Linear Algebra First Class

#2/8/2024

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Group

Definition

- $G \neq \phi$
- $*$ is an operation on G with following properties:
  - Closer property

    $* : G \times G \to G$
  - Associative property

    $a * (b * c) = (a * b) * c$
  - Existence of Identity

    $\exists e \in G \ni a * e = a = e * a, \forall a \in G$
  - Existence of Inverse

    $\forall a \in G, \exists b \in G \ni a * b = e = b * a \quad (b = a^{-1})$

Exercise

1) e is unique

Let, $\exists e_1, e_2 \ni a * e_1 = a = e_1 * a \,\&\, a * e_2 = a = e_2 * a, \forall a \in G$

$\Rightarrow a * e_1 = a = a * e_2$

$\Rightarrow a * e_1 = a * e_2$

$\Rightarrow a^{-1} * (a * e_1) = a^{-1} * (a * e_2)$ (Existence of Inverse)

$\Rightarrow (a^{-1} * a) * e_1 = (a^{-1} * a) * e_2$ (Associativity)

$\Rightarrow e_1 = e_2$

$\Rightarrow e$ is unique

2) $\forall a \in G, a^{-1}$ is unique

Let, $\forall a \in G, \exists b_1, b_2 \ni a * b_1 = e = b_1 * a \,\&\, a * b_2 = e = b_2 * a$
$\Rightarrow a * b_1 = e = a * b_2$
$\Rightarrow a * b_1 = a * b_2$
$\Rightarrow a^{-1} * (a * b_1) = a^{-1} * (a * b_2)$ (Existence of Inverse)
$\Rightarrow (a^{-1} * a) * b_1 = (a^{-1} * a) * b_2$ (Associativity)
$\Rightarrow b_1 = b_2$
$\Rightarrow b = a^{-1}$ is unique

Two notations for group

(G,$\cdot$)

$ab := a \cdot b$
$1 :=$ identity
$a^{-1} :=$ inverse of a
$k \in \mathbb{N} \Rightarrow a^k = a \cdot a \cdot \ \ldots \ \cdot a$ (k times)

$(H, +)$

$a + b :\neq ab$
$0 :=$ identity $:\neq 1$
$-a :=$ inverse of $a :\neq a^{-1}$
$k \in \mathbb{N} \Rightarrow ka := a + a + \cdots + a$ (k times) $:\neq a^k$

Abelian group

a group follows commutative $\Rightarrow$ it is an abelian group
$\forall a, b \in (G, *), a * b = b * a$