$[n] = \{1, 2, 3 \ldots n\}$

Definition: An one-to-one and onto fn.

$\pi : [n] \longrightarrow [n]$ is called permutation fn a simply Permutation

S_n: The set of all permutations from $[n] \to [n]$

$$= \{ \pi \mid \pi : [n] \xrightarrow[\text{onto}]{1-1} [n] \}$$

→ symmetric set of n symbols

Let an urn contains . N many ~~ball~~ labelled balls.

Task : choosing n. many labelled ball from the urn (Ω)

Impossible if $n > N$ & possible if $1 \leq n \leq N$

2 Styles of choices -

A Choose one ball at a time and repeat it for n-times

B Choose n many balls in one grip

$\{E \subset \Omega : |E| = n\}$

1. Style of Choosing

Additional : observe the label of the ball at each step.

$(\omega_1, \omega_2, \omega_3 \ldots \omega_n)$

Addition : You observe the label an put back in Ω

$$\{(\omega_1, \omega_2 \ldots \omega_n) : \omega_k \in \Omega \quad (k = 1, 2, 3 \ldots)\}$$

$$= \Omega^n$$

Alternatively : Observe the label and do not put it back to the Urn

$$\{(\omega_1, \omega_2, \omega_3 \ldots \omega_n) : \omega_1 \in \Omega, \omega_2 \in \Omega / \{\omega_1\}, \omega_3 \in \Omega / \{\omega_1, \omega_2$$

$$\{(\omega_1, \omega_2 \ldots \omega_n) : \forall i \in [n], \omega_i \in \Omega$$
$$\forall i, j \in [n], \text{with } i \neq j \ \omega_i \neq \omega_j\}$$

RE: Urn Ω Contain N many labeled balls and we are drawing n many balls from it Here $n \leq N$ & N is a positive integer $k > 0$ is an integer

Choice Style A3: We choose one ball and do not observe the label of the ball but observe all the n many balls at the end; and do not return it to Ω

$$A = \{ (\omega_1, \omega_2 \ldots \omega_n) : \forall\, i \in [n] \,,\ \omega_i \in \Omega \quad \forall i,j \in [n] \text{ with } i \neq j \ \ \omega_i \neq \omega_j \}$$

$$(\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n) \sim (\beta_1, \beta_2 \ldots \beta_n)$$

$$\Updownarrow$$

$$\exists\ \pi : [n] \to [n]$$

$$\text{s.t} = \alpha_1 = \beta_{\pi(1)}$$
$$\alpha_2 = \beta_{\pi(2)}$$
$$\vdots$$
$$\alpha_n = \beta_{\pi(n)}$$

Equivalence relation $\sim$ over A

$$(\alpha_1 \ldots \alpha_n) \sim (\beta_1 \ldots \beta_n) \iff \exists\ \pi : [n] \to [n] \quad \text{s.t.} \quad \alpha_i = \beta_{\pi(i)} \quad \forall i \in [n]$$

Example: Show that $\sim$ is an equivalence relation on A

Let $A = A_1 \sqcup A_2 \sqcup A_3 \ldots \sqcup A_L$

A_i forms an equivalent class wrt $\sim$ $\forall\ i \in [n]$

Task : $L = ?$

$A/\sim$ = The set formed by choosing <u>exactly</u> one element from each equivalent class A_i

= Residue or Quotient set of A wrt $\sim$

<u>Example</u> : equivalence relation on $\mathbb{Z}$

$$\alpha \sim \beta \iff 5 \mid \alpha - \beta$$

Lemma : $|A/\sim| = \left|\binom{\Omega}{n}\right| = \binom{N}{n}$

Proof : Let $A = A_1 \sqcup A_2 \sqcup A_3 \ldots \sqcup A_L$

$$A/\sim = \left\{ \begin{array}{l} \omega_1(1), \omega_2(1) \ldots \omega_n(1),\ (\omega_1(2), \omega_2(2) \ldots \omega_n(2)) \\ \qquad \ldots\ (\omega_1(L), \omega_2(L), \omega \ldots \omega_n(L) \end{array} \right\}$$

$$= \left((\omega_1(i), \omega_2(i) \ldots \omega_n(i) : i \in [L] \right)$$

We Construct a mapping from $A/\sim$ to $\binom{\Omega}{n}$

$$(\omega_1(i), \ldots \omega_n(i)) \xrightarrow{f} \{\omega_1(i) \ldots \omega_n(i)\}$$

Claim f : $A/\sim \longrightarrow \binom{\Omega}{n}$ is 1-1 and onto

Proof of claim : Let $E \in \binom{\Omega}{n}$ $E = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$.

we choose $i \in [L]$ s.t $(\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n) \in A_i$

choose $(\omega_1(i), \ldots \omega_n(i)) \in A/\omega$

Note that $f(\omega_1(i), \ldots \omega_n(1)) = \{\omega_1(i), \ldots \omega_n(i)\}$

$$= \{\alpha_1, \alpha_2 \ldots \alpha_n\}$$

$$= E$$

Thus f is onto

Let $i, j \in [L]$ with $i \neq j$ then

$$(\omega_1(i), \omega_2(i) \ldots \omega_n(i)) \neq (\omega_1(j), \omega_2(j) \ldots \omega_n(j))$$

$\exists\ k \in [n]$ s.t. $\forall\ \pi : [n] \xrightarrow[\text{onto}]{1-1} [n]$

$$\omega_k(i) = \omega_{\pi(k)}(j)$$

Hence, $\{\omega_1(i), \omega_2(i) \ldots \omega_n(i)\} \neq \{\omega_1(j), \ldots \omega_n(j)\}$

i.e. $f(\omega_1(i) \ldots \omega_n(i)) \neq f(\omega_1(j) \ldots \omega_n(j))$

Thus f is $1-1$

Hence $L = |A/\omega| = \left|\binom{\Omega}{n}\right| = \binom{N}{n}$

Th (Binomial Theorem) Let A be an a-set and B be an b-set
i.e. $|A| = a$ and $|B| = b$ with $A \cap B = \phi$

Then $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Let X be an n-set i.e. $|X| = n$

$\mathcal{P}(X)$ = The set of all subsets of X = $\{E : E \subset X\}$

$$2^X = \{f \mid f : X \to \{0,1\}\}$$

Lemma : $\exists$ 1-1 and onto fn$^{\underline{n}}$ from 2^X to $\mathcal{P}(X)$

Proof : Let $S = \{(\alpha_1, \ldots \alpha_n) : \forall\ i \in [n],\ \alpha_i \in \{0,1\}\}$

$\Rightarrow |S| = 2^n$

Let $X = \{x_1, x_2 \ldots x_n\}$

We construct a map from 2^X to S

$$f \longmapsto (f(x_1), f(x_2) \ldots f(x_n))$$

Let $(\alpha_1, \alpha_2 \ldots \alpha_n) \in S$ we construct the fn$^{\underline{c}}$

$f : X \to \{0,1\}$

$f(x_1) = \alpha_1$

$\vdots$

$f(x_n) = \alpha_n$

Hence map is onto

let $f, g \in 2^X$ with $f \neq g$

$\exists\ x \in X$ s.t $f(x) \neq g(x)$

$\exists\ i \in [n]$ s.t $x = x_i$, $f(x_i) \neq g(x_i)$

Note:

$$f \longmapsto (f(x_1), f(x_2) \ldots f(x_i) \ldots f(x_n))$$

$$g \longmapsto (g(x_1), g(x_2) \ldots g(x_i) \ldots g(x_n))$$

Then, $(f(x_1), f(x_2) \ldots f(x_n)) \neq (g(x_1)) \ldots g(x_n))$

i.e. such map is 1-1

We Construct an 1-1 & onto fn^{x} from S to $p(x)$

$$(\alpha_1, \alpha_2 \ldots \alpha_n) \longmapsto \{x_i : \alpha_i = 0\}$$

(Show that such map is 1-1 & onto)

$$2^X \longrightarrow \mathcal{P}(x)$$

An equivalence rel^n on $\mathcal{P}(x)$

$$x \sim y \Leftrightarrow |x| = |y|$$

$$\mathcal{P}(x) = A_0 \sqcup A_1 \sqcup A_2 \cdots \sqcup A_n$$

$$A_k = \{E \subset X : |E| = k\} = \binom{X}{k}.$$

$$2^n = |\mathcal{P}(x)| = |A_0| + |A_1| + \cdots |A_n|$$

$$= \left|\binom{X}{0}\right| + \left|\binom{X}{1}\right| + \cdots \left|\binom{X}{n}\right|$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n}$$

