HW #3: Solutions

Tuesday, March 13, 2018 11:18 AM

a) $H_0: Y \sim B(\lambda_0)$ $H_1: Y \sim B(1-\lambda_1)$

$$L(y) = \begin{cases} \lambda_1/1-\lambda_0 , & y=0 \\ 1-\lambda_1/\lambda_0 , & y=1 \end{cases}$$

$\delta(y)$ is zero when:

$y=0$ and $\frac{\lambda_1}{1-\lambda_0} < \eta$ or $y=1$ and $\frac{1-\lambda_1}{\lambda_0} < \eta$.

$\delta(y) = 1$ otherwise.

b) For threshold η, $\lambda_1 = \lambda_0$: $\frac{\lambda}{1-\lambda} < 1 < \frac{1-\lambda}{\lambda}$

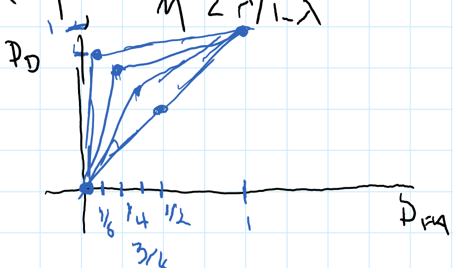
$$P_{FA} = \Pr[\delta(Y)=1 \mid H=0]$$

$$= \begin{cases} 0, & \text{if } \eta > 1-\lambda/\lambda \\ \lambda, & \text{if } \lambda/1-\lambda \le \eta \le 1-\lambda/\lambda . \\ 1, & \text{if } \eta < \lambda/1-\lambda \end{cases}$$

Only these three α's are possible w/out randomizing the test.

c) Need $P_D = \Pr[\delta(Y)=1 \mid H=1]$:

$$= \begin{cases} 0, & 1-\lambda/\lambda \\ 1-\lambda, & \lambda/1-\lambda > \eta > 1-\lambda/\lambda \\ 1, & \eta < \lambda/1-\lambda \end{cases}$$



d) For uniform priors, threshold is 1. Supposing $\lambda_0, \lambda_1 \le 1/2$:

$$\delta(y) = \begin{cases} 0, & y=0 \\ 1, & y=1 \end{cases}$$

e) For threshold $\pi_0/1-\pi_0$:

$$\delta(y) = \begin{cases} 0, & y=0, \ \frac{0.1}{1-0.2} < \frac{\pi_0}{1-\pi_0}, \ y=1, \ \frac{0.9}{0.2} < \frac{\pi_0}{1-\pi_0} \\ 1, & \text{otherwise} \end{cases}$$

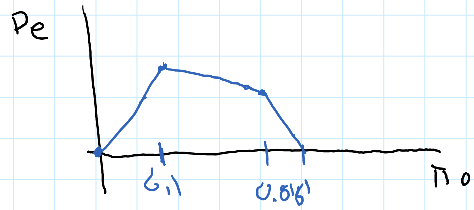
$$= \begin{cases} 0, & y=0, \ 0.125 \le \frac{\pi_0}{1-\pi_0}, \ y=1, \ 4.5 < \frac{\pi_0}{1-\pi_0} \\ 1, & \text{otherwise} \end{cases}$$

$\quad \quad 1 \quad \text{otherwise}$

So, $P_e = \begin{cases} \pi_0 \cdot 0 + (1-\pi_0)\cdot 1, & \pi_0/_{1-\pi_0} > 4.5 \\ \pi_0 \cdot \lambda_0 + (1-\pi_0)\lambda_1, & 0.125 < \pi_0/_{1-\pi_0} < 4.5 \\ \pi_0, & \pi_0/_{1-\pi_0} < 0.125 \end{cases}$

$$= \begin{cases} 1-\pi_0, & \pi_0 > 4.5(1-\pi_0) \Rightarrow 5.5\pi_0 > 4.5 \Rightarrow \pi_0 > 0.818 \\ \pi_0 \cdot 0.1 + (1-\pi_0)0.2 & 0.1 < \pi_0 < 0.818 \\ \pi_0, & \pi_0 \cdot (1.125) < 0.125 \Rightarrow \pi_0 < 0.1 \end{cases}$$

$$= \begin{cases} \pi_0, & \pi_0 < 0.1 \\ -0.1 \cdot \pi_0 + 0.2, & 0.1 < \pi_0 < 0.818 \\ 1-\pi_0, & \pi_0 > 0.818 \end{cases}$$



24. Need to calculate expected profits for each x:

$$E[3\cdot\theta|x_1] = \int 3\theta\, p(\theta|x_1)\, d\theta$$

$$= \int 3\theta \cdot \frac{p(x_1|\theta)\cdot p(\theta)}{p(x_1)} d\theta$$

$$= 3\int \theta \frac{0.8\exp(-\theta/100{,}000)\cdot\left(0.9\delta(\theta) + \frac{0.1}{300{,}000}\exp(-\frac{\theta}{300{,}000})\right)}{\int 0.8\exp(-\frac{\theta}{100{,}000})\left(0.9\delta(\theta) + \frac{0.1}{300{,}000}\exp(-\frac{\theta}{300{,}000}\right) d\theta} d\theta$$

$$= 3\int \frac{[p(x_1|\theta)p(\theta)]}{0.72 + \int_0^\infty \frac{0.08}{300{,}000}\exp(-\theta/300{,}000)d\theta}$$

