$[n] = \{1, 2, 3 \ldots n\}$

Definition : An one-to-one and onto f^n :-

$\pi : [n] \to [n]$ is called permutation f^n a simply Permutation

S_n : The set of all permutations from $[n] \to [n]$

$$= \left\{ \pi \;\middle|\; \pi : [n] \xrightarrow[\text{onto}]{1-1} [n] \right\}$$

symmetric set of n symbols

Let an urn contains. N many ~~ball~~ labelled balls.

Task : choosing n. many labelled ball from the urn (Ω)

Impossible if $n > N$ & possible if $1 \leq n \leq N$

2 Styles of choices -

A Choose one ball at a time and repeat it for n-times

B Choose n many balls in one grip

$$\{ E \subset \Omega : |E| = n \}$$

1. Style of Choosing

Additional : observe the label of the ball at each step.

$(\omega_1, \omega_2, \omega_3 \ldots \omega_n)$

Addition : You observe the label an put back in Ω

$$\{(\omega_1, \omega_2 \ldots \omega_n) : \omega_k \in \Omega \quad (k = 1, 2, 3 \ldots)\}$$

$$= \Omega^n$$

Alternatively : Observe the label and do not put it back to the Urn

$$\{(\omega_1, \omega_2, \omega_3 \ldots \omega_n) : \omega_1 \in \Omega, \omega_2 \in \Omega / \{\omega_1\}, \omega_3 \in \Omega / \{\omega_1, \omega_2 \ldots$$

$$\left\{(\omega_1, \omega_2 \ldots \omega_n) : \begin{array}{l} \forall i \in [n], \omega_i \in \Omega \\ \forall i, j \in [n], \text{with } i \neq j \ \omega_i \neq \omega_j \end{array}\right\}$$

BLANK PAGES means Class missed
~ Himanshu

RE: Urn Ω Contain N many labeled balls and we are drawing n many balls from it. Here $n \leq N$ & N is a positive integer. $k \geq 0$ is an integer.

Choice Style A3: We choose one ball and do not observe the label of the ball but observe all the n many balls at the end; and do not return it to Ω

$$A = \left\{ (\omega_1, \omega_2 \ldots \omega_n) : \forall i \in [n],\ \omega_i \in \Omega \ \ \forall i,j \in [n] \text{ with } i \neq j \ \ \omega_i \neq \omega_j \right\}$$

$$(\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n) \sim (\beta_1, \beta_2 \ldots \beta_n)$$

$$\Updownarrow$$

$$\exists\ \pi : [n] \to [n]$$

s.t. $\alpha_1 = \beta_{\pi(1)}$
$\alpha_2 = \beta_{\pi(2)}$
$\vdots$
$\alpha_n = \beta_{\pi(n)}$

Equivalence relation $\sim$ over A

$$(\alpha_1 \ldots \alpha_n) \sim (\beta_1 \ldots \beta_n) \iff \exists\ \pi : [n] \to [n] \text{ s.t. } \alpha_i = \beta_{\pi(i)} \ \forall i \in [n]$$

Example: Show that $\sim$ is an equivalence relation on A

Let $A = A_1 \sqcup A_2 \sqcup A_3 \dots \sqcup A_L$

A_i forms an equivalent class wrt $\sim$ $\forall\ i \in [n]$

Task : $L = ?$

$A/\sim$ = The set formed by choosing <u>exactly</u> one element from each equivalent class A_i

= Residue or Quotient set of A wrt $\sim$

<u>Example</u> : equivalence relation on $\mathbb{Z}$

$$\alpha \sim \beta \Leftrightarrow 5 \mid \alpha - \beta$$

Lemma : $\left| A/\sim \right| = \left| \binom{\Omega}{n} \right| = \binom{N}{n}$

Proof : Let $A = A_1 \sqcup A_2 \sqcup A_3 \dots \sqcup A_L$

$$A/\sim = \{ \omega_1(1), \omega_2(1) \dots \omega_n(1), (\omega_1(2), \omega_2(2) \dots \omega_n(2)) \dots (\omega_1(L), \omega_2(L), \dots \omega_n(L)) \}$$

$$= ((\omega_1(i), \omega_2(i) \dots \omega_n(i) : i \in [L])$$

We Construct a mapping from $A/\sim$ to $\binom{\Omega}{n}$

$$(\omega_1(i), \dots \omega_n(i)) \xrightarrow{f} \{\omega_1(i) \dots \omega_n(i)\}$$

Claim $f : A/\sim \longrightarrow \binom{\Omega}{n}$ is 1-1 and onto

Proof of claim : Let $E \in \binom{\Omega}{n}$ $E = \{\alpha_1, \alpha_2 \dots \alpha_n\}$.

we choose $i \in [L]$ s.t $(\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n) \in A_i$

choose $(\omega_1(i), \ldots \omega_n(i))' \in A/\sim$

Note that $f(\omega_1(i), \ldots \omega_n(i)) = \{\omega_1(i), \ldots \omega_n(i)\}$

$$= \{\alpha_1, \alpha_2 \ldots \alpha_n\}$$

$$= E$$

Thus f is onto

Let $i, j \in [L]$ with $i \neq j$ then

$$(\omega_1(i), \omega_2(i) \ldots \omega_n(i)) \neq (\omega_1(j), \omega_2(j) \ldots \omega_n(j))$$

$\exists\, k \in [n]$ s.t. $\forall\, \pi : [n] \xrightarrow[\text{onto}]{1-1} [n]$

$$\omega_k(i) = \omega_{\pi(k)}(j)$$

Hence, $\{\omega_1(i), \omega_2(i) \ldots \omega_n(i)\} \neq \{\omega_1(j), \ldots \omega_n(j)\}$

i.e. $f(\omega_1(i) \ldots \omega_n(i)) \neq f(\omega_1(j) \ldots \omega_n(j))$

Thus f is 1-1

Hence $L = |A/\sim| = \left|\binom{\Omega}{n}\right| = \binom{N}{n}$

Th (Binomial Theorem) Let A be an a-set and B be an b-set i.e. $|A| = a$ and $|B| = b$ with $A \cap B = \emptyset$

Then $$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

Let X be an n-set i.e. $|X| = n$

$\mathcal{P}(X)$ = The set of all subsets of X = $\{E : E \subset X\}$

$2^X = \{ f \mid f : X \to \{0,1\} \}$

<u>Lemma</u>: $\exists$ 1-1 and onto fn$^{\underline{c}}$ from 2^X to $\mathcal{P}(X)$

Proof: Let $S = \{ (\alpha_1, \dots \alpha_n) : \forall\, i \in [n], \alpha_i \in \{0,1\} \}$

$\Rightarrow |S| = 2^n$

Let $X = \{x_1, x_2 \dots x_n\}$

We construct a map from 2^X to S

$$f \longmapsto (f(x_1), f(x_2) \dots f(x_n))$$

Let $(\alpha_1, \alpha_2 \dots \alpha_n) \in S$ we construct the fn$^{\underline{c}}$

$f : X \to \{0,1\}$

$f(x_1) = \alpha_1$

$\vdots$

$f(x_n) = \alpha_n$

Hence map is onto

Let $f, g \in 2^X$ with $f \neq g$

$\exists\, x \in X$ s.t $f(x) \neq g(x)$

$\exists\, i \in [n]$ s.t $x = x_i$, $f(x_i) \neq g(x_i)$

Note:

$$f \longmapsto (f(x_1), f(x_2) \ldots f(x_i) \ldots f(x_n))$$
$$g \longmapsto (g(x_1), g(x_2) \ldots g(x_i) \ldots g(x_n))$$

Then, $(f(x_1), f(x_2) \ldots f(x_n)) \neq ((g(x_1)) \ldots g(x_n))$

i.e. such map is 1-1

We Construct an 1-1 & onto fn^x from S to $p(x)$

$$(\alpha_1, \alpha_2 \ldots \alpha_n) \longmapsto \{x_i : \alpha_i = 0\}$$

(Show that such map is 1-1 & onto)

$$2^X \longrightarrow \mathcal{P}(X)$$

An equivalence reln on $\mathcal{P}(X)$

$$x \sim y \Leftrightarrow |x| = |y|$$

$$\mathcal{P}(x) = A_0 \sqcup A_1 \sqcup A_2 \cdots \sqcup A_n$$

$$A_k = \{E \subset X : |E| = k\} = \binom{X}{k}$$

$$2^n = |\mathcal{P}(x)| = |A_0| + |A_1| + \cdots |A_n|$$

$$= \left|\binom{X}{0}\right| + \left|\binom{X}{1}\right| + \cdots \left|\binom{X}{n}\right|$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n}$$

Lemma: $\exists$ 1-1 & onto map between $\binom{X}{k}$ and $\binom{X}{n-k}$

Proof: Required map is $A \mapsto X \setminus A$

Lemma: $\exists$ 1-1 & onto map b/w

$$\binom{X \sqcup Y}{n} \text{ and } \bigsqcup_{k=0}^{n} \binom{X}{k} \times \binom{Y}{n-k}$$

where $X \cap Y = \emptyset$ & X and Y are finite sets

$$|X| + |Y| \geq n$$

Plug in $|X| = |Y| = n$

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$$

Lemma Pascal Identity For each positive integer n

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \quad \text{where } 1 \leq k \leq n$$

Proof : Let $A = \left\{ X \in \binom{[n+1]}{k} : n+1 \in X \right\}$

$$B = \left\{ X \in \binom{[n+1]}{k} : n+1 \notin X \right\}$$

Then $|A| + |B| = \left| \binom{[n+1]}{k} \right|$

Note that $\forall X \in B$, $X \subset [n]$, i.e. $B \subset \binom{[n]}{k}$

For each $y \in \binom{[n]}{k}$, then $y \in B$

Hence $B = \binom{[n]}{k} \Rightarrow |B| = \binom{n}{k}$

Again $X \longmapsto X \setminus \{n+1\}$ is an 1-1 & onto map from A to $\binom{[n]}{k-1}$

For $Y \in \binom{[n]}{k-1}$, note that $Y \sqcup \{n+1\} \in A$

i.e. such map is onto

For $X \neq Y$, where $X, Y \in A$, $\exists \alpha \in [n]$ s.t. $\alpha \in X$ but $\alpha \notin Y$,

Hence $X \setminus \{n+1\} \neq Y \setminus \{n+1\}$ i.e such

map is 1-1

$\Rightarrow |A| = \left| \binom{[n]}{k-1} \right|$

$\Rightarrow |A| + |B| = \left| \binom{[n+1]}{k} \right| \Rightarrow \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$

Theorem : Given two positive integers N and i then there is a unique way to expand N as a sum of binomial coefficients as follows

$$N = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \cdots \binom{n_j}{j}$$

where j is a positive integer

$$n_i \geq n_{i-1} \geq \cdots n_j \geq j \geq 1$$

Proof:

$$n_i = \max\left\{n : \binom{n}{i} \leq N\right\}$$

$$n_{i-1} = \max\left\{n : \binom{n}{i-1} \leq N - \binom{n_i}{i}\right\}$$

$\vdots$

$$n_{i-k} = \max\left\{n : \binom{n}{i-k} \leq N - \binom{n_i}{i} - \binom{n_{i-1}}{i-1} - \cdots - \binom{n_{i-k+1}}{i-k+1}\right\}$$

where $k \in \{0, 1, 2, 3, \ldots\}$

claim : $n_{i-1} \leq n_i$

Proof of claim : Note that $\binom{n_i}{i} \leq N < \binom{n_i+1}{i} = \binom{n_i}{i} + \binom{n_i}{i-1}$

Hence, $0 \leq N - \binom{n_i}{i} < \binom{n_i}{i-1}$

Since $i \geq 1$ $\forall$ $m \geq 1$ we have

$$\binom{m}{i-1} \leq \binom{m+1}{i-1}$$

Therefore $\max\left\{m : \binom{m}{i-1} \leq N - \binom{n_i}{i}\right\} \leq n_i$

i.e. $n_{i-1} \leq n_i$

Th (Binomial Theorem) Let A be an a-set and B be a b-set.

with $A \cap B = \emptyset$

$$(a+b)^n = \sum_{i=1}^{n} \binom{n}{i} a^i b^{n-i}$$

Proof: Let $S = \{(x_1, x_2 \dots x_n) : \forall i \in [n], x_i \in A \sqcup B\}$

we construct an equivalence relation $\sim$ overs

$(x_1, x_2 \dots, x_n) \sim (y_1, y_2, \dots y_n) \iff \exists (X, Y) \subset [n] \times [n]$

such that $|X| = |Y|$

$\forall i \in X \quad x_i \in A, \forall j \in [n] \setminus X, x_j \in$

$\forall i \in Y \quad y_i \in A, \forall j \in [n] \setminus Y, y_i \in B$

$S = A_0 \sqcup A_1 \sqcup A_2 \dots \sqcup A_n$

A_0

$\vdots$

$A_k = \{(x_1, x_2, x_3 \dots x_n) \in S$: exactly k element belongs to A

$\vdots \quad = \{(x_1, \dots x_n) \in S : \exists X \subset [n]$, with $|X| = k$ s.t $\forall i \in X \; x_i \in A$ &

