Linear Alge

02.08.24

Group

(i) $G \neq \phi$

(ii) $*$ is a operation on G

(A function from $G \times G \to G$ must exist)

$* : G \times G \to G$

$\Rightarrow \forall a, b \in G \Rightarrow a * b \in G$

(iii) $a * (b * c) = (a * b) * c$

(iv) $\exists$ an element "$e \in G$" st

$a * e = e * a = a \quad \forall a \in G$

$\hookrightarrow$ existance of identity

(v) For each $a \in G$ $\exists b \in G \ni$

$a * b = e = b * a$

$\hookrightarrow$ existance of inverse

$b \to$ inverse of $a = a^{-1}$ ($\neq \frac{1}{b}$ always)

Exercise

(i) Prove that e is unique for a Group.

Proof If possible assume that e is not unique

we can find more than 1 'e'.

e_1 is also an identity. & $e \neq e_1$

So,

So,

$$a * e = e * a = a \quad (\forall\, a \in G) \quad \text{--(i)}$$

Also,

$$a * e_1 = e_1 * a = a \quad (\forall\, a \in G) \quad \text{--(ii)}$$

<u>Now</u> $\forall\, a \in G$

$$a = a$$

using eqⁿ (i) & (ii),

$$a * e = a * e_1$$

operating a^{-1} both sides,

$$a^{-1} * a * e = a^{-1} * a * e_1$$

$$(a^{-1} * a) * e = (a^{-1} * a) * e_1 \quad \text{(Associative property)}$$

$$\Rightarrow e * e = e_1 * e_1$$

$$\Rightarrow e = e_1 \quad \text{(which contradicting our assumption)}$$

$\therefore$ identity is unique for a group.

(ii) claim: $\forall\, a \in G$, a^{-1} is unique.

Proof: If possible let assume that it is not unique.

$\Rightarrow$ an element $p \in G$ has more than one inverse. i.e p_1^{-1} & $p_2^{-1} \in G$

We have,

$$p * p_1^{-1} = e = p_1^{-1} * p \quad \text{--- (i)}$$

&

$$p * p_2^{-1} = e = p_2^{-1} * p \quad \text{--- (ii)}$$

Now take

$$e = e$$

using eqⁿ (i) & (ii)

$$p * p_1^{-1} = p * p_2^{-1}$$

operating inverse both sides

$$e * p_1^{-1} = e * p_2^{-1}$$

$$\Rightarrow p_1^{-1} = p_2^{-1}$$

this contradicts our assumption.

$\therefore \forall\, a \in G$, a^{-1} is unique.

Expression

$(G, *) \rightarrow$ A Group

$(H, +) \rightarrow$ A Group.

($a*b$ can be written as ab $\forall$ operations except addition operation)

* $a+b \in H \; \forall a, b \in H$
* $(a+b)+c = a+(b+c)$
* identity $= 0$
* inverse of $a = -a$

$(G, \cdot) \rightarrow$ Group

$\hookrightarrow$ abelian Group if $ab = ba \; \forall a, b \in G$

$(G, +) \rightarrow$ abelian if $a+b = b+a$

* $\underbrace{a+a+a+\cdots+a}_{k \text{ times}} = ka$

$\underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{k \text{ times}} = a^k$

just notations.

Q $(H, +)$

$a \in G$

what is $-5a$?

$\rightarrow$ Adding inverse of 'a', 5 times

or

$\rightarrow$ inverse of the number that we got after adding 'a', 5 times

05.08.2024

Exercise

$(ab)^{-1} = b^{-1}a^{-1}$

Proof

We know

$ab = ab \quad (a, b \in G)$

$\Rightarrow ab \cdot b^{-1} = ab \cdot b^{-1}$

$\Rightarrow a = ab \cdot b^{-1}$

$\Rightarrow aa^{-1} = ab\, b^{-1} \cdot a^{-1}$

$\Rightarrow 1 = (ab)(b^{-1}a^{-1})$ (Associative law) --(i)

also $(ab)(ab)^{-1} = 1$ $(\forall a, b \in G, ab \in G)$ --(ii)

from eqⁿ i & ii

$\Rightarrow (ab)(b^{-1}a^{-1}) = (ab)(ab)^{-1}$

$\Rightarrow (ab)^{-1}(ab)(b^{-1}a^{-1}) = (ab)^{-1}(ab)(ab)^{-1}$

$\Rightarrow (ab)^{-1} = b^{-1}a^{-1}$ -- $\boxed{}$

$a^0 = 1$

$a^1 = a$

$a^2 = aa$

$\vdots$

$a^k = \underbrace{a \cdot a \cdot a \cdots a}_{k \text{ times}}$

$a^{-1} = a^{-1}$

$a^{-2} = a^{-1} \cdot a^{-1}$

$\vdots$

$a^{-k} = \underbrace{a^{-1} \cdot a^{-1} \cdot a^{-1} \cdots a^{-1}}_{k \text{ times}}$

$X \neq \Phi$

$Sym(X) = \{f : X \to X \mid f$ is a bijective map $\}$

$(Sym(X), \circ) \to$ group

Symmetric group defined on X.

$|X| = n$

$Sym(X) = n!$

Additive

$(G, +) \to$ group

(properties)

$-(a+b) = -b + (-a)$

$(F, +, \cdot)$

$F \neq \Phi$

$(F, +) \to$ abelian group

$a+b \in F \ \forall \ a, b \in F$

$a+(b+c) = (a+b)+c$

$a+0 = a = 0+a$

$a \in F \Rightarrow \exists \ -a \in F \ni$

$a+(-a) = 0 = (-a)+a$

$a+b = b+a$

$(F, \cdot) \to$ abelian group

$ab \in F \ \forall \ a, b \in F$

$a(bc) = (ab)c$

$a \cdot 1 = a = 1 \cdot a$

$a \in F \ \exists \ b \in F \ni$

$ab = 1 = ba$

$ab = ba$

Distributive law

$a \cdot (b+c) = ab + ac \quad \forall a, b, c \in F$

★ Then $(F, +, \cdot)$ is called a field

Prove :- $a \cdot 0 = 0$

Proof

$a \cdot 0 = a(0+0) = a \cdot 0 + a \cdot 0$

$\Rightarrow a \cdot 0 + (a \cdot 0)^{-1} = (a \cdot 0) + (a \cdot 0) + (a \cdot 0)^{-1}$

$\Rightarrow \boxed{a \cdot 0 = 0}$ -- proved.

Ex

$(\mathbb{Z}, +, \cdot)$ ✗

$(\mathbb{Q}, +, \cdot)$

$(\mathbb{R}, +, \cdot)$

$(\mathbb{C}, +, \cdot)$

$(\mathbb{Q}(\sqrt{2}), +, \cdot)$

$(\mathbb{Q}\sqrt{p}, +, \cdot)$

$\mathbb{Q}(\sqrt{2}) = \{ a + b\sqrt{2} \mid a, b \in \mathbb{Q} \} \subsetneq \mathbb{R}$

$\mathbb{Q}(\sqrt{p}) = \{ a + b\sqrt{p} \mid a, b \in \mathbb{Q} \} \subsetneq \mathbb{R}$

p = prime

$Z_p = \{0, 1, 2, \dots\dots, p-1\}$

$(Z_p, \oplus, *) \longrightarrow$ field

$a, b \in Z_p$; $a \oplus b = a+b \pmod{p}$

$a * b = ab \pmod{p}$

F is field

$F[x]$ = the set of all polynomials with co-efficient from F

$P(x) = a_0 + a_1x^1 + a_2x^2 + \dots\dots + a_kx^k$

$\deg(P(x)) = k$ if $a_k \neq 0$

(*) $\left\{ \dfrac{P(x)}{g(x)} \,\middle|\, P(x), g(x) \in F[x],\ g(x) \neq 0 \right\} = F(x)$

rational function

function field.

