L.A

V, W – V.S over F

$L(V, W) = \{T : V \to W \mid T \text{ is a l.t}\}$

$\hookrightarrow$ V.S over F

Further if $\dim_F V = n$, $\dim_F W = m$

then $\dim_F L(V, W) = m.n$

—— x ——

$V = W$, $L(V, V) = L(V)$

$\left.\begin{array}{l} T_1 : U \longrightarrow V \\ T_2 : V \longrightarrow W \end{array}\right\}$ l.ts

$\Rightarrow T_2 T_1 : U \to W$ is a l.t

T_1, T_2 – bijective $\Rightarrow T_2 . T_1$ is bijective

in $L(V)$, $T_1 + T_2 : V \longrightarrow V$, $v \longmapsto T_1(v) + T_2(v)$

$\lambda T_1 : V \longrightarrow V$, $v \longmapsto \lambda(T_1(v))$

$T_1 T_2 : V \longrightarrow V$, $v \longmapsto (T_1 T_2)(v)$

$= T(T_2(v))$

<u>Exercise</u>

Let $T_1, T_2, S \in L(V)$

& $\lambda \in F$

Then (a) $IS = S = SI$, I – id on V.

(b) $S(T_1 + T_2) = ST_1 + ST_2$

$(T_1 + T_2)S = T_1 S + T_2 S$

(c) $S(T_1 T_2) = (ST_1)T_2$

(d) $\lambda(T_1 T_2) = (\lambda T_1)T_2 = T_1(\lambda T_2)$

~~$L(V)$ is an F-algebra with~~

(*) $L(V)$ is an associative F-algebra with multiplicative identity.

Matrix representation of linear transformations:

V, W - v. spaces over F

$\dim_F V = n$, $\dim_F W = m$

$B = \{v_1, v_2, \cdots, v_n\} \rightarrow$ ordered basis for V

$C = \{w_1, w_2, \cdots, w_m\} \rightarrow$ " " " W

$T : V \longrightarrow W$ is a l.t

for each $j \in \{1, 2, \cdots, n\}$

let $[T(v_j)]_C = \begin{bmatrix} \lambda_{1j} \\ \lambda_{2j} \\ \vdots \\ \lambda_{mj} \end{bmatrix}$

i.e $T(v_j) = \lambda_{1j} w_1 + \lambda_{2j} w_2 + \cdots + \lambda_{mj} w_m$

$$[T]_B^C = \begin{bmatrix} \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1k} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & & \lambda_{2k} & & \lambda_{2n} \\ \lambda_{31} & \lambda_{32} & & \lambda_{3k} & & \lambda_{3n} \\ \vdots & \vdots & & \vdots & & \vdots \\ \lambda_{m1} & \lambda_{m2} & & \lambda_{mk} & & \lambda_{mn} \end{bmatrix}$$

$[T]_B^C$ is called the matrix representation of T w.r.t the ordered bases B & C

* $T_1, T_2 : V \to W$ are l.ts
  if $[T_1]_B^C = [T_2]_B^C$; then $T_1 = T_2$.

Define

$$f : \mathcal{L}(V, W) \to M_{m \times n}(F)$$

$$T \mapsto f(T) = [T]_B^C$$

- f is one-one
- f is a l.t : $f(T_1 + T_2) = f(T_1) + f(T_2)$
  $f(\lambda T) = \lambda (f(T))$

- $\mathcal{L}(V, W) \simeq M_{m \times n}(F)$

Exercise

Given $X \in M_{m \times n}(F)$, find $T \in \mathcal{L}(V, W)$ st $f(T) = X$

$v \in V$, $T(v) = ?$

$$\underset{m \times n}{[T]_B^C} \; \underset{n \times 1}{[v]_B} = \underset{m \times 1}{X} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_m \end{bmatrix}$$

$$T(v) = \beta_1 w_1 + \beta_2 w_2 + \cdots + \beta_m w_m$$

Theorem

For $v \in V$, $[T(v)]_C = [T]_B^C [v]_B$

Proof

Let $[v]_B = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = [\alpha_1, \alpha_2, \ldots, \alpha_n]^T$

& $[T]_B^C = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

$$[T]_B^C [v]_B = \begin{bmatrix} \sum_{j=1}^n a_{1j}\alpha_j \\ \sum_{j=1}^n a_{2j}\alpha_j \\ \vdots \\ \sum_{j=1}^n a_{mj}\alpha_j \end{bmatrix}$$

$$v = \sum_{j=1}^n \alpha_j v_j$$

$$T(v) = \sum_{j=1}^n \alpha_j T(v_j) = \sum_{j=1}^n \alpha_j \left(\sum_{i=1}^m a_{ij} w_i \right)$$

$$= \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} \alpha_j \right) w_i$$

$$\mathcal{L}(V,W) \longrightarrow M_{m \times n}(F)$$

$$T_1 + T_2 \longmapsto [T_1]_B^C + [T_2]_B^C$$

$$\lambda T \longmapsto \lambda [T]_B^C$$

$$
\begin{array}{ccccc}
U & \xrightarrow{T_1} & V & \xrightarrow{T_2} & W \\
\downarrow & & \downarrow & & \downarrow \\
\dim = n & & \dim = m & & \dim = p
\end{array}
$$

ordered Basis = B ord. Basis = C ord. Basis = D

Theorem
$$[T_2 T_1]_B^D = [T_2]_C^D [T_1]_B^C$$

Proof Ex

$$B = \{u_1, u_2, \dots, u_n\}$$
$$C = \{v_1, v_2, \dots, v_m\}$$
$$P = \{w_1, w_2, \dots, w_p\}$$

