UNIT 1 INTRODUCTION TO PROBABILITY

Structure

1.1 INTRODUCTION

In our daily lives, we face many situations when we are unable to forecast the future with complete certainty. That is, in many decisions, the uncertainty is faced. Need to cope up with the uncertainty leads to the study and use of the probability theory. The first attempt to give quantitative measure of probability was made by Galileo (1564-1642), an Italian mathematician, when he was answering the following question on the request of his patron, the Grand Duke of Tuscany, who wanted to improve his performance at the gambling tables: "With three dice a total of 9 and 10 can each be produced by six different combinations, and yet experience shows that the number 10 is oftener thrown than the number 9?" To the mind of his patron the cases were (1, 2, 6), (1, 3, 5), (1, 4, 4), (2, 2, 5), (2, 3, 4), (3, 3, 3) for 9 and (1, 3, 6), (1, 4, 5), (2, 2, 6), (2, 3, 5), (2, 4, 4), (3, 3, 4) for 10 and hence he was thinking that why they do not occur equally frequently i.e. why there chances are not the same? Galileo makes a careful analysis of all the cases which can occur, and he showed that out of the 216 possible cases 27 are favourable to the appearance of the number 10 since permutations of (1, 3, 6) are (1, 3, 6), (1, 6, 3), (3, 1, 6), (3, 6, 1), (6, 1, 3), (6, 3, 1) i.e. number of permutations of (1, 3, 6) is 6; similarly, the number of permutations of (1, 4, 5), (2, 2, 6), (2, 3, 5), (2, 4, 4), (3, 3, 4) is 6, 3, 6, 3, 3 respectively and hence the total number of cases come out to be 6 + 6 + 3 + 6 + 3 + 3 = 27 whereas the number of favourable cases for getting a total of 9 on three dice are 6 + 6 + 3 + 3 + 6 + 1 = 25. Hence, this was the reason for 10 appearing oftener thrown than 9. But the first foundation was laid by the two mathematicians Pascal (1623-62) and Fermat (1601-65) due to a gambler's dispute in 1654 which led to the creation of a mathematical theory of probability by them. Later, important contributions were made by various researchers including Huyghens (1629 - 1695), Jacob Bernoulli (1654-1705), Laplace (1749-1827), Abraham De-Moivre (1667-1754), and Markov (1856-1922). Thomas Bayes (died in 1761, at the age of 59) gave an important technical result known as Bayes' theorem, published after his death in 1763, using which probabilities can be revised on the basis of some new information. Thereafter, the probability, an important branch of Statistics, is being used worldwide.

We will start this unit with very elementary ideas. In other words, we are assuming that reader knows nothing about probability. We will go step by step clearing the basic ideas which are required to understand the probability. In this unit, we will first present the various terms which are used in the definition of probability and then we will give the classical definition of probability and simple problems on it.

Objectives

After studing this unit, you should be able to:

- define and give examples of random experiment and trial;
- define and give examples of sample space, sample point and event;
- explain mutually exclusive, equally likely, exhaustive and favourable cases and why they are different in nature and how much these terms are important to define probability;
- explain the classical definition of probability;
- solve simple problems based on the classical definition of probability; and
- distinguish between odds in favour and odds against the happening of an event.

Random Experiment

An experiment in which all the possible outcomes are known in advance but we cannot predict as to which of them will occur when we perform the experiment, e.g. Experiment of tossing a coin is random experiment as the possible outcomes head and tail are known in advance but which one will turn up is not known.

Similarly, 'Throwing a die' and 'Drawing a card from a well shuffled pack of 52 playing cards 'are the examples of random experiment.

Trial

Performing an experiment is called trial, e.g.

(i) Tossing a coin is a trial.

(ii) Throwing a die is a trial.

1.2 SAMPLE SPACE, SAMPLE POINT AND EVENT

Sample Space

Set of all possible outcomes of a random experiment is known as sample space and is usually denoted by S, and the total number of elements in the sample space is known as size of the sample space and is denoted by n(S), e.g.

(i) If we toss a coin then the sample space is

$S = \{H, T\}$, where H and T denote head and tail respectively and $n(S) = 2$.

(ii) If a die is thrown, then the sample space is

$$S = \{1, 2, 3, 4, 5, 6\} \text{ and } n(S) = 6. \left[\begin{array}{l}\because \text{die has six faces} \\ \text{numbered } 1, 2, 3, 4, 5, 6\end{array}\right]$$

(iii) If a coin and a die are thrown simultaneously, then the sample space is
S = {H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6} and n(S) = 12.

where H1 denotes that the coin shows head and die shows 1 etc.