$(F(x), +, \cdot) \rightarrow$ field

Prove that

(i) $a(-b) = -(a \cdot b) = (-a) \cdot b \quad \forall\, a, b \in F$

(ii) $(-a)(-b) = ab$

\# $\underbrace{1+1+\cdots+1}_{k \text{ times}} = k\cdot 1 = 0$

F : field
if $k1 \neq 0$ $\forall k \in \mathbb{N}$ then we say that the characteristics of F is 0. ($\text{char}(F) = 0$)

Not in the syllabus

if $k1 = 0$ for some $k \in \mathbb{N}$ then we say the smallest k with $k1 = 0$ is the characteristics of F.
[$\text{char}(F) = k$]

k must be prime

Proof

we know,

$$a\cdot 0 = 0 \quad \forall a \in F$$

& $$b + (-b) = 0 \quad \forall b \in F$$

$$\Rightarrow a\cdot(b + (-b)) = 0$$

$$\Rightarrow a\cdot b + a(-b) = 0$$ (using distributive law)

$$\Rightarrow (-(ab)) + ab + a(-b) = -(ab)$$

$$\Rightarrow a(-b) = -(ab) \quad \cdots (i) \; \boxed{}$$

similarly, $(-a)\cdot b = -(ab)$ ... (ii) $\boxed{}$

$$\Rightarrow a(-b) = -(ab) = (-a)b \quad \cdots \boxed{}$$

(ii) $0 \cdot 0 = 0$

$\Rightarrow [a+(-a)][b+(-b)] = 0$

$\Rightarrow ab + a(-b) + (-a)b + (-a)(-b) = 0$

$\Rightarrow \underbrace{ab + [-(ab)]}_{0} + [-(ab)] + (-a)(-b) = 0$

$\Rightarrow (ab) + [-(ab)] + (-a)(-b) = ab$

$\Rightarrow (-a)(-b) = ab$ . . . . . $\boxed{a}$

06.08.2024

Vector Space

F - a field

$(F, +, \cdot)$

$V \neq \Phi$

$(V, +)$ - abelian group

$+ : V \times V \longrightarrow V$

$(x, y) \longrightarrow x+y$

{conditions to be a Group.}

$\cdot : F \times V \longrightarrow V$

$(\lambda, v) \longrightarrow \lambda . v = \lambda v$

(i) $1 v = v$, $\forall v \in V$.

(ii) $(\lambda \beta) v = \lambda (\beta v)$

(iii) $(\lambda + \beta) v = \lambda v + \beta v$

(iv) $\lambda (v_1 + v_2) = \lambda v_1 + \lambda v_2$

Then V is called a vector space over F. Elements of F are called scalars, that of V are called vectors.

For $v, w \in V$ $v+w$ is the sum of the vectors v & w.

For $\lambda \in F$, $v \in V$, λv is called the scalar multiplication of λ with v.

Ex $V = M_{m \times n}(F)$ = set of all $m \times n$ matrices with entrees from F

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$(M_{m\times n}(F), +, \cdot) \longrightarrow$ Vector space over F

$$B = (b_{ij})_{m\times n} = \begin{bmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ b_{m1} & \cdots & b_{mn} \end{bmatrix}$$

$$A+B = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix} = (a_{ij}+b_{ij})_{m\times n}$$

$(A+B)+C = (a_{ij}+b_{ij}) + c_{ij}$

$A+(B+C) \longrightarrow a_{ij} + (b_{ij} + c_{ij})$

* $(\lambda+\beta)A = \lambda A + \beta A$

Example

(2) $V = \mathbb{R}^n = \{ a = (a_1, a_2, \cdots, a_n) \mid a_i \in \mathbb{R},\ i = 1, 2, \cdots, n \}$

$a, b \in V, \quad a+b = (a_1+b_1, \cdots, a_n+b_n)$

$\lambda \in \mathbb{R}, \quad a \in V, \quad \lambda a = (\lambda a_1, \cdots, \lambda a_n)$

F - Field

$F^n = \{ a = (a_1, \cdots, a_n) \mid a_i \in F \}$

$(F^n, +, \cdot)$ - Vector space over F.

(3) $V = \mathbb{C}$, $F = \mathbb{Q}$

$\mathbb{C}$ is a vector space over $\mathbb{Q}$

$r \in \mathbb{Q}$, $z = a + ib \in \mathbb{C}$

$rz = ra + irb$

$F = \mathbb{R}$

$\mathbb{C}$ is a vector space over $\mathbb{R}$

(4) $V = \mathbb{C}^n$

$(\mathbb{C}^n, +) \to$ abelian group

$F = \mathbb{Q} / \mathbb{R} / \mathbb{Q}(\sqrt{2})$

$\mathbb{C}^n$ is a vector space over F

(5) $X \neq \Phi$, F = Field

$\mathcal{F}(X, F) = \{f : X \to F \mid f \text{ is a function}\}$

for $f, g \in \mathcal{F}(X, F)$

$f + g : X \to F$

$x \longmapsto (f+g)(x) := f(x) + g(x)$

$\lambda f : X \to F$

$x \longmapsto (\lambda f)(x) := \lambda(f(x))$

So, $(\mathcal{F}(X, F), +, \cdot)$ is a vector space over F.

(ii) F - field

$F[x] = \{a_0 + a_1 x + \cdots + a_k x^k \mid k \in \mathbb{N} \cup \{0\},\ a_0, \cdots, a_k \in F\}$

$(F[x], +, \cdot) \rightarrow$ Vector space over F

F - field

$F^2 = \{(a_1, a_2) \mid a_1, a_2 \in F\}$

operators

$(a_1, a_2) + (b_1, b_2) = (a_1 + a_2, 0)$

$\lambda(a_1, a_2) = (\lambda a_1, \lambda a_2),\ \lambda \in F$

Q/ tell why this F^2 is not a vector space.

Ans/ It doesn't follows the condition

$$\boxed{(\lambda + \beta) u = \lambda u + \beta u}$$

LHS

$(\lambda + \beta)(a_1, a_2) =$

$((\lambda + \beta) a_1, (\lambda + \beta) a_2)$

RHS

$\lambda(a_1, a_2) + \beta(a_1, a_2)$

$(\lambda a_1, \lambda a_2) + (\beta a_1, \beta a_2)$

$= (\lambda a_1 + \lambda a_2, 0)$

So LHS $\neq$ RHS $\rightarrow$ not a vector space

Also Additive identity & Additive inverse not exist.

V = vector space over F

$v_1, v_2, \dots, v_K \in V$

$v_1 + v_2 + \dots + v_K \in V$

$\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_K \in F$

$$\underbrace{\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \dots + \lambda_K v_K}_{} \in V$$

↳ A linear combination of $v_1, v_2, \dots, v_K$

$\lambda_1, \lambda_2, \dots, \lambda_K \rightarrow$ co-efficient of the linear combination.

$V = \mathbb{R}$, $F = \mathbb{R}$

$\mathbb{R}$ is a vector space over $\mathbb{R}$

$v_1 = 1$, $v_2 = \frac{1}{2}, \dots\dots$

$\lambda_1 = \lambda_2 = \dots = 1$

$$\lambda_1 v_1 + \lambda_2 v_2 + \dots = \sum_{n=1}^{\infty} \frac{1}{n}$$

this is divergent.

