

$$[n] = \{1, 2, 3, \dots, n\}$$

Definition: An one-to-one and onto ~~fn~~ f_n

$\pi: [n] \rightarrow [n]$ is called permutation f_n a simply Permutation

S_n : The set of all permutations from $[n] \rightarrow [n]$

$$= \left\{ \pi \mid \pi: [n] \xrightarrow[\text{onto}]{} [n] \right\}$$

→ Symmetric set of n symbols

Let an urn contains N many ~~but~~ labelled balls.

Task: choosing n many labelled ball from the urn (Ω)

Impossible if $n > N$ & possible if $1 \leq n \leq N$

2 Styles of choices -

A Choose one ball at a time and repeat it for n -times

B Choose n many balls in one grip

$$\{E \subset \Omega : |E| = n\}$$

A. Style of Choosing

Additional : observe the label of the ball at each step

$$(\omega_1, \omega_2, \omega_3, \dots, \omega_n)$$

Addition : You observe the label and put back in \mathcal{S}

$$\left\{ (\omega_1, \omega_2, \dots, \omega_n) : \omega_k \in \mathcal{S} \quad (k=1, 2, 3, \dots) \right\}$$

$$= \mathcal{S}^n$$

Alternatively : Observe the label and do not put it back to the box

$$\left\{ (\omega_1, \omega_2, \omega_3, \dots, \omega_n) : \omega_1 \in \mathcal{S}, \omega_2 \in \mathcal{S} / \{\omega_1\}, \omega_3 \in \mathcal{S} / \{\omega_1, \omega_2\}, \dots \right\}$$

$$\left\{ (\omega_1, \omega_2, \dots, \omega_n) : \begin{array}{l} \forall i \in [n], \omega_i \in \mathcal{S} \\ \forall i, j \in [n], \text{ with } i \neq j \quad \omega_i \neq \omega_j \end{array} \right\}$$

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~ Himanshu

But this missing Part is taken from Karan's Notes

Discrete Math

Aim: We are given a sequence of finite sets $\{A_n\}_{n=1}^{\infty}$ and wish to count the number of elements of each A_n .

$$1 \mapsto A_1, \quad \underbrace{\text{count}}_{f(1)}$$

$$2 \mapsto A_2, \quad \underbrace{\text{count}}_{f(2)}$$

$$n \mapsto A_n \rightarrow f(n)$$

Let $f(n)$ denote the number of elements of A_n \forall integer $n \geq 1$.

$$\text{i.e. } f(n) := |A_n|, \forall n \geq 1, n \in \mathbb{Z}$$

The problem is to determine $f(n)$

Examples

① For each positive integer $n \geq 1$, let $f(n)$ denote the no. of subsets of $[n] = \{1, 2, \dots, n\}$.

$$\text{Here } f(n) = 2^n \quad \forall \text{ integer } n \geq 1$$

② For each $n \geq 1$, let $f(n)$ denote the no. of n -tuple consisting of either zero or one.

$$f(n) = |\{(x_1, x_2, \dots, x_n) \mid \forall i \in [n], x_i \in \{0, 1\}\}| \\ = 2^n$$

$[n]$ is a set $[n] := \{1, 2, \dots, n\}$

B. Suppose there are n many letters and n envelopes.

Random experiment: Put a letter into a envelop.

Let $f(n)$ denote the number of ways that one letter put into one envelop so that no letters receives its own envelop.

$$\text{Here } f(n) = n! \sum_{k=0}^n (-1)^k \frac{1}{k!} \quad \forall \text{ integer } n \geq 1$$

↳ Derangement problem

C. Let $f(n)$ denote the numbers of $n \times n$ matrices over \mathbb{C} with zero or one entries.

Def': An $n \times n$ matrix over \mathbb{C} is a function

$$f : [n] \times [n] \rightarrow \mathbb{C}$$

$$[a_{ij}]_{i,j \in [n]}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & \ddots & \ddots & a_{2n} \\ a_{n1} & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}$$

$$\text{Here } f(n) = 2^{n^2}$$

E. Example: Count the number of $n \times n$ matrices over \mathbb{C} with zero one entries so that

(a) Each row consists of 3 ones

(b) Each row and each column consists of 3 ones.

Rule 0 :- Check if the given set is empty i.e. can you construct one element?

$$f(n) = |\{M : M = [a_{ij}]_{i,j \in [n]}, a_{ij} \in \{0,1\}\}| \\ = 2^n$$

D A recurrence for $f(n)$ may be given in terms of $f(1)$ or $f(2)$, or \dots or $f(n-1)$.

E2 Let $f(n)$ denote the no. of subsets of $[n]$ that do not contain consecutive integers

$$f(1) = 2 \quad \emptyset, \{1\}$$

$$f(2) = 3 \quad \emptyset, \{1\}, \{2\}$$

$$f(3) = 5 \quad \emptyset, \{1\}, \{2\}, \{1,3\}$$

Claim :- $f(n) = f(n-1) + f(n-2)$ A integer

$$\text{Hence } f(n) = \frac{1}{\sqrt{5}} \left(T^{n+2} - \frac{1}{T^{n+2}} \right) \text{ A integer } n \geq 1 \quad n \geq 3.$$

$$\text{where } T = \frac{1 + \sqrt{5}}{2}$$

E Let $f(n)$ count of some set A_n i.e. $f(n) = |A_n|$. The function $g(n)$ is called an estimate of $f(n)$ if

$$\lim_{n \rightarrow \infty} \frac{f(n)}{g(n)} = 1$$

E2 $g(n) = \begin{cases} \text{Polynomial} & \text{eg, } n, n^2, n^3, \dots \\ \text{Exponential} & 2^n, 3^n, e^n, n^n \end{cases}$

Oscillatory cases : $\sin \frac{n\pi}{2}, \cos \frac{n\pi}{2}$

$\alpha G R, e^{\frac{i\pi n}{2}}$

Stirling approximation (Eg's of asymptotics)

$$\text{Eg: } n! = n(n-1) \cdots 1 \sim \sqrt{2\pi} n^{(n+\frac{1}{2})} e^{-n}$$

$$\lim_{n \rightarrow \infty} \frac{n!}{\sqrt{2\pi} n^{(n+\frac{1}{2})} e^{-n}} = 1$$

For many cases estimation or asymptotic form of $f(n)$ for $f(n)$

E To understand methods of evaluating $f(n)$ is to supply its generating function.

$$\sum_{n=1}^{\infty} f(n) x^n$$

$$\sum_{n=1}^{\infty} f(n) \frac{x^n}{n!}$$

Exponential

"Formal Way": Ignore the convergent and divergent issues

$$[n] = \{1, 2, \dots, n\}$$

Definition: An one-to-one and onto function $\pi: [n] \rightarrow [n]$ is called permutation function or simply permutation.

Eg:- $n = 3$

$$\pi : [3] \rightarrow [3]$$

~~π =~~

$$\pi(1) = 2$$

$$\pi(2) = 3$$

$$\pi(3) = 1$$

$$\text{or } \pi = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

$$\pi : [n] \rightarrow [n]$$

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ \pi(1) & \pi(2) & \dots & \pi(n) \end{pmatrix}$$

S_n = The set of all permutations from $[n]$ to $[n]$

$$= \{ \pi \mid \pi : [n] \xrightarrow{\text{onto}} [n] \}$$

Symmetric set of n symbols.

Task: How many elements are there in S_n .

$$\text{i.e. } |S_n| = 2$$

2

(2)

Let an urn ~~contain~~ contain N many labelled balls.

Task: Choosing n many labelled balls from the urn (2)

Impossible if $n > N$ and ~~yes~~ possible if $1 \leq n \leq N$

2 ways to choose.

A. Choose one ball at a time and repeat it for n many times.

B. Choose n many balls in one grip.
 $\{E \subset \Omega : |E| = n\}$

A. Style of Choosing]

Additional: Observe the label of the ball at each step.

(w_1, w_2, \dots, w_n) .

Additional: You observe the label and put it back in the urn Ω .

$\{ (w_1, w_2, \dots, w_n) : w_i \in \Omega \forall i \in \mathbb{N} \}$

$= \Omega^n$.

Alternatively: Observe the label and don't put it back to Ω .

$\{ (w_1, w_2, w_3, \dots, w_n) : w_i \in \Omega, w_1 \in \Omega \setminus \{w_2, \dots, w_n\}, w_2 \in \Omega \setminus \{w_1, w_3, \dots, w_n\}, w_3 \in \Omega \setminus \{w_1, w_2, \dots, w_n\}, \dots, w_n \in \Omega \setminus \{w_1, w_2, \dots, w_{n-1}\} \}$

$\forall i, \forall i \in [n], w_i \in \Omega$

and

$\forall i, j \in [n], \text{ with } i \neq j \Rightarrow w_i \neq w_j$

$$= N(N-1) \dots (N-n+1)$$

$$= N!$$

$$(N-n)!$$

Absent:

what if $N=n$, $\rightarrow n!$

$\{(\omega_1, \omega_2, \dots, \omega_n) : \forall i, j \in [n], i \neq j \Rightarrow \omega_i \neq \omega_j\}$

Application :- Count the no. of Permutation of $[n]$ or

$$S_n = \{ \pi \mid \pi : [n] \xrightarrow[\text{onto}]{} [n] \}$$

$$|S_n| = ?$$

$$A \xrightarrow[\text{onto}]{1-1} B \Rightarrow |A| = |B|.$$

$$\text{Claim : } |S_n| = n!$$

Proof : Consider two sets : S_n and $\{(\omega_1, \omega_2, \dots, \omega_n) : \omega_i \in S\}$

$$B = \{(\omega_1, \omega_2, \dots, \omega_n) : \forall i, j \in [n] \text{ with}$$

$$i \neq j \Rightarrow \omega_i \neq \omega_j\}$$

Plan is to establish an one-one and onto function :-

$$\pi : S_n \rightarrow B$$

$$f : B \rightarrow S_n$$

$$f : (\omega_1, \omega_2, \dots, \omega_n) \rightarrow S_n$$

$\pi(1) = \text{label of } \omega_1$,

$\pi(2) = \text{label of } \omega_2$.

$\pi(n) = \text{label of } \omega_n$.

Claim :- f is one-one and onto.

Proof :- Let $\pi \in S_n$.

Choose $w_1 \in \Omega$ s.t. Label of $w_1 = \pi(1)$

Choose $w_2 \in \Omega$ s.t. Label of $w_2 = \pi(2)$

choose $w_n \in \Omega$ s.t. Label of $w_n = \pi(n)$

Thus $f(w_1, w_2, \dots, w_n) = \pi$

Since $\pi \in S_n$ has been chosen arbitrarily we have,

$\forall \pi \in S_n, \exists (w_1, w_2, \dots, w_n) \in B$ s.t.

$f(w_1, w_2, \dots, w_n) = \pi$ i.e. f is onto

If $f(w_1, w_2, \dots, w_n) = \pi = f(w'_1, w'_2, \dots, w'_n)$

Label of $w_1 = \pi(1) =$ Label of w'_1

Label of $w_n = \pi(n) =$ Label of w'_n

i.e. $(w_1, w_2, \dots, w_n) = (w'_1, \dots, w'_n)$

Hence Q.E.D.

Lemma 1 : If $\pi, \lambda \in S_n$, then $\pi \circ \lambda \in S_n$.

$$\begin{aligned}\pi \circ \lambda : [n] &\rightarrow [n] \\ \pi \circ \lambda(i) &= \pi(\lambda(i))\end{aligned}$$

$$\pi \circ \lambda(n) = \pi(\lambda(n))$$

$$\pi \circ \lambda \neq \lambda \circ \pi$$

\square might be true eg. find example
Just to show $\pi \circ \lambda$ is an $1-1$ and onto.

Lemma : $\forall \pi \in S_n, \exists$ unique $\lambda \in S_n$. $\lambda = \pi^{-1}$
s.t. $\pi \circ \lambda = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} = \lambda \circ \pi$.

Need to prove: $\pi \circ (\lambda \circ \mu) = (\pi \circ \lambda) \circ \mu$ $\left\{ \begin{array}{l} * \pi, \lambda, \mu \in S_n \end{array} \right.$

We can talk about,

$$\pi^1 = \pi \circ \pi = \pi^2$$

$$\pi \circ \pi \circ \pi = \pi^3$$

$$\begin{aligned}\pi^{-1} \\ \pi^{-1} \circ \pi^{-1} &= \pi^{-2} \\ \pi^{-1} \circ \pi^{-1} \circ \pi^{-1} &= \pi^{-3}\end{aligned}$$

$$\pi^k : k \in \mathbb{Z}$$

$$\pi^0 = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix} = i$$

We construct an equivalence relation \sim on $[n]$.

$$i \sim j \Leftrightarrow \exists k \in \mathbb{Z} \text{ s.t. } \pi^k(i) = j$$

Lemma: \sim is an equivalence relation on the set $[n]$

$${}^n C_{\gamma} \sim \binom{n}{\gamma}$$

$${}^n C_{\gamma} = \frac{n!}{(n-\gamma)! \gamma!}$$

$$= \frac{(n)(n-1)\cdots(n-\gamma+1)}{\gamma!}$$

Binomial and multinomial theorems

Defn:- Let X be an n -set ($i.e. |X| = n$)

Let $\binom{X}{k}$ denote the set of all k -subsets of X .

$$i.e. \binom{X}{k} = \{E \subset X : |E| = k\}$$

$$\text{we define } \binom{n}{k} = |\binom{X}{k}|$$

Here n is a positive integer :- i.e. $n \geq 1$

and $k \geq 0$ is an integer.

$$\text{Convention:- } \binom{X}{0} = \phi \quad |\phi| = 0.$$

Lemma : $\binom{n}{k}$ is a well defined integer

Proof :- Let X and Y be two n -sets i.e.

$$|X|=|Y|=n.$$

\exists an 1-1 and onto fn. $f: X \rightarrow Y$

Prove exercise. \leftarrow first step is to prove this

We construct (Lifting function)

$$f': \binom{X}{k} \rightarrow \binom{Y}{k}$$

$$f'(A) = f(A) \cap \{f(x) | x \in A\}$$

Claim: f' is 1-1 and onto.

Proof of claim' Let $B \in \binom{Y}{k}$.

$$\text{Then } A = f^{-1}(B) = \{a \in A : f(a) \in B\}$$

$$\text{Satisfy } f'(A) = B$$

i.e. f' is onto.

$$\text{Let } A_1, A_2 \in \binom{X}{k} \text{ s.t. } A_1 \neq A_2.$$

without loss of generality,

$$\exists \alpha \in A, \setminus A_2$$

$$f(\alpha) \notin f(A, \setminus A_2) = f(A_1) \cup f(A_2)$$

$$\text{i.e. } f'(A_1) \neq f'(A_2) \quad \text{and} \quad f(\alpha) \notin f(A_2)$$

Lemma 2 For each $A, B \in \binom{X}{k}$, \exists 1-1 and

onto $f: A \rightarrow B$ such that

$$\pi(A) = B$$

Proof :- Since $|A| = |B| = k$ and $|X - A| = |X - B| = n - k$.