Note: Unless stated the coin means an unbiased coin (i.e. the coin which favours neither head nor tail).

(iv) If a coin is tossed twice or two coins are tossed simultaneously then the sample space is

S = {HH, HT, TH, TT},

where HH means both the coins show head, HT means the first coin shows head and the second shows tail, etc. Here, n(S) = 4.

(v) If a coin is tossed thrice or three coins are tossed simultaneously, then the sample space is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT} and n(S) = 8.

(vi) If a coin is tossed 4 times or four coins are tossed simultaneously then the sample space is

S = {HHHH, HHHT, HHTH, HTHH, THHH, HHTT, HTHT, HTTH, THHT, THTH, TTHH, HTTT, THTT, TTHT, TTTH, TTTT} and n(S) = 16.

(vii) If a die is thrown twice or a pair of dice is thrown simultaneously, then sample space is

S = {(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6),
(2, 1), (2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6),
(4, 1), (4, 2), (4, 3), (4, 4), (5, 5), (4, 6),
(5, 1), (5, 2), (5, 3), (5, 4), (5, 5), (5, 6),
(6, 1), (6, 2), (6, 3), (6, 4), (6, 5), (6, 6)}

here, e.g., (1, 4) means the first die shows 1 and the second die shows 4.

Here, n(S) = 36.

(viii) If a family contains two children then the sample space is

S = $\{B_1B_2, B_1G_2, G_1B_2, G_1G_2\}$

where B_i denotes that i^{th} birth is of boy, i = 1, 2, and

G_i denotes that i^{th} birth is of girl, i = 1, 2.

This sample space can also be written as

S = {BB, BG, GB, GG}

(ix) If a bag contains 3 red and 4 black balls and

(a) One ball is drawn from the bag, then the sample space is

$\{R_1, R_2, B_1, B_2, B_3, B_4\}$, where R_1, R_2, R_3 denote three red balls and B_1, B_2, B_3, B_4

denote four black balls in the bag.

(b) Two balls are drawn one by one without replacement from the bag, then the sample space is

$$S = \{R_1R_2, R_1R_3, R_1B_1, R_1B_2, R_1B_3, R_1B_4, R_2R_1, R_2R_3, R_2B_1, R_2B_2, R_2B_3, R_2B_4, R_3R_1, R_3R_2, R_3B_1, R_3B_2, R_3B_3, R_3B_4, B_1R_1, B_1R_2, B_1R_3, B_1B_2, B_1B_3, B_1B_4, B_2R_1, B_2R_2, B_2R_3, B_2B_1, B_2B_3, B_2B_4, B_3R_1, B_3R_2, B_3R_3, B_3B_1, B_3B_2, B_3B_4, B_4R_1, B_4R_2, B_4R_3, B_4B_1, B_4B_2, B_4B_3\}$$

Note: It is very simple to write the above sample space – first write all other balls with R_1, then with R_2, then with R_3 and so on.

Remark 1: If a random experiment with x possible outcomes is performed n times, then the total number of elements in the sample is x^n i.e. $n(S) = x^n$, e.g. if a coin is tossed twice, then n(S) = 2^2 =4; if a die is thrown thrice, then n(S) = 6^3 = 216.

Now you can try the following exercise.

E1) Write the sample space if we draw a card from a pack of 52 playing cards.

Sample Point

Each outcome of an experiment is visualised as a sample point in the sample space. e.g.

(i) If a coin is tossed then getting head or tail is a sample point.

(ii) If a die is thrown twice, then getting (1, 1) or (1, 2) or (1, 3) or…or (6, 6) is a sample point.

Event

Set of one or more possible outcomes of an experiment constitutes what is known as event. Thus, an event can be defined as a subset of the sample space, e.g.

i) In a die throwing experiment, event of getting a number less than 5 is the set {1, 2, 3, 4},

which refers to the combination of 4 outcomes and is a sub-set of the sample space

= {1, 2, 3, 4, 5, 6}.

ii) If a card is drawn form a well-shuffled pack of playing cards, then the event of getting a card of a spade suit is

$\{1_S, 2_S, 3_S, ..., 9_S, 10_S, J_S, Q_S, K_S\}$

where suffix S under each character in the set denotes that the card is of spade and J, Q and K represent jack, queen and king respectively.

