Real Analysis 05-08-2024

Up till now

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function, one-one, onto, equivalence, finite, infinite, countable, uncountable, at most countable

Example

$\mathbb{N} \subsetneq \mathbb{Z}$

$$\exists f : \mathbb{N} \to \mathbb{Z} \quad f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ -\frac{n-1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

$\Rightarrow \mathbb{N} \sim \mathbb{Z}$

> Remark
> A finite set can't be equivalent to one of it's proper subsets. However, this is possible for infinite sets.

Definition: Sequence

By a sequence, we mean a function f defined on the set $\mathbb{N}$. If $f(n) = x_n, \forall n \in \mathbb{N}$, it is customary to denote the sequence f by the symbol $\langle x_n \rangle$, or sometimes by $x_1, x_2, x_3, \cdots$. The value of f, that is the element x_n are called the terms the sequence.
if $x_n \in A, \forall n \in \mathbb{N}, \{x_n\}$ is said to be a sequence in A or a sequence of elements A.

Theorem: Every infinite subset of countable set A is countable

Proof

Let, A is a countable set and E be a subset of A. i.e.,$E \subseteq A \sim \mathbb{N}$
we can define the elements of A in a sequence $\{a_n\}$ of distinct elements.
$g : \mathbb{N} \to E$
$g(k) = a_{n_k}$ where, n_k is the smallest integer greater than $n_{k-1} \ni a_{n_k} \in E$ and n_1 the smallest integer $\ni a_{n_1} \in E$

Definition

Let,A and Ω be sets, and $\forall \alpha \in A, \exists E_\alpha \subseteq \Omega$ (E_α is associated with α)
The set whose elements are from E_α will be denoted as $\{E_\alpha\}$
$x \in S = \bigcup_{\alpha \in A} E_\alpha \iff \exists \alpha \in A \ni x \in E_\alpha$
$A = \{1, 2, \cdots, n\} \Rightarrow S = \bigcup_{i=1}^{n} E_i = E_1 \cup E_2 \cup \cdots \cup E_n$
$A = \mathbb{N} \Rightarrow S = \bigcup_{i=1}^{\infty} E_i$
$x \in P = \bigcap_{\alpha \in A} E_\alpha \iff \forall \alpha \in A, x \in E_\alpha$
$A = \{1, 2, \cdots, n\} \Rightarrow P = \bigcap_{i=1}^{n} E_i = E_1 \cap E_2 \cap \cdots \cap E_n$
$A = \mathbb{N} \Rightarrow P = \bigcap_{i=1}^{\infty} E_i$

Example

1. $E_1 = \{1, 2, 3\}, E_2 = \{2, 3, 4\}$
   $E_1 \cup E_2 = \{1, 2, 3, 4\}$
   $E_1 \cap E_2 = \{2, 3\}$
2. $x \in A = (0, 1] \Rightarrow E_x = \{y \in \mathbb{R} | 0 < y < x\} = (0, x) \quad \Omega = (0, 1)$
   $\bigcup_{x \in A} E_x = E_1$
   $\bigcap_{x \in A} E_x = \phi$