LA - Exercises

2/8/2024 (group)

1. e is unique
2. $\forall a \in G, a^{-1}$ is unique

5/8/2024

$(ab)^{-1} = b^{-1}a^{-1}$

7/8/2024 (vector space)

1. $(-1)v = -v$
2. Intersection of any collection of subspaces of V is a subspace of V

8/8/2024

1. $\{\sum_{i=1}^{n} \lambda_i v_i \mid n \in \mathbb{N}, v_i \in X, \lambda_i \in F, 1 \leq i \leq n\} =: U_3 \subseteq V \Rightarrow U_3 \leq V$
2. $\Phi \neq X \subseteq V, A = \{W \leq V | X \subseteq W\} \Rightarrow \bigcap_{W \in A} W =$ smallest $W \in A = U_3$
3. $\langle \bigcup_{i=1}^{k} W_i \rangle = \sum_{i=1}^{k} W_i \leq V$

12/8/2024

1. $S \subseteq V \Rightarrow [S$ is L.I. $\iff$ every finite subset of S is L.I.$]$
2. $X \subseteq V$ is L.I., $v \in V \backslash X \Rightarrow X \cup \{v\}$ is $\begin{cases} L.D. \iff v \in \langle X \rangle \\ L.I. \iff v \notin \langle X \rangle \end{cases}$
3. $A \subseteq B \subseteq V = \langle B \rangle \neq \langle A \rangle \Rightarrow \exists v \in B \backslash \langle A \rangle$

16/8/2024

1. $B \subseteq V = \langle B \rangle$ and $|B| < \infty \Rightarrow B$ contains a basis of V
2. L.I.$\rightarrow A \subseteq V \rightarrow$ F.D. $\Rightarrow \exists$ a basis B of $V \ni A \in B$

3. $W \leq V \to$ F.D. $\Rightarrow$
    1. $W \to$ F.D.
    2. $dim_F W \leq dim_F V$
    3. $dim_F W = dim_F V \iff W = V$

19/8/2024

$V = W_1 + W_2 + \cdots + W_k$

1. $V = W_1 \oplus W_2 \oplus \cdots \oplus W_k$
2. $w_1 + w_2 + \cdots + w_k = 0, w_i \in W_i (1 \leq i \leq k) \Rightarrow w_i = 0, \forall i$
3. $\forall j \in \{1, 2, \cdots, k\}, W_j \bigcap (\sum_{j \neq i=1}^{k} W_i) = \{0\}$
4. $\forall j \in \{2, \cdots, k\}, W_j \bigcap (\sum_{i=1}^{j-1} W_i) = \{0\}$
5. B_i is a basis of $W_i \Rightarrow \bigcup_{i=1}^{k} B_i$ is a basis of V

20/8/2024

$B \subseteq V \Rightarrow [B$ is a basis of $V \iff B$ is a max L.I. subset of $V]$

23/8/2024 (Linear Transformation)

1. L.I. $\to S \subseteq V \Rightarrow \exists$ a basis B of $V \ni S \subseteq B$
2. $F = Q \Rightarrow [T(x+y) = T(x) + T(y) \Rightarrow T(\lambda x) = \lambda T(x)]$
3. T is L.T. $\Rightarrow$
    1. $T(0) = 0$
    2. $T(-x) = -T(x), \forall x \in V$
    3. $T(x - y) = T(x) - T(y)$
    4. $v_1, v_2, \cdots, v_t \in V$ and $\lambda_1, \lambda_2, \cdots, \lambda_t \in F \Rightarrow T(\sum_{i=1}^{t} \lambda_i v_i) = \sum_{i=1}^{t} \lambda_i T(v_i)$

26/8/2024

1. $R(T)$ is a subspace of W
2. $N(T)$ is a subspace of W
3. $T : V \to W$ is a L.T. $\Rightarrow [T$ is $1-1 \iff N(T) = \{0\}]$

27/8/2024

$T : V \to W$ is a L.T. $\Rightarrow$

1. T is $1-1 \iff T$ maps every L.I. subset of V to a L.I. subset of W
2. T is bijective $\iff T$ maps every basis of V to basis of W
3. $V = R(T) + N(T) \iff R(T) \cap N(T) = \{0\}$
4. $V = R(T) \oplus N(T)$