Lemma: $\exists$ 1-1 & onto map between $\binom{X}{k}$ and $\binom{X}{n-k}$

Proof: Required map is $A \longmapsto X \setminus A$

Lemma: $\exists$ 1-1 & onto map b/w

$$\binom{X \sqcup Y}{n} \text{ and } \bigsqcup_{k=0}^{n} \binom{X}{k} \times \binom{Y}{n-k}$$

where $X \cap Y = \emptyset$ & x and y are finite sets

$$|x| + |y| \geq n$$

Plug in $|x| = |y| = n$

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Lemma Pascal Identity For each positive integer n

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \quad \text{where } 1 \le k \le n$$

Proof : Let $\mathcal{A} = \left\{ X \in \binom{[n+1]}{k} : n+1 \in X \right\}$

$$\mathcal{B} = \left\{ X \in \binom{[n+1]}{k} : n+1 \notin X \right\}$$

Then $|A| + |B| = \left| \binom{[n+1]}{k} \right|$

Note that $\forall\ X \in \mathcal{B}$, $X \subset [n]$, i.e. $\mathcal{B} \subset \binom{[n]}{k}$

For each $y \in \binom{[n]}{k}$, then $y \in \mathcal{B}$

Hence $\mathcal{B} = \binom{[n]}{k} \Rightarrow |\mathcal{B}| = \binom{n}{k}$

Again $X \longmapsto X \setminus \{n+1\}$ is an 1-1 & onto map from $\mathcal{A}$ to $\binom{[n]}{k-1}$

For $Y \in \binom{[n]}{k-1}$, note that $Y \sqcup \{n+1\} \in \mathcal{A}$

i.e such map is onto

For $X \neq Y$, where $X, Y \in \mathcal{A}$, $\exists\ \alpha \in [n]$ s.t. $\alpha \in X$ but $\alpha \notin Y$,

Hence $X \setminus \{n+1\} \neq Y \setminus \{n+1\}$ i.e such map is 1-1

$\Rightarrow |\mathcal{A}| = \left| \binom{[n]}{k-1} \right|$

$\Rightarrow |\mathcal{A}| + |\mathcal{B}| = \left| \binom{[n+1]}{k} \right| \Rightarrow \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

Theorem: Given two positive integers N and i then there is a unique way to expand N as a sum of binomial coefficients as follows

$$N = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots \binom{n_j}{j}$$

where j is a positive integer.

$$n_i \geq n_{i-1} \geq \cdots n_j \geq j \geq 1$$

Proof: $n_i = \max \{n : \binom{n}{i} \leq N\}$

$$n_{i-1} = \max \{n : \binom{n}{i-1} \leq N - \binom{n_i}{i}$$

$$\vdots$$

$$n_{i-k} = \max \{n : \binom{n}{i-k} \leq N - \binom{n_i}{i} - \binom{n_{i-1}}{i-1} - \cdots - \binom{n_{i-k+1}}{i-k+1}\}$$

where $k \in \{0, 1, 2, 3, \ldots\}$

claim: $n_{i-1} \leq n_i$

Proof of claim: Note that $\binom{n_i}{i} \leq N < \binom{n_i+1}{i} = \binom{n_i}{i} + \binom{n_i}{i-1}$

Hence, $0 \leq N - \binom{n_i}{i} < \binom{n_i}{i-1}$

Since $p \geq 1 \ \forall \ m \geq 1$ we have

$$\binom{m}{i-1} \leq \binom{m+1}{i-1}$$

Therefore $\max \{m : \binom{m}{i-1} \leq N - \binom{n_i}{i}\} \leq n_i$

i.e. $n_{i-1} \leq n_i$

Th (Binomial Theorem) Let A be an a-set and B be a b-set.

with $A \cap B = \emptyset$

$$(a+b)^n = \sum_{i=1}^{n} \binom{n}{i} a^i b^{n-i}$$

Proof: Let $S = \{(x_1, x_2 \dots x_n) : \forall i \in [n],\ x_i \in A \sqcup B\}$

we construct an equivalence relation $\sim$ over

$(x_1, x_2 \dots, x_n) \sim (y_1, y_2, \dots y_n) \iff \exists (X, Y) \subset [n] \times [n]$

such that $|X| = |Y|$

$\forall i \in X \quad x_i \in A, \forall j \in [n] \setminus X, x_j \in$

$\forall i \in Y \quad y_i \in A, \forall j \in [n] \setminus Y, y_i \in B$

$S = A_0 \sqcup A_1 \sqcup A_2 \dots \sqcup A_n$

A_0

$\vdots$

$A_k = \{(x_1, x_2, x_3 \dots x_n) \in S$: exactly k element belongs to A

$\vdots = \{(x_1, \dots x_n) \in S : \exists X \subset [n]$, with $|X| = k$ s.t $\forall i \in X \quad x_i \in A$ &

$\forall j \in [n] \setminus X, x_j \in B$

A_n

$|S| = |A_0| + |A_1| + \dots + |A_n|$

$\therefore |A_0| = b^n$

$|A_1| = \binom{n}{1} a b^{n-1}$

$\vdots$

$|A_k| = \binom{[n]}{k} a^k b^{n-k}$ — k many elements $\in A$, $n-k$ many elements $\in B$

We Construct the tuplet

$$\left(\{n_1, n_2 \ldots n_k\}, (a_i)_{i \in \{n_1, n_2 \ldots n_k\}} (b_j)_{j \in [n] \setminus \{n_1, n_2 \ldots n_k\}}\right)$$

$$A_k \xrightarrow{f} \left\{ \left(X, (a_i)_{i \in X}, (b_j)_{j \in [n] \setminus X} : X \in \binom{[n]}{k} \right\}\right.$$

$$(x_1, x_2 \ldots x_n) \longmapsto \left(\{n_1 \ldots n_k\}, (a_i)_{i \in \{n_1, n_2 \ldots n_k\}}, (b_j)_{\in [n] \setminus \{n_1, n_2 \ldots n_k\}}\right)$$

* Bijective function

---

* Binomial

Let X be a n-set. and r, t be integers satisfying $0 \leq r+t \leq n$

$$\binom{X}{r,t} = \left\{ (A,B) : A \subset X, \text{ with } |A| = r, B \subset X \text{ with } |B| = t \right\}$$
$$A \cap B = \emptyset$$

Ex: show that $\left| \binom{X}{r,t} \right| = \dfrac{n!}{r!\, t!\, (n-r-t)!} = \dfrac{n!}{r!\,(n-r)!} \cdot \dfrac{(n-r)!}{t!\,(n-r-t)!}$

Th: Multinomial Theorem : Let for each $i \in [k]$, A_i be an a_i-set

& $\forall i, j \in [k]$, with $i \neq j$, $A_i \cap A_j = \emptyset$

Then $(a_1 + a_2 + \ldots + a_k)^n = \sum_{t_1 + t_2 + \ldots t_k = n} \dfrac{n!}{t_1!\, t_2! \ldots t_k!} a_1^{t_1} a_2^{t_2} \ldots a_k^{t_k}$

Here the summation extend over all non-negative integer soln-s. of $t_1, t_2 \ldots t_k$ satisfies $t_1 + t_2 + t_3 + \ldots t_k = n$

A Addition principle of counting

⟶ if a finite non-empty set S is partitioned into k many parts $B_1 \ldots B_k$ i.e $\forall i,j \in [k]$, $B_i \subset S$ and

$$B_i \cap B_j = \begin{cases} \emptyset & \text{if } i \neq j \\ B_i & \text{else } (i=j) \end{cases}$$

$$|S| = |B_1| + |B_2| + \ldots + |B_k|$$

B. Multiplication Principle of Counting

if a finite non-empty Set S can be expressed as

$$S_1 \times S_2 \times \ldots \times S_k$$

then

$$|S| = |S_1| \times |S_2| \times \ldots \times |S_k| = \prod_{i=1}^{k} |S_i|$$

C. One-One and Onto fn^{2} (principle of counting)

Let R, S be two finite non-empty sets if $\exists$ a 1-1 & onto fn^{2} $f: S \to R$ then $|S| = |R|$

D. The double counting principle

Let A & B be two non-empty finite sets & $S = A \times B$

The double counting principle is to count |S| in two ways

First way : We Count : $\sum_{a \in A} |\{b \in B : (a,b) \in S\}|$

Second way : We Count : $\sum_{b \in B} |\{a \in A) : (a,b) \in S\}|$

Now we have an identity

$$\sum_{a \in A} |\{b \in B : (a,b) \in S\}| = |S| = \sum_{b \in B} |\{a \in A : (a,b) \in S\}|$$

Example : (Handshaking Lemma) Suppose there are n many people at a party each of them will shake hands with everyone else

Q: How many handshakes will occur?