\exists 1-1 and onto $f_n: f: A \rightarrow B$ and $g: X \setminus A \rightarrow X \setminus B$

Consequently, we construct the map $\pi: X \rightarrow X$

$$\pi(x) = \begin{cases} f(x) & \text{if } x \in A \\ g(x) & \text{if } x \in X \setminus A \end{cases}$$

Here $\pi(A) = B$ and π is 1-1 and onto. ($\forall x$)

Given $A \in \binom{X}{K}$, using lemma 2, we have

$$\{\pi(A) : \pi \in S_n(X)\} = \binom{X}{K}.$$

$$S_n(X) = \left\{ \pi \mid \pi : X \xrightarrow[\text{onto}]{} X \right\}$$

Task: Count the no. of permutations

$\lambda: X \rightarrow X$ such that $\lambda(A) = A$

Count the number of permutations $\lambda: X \rightarrow X$ which fixes/stabilizes A i.e.

$$\#\{\lambda \in S_n(X) : \lambda(\lambda A) = A\}$$

$$\pi \in S_n(X)$$

$$\pi \mapsto (\pi|_A, \pi|_{X-A}) \in S_n(A) \times S_n(X-A)$$

$$\lambda: \{\pi \in S_n(X) : \pi(A) = A\} \rightarrow S_n(A) \times S_n(X-A)$$

$$\lambda(\pi) = (\pi|_A, \pi|_{X-A})$$

λ is 1-1 and onto (E_n)

~~10~~

$$E_n(A) \cong (X \setminus A)^{(n)}$$

$$\begin{aligned} |\{ \pi \in S_n(x) : \pi(A) = A \}| &= |S_n(A) \times S_n(x-A)| \\ &= |S_n(A)| \cdot |S_n(x-A)| \\ &= k! (n-k)! \end{aligned}$$

We construct an equivalence relation \sim over the set $S_n(x)$. $\pi \sim \mu$, where $\pi, \mu \in S_n(x)$

$$\pi, \mu \in S_n(x) \iff \pi(A) = \mu(A)$$

An equivalent class,

$$\{ \pi_1, \dots, \pi_L \} \subset S_n(x)$$

$$\{ \} = \{ \pi \in S_n(x) : \pi(A) = \pi_1(A) = \dots = \pi_L(A) \}$$

means that

$$\frac{\pi(x-a)}{c} = \{ \pi \in S_n(x) : \pi(A) = \text{BE}_{\binom{x}{k}} \}$$

c could be anything

Two equivalent classes.

$$\{ \pi \in S_n(x) : \pi(A) = c \} \text{ where } c \in \binom{x}{k}$$

$$\{ \pi \in S_n(x) : \pi(A) = 0 \}$$

$$\text{Lemma 3: } |\{ \pi \in S_n(x) : \pi(A) = c \}|$$

$$= |\{ \pi \in S_n(x) : \pi(A) = 0 \}|$$

Proof. $\exists \mu \in S_n(x)$ s.t. $\mu(c) = d$

$$f_\mu : \{ \pi \in S_n(x) : \pi(A) = C \} \rightarrow \{$$

↓

$$\{ x \in S_n(x) : \lambda(A) = B \}$$

$$f_\mu(\pi) = \underbrace{\text{_____} \times \text{_____}}_{\in M_p(n)} \in \{ \lambda(A) = B \}$$

Then f_μ is 1-1 and onto ($\exists x$)

$$S_n(x) = \bigcup_{B \in \binom{x}{k}} \{ \pi \in S_n(x) : \pi(A) = B \}$$

$$|S_n(x)| = \sum_{B \in \binom{x}{k}} |\{ \pi \in S_n(x) : \pi(A) = B \}|$$

$$= \sum_{B \in \binom{x}{k}} |\{ \pi \in S_n(x) : \pi(A) = A \}|$$

$$\binom{x}{k}^{\text{no. of time}} = \sum \left| \{ \pi \in S_n(x) : \pi(A) = A \} \right|$$

$$\text{i.e. } n! = \left| \binom{x}{k} \right| k! (n-k)!$$

Hence $|(\underline{X})| = \frac{n!}{k!(n-k)!}$

i.e. $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Theorem: Given an n -set X and the set of all k -subsets of X , denoted as

$\binom{X}{k}$ where n, k are integers with $n \geq 1$

and $k \geq 0$.

We define $\binom{n}{k} = |(\underline{X})|$, then

$$\binom{n}{k} = \begin{cases} 0 & \text{if } k > n+1 \\ \frac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n \end{cases}$$

If $A = \{A\} \cap (\underline{X})$, then $|A| = 1$

$A = \{A\} \cap (\underline{X})$ for all $A \in (\underline{X})$

$(A \cap A) \cap A = A$

Here ends Karan's Notesand Himan

RE: Urn Ω contains N many labeled balls and we are drawing n many balls from it. Here $n \leq N$ & N is a positive integer. $k > 0$ is an integer.

Choice Style A3: We choose one ball and do not observe the label of the ball but observe all the n many balls at the end; and do not return it to Ω .

$$A = \left\{ (\omega_1, \omega_2, \dots, \omega_n) : \forall i \in [n], \omega_i \in \Omega \quad \forall i, j \in [n] \text{ with } i \neq j \quad \omega_i \neq \omega_j \right\}$$

$$(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n) \sim (\beta_1, \beta_2, \dots, \beta_n)$$



$$\exists \pi: [n] \rightarrow [n]$$

$$\text{s.t. } \alpha_1 = \beta_{\pi(1)}$$

$$\alpha_2 = \beta_{\pi(2)}$$

⋮

$$\alpha_n = \beta_{\pi(n)}$$

Equivalence relation \sim over A

$$(\alpha_1, \dots, \alpha_n) \sim (\beta_1, \dots, \beta_n) \iff \exists \pi: [n] \rightarrow [n] \quad \text{s.t.} \quad \alpha_i = \beta_{\pi(i)} \quad \forall i \in [n]$$

Example: Show that \sim is an equivalence relation on A

Let $A = A_1 \cup A_2 \cup A_3 \dots \cup A_L$

A_i forms an equivalent class wrt \sim $\forall i \in [n]$

Task: $L = ?$

A/\sim = The set formed by choosing exactly one element from each equivalent class A_i

= Residue or Quotient set of A wrt \sim

Example: equivalence relation on \mathbb{Z}

$$\alpha \sim \beta \Leftrightarrow 5 | \alpha - \beta$$

Lemma: $|A/\sim| = \left| \binom{\mathbb{Z}}{n} \right| = \binom{N}{n}$

Proof: Let $A = A_1 \cup A_2 \cup A_3 \dots \cup A_L$

$$A/\sim = \left\{ (w_1(1), w_2(1) \dots, w_n(1)), (w_1(2), w_2(2) \dots, w_n(2)) \dots, (w_1(L), w_2(L), \dots, w_n(L)) \right\}$$
$$= \left\{ (w_1(i), w_2(i) \dots, w_n(i)) : i \in [L] \right\}$$

We construct a mapping from A/\sim to $\binom{\mathbb{Z}}{n}$

$$(w_1(i), \dots, w_n(i)) \xrightarrow{f} \{w_1(i), \dots, w_n(i)\}$$

Claim f: $A/\sim \rightarrow \binom{\mathbb{Z}}{n}$ is 1-1 and onto

Proof of claim: Let $E \in \binom{\mathbb{Z}}{n}$ $E = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$.

we choose $i \in [L]$ s.t $(\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n) \in A_i$

choose $(\omega_1(i), \dots, \omega_n(i)) \in A/\sim$

Note that $f(\omega_1(i), \dots, \omega_n(i)) = \{\omega_1(i), \dots, \omega_n(i)\}$

$$= \{x_1, x_2, \dots, x_n\}$$
$$= E$$

Thus f is onto

Let $i, j \in [L]$ with $i \neq j$ then

$$(\omega_1(i), \omega_2(i), \dots, \omega_n(i)) \neq (\omega_1(j), \omega_2(j), \dots, \omega_n(j))$$

$\exists k \in [n]$ s.t. $\forall \pi : [n] \xrightarrow[\text{onto}]{} [n]$

$$\omega_k(i) = \omega_{\pi(k)}(j)$$

Hence, $\{\omega_1(i), \omega_2(i), \dots, \omega_n(i)\} \neq \{\omega_1(j), \omega_2(j), \dots, \omega_n(j)\}$

i.e. $f(\omega_1(i), \dots, \omega_n(i)) \neq f(\omega_1(j), \dots, \omega_n(j))$

Thus f is 1-1

Hence $L = |A/\sim| = \left| \binom{N}{n} \right| = \binom{N}{n}$

Th (Binomial Theorem) Let A be an a-set and B be an b-set
i.e. $|A|=a$ and $|B|=b$ with $A \cap B = \emptyset$

Then $(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$

Let X be an n-set i.e. $|X|=n$

$\mathcal{P}(X) =$ The set of all subsets of $X = \{E : E \subseteq X\}$

$$2^X = \{f \mid f: X \rightarrow \{0,1\}\}$$

Lemma: \exists 1-1 and onto f_{n^x} from 2^X to $\mathcal{P}(X)$

Proof: Let $S = \{(\alpha_1, \dots, \alpha_n) : \forall i \in [n], \alpha_i \in \{0,1\}\}$

$$\Rightarrow |S| = 2^n$$

Let $X = \{x_1, x_2, \dots, x_n\}$

We construct a map from 2^X to S

$$f \longmapsto (f(x_1), f(x_2), \dots, f(x_n))$$

Let $(\alpha_1, \alpha_2, \dots, \alpha_n) \in S$ we construct the func

$$f: X \rightarrow \{0,1\}$$

$$f(x_1) = \alpha_1$$

Hence map is onto

$$f(x_n) = \alpha_n$$

Let $f, g \in 2^X$ with $f \neq g$

$$\exists x \in X \text{ s.t } f(x) \neq g(x)$$

$$\exists i \in [n] \text{ s.t } x=x_i, f(x_i) \neq g(x_i)$$

Note:

$$f \mapsto (f(x_1), f(x_2), \dots, f(x_i), \dots, f(x_n))$$

$$g \mapsto (g(x_1), g(x_2), \dots, g(x_i), \dots, g(x_n))$$

$$\text{Then, } (f(x_1), f(x_2), \dots, f(x_n)) \neq (g(x_1), \dots, g(x_n))$$

i.e. such map is 1-1

We construct an 1-1 & onto f_{n^k} from S to $P(S)$

$$(\alpha_1, \alpha_2, \dots, \alpha_n) \mapsto \{x_i : \alpha_i = 0\}$$

(Show that such map is 1-1 & onto)

$$2^x \longrightarrow \mathcal{P}(x)$$

An equivalence relation on $\mathcal{P}(n)$

$$x \sim y \Leftrightarrow |x| = |y|$$

$$\mathcal{P}(x) = A_0 \sqcup A_1 \sqcup A_2 \dots \sqcup A_n$$

$$A_k = \left\{ E \subset x : |E| = k \right\} = \binom{x}{k}$$

$$2^n = |\mathcal{P}(x)| = |A_0| + |A_1| + \dots + |A_n| \\ = |\binom{x}{0}| + |\binom{x}{1}| + \dots + |\binom{x}{n}|$$

$$2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n}$$

Lemma: \exists 1-1 & onto map between $\binom{x}{k}$ and $\binom{x}{n-k}$

Proof: Required map is $A \mapsto x \setminus A$

Lemma: \exists 1-1 & onto map b/w

$$\binom{x \sqcup y}{n} \text{ and } \bigsqcup_{k=0}^n \binom{x}{k} \times \binom{y}{n-k}$$

where $x \cap y = \emptyset$ & x and y are finite sets

$$|x| + |y| \geq n$$

Plug in $|x| = |y| = n$

$$\binom{2n}{n} = \sum_{k=0}^n \binom{n}{k} \binom{n}{n-k} = \sum_{k=0}^n \binom{n}{k}^2$$

Lemma Pascal Identity For each positive integer n

$$\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k} \quad \text{where } 1 \leq k \leq n$$

Proof : Let $A = \left\{ X \in \binom{[n]}{k} : n+1 \notin X \right\}$

$$B = \left\{ X \in \binom{[n]}{k} : n+1 \in X \right\}$$

Then $|A| + |B| = \left| \binom{[n+1]}{k} \right|$

Note that $\forall X \in B, X \subset [n]$, i.e. $B \subset \binom{[n]}{k}$

For each $y \in \binom{[n]}{k}$, then $y \in B$

Hence $B = \binom{[n]}{k} \Rightarrow |B| = \binom{n}{k}$

Again $X \mapsto X \setminus \{n+1\}$ is an 1-1 & onto map from A to $\binom{[n]}{k-1}$

For $y \in \binom{[n]}{k-1}$, note that $y \cup \{n+1\} \in A$

i.e. such map is onto

For $X \neq Y$, where $X, Y \in A$, $\exists \alpha \in [n]$ s.t. $\alpha \in X$ but $\alpha \notin Y$,

Hence $X \setminus \{n+1\} \neq Y \setminus \{n+1\}$ i.e. such

map is 1-1

$$\Rightarrow |A| = \left| \binom{[n]}{k-1} \right|$$

$$\Rightarrow |A| + |B| = \left| \binom{[n+1]}{k} \right| \Rightarrow \binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$$

Theorem : Given two positive integers N and i there is a unique way to expand N as a sum of binomial coefficients as follows

$$N = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \dots + \binom{n_j}{j}$$

where j is a positive integer.

$$n_i \geq n_{i-1} \geq \dots \geq n_j \geq j \geq 1$$

Proof: $n_i = \max \{n : \binom{n}{i} \leq N\}$

$$n_{i-1} = \max \{n : \binom{n}{i-1} \leq N - \binom{n}{i}\}$$

⋮

$$n_{i-k} = \max \left\{ n : \binom{n}{i-k} \leq N - \binom{n}{i} - \binom{n}{i-1} - \dots - \binom{n}{i-k+1} \right\}$$

where $k \in \{0, 1, 2, 3, \dots\}$

claim: $n_{i-1} \leq n_i$

Proof of claim: Note that $\binom{n_i}{i} \leq N < \binom{n_{i+1}}{i} = \binom{n_i}{i} + \binom{n_i}{i-1}$

$$\text{Hence, } 0 \leq N - \binom{n_i}{i} < \binom{n_i}{i-1}$$

Since $p \geq 1 \wedge m \geq 1$ we have

$$\binom{m}{i-1} \leq \binom{m+1}{i-1}$$

Therefore $\max \left\{ m : \binom{m}{i-1} \leq N - \binom{n_i}{i} \right\} \leq n_i$

i.e. $n_{i-1} \leq n_i$

Th (Binomial Theorem) Let A be an a -set and B be a b -set,
with $A \cap B = \emptyset$

$$(a+b)^n = \sum_{i=0}^n \binom{n}{i} a^i b^{n-i}$$

Proof: Let $S = \{(x_1, x_2, \dots, x_n) : \forall i \in [n], x_i \in A \cup B\}$

We construct an equivalence relation \sim over S

$$(x_1, x_2, \dots, x_n) \sim (y_1, y_2, \dots, y_n) \Leftrightarrow \exists (x, y) \in [n] \times [n]$$

such that $|x| = |y|$

$$\forall i \in X \quad x_i \in A, \forall j \in [n] \setminus X, x_j \in B$$

$$\forall i \in Y \quad y_i \in A, \forall j \in [n] \setminus Y, y_j \in B$$

$$S = A_0 \sqcup A_1 \sqcup A_2 \dots \sqcup A_n$$

A_0

\vdots

$A_k = \{(x_1, x_2, x_3, \dots, x_n) \in S : \text{exactly } k \text{ elements belong to } A\}$

$\vdots = \{(x_1, \dots, x_n) \in S : \exists X \subset [n], \text{ with } |X| = k \text{ s.t. } \forall i \in X \quad x_i \in A \text{ & } \forall j \in [n] \setminus X, x_j \in B\}$

A_n

$$|S| = |A_0| + |A_1| + \dots + |A_n|$$

$$\therefore |A_0| = b^n$$

$$|A_1| = \binom{n}{1} a b^{n-1}$$

$$|A_k| = \binom{n}{k} a^k b^{n-k} \quad \begin{array}{l} k \text{ many elements } \in A \\ n-k \text{ many elements } \in B \end{array}$$

Let's construct the triplet

$$\left(\{n_1, n_2, \dots, n_k\}, (a_i)_{i \in \{n_1, n_2, \dots, n_k\}}, (b_j)_{j \in [n] \setminus \{n_1, n_2, \dots, n_k\}} \right)$$

$$A_K \xrightarrow{f} \left\{ (x, (a_i)_{i \in x}, (b_j)_{j \in [n] \setminus x}) : x \in \binom{[n]}{k} \right\}$$

$$(x_1, x_2, \dots, x_n) \longmapsto \left(\{n_1, \dots, n_k\}, (a_i)_{i \in \{n_1, \dots, n_k\}}, (b_j)_{j \in [n] \setminus \{n_1, \dots, n_k\}} \right)$$

* Bijective function

* Binomial

Let X be a n -set, and r, t be integers satisfying $0 \leq r+t \leq n$

$$\binom{X}{r,t} = \left\{ (A, B) : A \subset X, \text{ with } |A|=r, B \subset X \text{ with } |B|=t \right\} \\ A \cap B = \emptyset$$

E.g. show that

$$\left| \binom{X}{r,t} \right| = \frac{n!}{r! t! (n-r-t)!} = \frac{n!}{r! (n-r)!} \cdot \frac{(n-r)!}{t! (n-r-t)!}$$

Th: Multinomial Theorem : Let for each $i \in [k]$, A_i be an a_i -set

& $\forall i, j \in [k]$, with $i \neq j \Rightarrow A_i \cap A_j = \emptyset$

$$\text{Then } (a_1 + a_2 + \dots + a_k)^n = \sum_{t_1+t_2+\dots+t_k=n} \frac{n!}{t_1! t_2! \dots t_k!} a_1^{t_1} a_2^{t_2} \dots a_k^{t_k}$$

Here the summation extend over all non-negative integer solns.
of t_1, t_2, \dots, t_k satisfies $t_1+t_2+t_3+\dots+t_k=n$.

