

$$|\tilde{Q}_i(k+1)| \le \Phi(k+1,2)|\tilde{Q}_i(1)| + \sum_{v=1}^{l}\left[\sigma_{j_v}^k\ E_{ij_v}(k+1)\mathbf{J}_{r_{j_v}\times r} + \sum_{\eta=1}^{k-1}\sigma_{j_v}^s\ \Phi(k+1,\eta+2)E_{ij_v}(\eta+1)\mathbf{J}_{r_{j_v}\times r}\right]. \qquad (38)$$

On the other hand, it follows from $E_{ij_v}(k), E_{ii}(k) > 0$ and $\sum_{v=1}^{l} E_{ij_v}(k)\mathbf{1}_{r_{j_v}} + E_{ii}(k)\mathbf{1}_{r_i} = \mathbf{1}_{r_i}$ that, $\forall k$,

$$E_{ij_v}(k)\mathbf{J}_{r_{j_v}\times r} \le \sigma'_{j_v}\ \mathbf{J}_{r_i\times r},\ \ E_{ii}(k)\mathbf{J}_{r_i\times r} \le \sigma'_i\ \mathbf{J}_{r_i\times r},\ \ \Phi(k_2,k_1)\mathbf{J}_{r_i\times r} \le (\sigma'_i)^{k_2-k_1+1}\mathbf{J}_{r_i\times r}, \qquad (39)$$

hold for some constants $\sigma'_{j_v}, \sigma'_i \in (0,1)$. Noting that $|\tilde{Q}_i(1)| \le \mathbf{J}_{r_i\times r}$ and substituting (39) into (38) yield

$$|\tilde{Q}_i(k)| \le \left\{(\sigma'_i)^k + \sum_{v=1}^{l}\left[\sigma_{j_v}^k + \sum_{\eta=1}^{k-1}\sigma_{j_v}^s\ (\sigma'_i)^{k-\eta}\right]\ \sigma'_{j_v}\right\}\mathbf{J}_{r_i\times r} \triangleq \sigma_i^k \mathbf{J}_{r_i\times r}, \qquad (40)$$

in which $0 < \sigma_i < 1$ since power sequence $\{\sigma_i^k\}$ is the sum of scalar power sequences of $\{\sigma_{j_v}^k\}$, $\{(\sigma'_i)^k\}$, and their convolutions (all convergent to zero). Obviously, inequality (40) implies the results in (36). □

V. Lemmas Needed to Establish Theorem III.9

Given a multiplicative sequence of squarely row stochastic and lower triangular matrices with positive diagonal blocks, the following lemma provides a necessary and sufficient condition for it to converge to a matrix of identical rows. In comparison, lemma A.5 only provides a sufficient condition which, as shown by the discussion preceding theorem III.7, is often too restrictive to be directly applied to the sequence in lemma A.10.

Lemma A.10: Consider the sequence of squarely row stochastic and lower triangular matrices $\{P_{\searrow}(k) : k \in \aleph^+\}$ defined by (34) and in lemma A.7. Then, $\prod_{k=1}^{\infty} P_{\searrow}(k) = \mathbf{1}_r c$ for some $c \in \Re^{1\times r}$ if and only if the product of $\prod_{k=1}^{\infty} P_{\searrow}(k)$ can be grouped into another of form $\prod_{\eta=1}^{\infty} E_{\searrow}(\eta)$ such that $\prod_{k=1}^{\infty} P_{\searrow}(k) = \prod_{\eta=1}^{\infty} E_{\searrow}(\eta)$ and $0 \le \lambda(E_{\searrow}(\eta)) \le c_e < 1$ for all η.

Proof: Sufficiency of $0 \le \lambda(E_{\searrow}(\eta)) \le c_e < 1$ for all η to ensure $\prod_{\eta=1}^{\infty} E_{\searrow}(\eta) = \mathbf{1}_r c$ follows directly from lemma A.5. Hence, $\prod_{k=1}^{\infty} P_{\searrow}(k) = \prod_{\eta=1}^{\infty} E_{\searrow}(\eta) = \mathbf{1}_r c$.