1.3 EXHAUSTIVE CASES, FAVOURABLE CASES, MUTUALLY EXCLUSIVE CASES AND EQUALLY LIKELY CASES

Exhaustive Cases

The total number of possible outcomes in a random experiment is called the exhaustive cases. In other words, the number of elements in the sample space is known as number of exhaustive cases, e.g.

(i) If we toss a coin, then the number of exhaustive cases is 2 and the sample space in this case is {H, T}.

(ii) If we throw a die then number of exhaustive cases is 6 and the sample space in this case is {1, 2, 3, 4, 5, 6}

Favourable Cases

The cases which favour to the happening of an event are called favourable cases. e.g.

(i) For the event of drawing a card of spade from a pack of 52 cards, the number of favourable cases is 13.

(ii) For the event of getting an even number in throwing a die, the number of favourable cases is 3 and the event in this case is {2, 4, 6}.

Mutually Exclusive Cases

Cases are said to be mutually exclusive if the happening of any one of them prevents the happening of all others in a single experiment, e.g.

(i) In a coin tossing experiment head and tail are mutually exclusive as there cannot be simultaneous occurrence of head and tail.

Equally Likely Cases

Cases are said to be equally likely if we do not have any reason to expect one in preference to others. If there is some reason to expect one in preference to others, then the cases will not be equally likely, For example,

(i) Head and tail are equally likely in an experiment of tossing an unbiased coin. This is because if someone is expecting say head, he/she does not have any reason as to why he/she is expecting it.

(ii) All the six faces in an experiment of throwing an unbiased die are equally likely.

You will become more familiar with the concept of “equally likely cases” from the following examples, where the non-equally likely cases have been taken into consideration:

(i) Cases of “passing” and “not passing” a candidate in a test are not equally likely. This is because a candidate has some reason(s) to expect “passing” or “not passing” the test. If he/she prepares well for the test, he/she will pass the test and if he/she does not prepare for the test, he/she will not pass. So, here the cases are not equally likely.

(ii) Cases of "falling a ceiling fan" and "not falling" are not equally likely. This is because, we can give some reason(s) for not falling if the bolts and other parts are in good condition.

1.5 CLASSICAL OR MATHEMATICAL PROBABILITY

Let there be 'n' exhaustive cases in a random experiment which are mutually exclusive as well as equally likely. Let 'm' out of them be favourable for the happening of an event A (say), then the probability of happening event A (denoted by P (A)) is defined as

$$P(A) = \frac{\text{Number of favourable cases for event A}}{\text{Number of exhaustive cases}} = \frac{m}{n} \qquad \ldots (1)$$

Probability of non-happening of the event A is denoted by $P(\overline{A})$ and is defined as

$$P(\overline{A}) = \frac{\text{Number of favourable cases for event } \overline{A}}{\text{Number of exhaustive cases}} = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(A)$$

So, $\quad P(A) + P(\overline{A}) = 1$

Therefore, we conclude that, the sum of the probabilities of happening an event and that of its complementary event is 1.

Let us now prove that $\mathbf{0 \le P(A) \le 1}$

Proof: We know that

$0 \le$ Number of favourable cases $\le$ No of exhaustive cases

[$\because$ Number of favourable cases can never be negative and can at the most be equal to the number of exhaustive cases.]

$\Rightarrow 0 \le m \le n$

Dividing both sides by n, we get

$\Rightarrow \frac{0}{n} = \frac{m}{n} = \frac{n}{n}$

$\Rightarrow 0 \le P(A) \le 1$

Remark 2: Probability of an impossible event is always zero and that of certain event is 1, e.g. probability of getting 7 when we throw a die is zero as getting 7 here is an impossible event and probability of getting either of the six faces is 1 as it is a certain event.

Classical definition of probability fails if

(i) The cases are not equally likely, e.g. probability of a candidate passing a test is not defined.

$$\left[\begin{array}{l}\text{Passing or failing in a test} \\ \text{are not equally likely cases.}\end{array}\right]$$

(ii) The number of exhaustive cases is indefinitely large, e.g.

probability of drawing an integer say 2 from the set of integers i.e.

$\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ by classical definition probability, is $\frac{1}{\infty} = 0.$

But, in actual, it is not so, happening of 2 is not impossible, i.e. there are some chances of drawing 2. Hence, classical definition is failed here also.

Before we give some examples on classical definition of probability, let us take up some examples which define the events as subsets of sample space.

Example 1: If a fair die is thrown once, what is the event of?