$$= 3\int \frac{\theta[p(x_1|\theta)p(\theta)]}{0.8} d\theta$$

$$= 3.75\cdot\left[0 + \int_0^\infty \frac{0.08}{300{,}000}\theta\exp\left(-\frac{\theta}{100{,}000} - \frac{\theta}{300{,}000}\right)d\theta\right]$$

$$= 3.75[1500] = 5625.$$

$$E[\theta|x_2] = 3\int_0^\infty \theta\cdot\frac{p(x_2|\theta)p(\theta)}{p(x_2)} d\theta = 3\int_0^\infty \theta\, p(\theta)\, d\theta \quad (x_2 \text{ and } \theta \text{ are ind.})$$

$$= 3\int_0^\infty \theta\left[\delta(\theta)\cdot 0.9 + \frac{0.1}{300{,}000}\exp\left(\frac{-\theta}{300{,}000}\right)\right] d\theta$$

$$= 90{,}000$$

$$E[\theta \mid x_3] = 3 \int \frac{\theta\, p(x\mid\theta)\, p(\theta)}{\int 0.8\left(1-\exp\left(-\frac{\theta}{100{,}000}\right)\right)\left(0.9\,\delta(\theta) + \frac{0.1}{300{,}000}\exp\left(-\frac{\theta}{300{,}000}\right)\right) d\theta}\, d\theta$$

$$= 3 \int \frac{\theta\, p(x\mid\theta)\, p(\theta)}{0 + \frac{0.08}{300{,}000}\int (1-\exp())\cdot \exp()\, d\theta}$$

$$= 5 \cdot \int \theta \left(0.8\left(1-\exp\left(-\frac{\theta}{100{,}000}\right)\right)\right)\left(0.9\,\delta(\theta) + \frac{0.1}{300{,}000}\exp\left(-\frac{\theta}{300{,}000}\right)\right) d\theta$$

$$= 5 \int \theta \cdot \frac{0.08}{300{,}000}(1-\exp())(\exp())\, d\theta$$

$$= 5 \cdot 225{,}000 = 1{,}125{,}000.$$

Now, the Bayes risk for each decision is:

H_0: $100{,}000 - \text{profit}$

H_1: $100{,}000 - 70{,}000 - \frac{1}{2}\cdot\text{profit} = 30{,}000 - \frac{1}{2}\cdot\text{profit}$

H_2: $100{,}000 - 120{,}000 - \frac{1}{10}\cdot\text{profit} = -20{,}000 - \frac{1}{10}\cdot\text{profit}$

H_0 is better than H_1 when: $100{,}000 - \text{profit} < 30{,}000 - \frac{1}{2}\text{profit}$

$$70{,}000 < \tfrac{1}{2}\text{profit}$$
$$\text{profit} > 35{,}000.$$

H_0 is better than H_2 when: $100{,}000 - \text{profit} < -20{,}000 - \frac{1}{10}\text{profit}$.

$$120{,}000 < \tfrac{9}{10}\cdot\text{profit}$$
$$\text{profit} > 133{,}333$$

H_2 is better than H_1 when:

$$-20{,}000 - \tfrac{1}{10}\cdot\text{profit} < 30{,}000 - \tfrac{1}{2}\cdot\text{profit}$$
$$-50{,}000 < -\tfrac{2}{5}\,\text{profit}$$
$$\text{profit} < 125{,}000$$

So: a) choose H_2

b) choose H_2

c) choose H_0

11.10-33: H_0: $X \sim N([m_0, m_0, \ldots m_0]^T, \sigma^2 I)$

H_1: $X \sim N([m_0, m_0, \ldots, m_1, \ldots, m_1]^T, \sigma^2 I)$

a)

$$H_1: \ x \sim N(\{m_0, m_0, \ldots, m_1, \ldots, m_1\}^T, \sigma^2 I)$$

a)

$$\ell(y) = \frac{-1}{2\sigma^2}\sum_{i=1}^{n_0}\left[(x_i - m_0)^2 - (x_i - m_0)^2\right] - \frac{1}{2\sigma^2}\sum_{i=n_0}^{n}(x_i - m_1)^2 - (x_i - m_0)^2$$

$$= \frac{-1}{2\sigma^2}\sum_{i=n_0}^{n}\left(x^2 + m_1^2 - 2x_i m_1 - x^2 - m_0^2 + 2x_i m_0\right)$$

$$= \frac{-1}{2\sigma^2}\sum_{i=n_0}^{n} m_1^2 - m_0^2 - 2x_i(m_1 - m_0)$$

$$= C \cdot \sum_i (x_i - m_0)(m_1 - m_0) + m_0(m_1 - m_0) + C$$

So, by translation:

$$\frac{1}{n - n_0 + 1}\sum_{i=n_0}^{n}(x_i - m_0) \gtrless \eta$$

is the LRT

b) Under H_0, each $x_i - m_0$ is $N(0, \sigma^2)$, so

$$T(x) \sim N\left(0, \frac{\sigma^2}{(n - n_0 + 1)^2}\right)$$

Under H_1, each x_i is $N(m_1 - m_0, \sigma^2 I)$, so

$$T(x) = N\left(\frac{m_1 - m_0}{n - n_0 + 1}, \frac{\sigma^2}{(n - n_0 + 1)^2}\right)$$

So,

$$\boxed{P_{FA} = Q\left(\frac{T}{n - n_0 + 1}\right)}$$