F - field

V - vector space over F

$0 \in F, \; v \in V$

$\Rightarrow 0 \cdot v = 0$

$\lambda \in F, \; 0 \in V$

$\Rightarrow \lambda 0 = 0$

$\lambda \in F, \; v \in V$

$\Rightarrow (-\lambda) v = -(\lambda v) = \lambda(-v)$ <u>(Exercise)</u>

$(-1) v = -v$

$\lambda \in F, \; v \in V$

$\lambda v = 0, \; \lambda \neq 0 \Rightarrow v = 0$

$\lambda v = 0, \; v \neq 0 \Rightarrow \lambda = 0$

Subspace

$(V, +) \rightarrow$ abelian group

$V \neq \phi$

$v+w \in V, \ \forall v, w \in V$

$(v+u)+w = v+(u+w)$

$u+v = v+u$

$v+0 = v = 0+v$

$v+(-v) = 0 = (-v)+v$

$(V, +)$ vector space over F

$\lambda v \in V, \lambda \in F$

$1v = v$

$(\lambda\beta)v = \lambda(\beta v)$

$(\lambda+\beta)v = \lambda v + \beta v$

$\lambda(v+w) = \lambda v + \lambda w$

Subspace

V - Vector space over F (Given)

A non empty subset W of V is called V if W itself is a vector space over F with vector addition & scalar multiplication as defined on V.

$W \leq V$ $\rightarrow$ subspace

$+ : V \times V \rightarrow V$

$+ : W \times W \rightarrow W$

- $W \neq \phi$
- Closure property
- $0 \in W$
- Additive inverse.
- $\lambda w \in W, \lambda \in F, w \in W$

satisfied by

so final conditions,

$W \subseteq V$

- $W \neq \phi$
- closure property
- $\lambda w \in W$, $\lambda \in F$, $w \in W$

<u>Theorm</u>

Let W be a non-empty subset of a vector space V over F. Then W is a subspace of V if & only if the following condition are satisfied:

(1) $w_1 + w_2 \in W$, $\forall\, w_1, w_2 \in W$

(2) $\lambda w \in W$, $\forall\, \lambda \in F$, $w \in W$

<u>Ex</u> V - v.s over F

$\left.\begin{array}{l}\{0\} \\ V\end{array}\right\} \longrightarrow$ subspaces of V (trivial).

$V \rightarrow$ Full subspace

$\{0\} \rightarrow$ Zero subspace.

$\rightarrow$ rest are proper

W is a subspace of V (i.e $W \leq V$) (notation)

<u>Exercise</u>

Let V be a vector space over F, Then the intersection of any collection of subspaces of V is again a subspace.

Prove

I index set

collection of subspaces $\{W_\alpha\}_{\alpha \in I}$

Claim

$\bigcap_{\alpha \in I} W_\alpha$ is a subspace

Ex

(1) $F[x]$

$n \in W \cup \{0\}$

$F_n[x] = \{ f(x) \in F[x] \mid \deg f(x) \leq n \}$

$\leq F[x]$

(2) F^n

For $1 \leq i \leq n$, $W_i = \{ (a_1, \dots, a_n) \in F^n \mid a_i = 0 \} \leq F^n$

(3) $M_{n \times n}(F)$

$W = \{ A \in M_{n \times n}(F) \mid A$ is a diagonal matrix$\}$

$\leq M_{n \times n}(F)$

(4) $M_{n \times n}(F)$

$W = \{ A \in M_{n \times n}(F) \mid A$ is a symmetric matrix$\}$

$A^T = A$

$A = (A_{ij}) = A_{ji}$

(5) $\mathcal{F}(R, R)$

$C(R,R) = \{ g \in \mathcal{F}(R, R) \mid g$ is continuous $\} \leq \mathcal{F}(R, R)$

$\mathcal{F}(R, R) \geq C(R, R) \geq R[x]$

V - vector space over F

$\Phi \neq X \in V$

$v \in V$ is a linear combination of vectors of X.

i.e $\exists$ $v_1, v_2, v_3, \dots, v_n \in X$ &

Scalars, $\lambda_1, \lambda_2, \dots, \lambda_n$ $\ni$

$$v = \sum_{i=1}^{n} \lambda_i v_i$$

Exercise

Let W be the subset of V consisting of all possible linear combination of the vectors of X

i.e $W = \{ \sum_{i=1}^{n} \lambda_i v_i \mid n \in N, v_i \in X, \lambda_i \in F \}$
$1 \leq i \leq n$

Prove that $W \leq V$,

#, $\Phi \neq X \subseteq V$

$A = \{ W \mid W \leq V, X \subseteq W \} \neq \Phi$

$V \in A$

$V_1 = \bigcap_{W \in A} W$

V_2 = the smallest subspace of V containing X as a subset

i.e

whenever W is a subspace of V containing X then $V_2 \subseteq W$

$U_3 = \left\{ \sum_{i=1}^{n} \lambda_i v_i \mid n \in \mathbb{N},\ v_1, v_2, \ldots, v_n \in X,\ \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n \in X \right\}$

Exercise

(1) Prove that, $U_1 = U_2 = U_3$

Notation: $\langle X \rangle := U_1 = U_2 = U_3$

$\langle X \rangle$ is called the subspace spanned by X

or

$\langle X \rangle$ is the subspace generated by X

if $X = \emptyset$, then $\langle X \rangle = \{0\}$.

$X = \{v_1, v_2, \ldots, v_k\}$

$\langle X \rangle = \langle v_1, v_2, \ldots, v_k \rangle$

↳ the subspace generated by the vectors $v_1, \ldots, v_k$

$X = \{(1,1)\}$

$\langle X \rangle = \{(\lambda, \lambda) \mid \lambda \in \mathbb{R}\}$

$X = \{(1,1), (1,0)\}$

$\langle X \rangle = \mathbb{R}^2$.

Sum of Subspaces

$W_1, W_2, \ldots, W_k$ + subspace of V $(k \geq 2)$

$$W_1 + W_2 + \cdots + W_k = \{ w_1 + w_2 + \cdots + w_k \mid w_i \in W_i, 1 \leq i \leq k \}$$

$\downarrow$

(Sum of $W_1, \ldots, W_k$) $= \sum_{i=1}^{k} W_i$

Exercise:

(1) Prove that $\sum_{i=1}^{k} W_i$ is a subspace of V

(2) $X = W_1 \cup W_2 \cup \cdots \cup W_k$ then

$\langle X \rangle = W_1 + W_2 + \cdots + W_k$

$F[x]$

$W_1 = F_{10}[x]$

$W_2 = F_{20}[x]$

$W_1 + W_2 = W_2$

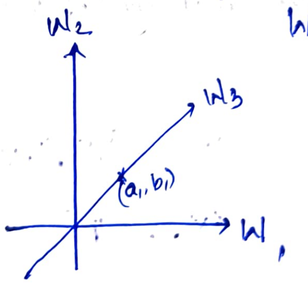
R^2

$W_1 = \langle (1,0) \rangle$

$W_2 = \langle (0,1) \rangle$

$W_1 + W_2 = R^2$

$W_1 + W_3 =$



$\Phi \neq X \subseteq V$

X is said to be linearly dependent (L.d) if $\exists$ distinct vectors $u_1, u_2, \dots, u_n \in X$ & Scalars $\lambda_1, \lambda_2, \dots, \lambda_n \in F$ (not all of them are zero)

s.t. $\lambda_1 u_1 + \lambda_2 u_2 + \lambda_3 u_3 + \dots + \lambda_k u_k = 0$

$$\Rightarrow \sum_{i=1}^{k} \lambda_i u_i = 0$$

X is not L.d means whenever,

$$\lambda_1 u_1 + \dots + \lambda_k u_k = 0$$

$$\Rightarrow \lambda_1 = \lambda_2 = \dots = \lambda_k = 0$$

A set X which is not l.d. is called a linearly independent (l.i.) set.