(i) getting an even number

(ii) getting a prime number

(iii) getting a number multiple of 3

(iv) getting an odd prime

(v) getting an even prime

(vi) getting a number greater than 4

(vii) getting a number multiple of 2 and 3

Solution: When a die is thrown, then the sample space is

$$S = \{1, 2, 3, 4, 5, 6\}$$

(i) Let E_1 be the event of getting an even number,

$\therefore$ $E_1 = \{2, 4, 6\}$

(ii) Let E_2 be the event of getting a prime number,

$\therefore$ $E_2 = \{2, 3, 5\}$

(iii) Let E_3 be the event of getting a number multiple of 3

$\therefore$ $E_3 = \{3, 6\}$

(iv) Let E_4 be the event of getting an odd prime,

$\therefore$ $E_4 = \{3, 5\}$

(v) Let E_5 be the event of getting an even prime,

$\therefore$ $E_5 = \{2\}$

(vi) Let E_6 be the event of getting a number greater than 4,

$\therefore$ $E_6 = \{5, 6\}$

(vii) Let E_7 be the event of getting a number multiple of 2 and 3,

$\therefore$ $E_7 = \{6\}$

Example 2: If a pair of a fair dice is thrown, what is the event of

(i) getting a doublet

(ii) getting sum as 11

(iii) getting sum less than 5

(iv) getting sum greater than 16

(v) getting 3 on the first die

(vi) getting a number multiple of 3 on second die

(vii) getting a number multiple of 2 on first die and a multiple of 3 on second die.

Solution: When two dice are thrown, then the sample space is already given in (vii) of Sec.1.3.

(i) Let E_1 be the event of getting a doublet.

$\therefore E_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$

(ii) Let E_2 be the event of getting sum 11.

$\therefore E_2 = \{(5, 6), (6, 5)\}$

(iii) Let E_3 be the event of getting sum less than 5 i.e. sum can be 2 or 3 or 4

$\therefore E_3 = \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1), (2, 2)\}$

(iv) Let E_4 be the event of getting sum greater than 16.

$\therefore E_4 = \{\ \}$ i.e. E_4 is a null event.

(v) Let E_5 be the event of getting 3 on the first die i.e. 3 on first die and second die may have any number

$\therefore E_5 = \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\}$

(vi) Let E_6 be the event of getting a number multiple of 3 on second die i.e. first die may have any number and the second has 3 or 6.

$$\therefore E_6 = \{(1, 3), (2, 3), (3, 3), (4, 3), (5, 3), (6, 3), (1, 6), (2, 6), (3, 6), (4, 6), (5, 6), (6, 6)\}$$

(vii) Let E_7 be the event of getting a multiple of 2 on the first die and a multiple of 3 on the second die i.e. 2 or 4 or 6 on first die and 3 or 6 on the second.

$\therefore E_7 = \{(2, 3), (4, 3), (6, 3), (2, 6), (4, 6), (6, 6)\}$

Now, you can try the following exercise.

E2) If a die and a coin are tossed simultaneously, write the event of getting

(i) head and prime number

(ii) tail and an even number

(iii) head and multiple

1.6 SIMPLE PROBLEMS ON PROBABILITY

Now let us give some examples so that you become familiar as to how and when the classical definition of probability is used.

Example 3: A bag contains 4 red, 5 black and 2 green balls. One ball is drawn from the bag. Find the probability that?

(i) It is a red ball
(ii) It is not black
(iii) It is green or black

Solution: Let R_1, R_2, R_3, R_4 denote 4 red balls in the bag. Similarly B_1, B_2, B_3, B_4, B_5 denote 5 black balls and G_1, G_2 denote two green balls in the bag. Then the sample space for drawing a ball is given by

$\{R_1, R_2, R_3, R_4, B_1, B_2, B_3, B_4, B_5, G_1, G_2\}$

(i) Let A be the event of getting a red ball, then A = $\{R_1, R_2, R_3, R_4\}$

$$\therefore P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{4}{11}$$

(ii) Let B be the event that drawn ball is not black, then
B = $\{R_1, R_2, R_3, R_4, G_1, G_2\}$

$$\therefore P(B) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{6}{11}$$

(iii) Let C be the event that drawn ball is green or black, then
C = $\{B_1, B_2, B_3, B_4, B_5, G_1, G_2\}$.

$$\therefore P(C) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{7}{11}$$

Example 4: Three unbiased coins are tossed simultaneously. Find the probability of getting