29/8/2024

1. $T : V \to W$ is a bij. L.T. $\Rightarrow T^{-1} : W \to V$ is a bij. L.T.
2. $T_1 : V \to W$ and $T_2 : U \to V$ are L.T.s $\Rightarrow T_1T_2 : U \to W$ is L.T.
3. $\mathbb{F}(X, W) = \{f | f : \Phi \neq X \to W \to \text{v.s. over } F\}$ is a V.S. over F w.r.t. the following addition and scalar multiplication
   $\forall f, g \in \mathbb{F}(X, W), \alpha \in F$
   - $f + g : X \to W \ni x \mapsto (f + g)(x) := f(x) + g(x)$
   - $\alpha f : X \to W \ni x \mapsto (\alpha f)(x) := \alpha f(x)$
4. $\mathbb{L}(V, W) = \{T : V \to W | V, W \text{ are v.s. over } F \text{ and } T \text{ is a L.T.}\}$ is a V.S. over F w.r.t. the following addition and scalar multiplication
   $\forall T_1, T_2 \in \mathbb{L}(V, W), \alpha \in F$
   - $T_1 + T_2 : V \to W \ni v \mapsto (T_1 + T_2)(v) := T_1(v) + T_2(v)$
   - $\alpha T_1 : V \to W \ni v \mapsto (\alpha T_1)(v) := \alpha T_1(v)$

30/8/2024

$W < V \Rightarrow \exists W' < V \ni W \oplus W' = V$

2/9/2024

1. $T_1, T_2, S \in \mathbb{L}(V), \lambda \in F \Rightarrow$
   1. $IS = S = SI$
   2. $S(T_1 + T_2) = ST_1 + ST_2$
      $(T_1 + T_2)S = T_1S + T_2S$
   3. $S(T_1T_2) = (ST_1)T_2$
   4. $\lambda(T_1T_2) = (\lambda T_1)T_2 = T_1(\lambda T_2)$

2. Given,$X \in M_{m\times n}(F)$, Find $T \in \mathbb{L}(V,W) \ni f(T) = X$
   $[f : \mathbb{L}(V,W) \to M_{m\times n}(F) \ni T \mapsto f(T) := [T]_B^C]$ is an $1-1$ L.T.
3. $[T_2T_1]_B^D = [T_2]_C^D[T_1]_B^C$
   $U - T_1 \to V - T_2 \to W$; dimension: n, m, p and ordered basis: B, C, D respectively

4/9/2024 EXAM QUESTION

Suppose that, $X, Y \in M_{n\times n}(F)$ are similar matrices, (i.e.,$Y = P^{-1}XP$)
Then, $\exists$ ordered basis B_1, B_2 for V and L.T. $T \in \mathbb{L}(V)$ s.t. $X = [T]_{B_1}$ and $Y = [T]_{B_2}$

(Date - 06/09/24)

Q.1 Let V be a n-dim V.space over $\mathbb{F}$. Then P.T $V \cong \mathbb{F}^n$

Q.2 Let V, W be two V.space of dim m & n. Then Prove that $L(V, W) \cong \mathbb{F}^{mn}$

Q.3 Let $V \to n$-dim V.space. $T \in L(V)$, and V has a subspace $W \leq V$ (T-invariant) s.t $\boxed{T(x) \in W,\ \forall x \in W}$

Then show that $\exists$ a basis $\mathcal{B}$ for V s.t

$$[T]_{\mathcal{B}} = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

Q.4 $V, W \to$ V.space. $T, U \in \mathcal{L}(V, W)$ with $R(T) \cap R(U) = \{0\}$

Then P.T $\{T, U\} \to$ L.I.D in $\mathcal{L}(V, W)$.

Q.5 $V, W \to$ V.spaces & $S \subseteq V$. Then

$$S^0 := \{T \in \mathcal{L}(V, W) \mid T(x) = 0 \ \forall x \in S\}$$ Then P.T

(i) $S^0 \leq \mathcal{L}(V, W)$

(ii) $S_1 \subseteq S_2 \Rightarrow S_2^0 \subseteq S_1^0$

(iii) $S_1 \subseteq V \Rightarrow S^0 = (\langle S \rangle)^0$

Q.6 $V \to$ f.d v.space & $T \in \mathcal{L}(V)$

If $\text{rank}(T) = \text{rank}(T^2)$ then P.T $V = R(T) \oplus N(T)$