$X = \{x_1, x_2, \dots, x_r\}$

X is l.d / l.i

$\Downarrow$

the vectors $x_1, x_2, \dots, x_r$ are l.d / l.i.

$X \subseteq V$

$0 \in X \Rightarrow X$ is l.d

$\lambda \in F, \lambda \neq 0$

$\lambda 0 = 0$

LA

$X \subseteq V$

$\langle X \rangle$

$X = \Phi$

$\langle \phi \rangle = \{0\}$

$\Phi \subseteq V$

↳ li / l.d ? → li

Exercise

V is over F

(1) $S \subseteq V$

then S is li iff every finite subset of S is l.i

(2) $X \subseteq V$, X is li, $v \in V \setminus X$

then

(a) $X \cup \{v\}$ is l.d. iff $v \in \langle X \rangle$

(b) $X \cup \{v\}$ is li iff $v \notin \langle X \rangle$

$X \subseteq V$

X spans V

$\Rightarrow \langle X \rangle = V$

Suppose, $B \subseteq V$, $|B| < \infty$; $\langle B \rangle = V$

Theorem (Replacement theorem)

V - V.S over F

$B \subseteq V$, $\langle B \rangle = V$, $|B| = n < \infty$

S - l.i subset of V, $|S| = m \geq 0$

Then

(i) $m \leq n$;

(ii) $\exists$ a subset H of $B \ni |H| = n-m$ & $\langle S \cup H \rangle = V$

Proof

By Mathematical induction on $m \geq 0$

$m = 0$: $S = \phi$ take $H = B$.

Assume for $m = k \geq 0$

To prove for $m = k+1$

Let $T = \{v_1, v_2, \dots, v_k, v_{k+1}\}$

be a l.i subset of V.

The set $\{v_1, v_2, \dots, v_k\}$ is l.i

By induction hypothesis,

(i) $k \leq n$

(ii) $\exists$ a subset $\{u_1, u_2, \dots, u_{n-k}\}$ of $B \ni$
$\langle \{v_1, \dots, v_k\} \cup \{u_1, \dots, u_{n-k}\} \rangle = V$

$k < n$ otherwise, $k = n$

then $\langle v_1, \cdots, v_k = v_n \rangle = V$

$\Rightarrow v_{k+1} \in \langle v_1, \cdots, v_k = v_n \rangle$.

$\Rightarrow \underbrace{\{v_1, \cdots, v_k = v_n\} \cup \{v_{k+1}\}}_{T}$ is l.d.

a contradiction.

$\langle \{v_1, \cdots, v_k\} \cup \{u_1, \cdots, u_{n-k}\} \rangle = V$

$\Rightarrow \quad v_{k+1} = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_k v_k + \beta_1 u_1 + \cdots + \beta_{n-k} u_{n-k}$

for some $\lambda_1, \cdots, \lambda_k$ & $\beta_1, \cdots, \beta_{n-k} \in F$

Not all the $\beta_1, \cdots, \beta_{n-k}$ are 0.

WLOG assume that $\beta_1 \neq 0$.

Then,

$u_1 = -\beta_1^{-1} \lambda_1 v_1 - \cdots - \beta_1^{-1} \lambda_k v_k + \beta_1^{-1} v_{k+1} - \beta_1^{-1} \beta_2 u_2 - \cdots - \beta_1^{-1} \beta_{n-k} u_{n-k}$

Take $H = \{u_2, u_3, \cdots, u_{n-k}\}$

Then $|H| = n-k-1 = n-(k+1)$

Claim $\Rightarrow$ $\langle T \cup H \rangle = V$

$u_1 \in \langle T \cup H \rangle$? = ✓

$v_1, \cdots, v_k \in T \cup H \subseteq \langle T \cup H \rangle$

$u_2, \cdots, u_{n-k} \in T \cup H \subseteq \langle T \cup H \rangle$

$\langle x_1, \ldots, x_{r}, u_1, \ldots, u_k \rangle \leq T \cup H \leq V$

V

$\Rightarrow \langle T \cup H \rangle = V$

<u>Defⁿ</u>

V - VS over F

A subset B of V is called a basis of vector V if

(i) B is l.i.

(ii) $\langle B \rangle = V$

* If V has a finite bases, then V is called finite dimensional.

Otherwise, Infinite-dimensional.

<u>Theorem</u>

Let V be a f.d. vector space over F

$\Rightarrow$ Then any two bases of V have the same no. of vectors.

<u>Proof</u>

B_1, B_2 - bases of V. s.t $|B_1| = n < \infty$

$\langle B_1 \rangle$ V

B_2 - li subset of V.

$|B_2| \leq n = |B_1|$

Let $|B_2| = m$ $\Rightarrow$ $\boxed{m \leq n}$ — (i)

Also $n \leq m$ by interchanging B_1 & B_2.

$\Rightarrow$ $m \leq n$

&

$n \leq m$

$\Rightarrow$ $m = n$

V - v.s over F

$B \subseteq V$, $\langle B \rangle = V$

if $A \subseteq B$ & $\langle A \rangle \neq V$

then prove that $\exists$ $v \in B \setminus \langle A \rangle$

Recall

A basis of V over F is a subset B of V s.t.

(i) B is l.i

(ii) $\langle B \rangle = V$

Suppose that,

$|B| < \infty$ & $\langle B \rangle = V$

$\Rightarrow$ Any l.i subset of V has at most $|B|$ vectors.

in particular, V is f.d

Suppose V is f.d

$\Rightarrow$ Any two basis have the same no. of vectors

Defn

Let V be a f.d vector space over F then the no. of vectors in a basis of V is called the dimension of V.

Notation: $\dim_F V$: dimension of V over F

Examples

(1) F is a vector space over F

$\dim_F F = 1$

Basis $\rightarrow \{1\}$, $\{\lambda \mid \lambda \neq 0\}$

(2) $F[x]$

Basis: $\{1, x, x^2, x^3, \cdots\}$. the standard basis.

$\dim_F F[x] = \infty$

$F_n[x] \leq F[x]$, $n \in \{0\} \cup \mathbb{N}$

Basis : $\{1, x, \cdots, x^n\}$ $\longrightarrow$ the standard basis.

$\dim_F F_n[x] = n+1$

$P_0(x) = 1 + x + x^2 + \cdots + x^n$

$P_1(x) = x + x^2 + \cdots + x^n$

$P_2(x) = x^2 + x^3 + \cdots + x^n$

$P_n(x) = x^n$

$\{P_0(x), P_1(x), \cdots, P_n(x)\}$ $\rightarrow$ basis of $F_n[x]$

(★ Langrang's polynomials)

$F \times F \times \cdots \times F$

(3) F^n is a v.s over F

$e_1 = (1, 0, \cdots, 0)$

$e_2 = (0, 1, 0, \cdots, 0)$

$\vdots$

$e_j = \{(\alpha_1, \alpha_2, \cdots, \alpha_n) \mid \alpha_j = 1$ else $\alpha_i = 0\}$

$\vdots$

$e_n = (0, \cdots, 0, 1)$

$\{e_1, e_2, \cdots, e_n\}$ $\rightarrow$ The standard basis of F^n.