(i) at least two heads

(ii) at most two heads

(iii) all heads

(iv) exactly one head

(v) exactly one tail

Solution: The sample space in this case is

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

(i) Let E_1 be the event of getting at least 2 heads, then
E_1 = {HHT, HTH, THH, HHH}

$$\therefore\ P(E_1) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{4}{8} = \frac{1}{2}$$

(ii) Let E_2 be the event of getting at most 2 heads then

E_2 = {TTT, TTH, THT, HTT, HHT, HTH, THH}

$$\therefore\ P(E_2) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{7}{8}$$

(iii) Let E_3 be the event of getting all heads, then

E_3 = {HHH}

$$\therefore\ P(E_3) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{1}{8}$$

(iv) Let E_4 be the event of getting exactly one head then

E_4 = {HTT, THT, TTH}

$$\therefore\ P(E_4) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{3}{8}$$

(v) Let E_5 be the event of getting exactly one tail, then

E_5 = {HHT, HTH, THH}

$$\therefore\ P(E_5) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{3}{8}$$

Example 5: A fair die is thrown. Find the probability of getting

(i) a prime number

(ii) an even number

(iii) a number multiple of 2 or 3

(iv) a number multiple of 2 and 3

(v) a number greater than 4

Solution: The sample space in this case is

S = {1, 2, 3, 4, 5, 6}

(i) Let E_1 be the event of getting a prime number, then E_1 = {2, 3, 5}.

$$\therefore\ P(E_1) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{3}{6} = \frac{1}{2}$$

(ii) Let E_2 be the event of getting an even number, then

$E_2 = \{2, 4, 6\}$

$$\therefore\ P(E_2) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{3}{6} = \frac{1}{2}$$

(iii) Let E_3 event of getting a multiple of 2 or 3, then

$E_3 = \{2, 3, 4, 6\}$

$$\therefore\ P(E_3) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{4}{6} = \frac{2}{3}$$

(iv) Let E_4 event of getting a number multiple of 2 and 3, then

$E_4 = \{6\}$

$$\therefore\ P(E_4) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{1}{6}$$

(v) Let E_5 be the event of getting a number greater than 4, then

$E_5 = \{5, 6\}$

$$\therefore\ P(E_5) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{2}{6} = \frac{1}{3}$$

Example 6: In an experiment of throwing two fair dice, find the probability of getting

(i) a doublet
(ii) sum 7
(iii) sum greater than 8
(iv) 3 on first die and a multiple of 2 on second die
(v) prime number on the first die and odd prime on the second die.

Solution: The sample space has already been given in (vii) of Sec. 1.3.

Here, the sample space contains 36 elements i.e. number of exhaustive cases is 36.

(i) Let E_1 be the event of getting a doublet (i.e. same number on both dice), then

$E_1 = \{(1, 1), (2, 2), (3, 3), (4, 4), (5, 5), (6, 6)\}$.

$$\therefore\ P(E_1) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{6}{36} = \frac{1}{6}$$

(ii) Let E_2 be the event of getting sum 7, then

$E_2 = \{(1, 6), (6, 1), (2, 5), (5, 2), (3, 4), (4, 3)\}$

$$\therefore \quad P(E_2) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{6}{36} = \frac{1}{6}$$

(iii) Let E_3 be the event of getting sum greater than 8, then

E_3 = {(3, 6), (6, 3), (4, 5), (5, 4), (4, 6),

(6, 4), (5, 5), (5, 6), (6, 5), (6, 6)}

$$\therefore \quad P(E_3) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{10}{36} = \frac{5}{18}$$

(iv) Let E_4 be the event of getting 3 on first die and multiple of 2 on second die, then

E_4 = {(3, 2), (3, 4), (3, 6)}

$$\therefore \quad P(E_4) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{3}{36} = \frac{1}{12}$$

(v) Let E_5 be the event of getting prime number on first die and odd prime on second die, then

E_5 = {(2, 3), (2, 5), (3, 3), (3, 5), (5, 3), (5, 5)}

$$\therefore \quad P(E_5) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{6}{36} = \frac{1}{6}$$

Example 7: Out of 52 well shuffled playing cards, one card is drawn at random. Find the probability of getting

(i) a red card

(ii) a face card

(iii) a card of spade

(iv) a card other than club

(v) a king

Solution: Here, the number of exhaustive cases is 52 and a pack of playing cards contains 13 cards of each suit (spade, club, diamond, heart).