A

Addition principle of counting

→ if a finite non-empty set S is partitioned into k many parts B_1, \dots, B_k i.e. $\forall i, j \in [k], B_i \cap B_j = \emptyset$ and

$$B_i \cap B_j = \begin{cases} \emptyset & \text{if } i \neq j \\ B_i & \text{else } (i=j) \end{cases}$$

$$|S| = |B_1| + |B_2| + \dots + |B_k|$$

Multiplication Principle of Counting

if a finite non-empty set S can be expressed as

$$S_1 \times S_2 \times \dots \times S_k$$

then

$$|S| = |S_1| \times |S_2| \times \dots \times |S_k| = \prod_{i=1}^k |S_i|$$

c. One-One and Onto fn² (principle of counting)

Let R, S be two finite non-empty sets if \exists a 1-1 & onto $f: S \rightarrow R$ then $|S| = |R|$

D. The double counting principle

Let $A \& B$ be two non-empty finite sets & $S = A \times B$

The double counting principle is to count $|S|$ in two ways

First way: We count: $\sum_{a \in A} |\{b \in B : (a, b) \in S\}|$

Second way: We count: $\sum_{b \in B} |\{a \in A : (a, b) \in S\}|$

Now we have an identity

$$\sum_{a \in A} |\{b \in B : (a, b) \in S\}| = |S| = \sum_{b \in B} |\{a \in A : (a, b) \in S\}|$$

Example : (Handshaking Lemma) Suppose there are n many people at a party each of them will shake hands with everyone else

Q: How many handshakes will occur?

Solⁿ: Let $[n] = \{1, 2, 3, \dots, n\}$ represent the set of one (right) hand of each people

we consider the pair $(x, \{x, y\})$

where $x \in [n]$, and $\{x, y\}$ represent the handshake let there be N no. of handshakes.

We count the set $S = \{(x, \{x, y\}) : x \in [n], y \in [n]\}$

First way: There are n many choices for x and chosen x there are $n-1$ many choices for $\{x, y\}$

$$|S| = n(n-1)$$

Second way: There are N many choices for $\{x, y\}$ Now chosen $\{x, y\}$, there are 2 choices such pairs namely $(x, \{x, y\})$ and $(y, \{x, y\}) \in S$

$$\text{i.e. } |S| = 2N$$

$$\text{Hence: } 2N = |S| = n(n-1)$$

$$\text{f.e. } N = \frac{n(n-1)}{2}$$

Problem: How many way one can distribute n identical object into k many distinguishable / labelled boxes

$0 \ 0 \ 0 \dots 0$
 ← n identical objects →

$\boxed{1} \ \boxed{2} \ \boxed{3} \ \dots \ \boxed{k}$
 ← k labelled boxes →

Special cases $0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$
 ← 10 biscuits →

$\boxed{} \ \boxed{} \ \boxed{}$
 Amit Naveen Raju

Solⁿ: step-1 $0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$
 ← 10 identical biscuits →
 — — 2 separators

step-2 Place those two separators among 10 biscuits eg. (a)

$0 \ 0 \ 0 \ | \ 0 \ 0 \ 0 \ 0 \ | \ 0 \ 0$ $\boxed{3B} \ \boxed{4B} \ \boxed{3B}$
 Amit Naveen Raju

$1 \ 0 \ 0 \ 0 \ 0 \ 0 \ | \ 0 \ 0 \ 0 \ 0 \ 0$ $\boxed{OB} \ \boxed{5B} \ \boxed{5B}$
 A N R

$1 \ 1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0$ $\boxed{OB} \ \boxed{OB} \ \boxed{10B}$
 A N R

Step-3 Consider 12 biscuits choose 2 of them and throw those two. Finally replace it by two separators.

$\binom{12}{2}$ ways

Theorem
One can distribute n identical objects into k labelled boxes into $\binom{n+k-1}{k-1}$ many ways.

Proof: Let B be the set of labelled boxes Note $|B|=k$
For each distribution D of n identical objects into B , we
associate a function $\gamma_D : B \rightarrow \mathbb{Z}$ as
 $\gamma_D(b) = \text{No. of identical objects received by the}$
box b w.r.t D

Note $\sum_{b \in B} \gamma_D(b) = n$

we note that for each $D \in \mathcal{D}$

$$\sum_{b \in B} \gamma_D(b) = n$$

where \mathcal{D} denote the set of all distribution of n identical objects into k many labelled boxes

$$|\mathcal{D}| = |\{\gamma_D : D \in \mathcal{D}\}|$$

(Ex: show that $D \mapsto \gamma_D$ from \mathcal{D} to $\{\gamma_D : D \in \mathcal{D}\}$ forms an
1-1 and onto fn)

$$\{\gamma_D : D \in \mathcal{D}\} = \{f : [k] \rightarrow \mathbb{Z} \cap [0, \infty) \mid f(1) + \dots + f(k) = n\}$$

claim: $|\{f : [k] \rightarrow \mathbb{Z} \cap [0, \infty) \mid f(1) + \dots + f(k) = n\}| = \left| \binom{n+k-1}{k-1} \right|$

Proof of claim : $\forall B \in \binom{[n+k-1]}{k-1}$, i.e $|B|=k-1$ say

$$B = \{i_1, i_2, \dots, i_{k-1}\} \subset [n+k-1]$$

we construct the function $f_B : [k] \rightarrow \mathbb{Z}[0, \infty)$

WLOG

$$i_1 < i_2 < \dots < i_{k-1}$$

$$f_B(1) = |\{i \in [n+k-1] : i < i_1\}|$$

$$f_B(2) = |\{i \in [n+k-1] : i_1 \leq i < i_2\}|$$

$$f_B(3) = |\{i \in [n+k-1] : i_2 < i < i_3\}|$$

:

$$f_B(k) = |\{i \in [n+k-1] : i_{k-1} < i\}|$$

Construct for a fn². $f : [k] \rightarrow \mathbb{Z}[0, \infty)$

$$\text{with } f(1) + \dots + f(k) = n$$

we construct the $k-1$ -set $\{i_1, i_2, \dots, i_{k-1}\}$ by setting

$$i_1 = f(1) + 1$$

$$i_2 = f(2) + f(1) + 2$$

$$i_{k-1} = f(1) + \dots + f(k-1) + k-1$$

Thus, $B \mapsto f_B$ is an 1-1 & onto welldefined function.

Corollary: One can distribute n identical objects into k labelled boxes such that each box contains atleast one identical object into $\binom{n-1}{k-1}$ many ways.

Proof: we first distribute one identical object into each boxes it results exactly $(n-k)$ ways identical objects are to be distributed among k labelled boxes

using the Thm we have $\binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$ many ways

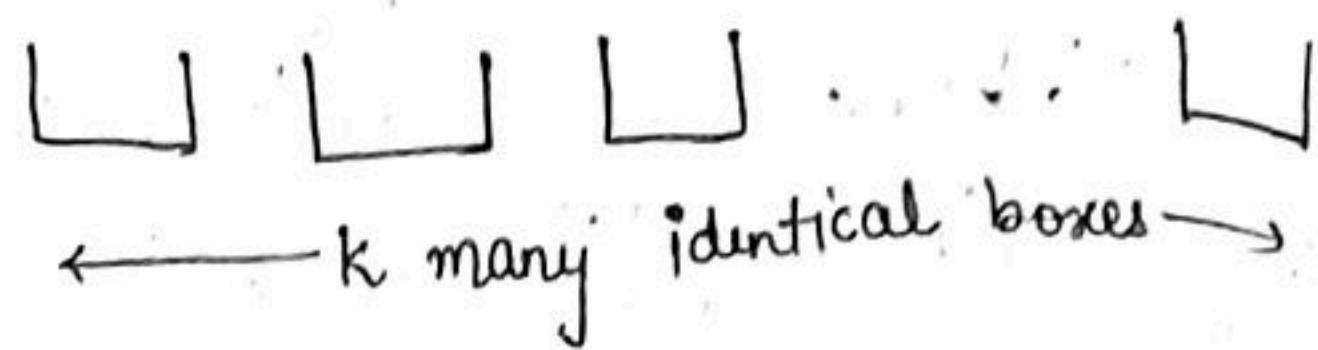
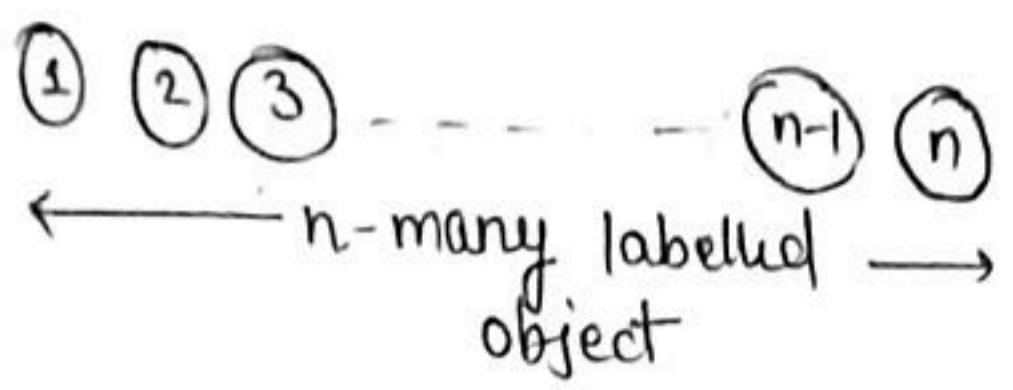
Exercise: Consider the eqn $\sum_{i=1}^k x_i = n$, where n and k are positive integer show that

① the number of non-negative integer sol's of the eqn is $\binom{n-k+1}{k-1}$

② the ~~no~~ number of positive integer sol's of the above eqn is $\binom{n-1}{k-1}$

Defn: Let n and k be positive integers with $n \geq k$ The stirling number of 2nd kind denoted as $S_k^{(n)} = S(n, k)$ is total no of partition of $[n]$ into k -many non-empty sets

The set $[n]$ is chopped into k -many parts



$$\text{Thm: } S(n, k) = S(n-1, k-1) + k \cdot S(n-1, k)$$

Proof: $\mathcal{S}([n], k) \rightarrow$ The set of all partition of $[n]$ into
k-many non-empty sets

$$S(n, k) = |\mathcal{S}([n], k)|$$

\mathcal{A} = The set of all partition of $[n]$ into k - non-empty
where n belongs is a singleton set
i.e. exactly one part of partition of $[n]$ is $\{n\}$.

\mathcal{B} = The set of all partitions of $[n]$ into k non-empty
sets where n belongs is not a singleton set

$$\text{claim 1: } |\mathcal{A}| = S(n-1, k-1)$$

Proof of claim: Let x_1, \dots, x_k be a partition of $[n]$ into
k non-empty parts and exactly one of
 x_1, \dots, x_k is $\{n\}$

WLOG, let $x_k = \{n\}$

$$\begin{aligned} \text{i.e. } [n] &= x_1 \sqcup x_2 \dots \sqcup x_{k-1} \sqcup x_k \\ &= x_1 \sqcup \dots \sqcup x_{k-1} \sqcup \{n\} \end{aligned}$$

$$\forall i \in [k-1], x_i \neq \emptyset \quad \& \quad x_k = \{n\}$$

This induces a partition of $[n-1]$ into $(k-1)$ -non-empty set
 namely if y_1, \dots, y_{k-1} is a partition of $[n-1]$ into $(k-1)$
 many non-empty parts, then $y_1, y_2, \dots, y_{k-1}, \{n\}$ is a partition
 of $[n]$ into k -many part with exactly one part
 equals $\{n\}$.

$$A \longleftrightarrow S([n-1], k-1)$$

Hence, \exists a bijection corresponds between A and $S([n-1], k-1)$
 This establish the claim

claim 2: $|B| = k \cdot s(n-1, k)$

Proof of claim: We note that

$$\{x_1, x_2, \dots, x_k\} \longmapsto \{x_1 \setminus \{n\}, \dots, x_k \setminus \{n\}\}$$

where x_1, \dots, x_k is a partition of $[n]$ into k many non-empty set and where n belongs is not a singleton set

Note: $x_1 \setminus \{n\}, \dots, x_k \setminus \{n\}$ yeilds a partition of $[n-1]$ into k -many non-empty sets

Again each partition of $[n-1]$ onto k -many non-empty sets

$$y_1, y_2, \dots, y_k$$

has exactly k many preimages w.r.t the said map namely

$$\{y_1 \sqcup \{n\}, y_2, \dots, y_k\}$$

$$\{y_1, y_2 \sqcup \{n\}, \dots, y_k\}$$

$$\longrightarrow \{y_1 \setminus \{n\}, y_2 \setminus \{n\}, \dots, y_k \setminus \{n\}\}$$

:

$$\{y_1, y_2, \dots, y_k \sqcup \{n\}\}$$

Hence such map is k to 1 & onto map

Hence, $|B| = k |\mathcal{S}([n-1], k)| = k S(n-1, k)$

Thm: for each integer $n \geq 2$ $S(n, 2) = 2^{n-1} - 1$

Proof-1: Note that $S(2, 2) = 1$

$$\begin{aligned} S(n, 2) &= 1 + 2 S(n-1, 2) \\ &= 1 + 2(1 + 2 S(n-2, 2)) \\ &= \vdots \\ &= 1 + 2 + 2^2 + \dots + 2^{n-2} S(2, 2) \\ &= 1 + 2 + 2^2 + \dots + 2^{n-2} \\ &= 2^{n-1} - 1 \end{aligned}$$

Proof-2 $\mathcal{S}([n], 2)$ = The set of all of $[n]$ into 2 non-empty sets

$$\mathcal{P} = \{(A, B) : A \cup B = [n], A \neq \emptyset, B \neq \emptyset\}$$

$$|\mathcal{P}| \rightarrow \text{Exercise} = 2^n - 2$$

if $(A, B) \in \mathcal{P}$, then $(B, A) \in \mathcal{P}$

but $(A, B) \& (B, A)$ represents one partition of $[n]$ namely

Hence such mapping from \mathcal{P} to $S([n], 2)$ is a
2-to-1 mapping and onto, hence $2|\mathcal{S}([n], 2)| = |\mathcal{P}|$

$$\text{i.e. } 2 S(n, 2) = 2^n - 2$$

$$\Leftrightarrow S(n, 2) = 2^{n-1} - 1$$

$$s(n, n-1) = \binom{n}{2}$$

Then:
Proof: Ex

Defⁿ The Bell no. $B(n)$, where n is a positive integer,
is the to number of partitions of $[n]$ i.e. $B(n)$