$\forall j \in [n] \setminus X, \; x_j \in B$

A_n

$|S| = |A_0| + |A_1| + \dots + |A_n|$

$\therefore |A_0| = b^n$

$|A_1| = \binom{n}{1} a b^{n-1}$

$\vdots$

$|A_k| = \binom{[n]}{k} a^k b^{n-k}$ $\quad$ k many elements $\in A$

$n-k$ many elements $\in B$

We Construct the triplet

$$\left(\{n_1, n_2 \ldots n_k\} , (a_i)_{i \in \{n_1, n_2 \ldots n_k\}} , (b_j)_{j \in [n] \setminus \{n_1, n_2 \ldots n_k\}} \right)$$

$$A_K \xrightarrow{f} \left\{ \left(X, (a_i)_{i \in X}, (b_j)_{j \in [n] \setminus X} \right) : X \in \binom{[n]}{K} \right\}$$

$$(x_1, x_2 \ldots x_n) \longmapsto \left(\{n_1 \ldots n_k\} , (a_i)_{i \in \{n_1, n_2 \ldots n_k\}} , (b_j)_{\in [n] \setminus \{n_1, n_2 \ldots n_k\}} \right)$$

* Bijective function

---

* Trinomial

Let X be a n-set, and r, t be integers satisfying $0 \leq r+t \leq n$

$$\binom{X}{r,t} = \left\{ (A,B) : A \subset X, \text{ with } |A| = r, \ B \subset X \text{ with } |B| = t \right.$$
$$\left. A \cap B = \emptyset \right\}$$

Ex: show that $\left| \binom{X}{r,t} \right| = \dfrac{n!}{r!\,t!\,(n-r-t)!} = \dfrac{n!}{r!\,(n-r)!} \cdot \dfrac{(n-r)!}{t!\,(n-r-t)!}$

Th: Multinomial Theorem : Let for each $i \in [K]$, A_i be an a_i-set & $\forall i, j \in [K]$, with $i \neq j$, $A_i \cap A_j = \emptyset$

Then $$(a_1 + a_2 + \ldots + a_K)^n = \sum_{t_1 + t_2 + \ldots t_K = n} \frac{n!}{t_1!\, t_2! \ldots t_K!} a_1^{t_1} a_2^{t_2} \ldots a_K^{t_K}$$

Here the summation extend over all non-negative integer sol^n-s. of $t_1, t_2 \ldots t_K$ satisfies $t_1 + t_2 + t_3 + \ldots t_K = n$.

A Addition principle of counting

→ if a finite non-empty set S is partitioned into k many
Parts $B_1 \ldots B_k$ i.e $\forall\, i,j \in [k]$, $B_i \subset S$ and

$$B_i \cap B_j = \begin{cases} \emptyset & \text{if } i \neq j \\ B_i & \text{else } (i=j) \end{cases}$$

$$|S| = |B_1| + |B_2| + \ldots + |B_k|$$

B. Multiplication Principle of Counting

if a finite non-empty Set S can be expressed as

$$S_1 \times S_2 \times \ldots \times S_k$$

then

$$|S| = |S_1| \times |S_2| \times \ldots |S_k| = \prod_{i=1}^{k} |S_i|$$

C. One-One and Onto fn$^{=}$ (principle of counting)

Let R, S be two finite non-empty sets if $\exists$ a 1-1 & onto fn$^{=}$ $f: S \rightarrow R$ then $|S| = |R|$

D. The double counting principle

Let A & B be two non-empty finite sets & $S = A \times B$

The double counting principle is to count $|S|$ in two ways

First way: We Count: $\sum_{a\in A} |\{b\in B : (a,b)\in S\}|$

Second way: We Count: $\sum_{b\in B} |\{a\in A : (a,b)\in S\}|$

Here we have an identity

$$\sum_{a\in A} |\{b\in B : (a,b)\in S\}| = |S| = \sum_{b\in B} |\{a\in A : (a,b)\in S\}|$$

Example: (Handshaking Lemma) Suppose there are n many people at a party each of them will shake hands with everyone else

Q: How many handshakes will occur?

Soln.: Let $[n] = \{1,2,3 \ldots, n\}$ represent the set of one (right) hand of each people

we consider the pair $(x, \{x,y\})$

where $x \in [n]$, and $\{x,y\}$ represent the handshake

Let there be N no. of handshakes.

We count the set $S = \{(x,\{x,y\}) : x\in [n], y\in [n]\}$

First way: There are n many choices for x and chosen x there are $n-1$ many choices for $\{x,y\}$

$|S| = n(n-1)$

Second way: There are N many choices for $\{x,y\}$ Now chosen $\{x,y\}$, there are 2 choices such pairs namely $(x,\{x,y\})$ and $(y,\{x,y\}) \in S$

i.e $|S| = 2N$

Hence;

$$2N = |S| = n(n-1)$$

i.e. $$N = \frac{n(n-1)}{2}$$

Problem : How many way one can distribute n identical object into k many distinguishable / labelled boxes

O O O . . . . O ← n identical objects →

$\lfloor 1 \rfloor$ $\lfloor 2 \rfloor$ $\lfloor 3 \rfloor$ . . . $\lfloor k \rfloor$ ← k . labelled boxes →

Special cases O O O O O O O O O O ← 10 biscuits →

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Sol^n : step-1 O O O O O O O O O O

← 10 identical biscuits →

____ ____ 2 seperators

step-2 Place those two seperators among 10 biscuits eg (a)

O O O | O O O O | O O O $\lfloor 3B \rfloor$ $\lfloor 4B \rfloor$ $\lfloor 3B \rfloor$

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| O O O O O | O O O O O $\lfloor 0B \rfloor$ $\lfloor 5B \rfloor$ $\lfloor 5B \rfloor$

A N R

| | O O O O O O O O O O $\lfloor 0B \rfloor$ $\lfloor 0B \rfloor$ $\lfloor 10B \rfloor$

A N R

step-3 Consider 12 biscuits choose 2 of them and throw those two Finally replace it by two seperators.

$\binom{12}{2}$ ways

Theorem: One can distribute n identical objects into k labelled boxes into $\binom{n+k-1}{k-1}$ many ways.

Proof: Let B be the set of labelled boxes. Note $|B| = k$. For each distribution D of n identical objects into B, we associate a function $\gamma_D : B \longrightarrow \mathbb{Z}$ as

$$\gamma_D(b) = \text{No. of identical objects received by the}$$

box b w.r.t D

Note $$\sum_{b \in B} \gamma_D(b) = n$$

we note that for each $D \in \mathcal{D}$

$$\sum_{b \in B} \gamma_D(b) = n$$

where $\mathcal{D}$ denote the set of all distribution of n identical objects into k many labelled boxes

$$|\mathcal{D}| = |\{\gamma_D : D \in \mathcal{D}\}|$$

(Ex: show that $D \mapsto \gamma_D$ from $\mathcal{D}$ to $\{\gamma_D : D \in \mathcal{D}\}$ forms an 1-1 and onto fn

$$\{\gamma_D : D \in \mathcal{D}\} = \{ f : [k] \rightarrow \mathbb{Z} \cap [0,\infty) \mid f(1) + \cdots f(k) = n\}$$

claim: $$|\{f : [k] \longrightarrow \mathbb{Z} \cap [0,\infty) \mid f(1) + \cdots\cdots + f(k) = n\}| = |\binom{n+k-1}{k-1}|$$

Proof of claim: $\forall B \in \binom{[n+k-1]}{k-1}$, i.e $|B| = k-1$ say

$B = \{i_1, i_2 \ldots i_{k-1}\} \subset [n+k-1]$

we construct the function $f_B : [k] \to \mathbb{Z} \cap [0,\infty)$

WLOG

$i_1 < i_2 < \ldots < i_{k-1}$

$$f_B(1) = |\{i \in [n+k-1] : i < i_1\}|$$
$$f_B(2) = |\{i \in [n+k-1] : i_1 < i < i_2\}|$$
$$f_B(3) = |\{i \in [n+k-1] : i_2 < i < i_3\}|$$
$$\vdots$$
$$f_B(k) = |\{i \in [n+k-1] : i_{k-1} < i\}|$$

Construct for a fn: $f : [k] \to \mathbb{Z} \cap [0,\infty)$

with $f(1) + \ldots + f(k) = n$

we construct the $k-1$-set $\{i_1, i_2, \ldots i_{k-1}\}$ by setting

$i_1 = f(1) + 1$

$i_2 = f(1) + f(2) + 2$

$\vdots$

$i_{k-1} = f(1) + \ldots + f(k-1) + k-1$

Thus, $B \mapsto f_B$ is an 1-1 & onto welldefined function.

Corollary : One can distribute n identical objects into k labelled boxes such that each box contains atleast one identical object into $\binom{n-1}{k-1}$ many ways.

Proof : We first distribute one identical object into each boxes it results exactly $(n-k)$ ways identical objects are to be distributed among k labelled boxes

using the Th^m, we have $\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$ many ways

Exercise : Consider the eq^n $\sum_{i=1}^{k} x_i = n$, where n and k are positive integer show that

ⓐ the number of non-negative integer sol^{n}s of the eq^n is $\binom{n-k+1}{k-1}$

ⓑ the number of positive integer sol^n of the above eq^n is $\binom{n-1}{k-1}$

Def^n : Let n and k be positive integers with $n \geq k$ The Stirling number of 2nd kind denoted as $S_k^{II}(n) = S(n,k)$ is total no of partition of $[n]$ into k-many non-empty sets

The set $[n]$ is choped into k-many parts

1 2 3 ---- n-1 n $\qquad\qquad \sqcup \ \sqcup \ \sqcup \ \cdots \ \sqcup$

$\longleftarrow$ n-many labelled object $\longrightarrow$ $\qquad$ $\longleftarrow$ k many identical boxes $\longrightarrow$

Thm. $S(n,k) = S(n-1,k-1) + k.S(n-1,k)$

Proof: $\mathcal{S}([n],k)$ = The set of all partion of $[n]$ into k-many non-empty sets

$S(n,k) = |\mathcal{S}([n],k)|$

$\mathcal{A}$ = The set of all partition of $[n]$ into k - non-empty where n belongs is a singleton set
i.e. exactly one part of partition of $[n]$ is $\{n\}$.

$\mathcal{B}$ = The set of all partitions of $[n]$ into k non-empty sets where n belongs is not a singleton set

claim 1: $|\mathcal{A}| = S(n-1,k-1)$

Proof of claim: Let $X_1, \dots X_k$ be a partition of $[n]$ into k non-empty parts and exactly one of $X_1, \dots X_k$ is $\{n\}$

WLOG, Let $X_k = \{n\}$

i.e. $[n] = X_1 \sqcup X_2 \cdots \sqcup X_{k-1} \sqcup X_k$

$= X_1 \sqcup \dots \sqcup X_{k-1} \sqcup \{n\}$

$\forall i \in [k-1]$, $X_i \neq \emptyset$ & $X_k = \{n\}$

This induces a partition of $[n-1]$ into $(k-1)$- non-empty set

conversely if $Y_1, \dots Y_{k-1}$ is a partition of $[n-1]$ into $(k-1)$ many non-empty parts, then $Y_1, Y_2 \dots Y_{k-1}, \{n\}$ is a partition of $[n]$ into k many part with exactly one part equals $\{n\}$

$$A \longleftrightarrow S([n-1], k-1)$$

Hence, $\exists$ a bijection corresponds between A and $S([n-1], k-1)$

This establish the claim

claim 2 : $|B| = k \cdot S(n-1, k)$

Proof of claim : We note that

$$\{X_1, X_2 \dots X_k\} \longmapsto \{X_1 \setminus \{n\}, \dots X_k \setminus \{n\}\}$$

where $X_1 \dots X_k$ is a partition of $[n]$ into k many non-empty set and where n belongs is not a singleton set

Note. $X_1 \setminus \{n\}, \dots X_k \setminus \{n\}$ yeilds a partition of $[n-1]$ into k-many non-empty sets

Again each partition of $[n-1]$ into k-many non-empty sets

$$Y_1, Y_2 \dots Y_k$$

has exactly k many preimages w.r.t the said map namely

$$\{Y_1 \sqcup \{n\}, Y_2 \dots Y_k\}$$

$$\{Y_1, Y_2 \sqcup \{n\} \dots Y_k\} \longrightarrow \{Y_1 \setminus \{n\}, Y_2 \setminus \{n\} \dots Y_k \setminus \{n\}\}$$

$$\vdots$$

$$\{Y_1, Y_2 \dots Y_k \sqcup \{n\}\}$$

Hence such map is k to 1 & onto map

Hence, $|B| = k\,|\mathcal{S}([n-1],k)| = k\,S(n-1,k)$

Th^m: for each integer $n \geq 2$ $S(n,2) = 2^{n-1} - 1$

Proof - 1 : Note that $S(2,2) = 1$

$$
\begin{aligned}
S(n,2) &= 1 + 2S(n-1,2) \\
&= 1 + 2(1 + 2 \cdot S(n-2,2)) \\
&= \vdots \\
&= 1 + 2 + 2^2 + \dots + 2^{n-2} S(2,2) \\
&= 1 + 2 + 2^2 + \dots + 2^{n-2} \\
&= 2^{n-1} - 1
\end{aligned}
$$