For each $j \in \{1, 2, \dots, n\}$

$$
\begin{aligned}
(T_2 T_1)(u_j) &= T_2(T_1(u_j)) \\
&= T_2\left(\sum_{k=1}^{m} \lambda_{kj} v_k\right) \\
&= \sum_{k=1}^{m} \lambda_{kj} T_2(v_k) \\
&= \sum_{k=1}^{m} \lambda_{kj} \left(\sum_{i=1}^{p} \beta_{ik} w_i\right) \\
&= \sum_{i=1}^{p} \left(\sum_{k=1}^{m} \beta_{ik} \lambda_{kj}\right) w_i
\end{aligned}
$$

$$
[T_1]_B^C = \begin{array}{c} j\text{th} \\ \downarrow \\ \begin{bmatrix} \lambda_{1j} \\ \lambda_{2j} \\ \vdots \\ \lambda_{mj} \end{bmatrix} \end{array}
\qquad
[T_2]_C^D = \begin{array}{c} k\text{th} \\ \downarrow \\ \begin{bmatrix} \beta_{1k} \\ \beta_{2k} \\ \vdots \\ \beta_{pk} \end{bmatrix} \end{array}
$$

$$
[T_2 T_1]_B^D = \begin{bmatrix} \sum_{k=1}^{m} \beta_{ik} \lambda_{kj} \\ \vdots \\ \end{bmatrix}
$$

L.A

V, W - V.s over F

dim V = n , dim W = m

$T : V \to W$ is a l.t

$[T]_B^C$ is of order $m \times n$

B → ordered bases of V
C → " " " W

$$\mathcal{L} \xrightarrow{\sim} M_{m\times n}(F)$$

$$T \longmapsto [T]_B^C$$

V = W

$T : V \longrightarrow W$
B C

$[T]_B^C \in M_{n \times n}(F)$

$$\mathcal{L}(V) \xrightarrow{\sim} M_{n\times n}(F)$$

$$T \longmapsto [T]_B^C$$

$$I \longmapsto$$

$[I]_B^C$

$I(v_j) \in V$
$= \sum_{i=1}^{n} \lambda_{ij} u_i$

$$\begin{bmatrix} \lambda_{1j} \\ \lambda_{2j} \\ \vdots \\ \lambda_{nj} \end{bmatrix}$$

B = C
$I(v_j) = v_j$

$$[I]_B^C = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

B → ordered basis for V
C → ordered basis for W

$T: V \to W$ is a l.t

Suppose T is bijective

Then $[T]_B^C$ is invertible

$f_0 = f_2 - 1$
$0 = 1 - 1 = 0$

$s_K = f_{K+2} - 1$

$s_{K+1} = f(K+2) - 1 + f(K+1)$
$f(K+3) - 1$

A l.t $T: V \to W$ is invertible if $\exists$ a l.t $S: W \to V$ $\ni$

$TS = I_W$ & $ST = I_V$

- T is invertible iff T is bijective. suppose dim V & dim W are finite

Then T is bijective iff

T is one-one & onto.

iff

T is invertible.

Suppose

dim V = dim W < ∞

Then, T is invertible

iff

T is bijective.

iff

T is one-one

iff

T is onto

iff

rank (T) = dim V

A matrix $X \in M_{n \times n}$ F is invertible.

if $\exists$ $Y \in M_{n \times n}(F)$ $\ni$

$XY = I_{n \times n} = YX$

Y = The inverse of X

$= X^{-1}$

Theorem

V, W – f.d vs over F

$T: V \to W$ is a l.t

B = ordered basis of V

C = " " " W

Then T is invertible

iff

dim V = dim W & $[T]_B^C$ is an invertible matrix

Further, $[T]_B^C = ([T^{-1}]_C^B)$

Proof

Assume that T is invertible

$\Rightarrow$ T is bijective

$\Rightarrow$ T maps a basis of V to a basis of W

$\Rightarrow$ dim V = dim W = n, say

$\Rightarrow [T]_B^C$ is a square matrix of order $n \times n$.

~~$T: W \to V$ is a l.t.~~

$T^{-1}: W \to V$ is a l.t

$T^{-1}T = I_V$ & $TT^{-1} = I_W$

$I_V = T^{-1}T : V \to V$

$I_W = TT^{-1} : W \to W$

$[T^{-1}T]_B^B = [I_V]_B^B = I_{n\times n}$

$\Rightarrow [T^{-1}]_C^B [T]_B^C = I_{n\times n}$

$[TT^{-1}]_C^C = [I_W]_C^C = I_{n\times n}$

$\Rightarrow [T]_B^C [T^{-1}]_C^B = I_{n\times n}$

so $[T]_B^C$ is invertible

& $\left([T]_B^C\right)^{-1} = [T^{-1}]_C^B$

conversely, assume that $\dim V = \dim W = n$, say

& that $[T]_B^C$ is invertible

i.e $\exists X \in M_{n\times n}(F)$ $\ni$

$$X[T]_B^C = I_{n\times n} = [T]_B^C X$$

claim

$\exists S: W \to V \ni TS = I_W$ & $ST = I_V$

Proof let $C = \{w_1, w_2, w_3, \dots, w_n\}$

$B = \{v_1, v_2, \dots, v_n\}$

define $S: W \to V$ by

$S(w_j) = y_j$

&

$$X = (x_{ij}) = \begin{bmatrix} & x_{1j} & \\ & x_{2j} & \\ & \vdots & \\ & x_{nj} & \end{bmatrix}_{n\times n}$$

$y_j = x_{1j}v_1 + x_{2j}v_2 + \dots + x_{nj}v_n$, $1 \le j \le n$

$\in V$

$y_1, y_2, \dots, y_n \in V$

The S is a l.t

$$[S]_C^B = \begin{bmatrix} & x_{1j} & \\ & x_{2j} & \\ & \vdots & \\ & x_{nj} & \end{bmatrix}_{n\times n} = X$$

$\quad\quad\quad\quad\quad\quad\quad\quad$ jth

$ST: V \to W \to V$

claim

$ST = I_V$

$[ST]_B^B = [S]_C^B [T]_B^C$

$= X[T]_B^C$

$= I_{n\times n} = [I_V]_B^C \Rightarrow ST = I_V$

$L(V, W) \longrightarrow M_{m\times n}(F)$

Similarly,

Proove tha $TS = I_V$

<u>Questions</u>

$\dim V = n$

B_1, B_2 - ordered basis for V

(1) For $v \in V$ how are the co-ordinate vectors $[v]_{B_1}$ & $[v]_{B_2}$ related?

(2) $T: V \longrightarrow V$ is a l.t.

How are the matrices $[T]_{B_1}^{B_1}$ & $[T]_{B_2}^{B_2}$ related?

