Problem 24 Show that the posterior density $f_{\theta|x}(\vartheta/x)$ of example 12.6.5 is a $\Gamma\left(\alpha+m, \frac{1}{(1/\beta+y)}\right)$

Soln:

Given; $X_1, \ldots X_m$ samples from exponential distribution.

$$f_{x|\theta}(x/\vartheta) = \begin{cases} \theta e^{-\theta x} & ; \quad x>0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

prior $$f_\theta(\vartheta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} \vartheta^{\alpha-1} e^{-\vartheta/\beta} & ; \quad \vartheta>0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Then;

$$f_{\theta|x}(\vartheta/x) = \frac{f_{x|\theta}(x/\vartheta) \cdot f_\theta(\vartheta)}{f_x(x)}$$

$$\propto f_\theta(\vartheta) \prod_{i=1}^{m} (x_i|\theta)$$

$$\propto \left(\vartheta^{\alpha-1} e^{-\vartheta/\beta}\right) \cdot \prod_{i=1}^{m} \theta e^{-\theta x_i}$$

$$\propto \vartheta^{\alpha-1} e^{-\vartheta/\beta} \cdot \vartheta^m e^{-\vartheta y} \qquad y = \sum_{i=1}^{m} x_i$$

$$\propto \vartheta^{\alpha-1+m} e^{-\vartheta/\beta - \vartheta y}$$

$$\propto \vartheta^{(\alpha+m)-1} e^{-\vartheta((1/\beta+y))}$$

$$\propto \vartheta^{\alpha'-1} e^{-\vartheta/\beta'} \quad ; \text{ when } \alpha' = \alpha+m, \quad \beta' = 1/\beta+y$$

$$= \frac{1}{\beta'^{\alpha'} \Gamma(\alpha')} \vartheta^{(\alpha'-1)} e^{-\vartheta/\beta'} = \Gamma(\alpha', \beta')$$