Proof - 2 $\mathcal{S}([n],2)$ = The set of all of $[n]$ into 2 non-empty sets

$$\mathcal{P} = \{(A,B) : A \sqcup B = [n],\ A \neq \emptyset,\ B \neq \emptyset\}$$

$|\mathcal{P}| \longrightarrow$ Exercise $= 2^n - 2$

if $(A,B) \in \mathcal{P}$, then $(B,A) \in \mathcal{P}$

but (A,B) & (B,A) represents one partition of $[n]$ namely

Hence such mapping from $\mathcal{P}$ to $S([n],2)$ is a 2-to-1 mapping and onto, Hence $2|\mathcal{S}([n],2)| = |\mathcal{P}|$

i.e. $2S(n,2) = 2^n - 2$

$\iff S(n,2) = 2^{n-1} - 1$

Theorem : $S(n, n-1) = \binom{n}{2}$

Proof : Ex

Defn: The Bell no. $B(n)$, where n is a positive integer, is the to number of partitions of $[n]$ i.e. $B(n)$

$$B(n) = \sum_{i=1}^{n} S(n,i)$$

Thm: One can distribute n labelled objects into k-identical boxes such that each box contains atleast one object, into $S(n,k)$ ways

Proof : Exercise

Thm : One can distribute n-labelled objects into k-identical boxes into $S(n,1) + S(n,2) + \dots S(n,k)$ many ways

Proof : Exercise

① ② ③ . . . . . ⓝ ($n \geq k$)

$\longleftarrow$ n labelled $\rightarrow$
objects

$\lfloor 1 \rfloor$ $\lfloor 2 \rfloor$ . . . $\lfloor k \rfloor$

$\longleftarrow$ k labelled boxes $\rightarrow$

Th: One can distribute n labelled objects into k labelled boxes into k^n ways

Proof: A distribution of "such" yields a fn^{c} $d : [n] \to [k]$

$d(i)$ = label of box where i object is distributed

Conversely if $f : [n] \to [k]$ is a fn^{c} then $f^{-1}(1), f^{-1}(2) \ldots f^{-1}(k)$ forms a partition of $[n]$ into k many labelled set

---

Thm: One can distribute n. labelled objects into k labelled boxes. show that each box contains <u>atleast</u> one object into $k!\, S(n,k)$ many ways

Proof: $O([n],[k]) = \{ f : [n] \to [k] \mid f \text{ is onto} \}$

we construct an equivalence relation $\sim$ over $O([n],[k])$

$f \sim g$, where $f, g \in O([n],[k])$, iff $\exists\, \pi : [k] \xrightarrow[\text{onto}]{1-1} [k]$ s.t $g = \pi \circ f$

Let $[f]$ denote an equivalent class containing $f \in O([n],[k])$

$|[f]| = k!$

$O([n],[k]) = A_1 \sqcup A_2 \sqcup \dots A_N$

$k! = |A| = |A_2| = \dots = |A_N|$

claim: There are $S(n,k)$ no. of equivalence classes wrt to $\sim$

$f \sim g :\iff \exists \pi, g = \pi \circ f$

$(f^{-1}(1), f^{-1}(2) \dots f^{-1}(k))$ unordered k-tuple

$(g^{-1}(1), g^{-1}(2) \dots g^{-1}(k))$

$\{f^{-1}(1) \dots f^{-1}(k)\} = \{g^{-1}(1), g^{-1}(2) \dots g^{-1}(k)\}$

WLOG $|f^{-1}(1)| \leq |f^{-1}(2)| \dots \leq |f^{-1}(k)|$

$|g^{-1}(i_1)| \leq |g^{-1}(i_2)| \leq \dots \leq |g^{-1}(i_k)|$

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & k \\ i_1 & i_2 & i_3 & \dots & i_k \end{pmatrix}$$

<u>Proof of claim</u>: We first establish that if $f, g \in O([n],[k])$ s.t. the unordered k-tuple subset of $[n]$ $(f^{-1}(1), f^{-1}(2) \dots f^{-1}(k))$ and $(g^{-1}(1), g^{-1}(2) \dots g^{-1}(k))$ are same i.e.

$[n] = f^{-1}(1) \sqcup f^{-1}(2) \sqcup \dots \sqcup f^{-1}(k)$

the set of $\{f^{-1}(1), f^{-1}(2) \dots f^{-1}(k)\} = \{g^{-1}(1), g^{-1}(2) \dots g^{-1}(k)\}$

Then $\exists$ a permutation on $\pi : [k] \to [k]$ s.t. $f = \pi \circ g$

To see this WLOG

$|f^{-1}(1)| \leq |f^{-1}(2)| \leq \dots \leq |f^{-1}(k)|$

Suppose $|g^{-1}(i_1)| \le |g^{-1}(i_2)| \le \dots \le |g^{-1}(i_k)|$

we construct $\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & k \\ i_1 & i_2 & i_3 & \dots & i_k \end{pmatrix}$

Here we have,

$$|f^{-1}(1)| = |g^{-1}(i_1)|$$
$$|f^{-1}(2)| = |g^{-1}(i_2)|$$
$$\vdots$$
$$|f^{-1}(k)| = |g^{-1}(i_k)|$$

Note that $\forall x \in [n]$

$x \in f^{-1}(\alpha)$ for some $\alpha \in [k]$

i.e $f(x) = \alpha$

$\pi \circ f(x) = \pi(\alpha) = i_\alpha$

i.e $x \xmapsto{\pi \circ f} i_\alpha$

Also $x \in g^{-1}(i_\alpha)$

i.e $g(x) = i_\alpha$

i.e $x \xrightarrow{g} i_\alpha$

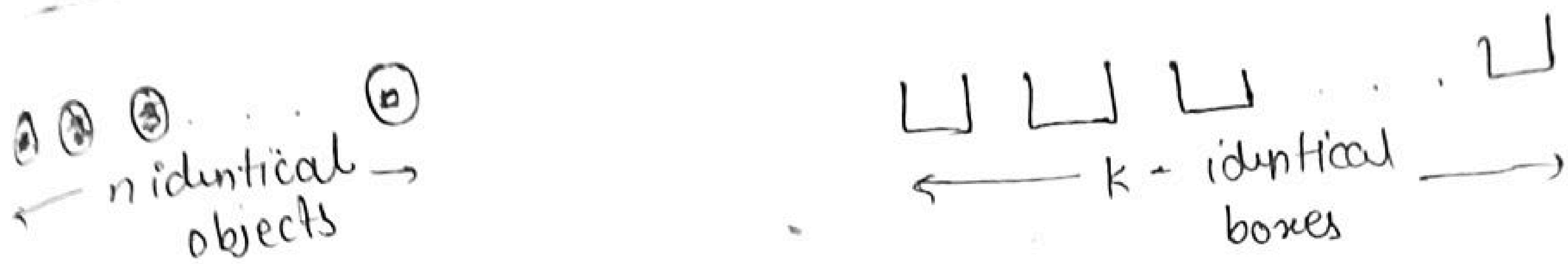
$g = \pi \circ f$

This means two different unordered k-tuples of subset of $[n]$ $(f^{-1}(1), f^{-1}(2) \dots f^{-1}(k))$ and $(g^{-1}(1) \dots g^{-1}(k))$ are not same $\Leftrightarrow \nexists$ $\pi : [k] \xrightarrow[\text{onto}]{1-1} [k]$ s.t $g = \pi \circ f \Leftrightarrow f \not\sim g$

Recall that unordered k tuple $(f^{-1}(1) \dots, f^{-1}(k))$ represents a partition of $[n]$ into k-non-empty sets

Hence the claim is established

So. no. of "such" distribution are $|O([n],[k])| = k!\,S(n,k)$



Defn: Let n, k be positive integer with $n \geq k$. An unordered k-tuple $(a_1, a_2 \ldots a_k)$ is said to be partition of integer n into k-positive integers if $a_1, a_2 \ldots a_k$ are positive integer satisfy

$$a_1 + a_2 + a_3 + \ldots a_k = n$$

The integer $P_k(n) = P(n,k)$ denotes the total no. of way in which the integer n can be partioned into k positive integer

Proof $\mathcal{P}(n,k)$ = The set of partitions of the integer n into k positive integers

$$= \left\{ (a_1, a_2 \ldots a_k) : (a_1 \ldots a_k) \text{ is an unordered } k\text{-tuple of positive integers satisfy } a_1 + a_2 + \ldots + a_k = n \right\}$$

$$A = \{ (a_1, a_2 \ldots a_k) \in \mathcal{P}(n,k) : a_i = 1, \text{ for some } i \in [k] \}$$

$$B = \{ (a_1, a_2 \ldots a_k) \in \mathcal{P}(n,k) : \forall i \in [k], a_i \geq 2 \}$$

$$A \longleftrightarrow \mathcal{P}(n-1, k-1)$$

let $(a_1, a_2 \ldots a_k) \in A$, WLOG $a_k = 1$

Note that $a_1 + a_2 + \ldots + a_{k-1} = n-1$

i.e unordered $k-1$ tuple $(a_1, a_2 \ldots a_{k-1})$ is a partition of integer $n-1$ into $k-1$ positive integers

i.e. $(a_1, a_2 \ldots a_{k-1}) \in \mathcal{P}(n-1, k-1)$

Conversely, if $(b_1, b_2 \ldots b_{k-1}) \in \mathcal{P}(n-1, k-1)$ then the unordered k-tuple $(b_1 \ldots b_{k-1}, 1) \in A$

Hence, There exist bijective correspondence exists b/w A and $\mathcal{P}(n-1, k-1)$. Hence,

$$|A| = |\mathcal{P}(n-1, k-1)| = P(n-1, k-1)$$

We note that,

$$(a_1, a_2 \ldots a_k) \longmapsto (a_1-1, a_2-1 \ldots a_k-1)$$

is an 1-1 & onto map from $\mathcal{B}$ to $\mathcal{P}(n-k, k)$ (Ex)

$$|\mathcal{B}| = |\mathcal{P}(n-k, k)| = P(n-k, k) \qquad \square$$

Thm: One can distribut n-identical objects into k identical boxes into $\sum_{i=1}^{k} P(n, i)$ many ways

Statistical Mechanics

Placing n particles into k-different energy levels

THREE different Statistics are obtained by making three different assumptions

- Maxwell - Boltzman ; Here ~~the~~ n labelled particles are distributed into k-labelled boxes (energy levels)

- Bose - Einstein : Here n identical particles are distributed into k labelled boxes (energy levels)

- Fermi - Dirac : Here n- identical particles are distributed into k labelled energy levels but no box can hold more than one particle

$\rightarrow \binom{k}{n}$

Subject

Date: / /

MON TUE WED THR FRI SAT SUN

* Principle of inclusion & exclusion

Th (IEP - Ver II) : Let $\mu : 2^{[n]} \longrightarrow \mathbb{R}$ be a fn and we construct

$$\nu : 2^{[n]} \longrightarrow \mathbb{R}$$

$$\nu(E) = \sum_{S \subset E} \mu(S) = \sum_{S \in 2^E} \mu(S)$$

Then, $$\mu(E) = \sum_{S \subset E} (-1)^{|E|-|S|} \nu(S)$$

Th (IEP - Ver III) Let $u : 2^{[n]} \rightarrow \mathbb{R}$ be a fn and

$$\nu : 2^{[n]} \rightarrow \mathbb{R}$$

$$\nu(E) = \sum_{E \subset S} \mu(S)$$

Then $$\mu(E) = \sum_{E \subset S} (-1)^{|S|-|E|} \nu(S) \qquad E \subset S$$

Proof: $$\sum_{E \subset S} (-1)^{|S|-|E|} \nu(S)$$

$$= \sum_{E \subset S} (-1)^{|S|-|E|} \left(\sum_{S \subset C} \mu(C) \right)$$

$$= \sum_{E \subset S} \sum_{S \subset C} (-1)^{|S|-|E|} \mu(C)$$

EL 60 X 84 CUT SINGLE LINE

Subject

Date: __/__/___

MON TUE WED THU FRI SAT SUN

* Principle of inclusion & exclusion

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$$\nu : 2^{[n]} \rightarrow \mathbb{R}$$

$$\nu(E) = \sum_{E \subseteq S} \mu(S)$$

Then $$\mu(E) = \sum_{E \subseteq S} (-1)^{|S| - |E|} \nu(S) \qquad E \subseteq S$$