05.09.2024

V - V.S over F

$\dim_F V = n$

$B = \{v_1, v_2, \cdots, v_n\}$
$C = \{u_1, u_2, \cdots, u_n\}$ → ordered basis for V

$v \in V$

$$[v]_B = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}, \quad \lambda_i, \beta_i \in F$$

$v = \sum \lambda_i v_i$

$v = \sum \beta_i u_i$

$$[v]_C = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

Theorem

$[v]_B = Q[v]_C$ where Q is an $n \times n$ invertible matrix over F

Proof Define $Q = \left[[u_1]_B [u_2]_B \cdots [u_n]_B\right]_{n\times n}$

j^{th} column of Q is the co-ordinate vector of $[u_j]_B$

$I_V : V \to V$

ordered basis is C → ordered basis is B

$u_j \longmapsto I_V(u_j) = u_j$

$[I_V]_C^B = \left[[u_1]_B \cdots [u_n]_B\right] = Q$

I_V is invertible ⇒ Q is invertible.

$[v]_B = [I_V(v)]_B$

$[v]_B = [I_V]_C^B [v]_C$

$\boxed{[v]_B = Q\,[v]_C}$ --(proved)

$T : \overset{B}{V} \longrightarrow \overset{C}{W}$

$v \longmapsto T(v)$

$[T]_B^C [v]_B = [T(v)]_C$

This Q is unique ?

$Q_1[v]_C = [v]_B$

$Q_2[v]_C = [v]_B$

$Q_1[v]_C = Q_2[v]_C$

$\Rightarrow \underline{Q_2^{-1} Q_1} [v]_C = [v]_C$

→ this works for every $n \times 1$ vector matrix so it should be Identity (I_V).

This matrix Q is called the change of co-ordinate matrix which changes C-coordinate to B-coordinates.

$S : V \longrightarrow V$

$v_1 \longmapsto u_1$

$v_2 \longmapsto u_2$

$\vdots$

$v_n \longmapsto u_n$

S is invertible

$[S]_B^B$

$[S(v_i)]_B = [u_i]_B$

$V \to$ n-dim V.S $\quad B$ - an ordered basis for V.

$\mathcal{L}(V) \longrightarrow M_{n\times n}(F)$

$T \longmapsto [T]_B^B =: [T]_B$

↳ an isomorphism

$I_V \longmapsto [I_V]_B = I_{n\times n}$

$T_1 + T_2 \longmapsto [T_1]_B + [T_2]_B$

$T_1 T_2 \longmapsto [T_1]_B [T_2]_B$

$\lambda T \longmapsto \lambda [T]_B$

$T \in \mathcal{L}(V)$ is invertible iff $[T]_B$ is an invertible matrix

$[T]_B^{-1} = [T^{-1}]_B$

Theorem Let B_1 & B_2 be two ordered basis for V, and $T \in \mathcal{L}(V)$, then

$$[T]_{B_2} = Q^{-1} [T]_{B_1} Q,$$

where Q is the change of co-ordinate matrix that changes B_2-coordinate to B_1-coordinate.

Proof

$B_1 = \{v_1, v_2, \cdots, v_n\}$

$B_2 = \{u_1, u_2, \cdots, u_n\}$

$Q = \left[[u_1]_{B_1} \cdots [u_n]_{B_1}\right]_{n\times n}$

$= [I_V]_{B_2}^{B_1}$

$$Q[T]_{B_2} = Q[T]_{B_2}^{B_2}$$

$$= [I_V]_{B_2}^{B_1}[T]_{B_2}^{B_2} = [I_V T]_{B_2}^{B_1}$$

$$[I_V T]_{B_2}^{B_1} = [T I_V]_{B_2}^{B_1} = [T]_{B_1}^{B_1}[I_V]_{B_2}^{B_1}$$

$$\begin{tikzcd} V_{B_2} \arrow[r, "I_V"] & V_{B_1} \arrow[r, "T"] & V_{B_1} \end{tikzcd}$$

$$= [T]_{B_1}^{B_1} Q = [T]_{B_1} Q$$

$$Q[T]_{B_2} = [T]_{B_1} Q$$

$$Q^{-1}Q[T]_{B_2} = Q^{-1}[T]_{B_1}Q$$

$$\Rightarrow \boxed{[T]_{B_2} = Q^{-1}[T]_{B_1}Q}$$

Def^n: Let $X, Y \in M_{n\times n}(F)$. We say that Y is similar to X if $\exists$ an invertible matrix $P \in M_{n\times n}(F) \ni Y = P^{-1}XP$

\# $T \in L(V)$

$B_1, B_2 \rightarrow$ ordered basis

$\Rightarrow [T]_{B_1}$ & $[T]_{B_2}$ are similar matrices

Exam Ques

Conversely, suppose that $X, Y \in M_{n\times n}(F)$ are similar matrices. Then $\exists$ ordered bases B_1, B_2 for V & a l.t. $T \in L(V)$ $\ni$ $X = [T]_{B_1}$ & $Y = [T]_{B_2}$

9/09/2019

Continuity

L.A

V, W v.s over F.

$\mathcal{L}(V,W) = \{T : V \to W \mid T \text{ is a l.t}\}$
↳ v.s over F

$\dim_F \mathcal{L}(V,W) = mn$ if $\dim V = n$, $\dim W = m$

$V = W$, $T \in \mathcal{L}(V) = \mathcal{L}(V,V)$

C, B - ordered basis for V.

$[T]_B = Q^{-1}[T]_C Q$, $Q = ?$

Ex Suppose $X, Y \in M_{n\times n}(F)$ & X & Y ~~are~~ similar. Then $\exists$ a l.t $T \in \mathcal{L}(V)$ and ordered bases B & C for V st $[T]_B = X$ & $[T]_C = Y$

Linear Functional & Dual spaces

V - v.s over F

$f : V \to F$

$V^* := \mathcal{L}(V, F)$
↳ v.s over F
↳ dual space of V.