$\dim_F F^n = n$

(4) $M_{m\times n} F \to$ vector space over F

For $1 \le i \le m$, $1 \le j \le n$ consider

$E_{ij} = ([a_{pq}]_{m\times n} \mid a_{pq} = 1$ if $p = i$ & $q = j$ else $0)$

standard Basis $= \{E_{ij} \mid 1 \le i \le m, 1 \le j \le n\}$

$\dim_F M_{m\times n}(F) = mn$.

(5) $W \subseteq M_{n\times n}(F)$

$W = \{A \in M_{n\times n}(F) \mid A = A^T\}$

Find $\dim_F W$ (Exercise)

(6) $\dim_F \{0\} = 0$

(7) $\dim_{\mathbb{C}} \mathbb{C} = 1$ S.b $\to \{1\}$

(8) $\dim_{\mathbb{R}} \mathbb{C} = 2$ S.b $\to \{1, i\}$

<u>Theorem</u> :→

V - f.d vector space over F.

$B = \{u_1, u_2, \dots, u_t\} \subseteq V$

Then B is a basis of V
iff
every element of V can be uniquely expressed as a linear combination of the vectors of B.

i.e For $v \in V$ $\exists$ $\lambda_1, \lambda_2, \dots, \lambda_t$ which are uniquely determined by v s.t

$v = \lambda_1 u_1 + \lambda_2 u_2 + \dots + \lambda_l u_l$

Proof

Exercise

Theorm

$B \subseteq V$, $|B| < \infty$ $\langle B \rangle V$

then $\exists$ a subset A of B s.t A is a basis of V.

in particular, V is f.d.

Proof

$\langle \phi \rangle = \{0\}$

$V = \{0\}$

$B = \phi$

$B = \{0\} = V$

Assume $V \neq \{0\}$ & $B \neq \phi$

$\exists u_1 \in B$, $u_1 \neq \phi$

16.08.2020

$B \subseteq V$, $\langle B \rangle = V$, $|B| < \infty$

$\Rightarrow$ B contains a basis of V

Proof

$B = \phi$, $\langle B \rangle = \{0\}$

$\Rightarrow V = \{0\}$

Assume $B \neq \{0\}, \phi$

$\Rightarrow 0 \neq u_1 \in B$

$\Rightarrow \{u_1\}$ li

if $\langle u_1 \rangle = V$, then $\{u_1\}$ is a basis of V

Otherwise, $\{u_1\} \neq V$ & so

$\exists\, u_2 \in B \setminus \langle u_1 \rangle$

Now

$\{u_1, u_2\} \rightarrow$ li

if $\langle u_1, u_2 \rangle = V$, then $\{u_1, u_2\}$ is a basis of V

otherwise

⋮

finite steps.

V - f.d (V.S)

A - l.i subset of V

then $\exists$ a basis T of $V \ni A \subseteq T$

<u>Proof</u>

if $\langle A \rangle = V$ ...

else $\langle A \rangle \subsetneq V$ then,

$\exists\ v_1 \in V \setminus \langle A \rangle$

$A_1 = A \cup \{v_1\}$

if $\langle A_1 \rangle = V$ then ...

else,

~~repeat~~ repeat

Every Linearly independent subset of a vector space can be extended to a basis of V. (or is a part of a basis of V)

<u>Exercise</u>

V - f.d V.S & $W \leq V$

Then

(i) W is f.d

(ii) $\dim_F W \leq \dim_F V$

(iii) if $\dim_F W = \dim_F V \Rightarrow W = V$

Linear Algebra

A - nonempty set

A relation $\sim$ on A is called a partial order if

(1) $x \sim x$, $\forall x \in A$ (reflexive)

(2) $x \sim y$, $y \sim z \Rightarrow x \sim z$ | (transitive)

(3) $x \sim y$ & $y \sim x \Rightarrow x = y$ | (anti-symmetric)

$(A, \sim)$ - partially ordered set

Replace,
$\sim$ by $\leq$

$(A, \leq) \to$ poset

Examples

A - a nonempty family of sets
$(A, \subseteq)$ - poset

$X = \emptyset$, $(P(X), \subseteq)$ - poset

$(\mathbb{N}, \leq)$, $(\mathbb{Q}, \leq)$, $(\mathbb{R}, \leq)$

(3) X = infinite set
$A = \{ B \mid B \subseteq X, |B| < \infty \}$
$(A, \subseteq) \to$ poset

Def^n

$(A, \leq) \to$ poset

(1) A subset C of A is called a chain if for every pair of elements x & y in C, we have $x \leq y$ or $y \leq x$

(Chain $\equiv$ totally ordered set)

(2) Let $X \subseteq A$. An upper bound for X is an element $u \in A$ such that $x \leq u$ $\forall x \in X$.

(u may not be in X)

(3) A maximal element in A is an element $m \in A$ such that if $m \leq x$ for some $x \in A$ then $m = x$

Examples:-

(1) $(P(\mathbb{N}), \subseteq)$ - poset

$A_n = \{1, 2, \dots, n\}$, $n \geq 1$

$C = \{A_n \mid n = 1, 2, \dots\}$

↳ chain ($A_m \subseteq A_n$ iff $m \leq n$)

↳ upper bound is $\mathbb{N} \notin C$

(2) $A =$ the collection of all proper subset of $\mathbb{N}$

$(A, \subseteq) \to$ poset

$A_n = \{1, 2, \dots, n\}$, $n \geq 1$

$C = \{A_n \mid n = 1, 2, \dots\}$

↳ chain

↳ upper bound doesn't exist

(3) $(\mathbb{R}, \leq)$ - poset

$C' = [2, 8)$ (u.b $\to$ 8)

$C'' = [2, 8]$ (u.b $\to$ 8)

Maximal

(4) $X \neq \phi$, $(P(X), \subseteq)$ - poset

$\hookrightarrow$ maximal element $= X$

(5) $X \neq \phi$, $Y \neq \phi$, $X \cap Y = \phi$

$A = P(X) \cup P(Y)$

$(A, \subseteq)$ - poset

$\hookrightarrow$ maximal elements are X & Y

(6) $X =$ infinite set

$A = \{B \mid B \subseteq X, |B| < \infty\}$

$(A, \subseteq)$ - poset

$\hookrightarrow$ no maximal element.

Zorn's Lemma :$\Rightarrow$

Let $(A, \leq)$ be a poset if every chain of A has an upper bound, then A has a maximal element.

* I = index set

$\{A_\alpha\}_{\alpha \in I}$ $\quad \prod_{\alpha \in I} A_\alpha \longrightarrow (x_\alpha)_{\alpha \in I}$

$x_\alpha \in A_\alpha$

Let $\mathcal{S}$ be a non empty family of sets.

(so $(\mathcal{S}, \subseteq)$ - is a poset)

If every chain of $\mathcal{S}$ has an upper bound (i.e if C is a chain of $\mathcal{S}$ then $\exists\ U \in \mathcal{S} \ni X \subseteq U, \forall X \in C$)

then $\mathcal{S}$ contains a maximal element i.e $\exists\ M \in \mathcal{S} \ni$ $M \leq X$ for $X \in \mathcal{S} \Rightarrow M = X$

<u>Recall</u>

V - V.S over F

X - l.i subset of V

& $v \in V$

Then

(1) $X \cup \{v\}$ is l.d iff $v \in \langle X \rangle$

(2) $X \cup \{v\}$ is l.i iff $v \notin \langle X \rangle$

Def^n Let $B \subseteq V$. Then B is called a maximal linearly independent subset of V if

(1) B is l.i

(2) if $B \subseteq X$, when X is a l.i subset of V

then $B = X$

<u>Theorem</u>

Let $B \subseteq V$. Then

B is a basis of V

iff

B is a max linearly independent subset of V.