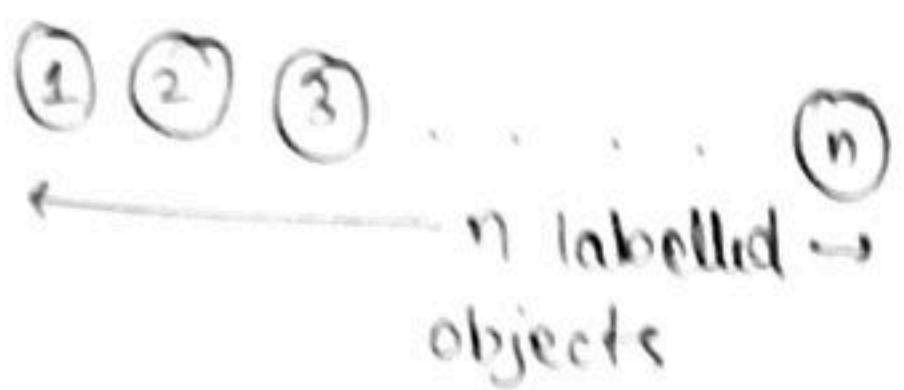
$$B(n) = \sum_{i=1}^n s(n, i)$$

Th^m: One can distribute n labelled objects into k -identical boxes
such that each box contains atleast one object, into $s(n, k)$ ways

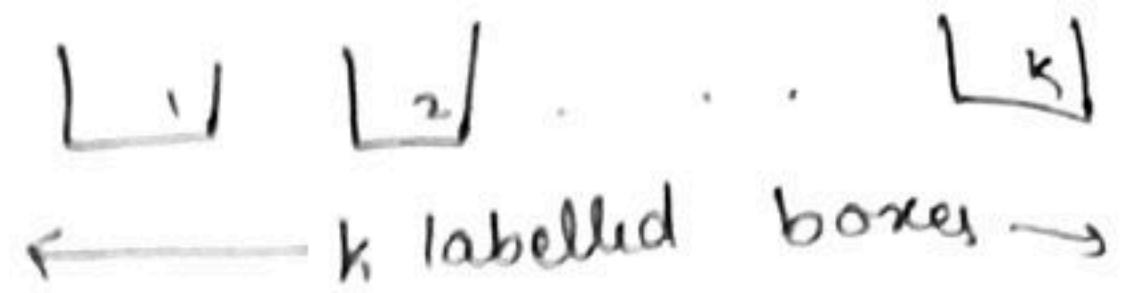
Proof: Exercise

Th^m: One can distribute n -labelled objects into k -identical
boxes, into $s(n, 1) + s(n, 2) + \dots + s(n, k)$ many ways

Proof: Exercise



$n \geq k$



Thm: One can distribute n labelled objects into k labelled boxes into k^n ways

Proof: A distribution of "such" yields a f^n $d: [n] \rightarrow [k]$

$d(i)$ = label of box where i object is distributed

Conversely if $f: [n] \rightarrow [k]$ is a f^n

then $f^{-1}(1), f^{-1}(2), \dots, f^{-1}(k)$ forms a partition of $[n]$ into k many labelled sets

Thm: One can distribute n. labelled objects into k labelled boxes show that each box contains at least one object into $k! s(n, k)$ many ways

Proof: $O([n], [k]) = \{f: [n] \rightarrow [k] \mid f \text{ is onto}\}$

we construct an equivalence relation \sim over $O([n], [k])$

$f \sim g$, where $f, g \in O([n], [k])$, iff $\exists \pi: [k] \xrightarrow{\text{onto}} [k]$
 $s.t. -g = \pi \circ f$

Let $[f]$ denote an equivalent class containing $f \in O([n], [k])$

$$|[f]| = k!$$

$$\Omega([n], [k]) = A_1 \sqcup A_2 \sqcup \dots \sqcup A_N$$

$$|A| = |A_1| = |A_2| = \dots = |A_N|$$

claim: There are $S(n, k)$ no. of equivalence classes wrt to \sim

$$f \sim g : \Leftrightarrow \exists \pi, g = \pi \circ f$$

$$(f^{-1}(1), f^{-1}(2), \dots, f^{-1}(k))$$

$$(g^{-1}(1), g^{-1}(2), \dots, g^{-1}(k))$$

unordered k -tuple

$$\{f^{-1}(1), \dots, f^{-1}(k)\} = \{g^{-1}(1), g^{-1}(2), \dots, g^{-1}(k)\}$$

$$\text{WLOG } |f^{-1}(1)| \leq |f^{-1}(2)| \leq \dots \leq |f^{-1}(k)|$$

$$|g^{-1}(1)| \leq |g^{-1}(2)| \leq \dots \leq |g^{-1}(k)|$$

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & k \\ i_1 & i_2 & i_3 & \dots & i_k \end{pmatrix}$$

Proof of claim: We first establish that if $f, g \in \Omega([n], [k])$

s.t. the unordered k -tuple subset of $[n]$ $(f^{-1}(1), f^{-1}(2), \dots, f^{-1}(k))$ and $(g^{-1}(1), g^{-1}(2), \dots, g^{-1}(k))$ are same i.e.

$$[n] = f^{-1}(1) \sqcup f^{-1}(2) \sqcup \dots \sqcup f^{-1}(k)$$

the set of $\{f^{-1}(1), f^{-1}(2), \dots, f^{-1}(k)\} = \{g^{-1}(1), g^{-1}(2), \dots, g^{-1}(k)\}$

Then \exists a permutation on $\pi: [k] \rightarrow [k]$ s.t. $f = \pi \circ g$

To see this WLOG

$$|f^{-1}(1)| \leq |f^{-1}(2)| \leq \dots \leq |f^{-1}(k)|$$

Suppose $|g^{-1}(i_1)| \leq |g^{-1}(i_2)| \leq \dots \leq |g^{-1}(i_k)|$

We construct $\pi = \begin{pmatrix} 1 & 2 & 3 & \dots & k \\ i_1 & i_2 & i_3 & \dots & i_k \end{pmatrix}$

Here we have,

$$|f^{-1}(1)| = |g^{-1}(i_1)|$$

$$|f^{-1}(2)| = |g^{-1}(i_2)|$$

$$\vdots$$

$$|f^{-1}(k)| = |g^{-1}(i_k)|$$

Note that $\forall x \in [n]$

$$x \in f^{-1}(\alpha) \text{ for some } \alpha \in [k]$$

i.e. $f(x) = \overset{\circ}{\alpha}$

$$\pi \circ f(\alpha) = \pi(\alpha) = i_\alpha$$

i.e. $x \xrightarrow{\pi \circ f} i_\alpha$

Also $x \in g^{-1}(i_\alpha)$

$$\text{i.e. } g(x) = i_\alpha$$

i.e. $x \xrightarrow{g} i_\alpha$

$$g = \pi \circ f$$

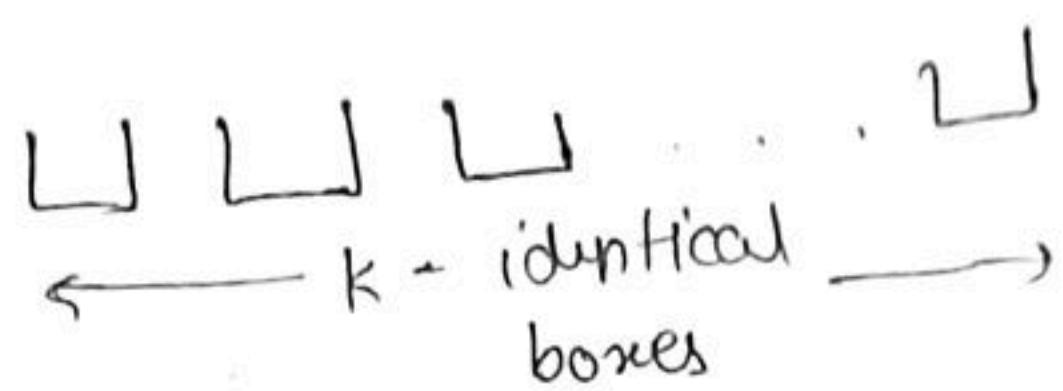
This means two different undirected k -tuples of subset of $[n]$ $(f^{-1}(1), f^{-1}(2), \dots, f^{-1}(k))$ and $(g^{-1}(1), \dots, g^{-1}(k))$ are not same $\Leftrightarrow \# \pi: [k] \xrightarrow{\text{onto}} [k] \text{ s.t. } g = \pi \circ f \Leftrightarrow f \neq g$

Recall that undirected k -tuple $(f^{-1}(1), \dots, f^{-1}(k))$ represents a partition of $[n]$ into k -non-empty sets

Hence the claim is established

So. no. of "such" distribution are $|P([n], [k])| = k! s(n, k)$

Ⓐ Ⓛ Ⓜ ... Ⓞ
n identical → objects



Defⁿ: Let n, k be positive integer with $n \geq k$. An unordered k -tuple (a_1, a_2, \dots, a_k) is said to be partition of integer n into k -positive integers if a_1, a_2, \dots, a_k are positive integer satisfy

$$a_1 + a_2 + a_3 + \dots + a_k = n$$

The integer $P_{\leq k}(n) = P(n, k)$ denotes the total no. of way in which the integer n can be partitioned into k positive integer

Proof $\mathcal{P}(n, k) = \text{The set of partitions of the integer } n \text{ into } k \text{ positive integers}$

$$= \left\{ (a_1, a_2, \dots, a_k) : (a_1, \dots, a_k) \text{ is an unordered } k\text{-tuple of positive integers satisfy } a_1 + a_2 + \dots + a_k = n \right\}$$

$$A = \left\{ (a_1, a_2, \dots, a_k) \in \mathcal{P}(n, k) : a_i = 1, \text{ for some } i \in [k] \right\}$$

$$B = \left\{ (a_1, a_2, \dots, a_k) \in \mathcal{P}(n, k) : \forall i \in [k], a_i > 2 \right\}$$

$$A \longleftrightarrow \mathcal{P}(n-1, k-1)$$

let $(a_1, a_2, \dots, a_k) \in A$, WLOG $a_k = 1$

Note that $a_1 + a_2 + \dots + a_{k-1} = n-1$

i.e. unordered $k-1$ tuple $(a_1, a_2, \dots, a_{k-1})$ is a partition of

Integer $n-1$ into $k-1$ positive integers

i.e. $(a_1, a_2, \dots, a_{k-1}) \in \wp(n-1, k-1)$

Conversely, if $(b_1, b_2, \dots, b_{k-1}) \in \wp(n-1, k-1)$ then the unordered k -tuple $(b_1, \dots, b_{k-1}, 1) \in \mathcal{X}$

Hence, There exist bijective correspondence exists b/w \mathcal{X} and $\wp(n-1, k-1)$. Hence,

$$|\mathcal{A}| = |\wp(n-1, k-1)| = p(n-1, k-1)$$

We note that,

$$(a_1, a_2, \dots, a_k) \mapsto (a_1-1, a_2-1, \dots, a_{k-1})$$

is an 1-1 & onto map from \mathcal{B} to $\wp(n-k, k)$ (Ex)

$$|\mathcal{B}| = |\wp(n-k, k)| = p(n-k, k)$$

□

Thm: One can distribute n -identical objects into k identical boxes into $\sum_{i=1}^k p(n, i)$ many ways

statistical Mechanics

Placing n particles into k -different energy levels

THREE different statistics are obtained by making three different assumptions

• Maxwell - Boltzman : Here ~~n~~ n labelled particles are distributed into k -labelled boxes (energy levels)

• Bose - Einstein : Here n identical particles are distributed into k labelled boxes (energy levels)

• Fermi - Dirac : Here n -identical particles are distributed into k labelled energy levels but no box can hold more than ~~one~~ one particle

$$\rightarrow \binom{k}{n}$$

* Principle of inclusion & exclusion

Th (IEP - Ver II) : Let $\mu: 2^{[n]} \rightarrow \mathbb{R}$ be a fn and we construct
 $v: 2^{[n]} \rightarrow \mathbb{R}$

$$v(E) = \sum_{S \subseteq E} \mu(S) = \sum_{S \in 2^E} \mu(S)$$

$$\text{Then, } \mu(E) = \sum_{S \subseteq E} (-1)^{|E|-|S|} v(S)$$

Th (IEP - Ver III) Let $\mu: 2^{[n]} \rightarrow \mathbb{R}$ be a fn and

$$v: 2^{[n]} \rightarrow \mathbb{R}$$

$$v(E) = \sum_{S \subseteq E} \mu(S)$$

$$\text{Then } \mu(E) = \sum_{S \subseteq E} (-1)^{|S|-|E|} v(S) \quad S \subseteq E$$

$$\text{Proof: } \sum_{S \subseteq E} (-1)^{|S|-|E|} v(S)$$

$$= \sum_{S \subseteq E} (-1)^{|S|-|E|} \left(\sum_{C \subseteq S} \mu(C) \right)$$

$$= \sum_{S \subseteq E} \sum_{C \subseteq S} (-1)^{|S|-|E|} \mu(C)$$

* Principle of inclusion & exclusion

Theorem - Ver II : Let $\mu : 2^{[n]} \rightarrow \mathbb{R}$ be a fn and we construct
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$$v(E) = \sum_{S \subseteq E} \mu(S) = \sum_{S \in 2^E} \mu(S)$$

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Theorem - Ver III : Let $\mu : 2^{[n]} \rightarrow \mathbb{R}$ be a fn and

$$v : 2^{[n]} \rightarrow \mathbb{R}$$

$$v(E) = \sum_{E \subseteq S} \mu(S)$$

$$\text{Then } \mu(E) = \sum_{E \subseteq S} (-1)^{|S| - |E|} v(S)$$

Proof : $\sum_{E \subseteq S} (-1)^{|S| - |E|} v(S)$

$$= \sum_{E \subseteq S} (-1)^{|S| - |E|} \left(\sum_{S \subseteq C} \mu(C) \right)$$

$$= \sum_{E \subseteq S} \sum_{S \subseteq C} (-1)^{|S| - |E|} \mu(C)$$

$$(c, s) \longrightarrow ((-1)^{|E|-|S|}, \mu(c))$$

$$\mu(c) \sum_{S \in c} (-1)^{|S| - |E|}$$

$$\sum_{c \in S} \mu(c) \sum_{S \in c} (-1)^{|E|-|S|}$$

$$\Rightarrow \mu(c) \sum_{S \in c} (-1)^{|E|-|S|} = \mu(c) \sum_{Z \subseteq c \setminus E} (-1)^{|E|-|E|-|Z|}$$

$S = E \sqcup Z$

$$Z \subseteq c \setminus E$$

$$= \mu(c) \sum_{Z \subseteq c \setminus E} (-1)^{|Z|}$$

$$= \mu(c) \cdot 0$$

Hence, by DCP, we have

$$\mu(E) = \sum_{S \subseteq E} (-1)^{|E|-|S|} v(S)$$

$$\sum_{S \subseteq [n]} (-1)^{|S|} = \sum_{i=0}^n \binom{n}{i} (-1)^i$$

$$2 \sum_{i=0}^n \binom{n}{i} (-1)^i (1)^{n-i}$$

$$2 - (n+1)^n = 0$$

Theorem (IEP - Ver IV) Let A be a finite non-empty set and $\forall i \in [m]$, P_i denote denote a property for each $x \in A$, and $i \in [m]$ either x satisfies the property P_i or (exclusive or) x does not satisfy property P_i .

let $S \subseteq [m]$

$$N(S) = \{x \in A : x \text{ satisfies property } P_i \forall i \in S\}$$

The no. of elements of A that satisfy none of the properties P_1, P_2, \dots, P_m is given by

$$\sum_{S \subseteq [m]} (-1)^{|S|} |N(S)|$$

APP-I ONTO FUNCTION COUNTING

Let $s(n, k)$ denote the stirling no. of 2nd kind where $n \geq k$ are positive integers with $n \geq k$ the $s(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{n}{i} (k)_i^{(n)}$

Proof $A([n], [k])$ = The set of all $f^{n \times k}$ from $[n]$ to $[k]$

$$|A([n], [k])| = k^n$$

$O([n], [k])$ = The set of all onto $f^{n \times k}$ from $[n]$ to $[k]$

we define the properties P_1, P_2, \dots, P_k

$\forall i \in [k]$ P_i denote the property that i does not belong to image of $f \in A([n], [k])$

Note that the set of all elements of $A([n], [k])$ that satisfy none of the properties P_1, P_2, \dots, P_k is that set of all onto $f^{n \times k}$