Soln: Let $[n] = \{1, 2, 3 \ldots, n\}$ represent the set of one (right) hand of each people

we consider the pair $(x, \{x, y\})$

where $x \in [n]$, and $\{x, y\}$ represent the handshake

Let there be N no. of handshakes.

We count the set $S = \{(x, \{x,y\}) : x \in [n], y \in [n]\}$

First way : There are n many choices for x and chosen x there are $n-1$ many choices for $\{x,y\}$

$$|S| = n(n-1)$$

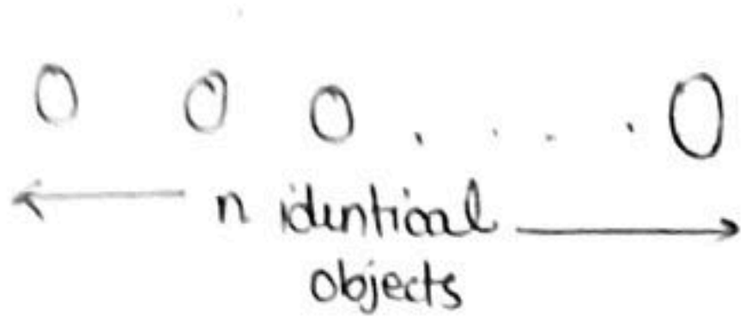
Second way : There are N many choices for $\{x,y\}$ Now chosen $\{x,y\}$, there are 2 choices such pairs namely $(x, \{x,y\})$ and $(y, \{x,y\}) \in S$

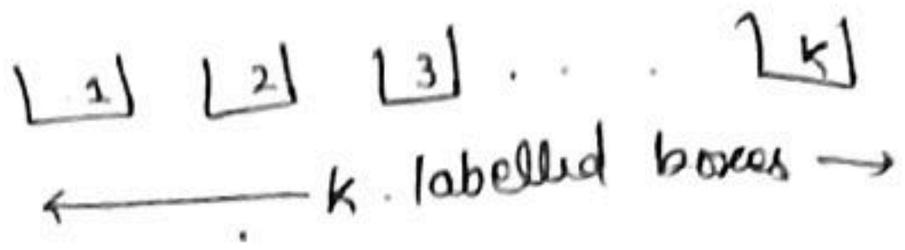
i.e $|S| = 2N$

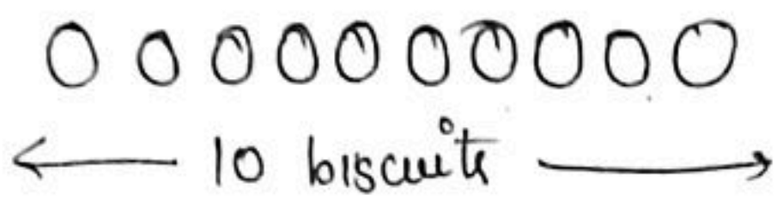
Hence : $2N = |S| = n(n-1)$

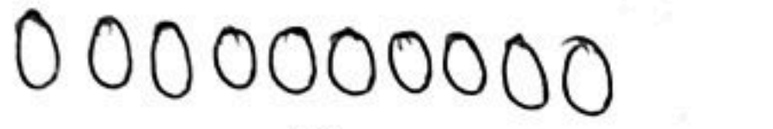
i.e. $N = \frac{n(n-1)}{2}$

Problem: How many way one can distribute n identical object into k many distinguishable / labelled boxes

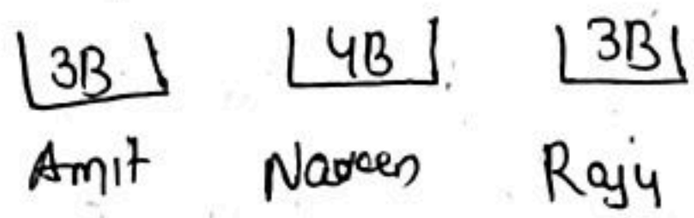


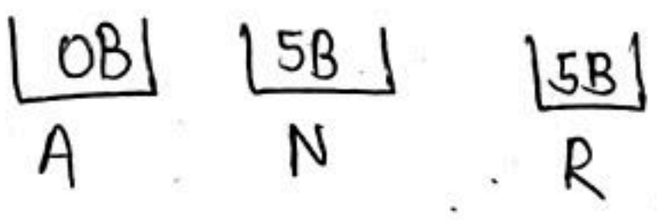


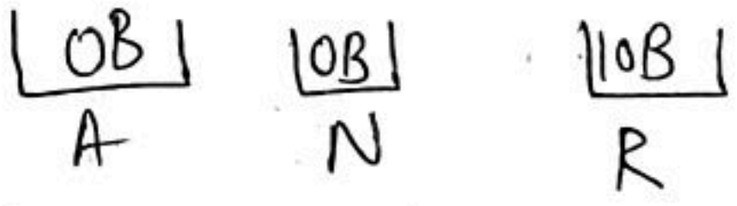
Special cases 

Soln: step-1 

step-2 Place those two seperators among 10 biscuits eg (a)

000 | 0000 | 000 

| 00000 | 00000 

| | 0000000000 

step-3 Consider 12 biscuits choose 2 of them and throw those two Finally replace it by two seperators.

$\binom{12}{2}$ ways

Theorem One can distribute n identical objects into k labelled boxes into $\binom{n+k-1}{k-1}$ many ways.

Proof: Let B be the set of labelled boxes Note $|B| = k$

For each distribution $\mathcal{D}$ of n identical objects into B, we associate a function $\gamma_D : B \longrightarrow \mathbb{Z}$ as

$\gamma_D(b)$ = No. of identical objects received by the box b w.r.t $\mathcal{D}$

Note $\sum_{b \in B} \gamma_D(b) = n$

we note that for each $\mathcal{D} \in \mathfrak{D}$

$$\sum_{b \in B} \gamma_D(b) = n$$

where $\mathfrak{D}$ denote the set of all distribution of n identical objects into k many labelled boxes

$$|\mathfrak{D}| = |\{\gamma_D : \mathcal{D} \in \mathfrak{D}\}|.$$

(Ex: show that $\mathcal{D} \longmapsto \gamma_D$ from $\mathfrak{D}$ to $\{\gamma_D : \mathcal{D} \in \mathfrak{D}\}$ forms an 1-1 and onto fn^{n}

$$\{\gamma_D : \mathcal{D} \in \mathfrak{D}\} = \{f : [k] \to \mathbb{Z} \cap [0,\infty) \mid f(1) + \cdots f(k) = n\}$$

claim: $|\{f : [k] \longrightarrow \mathbb{Z} \cap [0,\infty) \mid f(1) + \ldots\ldots + f(k) = n\}| = \left|\binom{[n+k-1]}{k-1}\right|$

Proof of claim : $\forall\ B \in \binom{[n+k-1]}{k-1}$. i.e $|B| = k-1$ say

$$B = \{i_1, i_2 \ldots i_{k-1}\} \subset [n+k-1]$$

we construct the function $f_B : [k] \longrightarrow \mathbb{Z} \cap [0,\infty)$

WLOG

$i_1 < i_2 < \ldots < i_{k-1}$

$$f_B(1) = |\{i \in [n+k-1] : i < i_1\}|$$

$$f_B(2) = |\{i \in [n+k-1] : i_1 \leq i < i_2\}|$$

$$f_B(3) = |\{i \in [n+k-1] : i_2 < i < i_3\}|$$

$$\vdots$$

$$f_B(k) = |\{i \in [n+k-1] : i_{k-1} < i\}|$$

Construct for a $fn^{\underline{2}}$. $f : [k] \longrightarrow \mathbb{Z} \cap [0,\infty)$

with $f(1) + \ldots + f(k) = n$

we construct the $k-1$ - set $\{i_1, i_2, \ldots i_{k-1}\}$ by setting

$$i_1 = f(1) + 1$$

$$i_2 = f(1) + f(2) + 2$$

$$\vdots$$

$$i_{k-1} = f(1) + \ldots + f(k-1) + k-1$$

Thus, $B \longmapsto f_B$ is an 1-1 & onto welldefined function.

Corollary: One can distribute n identical objects into k labelled boxes such that each box contains atleast one identical object into $\binom{n-1}{k-1}$ many Ways.