$(V^*)^* = V^{**} = \{L : V^* \to F \mid L \text{ is a l.t on } V^*\}$
↳ Vector space over F
↳ double dual

$L \in V^{**}$

$L : V^* \to F$

$f \longmapsto L(f)$

Suppose $\dim_F V < \infty$

$\Rightarrow \dim V^* = \dim V$

& $\dim V^{**} = \dim V^* = \dim V$

So $V^{**} \simeq V^* \simeq V$

Examples

(1) $A \in M_{n\times n}(F)$

$tr(A) =$ Sum of the diagonal entries of A $\in F$

$f : M_{n\times n}(F) \longrightarrow F$

$A \longmapsto f(A) := tr(A)$

(2) $V = F^n$, Fix $a_1, a_2, \cdots, a_n \in F$

Define $f : F^n \to F$ by

$(x_1, x_2, \cdots, x_n) \mapsto f(x_1, x_2, x_3, \cdots, x_n) := a_1x_1 + a_2x_2 + \cdots + a_nx_n$

Then f is a l.f on F^n.

(*) In fact every l.f on F^n is obtained in this way.

i.e given a l.f $g : F^n \longrightarrow F$, $\exists \lambda_1, \lambda_2, \cdots, \lambda_n \in F \ni$

$g(x_1, \cdots, x_n) = \lambda_1x_1 + \cdots + \lambda_nx_n \quad \forall (x_1, x_2, \cdots, x_n) \in F^n$

Proof * Ex

Hint. $B = \{e_1, \cdots, e_n\}$ - stand. ordered basis of F^n.

Put $\lambda_i = g(e_i)$, $1 \leq i \leq n$.

(3) Fix $t \in F$

(2) $f: F[x] \to F$

$P(x) \longmapsto f(P(x)) := P(t)$

$\hookrightarrow$ is a l.f on $F[x]$

(4) $\mathcal{F}(F,F) = \{f : F \to F \mid f \text{ is a func}^n\}$

$\hookrightarrow$ v.s over F.

Fix $t \in F$

define $T : \mathcal{F}(F,F) \to F$ by

$T(f) = f(t)$, $f \in \mathcal{F}(F,F)$

then, T is a lf on $\mathcal{F}(F,F)$

(5) $a, b \in \mathbb{R}$ $a < b$

$C([a,b]) = \{f : [a,b] \to \mathbb{R} \mid f \text{ is continuous}\}$

$\hookrightarrow$ v.s over $\mathbb{R}$.

Define $T : C([a,b]) \to \mathbb{R}$ by

$f \longmapsto T(f) = \int_a^b f(x)\,dx$

then T is a l.f

(6) $V \to$ v.s over F

$\dim_F V = n < \infty$

$B = \{v_1, v_2, v_3, \dots, v_n\}$ — ordered basis for V

$f_j : V \to F$

$v \longmapsto f(v) = \lambda_j$, $1 \le j \le n$

$$[v]_B = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$$

$\underbrace{f_1, f_2, \dots, f_n}$ are linear func^{nal} on V

$\hookrightarrow$ pairwise distinct.

for $j \in \{1, \dots, n\}$

$$f_j(v_i) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

$f_1 :$ $v_1 \mapsto 1$
$v_2 \mapsto 0$
$\vdots$
$v_n \mapsto 0$

Claim

$B^* := \{f_1, f_2, \dots, f_n\}$ is a basis for V^*

Proof

Let $g = \sum_{i=1}^{n} \lambda_i f_i = \underset{\hookrightarrow \text{zero l.t}}{0}$

$0 = g(v_j) = \left(\sum_{i=1}^{n} \lambda_i f_i\right)(v_j)$

$0 = \sum_{i=1}^{n} \lambda_i f_i(v_j) = \lambda_j$

$\Rightarrow \lambda_j = 0, \forall j \in \{1, 2, \dots, n\}$

$\boxed{\# \; B^* = \{f_1, f_2, \dots, f_n\} \text{ is called the dual basis of } B.}$

Thus for every basis B of V, B^* is a basis for V^* is the converse true?

i.e given a basis C of V^* does $\exists$ a basis $\bar{C}$ of V s.t. $C^* = \bar{C}$?

Ans : = Yes

Given

$g \in V^*$

$$g = \sum_{i=1}^{n} \beta_i f_i = \sum_{i=1}^{n} g(v_i) f_i$$

Given

$v \in V$

$$v = \sum_{i=1}^{n} \alpha_i v_i = \sum_{i=1}^{n} f_i(v) v_i$$

$$\boxed{f_j(v) = \sum_{i=1}^{n} \alpha_i f_j(v_i) = \alpha_j}$$

L.A

V - n-dim v.s over F

B - basis for V, $= \{v_1, v_2, \dots, v_n\}$

B^* - dual basis of B for V^*

$= \{f_1, f_2, \dots, f_n\}$

$f_j(v_i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ for $j \in \{1, 2, \dots, n\}$

$1 \leq i \leq n$

$g \in V^*$, $g = \sum_{i=1}^{n} \lambda_i f_i$, $\lambda_i \in F$

$= \sum_{i=1}^{n} g(v_i) f_i$

$v \in V$, $v = \sum_{i=1}^{n} \beta_i v_i$

$= \sum_{i=1}^{n} f_i(v) v_i$

Aim

Every basis of V^* is the dual basis of some bases for V, when $\dim_F V < \infty$

Theorm

The map $\theta : V \longrightarrow V^{**}$

by $v \longmapsto \theta(v) := L_v$,

where $L_v : V^* \longrightarrow F$

$f \longmapsto L_v(f) := f(v)$

is an isomorphism from V to V^*.

Claim-1

L_v is a l.t

$f, g \in V^*$, $\lambda \in F$

$L_v(f+g) = (f+g)v$

$= f(v) + g(v)$

$= L_v(f) + L_v(g)$

$L_v(\lambda f) = (\lambda f)(v)$

$= \lambda(f(v)) = \lambda . L_v(f)$

So L_v is a l.t so $L_v \in V^{**}$

Claim-2

θ is l.t

Let $u, v \in V$ & $\lambda \in F$.

By defn of θ,

$\theta(u+v) = L_{u+v} = L_u + L_v = \theta(u) + \theta(v)$

for $g \in V^*$

$L_{u+v}(g) = g(u+v)$

$= g(u) + g(v)$

$= L_u(g) + L_v(g)$.