Proof: $$\sum_{E \subseteq S} (-1)^{|S| - |E|} \nu(S)$$

$$= \sum_{E \subseteq S} (-1)^{|S| - |E|} \left(\sum_{S \subseteq C} \mu(C) \right)$$

$$= \sum_{E \subseteq S} \sum_{S \subseteq C} (-1)^{|S| - |E|} \mu(C)$$

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Subject

$$(C, S) \longrightarrow \left((-1)^{|E|-|S|}, \mu(C) \right)$$

$$\mu(C) \sum_{S's} (-1)^{|S|-|E|}$$

$$\sum_{C's} \mu(C) \sum_{S's} (-1)^{|E|-|S|}$$

$$\Rightarrow \mu(C) \sum_{S's} (-1)^{|E|-|S|} \qquad = \mu(C) \sum_{Z \subset C \setminus E} (-1)^{\cancel{|E|-|E|}-|Z|}$$

$$S = E \sqcup Z$$

$$Z \subset C \setminus E \qquad = \mu(C) \sum_{Z \subset C \setminus E} (-1)^{|Z|}$$

$$= \mu(C) \cdot 0$$

Hence, by DCP, we have

$$\mu(E) = \sum_{E \subset S} (-1)^{|E|-|S|} \nu(S)$$

$$\sum_{S \subset [n]} (-1)^{|S|} = \sum_{i=0}^{n} \binom{n}{i} (-1)^{i}$$

$$= \sum_{i=0}^{n} \binom{n}{i} (-1)^{i} (1)^{n-i}$$

$$= (\cancel{-1}+1)^{n} = 0$$

Theor (IEP - Ver IV) Let A be a finite non-empty set and $\forall i \in [m]$, P_i denote ~~dinote~~ a property for each $x \in A$, and $i \in [m]$ either x satisfies the property P_i or (exclusive or) x does not satisfy property P_i

let $S \subset [m]$

$$N(S) = \{x \in A : x \text{ satisfies property } P_i \ \forall i \in S\}$$

The no. of element of A that satisfy none of the properties $P_1, P_2 \dots P_m$ is given by

$$\sum_{S \subset [m]} (-1)^{|S|} |N(S)|$$

APP-I ONTO FUNCTION COUNTING

Let $S(n,k)$ denote the stirling no. of 2nd kind where n & k are positive integers with $n \geq k$ the $S(n,k) = \frac{1}{K!} \sum_{i=0}^{k} (-1)^i \binom{n}{i} (k-i)$

Proof $A([n],[k])$ = The set of all fn^2 from $[n]$ to $[k]$

$$|A([n],[k])| = k^n$$

$O([n],[k])$ = The set of all onto fn^{s} from $[n]$ to $[k]$

we define the properties $P_1, P_2 \dots P_K$

$\forall i \in [k]$ P_i denote the property that i does not belong to image of $f \in A([n],[k])$

Note that the set of all elements of $A([n],[k])$ that satisfy none of the properties $P_1, P_2 \dots P_K$ is that set of all onto fn^{s}

For $S \subset [K]$

$N(S) = \{ f \in A([n],[K]) : f$ satisfies property $P_i \ \forall i \in S \}$

claim : for $S \subset [K]$, $|N(S)| = (K - |S|)^n$

Proof of claim : we note the following

$f \in N(S) \Leftrightarrow f$ satisfies Prop. $P_i \ \forall i \in S$

$\Leftrightarrow i \notin \text{Im} f \quad \forall i \in S$

$\Leftrightarrow S$ not in the Im f

$\Leftrightarrow S$ is disjoint from im f

$\Leftrightarrow f$ is a fn^x from $[n]$ to $[K] \setminus S$

Hence, $|N(S)| = (K - |S|)^n$ This establishes the claim

Using the IEP - Ver IV $\Rightarrow$

$$|O([n],[K])| = \sum_{S \subset [K]} (-1)^{|S|} |N(S)| = \sum_{S \subset [K]} (-1)^{|S|} (K - |S|)^n$$

$$= \sum_{i=0}^{K} (-1)^i \binom{n}{i} (K-i)^n$$

Then, we know, $|O([n],[K])| = K! \, S(n,K)$

$$S(n,K) = \frac{1}{K!} \sum_{i=0}^{K} (-1)^i \binom{n}{i} (K-i)^n$$

Proof : let $x \in A$, we construct $S_x \{ i \in [m] : x$ satisfies prop $P_i\}$

$$N(S) = \{x \in A : S \subset S_x\}$$

$$= \sum_{S \subset [m]} (-1)^{|S|} |N(S)|$$

$$= \sum_{S \subset [m]} (-1)^{|S|} |\{x \in A : S \subset S_x\}|$$

$$= \sum_{S \subset [m]} (-1)^{|S|} \left(\sum_{\substack{x \in A \\ S \subset S_x}} 1 \right)$$

$$= \sum_{S \subset [m]} \sum_{\substack{x \in A \\ S \subset S_x}} (-1)^{|S|} = \sum_{S \subset S_x \subset [m]} \sum_{x \in A} (-1)^{|S|}$$

$$= \sum_{x \in A} \sum_{S \subset S_x} (-1)^{|S|}$$

$$= \sum_{x \in A} \sum_{S \subset S_x = \phi} (-1)^{|S|} + \sum_{x \in A} \sum_{S \subset S_x \neq \phi} (-1)^{|S|}$$

$$= \sum_{x \in A} \sum_{S_x = \phi} 1 + \sum_{x \in A} (-1+1)^{|S_x|}$$

$$= \sum_{x \in A} \sum_{S_x = \phi} 1$$

$$= |\{x \in A : S_x = \phi\}|$$

$$= |\{x \in A : x \text{ does not satisfy property } P_1, P_2 \ldots P_m\}|$$

App-II : Dearangement Permutation counting

Defn : A permutation $\pi : [n] \xrightarrow[\text{onto}]{1-1} [n]$ is called derangement if $\forall i \in [n]$, $\pi(i) \neq i$

The set of all derangements on $[n]$ is denoted by D_n

claim $S(v_j) = T(v_j) \quad \forall j \in \{1, 2, \dots n\}$

APP-III Establishment of EULER PHI Function

Th: if $n = p_1^{n_1} p_2^{n_2} \dots p_n^{n_k}$ where $\forall i \in [k]$. $n_i \geq 1$ $n > 1$ are integer and p_i are positive prime integers, then

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right) \dots \left(1-\frac{1}{p_k}\right)$$

Here $\phi(n)$ denotes Euler's phi function defined by

$$\phi(n) = \left|\{x \in [n] : \gcd(x, n) = 1\}\right|$$

Proof: For each $i \in [k]$, let P_i denote the property that x is divisible by the positive prime integer p_i

claim: The set of elements of $[n]$ that satisfies the none of the properties of $P_1, P_2 \dots P_i$ then

$$R = \{x \in [n] : \gcd(x, n) = 1\}$$

Proof of claim: Note that if x does not satisfy the P_i for some $i \in [n]$ then $\gcd(x, p_i) = 1$

$\Rightarrow \exists u, v \in \mathbb{Z}$ s.t $xu + p_i v = 1$

$$\Rightarrow (xu + p_i v)^{n_i} = 1$$

$$\Leftrightarrow x(\quad) + p^{n_i}(\quad) = 1$$

$$\Rightarrow \gcd(x, p_i^{n_i}) = 1$$

Therefore x does not satisfies property ~~P~~ P_i then $\gcd(x, p_i^{n_i}) = 1$

So, if x does not satisfy $P_1, P_2 \dots P_k$ then $\exists u_i, v_i \in \mathbb{Z}$ $\forall i \in [k]$ s.t. $(xu_1 + p_1^{n_1} v)(xu_2 + p_2^{n_2} v_2) + \dots + (xu_k + p_k^{n_k} v_k) = 1$

$\Leftrightarrow xU + nV = 1$ for some $U, V \in \mathbb{Z}$

claim $S(v_j) = T(v_j)$ $\forall\, i \in \{1, 2, \dots n\}$

$\Leftrightarrow \gcd(x, n) = 1$

$\Leftrightarrow x \in R$ (this establishes the claim)

claim : $S \subset [K]$

$$|N(S)| = \frac{n}{\prod_{i \in S} p_i}$$

Proof of claim : $x \in N(S) \Leftrightarrow x$ satisfies property P_i $\forall\, i \in S$

$\Leftrightarrow p_i \mid x \quad \forall\, i \in S$

$\Leftrightarrow \prod_{i \in S} p_i \mid x$

Hence, $N(S) = \left\{ x \in [n],\ x = r\left(\prod_{i \in S} p_i\right),\ r = 1, 2, \dots \frac{n}{\prod_{i \in S} p_i} \right\}$

Using IEP - Vertex and the claim we have

$$\phi(n) = |R| = \sum_{S \subset [K]} (-1)^{|S|} |N(S)|$$

$$= \sum_{S \subset [K]} (-1)^{|S|} \frac{n}{\prod_{i \in S} p_i} = n \prod_{i=1}^{K} \left(1 - \frac{1}{p_i}\right)$$

$$\left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right) = 1 + (-1)\left(\frac{1}{p} + \frac{1}{q}\right) + \frac{1}{pq}$$

Th: For any integer $n \geq 1$

$$n = \sum_{d|n} \phi(d)$$

$$\sum_{d|n} \phi(d) = \sum_{d|n} \phi\left(\frac{n}{d}\right)$$

Th: for each positive integer n

$$n = \sum_{d|n} \phi(d)$$

Proof : Let $\gcd(x,n) = d$

Then $\exists\ u, v \in \mathbb{Z}$ s.t. $xu + nv = d$

$$= \frac{x}{d}u + \frac{n}{d}v = 1$$

so $x \longmapsto \frac{x}{d}$ forms a bijective mapping between set $\{x \in [n] : \gcd(x,n) = d\}$

and $\{y \in [\frac{n}{d}] : \gcd(y, \frac{n}{d}) = 1\}$

Hence,

$$|\{x \in [n]\ \gcd(x,n) = d\}| = \phi\left(\frac{n}{d}\right)$$

Now, $x \sim y$, where $x, y \in [n]$ for a equivalence r/n

iff $\gcd(x,n) = \gcd(y,n)$

Hence $n = |[n]| = \left| \bigsqcup_{d=1}^{n} \{x \in [n] : \gcd(x,n) = d\} \right|$

$$\sum_{d=1}^{n} |\{x \in [n] : \gcd(x,n) = d\}|$$

$$n = \sum_{d/n} \phi\left(\frac{n}{d}\right) = \sum_{d/n} |\{x \in [n] : \gcd(x,n) = d\}|$$

$d \nmid n \quad \{x \in [n] : \gcd(x,n) = d\} = \phi$

MOBIUS FUNCTION

Def^n: The Mobius fn^x $\mu : \{n \in \mathbb{Z} : n \geq 1\} \to \{-1, 0, 1\}$ is defined by

$$\mu(1) = 1$$

If $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n_1 = 1 = n_2 = \dots n_k \\ 0 & \text{if } \exists i \in [k] \ \ n_i \geq 2 \ \text{(else)} \end{cases}$$

Th: for each integer n

$$\sum_{d \mid n} \mu(d) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{else } n \neq 1 \end{cases}$$

Proof: if $n = 1$ then only divisor of $n = 1$ is $d = 1$ Hence

$$\sum_{d \mid n} \mu(d) = 1 = \mu(1) = 1$$

So suppose $n \geq 2$ & $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$

Then

$$\sum_{d \mid n} \mu(d) = \sum_{d \mid p_1 p_2 \dots p_k} \mu(d) = \sum_{S \subset [k]} (-1)^{|S|} = 0$$

Corollary: For each integer $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$$

Proof: $$\phi(n) = n \prod_{i=1}^{k} \left(1 - \frac{1}{p_i}\right)$$

$$= n \sum_{S \subset [k]} \frac{(-1)^{|S|}}{\prod_{i \in S} p_i} = n \sum_{d | p_1 p_2 p_3 \cdots p_k} \frac{\mu(d)}{d}$$

Def^n: A real valued sequence $\{x_n\}_{n=0}^{\infty}$ is said to satisfy k- ~~term~~ recurrence relation or simply recurrence relation

if $\exists \lambda \in \mathbb{R}^k \to \mathbb{R}$ s.t $x_0, x_1 \ldots x_{k-1}$ is known (referred as initial Cond^n) and $\forall$ integer $n \geq 0$

$$x_{n+k} = \lambda(x_n, x_{n+1} \ldots x_{n+k-1})$$

$x_k = \lambda(x_0, x_1 \ldots x_{k-1}\}$
$x_{k+1} = \lambda(x_1, x_2 \ldots x_k\}$
$x_{k+2} = \lambda(x_2, x_3 \ldots x_{k+1}\}$

We say the fn^x λ as k-term recurrence fn^x is simply recurrence fn^x

If f λ is Linear i.e.