Proof: we first distribute one identical object into each boxes it results exactly $(n-k)$ ways identical objects are to be distributed among k labelled boxes

using the Th^m we have $\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$ many ways

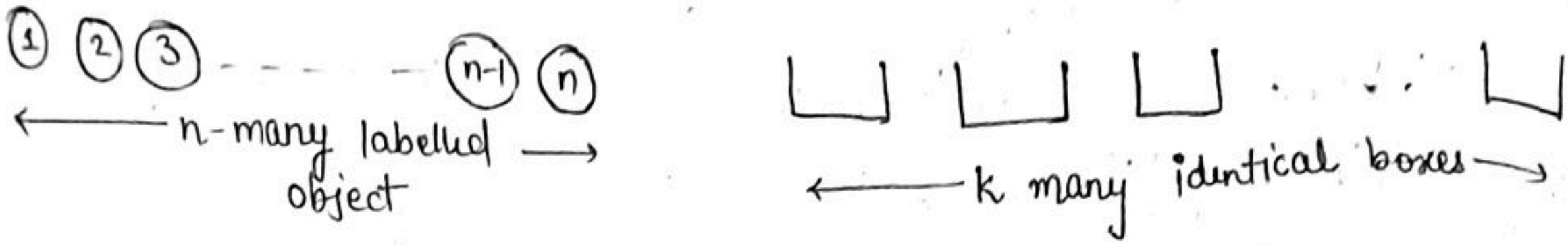
Exercise: Consider the eqn $\sum_{i=1}^{k} x_i = n$, where n and k are positive integer show that

(a) the number of non-negative integer sol^{n}s of the eq^n is $\binom{n-k+1}{k-1}$

(b) the number of positive integer soln of the above eq^n is $\binom{n-1}{k-1}$

Defn: Let n and k be positive integers with $n \geq k$ The Stirling number of 2nd kind denoted as $S_k^{II}(n) = S(n,k)$ is total no of partition of $[n]$ into k-many non-empty sets

The set $[n]$ is choped into k-many parts



Thm $S(n,k) = S(n-1,k-1) + k.S(n-1,k)$

Proof: $\mathcal{S}([n],k)$ = The set of all partion of $[n]$ into k-many non-empty sets

$S(n,k) = |\mathcal{S}([n],k)|$

$\mathcal{A}$ = The set of all partition of $[n]$ into k - non-empty where n belongs is a singleton set
i.e. exactly one part of partition of $[n]$ is $\{n\}$.

$\mathcal{B}$ = The set of all partitions of $[n]$ into k non-empty sets where n belongs is not a sigleton set

claim 1: $|\mathcal{A}| = S(n-1,k-1)$

Proof of claim: Let $X_1, \ldots X_k$ be a partition of $[n]$ into k non-empty parts and exactly one of $X_1, \ldots X_k$ is $\{n\}$

WLOG, Let $X_k = \{n\}$

i.e. $[n] = X_1 \sqcup X_2 \ldots \sqcup X_{k-1} \sqcup X_k$

$= X_1 \sqcup \ldots \sqcup X_{k-1} \sqcup \{n\}$

$\forall i \in [n-1]$, $X_i \neq \emptyset$ & $X_k = \{n\}$

This induces a partition of $[n-1]$ into $(k-1)$- non-empty set conversely if $Y_1, \ldots Y_{k-1}$ is a partition of $[n-1]$ into $(k-1)$ many non-empty parts, then $Y_1, Y_2 \ldots Y_{k-1}, \{n\}$ is a partition of $[n]$ into k many part with exactly one part equals $\{n\}$

$$A \longleftrightarrow \mathcal{S}([n-1], k-1)$$

Hence, $\exists$ a bijection corresponds between A and $\mathcal{S}([n-1], k-1)$

This establish the claim

claim 2 : $|B| = k \cdot S(n-1, k)$

Proof of claim : We note that

$$\{X_1, X_2 \ldots X_k\} \longmapsto \{X_1 \setminus \{n\}, \ldots X_k \setminus \{n\}\}$$

where $X_1 \ldots X_k$ is a partition of $[n]$ into k many non-empty set and where n belongs is not a singleton set

Note: $X_1 \setminus \{n\}, \ldots X_k \setminus \{n\}$ yeilds a partition of $[n-1]$ into k-many non-empty sets

Again each partition of $[n-1]$ into k-many non-empty sets

$$Y_1, Y_2 \ldots Y_k$$

has exactly k many preimages w.r.t the said map namely

$$\{Y_1 \sqcup \{n\}, Y_2 \ldots Y_k\}$$

$$\{Y_1, Y_2 \sqcup \{n\} \ldots Y_k\} \longrightarrow \{Y_1 \setminus \{n\}, Y_2 \setminus \{n\} \ldots Y_k \setminus \{n\}\}$$

$$\vdots$$

$$\{Y_1, Y_2 \ldots Y_k \sqcup \{n\}\}$$

Hence such map is k to 1 & onto map

Hence, $|B| = k\,|S([n-1],k)| = k\,S(n-1,k)$

Th^m: for each integer $n \geq 2$ $S(n,2) = 2^{n-1} - 1$

Proof - 1 : Note that $S(2,2) = 1$

$$S(n,2) = 1 + 2\,S(n-1,2)$$
$$= 1 + 2\,(1 + 2 \cdot S(n-2,2))$$
$$= \vdots$$
$$= 1 + 2 + 2^2 + \ldots + 2^{n-2} S(2,2)$$
$$= 1 + 2 + 2^2 + \ldots + 2^{n-2}$$
$$= 2^{n-1} - 1$$

Proof - 2 $\mathcal{S}([n],2)$ = The set of all of $[n]$ into 2 non-empty sets

$$\mathcal{P} = \{(A,B) : A \sqcup B = [n],\ A \neq \emptyset,\ B \neq \emptyset\}$$

$|\mathcal{P}| \longrightarrow$ Exercise $= 2^n - 2$

if $(A,B) \in \mathcal{P}$, then $(B,A) \in \mathcal{P}$

but (A,B) & (B,A) represents one partition of $[n]$ namely

Hence such mapping from $\mathcal{P}$ to $S([n],2)$ is a 2-to-1 mapping and onto, Hence $2\,|\mathcal{S}([n],2)| = |\mathcal{P}|$

i.e. $2\,S(n,2) = 2^n - 2$

$\Leftrightarrow$ $S(n,2) = 2^{n-1} - 1$

Then : $S(n, n-1) = \binom{n}{2}$

Proof : Ex

Defn The Bell no. $B(n)$, where n is a positive integer, is the to number of partitions of $[n]$ i.e. $B(n)$

$$B(n) = \sum_{i=1}^{n} S(n,i)$$

Thm : One can distribute n labelled objects into k-identical boxes such that each box contains atleast one object, into $S(n,k)$ ways

Proof : Exercise

Thm : One can distribute n-labelled objects into k-identical boxes into $S(n,1) + S(n,2) + \ldots S(n,k)$ many ways

Proof : Exercise

Th: One can distribute n labelled objects into k labelled boxes into k^n ways

Proof: A distribution of "such" yields a fn^{v} $d : [n] \to [k]$

$d(i)$ = label of box where i object is distributed

Conversely if $f : [n] \to [k]$ is a fn^{v}

then $f^{-1}(1), f^{-1}(2) \ldots f^{-1}(k)$ forms a partition of $[n]$ into k many labelled set

---

Th^{m}: One can distribute n labelled objects into k labelled boxes show that each box contains atleast one object into

$k!\, S(n,k)$ many ways

Proof: $O([n],[k]) = \{ f : [n] \to [k] \mid f \text{ is onto} \}$

we construct an equivalence relation $\sim$ over $O([n],[k])$

$f \sim g$, where $f, g \in O([n],[k])$, iff $\exists\, \pi : [k] \xrightarrow[\text{onto}]{1-1} [k]$

s.t $g = \pi \circ f$

Let $[f]$ denote an equivalent class containing $f \in O([n],[k])$

$|[f]| = k!$

$O([n],[k]) = A_1 \sqcup A_2 \sqcup \dots A_N$

$k! = |A_1| = |A_2| = \dots = |A_N|$

claim: There are $S(n,k)$ no. of equivalence classes wrt to $\sim$

$f \sim g : \iff \exists \pi, g = \pi \circ f$

$(f^{-1}(1), f^{-1}(2) \dots f^{-1}(k))$ unordered k-tuple

$(g^{-1}(1), g^{-1}(2) \dots g^{-1}(k))$

$\{f^{-1}(1) \dots f^{-1}(k)\} = \{g^{-1}(1), g^{-1}(2) \dots g^{-1}(k)\}$

WLOG $|f^{-1}(1)| \leq |f^{-1}(2)| \dots \leq |f^{-1}(k)|$

$|g^{-1}(i_1)| \leq |g^{-1}(i_2)| \leq \dots \leq |g^{-1}(i_k)|$

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & k \\ i_1 & i_2 & i_3 & \dots & i_k \end{pmatrix}$$

Proof of claim: We first establish that if $f, g \in O([n],[k])$ s.t. the unordered k-tuple subset of $[n]$ $(f^{-1}(1), f^{-1}(2) \dots f^{-1}(k))$ and $(g^{-1}(1), g^{-1}(2) \dots g^{-1}(k))$ are same i.e.