$= (L_u + L_v)(g)$

Also, $\theta(\lambda u) = L_{\lambda u} = \lambda L_u = \lambda . \theta(u)$

for $f \in V^*$

$L_{\lambda u}(f) = f(\lambda u) = \lambda f(u) = \lambda . L_u(f)$

$\therefore$ θ is a l.t.

Claim-3

θ is one-one.

> **Ex** $\dim V < \infty$, $v \in V$, $v \neq 0$
> Prove that $\exists\, f \in V^*$ $\ni$ $f(v) \neq 0$

Let $u \in V$ st $\theta(u) = 0$

$\Rightarrow L_u = 0$

$\Rightarrow L_u(f) = 0, \ \forall f \in V^*$

$\Rightarrow f(u) = 0, \ \forall f \in V^*$

$\Rightarrow u = 0$ (by the ex)

θ is one-one & $\dim V = \dim V^*$

$\Rightarrow \theta$ is onto $\Rightarrow$ is an

isomorphism from $V \to V^{**}$

Corollary

$\dim V < \infty$

L is a linear ~~...~~ functional on V^*

(i.e $L \in V^{**}$)

$\Rightarrow \exists!\ v \in V \ni$

$\boxed{L(f) = f(v)}$ $\forall f \in V^*$

Theorem: $\dim_F V < \infty$

$\Rightarrow$ Every basis for V^* is the dual basis of some basis for V.

Proof Let $\dim_F V = n < \infty$

$\Rightarrow \dim_F V^* = n$

Consider a basis $\bar{B} = \{f_1, f_2, \dots, f_n\}$ for V^*

Let $\bar{\bar{B}} = \{L_1, L_2, \dots, L_n\}$ be the dual basis of $\bar{B}$ for V^{**}.

Then, for $j \in \{1, 2, \dots, n\}$

$$L_i(f_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad 1 \leq i \leq n$$

For each L_i, $1 \leq i \leq n$, $\exists\, v_i \in V \ni L_i = L_{v_i}$

Then $\boxed{L_i(f) = f(v_i),}$ $\forall f \in V^*$

Consider $B = \{v_1, v_2, \dots, v_n\} \subseteq V$

$\hookrightarrow$ is a basis for V

$$\theta : V \xrightarrow[\text{L.T}]{\text{bij}} V^{**}$$

$$\theta^{-1} : V^{**} \xrightarrow[\text{L.T}]{\text{bij}} V$$

$$\begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix} \longmapsto \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Claim $\bar{B} = B^*$

$B^* = \{g_1, g_2, \cdots, g_n\}$

$g_j(v_i) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases}$

$\boxed{f_i(v_j) = L_i(f_j) = \begin{cases} 1 & \text{if } j=i \\ 0 & \text{else} \end{cases}}$

$\Rightarrow g_j = f_j$

Def^n

V - any v.s over F

A hyperplane of V is a maximal proper ~~sup~~ subspace of V.

- $\dim_F V = n < \infty$

A subspace of V is a hyper plane iff its dim is $n-1$.

$f : V \xrightarrow{\text{L.f}} F$

$\omega \longmapsto f(\omega) \neq 0$

$f \equiv 0 \Rightarrow Ker(f) = V$

$f \not\equiv \Rightarrow Range(f) = F$

$\Rightarrow rank(f) = 1$

$\Rightarrow nullity(f) = n-1$

$\Rightarrow Ker(f)$ is a hypespace.

12.09.2024

$\dim V = n < \infty$

$f: V \to F$ is a l.t

if $f \neq 0$, then $\ker(f)$ is a hyperplane of V

Q Given a hyperplane H of V, does ~~the~~ $\exists$ a l.t $f: V \to F \ni$ $\ker(f) = H$?

Recall

$\dim V = n < \infty$

$B = \{v_1, v_2, \cdots, v_n\}$ - basis for V

$B^* = \{f_1, f_2, \cdots, f_n\}$ - dual basis of B for V^*

$g \in V^*$

$\Rightarrow g = \sum_{i=1}^{n} \beta_i f_i \ , \ \beta_i \in F$

$= \sum_{i=1}^{n} g(v_i) f_i \quad \beta_i = g(v_i)$

V, W - l.t.s over F

(need not be f.d)

$T: V \to W$ is a l.t

Define a map $T^*: W^* \to V^*$

by $g \mapsto T^*(g) :\equiv gT$

$V \xrightarrow{T} W \xrightarrow{g} F$

$g(T): V \longrightarrow F$

Claim T^* is a l.t

Let $g_1, g_2 \in W^*$

$\Rightarrow g_1 + g_2 \in W^*$

To show

$T^*(g_1+g_2) = T^*(g_1) + T^*(g_2)$

i.e $(g_1+g_2)T = g_1T + g_2T$

Let $v \in V$, we have

$(g_1+g_2)T : V \to F$

$((g_1+g_2)T)(v) = (g_1+g_2)\underbrace{(T(v))}_{\in W}$

$= g_1(T(v)) + g_2(T(v))$

$= (g_1T)(v) + (g_2T)(v)$

$= (g_1T + g_2T)(v)$

$\Rightarrow (g_1+g_2)T = g_1T + g_2T$

$\Rightarrow T^*(g_1+g_2) = T^*(g_1) + T^*(g_2)$

Similarly prove that,

$T^*(\lambda g) = \lambda T^*(g)$ for $\lambda \in F$ & $g \in W^*$

Def^n

The map $T^*: W^* \to V^*$ is called the adjoint of T

if $\dim(V) < \infty$ & $\dim(W) < \infty$ the T^t is called the transpose of T.

B- ordered basis for V

C " " " W

$\left([T]_B^C\right)^t = [T^t]_{C^*}^{B^*}$

$T^t : W^* \longrightarrow V^*$

$g \longmapsto T^t(g) := gT$

Theorem

$V, W \to$ v.s over F

$\dim V = n < \infty$, $\dim W = m < \infty$

$B \to$ or. basis for V

$C \to$ " " " W

$B^* \to$ dual basis of B for V^*

$C^* \to$ " " " W^*

$T : V \longrightarrow W$ is a l.t

if $A = [T]_B^C$, then $A^t = [T^t]_{C^*}^{B^*}$

$\downarrow$ transpose of a matrix.