$sc[k]$

$$N(s) = \{f \in A([n], [k]) : f \text{ satisfies property } P_i \forall i \in s\}$$

claim: for $sc[k]$, $|N(s)| = (k - |s|)^n$

Proof of claim: we note the following

$f \in N(s) \Leftrightarrow f \text{ satisfies Prop. } P_i \forall i \in s$

$\Leftrightarrow i \notin \text{Im } f \quad \forall i \in s$

$\Leftrightarrow s \text{ not in the Im } f$

$\Leftrightarrow s \text{ is disjoint from } \text{im } f$

$\Leftrightarrow f \text{ is a fn}^* \text{ from } [n] \text{ to } [k] \setminus s$

Hence, $|N(s)| = (k - |s|)^n$ This establishes the claim

Using the IEP-Vari \Rightarrow

$$|O([n], [k])| = \sum_{\substack{\text{sc}[k] \\ sc[k]}} (-1)^{|s|} |N(s)| = \sum_{sc[k]} (-1)^{|s|} (k - |s|)^n$$

$$= \sum_{i=0}^k (-1)^i \binom{n}{i} (k-i)^n$$

Then, we know, $|O([n], [k])| = k! s(n, k)$

$$s(n, k) = \frac{1}{k!} \sum_{i=0}^k (-1)^i \binom{n}{i} (k-i)^n$$

Proof : let $x \in A$, we construct $S_x \{ i \in [m] : x \text{ satisfies } p_{i,p} \}$

$$N(S) = \{x \in A : S \subseteq S_x\}$$

$$= \sum_{S \subseteq [m]} (-1)^{|S|} |N(S)|$$

$$= \sum_{S \subseteq [m]} (-1)^{|S|} |\{x \in A : S \subseteq S_x\}|$$

$$= \sum_{S \subseteq [m]} (-1)^{|S|} \left(\sum_{\substack{x \in A \\ S \subseteq S_x}} 1 \right)$$

$$= \sum_{S \subseteq [m]} \sum_{\substack{x \in A \\ S \subseteq S_x}} (-1)^{|S|} = \sum_{S \subseteq S_x \subseteq [m]} \sum_{x \in A} (-1)^{|S|}$$

$$= \sum_{x \in A} \sum_{S \subseteq S_x} (-1)^{|S|}$$

$$= \sum_{x \in A} \sum_{S \subseteq S_x = \emptyset} (-1)^{|S|} + \sum_{x \in A} \sum_{S \subseteq S_x \neq \emptyset} (-1)^{|S|}$$

$$= \sum_{x \in A} \sum_{S_x = \emptyset} 1 + \sum_{x \in A} (-1+1)^{|S_x|}$$

$$= \sum_{x \in A} \sum_{S_x \neq \emptyset} 1$$

$$= |\{x \in A : S_x = \emptyset\}|$$

$$= |\{x \in A : x \text{ does not satisfy property } p_1, p_2, \dots, p_m\}|$$

App-II : Derangement Permutation counting

Def: A permutation $\pi: [n] \xrightarrow{\text{onto}} [n]$ is called derangement if $\forall i \in [n], \pi(i) \neq i$

The set of all derangements on $[n]$ is denoted by D_n

claim $S(v_j) = T(v_j) \quad \forall j \in \{1, 2, \dots, n\}$

APP-III Establishment of Euler PHI Function

Th: if $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ where $\forall i \in [k]$, $n_i \geq 1$

$n > 1$ are integer and p_i are positive prime integers, then

$$\phi(n) = n \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_k}\right)$$

Here $\phi(n)$ denotes Euler's phi function defined by

$$\phi(n) = |\{x \in [n] : \gcd(x, n) = 1\}|$$

Proof: For each $i \in [k]$, let P_i denote the property that x is divisible by the positive prime integer p_i

claim: The set of elements of $[n]$ that satisfies the none of the properties of P_1, P_2, \dots, P_k

then

$$R = \{x \in [n] : \gcd(x, n) = 1\}$$

Proof of claim: Note that if x does not satisfy the P_i for some $i \in [n]$ then $\gcd(x, p_i) = 1$

$$\begin{aligned} \Rightarrow \exists u, v \in \mathbb{Z} \text{ s.t } xu + p_i v = 1 \\ \Rightarrow (xu + p_i v)^{n_i} = 1 \\ \Leftrightarrow x^{n_i} + p_i^{n_i} v^{n_i} = 1 \\ \Rightarrow \gcd(x, p_i^{n_i}) = 1. \end{aligned}$$

Therefore x does not satisfies property $\# P_i$ then $\gcd(x, p_i^{n_i}) = 1$

so, if x does not satisfy P_1, P_2, \dots, P_k then $\exists u_i, v_i \in \mathbb{Z}$

$$\begin{aligned} \forall i \in [k] \text{ s.t. } (xu_1 + p_1^{n_1} v_1)(xu_2 + p_2^{n_2} v_2) + \dots + (xu_k + p_k^{n_k} v_k) = 1 \\ \Leftrightarrow xv + nv = 1 \quad \text{for some } v, v \in \mathbb{Z} \end{aligned}$$

$$\text{claim } s(v_j) = \prod_{i \in j} p_i \quad \forall j \in \{1, 2, \dots, n\}$$

$$\Leftrightarrow \gcd(x, n) = 1$$

$$\Leftrightarrow x \in \mathbb{Z}/n\mathbb{Z} \quad (\text{this establishes the claim})$$

claim: $s \in [k]$

$$|N(s)| = \frac{n}{\prod_{i \in s} p_i}$$

Proof of claim: $x \in N(s) \Leftrightarrow x \text{ satisfies property } p_i \forall i \in s$

$$\Leftrightarrow p_i | x \forall i \in s$$

$$\Leftrightarrow \prod_{i \in s} p_i | x$$

$$\text{Hence, } N(s) = \left\{ x \in [n], x = \prod_{i \in s} p_i, g_i = 1, 2, \dots, \frac{n}{\prod_{i \in s} p_i} \right\}$$

Using IEP - Verteilung and the claim we have

$$\begin{aligned} \phi(n) &= |R| = \sum_{s \subseteq [k]} (-1)^{|s|} |N(s)| \\ &= \sum_{s \subseteq [k]} (-1)^{|s|} \frac{n}{\prod_{i \in s} p_i} = n \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right) \end{aligned}$$

$$\left(1 - \frac{1}{p}\right) \left(1 - \frac{1}{q}\right) = 1 + (-1) \left(\frac{1}{p} + \frac{1}{q}\right) + \frac{1}{pq}$$

Th: For any integer $n \geq 1$

$$n = \sum_{d|n} \phi(d)$$

Th: for each positive integer n

$$n = \sum_{d|n} \phi(d)$$

$$\sum_{d|n} \phi(d) = \sum_{d|n} \phi\left(\frac{n}{d}\right)$$

Proof : Let $\gcd(x, n) = d$

Then $\exists u, v \in \mathbb{Z}$ s.t. $xu + nv = d$

$$= \frac{x}{d}u + \frac{n}{d}v = 1$$

so $x \mapsto \frac{x}{d}$ forms a bijective

mapping between set $\{x \in [n] : \gcd(x, n) = d\}$

and $\{y \in \left[\frac{n}{d}\right] : \gcd(y, \frac{n}{d}) = 1\}$

Hence,

$$|\{x \in [n] : \gcd(x, n) = d\}| = \phi\left(\frac{n}{d}\right)$$

Now, $x \sim y$, where $x, y \in [n]$ for a equivalence reln

iff $\gcd(x, n) = \gcd(y, n)$

$$\text{Hence } n = |[n]| = \left| \bigsqcup_{d=1}^n \{x \in [n] : \gcd(x, n) = d\} \right|$$

$$\sum_{d=1}^n |\{x \in [n] : \gcd(x, n) = d\}|$$

$$n = \sum_{d|n} \phi\left(\frac{n}{d}\right) = \sum_{d|n} |\{x \in [n] : \gcd(x, n) = d\}|$$

$$d \nmid n \quad \{x \in [n] : \gcd(x, n) = d\} = \emptyset = \phi(d)$$

MOBIUS FUNCTION

Defⁿ: The Mobius fn^x $\mu: \{n \in \mathbb{Z}: n \geq 1\} \rightarrow \{-1, 0, 1\}$ is defined by

$$\mu(1) = 1$$

If $n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$ then

$$\mu(n) = \begin{cases} (-1)^k & \text{if } n_1 = n_2 = \dots = n_k \\ 0 & \text{if } \exists i \in [k] \ n_i \geq 2 \ (\text{else}) \end{cases}$$

Th: for each integer n

$$\sum_{d|n} \mu(d) = \begin{cases} 1 & \text{if } n=1 \\ 0 & \text{else } n \neq 1 \end{cases}$$

Proof: if $n=1$ then only divisor of $n=1$ is $d=1$ Hence

$$\sum_{d|n} \mu(d) = 1 = \mu(1) = 1$$

$$\text{So suppose } n \geq 2 \text{ & } n = p_1^{n_1} p_2^{n_2} \dots p_k^{n_k}$$

Then

$$\sum_{d|n} \mu(d) = \sum_{d|p_1 p_2 \dots p_k} \mu(d) = \sum_{S \subseteq [k]} (-1)^{|S|} = 0$$

Corollary: For each integer $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$

$$\frac{\phi(n)}{n} = \sum_{d|n} \frac{\mu(d)}{d}$$

Proof: $\phi(n) = n \prod_{i=1}^k \left(1 - \frac{1}{p_i}\right)$

$$= n \sum_{S \subseteq [k]} \frac{(-1)^{|S|}}{\prod_{i \in S} p_i} = n \sum_{d|p_1 p_2 \cdots p_k} \frac{\mu(d)}{d}$$

Def^r: A real valued sequence $\{x_n\}_{n=0}^{\infty}$ is said to satisfy k -term recurrence relation or simply recurrence relation

if $\exists \lambda \in \mathbb{R}^k \rightarrow \mathbb{R}$ s.t x_0, x_1, \dots, x_{k-1} is known (referred as initial condⁿ) and \forall integer $n \geq 0$

$$x_{n+k} = \lambda(x_n, x_{n+1}, \dots, x_{n+k-1})$$

$$x_k = \lambda(x_0, x_1, \dots, x_{k-1})$$

$$x_{k+1} = \lambda(x_1, x_2, \dots, x_k)$$

$$x_{k+2} = \lambda(x_2, x_3, \dots, x_{k+1})$$

We say the fn^x λ as k -term recurrence fn^x is simply recurrence fn^x

If λ is linear i.e.

$$\left. \begin{array}{l} \lambda(\bar{x} + \alpha\bar{y}) = \lambda(\bar{x}) + \alpha\lambda(\bar{y}) \\ \bar{x} = (x_1, x_2, \dots, x_k) \in \mathbb{R}^k \\ \bar{y} \in \mathbb{R}^k \\ \alpha \in \mathbb{R} \end{array} \right\}$$

& λ is a k -term recurrence relation

then such λ is called k -term linear recurrence relation

if λ is homogeneous of degree n

$$\begin{aligned} \bar{x} &= \{x_1, x_2, \dots, x_k\} \\ \alpha\bar{x} &= \{\alpha x_1, \alpha x_2, \dots, \alpha x_k\} \\ \lambda(\alpha\bar{x}) &= \lambda(\alpha x_1, \alpha x_2, \dots, \alpha x_k) = \alpha^n \lambda(x_1, x_2, \dots, x_k) \\ &= \alpha^n \lambda(\bar{x}) \end{aligned}$$

$$\lambda(\alpha\bar{x}) = \alpha^n \lambda(\bar{x})$$

& λ is a k -term recurrence relation
then such λ is called k -term homogeneous recurrence relation

Example : $\lambda: \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\lambda(x, y) = x + y \quad \leftarrow \text{Linear}$$

$$\lambda(x, y) = (x+y)^2 \quad \leftarrow \text{homogeneous of degree 2}$$

problem: Solve the linear recurrence relation

$$x_0 = 0 \quad x_1 = 1 \quad \text{and} \quad x_{n+2} = x_{n+1} + x_n \quad \forall \text{ integer } n \geq 0 \quad \} \rightarrow *$$

Soln

We first note that

$$(R^2 - R - I) (\{x_n\}_{n=0}^{\infty}) = \{0_n (=0)\}_{n=0}^{\infty}$$

step-1 : Identity the characteristic polynomial with the recurrence relation Here it is $x^2 - x - 1$

step 2: find the roots of characteristic polynomial

$$\text{Here } x^2 - x - 1 = 0$$

$$\Leftrightarrow (x - \tau)(x + \frac{1}{\tau}) = 0 \quad \tau = \frac{1 + \sqrt{5}}{2}$$

Roots are τ and $-\frac{1}{\tau}$,

step 3: formation of general soln.

$$\text{Here it is } x_n = c_1 \tau^n + c_2 \left(-\frac{1}{\tau}\right)^n$$

where $c_1, c_2 \in \mathbb{C}$

step 4: Apply initial cond'n

$$\text{Here } x_0 = 0 \quad x_1 = 1$$

$$0 = c_1 + c_2$$

$$1 = c_1 \tau + c_2 \left(-\frac{1}{\tau}\right)$$

Final : Unique solution of * is : $x_n = \left(-\frac{1}{\sqrt{5}}\right) \tau^n + \left(\frac{1}{\sqrt{5}}\right) \left(-\frac{1}{\tau}\right)^n$

Problem: Solve the linear recurrence relation

$$x_0 = 0, x_1 = 1, x_2 = 3$$
$$x_{n+3} = 5x_{n+2} - 8x_{n+1} + 4x_n \quad \forall \text{ integer } n \geq 0 \quad \star$$

$$\text{Soln: } (R^3 - 5R^2 + 8R - 4I) (\{x_n\}_{n=0}^{\infty}) = \{0_n (=0)\}_{n=0}^{\infty}$$

Step 1 - identify the characteristic polynomial associated with the recurrence relation

$$x^3 - 5x^2 + 8x - 4$$

Step 2 roots of the eqn

$$x^3 - 5x^2 + 8x - 4 = 0$$

$$(x-2)^2(x-1) = 0$$

Roots 2, 2, 1

Step 3 formation of general soln

$$x_n = (c_0 + c_1 n) 2^n + (c_3) 1^n$$

Step 4 apply initial cond'n of *

$$x_0, x_1 = 1 \quad x_2 = 3$$

Rough

$$c_0 + c_3 = 0$$

$$2(c_0 + c_1) + c_3 = 1$$

$$(c_0 + 2c_1) 4 + c_3 = 3$$

$$4\underbrace{(c_0 + 2c_1)}_{6c_1 + 2c_0} + 1 - 2c_0 - 2c_1 = 3$$

$$6c_1 + 2c_0 = 2$$

Let $\alpha_1, \alpha_2, \dots, \alpha_k$ are roots with multiplicities

Remark sup $(R - \alpha_1 I)^{\alpha_1} (R - \alpha_2 I)^{\alpha_2} \cdots (R - \alpha_k I)^{\alpha_k} (\{x_n\}_{n=0}^{\infty}) = \{0_k (=0)\}_{n=0}^{\infty}$

general solution = $(\underbrace{\quad}_{\substack{\text{Polynomial} \\ \text{of } n \text{ with degree} \\ \alpha_1-1}}) \alpha_1^n + (\underbrace{\quad}_{\substack{\text{Polynomial} \\ \text{of } n \text{ with} \\ \text{degree} \\ \alpha_2-1}}) \alpha_2^n + \dots$

$S(\mathbb{C})$ = The set of all Complex valued seq

$$\left\{ \{x_n\}_{n=0}^{\infty} \mid \forall \text{ integer } n \geq 0, x_n \in \mathbb{C} \right\}$$

claim: $S(\mathbb{C})$ form a vector space over \mathbb{C}

$$\alpha \in \mathbb{C}$$

$$\alpha \{x_n\}_{n=0}^{\infty} = \{\alpha x_n\}_{n=0}^{\infty}$$

Defn: A linear operator $T: S(\mathbb{C}) \rightarrow S(\mathbb{C})$ is a map
satisfies

$$\textcircled{a} \quad T\left(\{x_n\}_{n=0}^{\infty} + \{y_n\}_{n=0}^{\infty}\right) = T\left(\{x_n\}_{n=0}^{\infty}\right) + T\left(\{y_n\}_{n=0}^{\infty}\right)$$

$$\textcircled{b} \quad T\left(\alpha \{x_n\}_{n=0}^{\infty}\right) = \alpha T\left(\{x_n\}_{n=0}^{\infty}\right)$$