$[n] = f^{-1}(1) \sqcup f^{-1}(2) \sqcup \dots \sqcup f^{-1}(k)$

the set of $\{f^{-1}(1), f^{-1}(2) \dots f^{-1}(k)\} = \{g^{-1}(1), g^{-1}(2) \dots g^{-1}(k)\}$

Then $\exists$ a permutation on $\Pi : [k] \to [k]$ s.t. $f = \pi \circ g$

To see this WLOG

$|f^{-1}(1)| \leq |f^{-1}(2)| \leq \dots \leq |f^{-1}(k)|$

Suppose $|g^{-1}(i_1)| \leq |g^{-1}(i_2)| \leq \ldots \leq |g^{-1}(i_k)|$

we construct $\pi = \begin{pmatrix} 1 & 2 & 3 & \ldots & k \\ i_1 & i_2 & i_3 & \ldots & i_k \end{pmatrix}$

Here we have,

$$|f^{-1}(1)| = |g^{-1}(i_1)|$$
$$|f^{-1}(2)| = |g^{-1}(i_2)|$$
$$\vdots$$
$$|f^{-1}(k)| = |g^{-1}(i_k)|$$

Note that $\forall x \in [n]$

$x \in f^{-1}(\alpha)$ for some $\alpha \in [k]$

i.e $f(x) = \alpha$

$\pi \circ f(x) = \pi(\alpha) = i_\alpha$

i.e $x \xmapsto{\pi \circ f} i_\alpha$

Also $x \in g^{-1}(i_\alpha)$

i.e $g(x) = i_\alpha$

i.e $x \xrightarrow{g} i_\alpha$

$g = \pi \circ f$

This means two different unordered k-tuples of subset of $[n]$ $(f^{-1}(1), f^{-1}(2) \ldots f^{-1}(k))$ and $(g^{-1}(1) \ldots g^{-1}(k))$ are not same $\Leftrightarrow \nexists\ \pi : [k] \xrightarrow[\text{onto}]{1\text{-}1} [k]$ s.t $g = \pi \circ f \Leftrightarrow f \not\sim g$

Recall that unordered k tuple $(f^{-1}(1) \ldots, f^{-1}(k))$ represents a partition of $[n]$ into k-non-empty sets

Hence the claim is established

So. no. of "such" distribution are $|O([n],[k])| = k!\, S(n,k)$

Defn: Let n, k be positive integer with $n \geq k$ An unordered k-tuple $(a_1, a_2 \ldots a_k)$ is said to be partition of integer n into k-positive integers if $a_1, a_2 \ldots a_k$ are positive integer satisfy

$$a_1 + a_2 + a_3 + \ldots a_k = n$$

The integer $P_k(n) = P(n,k)$ denotes the total no. of way in which the integer n can be partioned into k positive integer

Proof $\mathcal{P}(n,k)$ = The set of partitions of the integer n into k positive integers

$$= \left\{ (a_1, a_2 \ldots a_k) : (a_1 \ldots a_k) \text{ is an unordered } k\text{-tuple of positive integers satisfy } a_1 + a_2 + \ldots + a_k = n \right\}$$

$$\mathcal{A} = \{ (a_1, a_2 \ldots a_k) \in \mathcal{P}(n,k) : a_i = 1 \text{ , for some } i \in [k] \}$$

$$\mathcal{B} = \{ (a_1, a_2 \ldots a_k) \in \mathcal{P}(n,k) : \forall\, i \in [k], a_i \geq 2 \}$$

$$\mathcal{A} \longleftrightarrow \mathcal{P}(n-1, k-1)$$

Let $(a_1, a_2 \ldots a_k) \in \mathcal{A}$, WLOG $a_k = 1$

Note that $a_1 + a_2 + \ldots + a_{k-1} = n-1$

i.e unordered $k-1$ tuple $(a_1, a_2 \ldots a_{k-1})$ is a partition of

integer $n-1$ into $k-1$ positive integers

i.e. $(a_1, a_2 \ldots a_{k-1}) \in \wp(n-1, k-1)$

Conversely, if $(b_1, b_2 \ldots b_{k-1}) \in \wp(n-1, k-1)$ then the unordered k-tuple $(b_1 \ldots b_{k-1}, 1) \in \mathcal{A}$

Hence, There exist bijective correspondence exists b/w $\mathcal{A}$ and $\wp(n-1, k-1)$. Hence,

$$|A| = |\wp(n-1, k-1)| = P(n-1, k-1)$$

We note that,

$$(a_1, a_2 \ldots a_k) \longmapsto (a_1-1, a_2-1 \ldots a_k-1)$$

is an 1-1 & onto map from $\mathcal{B}$ to $\wp(n-k, k)$ (Ex)

$$|\mathcal{B}| = |\wp(n-k, k)| = P(n-k, k) \qquad \square$$

Thm: One can distribut n-identical objects into k identical boxes into $\sum_{i=1}^{k} P(n, i)$ many ways

Statistical Mechanics

Placing n particles into k-different energy levels

THREE different Statistics are obtained by making three different assumptions

- Maxwell - Boltzman ; Here n labelled particles are distributed into k-labelled boxes (energy levels)
- Bose - Einstein : Here n identical particles are distributed into k labelled boxes (energy levels)
- Fermi - Dirac :- Here n- identical Particles are distributed into k labelled energy levels but no box can hold more than one particle

$$\hookrightarrow \binom{k}{n}$$

Date: __/__/____

Subject _____

MON TUE WED THR FRI SAT SUN

* Principle of inclusion & exclusion

Th (IEP - VerII) : Let $\mu : 2^{[n]} \longrightarrow \mathbb{R}$ be a fn and we construct

$$\nu : 2^{[n]} \longrightarrow \mathbb{R}$$

$$\nu(E) = \sum_{S \subseteq E} \mu(S) = \sum_{S \in 2^E} \mu(S)$$

Then, $$\mu(E) = \sum_{S \subseteq E} (-1)^{|E|-|S|} \nu(S)$$

Th (IEP - VerIII) Let $u : 2^{[n]} \longrightarrow \mathbb{R}$ be a fn and

$$\nu : 2^{[n]} \longrightarrow \mathbb{R}$$

$$\nu(E) = \sum_{E \subseteq S} \mu(S)$$

Then $$\mu(E) = \sum_{E \subseteq S} (-1)^{|S|-|E|} \nu(S) \qquad E \subseteq S$$

Proof: $$\sum_{E \subseteq S} (-1)^{|S|-|E|} \nu(S)$$

$$= \sum_{E \subseteq S} (-1)^{|S|-|E|} \left(\sum_{S \subseteq C} \mu(C) \right)$$

$$= \sum_{E \subseteq S} \sum_{S \subseteq C} (-1)^{|S|-|E|} \mu(C)$$

Date: __/__/____

MON TUE WED THR FRI SAT SUN

Subject _____

* Principle of Inclusion & exclusion

Th (PIE - Ver I) : Let $\mu : 2^{[n]} \longrightarrow \mathbb{R}$ be a fn and we construct

$$\nu : 2^{[n]} \longrightarrow \mathbb{R}$$

$$\nu(E) = \sum_{S \subseteq E} \mu(S) = \sum_{S \in 2^E} \mu(S)$$

Then, $$\mu(F) = \sum_{S \subseteq F} (-1)^{|E|-|S|} \nu(S)$$

Th (PIE - Ver II) Let $u : 2^{[n]} \to \mathbb{R}$ be a fn and

$$\nu : 2^{[n]} \to \mathbb{R}$$

$$\nu(E) = \sum_{E \subseteq S} \mu(S)$$

Then $$\mu(E) = \sum_{E \subseteq S} (-1)^{|S|-|E|} \nu(S) \qquad E \subseteq S$$