Proof

Let $B = \{v_1, v_2, \dots, v_n\}$

$C = \{w_1, w_2, \dots, w_m\}$

$B^* = \{f_1, f_2, \dots, f_n\}$

$C^* = \{g_1, g_2, \dots, g_m\}$

$A = (a_{ij})_{m\times n}$

$$A = \begin{bmatrix} & a_{1j} & \\ & a_{2j} & \\ a_{i1}\ a_{i2} & \cdots & a_{in} \\ & a_{mj} & \end{bmatrix}$$

By def^n $T(v_j) = \sum_{k=1}^{n} a_{kj} w_k$

The ith column of A^t = the ith row of A $= (a_{i1}, a_{i2}, \dots, a_{in})$

claim

$T^t(g_i) = \sum_{k=1}^{n} a_{ik} f_k$, $1 \le i \le m$

$T^t : W^* \longrightarrow V^*$, $B^* = \{f_1, \dots, f_n\}$

$C^* = \{g_1, \dots, g_m\}$

$T^t(g_i) = g_i T : V \longrightarrow F$

$v \longmapsto g_i T(v)$

$g_i T \in V^* \Rightarrow g_i T = \sum_{k=1}^{n} [(g_i T)(v_k)] f_k$

$T^t(g_i) = g_i \circ T = \sum_{k=1}^{n} \underbrace{(g_i T)(v_k)}_{\in F} f_k$

$(g_i T)(v_k) = g_i(T(v_k))$

$= g_i\left(\sum_{j=1}^{m} a_{jk} w_j\right)$

$= \sum_{j=1}^{m} a_{jk}\, g_i(w_j) = a_{ik}$ $\boxed{1 \le k \le n}$

$T^t(g_i) = \sum_{k=1}^{n} a_{ik} f_k$

$= a_{i1} f_1 + a_{i2} f_2 + \dots + a_{in} f_n$

ith column of $[T^t]_{C^*}^{B^*}$

$$= \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix} = (i\text{th row of } A)^t$$

So $A^t = [T^t]_{C^*}^{B^*}$

Ex

$V = \mathbb{R}^3 \longrightarrow$ v.s over $\mathbb{R}$

$B = \{(1,0,1), (1,2,1), (0,0,1)\}$
↳ basis for $\mathbb{R}^3$

$B^* = \{f_1, f_2, f_3\} \rightarrow$ dual basis of B

$f_j : \mathbb{R}^3 \longrightarrow F, \quad 1 \leq j \leq 3$

Find f_1, f_2, f_3

i.e give $(x_1, x_2, x_3) \in \mathbb{R}^3$, what are $f_1(x_1, x_2, x_3)$

$f_2(x_1, x_2, x_3)$

& $f_3(x_1, x_2, x_3)$?

$(x_1, x_2, x_3) = x_1 e_1 + x_2 e_2 + x_3 e_3$

$f_1(x_1, x_2, x_3) = x_1 f_1(e_1) + x_2 f_1(e_2) + x_3 f_1(e_3)$

$1 = f_1(1,0,1)$

$= f_1(1e_1 + 0 \cdot e_2 + 1e_3)$

$= f(e_1) + f(e_3)$

$0 = f_1(1,2,1)$

$= f_1(1e_1 + 2e_2 + 1e_3)$

$= f_1(e_1) + 2f_1(e_2) + f_1(e_3)$

$0 = f_1(0,0,1) = f_1(e_3)$

Linear Algebra 17.09.2024

Elementary row operation

$A \in M_{m \times n}(F)$

Defn

Any of the following three type of row-operations is called an elementary row operation.

(I) interchanging any two rows.

(II) multiplying any row of A by a nonzero scalar.

(III) Adding any scalar multiple of a row to another row of A.

$$A = \begin{array}{c} \\ \\ \\ \\ \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{j1} & a_{j2} & \cdots & a_{jn} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \begin{array}{l} \\ \\ \rightarrow i\text{th row} \\ \\ \rightarrow j\text{th row} \\ \\ \end{array}$$

$$B = \begin{bmatrix} a_{j1} & a_{j2} & \cdots & a_{jn} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix}_{m \times n}$$

R_{ij} = the matrix obtained from $I_{m \times m}$ by interchanging ith & jth rows of $I_{m \times m}$.

$$R_{ij} = \begin{array}{c} \\ \end{array} \begin{array}{c} i\text{th column} \quad j\text{th column} \\ \begin{bmatrix} 0 & \cdots & 0\,1\,0 & \cdots & 0 \\ 0 & \cdots\, 0\,1\,0 & \cdots & & 0 \end{bmatrix} \end{array} \begin{array}{l} \rightarrow i\text{th row} \\ \rightarrow j\text{th row} \end{array}$$

$$\boxed{R_{ij} A = B}$$

Suppose ~~given~~ B is given,

$$\boxed{A = R_{ij} B}$$

#

$$C = \begin{bmatrix} & & & \\ \lambda a_{j1} & \lambda a_{j2} & \cdots & \lambda a_{jn} \\ & & & \end{bmatrix} \begin{matrix} \\ \rightarrow j\text{th row} \\ \\ \end{matrix}$$

$(\lambda \neq 0)$

$R_j(\lambda)$ – the matrix obtained from $I_{m\times m}$ by multiplying λ to the j^{th} row of $I_{m\times m}$.

$$= \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & \lambda & \\ & & & & \ddots \\ & & & & & 1 \end{bmatrix} \rightarrow j\text{th row.}$$

$R_j(\lambda) A = C$

Suppose C is given then,

$A = R_j(\frac{1}{\lambda}) C$

#

$$D = \begin{bmatrix} a_{11} & \cdots & a_{12} & \cdots & a_{1n} \\ \vdots & & \vdots & & \vdots \\ a_{i1}+\lambda a_{j1} & \cdots & a_{i2}+\lambda a_{i2} & \cdots & a_{in}+\lambda a_{jn} \\ \vdots & & \vdots & & \vdots \\ a_{m1} & \cdots & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$R_{i,j}^{(\lambda)}$ · the matrix ~~multiply by~~ obtained from $I_{m\times m}$ by adding λ times of jth row to the ith row of $I_{m\times m}$.