The proof of necessity is based on the following two facts. First, for any lower triangular matrix $E_{\searrow}(k)$ with positive diagonal blocks, inequality $0 \le \lambda(E_{\searrow}(k)) \le c_e < 1$ holds if and only if all the blocks in the first block column are all non-zero. Second, theorem III.7 is used to show by construction that, in order to have $\mathbf{1}_r c$ as the limit for the product of sequence $\{E_{\searrow}(k)\}$, the spread of positive elements (which has been studied in lemma A.7) must fill up all the lower-triangular blocks in the first block column. In what follows, sequence $\{E_{\searrow}(\eta)\}$ is constructively found by grouping $\prod_k P_{\searrow}(k)$ into $\prod_{\eta_a} E_{\searrow}(\eta_a)$, then by grouping the first generation $\prod_{\eta_a} E_{\searrow}(\eta_a)$ into the second generation $\prod_{\eta_b} E_{\searrow}(\eta_b)$, and this process continues for a finite iterations. At the end, the last generation becomes $\prod_{\eta=1}^{\infty} E_{\searrow}(\eta) = \prod_{k=1}^{\infty} P_{\searrow}(k)$. Note that, although $E_{\searrow}(\cdot)$ is used in all the steps, its argument (η_a, η_b, and so on) defines the current generation of sequence $\{E_{\searrow}(\cdot)\}$. As before, denote $Q(\kappa) = \prod_{\eta=1}^{\kappa} E_{\searrow}(\eta)$ and $\lim_{\kappa\to\infty} Q(\kappa) = Q^{ss}$, their ith rows are $Q_i(\kappa)$ and Q_i^{ss}, and their (i,j)th blocks are $Q_{ij}(\kappa)$ and Q_{ij}^{ss}, respective.

Let us begin with the first block row of Q^{ss}. It follows that, by simply setting $E_{\searrow}(\eta_a) = P_{\searrow}(k)$ for $\eta_a = k$, block row $E_1(\eta_a)$ is zero except that $E_{11}(\eta_a) > 0$ for all η_a. Hence, $Q_1^{ss} = \mathbf{1}_{r_1} c$ where the limiting row vector c is of form $c = [c_1 \ 0 \ \cdots \ 0]$ where $c_1 \in \Re^{1\times r_1}$.

Next, consider the second block row of Q^{ss}. It follows from case 1 of theorem III.7 that, if $\{P_{21}(k)\}$ is vanishing, $Q_{22}^{ss} \neq 0$ or simply $Q_2^{ss} \neq \mathbf{1}_{r_2} c$. Therefore, we know that $\{P_{21}(k)\} \succ 0$ and that, by lemma A.7, the second generation $\{E_{\searrow}(\eta_b)\}$ can be found by grouping $\{E_{\searrow}(\eta_a)\}$ such that $E_{21}(\eta_b) > 0$ uniformly for all η_b. Hence, for rectangular matrix $E_{1\to 2}(\eta_b) \triangleq [\ E_1^T(\eta_b)\ E_2^T(\eta_b)\ ]^T$, $\lambda(E_{1\to 2}(\eta_b)) < 1$ holds uniformly for all η_b.

Further, consider the third block row of Q^{ss}. Similar to the second block row, we know from case 1 of theorem III.7 that either $\{P_{31}(k)\} \succ 0$ or $\{P_{32}(k)\} \succ 0$, which implies that either $\{E_{31}(\eta_b)\} \succ 0$ or $\{E_{32}(\eta_b)\} \succ 0$. If $\{E_{31}(\eta_b)\} \succ 0$, we know again from lemma A.7 that the third generation $\{E_{\searrow}(\eta_c)\}$ can be found by grouping $\{E_{\searrow}(\eta_b)\}$ such that $E_{31}(\eta_c) > 0$ uniformly for all η_c. Note that $E_{21}(\eta_b) > 0$ uniformly for all η_b implies $E_{21}(\eta_c) > 0$ uniformly for all η_c. On the other hand, if $\{E_{32}(\eta_b)\} \succ 0$, we know from $E_{21}(\eta_b) > 0$ for all η_b and from $[E_{\searrow}(\eta_b)E_{\searrow}(\eta_b - 1)]_{31} \geq E_{32}(\eta_b)E_{21}(\eta_b - 1)$ that $\{[E_{\searrow}(\eta_b)E_{\searrow}(\eta_b - 1)]_{31}\} \succ 0$, which brings us back to the previous case of generating a new re-grouped sequence. In short, the third generation $\{E_{\searrow}(\eta_c)\}$ can be generated from grouping $\{E_{\searrow}(\eta_b)\}$ such that $E_{21}(\eta_c) > 0$ and $E_{31}(\eta_c) > 0$ holds uniformly for all η_c. Hence, for rectangular matrix $E_{1\to 3}(\eta_c) \triangleq [\ E_1^T(\eta_c)\ E_2^T(\eta_c)\ E_3^T(\eta_c)\ ]^T$, $\lambda(E_{1\to 3}(\eta_c)) < 1$ is valid uniformly for all η_c.