$$\left.\begin{aligned}&\lambda(\bar{x}+\alpha\bar{y}) = \lambda(\bar{x}) + \alpha\lambda(\bar{y})\\ &\bar{x} = (x_1, x_2 \ldots x_K) \in \mathbb{R}^K\\ &\bar{y} \in \mathbb{R}^K\\ &\alpha \in \mathbb{R}\end{aligned}\right\}$$ & λ is a k-term recurrence relation

then such λ is called k-term Linear recurrence relation

if λ is homogeneous of degree n

$$\bar{x} = \{x_1, x_2 \ldots x_K)$$

$$\alpha\bar{x} = \{\alpha x_1, \alpha x_2 \ldots \alpha x_K\}$$

$$\lambda(\alpha\bar{x}) = \lambda(\alpha x_1, \alpha x_2 \ldots \alpha x_K\} = \alpha^n \lambda(x_1, x_2 \ldots x_K)$$

$$= \alpha^n \lambda(\bar{x})$$

$$\lambda(\alpha\bar{x}) = \alpha^n \lambda(\bar{x})$$

& λ is a K-term recurrence relation
then such λ is called K-term homogeneous recurrence relation

Example : $\lambda : \mathbb{R}^2 \to \mathbb{R}$

$\lambda(x,y) = x+y$ $\longleftarrow$ Linear

$\lambda(x,y) = (x+y)^2$ $\longleftarrow$ homogeneous of degree 2

Problem : Solve the Linear recurrence relation

$x_0 = 0$ $x_1 = 1$ and $x_{n+2} = x_{n+1} + x_n$ $\forall$ integer $n \geq 0 \} \to *$

Soln We first note that

$$(R^2 - R - I)\left(\{x_n\}_{n=0}^{\infty}\right) = \{0_n (=0)\}_{n=0}^{\infty}$$

step-1 : Identity the characteristic polynomial with the recurrence relation Here it is $x^2 - x - 1$

step 2 : find the roots of characteristic polynomial

Here $x^2 - x - 1 = 0$

$$\Leftrightarrow (x - \tau)\left(x + \frac{1}{\tau}\right) = 0 \qquad \tau = \frac{1+\sqrt{5}}{2}$$

Roots are τ and $-\frac{1}{\tau}$

step 3 : formation of general soln

Here it is $x_n = c_1 \tau^n + c_2\left(-\frac{1}{\tau}\right)^n$

where $c_1, c_2 \in \mathbb{C}$

step 4 : Apply initial condn

Here $x_0 = 0$ $x_1 = 1$

$$0 = c_1 + c_2$$

$$1 = c_1 \tau + c_2\left(-\frac{1}{\tau}\right)$$

Final : Unique solution of * is : $x_n = \left(-\frac{1}{\sqrt{5}}\right)\tau^n + \left(\frac{1}{\sqrt{5}}\right)\left(-\frac{1}{\tau}\right)^n$

Problem : Solve the Linear recurrence relation

$x_0 = 0, x_1 = 1, x_2 = 3$

$x_{n+3} = 5x_{n+2} - 8x_{n+1} + 4x_n \quad \forall$ integer $n \geq 0 \} \; (*)$

Sol^n : $(R^3 - 5R^2 + 8R - 4I)(\{x_n\}_{n=0}^{\infty}) = \{0_n (=0)\}_{n=0}^{\infty}$

step1 - identify the characterstic polynomial associatited with the recurrence relation

$x^3 - 5x^2 + 3x - 4$

step-2 roots of the eq^n

$x^3 - 5x^2 + 8x - 4 = 0$

$(x-2)^2(x-1) = 0$

Roots 2, 2, 1

step-3 formation of generale sol^n

$x_n = (c_0 + c_1 n) 2^n + (c_3) 1^n$

Step 4 Apply initial cond^n of *

$x_0, x_1 = 1 \quad x_2 = 3$

Rough

$c_0 + c_3 = 0$

$2(c_0 + c_1) + c_3 = 1$

$(c_0 + 2c_1) 4 + c_3 = 3$

$\underbrace{4c_0 + 8c_1 + 1 - 2c_0 - 2c_1}$ $= 3$

$6c_1 + 2c_0 = 2$

Remark Let $\alpha_1, \alpha_2 \dots \alpha_k$ are roots with multiplities γ_1, γ_2 resp

$$(R-\alpha_1 I)^{\gamma_1} (R-\alpha_2 I)^{\gamma_2} \dots (R-\alpha_k I)^{\gamma_k} (\{x_n\}_0^\infty) = \{0_K (=0)\}_{n=0}^\infty$$

general solution $= \underbrace{(\quad)}_{\text{Polynomial of } n \text{ with degree } \gamma_1 - 1} \alpha_1^n + \underbrace{(\quad)}_{\text{Polynomial of } n \text{ with degree } \gamma_2 - 1} \alpha_2^n + \dots$

$S(\mathbb{C})$ = The set of all Complex valued soln

$$\left\{ \{x_n\}_{n=0}^\infty \mid \forall \text{ integer } n \geq 0, \ x_n \in \mathbb{C} \right\}$$

claim: $S(\mathbb{C})$ form a vector space over $\mathbb{C}$

$\alpha \in \mathbb{C}$

$$\alpha \{x_n\}_{n=0}^\infty = \{\alpha x_n\}_{n=0}^\infty$$

Defn: A Linear operator $T: S(\mathbb{C}) \longrightarrow S(\mathbb{C})$ is a map satisfies

(a) $T(\{x_n\}_{n=0}^\infty + \{y_n\}_{n=0}^\infty) = T(\{x_n\}_{n=0}^\infty) + T(\{y_n\}_{n=0}^\infty)$

(b) $T(\alpha \{x_n\}_{n=0}^\infty) = \alpha T(\{x_n\}_{n=0}^\infty)$

Example: $\{x_n\}_{n=0}^\infty \longmapsto \{x_{1+n}\}_{n=0}^\infty$

$$\{x_0, x_1, x_2 \dots\} \longrightarrow \{x_1, x_2 \dots x_n\}$$

claim. $R: S(\mathbb{C}) \longrightarrow S(\mathbb{C})$ is a linear operator

$$\{x_0, x_1, x_2 \dots\} \xrightarrow{R} \{x_1, x_2 \dots\} \xrightarrow{R} \{x_2, x_3, x_4, \dots\}$$

$R^2 = R \circ R$

$$R^2(\{x_n\}_{n=0}^{\infty}) = \{x_{2+n}\}_{n=0}^{\infty}$$

claim: R^2 is also a linear operator

claim: R^K

$$(R^3 - 5R^2 + 8R - 4I)(\{x_n\}_0^{\infty})$$

$$= \{x_{3+n}\}_{n=0}^{\infty} - 5\{x_{2+n}\}_{n=0}^{\infty} + 8\{x_{1+n}\}_{n=0}^{\infty} - 4\{x_n\}_{n=0}^{\infty} = \{0_n\}_{n=1}^{\infty}$$

Defn The Set

$$\left\{ \{x_n\}_{n=0}^{\infty} \in S(\mathbb{C}) \mid T(\{x_n\}_{n=0}^{\infty}) = \{0_n\}_{n=0}^{\infty} \right\}$$

is called kernel of T denoted as $\ker T$

Here, $S(\mathbb{C})$ is an example of infinite dim. vector space

* $R(\{\alpha^n\}_{n=0}^{\infty}) = \alpha\{\alpha^n\}_{n=0}^{\infty}$

$R\left(<\{\alpha^n\}_{n=0}^{\infty}>\right) = <\{\alpha^n\}_{n=0}^{\infty}>$

Def^n : Let H be an vector space over $\mathbb{C}$ and $T : H \to H$ be a linear operator. A non-zero vector $x \in H$ is called eigen (I-gen) vector if $\exists \alpha \in \mathbb{C}$ s.t.

$$T(x) = \alpha x$$

such α is called eigen value.

Note : $T(<x>) = <x>$

Note : $R^k(\{\alpha^n\}_{n=0}^{\infty}) = \alpha^k(\{\alpha^n\}_{n=0}^{\infty})$

Thus $\forall \alpha \in \mathbb{C}$, and $\forall$ integer $k \geq 0$ $\{\alpha^n\}_{n=0}^{\infty}$ is the eigen vector of ~~$\{\alpha^n\}_{n=0}^{\infty}$~~ R^k with eigen value α^k

Lemma : Let H be a vector space over $\mathbb{C}$ and $T : H \to H$ be a linear operator. Then $\ker T = \{x \in H : T(x) = 0\}$ for a vector subspace over $\mathbb{C}$

Proof Let $x, y \in \ker T$ and $\alpha \in \mathbb{C}$

The $T(x + \alpha y) = T(x) + T(\alpha y) = T(x) + \alpha T(y) = 0 + 0 = 0$

i.e. $x + \alpha y \in \ker T$

$\square$

Hence the result

Cor: Let $p(x) \in \mathbb{C}[x]$ be a polynomial

$R : S(\mathbb{C}) \longrightarrow S(\mathbb{C})$ be a right shift linear operator

$$R(\{x_n\}_{n=0}^{\infty}) = \{x_{1+n}\}_{n=0}^{\infty}$$

Then: $p(R) : S(\mathbb{C}) \longrightarrow S(\mathbb{C})$ form a Linear operator & $\ker p(R)$ forms a vector space over $\mathbb{C}$

Proof: Exercise

Hint: $T_1 : H \longrightarrow H$
$T_2 : H \longrightarrow H$
then $T_1 + T_2$ is a Linear operator

* Exercise $\operatorname{Im} T = \{T(x) : x \in H\}$ is vector subspace of H over $\mathbb{C}$

Lemma: Let $x_{n+k} = \sum_{i=1}^{k} c_i x_{n+k-i}$

$$= c_1 x_{n+k-1} + c_2 x_{n+k-2} + \cdots + c_k x_n$$

Where $\forall\, i \in [k]$, $c_i \in \mathbb{C}$, be a k-term Linear homogeneous recurrence relation with initial Condition $x_0, x_1, \ldots x_{k-1}$

Then $p(R)(\{x_n\}_{n=0}^{\infty}) = \{0_n (=0)\}_{n=0}^{\infty}$ $\qquad R^0 = I$

Where $p(R) = R^k - \sum_{i=1}^{k} c_i R^{k-i}$

Moreover $\ker p(R)$ is a k-dimensional vector subspace of $S(\mathbb{C})$ over $\mathbb{C}$

Proof : $p(R)(\{x_n\}_{n=0}^{\infty}) = \{0_n(=0)\}_{n=0}^{\infty}$

$\Rightarrow (R^k - \sum_{i=1}^{k} c_i R^{k-i})(\{x_n\}_{n=0}^{\infty}) = \{0_n(=0)\}_{n=0}^{\infty}$

we construct the map

$$\Lambda : \ker p(R) \to \mathbb{C}^k$$

$$\Lambda(\{x_n\}_{n=0}^{\infty}) = (x_0, x_1 \dots x_{k-1})$$

For each $\alpha \in \mathbb{C}$ and $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \in \ker p(R)$

$$\Lambda(\{x_n\}_{n=0}^{\infty} + c\{y_n\}_{n=0}^{\infty})$$

$$\Lambda(\{x_n + cy_n\}_{n=0}^{\infty}) = (x_0 + cy_0 + x_1 + cy_1 + \dots x_{k-1} + cy_{k-1})$$

$$\Lambda(\{x_n\}_{n=0}^{\infty}) + c\Lambda(\{y_n\}_{n=0}^{\infty}) = (x_0 + x_1 + \dots x_{k-1}) + c(y_0, y_1, \dots y_{k-1})$$

$$= (x_0 + cy_0, x_1 + cy_1 + \dots x_{k-1} + cy_{k-1})$$

i.e. $\Lambda(\{x_n\}_{n=0}^{\infty} + c\{y_n\}_{n=0}^{\infty}) = \Lambda(\{x_n\}_{n=0}^{\infty}) + c\Lambda(\{y_n\}_{n=0}^{\infty})$

Thus Λ is a linear map

if $\Lambda(\{x_n\}_{n=0}^{\infty}) = \Lambda(\{y_n\}_{n=0}^{\infty})$

$$(x_0, x_1 \dots x_{k-1}) = (y_0, y_1 \dots y_{k-1})$$

$$\Leftrightarrow x_0 = y_0, \ y_1 = x_1 \dots y_{k-1} = x_{k-1}$$

Λ is 1-1

Let $(a_0, a_1, \dots, a_{k-1}) \in \mathbb{C}^k$

$$a_{n+k} = \sum_{i=1}^{k} c_i a_{n+k-i}$$

Then $\{a_n\}_{n=0}^{\infty} \in \ker p(R)$

$$\Lambda\left(\{a_n\}_{n=0}^{\infty}\right) = (a_0, a_1, \dots a_{k-1})$$

Λ is onto

Thus Λ establishes the vector space isomorphism between $\ker p(R)$ and $\mathbb{C}^k$

Since $\mathbb{C}^k$ is a k-dimensional vector space over $\mathbb{C}$

we have $\ker p(R)$ is k-dimensional V.S over $\mathbb{C}$

(Vandermonde Determinant). Let $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ be $(n+1)$ many distinct non-zero complex no.