$$R_{ij} = \begin{array}{c} \begin{bmatrix} & & & & & \\ 0 & \cdots & 0\,1\,0 & \cdots & \lambda & \cdots & 0 \\ & & & & & \end{bmatrix} \rightarrow i\text{th row.} \\ \quad\quad i\text{th column} \quad j\text{th column} \end{array}$$

$R_{i,j}^{(\lambda)} A = D$

Suppose D is given,

$R_{i,j}^{(-\lambda)} D = A$

Def^n $A, X \in M_{m\times n}(F)$

We say that X is row equivalent to A if X can be obtained from A by a finite sequence elementary row operations.

i.e

$X = E_K \cdots E_2 E_1 A \qquad (E_i \in \{R_{ij}, R_j(\lambda), R_{i,j}^{(\lambda)}\})$

Def^n An $m\times m$ matrix is said to be an elementary ~~row~~ matrix if it is obtained from $I_{m\times m}$ by applying only one elementary row operations.

Recall: $X \in M_{m\times m}(F)$

X is invertible if $\exists\ Y \in M_{m\times m}(F) \ni XY = I_{m\times m} = YX$.

$YX = I_{m\times m} \rightarrow Y$ is a left inverse of X

$XY = I_{m\times m} \rightarrow Y$ is a right inverse of X

$AX = I_{m\times m}$ | $A, B \in M_{m\times m}(F)$

$XB = I_{m\times m}$

claim $A = B$

Proof

$AX = I \Rightarrow AXB = B$

$\Rightarrow A(XB) = B \Rightarrow AI = B \Rightarrow A = B$ ☐

Row reduced matrix :→

$A \in M_{m\times n}(F)$

A is called row-reduced matrix

if (a) the first non-zero entry (from the left) in each non zero row is equal to 1 (called the leading non zero entry)

(b) each column of A which ~~containing~~ contains the leading non-zero entry of some row has all its other entries 0.

20.09.2024

LA

Row reduced matrix

Cond^n

①

②

Ex

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Theorem: Every $m\times n$ matrix over F is row reduced equivalent to a row reduced matrix.

Proof

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Row reduced Echelon matrix :-

$A \in M_{m\times n}(F)$

A is a row reduced echlon matrix.

if

(a) A is a row reduced matrix.

(b) Every zero row of A occurs below the nonzero row of A

(c) Let row $1, 2, \ldots, r \leq m$ be the nonzero rows of A if the leading non-zero entry of row i appears in column K_i, for $i = 1, 2, \dots, r$, then

$$K_1 < K_2 < \dots < K_r$$

Theorem

Every $m \times n$ matrix over F is row equivalent to a row reduced echlon matrix.

Application

$A \in M_{n \times n}(F) \longrightarrow$ sq. matrix.

A is a row reduced echlon matrix.

$\Rightarrow$ A is invertible iff every row of A contains a non zero entry iff $A = I_{n \times n}$

Elementary Matrices.

R_{ij}, $1 \leq i \neq j \leq m$

$R_j(\lambda)$, $1 \leq j \leq m$, $\lambda \neq 0$, $\lambda \in F$

$R_{ij}(\lambda)$, $1 \leq i \neq j \leq m$, $\lambda \in F$.

All elementary matrices are invertible.

(i) $(R_{ij})^{-1} = R_{ij}$ (involution)

(ii) $(R_k(\lambda))^{-1} = R_k(\frac{1}{\lambda})$, $\lambda \neq 0$

(iii) $(R_{ij}(\lambda))^{-1} = R_{ij}(-\lambda)$

Theorem

$A \in M_{n \times n}(F) \rightarrow$ sq. matrix

[redacted]

T.F.A.E

(1) A is invertible

(2) A is row equivalent to $I_{n \times n}$

(3) A is a finit product of elementary matrices

Proof (i)

A is row equivalent to a row reduced echlon matrix B.

$B = E_k \cdots E_2 E_1 A$

$E_j \rightarrow$ elementary matrix.

$A = E_1^{-1} E_2^{-1} \cdots E_{k-1}^{-1} E_k^{-1} B$

B is invertible iff A is invertible.

B is invertible iff $B = I_{n \times n}$.

A is invertible

$\Rightarrow$ A is row equivalent to $B = I_{n \times n}$

$\Rightarrow$ $A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$, is a product of elementary matrices.

$\Rightarrow$ A is invertible.

Corollary

$A \in M_{n \times n}(F)$ is invertible

if a seq. of elementary row operat^n reduces A to $I_{n \times n}$, then the same seq. of elementary row operation when applie to $I_{n \times n}$ yields A^{-1}

L.A

$A \in M_{n\times n}(F)$

A is invertible
iff
A is row equivalent to $I_{n\times n}$
iff
A is a product of elementary matrices.

System of linear eq^{n}s

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \longrightarrow (*)$$

The system is homogeneous if $b_1 = \cdots = b_m = 0$
Otherwise, non homogeneous.

$(\lambda_1, \lambda_2, \cdots, \lambda_n)$ is a sol^n to (*) if (*) holds when we replace x_i by λ_i $i \in \{1, 2, \cdots, n\}$

(*) can be written as,

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}}_{b}$$

$Ax = b$

The set $\{x \in F^{n,1} / Ax = b\}$ is called the solution set of (*)

The system (*) is called consistent if its sol^n set is non-empty. otherwise, the system is inconsistent.

Ex ① $\left.\begin{array}{l} x_1 + x_2 = 0 \\ x_1 + x_2 = 1 \end{array}\right| \to$ inconsistent.

(2) $2x_1 + x_2 = 5 \to$ consistent & more than one sol^n.

Exercise

Let $A \in M_{m\times n}(F)$
the sol^n set of $Ax = 0$ is a subspace of $F^{n,1}$

$(*) \to Ax = b$ $\quad \bar{A} = [A | b]_{m\times(n+1)}$

$\bar{A} \to E, \bar{A}$

$$\begin{bmatrix} d_{11} & \cdots & d_{1n} & a_1 \\ & & & a_2 \\ \vdots & & & \vdots \\ d_{m1} & \cdots & d_{mn} & a_m \end{bmatrix}_{m\times n}$$

Theorem
Let $\bar{B} \in M_{m\times(n+1)}$ & $\bar{A}$ as before & $\bar{A}$ & $\bar{B}$ are row equiva
Then the corresponding systems of linear equation have the same solutions.