For the rest of the rows of Q^{ss}, the proof can be done by repeating the basic argument for the second row except that there are more ways for a positive element to propagate to $E_{i1}(\cdot)$ and then fill in all its entries. At the end, we obtain sequence $\{E_{\searrow}(\eta)\}$ with $E_{i1}(\eta) > 0$ uniformly for all η and for all i, which yields $\lambda(E_{\searrow}(\eta)) < 1$ uniformly for all η. □

Remark 7: Sufficiency of lemma A.10 holds for any row stochastic sequence, but its necessity is valid only for sequences whose matrices are row stochastic and lower-triangular and have positive diagonal blocks. For example, consider sequence $\{P(k)\}$ defined by

$$P(1) = \mathbf{1}_3 c_1, \quad c_1 = \left[\begin{array}{ccc} 0.4 & 0.3 & 0.3 \end{array}\right]; \quad \text{and} \quad P(k) = \left[\begin{array}{ccc} 0.1 & 0 & 0.9 \\ 0 & 1 & 0 \\ 0.2 & 0 & 0.8 \end{array}\right] \quad \text{for all } k \geq 2.$$

Then, it follows that $\prod_{k=1}^{\infty} P(k) = \mathbf{1}_3 c_1$. However, the product of $\prod_{k=1}^{\infty} P(k)$ cannot be grouped into $\prod_{\eta=1}^{\infty} E(\eta)$ with $\lambda(E(\eta)) < 1$ since $\lambda\left(\prod_{k=l_1}^{l_2} P(k)\right) = 1$ for any $l_2 \geq l_1 > 1$. $\diamond$

The following lemma shows that the property of $\lambda(\cdot) < 1$ is always preserved no matter how a row stochastic matrix with positive diagonal elements is introduced into a product of row stochastic matrices.

Lemma A.11: Consider two squarely row stochastic matrices $E, F \in \Re^{r \times r}$ satisfying $\lambda(EF) < 1$. Then, if row stochastic matrix $W \in \Re^{r \times r}$ has positive diagonal elements (i.e., $w_{ii} > 0$ for all $i = 1, \cdots, r$), $\lambda(EWF) < 1$, $\lambda(WEF) < 1$, and $\lambda(EFW) < 1$.

Proof: Let us first show that $\lambda(EF) < 1$ implies $\lambda(EWF) < 1$. By definition, $\lambda(EF) < 1$ says that, for any i_1 and i_2, there exists j (depending on i_1 and i_2) such that $h_{i_1 j} > 0$ and $h_{i_2 j} > 0$ where $H = EF$,

$$h_{i_1 j} = \sum_{k=1}^{r} e_{i_1 k} f_{kj}, \quad \text{and} \quad h_{i_2 j} = \sum_{k=1}^{r} e_{i_2 k} f_{kj}.$$

In other words, there exist k_1, k_2 and j (all depending on i_1 and i_2) such that $e_{i_1 k_1}, f_{k_1 j}, e_{i_2 k_2}, f_{k_2 j} > 0$. On the other hand, we have

$$[EWF]_{i_1 j} = \sum_{k=1}^{r} [EW]_{i_1 k} f_{kj} \geq [EW]_{i_1 k_1} f_{k_1 j} = \left[\sum_{\mu=1}^{r} e_{i_1 \mu} w_{\mu k_1}\right] f_{k_1 j} \geq e_{i_1 k_1} w_{k_1 k_1} f_{k_1 j} > 0$$

and similarly $[EWF]_{i_2 j} \geq e_{i_2 k_2} w_{k_2 k_2} f_{k_2 j} > 0$. Since i_1 and i_2 are arbitrary, $\lambda(EWF) < 1$ is readily concluded.

Since $\lambda(EF) < 1$ implies $\lambda(EWF) < 1$, we know that, by setting $E' = EF$ and $F' = I$, $\lambda(EF) = \lambda(E'F') < 1$ implies $\lambda(EFW) = \lambda(E'WF') < 1$. Similarly, $\lambda(WEF) < 1$ can be shown. $\square$