1 OBJECTIVE

- Attitude Estimation based on Extended Kalman Filtering method.
- Gyro bias estimation to improve the robustness of the algorithm.
- Attitude prediction to be done using gyro output.
- Attitude is updated using accelerometer and magnetometer output.

2 GENERALIZED MODEL

- Prediction model: $X_k = f(X_{k-1}, u) + N(0, Q_k)$
- Measurement model: $Z = h(X_k) + N(0, R_k)$

Here Q and R are the process and measurement co-variance noises (Usually assumed to be zero mean Gaussian noise). X is the state vector (our case: $[q_0\ q_1\ q_2\ q_3\ wx_b\ wy_b\ wz_b]$). f and h are the jacobian matrices.

2.1 INITIAL CONDITION

Initially quadrotor is assumed to be in level condition. So the initial states are,

- Initial state vector: $X_0 = [1\ 0\ 0\ 0\ wx_b\ wy_b\ wz_b]$

Here gyro biases are obtained from sensor calibration.

2.2 CO-VARIANCE MATRICES

Measurement co-variance matrix:

$$R = \begin{bmatrix} 100 & 0 & 0 & 0 & 0 & 0 \\ 0 & 100 & 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 10000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 10000 & 0 \\ 0 & 0 & 0 & 0 & 0 & 10000 \end{bmatrix} \tag{2.1}$$

Process co-variance matrix:

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8.e^{-7} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 8.e^{-7} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 8.e^{-7} \end{bmatrix} \tag{2.2}$$

Error co-variance matrix:

$$P = \begin{bmatrix} 1000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.3}$$

P takes a large value initially because it is assumed that we don't have the knowledge of current orientation and gyro biases.

3 Prediction model

- Quaternion prediction: $q_k \;=\; q_{k-1} \;+\; dt\,\dot{q}_k$

$$\implies \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}_k = \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ q_3 \end{bmatrix}_{k-1} + \frac{dt}{2} \begin{bmatrix} -q_1 & -q_2 & -q_3 \\ q_0 & -q_3 & q_2 \\ q_3 & q_0 & -q_1 \\ -q_2 & q_1 & q_0 \end{bmatrix} \begin{bmatrix} wx - wx_b \\ wy - wy_b \\ wz - wz_b \end{bmatrix} \tag{3.1}$$

3.1 State vector matrix

$$f(X_{k-1}, u) = \begin{bmatrix} q_0 + \frac{dt}{2}(-q_1(wx - wx_b) - q_2(wy - wy_b) - q_3(wz - wz_b)) \\ q_1 + \frac{dt}{2}(q_0(wx - wx_b) - q_3(wy - wy_b) + q_2(wz - wz_b)) \\ q_2 + \frac{dt}{2}(q_3(wx - wx_b) + q_0(wy - wy_b) - q_1(wz - wz_b)) \\ q_3 + \frac{dt}{2}(-q_2(wx - wx_b) + q_1(wy - wy_b) + q_0(wz - wz_b)) \\ wx_b \\ wy_b \\ wz_b \end{bmatrix} \tag{3.2}$$

3.1.1 JACOBIAN MATRIX (F) - DERIVATIVE OF STATES

$$F = \begin{bmatrix} 1 & -\frac{dt}{2}(wx-wx_b) & -\frac{dt}{2}(wy-wy_b) & -\frac{dt}{2}(wz-wz_b) & \frac{q_1 dt}{2} & \frac{q_2 dt}{2} & \frac{q_3 dt}{2} \\ \frac{dt}{2}(wx-wx_b) & 1 & \frac{dt}{2}(wz-wz_b) & -\frac{dt}{2}(wy-wy_b) & -\frac{q_0 dt}{2} & \frac{q_3 dt}{2} & -\frac{q_2 dt}{2} \\ \frac{dt}{2}(wy-wy_b) & -\frac{dt}{2}(wz-wz_b) & 1 & \frac{dt}{2}(wx-wx_b) & -\frac{q_3 dt}{2} & -\frac{q_0 dt}{2} & \frac{q_1 dt}{2} \\ \frac{dt}{2}(wz-wz_b) & \frac{dt}{2}(wy-wy_b) & -\frac{dt}{2}(wx-wx_b) & 1 & \frac{q_2 dt}{2} & -\frac{q_1 dt}{2} & -\frac{q_0 dt}{2} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.3}$$

4 SENSOR MODEL FORMULATION

The measurement vector $Z = [ax\ ay\ az\ mx\ my\ mz]^T$.

4.1 ACCELEROMETER MAPPING

$$\begin{bmatrix} ax \\ ay \\ az \end{bmatrix} = R(q).\vec{g} = \begin{bmatrix} -2(q_1q_3 - q_0q_2) \\ -2(q_2q_3 + q_0q_1) \\ -(1-2{q_1}^2 - 2{q_2}^2) \end{bmatrix} \tag{4.1}$$

4.2 MAGNETOMETER MAPPING

$$\begin{bmatrix} mx \\ my \\ mz \end{bmatrix} = \begin{bmatrix} (1-2{q_2}^2-2{q_3}^2) & 2(q_1q_2 - q_3q_0) & 2(q_1q_3+q_2q_0) \\ 2(q_1q_2+q_3q_0) & (1-2{q_1}^2-2{q_3}^2) & 2(q_2q_3-q_1q_0) \\ 2(q_1q_3-q_2q_0) & 2(q_2q_3+q_1q_0) & (1-2{q_1}^2-2{q_2}^2) \end{bmatrix} \begin{bmatrix} bx \\ by \\ bz \end{bmatrix} \tag{4.2}$$