Proof: $$\sum_{E \subseteq S} (-1)^{|S|-|E|} \nu(S)$$

$$= \sum_{E \subseteq S} (-1)^{|S|-|E|} \left(\sum_{S \subseteq C} \mu(C) \right)$$

$$= \sum_{E \subseteq S} \sum_{S \subseteq C} (-1)^{|S|-|E|} \mu(C)$$

$$(C,S) \longrightarrow \left((-1)^{|E|-|S|}, \mu(C) \right)$$

$$\mu(C) \sum_{S's} (-1)^{|S|-|E|}$$

$$\sum_{C's} \mu(C) \sum_{S's} (-1)^{|E|-|S|}$$

$$\Rightarrow \mu(C) \sum_{S's} (-1)^{|E|-|S|}$$

$$S = E \sqcup Z$$

$$Z \subset C \setminus E$$

$$= \mu(C) \sum_{Z \subset C \setminus E} (-1)^{|E|-|E|-|Z|}$$

$$= \mu(C) \sum_{Z \subset C \setminus E} (-1)^{|Z|}$$

$$= \mu(C) \cdot 0$$

Hence, by DCP, we have

$$\mu(E) = \sum_{E \subset S} (-1)^{|E|-|S|} \nu(S)$$

$$\sum_{S \subset [n]} (-1)^{|S|} = \sum_{i \geq 0}^{n} \binom{n}{i} (-1)^{i}$$

$$= \sum_{i \geq 0}^{n} \binom{n}{i} (-1)^{i} (1)^{n-i}$$

$$= (1+1)^n = 0$$

Ther (IEP - Ver IV) Let A be a finite non-empty set and

$\forall i \in [m]$, P_i denote dinote a property for each $x \in A$, and $i \in [m]$ either x satisfies the property P_i or (exclusive or) x does not satisfy property P_i

let $S \subseteq [m]$

$$N(S) = \{x \in A : x \text{ satisfies property } P_i \ \forall i \in S\}$$

The no. of element of A that satisfy none of the properties $P_1, P_2 \ldots P_m$ is given by

$$\sum_{S \subseteq [m]} (-1)^{|S|} |N(S)|$$

APP-I ONTO FUNCTION COUNTING

let $S(n,k)$ dinote the stirling no. of 2nd kind where $n \& k$ are positive. Integers with $n \geq k$ the $S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{n}{i} (k-i)^n$

Proof $A([n],[k])$ = The set of all fn^2 from $[n]$ to $[k]$

$$|A([n],[k])| = k^n$$

$O([n],[k])$ = The set of all onto $fn^{\underline{x}}$ from $[n]$ to $[k]$

we define the properties $P_1, P_2 \ldots P_k$

$\forall i \in [k]$ P_i dinote the property that i does not belong to image of $f \in A([n],[k])$

Note that the set of all elements of $A([n],[k])$ that satisfy none of the properties $P_1, P_2 \ldots P_k$ is that set of all onto $fn^{\underline{x}}$

for $SC[K]$

$N(S) = \{f \in A([n],[k]) : f \text{ satisfies property } P_i \ \forall i \in S\}$

claim : for $SC[K]$, $|N(S)| = (k-|S|)^n$

Proof of claim : we note the following

$f \in N(S) \Leftrightarrow f$ satisfies Prop. $P_i \ \forall i \in S$

$\Leftrightarrow i \notin \text{Im } f \quad \forall i \in S$

$\Leftrightarrow S$ not in the $\text{Im } f$

$\Leftrightarrow S$ is disjoint from $\text{im } f$

$\Leftrightarrow f$ is a fn^x from $[n]$ to $[k] \setminus S$

Hence, $|N(S)| = (k-|S|)^n$ Thus establishes the claim

Using the IEP-VerIII $\Rightarrow$

$$|O([n],[k])| = \sum_{SC[k]} (-1)^{|S|} |(N(S)|| = \sum_{SC[k]} (-1)^{|S|} (k-|S|)^n$$

$$= \sum_{i=0}^{k} (-1)^i \binom{n}{i} (k-i)^n$$

Then, we know, $|O([n],[k])| = k!\, S(n,k)$

$$S(n,k) = \frac{1}{k!} \sum_{i=0}^{k} (-1)^i \binom{n}{i} (k-i)^n$$

Proof : Let $x \in A$, we Construct $S_x \{ i \in [m] : x$ satisfies prop

$$N(S) = \{x \in A : S \subseteq S_x\}$$

$$= \sum_{S \subseteq [m]} (-1)^{|S|} |N(S)|$$

$$= \sum_{S \subseteq [m]} (-1)^{|S|} \left| \{x \in A : S \subseteq S_x\} \right|$$

$$= \sum_{S \subseteq [m]} (-1)^{|S|} \left(\sum_{\substack{x \in A \\ S \subseteq S_x}} 1 \right)$$

$$= \sum_{S \subseteq [m]} \sum_{\substack{x \in A \\ S \subseteq S_x}} (-1)^{|S|} = \sum_{S \subseteq S_x \subseteq [m]} \sum_{x \in A} (-1)^{|S|}$$

$$= \sum_{x \in A} \sum_{S \subseteq S_x} (-1)^{|S|}$$

$$= \sum_{x \in A} \sum_{S \subseteq S_x = \phi} (-1)^{|S|} + \sum_{x \in A} \sum_{S \subseteq S_x \neq \phi} (-1)^{|S|}$$

$$= \sum_{x \in A} \sum_{S_x = \phi} 1 + \sum_{x \in A} (-1+1)^{|S_x|}$$

$$= \sum_{x \in A} \sum_{S_x = \phi} 1$$

$$= \left| \{x \in A : S_x = \phi\} \right|$$

$$= |\{x \in A : x \text{ does not satisfy property } P_1, P_2 \ldots P_m\}|$$

App-II : Derangement Permutation counting

Def" : A permutation $\pi : [n] \xrightarrow[\text{onto}]{1-1} [n]$ is called derangement if $\forall i \in [n]$, $\pi(i) \neq i$

The set of all derangements on $[n]$ is denoted by D_n

claim $S(v_j) = T(v_j) \quad \forall\, j \in \{1, 2, \ldots n\}$

APP-III Establishment of EULER PHI Function

Th: if $n = p_1^{n_1} p_2^{n_2} \ldots p_n^{n_k}$ where $\forall i \in [k] \;.\; n_i \geq 1$

$n > 1$ are integer and p_i are positive prime integers, then

$$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \ldots \left(1 - \frac{1}{p_k}\right)$$

Here $\phi(n)$ denotes Euler's phi function defined by

$$\phi(n) = \left|\{x \in [n] : \gcd(x, n) = 1\}\right|$$

Proof: For each $i \in [k]$, let P_i denote the property that x is divisible by the positive prime integer p_i

claim: The set of elements of $[n]$ that satisfies the none of the properties of $P_1, P_2 \ldots P_i$

then

$$R = \{x \in [n] : \gcd(x, n) = 1\}$$

Proof of claim: Note that if x does not satisfy the P_i for some $i \in [n]$ then $\gcd(x, p_i) = 1$

$\Rightarrow \quad \exists\, u, v \in \mathbb{Z} \quad$ s.t $\quad xu + p_i v = 1$

$\Rightarrow \quad (xu + p_i v)^{n_i} = 1$

$\Leftrightarrow \quad x(\;) + p^{n_i}(\;) = 1$

$\Rightarrow \quad \gcd(x, p_i^{n_i}) = 1$

Therefore x does not satisfies property $\not\in P_i$ then $\gcd(x, p_i^{n_i}) = 1$

So, if x does not satisfy $P_1, P_2 \ldots P_k$ then $\exists\, u_i, v_i \in \mathbb{Z}$

$\forall i \in [k]$ s.t. $(xu_1 + p_i^{m_1} v)(xu_2 + p_2^{n_2} v_2) + \ldots + (xu_k + p_k^{n_k} v_k) = 1$

$\Leftrightarrow \quad xU + nV = 1 \quad$ for some $U, V \in \mathbb{Z}$

claim $S(v_j) = \{i : \gcd(i,n) = 1\}$ $\forall\, j \in \{1,2,\ldots n\}$

$\Leftrightarrow \gcd(x,n) = 1$

$\Leftrightarrow x \in R$ (this establishes the claim)

claim: $SC[k]$

$$|N(s)| = \frac{n}{\prod_{i \in s} p_i}$$

Proof of claim: $x \in N(s) \Leftrightarrow$ x satisfies property P_i $\forall\, i \in S$

$\Leftrightarrow p_i \mid x \quad \forall\, i \in S$

$\Leftrightarrow \prod_{i \in s} p_i \mid x$

Hence, $N(s) = \left\{ x \in [n],\ x = r\left(\prod_{i \in s} p_i\right),\ r = 1, 2, \ldots \frac{n}{\prod_{i \in s} p_i} \right\}$

Using IEP - Verify and the claim we have

$$\phi(n) = |R| = \sum_{SC[k]} (-1)^{|S|} |N(S)|$$

$$= \sum_{SC[k]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i} = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

$$\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right) = 1 + (-1)\left(\frac{1}{p} + \frac{1}{q}\right) + \frac{1}{pq}$$