Then
$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & \cdots & \alpha_n \\ \alpha_0^2 & \alpha_1^2 & & \alpha_n^2 \\ \vdots & \vdots & & \vdots \\ \alpha_0^n & \alpha_1^n & \cdots & \alpha_n^n \end{vmatrix}_{n+1 \times n+1} = \prod_{\substack{j=0 \\ 0 \le i < j \le n}}^{n} \prod_{i=0}^{n} (\alpha_j - \alpha_i)$$

Proof: We use backward induction on n and the fact
$$\alpha_i^p - \alpha_0^p - \alpha_0(\alpha_i^{p-1} - \alpha_0^{p-1}) = (\alpha_i - \alpha_0)(\alpha_i^{p-1}) \quad \forall\, p \geq 1,\ p \in \mathbb{Z}$$

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & & \alpha_n \\ \alpha_0^2 & \alpha_1^2 & & \alpha_n^2 \\ \vdots & \vdots & & \vdots \\ \alpha_0^n & \alpha_1^n & & \alpha_n^n \end{vmatrix} = \prod_{i=1}^{n} (\alpha_i - \alpha_0) \underbrace{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & & \alpha_n^2 \\ \vdots & \vdots & & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & & \alpha_n^{n-1} \end{vmatrix}}$$

$$\parallel$$

$$\prod_{i=2}^{n} (\alpha_i - \alpha_1) \; \bigcirc$$

another determinant

Th: Let $\alpha_0, \ldots \alpha_{k-1}$ be distinct non-zero Complex no. Then $\ker((R - \alpha_0 I) \circ (R - \alpha_1 I) \ldots \circ (R - \alpha_{k-1} I))$ is a k-dim vector subspace of $S(\mathbb{C})$

Proof: claim: The vectors $\{\alpha_0^n\}_{n=0}^{\infty}$ $\{\alpha_1^n\}_{n=0}^{\infty}$ $\ldots$ $\{\alpha_{k-1}^n\}_{n=0}^{\infty}$ are Linearly independent

Proof :- Let $c_0, c_1, \ldots c_{k-1} \in \mathbb{C}$ s.t.

$$c_0\{\alpha_0^n\}_{n=0}^{\infty} + c_1\{\alpha_1^n\}_{n=0}^{\infty} + \ldots + c_{k-1}\{\alpha_{k-1}^n\}_{n=0}^{\infty} = \{0_n\}_{n=0}^{\infty}$$

$$\Rightarrow \quad c_0 + c_1 + c_2 + \ldots c_{k-1} = 0$$

$$c_0\alpha_0 + c_1\alpha_1 + c_2\alpha_2 + \ldots c_{k-1}\alpha_{k-1} = 0$$

$$c_0\alpha_0^2 + c_1\alpha_1^2 + c_2\alpha_2^2 + \ldots c_{k-1}\alpha_{k-1}^2 = 0$$

$$\vdots$$

$$c_0\alpha_0^{k-1} + c_1\alpha_1^{k-1} + c_2\alpha_2^{k-1} + \ldots + c_{k-1}\alpha_{k-1}^{k-1} = 0$$

$$\Longleftrightarrow \begin{bmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & & \alpha_{k-1} \\ \vdots & \vdots & & \vdots \\ \alpha_0^{k-1} & \alpha_1^{k-1} & & \alpha_{k-1}^{k-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$A\bar{c} = 0$$

$$A^{-1}A\bar{c} = A^{-1}0 = 0$$

$$\bar{c} = 0$$

Since $\alpha_0, \alpha_1, \ldots \alpha_{k-1}$ are non-zero distinct complex no.s we have the vandermonde det is non-zero

i.e. $$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \alpha_0 & \alpha_1 & & \alpha_n \\ \vdots & \vdots & & \vdots \\ \alpha_0^{k-1} & \alpha_1^{k-1} & & \alpha_n^{k-1} \end{vmatrix} \neq 0$$

Therefore,

$$c_0 = 0 = c_1 = \cdots = c_{k-1}$$

This establishes the claim

$(R-\alpha_i I)\circ(R-\alpha_j I)$
$= (R-\alpha_j I)\circ(R-\alpha_i I)$

commutativity

Therefore, we have for each integer i, with $0 \le i \le k-1$

$$(R-\alpha_0 I)(R-\alpha_1 I) \cdots (R-\alpha_{k-1} I)\left(\{\alpha_i^n\}_{n=0}^{\infty}\right)$$

$$= (R-\alpha_0 I) \cdots (R-\alpha_{k-1} I)(R-\alpha_i I)\left(\{\alpha_i^n\}_{n=0}^{\infty}\right)$$

$$= \left(\prod_{\substack{j=0 \\ j\neq i}}^{k-1} (R-\alpha_j I)\right)\left(\{0_n(=0)\}_{n=0}^{\infty}\right)$$

$$= \{0_n(=0)\}_{n=0}^{\infty}$$

$$\{\alpha_i^n\}_{n=0}^{\infty} \in \ker\left(\bigcirc_{j=0}^{k-1} (R-\alpha_j I)\right)$$

Thus, $\left\{\{\alpha_0^n\}_{n=0}^{\infty}, \ldots \{\alpha_{k-1}^n\}_{n=0}^{\infty}\right\} \subset \ker((R-\alpha_0 I)\circ \cdots (R-\alpha_{k-1})$

$$\langle \{\alpha_0^n\}_{n=0}^{\infty}, \ldots \{\alpha_{k-1}^n\}_{n=0}^{\infty} \rangle = \ker(\quad " \quad)$$

General soln of the respective recurrence relation

is $c_0\{\alpha_0^n\}_{n=0}^{\infty} + c_1\{\alpha_1^n\}_{n=0}^{\infty} + \cdots + c_{k-1}\{\alpha_{k-1}^n\}_{n=0}^{\infty}$

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This means $\{\alpha_i^n\}_{n=0}^{\infty} \in \ker((R-\alpha_0 I)\circ \cdots \circ(R-\alpha_{k-1}I)) \ \forall\, i=0,1,2\ldots k-1$

Hence, $\{\{\alpha_i^n\}_{n=0}^{\infty} : i=0,1,\ldots k-1\} \subset \ker((R-\alpha_0 I)\circ\cdots\circ(R-\alpha_{k-1}I))$

This means $\langle \{\alpha_0^n\}_{n=0}^{\infty}, \ldots \{\alpha_{k-1}^n\}_{n=0}^{\infty} \rangle$ is a vector subspace of $\ker((R-\alpha_0 I)\circ\cdots\circ(R-\alpha_{k-1}I))$ over $\mathbb{C}$

Using the claim $\dim\{\langle \{\alpha_0^n\}_{n=0}^{\infty} \ldots \{\alpha_{k-1}^n\}_{n=0}^{\infty}\rangle\} = k$

Using the previos lemma

$$\dim\{\ker((R-\alpha_0 I)\circ\cdots\circ(R-\alpha_{k-1}I))\} = k$$

Thus,

$$\ker((R-\alpha_0 I)\circ\cdots\circ(R-\alpha_{k-1}I)) = \langle \{\alpha_0^n\}_{n=0}^{\infty}, \{\alpha_1^n\}_{n=0}^{\infty} \ldots \{\alpha_{k-1}^n\}_{n=0}^{\infty} \rangle$$

Remark: $\ker(R-3I)(R-2I)$

$$= \langle \{3^n\}_{n=0}^{\infty}, \{2^n\}_{n=0}^{\infty} \rangle$$

Hence the general sol^n

$$c_1\{3^n\}_{n=0}^{\infty} + c_2\{2^n\}_{n=0}^{\infty}$$

where, $c_1, c_2 \in \mathbb{C}$

$$\{c_1 3^n + c_2 2^n\}_{n=0}^{\infty}$$

i.e. $x_n = c_1 3^n + c_2 2^n \ \forall$ integer $n \geq 0$

Th: Let α be non-zero complex number Then $\ker((R-\alpha I)^k)$ is a k-dim subspace of $S(\mathbb{C})$

$\{x,$

Moreover $\ker((R-\alpha I)^k)$

$$= \langle \{\alpha^n\}_{n=0}^{\infty}, \{n\alpha^n\}_{n=0}^{\infty} \dots \{n^{k-1}\alpha^n\}_{n=0}^{\infty} \rangle$$

Remark $\ker((R-2I)^2) = \langle \{2^n\}_{n=0}^{\infty}, \{n2^n\}_{n=0}^{\infty} \rangle$

$$x_n = c_1 2^n + c_2 n 2^n = (c_1 + c_2 n)2^n \quad \text{where } c_1, c_2 \in \mathbb{C}$$

Claim 1: The vectors $\{\alpha^n\}_{n=0}^{\infty}, \{n\alpha^n\}_{n=0}^{\infty} \dots \{n^{k-1}\alpha^n\}_{n=0}^{\infty}$ are Linearly independent vector in $S(\mathbb{C})$

Proof of claim: Let $c_0, c_1 \dots c_{k-1} \in \mathbb{C}$ such that

$$c_0(\{\alpha^n\}_{n=0}^{\infty}) + c_1(\{n\alpha^n\}_{n=0}^{\infty}) + \dots + c_{k-1}(\{n^{k-1}\alpha^n\}_{n=0}^{\infty})$$
$$= \{0_n\}_{n=0}^{\infty}$$

$\Rightarrow \quad c_0 = 0$

$\Rightarrow \quad c_0\alpha + c_1\alpha + \dots c_{k-1}\alpha = 0$

$\Rightarrow \quad c_0\alpha^2 + 2c_1\alpha^2 + \dots + 2^{k-1}c_{k-1}\alpha^2 = 0$

$\vdots$

$\Rightarrow c_0\alpha^{k-1} + c_1(k-1)\alpha^{k-1} + c_2(k-1)^2\alpha^{k-1} + \dots c_{k-1}\alpha^{k-1}(k-1)^{k-1} = 0$

$\Leftrightarrow \quad c_0 = 0$

$c_1 + c_2 + \dots c_{k-1} = 0$

$2c_1 + 2^2c_2 + \dots 2^{k-1}c_{k-1} = 0$

$(k-1)c_1 + (k-1)^2c_2 + \dots (k-1)^{k-1}c_{k-1} = 0$

$$\Rightarrow \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 2 & 2^2 & & 2^{k-1} \\ \vdots & \vdots & & \vdots \\ k-1 & (k-1)^2 & & (k-1)^{k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$= |A|$$

non-zero

$\therefore$

$\Rightarrow c_1 = 0 = c_2 = \cdots c_{k-1}$

$\Rightarrow \left\{ \{\alpha^n\}_{n=0}^{\infty}, \{n\alpha^n\}_{n=0}^{\infty} \cdots \{n^{k-1}\alpha^n\}_{n=0}^{\infty} \right\}$ is linearly independent

claim: $\{n^p \alpha^n\}_{n=0}^{\infty} \in \operatorname{Ker}\left((R-\alpha I)^k\right) \quad p = 0, 1, 2 \ldots k-1$

$$\sum_{S \subset [n]} (-1)^{|S|} = 0$$

$$\sum_{S \subset [n]} (-1)^{|S|} |S| = \sum_{i=0}^{n} \binom{n}{i} (-1)^i i = \sum_{i=0}^{n} \frac{n!}{i!(n-i)!} (-1)^i i$$

$$= \sum_{i=1}^{n} \frac{n!}{i!(n-i)!} (-1)^i i$$

$$= n \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!(n-i)!} (-1)^i$$

$$= n \sum_{i-1=0}^{n-1} \binom{n-1}{i-1} (-1)^i$$

$$= n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j = 0$$

(b) $\Rightarrow \sum_{S \subseteq [k]} (k+n-|S|)(-1)^{|S|} = 0 \quad \forall$ integer $k \geq 0$

(c) $\Rightarrow \sum_{S \subseteq [k]} (k+n-|S|)^t (-1)^{|S|} = 0 \quad \forall$ integer $k \geq 0$
$\forall$ integer $t \geq 1$

Claim 2: $\forall$ integer $t \in \{0, 1, \ldots, k-1\}$
$\{n^t \alpha^n\}_{n=0}^{\infty} \in \ker((R-\alpha I)^k)$

Proof of claim: Note that

$$\begin{tikzcd} \{n^t \alpha^n\}_{n=0}^{\infty} \arrow[r, "R"] & \{(1+n)^t \alpha^{n+1}\}_{n=0}^{\infty} \arrow[d, "R"] \\ \{(k+n)^t \alpha^{k+n}\}_{n=0}^{\infty} & \cdots \{(2+n)^t \alpha^{2+n}\}_{n=0}^{\infty} \arrow[l, "R^k"'] \end{tikzcd}$$

$$\begin{tikzcd} \{n^t \alpha^n\}_{n=0}^{\infty} \arrow[r, mapsto, "\alpha I"] & \{n^t \alpha^{1+n}\}_{n=0}^{\infty} \arrow[r, mapsto, "\alpha I"] & \{n^t \alpha^{2+n}\}, \arrow[dl] \\ & \{n^t \alpha^{k+n}\}_{n=0}^{\infty} & \end{tikzcd}$$

with the last arrow labelled $(\alpha I)^k$.