AIMS

To prove that every vector space has a basis

Equivalently, to prove that every vector space has a max. l.i subset.

Define

$\mathcal{A} = \{ B \mid B \subseteq V, B \text{ is l.i} \}$

$(\mathcal{A}, \subseteq)$ - poset

$\hookrightarrow$ max element exists.

23.08.2024

Zorn's lemma

$(A, \leq) \to$ poset

if every chain in A has an upper bound, then A has a maximal element.

($\exists\, m \in A \ni$ whenever $m \leq x \Rightarrow m = x$)

V - v.s over F

$\Rightarrow V$ has a basis

Proof

$\mathcal{J} = \{B \mid B \subseteq V, B \text{ is l.i}\} \neq \phi$

as $\phi \in \mathcal{J}$

$(\mathcal{J}, \subseteq)$ - poset

claim

$\mathcal{J}$ has a maximal element. i.e V has a maximal L.i subset.

$\Rightarrow V$ has a basis.

Proof

By Zorn's lemma, it is enough to prove that every chain in $\mathcal{J}$ has an upper bound.

Let C be an chain in $\mathcal{J}$ so for $X, Y \in C$

we have $X \subseteq Y$

or $Y \subseteq X$

Define

$$B = \bigcup_{X \in C} X \subseteq V$$

Then $Y \subseteq B \quad \forall\, Y \in C$

claim

$B \in \mathcal{Y}$

enough to check that B is l.i

let $\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_t v_t = 0$

for some $v_1, v_2, \ldots, v_t \in B$

& $\lambda_1, \lambda_2, \ldots, \lambda_t \in F$.

$v_1 \in X_1$ for some $X_1 \in C$

$\vdots$

$v_i \in X_i$ forsome $X_i \in C$, $1 \leq i \leq t$

$X_1, X_2, \ldots, X_t \in C$

~~Cchain~~

C-chain $\Rightarrow \exists\, j \in \{1, \ldots, t\} \ni X_i \subseteq X_j \quad \forall\, 1 \leq i \leq t$

$\Rightarrow v_1, v_2, \ldots, v_t \in X$

X_i - l.i & $\lambda_1 v_1 + \cdots + \lambda_t v_t = 0$

$\Rightarrow \lambda_1 = \lambda_2 = \cdots = \lambda_t = 0$

Exercise

V - v.s over F

$S \subseteq V$, S is l.i

$\Rightarrow$ $\exists$ a basis B of V such that $S \subseteq B$

~~Odered~~

Ordered Basis

V = f.d vector space over F

A basis B of V is called an ordered basis if a specific order of the vectors in B is given

(if $|B| = K$, then consider the vectors in B as a seq of K vectors)

$B = \{v_1, v_2, \dots, v_k\}$

Linear Transformation

V, W - Vector space over the same field F

(V, W need not be distinct)

A function $T: V \to W$ is called a linear transformation

if (1) $T(x+y) = T(x) + T(y)$, $\forall x, y \in V$

(2) $T(\lambda x) = \lambda T(x)$, $\forall x \in V, \lambda \in F$

(i) [illegible] $: V \to W$

[illegible]

is called the zero transformation.

(ii) [illegible] $: V \to W$ $(V = W)$

[illegible]

is identity transformation

(iii) $D : F[x] \longrightarrow F[x]$

$P(x) = \lambda_n x^n + \lambda_{n-1} x^{n-1} + \cdots + \lambda_1 x + \lambda_0$ $(\lambda_0, \ldots, \lambda_n \in F)$

$P(x) \longmapsto n \lambda_n x^{n-1} + \cdots + 2\lambda_2 x + \lambda_1$

D is a l.t

<u>check</u>

$D_n : F_n[x] \longrightarrow F_{n-1}[x]$

(4) $T_x : R^2 \to R^2$

$(a, b) \to (a, -b)$

$\hookrightarrow$ reflection w.r.t. x-axis

$T_y : R^2 \to R^2$

$(a, b) \longmapsto (-a, b)$

$\hookrightarrow$ reflect^n w.r.t y-axis.

(5) $\theta \to$ a given angle.

$T: \mathbb{R}^2 \to \mathbb{R}^2$

$$(a_1, b_1) \longmapsto \begin{cases} (0,0), \text{ if } (a_1,b_1) = (0,0) \\ (x_1, y_1), \text{ if } (a_1,b_1) \neq (0,0) \end{cases}$$

where (x_1, y_1) is obtained from (a_1, b_1) with a rotation of $\theta°$ counter clockwise.

(6) $T: M_{m\times n}(F) \longrightarrow M_{n\times m}(F)$

$A \longmapsto A^T$

$\hookrightarrow$ is a linear transformation.

(7) $\mathcal{C}(\mathbb{R}, \mathbb{R})$ - the v.s of all real valued continuous function defined on $\mathbb{R}$.
$(a, b \in \mathbb{R}, a < b)$
(V.s over $\mathbb{R}$)

$T: \mathcal{C}(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$; $f \mapsto \int_a^b f(t)\,dt$

* If $T: V \to V$ is a l.t the T is called a linear operator.

* If $T: V \to F$ is a l.t, then T is called a linear functional

Lets define,

$$V^* = \{T : V \to F \mid T \text{ is a l.t}\}$$

$T : V \to V^*$ is a l.t

if $\dim_F V = n < \infty$, then

$$\dim_F V^* = n$$

&

$$\boxed{\begin{array}{c} T \text{ is one-one} \\ \text{iff} \\ T \text{ is onto} \end{array}}$$

<u>Recall</u>

$T : V \to W$ is a l.t if

$$\left.\begin{array}{l} (1)\ T(x+y) = T(x) + T(y) \\ (2)\ T(\lambda x) = \lambda T(x) \end{array}\right\} \forall\ x, y \in V,\ \lambda \in F$$

<u>Ex</u>

① if $F = \mathbb{Q}$, then prove that
cond^n (1) $\longrightarrow$ cond^n (2)

② $T(0) = 0$

③ $T(-x) = -T(x)$, $\forall\ x \in V$

(4) $T(x-y) = T(x) - T(y)$

(5) if $x_1, \dots, x_t \in V$ & $\lambda_1, \dots, \lambda_t \in F$, then
$$T\left(\sum_{i=1}^{t} \lambda_i x_i\right) = \sum_{i=1}^{t} \lambda_i T(x_i)$$

Linear Algebra

26.08.2024

Recall

$V, W \to$ V.S of F

$T: V \to W$ is a l.t

if (1) $T(v+w) = T(v) + T(w)$
$\forall\, v, w \in V$

(2) $T(\lambda v) = \lambda T(v)$
$\forall\, v \in V, \lambda \in F$

Suppose $\dim_F V < \infty$

& $T_1, T_2 : V \to W$ are l.t

$B = \{v_1, v_2, \dots, v_n\}$ - basis for V, $\dim_F V = n$

Then $T_1 = T_2$ iff $T_1(v_j) = T_2(v_j)$, $\forall\, j \in \{1, 2, \dots, n\}$

This Theorem is a Consequence of the next result.