Example: $\{x_n\}_{n=0}^{\infty} \xrightarrow{} \{x_{i+n}\}_{n=0}^{\infty}$

$$\{x_0, x_1, x_2, \dots\} \xrightarrow{} \{x_1, x_2, \dots, x_n\}$$

claim: $R: S(\mathbb{C}) \rightarrow S(\mathbb{C})$
is a linear operator

$$\{x_0, x_1, x_2, \dots\} \xrightarrow{R} \{x_1, x_2, \dots\} \xrightarrow{R} \{x_2, x_3, x_4, \dots\}$$

R^2 , $R \circ R$

$$R^2 \left(\{x_n\}_{n=0}^{\infty} \right) = \{x_{2+n}\}_{n=0}^{\infty}$$

claim: R^2 is also a linear operator

claim: R^K

$$(R^3 - 5R^2 + 8R - 4I) \left(\{x_n\}_{n=0}^{\infty} \right)$$

$$= \{x_{3+n}\}_{n=0}^{\infty} - 5\{x_{2+n}\}_{n=0}^{\infty} + 8\{x_{1+n}\}_{n=0}^{\infty} - 4\{x_n\}_{n=0}^{\infty} = \{0_n\}_{n=0}^{\infty}$$

Def' The set

$$\left\{ \{x_n\}_{n=0}^{\infty} \in S(c) \mid T(\{x_n\}_{n=0}^{\infty}) = \{0_n\}_{n=0}^{\infty} \right\}$$

is called kernel of T denoted as $\ker T$

Here, $S(c)$ is an example of infinite dim. vector space

$$R(\{\alpha^n\}_{n=0}^{\infty}) = \alpha \{\alpha^n\}_{n=0}^{\infty}$$

$$R(\langle \{\alpha^n\}_{n=0}^{\infty} \rangle) = \langle \{\alpha^n\}_{n=0}^{\infty} \rangle$$

Defⁿ: Let H be an vector space over \mathbb{C} and $T: H \rightarrow H$ be a linear operator. A non-zero vector $x \in H$ is called eigen (I-gen) vector if $\exists \alpha \in \mathbb{C}$ s.t.

$$T(x) = \alpha x$$

such α is called eigen value.

$$\text{Note: } T(\langle x \rangle) = \langle x \rangle$$

Note: $R^k(\{\alpha^n\}_{n=0}^{\infty}) = \alpha^k (\{\alpha^n\}_{n=0}^{\infty})$
 Thus $\forall \alpha \in \mathbb{C}$, and \forall integer $k \geq 0$ $\{\alpha^n\}_{n=0}^{\infty}$ is the eigen vector of ~~$\{\alpha^{nk}\}_{n=0}^{\infty}$~~ R^k with eigen value α^k

Lemma: Let H be a vector space over \mathbb{C} and $T: H \rightarrow H$ be a linear operator. Then $\ker T = \{x \in H : T(x) = 0\}$ for a vector subspace

over \mathbb{C}

Proof: let $x, y \in \ker T$ and $\alpha \in \mathbb{C}$

$$\text{The } T(x + \alpha y) = T(x) + T(\alpha y) = T(x) + \alpha T(y) = 0 + 0 = 0.$$

i.e. $x + \alpha y \in \ker T$

Hence the result □

Cor: Let $p(x) \in \mathbb{C}[x]$ be a polynomial
 $R : S(\mathbb{C}) \rightarrow S(\mathbb{C})$ be a right shift linear operator
 $R(\{x_n\}_{n=0}^{\infty}) = \{x_{1+n}\}_{n=0}^{\infty}$

Then: $p(R) : S(\mathbb{C}) \rightarrow S(\mathbb{C})$ form a Linear operator &
 $\ker p(R)$ forms a vector space over \mathbb{C}

Proof: Exercise

Hint: $T_1 : H \rightarrow H$
 $T_2 : H \rightarrow H$
 then $T_1 + T_2$ is a Linear operator

* Exercise

$$\text{Im } T = \{T(x) : x \in H\}$$

is vector subspace of H over \mathbb{C}

Lemma: Let $x_{n+k} = \sum_{i=1}^k c_i x_{n+k-i}$

$$= c_1 x_{n+k-1} + c_2 x_{n+k-2} + \dots + c_k x_n$$

where $\forall i \in [k]$, $c_i \in \mathbb{C}$, be a k -term linear homogeneous recurrence relation with initial condition x_0, x_1, \dots, x_{k-1}

Then $p(R)(\{x_n\}_{n=0}^{\infty}) = \{0_n (=0)\}_{n=0}^{\infty}$ $R^0 = I$

Where $p(R) = R^k - \sum_{i=1}^k c_i R^{k-i}$

Moreover $\ker p(R)$ is a k -dimensional vector subspace of $S(\mathbb{C})$ over \mathbb{C}

$$\beta(R)(\{x_n\}_{n=0}^{\infty}) = \{0_n(=0)\}_{n=0}^{\infty}$$

$$(R^k - \sum_{i=1}^k c_i R^{k-i})(\{x_n\}_{n=0}^{\infty}) = \{0_n(=0)\}_{n=0}^{\infty}$$

we construct the map

$$\Lambda : \ker \beta(R) \rightarrow \mathbb{C}^k$$

$$\Lambda(\{x_n\}_{n=0}^{\infty}) = (x_0, x_1, \dots, x_{k-1})$$

For each $\alpha \in \mathbb{C}$ and $\{x_n\}_{n=0}^{\infty}, \{y_n\}_{n=0}^{\infty} \in \ker \beta(R)$

$$\Lambda(\{x_n\}_{n=0}^{\infty} + c\{y_n\}_{n=0}^{\infty})$$

$$\Lambda(\{x_n + cy_n\}_{n=0}^{\infty}) = (x_0 + cy_0, x_1 + cy_1, \dots, x_{k-1} + cy_{k-1})$$

$$\begin{aligned} \Lambda(\{x_n\}_{n=0}^{\infty}) + c\Lambda(\{y_n\}_{n=0}^{\infty}) &= (x_0, x_1, \dots, x_{k-1}) + c(y_0, y_1, \dots, y_{k-1}) \\ &= (x_0 + cy_0, x_1 + cy_1, \dots, x_{k-1} + cy_{k-1}) \end{aligned}$$

$$\text{i.e. } \Lambda(\{x_n\}_{n=0}^{\infty} + c\{y_n\}_{n=0}^{\infty}) = \Lambda(\{x_n\}_{n=0}^{\infty}) + c\Lambda(\{y_n\}_{n=0}^{\infty})$$

Thus Λ is a linear map

$$\text{if } \Lambda(\{x_n\}_{n=0}^{\infty}) = \Lambda(\{y_n\}_{n=0}^{\infty})$$

$$(x_0, x_1, \dots, x_{k-1}) = (y_0, y_1, \dots, y_{k-1})$$

$$\Leftrightarrow x_0 = y_0, y_1 = x_1, \dots, y_{k-1} = x_{k-1}$$

Λ is 1-1

Let $(a_0, a_1, \dots, a_{k-1}) \in \mathbb{C}^k$

$$a_{n+k} = \sum_{i=1}^k c_i a_{n+k-i}$$

Then $\{a_n\}_{n=0}^{\infty} \in \text{ker } \phi(R)$

$$\Lambda(\{a_n\}_{n=0}^{\infty}) = (a_0, a_1, \dots, a_{k-1})$$

Λ is onto

Thus Λ establishes the vector space isomorphism between $\text{ker } \phi(R)$ and \mathbb{C}^k

Since \mathbb{C}^k is a k -dimensional vector space over \mathbb{C}

we have $\text{ker } \phi(R)$ is k -dimensional vs over \mathbb{C}

(Vandermonde Determinant) . Let $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$ be (n+1) many distinct non-zero complex no.

Then

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \alpha_0 & \alpha_1 & \dots & \alpha_n \\ \alpha_0^2 & \alpha_1^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_0^n & \alpha_1^n & \dots & \alpha_n^n \end{vmatrix}_{n+1 \times n+1} = \prod_{j=0}^n \prod_{i=0}^n (\alpha_i - \alpha_j) \quad 0 \leq i < j \leq n$$

Proof: We use backward induction on n , and the fact

$$\alpha_i^p - \alpha_0^p - \alpha_0(\alpha_i^{p-1} - \alpha_0^{p-1}) = (\alpha_i - \alpha_0)(\alpha_i^{p-1}) \quad \forall p \geq 1 \quad p \in \mathbb{Z}$$

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \alpha_0 & \alpha_1 & \dots & \alpha_n \\ \alpha_0^2 & \alpha_1^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_0^n & \alpha_1^n & \dots & \alpha_n^n \end{vmatrix} = \prod_{i=1}^n (\alpha_i - \alpha_0) \begin{vmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \alpha_1^2 & \alpha_2^2 & \dots & \alpha_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{vmatrix} \underbrace{\prod_{l=2}^n (\alpha_l - \alpha_1)}_{\text{another determinant}}$$

Th: Let $\alpha_0, \dots, \alpha_{k-1}$ be distinct non-zero complex no. Then

$\ker((R - \alpha_0 I) \circ (R - \alpha_1 I) \circ \dots \circ (R - \alpha_{k-1} I))$ is a k-dim vector subspace of $S(\mathbb{C})$

Proof: claim: The vectors $\{\alpha_0^n\}_{n=0}^{\infty}$, $\{\alpha_1^n\}_{n=0}^{\infty}$, \dots , $\{\alpha_{k-1}^n\}_{n=0}^{\infty}$ are linearly independent

Proof: Let $\alpha_0, \alpha_1, \dots, \alpha_{k-1} \in \mathbb{C}$ s.t. non-trivial solution

$$c_0\{\alpha_0^n\}_{n=0}^{\infty} + c_1\{\alpha_1^n\}_{n=0}^{\infty} + \dots + c_{k-1}\{\alpha_{k-1}^n\}_{n=0}^{\infty} = \{0_n\}_{n=0}^{\infty}$$

$$\Rightarrow c_0 + c_1 + c_2 + \dots + c_{k-1} = 0$$

$$c_0\alpha_0 + c_1\alpha_1 + c_2\alpha_2 + \dots + c_{k-1}\alpha_{k-1} = 0$$

$$c_0\alpha_0^2 + c_1\alpha_1^2 + c_2\alpha_2^2 + \dots + c_{k-1}\alpha_{k-1}^2 = 0$$

$$c_0\alpha_0^{k-1} + c_1\alpha_1^{k-1} + c_2\alpha_2^{k-1} + \dots + c_{k-1}\alpha_{k-1}^{k-1} = 0$$

$$\Leftrightarrow \begin{bmatrix} 1 & 1 & \dots & 1 \\ \alpha_0 & \alpha_1 & & \alpha_{k-1} \\ \vdots & \vdots & & \vdots \\ \alpha_0^{k-1} & \alpha_1^{k-1} & & \alpha_{k-1}^{k-1} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{k-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\bar{A}\bar{C} = 0$$

$$A^{-1}A\bar{C} = A^{-1}0 = 0$$

$$\bar{C} = 0$$

and we will now prove that A^{-1} exists.

Since $\alpha_0, \alpha_1, \dots, \alpha_{k-1}$ are non-zero distinct complex no.s we have the Vandermonde det is non-zero

i.e.

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \alpha_0 & \alpha_1 & & \alpha_{k-1} \\ \vdots & \vdots & & \vdots \\ \alpha_0^{k-1} & \alpha_1^{k-1} & & \alpha_{k-1}^{k-1} \end{vmatrix}$$

$\neq 0$

Therefore,

$$c_0 = 0 = c_1 = \dots = c_{k-1}$$

This establishes the claim

$$\begin{aligned} & (R - \alpha_i I) \circ (R - \alpha_j I) \\ &= (R - \alpha_j I) \circ (R - \alpha_i I) \end{aligned}$$

commutativity

Therefore, we have for each integer i , with $0 \leq i \leq k-1$

$$(R - \alpha_0 I) (R - \alpha_1 I) \dots (R - \alpha_{k-1} I) (\{\alpha_i^n\}_{n=0}^{\infty})$$

$$= (R - \alpha_0 I) \dots (R - \alpha_{k-1} I) (R - \alpha_i I) (\{\alpha_i^n\}_{n=0}^{\infty})$$

$$= \left(\prod_{\substack{j=0 \\ j \neq i}}^{k-1} (R - \alpha_j I) \right) \left(\{0_n (= 0)\}_{n=0}^{\infty} \right)$$

$$= \{0_n (= 0)\}_{n=0}^{\infty}$$

$$\{\alpha_i^n\}_{n=0}^{\infty} \in \ker \left(\prod_{j=0}^{k-1} (R - \alpha_j I) \right)$$

$$\text{Thus, } \left\{ \{\alpha_0^n\}_{n=0}^{\infty}, \dots, \{\alpha_{k-1}^n\}_{n=0}^{\infty} \right\} \subset \ker ((R - \alpha_0 I) \circ \dots \circ (R - \alpha_{k-1} I))$$

$$\langle \{\alpha_0^n\}_{n=0}^{\infty}, \dots, \{\alpha_{k-1}^n\}_{n=0}^{\infty} \rangle = \ker ("")$$

General solⁿ of the respective recurrence relation

$$\text{is } c_0 \{\alpha_0^n\}_{n=0}^{\infty} + c_1 \{\alpha_1^n\}_{n=0}^{\infty} + \dots + c_{k-1} \{\alpha_{k-1}^n\}_{n=0}^{\infty}$$

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This means $\{\alpha_i^n\}_{n=0}^{\infty} \in \ker((R-\alpha_0I) \circ \dots \circ (R-\alpha_{k-1}I)) \quad \forall i = 0, 1, 2, \dots, k-1$

Hence, $\{\{\alpha_i^n\}_{n=0}^{\infty} : i = 0, 1, \dots, k-1\} \subset \ker((R-\alpha_0I) \circ \dots \circ (R-\alpha_{k-1}I))$

This means $\langle \{\alpha_0^n\}_{n=0}^{\infty}, \dots, \{\alpha_{k-1}^n\}_{n=0}^{\infty} \rangle$ is a vector subspace of $\ker((R-\alpha_0I) \circ \dots \circ (R-\alpha_{k-1}I))$ over \mathbb{C}

Using the claim $\dim \{\langle \{\alpha_0^n\}_{n=0}^{\infty}, \dots, \{\alpha_{k-1}^n\}_{n=0}^{\infty} \rangle\} = k$

Using the previous lemma

$$\dim \{\ker((R-\alpha_0I) \circ \dots \circ (R-\alpha_{k-1}I))\} = k$$

Thus,

$$\begin{aligned} \ker((R-\alpha_0I) \circ \dots \circ (R-\alpha_{k-1}I)) \\ = \langle \{\alpha_0^n\}_{n=0}^{\infty}, \{\alpha_1^n\}_{n=0}^{\infty}, \dots, \{\alpha_{k-1}^n\}_{n=0}^{\infty} \rangle \end{aligned}$$

Remark: $\ker(R-3I)(R-2I)$

$$= \langle \{3^n\}_{n=0}^{\infty}, \{2^n\}_{n=0}^{\infty} \rangle$$

Hence the general solⁿ

$$c_1 \{3^n\}_{n=0}^{\infty} + c_2 \{2^n\}_{n=0}^{\infty}$$

where, $c_1, c_2 \in \mathbb{C}$

$$\{c_1 3^n + c_2 2^n\}_{n=0}^{\infty}$$

i.e. $x_n = c_1 3^n + c_2 2^n \quad \forall \text{ integer } n \geq 0$

Let α be non-zero complex number Then, $\ker((R-\alpha I)^k)$
is a k -dim subspace of $S(\mathbb{C})$.