Th: For any integer $n \geq 1$

$$n = \sum_{d|n} \phi(d)$$

Th: For each positive integer n

$$n = \sum_{d|n} \phi(d)$$

$$\sum_{d|n} \phi(d) \sum_{d|n} \phi\left(\frac{n}{d}\right)$$

Proof: Let $gcd(x,n) = d$

Then $\exists\ u, v \in \mathbb{Z}$ s.t. $xu + nv = d$

$$= \frac{x}{d}u + \frac{n}{d}v = 1$$

so $x \longmapsto \frac{x}{d}$ forms a bijective

mapping between set $\{x \in [n] : gcd(x,n) = d\}$

and $\left\{ y \in \left[\frac{n}{d}\right] : gcd\left(y, \frac{n}{d}\right) = 1 \right\}$

Hence,

$$\left| \{x \in [n] \ gcd(x,n) = d\} \right| = \phi\left(\frac{n}{d}\right)$$

Now, $x \sim y$, where $x, y \in [n]$ for a equivalence $\pmod n$

iff $gcd(x,n) = gcd(y,n)$

Hence $n = |[n]| = \left| \bigsqcup_{d=1}^{n} \{x \in [n] : gcd\{x,n) = d\} \right|$

$$\sum_{d=1}^{n} \left| \{x \in [n] : gcd(x,n) = d\} \right|$$

$$n = \sum_{d/n} \phi\left(\frac{n}{d}\right) = \sum_{d/n} \left| \{x \in [n] : gcd(x,n) = d\} \right|$$

$d \nmid n \quad \{x \in [n] : gcd(x,n) = d\} = \phi$

MOBIUS FUNCTION

Def^n: The Mobius fn^x $\mu : \{n \in \mathbb{Z} : n \geq 1\} \longrightarrow \{-1, 0, 1\}$ is defined by

$$\mu(1) = 1$$

If $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$ then

$$\mu(n) = \begin{cases} (-1)^k & \text{if} \quad n_1 = 1 = n_2 = \cdots n_k \\ 0 & \text{if} \quad \exists\, i \in [k] \quad n_i \geq 2 \quad \text{(else)} \end{cases}$$

Th: for each integer n

$$\sum_{d/n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{else } n \neq 1 \end{cases}$$

Proof: if $n = 1$ then only divisor of $n = 1$ is $d = 1$ Hence

$$\sum_{d/n} \mu(d) = 1 = \mu(1) = 1$$

So suppose $n \geq 2$ & $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$

Then $$\sum_{d/n} \mu(d) = \sum_{d/p_1 p_2 \cdots p_k} \mu(d) = \sum_{S \subseteq [k]} (-1)^{|S|} = 0$$

Corollary: For each integer $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$$

Proof: $$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

$$= n \sum_{S \subseteq [k]} \frac{(-1)^{|S|}}{\prod_{i \in S} p_i} = n \sum_{d | p_1 p_2 p_3 \cdots p_k} \frac{\mu(d)}{d}$$

Deff: A real valued sequence $\{x_n\}_{n=0}^{\infty}$ is said to satisfy k- ~~term~~ recurrence relation or simply recurrence relation

if $\exists\ \lambda \in \mathbb{R}^k \to \mathbb{R}$ s.t $x_0, x_1 \cdots x_{k-1}$ is known (referred as initial Cond^n) and $\forall$ integer $n \geq 0$

$$x_{n+k} = \lambda(x_n, x_{n+1} \cdots x_{n+k-1})$$

$x_k = \lambda(x_0, x_1 \cdots x_{k-1}\}$

$x_{k+1} = \lambda(x_1, x_2 \cdots x_k\}$

$x_{k+2} = \lambda(x_2, x_3 \cdots x_{k+1}\}$

We say the fn^x λ as k-term recurrence fn^x is simply recurrence fn^x

If λ is Linear i.e.

$$\lambda(\bar{x} + \alpha\bar{y}) = \lambda(\bar{x}) + \alpha\lambda(\bar{y})$$

$\bar{x} = (x_1, x_2 \ldots x_k) \in \mathbb{R}^k$

$\bar{y} \in \mathbb{R}^k$

$\alpha \in \mathbb{R}$

} & λ is a k-term recurrence relation

thn such λ is called k-term Linear recurrence relation

if λ is homogeneous of degree n

$$\bar{x} = \{x_1, x_2 \ldots x_k)$$

$$\alpha\bar{x} = \{\alpha x_1, \alpha x_2 \ldots \alpha x_k\}$$

$$\lambda(\alpha\bar{x}) = \lambda(\alpha x_1, \alpha x_2 \ldots \alpha x_k\} = \alpha^n \lambda(x_1, x_2 \ldots x_k)$$

$$= \alpha^n \lambda(\bar{x})$$

$$\lambda(\alpha\bar{x}) = \alpha^n \lambda(\bar{x})$$

& λ is a k-term recurrence relation
thn such λ is called k-term homogeneous recurrence relation

Example : $\lambda : \mathbb{R}^2 \to \mathbb{R}$

$\lambda(x,y) = x+y$ ← Linear

$\lambda(x,y) = (x+y)^2$ ← homogeneous of dgee 2

Problem: Solve the Linear recurrence relation

$x_0 = 0 \quad x_1 = 1 \quad$ and $x_{n+2} = x_{n+1} + x_n \quad \forall$ integer $n \geq 0 \} \rightarrow *$

Sol^n We first note that

$$(R^2 - R - I)\left(\{x_n\}_{n=0}^{\infty}\right) = \{0_n (=0)\}_{n=0}^{\infty}$$

step-1 : Identify the characteristic polynomial with the recurrence relation Here it is $x^2 - x - 1$

step 2: find the roots of characteristic polynomial

Here $x^2 - x - 1 = 0$

$$\Leftrightarrow (x - \tau)\left(x + \frac{1}{\tau}\right) = 0 \qquad \tau = \frac{1+\sqrt{5}}{2}$$

Roots are τ and $-\frac{1}{\tau}$.

step 3 : formation of general soln

Here it is $x_n = c_1 \tau^n + c_2 \left(-\frac{1}{\tau}\right)^n$

where $c_1, c_2 \in \mathbb{C}$

step 4 : Apply initial cond^n

Here $x_0 = 0 \quad x_1 = 1$

$$0 = c_1 + c_2$$

$$1 = c_1 \tau + c_2 \left(-\frac{1}{\tau}\right)$$

Final : Unique solution of * is : $x_n = \left(-\frac{1}{\sqrt{5}}\right)\tau^n + \left(\frac{1}{\sqrt{5}}\right)\left(-\frac{1}{\tau}\right)^n$

Problem : Solve the linear recurrence relation

$$x_0 = 0, x_1 = 1, x_2 = 3$$

$$x_{n+3} = 5x_{n+2} - 8x_{n+1} + 4x_n \quad \forall \text{ integer } n \geq 0 \} \text{ (*)}$$

Soln : $(R^3 - 5R^2 + 8R - 4I)(\{x_n\}_{n=0}^{\infty}) = \{0_n (=0)\}_{n=0}^{\infty}$

step1 - identify the characteristic polynomial associated with the recurrence relation

$$x^3 - 5x^2 + 3x - 4$$

step-2 roots of the eqn

$$x^3 - 5x^2 + 8x - 4 = 0$$
$$(x-2)^2(x-1) = 0$$

Roots 2, 2, 1

step-3 formation of general soln

$$x_n = (C_0 + C_1 n)2^n + (C_3)1^n$$

step 4 Apply initial cond^n of *

$$x_0, x_1 = 1 \quad x_2 = 3$$

Rough

$$C_0 + C_3 = 0$$

$$2(C_0 + C_1) + C_3 = 1$$

$$(C_0 + 2C_1)4 + C_3 = 3$$

$$4C_0 + 8C_1 + 1 - 2C_0 - 2C_1 = 3$$

$$6C_1 + 2C_0 = 2$$

Remark Let $\alpha_1, \alpha_2 \ldots \alpha_k$ are roots with mutiplities $\gamma_1 \ldots \gamma_k$ resp

$$(R-\alpha_1 I)^{\gamma_1} (R-\alpha_2 I)^{\gamma_2} \ldots (R-\alpha_k I)^{\gamma_k} \{x_n\}_0^\infty = \{0_k (=0)\}_{n=0}^\infty$$

general solution = ( ) α_1^n + ( ) α_2^n + ...