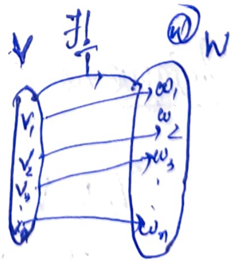
Theorem

Suppose $\dim_F V = n < \infty$

$B = \{v_1, v_2, \dots, v_n\}$ - basis of V

W - any vector space

let $w_1, w_2, \dots, w_n \in W$
(Need not to be distinct)



Then $\exists$ a unique l.t
$T : V \to W \ni$
$T(v_j) = w_j$ for $j \in \{1, 2, \dots, n\}$

Proof

existance To define a l.t

$T: V \to W$

$v \mapsto T(v) = ?$

Given

$v \in V$

$\Rightarrow v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$

for uniquely determined scalars $\lambda_1, \lambda_2, \ldots, \lambda_n \in F$

Define

$T(v) = \lambda_1 w_1 + \lambda_2 w_2 + \cdots + \lambda_n w_n \in W$

$u \in V \Rightarrow u = \beta_1 v_1 + \cdots + \beta_n v_n$

$T(u) = \beta_1 w_1 + \cdots + \beta_n w_n$

show

$T(u+v) = T(u) + T(v)$

similarly

$T(\alpha v) = \alpha T(v)$

Now $T(v_1) = w_1$

$T(v_2) = w_2$

Uniqueness

$T_1 : V \to W$ l.t

$T_1(v_j) = w_j, \ 1 \leq j \leq n$

claim $T_1 = T$

let $v \in V \Rightarrow v = \lambda_1 v_1 + \cdots + \lambda_n v_n$, $\lambda_i \in F$

$$T_1(v) = T_1\left(\sum_{i=1}^{n} \lambda_i v_i\right)$$

$$= \sum_{i=1}^{n} T_1(\lambda_i v_i)$$

$$= \sum_{i=1}^{n} \lambda_i (T_1 v_i)$$

$$= \sum_{i=1}^{n} \lambda_i w_i$$

$$= T(v)$$

$\Rightarrow T_1 = T$

$V, W \to$ V.S. of F

$T : V \to W$ is a l.t.

$R(T) = \{T(v) \mid v \in V\} \to$ image of T / range of T
$\subseteq W$

$N(T) = \{v \in V \mid T(v) = 0\} \to$ kernel of T
$\subseteq V$

Exercise

Prove that

(1) $R(T)$ is a subspace of W

(2) $N(T)$ is a subspace of V

$R(T) \to$ called range space of T

$N(T) \to$ " null space of T

Defn

if $R(T)$ is a f.d subspace of W, then the dimension of $R(T)$ is called the rank of T.

(denoted by rank (T))

if $N(T)$ is a f.d subspace of ~~[illegible]~~ V then the $\dim_F N(T)$ is called the nullity of T (nullity (T))

Theorem

$T: V \to W$ is a l.t

where $\dim_F V < \infty$

Then $\boxed{\text{rank}(T) + \text{nullity of }(T) = \dim_F V}$

Exercise

$T: V \to W$ is a l.t

Prove that T is one-one

iff

$N(T) = \{0\}$

L.A

Rank - nullity theorem

$T: V \to W$ is a l.t
$\dim_F V < \infty$

$\Rightarrow \text{rank}(T) + \text{nullity}(T) = \dim_F V$

~~$\dim_F V$~~

Th

$T: V \to W$ is a l.t
B - basis for V

$\Rightarrow R(T) = \langle T(B) \rangle$

Proof

$T(B) = \{T(x) \mid x \in B\} \subseteq R(T)$
$\downarrow$ subspace of W

$\Rightarrow \langle T(B) \rangle \subseteq R(T)$

claim

$R(T) \subseteq \langle T(B) \rangle$

let $w \in R(T)$

$\Rightarrow \exists\, v \in V$ st $T(v) = w$

$v \in V$, Basis is B for V

$\Rightarrow \exists\, v_1, v_2, \ldots, v_t \in B$ & $\lambda_1, \lambda_2, \ldots, \lambda_t \in F$

$\Rightarrow v = \lambda_1 v_1 + \cdots + \lambda_t v_t$

$T(v) = \sum_{i=1}^{t} T(\lambda_i v_i)$

$= \sum_{i=1}^{t} \lambda_i (T(v_i))$

$T(v_1), \ldots, T(v_t) \in T(B)$ so $w \in \langle T(B) \rangle$

$\Rightarrow R(T) \subseteq \langle T(B) \rangle$

Proof of R-N thm

Let $\dim_F V = n < \infty$ & $\dim_F N(T) = K \; (0 \le K \le n)$

Let $B_1 = \{v_1, v_2, \dots, v_k\}$ be a basis for $N(T)$

↳ l.i subset of V

Extend B_1 to a basis B for V

Let $B = \{v_1, \dots, v_k\} \cup \{u_{k+1}, \dots, u_n\}$

B - Basis for V

$\Rightarrow R(T) = \langle T(B) \rangle$

$= \langle T(v_1), \dots, T(v_k), T(u_{k+1}), \dots, T(u_n) \rangle$

$= \langle T(u_{k+1}), \dots, T(u_n) \rangle$

$C = \{T(u_{k+1}), \dots, T(u_n)\} \longrightarrow$ generating set for $R(T)$

Claim: C is l.i

Proof

Let $\beta_{k+1} T(u_{k+1}) + \beta_{k+2} T(u_{k+2}) + \dots + \beta_n T(u_n) = 0$

$\Rightarrow T(\beta_{k+1} u_{k+1} + \beta_{k+2} u_{k+2} + \dots + \beta_n u_n) = 0$

$\Rightarrow \beta_{k+1} u_{k+1} + \beta_{k+2} u_{k+2} + \dots + \beta_n u_n \in N(T)$

$\Rightarrow \beta_{k+1} u_{k+1} + \beta_{k+2} u_{k+2} + \dots + \beta_n u_n = \lambda_1 v_1 + \dots + \lambda_k v_k$

for some $\lambda_1, \dots, \lambda_k \in F$

$\Rightarrow \lambda_1 v_1 + \dots + \lambda_k v_k - \beta_{k+1} u_{k+1} - \beta_{k+2} u_{k+2} - \dots - \beta_n u_n = 0$

But as B is l.i $\Rightarrow \lambda_1 = \dots = \lambda_k = 0$

$\beta_{k+1} = \dots = \beta_n = 0$

So C is l.i

Thus C is a basis for $R(T)$ containing $n-k$ vectors.

$\Rightarrow \dim_F R(T) = n-k$

$\Rightarrow \text{Rank}(T) = n-k$

—— x ——

Theorem

$\dim_F V = \dim_F W < \infty$

$T : V \longrightarrow W$ is a l.t

$\Rightarrow T$ is one-one iff T is onto

Proof

$\text{Rank}(T) + \text{Nullity}(T) = \dim_F V$

T is one-one $\Rightarrow N(T) = \{0\}$

$\Leftrightarrow \text{Nullity}(T) = 0$

$\Leftrightarrow \text{Rank}(T) = \dim_F V$

$\Leftrightarrow \text{Rank}(T) = \dim_F W$

$\Rightarrow \dim_F R(T) = \dim_F W$ ---(i)

We know $R(T) \leq W$ ---(ii)

from (i) & (ii) we get,

$R(T) = W$

$\Leftrightarrow T$ is onto.

Ex

$V = W = \mathbb{R}[x]$

$T : \mathbb{R}[x] \longrightarrow \mathbb{R}[x]$

$\sum_{i=1}^{n} \lambda_i x^i \longmapsto \lambda_1 + 2\lambda_2 x + \cdots + n\lambda_n x^{n-1}$

→ Not one-one

→ But onto.