In particular,

<u>Theorem</u>

Let B & C be two $m \times n$ matrices which are row equivalent.

Then the systems

$Bx = 0$ & $Cx = 0$ have the same sol?

Let $A \in M_{m \times n}(F)$ & consider,

$$Ax = 0$$

$R \to$ is a row-reduced-echlon matrix which is row equivalent to A.

Let rows $1, 2, \ldots, r$ be the non zero rows of R.

Then $r \leq m$.

Let the leading non-zero entry 1 of row i occures in column k_i, $1 \leq i \leq r$.

Then, $k_1 \leq k_2 \leq \cdots \leq k_r$

Consider

$$Rx = 0$$

There are r non trivial equations in $Rx = 0$

Among the n unknown $x_1, x_2, \ldots, x_n$, the unknown x_{k_i} $(1 \leq i \leq r)$ will occure (with co-efficient 1) only in the i^{th} equation of $Rx = 0$

Let $J = \{1, 2, \ldots, n\} - \{k_1, k_2, \ldots, k_r\}$

then,

$$\left.\begin{aligned} x_{k_1} + \sum_{j \in J} c_{1j} x_j &= 0 \\ x_{k_2} + \sum_{j \in J} c_{2j} x_j &= 0 \\ x_{k_r} + \sum_{j \in J} c_{rj} x_j &= 0 \end{aligned}\right| \quad Rx = 0$$

Ex The sol set of $Ax = 0$ has dimension $n - r$

LA

$Ax = b$, $A \in M_{m\times n}(F)$ $\to (*)$

$A = (a_{ij})_{m\times n}$

R = a row reduced echelon matrix which is row equivalent to A.

$Rx = 0$ & $Ax = 0$ have same solutions.

Row $1, 2, \dots, r$ are the non zero rows of R $(r \le m)$.

$k_1, k_2, \dots k_r$ the column of $R \ni$ the leading non zero entry in the ith row occurs in the k_ith column.

$$
\begin{array}{cc}
 & \begin{array}{ccc} k_1 & k_2 & \quad k_r \end{array} \\
R = & \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & & & \\ & & & 1 & \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}
\end{array}
$$

r^{th} row

$X = (x_1, x_2, \dots, x_n)^t$.

$x_{k_1} + \sum_{\substack{j=1 \\ j\ne k_i, \forall i}}^{n} c_{1j} x_j = 0$

$x_{k_2} + \sum_{\substack{j=1 \\ j\ne k_i, \forall i}}^{n} c_{2j} x_j$

$\vdots$

$x_{k_r} + \sum_{\substack{j=1 \\ j\ne k_i, \forall i}}^{n} c_{rj} x_j$

Assign values to all x_j's with $j \in \{1, 2, \dots, n\} - \{k_1, k_2, \dots, k_r\}$

& then get $x_{k_1}, x_{k_2}, \dots, x_{k_r}$

if r is less than n, then $Rx = 0$ has an non zero solution & vice versa.

$\boxed{\text{Doubt} \to \text{Solved}}$

Theorem

Suppose $m < n$.

Then $Ax = 0$ has a non-trivial solⁿ.

Application

Theorem

$A \in M_{n\times n}(F) \to$ sq. matrix

Then A is row-equivalent to $I_{n\times n}$.

iff

The system $Ax = 0$ has only trivial solution.

Proof

Suppose A is row equivalent to $I_{n\times n}$

$\Rightarrow A$ is invertible.

$\Rightarrow Ax = 0$ has only trivial solⁿ.

Conversely

$Ax = 0$ has only trivial solⁿ.

R = row reduced echelon matrix which is row equivalent to A.

$Rx = 0$ & $AX = 0$ have the same solⁿ.

Claim

$R = I_{n\times n}$

$Rx = 0$ has only trivial sol^n

$\Rightarrow$ r (no. of nonzero rows of R) $\geq n$

~~and~~ on the other hand
$r \leq n$.

So $r = n$ $\Rightarrow$ $\boxed{R = I_{n\times n}}$

$\Rightarrow$ A is row equivalent to $I_{n\times n}$.

Non-homogeneous System of equations :-

$Ax = b$ — ⊛ , $A \in M_{m\times n}(F)$

$b \neq 0$

$\bar{A} = [A|b]_{m\times(n+1)}$

R = row reduced echelon matrix which is row equivalent to A.

$= E_k \cdots E_2 E_1 A$

Rows $1, 2, \ldots, r$ are the non-zero rows of R $(r \leq m)$

k_i : the column number in which the leading non-zero entry of ith row occurs $1 \leq i \leq r$)

$k_1 < k_2 < \cdots < k_r$

Perform the same seq $E_1, \ldots, E_k$ of elementary row operation on $\bar{A}$ to get a matrix $\bar{R}$.

R $\bar{R} = (R|c]$, $c \in F^{m,1}$

The first n columns of $\bar{R}$ are, precisely the columns of R.

$AX = b$ & $RX = C$ have same sol^n.

$C = (C_1, C_2, \ldots, C_n)^t$

$R = (d_{ij})_{m\times n}$

$$x_{k_1} + \sum_{\substack{j=1 \\ j\neq k_i, \forall i}}^{n} d_{1j} x_j = C_1$$

$$x_{k_2} + \sum_{\substack{j=1 \\ j\neq k_i, \forall i}}^{n} d_{2j} x_j = C_2$$

$\vdots$

$$x_{k_r} + \sum_{\substack{j=1 \\ j\neq k_i, \forall i}}^{n} d_{rj} x_j = C_r$$

$0 = C_{r+1}$

$\vdots$

$0 = C_m$

$Rx = C$ has a sol^n only when $C_{r+1} = \cdots = C_m = 0$

Assume that $C_{r+1} = \cdots = C_m = 0$

The ⓔ. $Rx = C$ must have a sol^n.

Ex

$$A = \begin{bmatrix} 3 & -6 & 2 & 7 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

for which $b \in F^{4,1}$, $Ax = b$ has a sol^n?