( ) : Polynomial of n with degree $\gamma_1 - 1$

( ) : Polynomial of n with degree $\gamma_2 - 1$

$S(\mathbb{C})$ = The set of all Complex valued soln

$$\left\{ \{x_n\}_{n=0}^\infty \mid \forall \text{ integer } n \geq 0 , x_n \in \mathbb{C} \right\}$$

Claim : $S(\mathbb{C})$ form a vector space over $\mathbb{C}$

$\alpha \in \mathbb{C}$

$$\alpha \{x_n\}_{n=0}^\infty = \{\alpha x_n\}_{n=0}^\infty$$

Defⁿ : A linear operator $T : S(\mathbb{C}) \longrightarrow S(\mathbb{C})$ is a map satisfies

(a) $T\left(\{x_n\}_{n=0}^\infty + \{y_n\}_{n=0}^\infty\right) = T\left(\{x_n\}_{n=0}^\infty\right) + T\left(\{y_n\}_{n=0}^\infty\right)$

(b) $T\left(\alpha \{x_n\}_{n=0}^\infty\right) = \alpha T\left(\{x_n\}_{n=0}^\infty\right)$

Example : $\{x_n\}_{n=0}^\infty \longmapsto \{x_{1+n}\}_{n=0}^\infty$

$\{x_0, x_1, x_2 \ldots\} \longrightarrow \{x_1, x_2 \ldots x_n\}$

claim : $R : S(\mathbb{C}) \longrightarrow S(\mathbb{C})$

is a linear operator

$$\{x_0, x_1, x_2 \ldots\} \xrightarrow{R} \{x_1, x_2 \ldots\} \xrightarrow{R} \{x_2, x_3, x_4, \ldots\}$$

$R^2 = ROR$

$$R^2\left(\{x_n\}_{n=0}^{\infty}\right) = \{x_{2+n}\}_{n=0}^{\infty}$$

claim: R^2 is also a linear operator

claim : R^k

$$(R^3 - 5R^2 + 8R - 4I)\left(\{x_n\}_0^{\infty}\right)$$

$$= \{x_{3+n}\}_{n=0}^{\infty} - 5\{x_{2+n}\}_{n=0}^{\infty} + 8\{x_{1+n}\}_{n=0}^{\infty} - 4\{x_n\}_{n=0}^{\infty} = \{0_n\}_{n=1}^{\infty}$$

Def^n The Set

$$\left\{\{x_n\}_{n=0}^{\infty} \in S(\mathbb{C}) \mid T\left(\{x_n\}_{n=0}^{\infty}\right) = \{0_n\}_{n=0}^{\infty}\right\}$$

is called kernel of T denoted as $\ker T$

Here, $S(\mathbb{C})$ is an example of infinite dim. vector space

* $R(\{\alpha^n\}_{n=0}^{\infty}) = \alpha\{\alpha^n\}_{n=0}^{\infty}$

$R\left(<\{\alpha^n\}_{n=0}^{\infty}>\right) = <\{\alpha^n\}_{n=0}^{\infty}>$

Def^n : Let H be an vector space over $\mathbb{C}$ and $T: H \to H$ be a linear operator. A non-zero vector $x \in H$ is called eigen (I-gen) vector if $\exists \alpha \in \mathbb{C}$ s.t.

$$T(x) = \alpha x$$

such α is called eigen value.

Note : $T(<x>) = <x>$

Note : $R^k\left(\{\alpha^n\}_{n=0}^{\infty}\right) = \alpha^k\left(\{\alpha^n\}_{n=0}^{\infty}\right)$

Thus $\forall \alpha \in \mathbb{C}$, and $\forall$ integer $k \geq 0$ $\{\alpha^n\}_{n=0}^{\infty}$ is the eigen vector of ~~$\{\alpha^n\}_{n=0}^{\infty}$~~ R^k with eigen value α^k

Lemma : Let H be a vector space over $\mathbb{C}$ and $T: H \to H$ be a Linear operator Then $\ker T = \{x \in H : T(x) = 0\}$ for a vector subspace over $\mathbb{C}$

Proof Let $x, y \in \ker T$ and $\alpha \in \mathbb{C}$

Thu $T(x+\alpha y) = T(x) + T(\alpha y) = T(x) + \alpha T(y) = 0 + 0 = 0$.

i.e. $x + \alpha y \in \ker T$

Hence the result $\square$

Cor: Let $p(x) \in \mathbb{C}[x]$ be a polynomial

$R : S(\mathbb{C}) \longrightarrow S(\mathbb{C})$ be a right Shift linear operator

$$R\left(\{x_n\}_{n=0}^{\infty}\right) = \{x_{1+n}\}_{n=0}^{\infty}$$

Then: $p(R) : S(\mathbb{C}) \longrightarrow S(\mathbb{C})$ form a Linear operator & $\ker p(R)$ forms a vector space over $\mathbb{C}$

Proof: Exercise

Hint: $T_1 : H \longrightarrow H$

$T_2 : H \longrightarrow H$

then $T_1 + T_2$ is a Linear operator

* Exercise $\operatorname{Im} T = \{T(x) : x \in H\}$ is vector subspace of H over $\mathbb{C}$

Lemma: Let $x_{n+k} = \sum_{i=1}^{k} c_i x_{n+k-i}$

$$= c_1 x_{n+k-1} + c_2 x_{n+k-2} + \ldots + c_k x_n$$

Where $\forall\, i \in [k]$, $c_i \in \mathbb{C}$, be a k-term Linear homogenous recurence relation with initial Condition $x_0, x_1 \ldots x_{k-1}$

Then $p(R)\left(\{x_n\}_{n=0}^{\infty}\right) = \{0_n (=0)\}_{n=0}^{\infty}$ $\qquad R^0 = I$

Where $p(R) = R^k - \sum_{i=1}^{k} c_i R^{k-i}$

Moreover $\ker p(R)$ is a k-dimensional vector subspace of $S(\mathbb{C})$ over $\mathbb{C}$

$$p(R)\left(\{x_n\}_{n=0}^{\infty}\right) = \{0_n (=0)\}_{n=0}^{\infty}$$

Proof:

$$\left(R^k - \sum_{i=1}^{k} c_i R^{k-i}\right)\left(\{x_n\}_{n=0}^{\infty}\right) = \{0_n (=0)\}_{n=0}^{\infty}$$

We construct the map

$$\Lambda : \operatorname{Ker} p(R) \longrightarrow \mathbb{C}^k$$

$$\Lambda\left(\{x_n\}_{n=0}^{\infty}\right) = (x_0, x_1 \cdots x_{k-1})$$

For each $\alpha \in \mathbb{C}$ and $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \in \operatorname{Ker} p(R)$

$$\Lambda\left(\{x_n\}_{n=0}^{\infty} + c\{y_n\}_{n=0}^{\infty}\right)$$

$$\Lambda\left(\{x_n + cy_n\}_{n=0}^{\infty}\right) = (x_0 + cy_0 + x_1 + cy_1 + \cdots x_{k-1} + cy_{k-1})$$

$$\Lambda\left(\{x_n\}_{n=0}^{\infty}\right) + c\Lambda\left(\{y_n\}_{n=0}^{\infty}\right) = (x_0 + x_1 \cdots x_{k-1}) + c(y_0, y_1 \cdots y_{k-1})$$

$$= (x_0 + cy_0, x_1 + cy_1 + \cdots x_{k-1} + cy_{k-1})$$

i.e. $$\Lambda\left(\{x_n\}_{n=0}^{\infty} + c\{y_n\}_{n=0}^{\infty}\right) = \Lambda\left(\{x_n\}_{n=0}^{\infty}\right) + c\Lambda\left(\{y_n\}_{n=0}^{\infty}\right)$$

Thus Λ is a linear map

if $$\Lambda\left(\{x_n\}_{n=0}^{\infty}\right) = \Lambda\left(\{y_n\}_{n=0}^{\infty}\right)$$

$$(x_0, x_1 \cdots x_{k-1}) = (y_0, y_1 \cdots y_{k-1})$$

$$\Leftrightarrow x_0 = y_0, y_1 = x_1 \cdots y_{k-1} = x_{k-1}$$

Λ is 1-1

Let $(a_0, a_1, \ldots a_{k-1}) \in \mathbb{C}^k$

$$a_{n+k} = \sum_{i=1}^{k} c_i \cdot a_{n+k-i}$$

Then $\{a_n\}_{n=0}^{\infty} \in \ker p(R)$

$$\wedge(\{a_n\}_{n=0}^{\infty}) = (a_0, a_1, \ldots a_{k-1})$$

$\wedge$ is onto

Thus $\wedge$ establishes the vector space isomorphism between $\ker p(R)$ and $\mathbb{C}^k$

Since $\mathbb{C}^k$ is a k-dimensional vector space over $\mathbb{C}$

We have $\ker p(R)$ is k-dimenstional V.S over $\mathbb{C}$