Ex

$T : V \longrightarrow W$ is a l.t

Prove the following:

(1) T is one-one iff T maps every l.i subset of V to a l.i subset of W

(2) T is bijective iff T maps every basis of V to a basis of ~~V~~ W.

(3) (V)-Q18

Ex V - v.s over F

$T : V \to V$ is a linear operator

Prove that $V = R(T) + N(T)$

iff

$R(T) \cap N(T) = \{0\}$.

Def[n] Two vector spaces V & W over the same field F are called isomorphic if $\exists$ a l.t

$$T : V \longrightarrow W$$ which is bijective.

In this case we write,

$$V \simeq W$$

Ex

$$F_n[x] \simeq F^{n+1}$$

$$T : F_n[x] \longrightarrow F^{n+1}$$

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \longmapsto (a_0, a_2, \cdots, a_n)$$

$\hookrightarrow$ is a l.t + one-one + onto

L.A 29.08.2024

V, W are v.s over F

An isomorphism from V to W is a map $T: V \longrightarrow W$ ∋

(i) T is linear

(ii) T is one-one

(iii) T is onto.

$W \leq V$ $v \in V$, $v + W = \{v + x \mid x \in W\} \subseteq V$

$\frac{V}{W} := \{v + W \mid v \in V\}$ $\boxed{\dim \frac{V}{W} = n}$

$\frac{V}{W} \cong F^n$

Theorm

Let V be a v.s over F of dimension n

then $V \cong F^n \cong F^{n \times 1} = \left\{ \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix} \middle| \lambda_i \in F \right\}$

Fix an ordered basis B for V, say $B = \{v_1, v_2, \dots, v_n\}$

$v \in V \Rightarrow v = \lambda_1 v_1 + \lambda_2 v_2 + \dots + \lambda_n v_n$

$\underbrace{\lambda_1, \lambda_2, \dots, \lambda_n}_{\text{unique}} \in F$

$v \longmapsto \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$

$[v]_B = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{bmatrix}$ → the co-ordinate vector of v w.r.t B

$w \in V \Rightarrow w = \beta_1 v_1 + \beta_2 v_2 + \dots + \beta_n v_n$

$$[w]_B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$[v+w]_B = \begin{bmatrix} \lambda_1 + \beta_1 \\ \lambda_2 + \beta_2 \\ \vdots \\ \lambda_n + \beta_n \end{bmatrix} = [v]_B + [w]_B$$

$$[\alpha v]_B = \begin{bmatrix} \alpha\lambda_1 \\ \alpha\lambda_2 \\ \vdots \\ \alpha\lambda_n \end{bmatrix} = \alpha [v]_B$$

Define : $T : V \longrightarrow F^{n \times 1}$ by

$v \longmapsto T(v) := [v]_B$

$T(v+w) = T(v) + T(w)$

$T(\alpha v) = \alpha T(v)$

$\Rightarrow$ T is a linear transf

also T is one-one $\Leftrightarrow$ T is onto

$\Rightarrow$ T is bijective

$\Rightarrow \quad V \simeq F^{n \times 1}$

Exercise

(1) $T: V \to W$ is a bijective L.t.

$\Rightarrow T: W \to V$ is a bijective L.t

(2) $\left.\begin{array}{l} T_1 : V \to W \\ T_2 : U \to V \end{array}\right\}$ are l.ts

$\Rightarrow T_1 T_2 : U \to W$ is also a l.t.

Exercise

$X \neq \Phi$, W - v.s over F

$\mathcal{F}(X, W) \{ f : X \to W \mid f$ is a function $\}$

↳ Is a vector space w.r.t the following addition & scalar multiplication:

For $f, g \in \mathcal{F}(X, W)$, $\alpha \in F$

$f+g : X \to W$

$x \longmapsto (f+g)(x) := f(x) + g(x)$

$\alpha f : X \to W$

$x \longrightarrow (\alpha f)(x) = \alpha(f(x))$

$0 : X \to W$

$f + 0 = f \quad \forall f$

$0(x) = 0$

Ex Take $X = V$, a v.s over F

W - a v.s over F

$$\mathcal{L}(V, W) = \{T : V \to W \mid T \text{ is a l.t.}\}$$

↳ v.s over F w.r.t the following operations.

For $T_1, T_2 \in \mathcal{L}(V, W)$, $\alpha \in F$

$$T_1 + T_2 : V \to W$$

$$v \mapsto (T_1 + T_2)v := T_1(v) + T_2(v)$$

$$\alpha T_1 : V \to W$$

$$v \mapsto (\alpha T_1)(v) := \alpha (T_1(v))$$

Theorem:

V, W - are v.s over F

$\dim_F V = n$, $\dim_F W = m$

$\Rightarrow \mathcal{L}(V, W)$ is a f.d v.s

& $\dim_F \mathcal{L}(V, W) = mn$

Proof

$$B = \{v_1, v_2, \ldots, v_n\}$$

$$C = \{w_1, w_2, \ldots, w_m\}$$

→ ordered basis

for $k \in \{1,2,\cdots,m\}$ & $l \in \{1,2,\cdots,n\}$

define $T_{kl} : V \longrightarrow W$

$v_j \longmapsto T_{kl}(v_j) := \begin{cases} 0 & \text{if } j \neq l \\ w_k & \text{if } j = l \end{cases}$

$$\boxed{\begin{aligned} T_{k3} :\ & v_1 \longmapsto 0 \\ & v_2 \longmapsto 0 \\ & v_3 \longmapsto w_k \\ & v_4 \longmapsto 0 \\ & \vdots \\ & v_n \longmapsto 0 \end{aligned}}$$

Let $X = \{T_{kl} \mid 1 \leq k \leq m,\ 1 \leq l \leq n\}$

<u>claim:-</u> X is a basis for $\mathcal{L}(V, W)$

<u>claim</u> $\langle X \rangle = \mathcal{L}(V, W)$

let $T \in \mathcal{L}(V, W)$

<u>Aim</u> :- $T = \sum_{k=1}^{m} \sum_{l=1}^{n} \lambda_{kl} T_{kl}$, $\lambda_{kl} \in F$

$[T(v_1)]_C = \begin{bmatrix} \beta_{11} \\ \beta_{21} \\ \vdots \\ \beta_{n1} \end{bmatrix}$. $T(v_1) = \beta_{11} w_1 + \beta_{21} w_2 + \cdots + \beta_{n1} w_n$

$[T(v_2)]_C = \begin{bmatrix} \beta_{12} \\ \beta_{22} \\ \beta_{32} \\ \vdots \\ \beta_{n2} \end{bmatrix}$

$[T(v_j)]_C = \begin{bmatrix} \beta_{1j} \\ \beta_{2j} \\ \vdots \\ \beta_{nj} \end{bmatrix}$ for $1 \leq j \leq n$

Now let $S = \sum_{k=1}^{m} \sum_{l=1}^{n} \beta_{kl} T_{kl}$

claim

$S(v_j) = T(v_j)$, $\forall j \in \{1, 2, \dots, n\}$

$$S(v_j) = \left(\sum_{k=1}^{m} \sum_{l=1}^{n} \beta_{kl} T_{kl} \right) v_j$$

$$= \sum_{k=1}^{m} \sum_{l=1}^{n} \beta_{kl} T_{kl} v_j$$

$$= \sum_{k=1}^{m} \beta_{kj} w_k = T(v_j)$$