Moreover $\ker((R-\alpha I)^k)$

$$= \left\langle \{\alpha^n\}_{n=0}^{\infty}, \{n\alpha^n\}_{n=0}^{\infty}, \dots, \{n^{k-1}\alpha^n\}_{n=0}^{\infty} \right\rangle$$

Remark $\ker((R-\alpha I)^k) = \left\langle \{2^n\}_{n=0}^{\infty}, \{n2^n\}_{n=0}^{\infty} \right\rangle$

$$x_n = c_1 2^n + c_2 n 2^n = (c_1 + c_2 n) 2^n \text{ where } c_1, c_2 \in \mathbb{C}$$

Claim 1: The vectors $\{\alpha^n\}_{n=0}^{\infty}, \{n\alpha^n\}_{n=0}^{\infty}, \dots, \{n^{k-1}\alpha^n\}_{n=0}^{\infty}$ are linearly independent vectors in $S(\mathbb{C})$

Proof of claim: let $c_0, c_1, \dots, c_{k-1} \in \mathbb{C}$ such that

$$c_0(\{\alpha^n\}_{n=0}^{\infty}) + c_1(\{n\alpha^n\}_{n=0}^{\infty}) + \dots + c_{k-1}(\{n^{k-1}\alpha^n\}_{n=0}^{\infty}) \\ = \{0_n\}_{n=0}^{\infty}$$

$$\Rightarrow c_0 = 0$$

$$\Rightarrow c_0\alpha + c_1\alpha + \dots + c_{k-1}\alpha = 0$$

$$\Rightarrow c_0\alpha^2 + 2c_1\alpha^2 + \dots + 2^{k-1}c_{k-1}\alpha^2 = 0$$

$$\Rightarrow c_0\alpha^{k-1} + c_1(k-1)\alpha^{k-1} + c_2(k-1)^2\alpha^{k-1} + \dots + c_{k-1}(k-1)^{k-1} = 0$$

$$\Leftrightarrow c_0 = 0$$

$$c_1 + c_2 + \dots + c_{k-1} = 0$$

$$2c_1 + 2^2 c_2 + \dots + 2^{k-1} c_{k-1} = 0$$

$$(k-1)c_1 + (k-1)^2 c_2 + \dots + (k-1)^{k-1} c_{k-1} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & \frac{1}{2^2} & \cdots & \frac{1}{2^{k-1}} \\ \vdots & \vdots & & \vdots \\ k-1 & (k-1)^2 & \cdots & (k-1)^{k-1} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_{k-1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\therefore \text{S1}$ $\xrightarrow{\text{non-zero}}$ $\|A\|$

$$\Rightarrow c_1 = 0 = c_2 = \cdots = c_{k-1}$$

$\Rightarrow \left\{ \{\alpha^n\}_{n=0}^{\infty}, \{n\alpha^n\}_{n=0}^{\infty}, \dots, \{n^{k-1}\alpha^n\}_{n=0}^{\infty} \right\}$ is linearly independent

claim: $\{n^p \alpha^n\}_{n=0}^{\infty} \in \ker((R - \alpha I)^k)$ $p = 0, 1, 2, \dots, k-1$

$$\sum_{S \subseteq [n]} (-1)^{|S|} = 0$$

$$\sum_{S \subseteq [n]} (-1)^{|S|} |S| = \sum_{i=0}^n \binom{n}{i} (-1)^i i = \sum_{i=0}^n \frac{n!}{i!(n-i)!} (-1)^i i$$

$$= \sum_{i=1}^n \frac{n!}{i!(n-i)!} (-1)^i i$$

$$= n \sum_{i=1}^n \frac{(n-1)!}{(i-1)!(n-i)!} (-1)^i i$$

$$= n \sum_{i-1=0}^{n-1} \left(\frac{n-1}{i-1}\right) (-1)^i$$

$$= n \sum_{j=0}^{n-1} \binom{n-1}{j} (-1)^j = 0$$

$$b) \Rightarrow \sum_{sc[k]} (k+n-|s|) (-1)^{|s|} = 0 \quad \forall \text{ integer } k \geq 0$$

$$c) \Rightarrow \sum_{sc[k]} (k+n-|s|)^t (-1)^{|s|} = 0 \quad \forall \text{ integer } k \geq 0$$

$$\quad \quad \quad \forall \text{ integer } t \geq 1.$$

claim 2: $\forall \text{ integer } t \in \{0, 1, \dots, k-1\}$

$$\left\{ n^t \alpha^n \right\}_{n=0}^{\infty} \in \ker((R - \alpha I)^k)$$

Proof of claim: Note that

$$\begin{array}{ccc} \left\{ n^t \alpha^n \right\}_{n=0}^{\infty} & \xrightarrow{R} & \left\{ (1+n)^t \alpha^{n+1} \right\}_{n=0}^{\infty} \\ & & \downarrow R \\ \left\{ (k+n)^t \alpha^{k+n} \right\}_{n=0}^{\infty} & \xleftarrow{R^K} & \dots \left\{ (2+n)^t \alpha^{2+n} \right\}_{n=0}^{\infty} \end{array}$$

$$\left\{ n^t \alpha^n \right\}_{n=0}^{\infty} \xleftrightarrow{\alpha I} \left\{ n^t \alpha^{1+n} \right\}_{n=0}^{\infty} \xleftrightarrow{\alpha I} \left\{ n^t \alpha^{2+n} \right\}_{n=0}^{\infty}$$

$$\left\{ n^t \alpha^{k+n} \right\}_{n=0}^{\infty} \xleftarrow{(\alpha I)^K}$$

$$\Rightarrow \left\{ n^t \alpha^n \right\}_{n=0}^{\infty} \xrightarrow{R - \alpha I} \left\{ [(1+n)^t - n^t] \alpha^{1+n} \right\}_{n=0}^{\infty}$$

$$\quad \quad \quad \downarrow R - \alpha I$$

$$\left\{ [(2+n)^t - 2(1+n)^t + n^t] \alpha^{2+n} \right\}_{n=0}^{\infty}$$

$$\xrightarrow{R - \alpha I} \left\{ \left[\sum_{i=0}^k \binom{k}{k-i} (k+n-i)^t (-1)^i \right] \alpha^{k+n} \right\}_{n=0}^{\infty}$$

$$= \left\{ \left[\sum_{sc[k]} (k+n-|s|)^t (-1)^{|s|} \right] \alpha^{k+n} \right\}_{n=0}^{\infty}$$

$$= \left\{ 0_n (=0) \right\}_{n=0}^{\infty}$$

Hence the claim is established and

$$\langle \{\alpha^n\}_{n=0}^{\infty}, \{n\alpha^n\}_{n=0}^{\infty}, \dots, \{n^{k+1}\alpha^n\}_{n=0}^{\infty} \rangle \subseteq \ker((R - \alpha I))$$

$$(R - 2I)^3 (\{x_n\}_{n=0}^{\infty}) = \{0_{n(=0)}\}_{n=0}^{\infty}$$

Here

$$\langle \{2^n\}_{n=0}^{\infty}, \{n2^n\}_{n=0}^{\infty}, \{n^22^n\}_{n=0}^{\infty} \rangle \subseteq \ker((R - 2I)^3)$$

Hence the general soln of such recurrence relation

$$x_n = c_0 2^n + c_1 n 2^n + c_2 n^2 2^n = (c_0 + c_1 n + c_2 n^2) 2^n$$

where, $c_1, c_2, c_3 \in \mathbb{C}$

* Finding Unique soln. plugin x_0, x_1, x_2 to determine c_0, c_1, c_2

We prove

\forall integers $n > 0, k \geq 1$ and $t \geq 1$

$$\sum_{S \subseteq [k]} (k+n-|S|)^t (-1)^{|S|} = 0$$

We prove (c) using induction on 't'. Note that (b) is the "Base case". To see the "step case" we compute

$$= \sum_{S \subseteq [k]} (k+n-|S|)^t (-1)^{|S|}$$

$$= \sum_{S \subseteq [k]} (k+n-|S|)^{t-1} (k+n-|S|) (-1)^{|S|}$$

$$= \sum_{S \subseteq [k]} (k+n-|S|)^{t-1} \cancel{(k+n)}^0 (-1)^{|S|} - \sum_{S \subseteq [k]} (k+n-|S|)^{t-1} |S| (-1)^{|S|}$$

$$\therefore \sum_{S \subseteq [k]} (k+n-|S|)^{t-1} (-1)^{|S|}$$

$$\text{Induction hypothesis : } \forall \alpha \in \{1, 2, \dots, t-1\}$$

$$\forall k \geq 1, \forall n \geq 0 \quad \sum_{s \in [k]} (k+n-|s|)^{\alpha} (-1)^{|s|} = 0$$

$$= - \sum_{s \in [k]} (k+n-|s|)^{t-1} |s| (-1)^{|s|}$$

$$= \sum_{s \in [k]} (k+n-|s|)^{t-1} |s| (-1)^{|s|-1}$$

$$= \sum_{i=0}^k (k+n-i)^{t-1} \binom{k}{i} i (-1)^{i-1} = \sum_{i=1}^k (k+n-i)^{t-1} \frac{(k)_i}{i!} i (-1)^{i-1}$$

$$= k \sum_{i=1}^k (k+n-i)^{t-1} \frac{(k-1)_i}{(i-1)! (k-i)!} (-1)^{i-1}$$

$$= k \sum_{i=1}^{k-1} (k-1+n-(i-1))^{t-1} \frac{(k-1)_i}{(i-1)! (k-1-(i-1))!} (-1)^{i-1}$$

$$= k \sum_{j=0}^{k-1} (k-1+n-j)^{t-1} \binom{k-1}{j} (-1)^j$$

(By induction Hypothesis)

Th: Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be non-zero distinct complex numbers,
 m_1, m_2, \dots, m_k be positive integers.

Then, $\ker((R - \alpha_1 I)^{m_1} (R - \alpha_2 I)^{m_2} \dots (R - \alpha_k I)^{m_k})$
 $= \langle s_1, s_2, \dots, s_k \rangle$

$$s_i = \left\{ \begin{array}{l} \{\alpha_i^n\}_{n=0}^{\infty}, \{\alpha_i n\}_{n=0}^{\infty}, \dots, \{\alpha_i^{m_i-1} n^{m_i-1}\}_{n=0}^{\infty} \end{array} \right\}$$

$$\forall i \in [k]$$

Remark: $(R - 2I)^2 (R - 3I)^3 (\{x_n\}_{n=0}^{\infty}) = \{0_n (=0)\}_{n=0}^{\infty}$

$$\ker((R - 2I)^2 (R - 3I)^3) = \langle s_2, s_3 \rangle$$

$$\ker((R - 2I)^2 (R - 3I)^3) = \left\langle \begin{array}{l} \{2^n\}_{n=0}^{\infty}, \{n2^n\}_{n=0}^{\infty}, \{3^n\}_{n=0}^{\infty}, \{n3^n\}_{n=0}^{\infty} \\ \{n^2 3^n\}_{n=0}^{\infty} \end{array} \right\rangle$$

general solution

$$x_n = c_0 2^n + c_1(n) 2^n + c_2 3^n + c_3(n) 3^n + c_4(n^2) 3^n$$

$$= (c_0 + c_1 n) 2^n + (c_2 + c_3 n + c_4 n^2) 3^n$$

where $c_0, c_1, c_2, c_3, c_4 \in \mathbb{C}$

$T: H \rightarrow H$

$$\ker T = \{x \in H : T(x) = 0\}$$

Find a soln of $T(x) = b$

if y is a soln i.e. $T(y) = b$

$$\text{Then } T(y+\alpha) = T(y) + T(\alpha)$$

$$= b + 0 = b$$

then $y + \alpha$ is a soln of $T(x) = b$

$\forall \alpha \in \ker T$, $y + \alpha$ is a soln of $T(x) = b$

Problem : solve the recurrence x_n with initial condn x_0, x_1

$$(R - 2I)^2 (\{x_n\}_{n=0}^{\infty}) = \{\alpha_n (= 3^n + 2^n)\}_{n=0}^{\infty} \quad \text{with the initial condn } x_0, x_1$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ T & x & b \end{matrix} \quad \text{--- } *$$

soln Step-1 : we need to solve (formation of general soln)

$$(R - 2I)^2 (\{x_n\}_{n=0}^{\infty}) = \{(0_n = 0)\}_{n=0}^{\infty} \quad \text{--- } **$$

The general soln $= (c_0 + q_n)2^n$, where $c_0, q \in \mathbb{C}$

Step 2 : formation of particular soln / trial soln

let $\{y_n\}_{n=0}^{\infty}$ be a particular soln i.e.

$$(R - 2I)^2 (\{y_n\}_{n=0}^{\infty}) = \{3^n + 2^n\}_{n=0}^{\infty}$$

suppose $y_n = \underbrace{d_1 3^n + d_2 n + d_3}_\text{Trial soln} \quad \forall \text{ integer } n \geq 0$

Then we have

$$3^n + 2^n = y_{2+n} - 4y_{1+n} + 4y_n$$

$$3^{n+2n} = (d_1 3^{2+n} + d_2(2+n) + d_3) - 4(d_1 3^{1+n} + d_2(1+n) + d_3) + 4(d_1 3^n + d_2 n + d_3)$$

$$= d_1 3^n + d_2 n + (d_3 - 2d_2)$$

Step-3 : Formation of particular soln of $\textcircled{*}$ equation

Hence equating the coefficients of the respective

$$d_1 = 1$$

$$d_2 = 2$$

$$d_3 - 2d_2 = 0 \Rightarrow d_3 = 2d_2 = 4$$

Hence, $y_n = 3^n + 2n + 4$ forms a particular soln of $\textcircled{*}$

Step-4 : Formation of general soln of $\textcircled{*}$

$$x_n = (c_0 + c_1 n) 2^n + (3^n + 2n + 4)$$

where, $c_0, c_1 \in \mathbb{C}$

Final: Using the initial cond'n (x_0, x_1) find value of c_0, c_1

$$\text{Hence } c_0 = x_0 - 5$$

$$c_1 = \frac{x_1 - 2x_0 + 1}{2}$$

Defn: A k -term recurrence relation is said to be linear - non homogeneous, is of the form

$$x_{n+k} = a_n + \sum_{i=1}^k c_i x_{n+k-i} \quad \forall \text{ integers } n \geq 0 \quad \textcircled{*}$$

where $\{x_n\}_{n=0}^{\infty} \in S(C)$, $c_i \in C \quad \forall i \in [k]$ & $x_0, x_1, \dots, x_{k-1} \in C$

check that $\textcircled{*}$ can be expressed as

$$\textcircled{i} \quad P(R) (\{x_n\}_{n=0}^{\infty}) = \{x_n\}_{n=0}^{\infty}$$

$$\text{where } P(R) = R^k - \sum_{i=1}^k c_i R^{k-i}$$

Given $\{x_n\}_{n=0}^{\infty}, k \geq 0$ is an integer and $C \subseteq C$

$$x_n = c n^k \gamma^n \quad P(\gamma) = 0$$

Eg $\textcircled{a} n \gamma^n$

$\textcircled{b} n^2 \gamma^n$

root with multiplicity 1 of $P(z)$

TRIAL SOLⁿ

$$(a_0 + a_1 n + \dots + a_k n^k) \gamma^n$$

$a_0, a_1, \dots, a_k \in C$

$$\textcircled{a} (a_0 + a_1 n) \gamma^n$$

$$\textcircled{b} (a_0 + a_1 n + a_2 n^2) \gamma^n$$

$x_n = c n^k \gamma^n$
if root γ has multiplicity of m

$$(R - \gamma I)^m$$

$$x_n = \begin{cases} c \gamma^n \cos \beta n \\ c \gamma^n \sin \beta n \end{cases}$$

$$(a \cos \beta n + b \sin \beta n) \gamma^n \quad a, b \in \mathbb{R}$$

$$(37)(28)(51) = (5)(2)(8)(51)$$

STIRLING'S NUMBER OF FIRST KIND

Defn . let $n \geq 1$ and $k \geq 1$ be positive integers with $n \geq k$

A cycle of length l is a permutation $\pi: [n] \xrightarrow{\text{onto}} [n]$ such that there are l integers $i_1, i_2, i_3, \dots, i_l \in [n]$

s.t. $\pi(i_1) = i_2$

$\pi(i_2) = i_3$

and $\forall j \in [n] \setminus \{i_1, i_2, \dots, i_l\}$

$\pi^l(j) = j$

$\pi(i_{l+1}) = i_1$

$\pi(i_1) = i_1$

we denote such cycle of length l by (i_1, i_2, \dots, i_l)

we call two cycle $(i_1 \dots i_l)$ (j_1, j_2, \dots, j_m) are disjoint if the corresponding sets $\{i_1, i_2, \dots, i_l\}$ are $\{j_1, \dots, j_m\}$ are disjoint with each other