$$\Rightarrow \{n^t \alpha^n\}_{n=0}^{\infty} \xmapsto{R-\alpha I} \{[(1+n)^t - n^t]\alpha^{1+n}\}_{n=0}^{\infty}$$

$$\downarrow R-\alpha I$$

$$\{[(2+n)^t - 2(1+n)^t + n^t]\alpha^{2+n}\}_{n=0}^{\infty}$$

k times

$$\xrightarrow{R-\alpha I} \left\{\left[\sum_{i=0}^{k} \binom{k}{k-i} (k+n-i)^t (-1)^i\right] \alpha^{k+n}\right.$$

$$= \left\{\left[\sum_{S \subseteq [k]} (k+n-|S|)^t (-1)^{|S|}\right] \alpha^{k+n}\right\}_{n=0}^{\infty}$$

$$= \{0_n (=0)\}_{n=0}^{\infty}$$

Hence the claim is established and

$$\langle \{\alpha^n\}_{n=0}^{\infty}, \{n\alpha^n\}_{n=0}^{\infty}, \dots \{n^{k-1}\alpha^n\}_{n=0}^{\infty} \rangle \ \mathrm{Ker}((R-\alpha I)^k)$$

$$(R-2I)^3(\{x_n\}_{n=0}^{\infty}) = \{0_n(=0)\}_{n=0}^{\infty}$$

Here $\langle \{2^n\}_{n=0}^{\infty}, \{n2^n\}_{n=0}^{\infty}, \{n^2 2^n\}_{n=0}^{\infty} \rangle = \mathrm{Ker}((R-2I)^3)$

Hence the general soln of such recurrence relation

$$x_n = C_0 2^n + C_1 n 2^n + C_2 n^2 2^n = (C_0 + C_1 n + C_2 n^2) 2^n$$

where, $C_1, C_2, C_3 \in \mathbb{C}$

* Finding Unique soln, plugin x_0, x_1, x_2 to determine C_0, C_1, C_2

We prove

$\forall$ integers $n \geq 0$, $k \geq 1$ and $t \geq 1$

$$\sum_{S \subset [k]} (k+n-|S|)^t (-1)^{|S|} = 0$$

We prove ⓒ using induction on 't'. Note that ⓑ is the "Base case". To see the "step case" we compute

$$= \sum_{S \subset [k]} (k+n-|S|)^t (-1)^{|S|}$$

$$= \sum_{S \subset [k]} (k+n-|S|)^{t-1} (k+n-|S|) (-1)^{|S|}$$

$$= \sum_{S \subset [k]} (k+n-|S|)^{t-1} (k+n)(-1)^{|S|} - \sum_{S \subset [k]} (k+n-|S|)^{t-1} |S| (-1)^{|S|}$$

$$\Downarrow$$

$$k+n \sum_{S \subset [k]} (k+n-|S|)^{t-1} (-1)^{|S|}$$

Induction hypothesis : $\forall \alpha \in \{1, 2, \dots t-1\}$

$\forall k \geq 1, \forall n \geq 0$ $\displaystyle\sum_{S \subset [k]} (k+n-|S|)^{\alpha} (-1)^{|S|} = 0$

$$= -\sum_{S \subset [k]} (k+n-|S|)^{t-1} |S| (-1)^{|S|}$$

$$= \sum_{S \subset [k]} (k+n-|S|)^{t-1} |S| (-1)^{|S|-1}$$

$$= \sum_{i=0}^{k} (k+n-i)^{t-1} \binom{k}{i} i (-1)^{i-1} = \sum_{i=1}^{k} (k+n-i)^{t-1} \frac{k!}{i!(k-i)!} i (-1)^{i-1}$$

$$= k \sum_{i=1}^{k} (k+n-i)^{t-1} \frac{(k-1)!}{(i-1)!(k-i)!} (-1)^{i-1}$$

$$= k \sum_{i-1=0}^{k-1} (k-1+n-(i-1))^{t-1} \frac{(k-1)!}{(i-1)!(k-1-(i-1))!} (-1)^{i-1}$$

$$= k \sum_{j=0}^{k-1} (k-1+n-j)^{t-1} \binom{k-1}{j} (-1)^{j}$$

$= k \cdot 0$ (By induction Hypothesis)

Th: let $\alpha_1, \alpha_2 \ldots \alpha_K$ be non-zero distinct complex number $m_1, m_2 \ldots m_K$ be positive integers.

Then, $\ker\left((R-\alpha_1 I)^{m_1}(R-\alpha_2 I)^{m_2} \cdots (R-\alpha_k I)^{m_K}\right)$

$= \langle S_1, S_2 \ldots S_K \rangle$

$$S_i = \left\{ \{\alpha_i^n\}_{n=0}^{\infty}, \{n\alpha_i^n\}_{n=0}^{\infty} \cdots \{n^{m_i-1}\alpha_i^n\}_{n=0}^{\infty} \right\}$$

$\forall i \in [K]$

<u>Remark</u>: $(R-2I)^2(R-3I)^3\left(\{x_n\}_{n=0}^{\infty}\right) = \{0_n (=0)\}_{n=0}^{\infty}$

$\ker\left((R-2I)^2(R-3I)^3\right) = \langle S_2\ S_3 \rangle$

$\ker\left((R-2I)^2(R-3I)^3\right) = \langle \{2^n\}_{n=0}^{\infty} \{n2^n\}_{n=0}^{\infty} \{3^n\}_{n=0}^{\infty} \{n3^n\}_{n=0}^{\infty}$
$\{n^2 3^n\}_{n=0}^{\infty} \rangle$

general solution

$x_n = c_0 2^n + c_1(n)2^n + c_2 3^n + c_3(n)3^n + c_4(n^2)3^n$

$= (c_0 + c_1 n)2^n + (c_2 + c_3 n + c_4 n^2)3^n$

where $c_0, c_1, c_2, c_3, c_4 \in \mathbb{C}$

$T: H \longrightarrow H$

$\ker T = \{x \in H : T(x) = 0\}$

Find a soln of $T(x) = b$

if y is a soln i.e $T(y) = b$

Then $T(y+\alpha) = T(y) + T(\alpha)$

$= b + 0 = b$

then $y+\alpha$ is a soln

$\forall \alpha \in \ker T$, $y+\alpha$ is a soln of $T(x) = b$

Problem : solve the recurrence reln

$$\underbrace{(R-2I)^2}_{T}\Big(\underbrace{\{x_n\}_0^\infty}_{x}\Big) = \underbrace{\{\alpha_n (= 3^n + 2n)\}_{n=0}^\infty}_{b}$$ with the initial condn x_0, x_1 — (*)

Soln Step-1 : we need to solve (formation of general soln)

$$(R-2I)^2(\{x_n\}_{n=0}^\infty) = \{(0_n = 0)\}_{n=0}^\infty \quad \text{— (**)}$$

The general soln $= (c_0 + c_1 n)2^n$, where $c_0, c_1 \in \mathbb{C}$

Step 2 : formation of particular soln / trial soln

let $\{y_n\}_{n=0}^\infty$ be a particular soln i.e.

$$(R-2I)^2(\{y_n\}_{n=0}^\infty) = \{3^n + 2n\}_{n=0}^\infty$$

suppose $y_n = \underbrace{d_1 3^n + d_2 n + d_3}_{\text{Trial soln}}$ $\forall$ integer $n \geq 0$

Then we have

$$3^n + 2n = y_{2+n} - 4y_{1+n} + 4y_n$$

$$3^n + 2n = \left(d_1 3^{2+n} + d_2(2+n) + d_3\right) - 4\left(d_1 3^{1+n} + d_2(1+n) + d_3\right)$$
$$+ 4(d_1 3^n + d_2 n + d_3$$
$$= d_1 3^n + d_2 n + (d_3 - 2d_2)$$

<u>step-3</u> : Formation of particular soln of ⊛ equation
Hence equating the coefficients the respective coefficients

$$d_1 = 1$$
$$d_2 = 2$$

$$d_3 - 2d_2 = 0 \Rightarrow d_3 = 2d_2 = 4$$

Hence, $y_n = 3^n + 2n + 4$ forms a particular soln of ⊛

<u>Step-4</u> : Formation of general sol^n of ⊛

$$x_n = (c_0 + c_1 n)2^n + (3^n + 2n + 4)$$

where, $c_0, c_1 \in \mathbb{C}$

Final: Using the initial condn x_0, x_1 find value of c_0, c_1

Here $c_0 = x_0 - 5$

$$c_1 = \frac{x_1 - 2x_0 + 1}{2}$$

□

Def^n : A k-term recurrence relation is said to be linear - non homogeneous is of the form

$$x_{n+k} = \alpha_n + \sum_{i=1}^{k} c_i x_{n+k-i} \quad \forall \text{ integer } n \geq 0 \quad (*)$$

where $\{\alpha_n\}_{n=0}^{\infty} \in S(\mathbb{C})$, $c_i \in \mathbb{C}$ $\forall i \in [k]$ & $x_0, x_1 \ldots x_{k-1} \in \mathbb{C}$

check that (*) can be expressed as

$$P(R)\left(\{x_n\}_{n=0}^{\infty}\right) = \{\alpha_n\}_{n=0}^{\infty}$$

where $p(R) = R^k - \sum_{i=1}^{k} c_i R^{k-i}$

Given $\{\alpha_n\}_{n=0}^{\infty}$ $k \geq 0$ is an integer and $c \in \mathbb{C}$	TRIAL SOL^n
$\alpha_n = cn^k r^n$ $\quad P(r) = 0$ → root with multiplicity 1 of $P(x)$ Eg (a) nr^n (b) $n^2 r^n$	$(a_0 + a_1 n + \ldots a_k n^k) r^n$ $\quad a_0, a_1 \ldots a_k \in \mathbb{C}$ (a) $(a_0 + a_1 n) r^n$ (b) $(a_0 + a_1 n + a_2 n^2) r^n$
$\alpha_n = cn^k r^n$ If root r has multiplicity of m $\quad (R - rI)^m$	$(a_0 + a_1 n + \ldots + a_n n^k) n^m r^n$
$\alpha_n = \begin{cases} cr^n \cos\beta n \\ cr^n \sin\beta n \end{cases}$	$(a\cos\beta n + b\sin\beta n) r^n$ $\quad a, b \in \mathbb{R}$

STIRLING'S NUMBER OF FIRST KIND

Defn. Let $n \geq 1$ and $k \geq 1$ be positive integers with $n \geq k$

A cycle of length l is a permutation $\pi : [n] \xrightarrow[\text{onto}]{1-1} [n]$ such that there are l integer $i_1, i_2, i_3 \cdots i_l \in [n]$

s.t.
$$\pi(i_1) = i_2$$
$$\pi(i_2) = i_3$$
$$\vdots$$
$$\pi(i_{l-1}) = i_l$$
$$\pi(i_l) = i_1$$

and $\forall j \in [n] \setminus \{i_1, i_2 \cdots i_l\}$ $\pi(j) = j$

we denote such cycle of length l by $(i_1, i_2 \cdots i_l)$

we call two cycle $(i_1 \cdots i_l)$ $(j_1 j_2 \cdots j_m)$ are disjoint if the corresponding sets $\{i_1, i_2 \cdots i_l\}$ are $\{j_1 \cdots j_m\}$ are disjoint with each other

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Th (cycle Decomposition Theorem) : For each permutation $\pi : [n] \xrightarrow[\text{onto}]{1-1} [n]$ there exist a positive integer $k = k(\pi)$ (i.e.) k depends on π s.t. π can be written as a product of k many disjoint cycle of ~~different~~ respective lengths

may be same

They have same length

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 3 & 2 & 7 & 6 & 5 & 1 \end{pmatrix} = \underbrace{(1\,4\,7)(2\,3)(5\,6)}_{k=3}$$

Proof : We construct an equivalence 'relation' ~~on~~ on over $[n]$ by relating $i \sim j \iff \exists\ \alpha \in \mathbb{Z}$ s.t $\pi^{\alpha}(i) = j$

Since, $[n]$ is a finite set, $\exists$ 1 positive integer $k \geq 1$ s.t.

$[n] = A_1 \sqcup A_2 \sqcup \dots \sqcup A_k$ where, $\forall \beta \in [k]$, A_β is an equivalence class

$$A_\beta = \{i_1, \dots i_l\}$$
$$= \{i_1, \pi(i_1), \pi^2(i_1) \dots \pi^{l-1}(i_1)\}$$

Defn :- The stirling no. of first kind is defined by

$$s_k'(n) = s(n,k) = (-1)^{n-k} c(n,k)$$

$c(n,k)$ denotes the total number of permutations of $[n]$, which can be written as product of k disjoint cycles of respective length

$$c(n,k) = \left| \left\{ \pi : [n] \xrightarrow[\text{onto}]{1-1} [n] \;\middle|\; \begin{array}{l} \pi \text{ can be written as a} \\ \text{product of } k \text{ many disjoint cycles} \\ \text{(of respective length} \end{array} \right\} \right|$$

Recall (i) is a cycle of length 1

Th: Let n & k be positive integers with $n \geq k$ then,

$$c(n,k) = c(n-1, k-1) + (n-1)c(n-1, k)$$

Proof : Exercise

PIGEON HOLE PRINCIPLE

Th (PHP-Ver-I) if $(n+1)$ many objects (balls) are distributed "among" n many boxes, where n is a positive integer then $\exists$ at least one box which contains TWO or MORE objects (balls)

Proof: Suppose (if possible) none of the n many boxes contains two or more objects (balls) i.e each of them contains at most 1 ball (object)

Then there are at most n many objects (balls)

However, we are distributing $(n+1)$ many objects (balls)

$\Rightarrow\Leftarrow$ contradiction

Th (PHP-Ver-II) - if $(mn+1)$ many objects (balls) are distributed among n many boxes where m, n are positive integers, then $\exists$ at least one box, which contains at least $(m+1)$ many objects (balls)

Proof:

Th (PHP - strong - version) : if $m_1 + m_2 + \dots + m_n - n + 1$ many objects distributed among n labelled boxes with labelling $1, 2, 3, \dots, n$ where $m_1, \dots, m_n$ are positive integers then $\exists$ at least one label $i \in [n]$ such that box with label i contains at least m_i object (balls)

Proof: Ex.