It is clear from the Eq.4.2 that if this model is used then there will be an issue in estimating the roll and pitch angles accurately since the magnetometer readings gets affected by the magnetic interference. Hence in order to estimate roll and pitch angles precisely, the measured magnetic field vector $[mx\ my\ mz]^T$ is transformed from body frame to inertial frame $[mx'\ my'\ mz']^T$. Then the new reference frame $[bx\ by\ bz]^T$ is computed assuming declination to be zero[1].

$$\begin{bmatrix} mx' \\ my' \\ mz' \end{bmatrix} = \begin{bmatrix} (1-2{q_2}^2-2{q_3}^2) & 2(q_1q_2 - q_3q_0) & 2(q_1q_3+q_2q_0) \\ 2(q_1q_2+q_3q_0) & (1-2{q_1}^2-2{q_3}^2) & 2(q_2q_3-q_1q_0) \\ 2(q_1q_3-q_2q_0) & 2(q_2q_3+q_1q_0) & (1-2{q_1}^2-2{q_2}^2) \end{bmatrix} \begin{bmatrix} mx \\ my \\ mz \end{bmatrix} \tag{4.3}$$

$$\begin{bmatrix} bx \\ by \\ bz \end{bmatrix} = \begin{bmatrix} \sqrt{mx'^2 + mx'^2} \\ 0 \\ mz' \end{bmatrix} \tag{4.4}$$

These new reference values are then substituted into the original mapping as shown in Eq 4.5,

$$\begin{bmatrix} mx \\ my \\ mz \end{bmatrix} = \begin{bmatrix} bx(1-2q_2{}^2-2q_3{}^2)+2bz(q_1q_3-q_0q_2) \\ 2bx(q_1q_2-q_0q_3)+2bz(q_2q_3+q_0q_1) \\ 2bx(q_1q_3+q_0q_2)+bz(1-2q_1{}^2-2q_2{}^2) \end{bmatrix} \tag{4.5}$$

4.3 Measurement model

$$h(X_k) = \begin{bmatrix} ax \\ ay \\ az \\ mx \\ my \\ mz \end{bmatrix} = \begin{bmatrix} -2(q_1q_3-q_0q_2) \\ -2(q_2q_3+q_0q_1) \\ -(1-2q_1{}^2-2q_2{}^2) \\ bx(1-2q_2{}^2-2q_3{}^2)+2bz(q_1q_3-q_0q_2) \\ 2bx(q_1q_2-q_0q_3)+2bz(q_2q_3+q_0q_1) \\ 2bx(q_1q_3+q_0q_2)+bz(1-2q_1{}^2-2q_2{}^2) \end{bmatrix} \tag{4.6}$$

4.3.1 Jacobian matrix (H)- derivative of states

$$H = \begin{bmatrix} 2q_2 & -2q_3 & 2q_0 & -2q_1 & 0 & 0 & 0 \\ -2q_1 & -2q_0 & -2q_3 & -2q_2 & 0 & 0 & 0 \\ 0 & 4q_1 & 4q_2 & 0 & 0 & 0 & 0 \\ -2q_2bz & 2q_3bz & -4q_2bx-2q_0bz & -4q_3bx+2q_1bz & 0 & 0 & 0 \\ -2q_3bx+2q_1bz & 2q_2bx+2q_0bz & 2q_1bx+2q_3bz & -2q_0bx+2q_2bz & 0 & 0 & 0 \\ 2q_2bx & 2q_3bx-4q_1bz & 2q_0bx-4q_2bz & 2q_1bx & 0 & 0 & 0 \end{bmatrix} \tag{4.7}$$

5 EKF algorithm

5.1 Prediction

Estimated state:

$$X_k = f(X_{k-1}, u) \qquad Ref: Eq\ 3.2 \tag{5.1}$$

Predict error co-variance matrix:

$$P = F\,P\,F^T + Q \qquad Ref: Eq\ 3.3 \tag{5.2}$$

5.2 Update

Measurement residual:

$$y = Z - h(X_k) \qquad Ref: Eq\ 4.6 \tag{5.3}$$

Residual co-variance:

$$S = H\,P\,H^T + R \qquad Ref : Eq\ 4.7, 5.2 \tag{5.4}$$

Kalman Gain:

$$K = P\,H^T\,S^{-1} \qquad Ref : Eq\ 4.7, 5.2, 5.4 \tag{5.5}$$

State estimate update:

$$X_k = X_k + K\,y \qquad Ref : Eq\ 5.1, 5.5, 5.3 \tag{5.6}$$

Error Co-variance update:

$$P = (I - K\,H)\,P \qquad Ref : Eq\ 5.5, 4.7, 5.2 \tag{5.7}$$

The prediction and update step is repeated continuously to obtain the instantaneous states.

6 Quaternion to Euler angle conversion

$$\begin{bmatrix} Roll\ (\phi) \\ Pitch\ (\theta) \\ Yaw\ (\psi) \end{bmatrix} = \frac{180}{\pi} \begin{bmatrix} tan^{-1}\left(\frac{2q_0q_1+q_2q_3}{1-2(q_1{}^2+q_2{}^2)}\right) \\ sin^{-1}(2q_0q_2 - q_1q_3) \\ tan^{-1}\left(\frac{2q_1q_2+q_0q_3}{1-2(q_2{}^2+q_3{}^2)}\right) \end{bmatrix} \tag{6.1}$$

Note: This filter was designed using various sources. Few errors has been found in the equations provided in the reference [1]. Those are rectified in this document and the validated C code is provided in the Appendix segment.

REFERENCES

[1] Matthew Watson, “The-Design-and-Implementation-of-a-Robust-AHRS-for-Implementation-on-a-Quadrotor".