Th (cycle Decomposition Theorem) : For each permutation

$\pi: [n] \xrightarrow{\text{onto}} [n]$ there exist a positive integer $k = k(\pi)$

(i.e.) k depends on π s.t. π can be written as a product of k many disjoint cycle of

lengths

They have length some

~~different respective~~
may be same

$$(1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7) = \underbrace{(1 \ 4 \ 7)(2 \ 3)(5 \ 6)}_{k=3}$$

Proof: We construct an equivalence relation α on $[n]$ over $[n]$ by relating $i \sim j \Leftrightarrow \exists \alpha \in \mathbb{Z}$ s.t. $\pi^\alpha(i) = j$. Since $[n]$ is a finite set, \exists 1 positive integer $K > 1$ s.t. $[n] = A_1 \sqcup A_2 \sqcup \dots \sqcup A_K$ where, $\forall \beta \in [K]$, A_β is an equivalence class

$$A_\beta = \{i_1, \dots, i_k\} \\ = \{i_1, \pi(i_1), \pi^2(i_1), \dots, \pi^{k-1}(i_1)\}$$

Defn: The stirling no. of first kind is defined by

$$s_k(n) = s(n, k) = (-1)^{n-k} c(n, k)$$

$c(n, k)$ denotes the total number of permutations of $[n]$, which can be written as product of k disjoint cycles of respective length

$$c(n, k) = |\{\pi: [n] \xrightarrow{\text{onto}} [n] \mid \pi \text{ can be written as a product of } k \text{ many disjoint cycles (of respective length)}\}|$$

Recall (i) is a cycle of length 1

Th: Let n & k be positive integers with $n \geq k$ then,

$$c(n, k) = c(n-1, k-1) + (n-1)c(n-1, k)$$

Proof: Exercise

PIGEON HOLE PRINCIPLE

Th (PHP-Ver-I) if $(n+1)$ many objects (balls) are distributed "among" n many boxes, where n is a positive integer, then \exists at least one box which contains TWO or more objects (balls)

Proof : Suppose (if possible) none of the n -many boxes contains two or more objects (balls) i.e each of them contains at most 1 ball (object)

Then there are at most n many objects (balls)

However, we are distributing $(n+1)$ many objects (balls)
 $\Rightarrow \Leftarrow$ contradiction

Th (PHP-Ver-II) - if $(m+n+1)$ many objects (balls) are distributed among n many boxes where m, n are positive integers, then \exists at least one box, which contains at least $(m+1)$ many objects (balls)

Proof :

(PHP - strong version) : if $m_1 + m_2 + \dots + m_n = n+1$ many objects distributed among n labelled boxes with labelling $1, 2, 3, \dots, n$, where m_1, \dots, m_n are positive integers, then \exists at least one label $i \in [n]$, such that box with label i contains at least m_i object (balls)

Ex.

Comment : "objects" "(balls)" are "pigeons" and boxes are "pigeonholes".

PP-I Let m, n be positive integers with $\gcd(m, n) = 1$
Let a, b be two integers with $0 \leq b \leq n-1, 0 \leq a \leq m-1$

Then \exists positive integer x satisfies

$$x \equiv a \pmod{m}$$

$$x \equiv b \pmod{n}$$

Solⁿ: We apply PHP.

BALL CONSTRUCTION : We construct the following n nos.

$$\text{Let } a = \gamma_1 \pmod{n}, a+m = \gamma_2 \pmod{n} \dots a+(n-1)m = \gamma_n \pmod{n}$$

claim : if $a+im = \gamma_{i+1} \pmod{n} \quad \forall i \in \{0, \dots, n-1\}$ then

$$i+1, j+1 \in [n] \quad \gamma_{i+1} \neq \gamma_{j+1}$$

Proof : Suppose $\exists i+1, j+1 \in [n]$ with $i+1 \leq j+1$, s.t.

$$\gamma_{i+1} = \gamma_{j+1}$$

$$\Rightarrow n \mid \gamma_{j+1} - \gamma_{i+1}$$

$$\Rightarrow n \mid (a+jm) - (a+im) \Rightarrow n \mid (j-i)m \Rightarrow n \mid j-i$$

$\downarrow \gcd(m, n)$

$$\Rightarrow \text{Prob} : i-j=0 \Rightarrow j=1$$

\Leftrightarrow because $i+1 < j+1$

* Required $(n+1)$ many balls are x_1, x_2, \dots, x_n, b

Box Construction : Required n many boxes are

$0, 1, 2, \dots, n-1$

which are all the remainder of modulo n

Note $\{0, 1, \dots, n-1\} = \{x_1, x_2, \dots, x_n\}$

Hence, $\exists k \in [n]$ such that

$$ak \equiv b \pmod{n}$$

$$\text{i.e. } a + (k-1)m \equiv b \pmod{n}$$

Required sol'n for x is $a + (k-1)m$

$$a + (k-1)m \equiv a \pmod{m}$$

$$a + (k-1)m \equiv b \pmod{n}$$

This is Chinese Remainder Theorem

BALL CONSTRUCTION \longleftrightarrow sample space

BOX CONSTRUCTION \longleftrightarrow Random variable

Q: What is sample space and How it is constructed?

Ans: Performing a Random experiment

RE: choose an element from A

$$|A| \geq 2$$

$$\Omega = A$$

sample space is set of all outputs of the RE

Defⁿ: Let Ω be a sample space.

The set of subsets of Ω , say F is called σ -algebra or informations if F satisfies the following properties

- (a) $\Omega \in F$ $F \subset 2^\Omega$
- (b) if $A \in F$, then $A^c \in F$
- (c) if \forall integer $n \geq 1$, $A_n \in F$, then $\bigcup_{n=1}^{\infty} A_n \in F$

Ex: (i) $F = 2^\Omega$

(ii) $F = \{\emptyset, A, A^c, \Omega\}$

Defⁿ: A fn^r $m: F \rightarrow [0, \infty]$ is called ^{measure} if it satisfies the following properties

- (a) $m(\emptyset) = 0$
- (b) $m\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} m(A_n)$

$$\{m(A_1) + \dots + m(A_n)\}_{n=1}^{\infty}$$

The pair (Ω, F) is called measurable space and the triplet (Ω, F, m) is called the measure space

Defn: Let (Ω, \mathcal{F}) be a measurable space, and $P_\Omega: \mathcal{F} \rightarrow [0, \infty]$ be a measure such measure is called probability measure if

$$P_\Omega(\Omega) = 1$$

The triplet $(\Omega, \mathcal{F}, P_\Omega)$ is called the probability space

Th (PHP-Dual Version) if $(n-1)$ -many objects (balls) are distributed among n boxes, i.e., among n many boxes, then \exists at least one empty box (i.e., \exists a box which contains no ball)

Proof: Exercise

Ex: RE: choose a point from $[n]$. This yields the sample space $\Omega = [n]$

\mathcal{F} = The σ -algebra = $2^\Omega = \langle \{\{x\}: x \in \Omega\} \rangle$

$P_\Omega: \mathcal{F} \rightarrow [0, 1]$

$$\begin{aligned} P_\Omega(\{x\}) &\stackrel{\text{def}}{=} \frac{1}{n} \quad \forall x \in \Omega \iff \{x\} \in \mathcal{F} \\ &= \frac{1}{|\Omega|}. \end{aligned}$$

$$P_\Omega(A) = P_\Omega\left(\bigcup_{x \in A} \{x\}\right) = \sum_{x \in A} P_\Omega(\{x\}) = \sum_{x \in A} \frac{1}{|\Omega|} = \frac{|A|}{|\Omega|}$$

$(\Omega, \mathcal{F}, P_\Omega)$ is a probability space

$$\Omega = \{1, 2, 3, 4\}$$

$$\begin{aligned} P(\{1\}) &= \frac{1}{4} && \left. \begin{array}{l} \text{Uniform} \\ \text{distribution} \end{array} \right\} & P_\Omega(\{2\}) &= 0 \\ P(\{2\}) &= \frac{1}{4} && \left. \begin{array}{l} \text{Uniform} \\ \text{distribution} \end{array} \right\} & P_\Omega(\{3\}) &= \frac{2}{4} \\ P(\{3\}) &= \frac{1}{4} && \left. \begin{array}{l} \text{Non-uniform} \\ \text{distribution} \end{array} \right\} & P_\Omega(\{4\}) &= \frac{1}{4} \end{aligned}$$

Defⁿ: Let $(\Omega, \mathcal{F}, P_r)$ be a probability space
A fn^x $X: \Omega \rightarrow \mathbb{R}$ is called discrete random variable
if $X(\Omega)$ is a countable subset of \mathbb{R} and $\forall x \in X(\Omega)$,
 $X^{-1}(\{x\}) \in \mathcal{F}$

Ex: a) $X: \Omega \rightarrow \{0, 1\}$, $A \subset \Omega$

$$X(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{else} \end{cases}$$

$$X(\Omega) = \{0, 1\}$$

$$X^{-1}(\{0\}) = A^c \in \mathcal{F}$$

$$X^{-1}(\{1\}) = A \in \mathcal{F}$$

$$X^{-1}(\{0\}) = \{\omega \in \Omega : X(\omega) = 0\}$$

$$P_r(\{\omega \in \Omega : X(\omega) = 1\}) = P_r(X^{-1}(\{1\})) = P_r(A) = \frac{|A|}{n} = P_r(X=1)$$

$$P_r(\{\omega \in \Omega : X(\omega) = 0\}) = P_r(X^{-1}(\{0\})) = P_r(A^c) = \frac{|A^c|}{n} = P_r(X=0)$$

b) $Y: \Omega \rightarrow \{-1, 1\}$

$$Y(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ -1 & \text{else} \end{cases}$$

$$P_r(Y=1) = \frac{|A|}{n}$$

$$P_r(Y=-1) = \frac{|A^c|}{n}$$

Def : Let (Ω, \mathcal{F}, P) be a probability space
and $X: \Omega \rightarrow \mathbb{R}$ be a discrete random variable

Note that $|X|: \Omega \rightarrow \mathbb{R}$ is also a discrete random variable

$$|X|(\omega) = (|X(\omega)|) \quad \forall \omega \in \Omega$$

$$E[X] = \sum_{k=1}^{\infty} x_k P_{\mathcal{F}}(X=x_k) = \sum_{x \in X(\Omega)} x P_{\mathcal{F}}(X=x) \in \mathbb{R}, \{\infty, -\infty\}$$

$$E[|X|] = \sum_{k=1}^{\infty} |x_k| P_{\mathcal{F}}(X=x_k) < \infty$$



RP: choose a labelled box with labelling $1, 2, \dots, n$

$\Omega =$ The set of labelled boxes = $\{1, 2, 3, \dots, n\} = [n]$

$$F = 2^n$$

$$Pr(\{\omega\}) = \frac{1}{|\Omega|} = \frac{1}{n} \quad \forall \omega \in \Omega$$

(BOX CONSTRUCTION)

Here we construct probability space (Ω, F, Pr)

$B =$ set of balls, $|B| = n-1$

[Here we "distribute" $(n-1)$ many balls into the boxes]

We construct $X: \Omega \rightarrow \mathbb{R}$

$X(\omega) =$ No. of balls in the labelled box ω

$$X(\Omega) \subset \{0, 1, \dots, n-1\}$$

$$X(\Omega) = \{X(\omega) : \omega \in \Omega\}$$

BALL Distribution
construction

AIM: To compute $E[X]$

Let $b \in B$, $X_b: \Omega \rightarrow \{0, 1\}$

$$X_b(\omega) = \begin{cases} 1 & \text{if } b \in \omega \\ 0 & \text{else} \end{cases}$$

$$Pr(X_b=1) = Pr(\{\omega \in \Omega : X_b(\omega)=1\}) = \frac{1}{n}$$

$$X = \sum_{b \in B} X_b$$

$$\text{i.e. } X(\omega) = \sum_{b \in B} X_b(\omega) \quad \forall \omega \in \Omega$$

$$\begin{aligned} \text{Hence, } E[X] &= E\left(\sum_{b \in B} X_b\right) = \sum_{b \in B} E[X_b] = \sum_{b \in B} Pr(X_b=1) \\ &= \sum_{b \in B} \frac{1}{n}, \\ &= \frac{1}{n} \sum_{b \in B} 1 = \frac{1}{n} |B| = \frac{|B|}{n} \\ &= \frac{n-1}{n} = 1 - \frac{1}{n} < 1 \end{aligned}$$

First Moment method : if $E[X] < 1$, then $\Pr(X=0) > 0$

Using Markov's Inequality, we have

$$\Pr(X \geq 1) \leq E[X] < 1$$

$$\text{we know } \Pr(X=0) + \Pr(X \geq 1) = 1$$

$$\text{Hence, } \Pr(X=0) > 0$$

$$\text{i.e. } \Pr(\{\omega \in \Omega : X(\omega)=0\}) > 0$$

$$\exists \omega \in \Omega \text{ s.t. } X(\omega)=0 \quad \text{i.e. } \forall b \in B, X_b(\omega)=0$$

In other words \exists an empty box

→ Conclusion of PHP

RE: choose a permutation $\pi: [n] \rightarrow [n]$

Ω : The set of all permutations

$$\left\{ \pi \mid \pi: [n] \xrightarrow[\text{onto}]{1-1} [n] \right\} \quad |\Omega| = n!$$

$$F = 2^{\Omega}$$

$$\Pr(\{\pi\}) = \frac{1}{n!}$$

Here we construct the probability space (Ω, F, \Pr)

Let $k \in [n]$

BALLS

$$X_k: \Omega \rightarrow \mathbb{R}$$

$$X_k(\pi) = \begin{cases} 1 & \text{if } \pi(k) = k \\ 0 & \text{else} \end{cases}$$

$$\Pr(X_k=1) = \Pr(\{\pi \in \Omega \mid X_k(\pi) = 1\}) \\ = \frac{(n-1)!}{n!} = \frac{1}{n}$$

We construct $X: \Omega \rightarrow \mathbb{R}$

$$X(\pi) = \sum_{k \in [n]} X_k(\pi) \quad \forall \pi \in \Omega$$

$$\text{i.e. } X = X_1 + X_2 + \dots + X_n$$

$$\text{Hence } E[X] = E[X_1 + X_2 + X_3 + \dots + X_n] = E[X_1] + E[X_2] + \dots + E[X_n] \\ = \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} \\ = 1$$

$$\pi = \begin{pmatrix} 1 & 2 & \dots & n \\ 1 & 2 & \dots & n \end{pmatrix}$$

$$X(\pi) = X_1(\pi) + X_2(\pi) + \dots + X_n(\pi) = n$$

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ 2 & 1 & 3 & \dots & n \end{pmatrix} \Rightarrow X(\sigma) = n-2$$

Hence, X is a non-constant random variable

$$\text{Also, } X(\omega) \subset \{0, 1, \dots, n\}$$

Since, $E[X] = 1$ & $X(\omega) \subset \{0, 1, \dots, n\}$ & X is non-constant. $\exists \pi \in \Omega$ s.t. $X(\pi) = 0$

$$\Leftrightarrow X_1(\pi) = 0 = X_2(\pi) = \dots = X_n(\pi)$$

Hence $\exists \pi \in \Omega$ s.t. $\forall k \in [n], \pi(k) \neq k$

↑ Existence of Derangement Permutation