Comment: "objects" "(balls)" are "pigeons" and boxes are "pigeonholes".

APP-I Let m, n be positive integers with $\gcd(m, n) = 1$
Let a, b be two integers with $0 \le b \le n-1$, $0 \le a \le m-1$

Then $\exists$ positive integer x satisfies

$$x \equiv a \bmod m$$
$$x \equiv b \bmod n$$

Soln: We apply PHP.

BALL CONSTRUCTION : We construct the following n nos.

$$a, a+m, a+2m, \dots, a+(n-1)m$$

Let $a = r_1 \bmod n$, $a+m = r_2 \bmod n$ $\dots$ $a+(n-1)m = r_n \bmod n$

claim : if $a + im = r_{i+1} \bmod n$ $\forall i \in \{0, \dots, n-1\}$ then $i+1, j+1 \in [n]$, $r_{i+1} \neq r_{j+1}$

Proof : Suppose $\exists$ $i+1, j+1 \in [n]$ with $i+1 < j+1$ s.t.

$$r_{i+1} = r_{j+1}$$

$$\Rightarrow n \mid r_{j+1} - r_{i+1}$$

$$\Rightarrow n \mid (a+jm) - (a+im) \Rightarrow n \mid (j-i)m \Rightarrow n \mid j-i$$

$\gcd(m, n)$

$\Rightarrow$ ~~Rara~~ $\cancel{\rightarrow}$ $i-j=0 \Rightarrow j=1$

$\Rightarrow\Leftarrow$ because $i+1 < j+1$

* Required $(n+1)$ many balls are $r_1, r_2 \ldots r_n, b$

Box Construction : Required n many boxes are

$0, 1, 2 \ldots n-1$

which are all the remainder of modulo n

Note $\{0, 1 \ldots n-1\} = \{r_1 \ldots r_n\}$

Hence $\exists \, K \in [n]$ such that

$r_K \equiv b \bmod n$

i.e. $a + (K-1)m \equiv b \bmod n$

Required soln for x is $a+(K-1)m$

$a + (K-1)m \equiv a \bmod m$

$a + (K-1)m \equiv b \bmod n$

This is chinese Remainder Theorem

BALL CONSTRUCTION $\longleftrightarrow$ sample space

BOX CONSTRUCTION $\longleftrightarrow$ Random variable

Q: What is sample space and How it is constructed?

Ans: Performing a Random experiment

RE: choose an element from A $\quad |A| \geq 2$

$\Omega = A$

sample space is set of all outputs of the RE

Def^n: Let Ω be a sample space.

The set of subsets of Ω, say $\mathcal{F}$ is called σ-algebra (sigma) or informations if $\mathcal{F}$ satisfies the following properties

(a) $\Omega \in \mathcal{F}$ $\quad \mathcal{F} \subset 2^{\Omega}$

(b) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$

(c) if $\forall$ integer $n \geq 1$, $A_n \in \mathcal{F}$, then $\bigcup_{n=1}^{n=\infty} A_n \in \mathcal{F}$

Ex: (i) $\mathcal{F} = 2^{\Omega}$

(ii) $\mathcal{F} = \{\emptyset, A, A^c, \Omega\}$

Def^n: A fn $m: \mathcal{F} \longrightarrow [0, \infty]$ is called measure if it satisfies the following properties

(a) $m(\emptyset) = 0$

(b) $m\left(\bigsqcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$

$$\{m(A_1) + \dots + m(A_n)\}_{n=1}^{\infty}$$

The pair $(\Omega, \mathcal{F})$ is called measurable space and the triplet $(\Omega, \mathcal{F}, m)$ is called the measure space

<u>Defn</u> : Let $(\Omega, \mathcal{F})$ be a measurable space and $P_r : \mathcal{F} \to [0,\infty]$ be a measure

such measure is called probability measure if

$$P_r(\Omega) = 1$$

The triplet $(\Omega, \mathcal{F}, P_r)$ is called the probability space

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Th (PHP - Dual Version) if $(n-1)$ - many objects (balls) are distributed among n many boxes, then $\exists$ a empty box (i.e., $\exists$ a box which contains no ball)

Proof: Exercise

Ex: RE: choose a point from $[n]$

This yields the sample space $\Omega = [n]$ $\quad |\Omega| < \infty$

$$\mathcal{F} = \text{The } \sigma\text{-algebra} = 2^{\Omega} = \langle \{\{x\} : x \in \Omega\} \rangle$$

$$P_r : \mathcal{F} \longrightarrow [0,1]$$

$$P_r(\{x\}) \triangleq \frac{1}{n} \quad \forall x \in \Omega$$

$$= \frac{1}{|\Omega|}$$

$$P_r(A) = P_r\left(\bigsqcup_{x\in A} \{x\}\right) = \sum_{x \in A} P_r(\{x\}) = \sum_{x \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}$$

$(\Omega, \mathcal{F}, P_r)$ is a probability space

$\Omega = \{2,3,4\}$

$$\left.\begin{aligned} P(\{2\}) &= \tfrac{1}{3} \\ P(\{3\}) &= \tfrac{1}{3} \\ P(\{4\}) &= \tfrac{1}{3} \end{aligned}\right\} \text{Uniform distribution} \qquad \left.\begin{aligned} P_r'(\{2\}) &= 0 \\ P_r'(\{3\}) &= \tfrac{2}{3} \\ P_r'(\{4\}) &= \tfrac{1}{3} \end{aligned}\right\} \text{Non-uniform distribution}$$

Def^n: Let $(\Omega, \mathcal{F}, P_r)$ be a probability space

A fn^x $X : \Omega \longrightarrow \mathbb{R}$ is called discreate random variable if $X(\Omega)$ is a countable subset of $\mathbb{R}$ and $\forall x \in X(\Omega)$, $X^{-1}(\{x\}) \in \mathcal{F}$

Ex: ⓐ $X : \Omega \longrightarrow \{0,1\}$, $A \subset \Omega$

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{else} \end{cases}$$

$X(\Omega) = \{0,1\}$

$X^{-1}(\{0\}) = \{\omega \in \Omega : X(\omega) = 0\}$

$X^{-1}(\{0\}) = A^c \in \mathcal{F}$

$X^{-1}(\{1\}) = A \in \mathcal{F}$

$$P_r(\{\omega \in \Omega : X(\omega) = 1\}) = P_r(X^{-1}(\{1\})) = P_r(A) = \frac{|A|}{n} = P_r(X=1)$$

$$P_r(\{\omega \in \Omega : X(\omega) = 0\}) = P_r(X^{-1}(\{0\})) = P_r(A^c) = \frac{|A^c|}{n} = P_r(X=0)$$

ⓑ $Y : \Omega \longrightarrow \{-1,1\}$

$$Y(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ -1 & \text{else} \end{cases}$$

$P_r(Y=1) = \frac{|A|}{n}$

$P_r(Y=-1) = \frac{|A^c|}{n}$

Defn : Let $(\Omega, \mathcal{F}, P_r)$ be a probability space and $X: \Omega \to \mathbb{R}$ be a discreate random variable

Note that $|X|: \Omega \to \mathbb{R}$ is also a discrete random variable

$|X|(\omega) = (|X(\omega)|) \ \forall \omega \in \Omega$

$$E[X] = \sum_{k=1}^{\infty} x_k P_r(X = x_k) = \sum_{x \in X(\Omega)} x P_r(X = x) \in \mathbb{R}, \{\infty, -\infty\}$$

$$E[|X|] = \sum_{k=1}^{\infty} |x_k| P_r(X = x_k) < \infty$$

RE: choose a labelled box with labelling $1, 2, \ldots n$

Ω = The set of labelled boxes = $\{1, 2, 3 \ldots n\} = [n]$

$\mathcal{F} = 2^{\Omega}$

$Pr(\{\omega\}) = \frac{1}{|\Omega|} = \frac{1}{n} \quad \forall \omega \in \Omega$

(BOX CONSTRUCTION)

Here we construct probability space $(\Omega, \mathcal{F}, Pr)$

B = set of balls, $|B| = n-1$

[Here we "distribute" $(n-1)$ many balls into the boxes]

we construct $X : \Omega \to \mathbb{R}$

$X(\omega)$ = No. of balls in the labelled box ω

$X(\Omega) \subset \{0, 1, \ldots n-1\}$ $\qquad X(\Omega) = \{X(\omega) : \omega \in \Omega\}$

BALL Distribution construction

AIM: To compute $E[X]$

Let $b \in B$, $X_b : \Omega \to \{0, 1\}$

$$X_b(\omega) = \begin{cases} 1 & \text{if } b \in \omega \\ 0 & \text{else} \end{cases}$$

$$Pr(X_b = 1) = Pr(\{\omega \in \Omega : X_b(\omega) = 1\}) = \frac{1}{n}$$

$$X = \sum_{b \in B} X_b$$

i.e. $X(\omega) = \sum_{b \in B} X_b(\omega) \quad \forall \omega \in \Omega$

Hence, $E[X] = E\left(\sum_{b \in B} X_b\right) = \sum_{b \in B} E[X_b] = \sum_{b \in B} Pr(X_b = 1)$

$$= \sum_{b \in B} \frac{1}{n}$$

$$= \frac{1}{n} \sum_{b \in B} 1 = \frac{1}{n} |B| = \frac{|B|}{n}$$

$$= \frac{n-1}{n} = 1 - \frac{1}{n} < 1$$

First Moment method : if $E[X] < 1$, then $Pr(X=0) > 0$

Using Markov's inequality, we have

$$Pr(X \geq 1) \leq E[X] < 1$$

we know $Pr(X=0) + P(X \geq 1) = 1$

Hence, $Pr(X=0) > 0$

i.e $Pr(\{\omega \in \Omega : X(\omega) = 0\}) > 0$

$\exists\ \omega \in \Omega$ s.t. $X(\omega) = 0$ i.e. $\forall b \in B$, $X_b(\omega) = 0$

In other words $\exists$ an empty box

Conclusion of PHP

RE: choose a permutation $\pi : [n] \longrightarrow [n]$.

Ω : The set of all permutations

$$\{\pi \mid \pi : [n] \xrightarrow[\text{onto}]{1-1} [n]\} \qquad |\Omega| = n!$$

$\mathcal{F} = 2^{\Omega}$

$Pr(\{\pi\}) = \frac{1}{n!}$

Here we construct the probabity space $(\Omega, \mathcal{F}, Pr)$

Let $\forall\ k \in [n]$ BALLS

$$X_k : \Omega \rightarrow \mathbb{R}$$

$$X_k(\pi) = \begin{cases} 1 & \text{if } \pi(k) = k \\ 0 & \text{else} \end{cases}$$

$$Pr(X_k = 1) = Pr(\{\pi \in \Omega \mid X_k(\pi) = 1\})$$
$$= \frac{(n-1)!}{n!} = \frac{1}{n}$$

We construct $X : \Omega \to \mathbb{R}$

$$X(\pi) = \sum_{k \in [n]} X_k(\pi) \qquad \forall \pi \in \Omega$$

i.e. $X = X_1 + X_2 + \dots + X_n$

Hence $E[X] = E[X_1 + X_2 + X_3 + \dots X_n] = E[X_1] + E[X_2] + \dots + E[X_n]$

$$= \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}$$
$$= 1$$

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$$

$$X(\pi) = X_1(\pi) + X_2(\pi) + \dots X_n(\pi) = n$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 1 & 3 & \dots & n \end{pmatrix} \Rightarrow X(\sigma) = n-2$$

Hence, X is a non-constant random variable

Also, $X(\Omega) \subset \{0, 1, \dots n\}$

Since, $E[X] = 1$ & $X(\Omega) \subset \{0, 1, \dots n\}$ & X is non-constant $\exists \pi \in \Omega$ s.t. $X(\pi) = 0$

$$\Leftrightarrow X_1(\pi) = 0 = X_2(\pi) = \dots X_n(\pi)$$

Hence $\exists \pi \in \Omega$ s.t. $\forall k \in [n]$, $\pi(k) \neq k$

Existence of Derangement Permutation