(i) Let A be the event of getting a red card. We know that there are 26 red cards,

$$\therefore \quad P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{26}{52} = \frac{1}{2}$$

(ii) Let B be the event of getting a face card. We know that there are 12 face cards (jack, queen and king in each suit),

$$\therefore \quad P(B) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{12}{52} = \frac{3}{13}$$

(iii) Let C be the event of getting a card of spade
We know that there are 13 cards of spade

$$\therefore \ P(C) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{13}{52} = \frac{1}{4}$$

(iv) Let D be the event of getting a card other than club.
As there are 39 cards other than that of club,.

$$\therefore \ P(D) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{39}{52} = \frac{3}{4}$$

(v) Let E be the event of getting a king.
We know that there are 4 kings,

$$\therefore \ P(E) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{4}{52} = \frac{1}{13}$$

Example 8: In a family, there are two children. Write the sample space and find the probability that

(i) the elder child is a girl

(ii) younger child is a girl

(iii) both are girls

(iv) both are of opposite sex

Solution: Let G_i denotes that i^{th} birth is of girl (i = 1, 2) and B_i denotes that i^{th} birth is of boy, (i =1, 2).

$$\therefore \ S = \{G_1G_2, G_1B_2, B_1G_2, B_1B_2\}$$

(i) Let A be the event that elder child is a girl

$$\therefore \ A = \{G_1G_2, G_1B_2\}$$

$$\text{and } P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{2}{4} = \frac{1}{2}$$

(ii) Let B be the event that younger child is a girl

$$\therefore \ B = \{G_1G_2, B_1G_2\}$$

$$\text{and } P(B) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{2}{4} = \frac{1}{2}$$

(iii) Let C be the event that both the children are girls

$$\therefore \ C = \{G_1G_2\}$$

$$\text{and } P(C) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{1}{4}$$

(iv) Let D be the event that both children are of opposite sex

$\therefore$ D = $\{G_1B_2, B_1G_2\}$

and $P(D) = \dfrac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \dfrac{2}{4} = \dfrac{1}{2}$

Example 9: Find the probability of getting 53 sundays in a randomly selected non-leap year.

Solution: We know that there are 365 days in a non-leap year.

$\dfrac{365}{7} = 52\dfrac{1}{7}$ weeks

i.e. one non-leap year = (52 complete weeks + one over day). This over day may be one of the days

Sunday, Monday, Tuesday, Wednesday, Thursday, Friday, Saturday

So, the number of exhaustive cases = 7

Let A be the event of getting 53 Sundays

There will be 53 Sundays in a non leap year if and only if the over day is Sunday.

$\therefore$ Number of favourable cases for event A= 1

$\therefore \quad P(A) = \dfrac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \dfrac{1}{7}$

Example 10: A single letter selected at random from the word 'STATISTICS'. What is the probability that it is a vowel?

Solution: Here, as the total number of letters in the word 'STATISTICS' is n = 10, and the number of vowels in the word is m = 3 (vowels are a, i, i),

$\therefore$ The required probability = $\dfrac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \dfrac{m}{n} = \dfrac{3}{10}$

Example 11: Three horses A, B and C are in a race. A is twice as likely to win as B and B is twice as likely to win as C. What are the respective probabilities of their winning?

Solution: Let p be the probability that A wins the race.

$\therefore$ Probability that B wins the race = twice the probability of A's winning

= 2(p) = 2 p

and probability of C's winning = twice the probability of B's winning

= 2(2p) = 4p

Now, as the sum of the probability of happening an event and that of its complementary event(s) is 1. Here, the complementary of A is the happening of B or C]

$\therefore \qquad p + 2p + 4p = 1$, and hence $p = \dfrac{1}{7}$.

Therefore, the respective chances of winning A, B and C are $\frac{1}{7}, \frac{2}{7}$ and $\frac{4}{7}$.

Now, let us take up some problems on probability which are based on permutation/combination which you have already studied in Unit 4 of Course MST-001.

Example 12: Out of 52 well shuffled playing cards, two cards are drawn at random. Find the probability of getting.

(i) One red and one black

(ii) Both cards of the same suit

(iii) One jack and other king

(iv) One red and the other of club

Solution: Out of 52 playing cards, two cards can be drawn in ${}^{52}C_2$ ways i.e.

$$\frac{52\times51}{2!} = 26\times51 \text{ ways}$$

(i) Let A be the event of getting one red and one black card, then the number of favourable cases for the event A are ${}^{26}C_1 \times {}^{26}C_1$ [As one red card out of 26 red cards can be drawn in ${}^{26}C_1$ ways and one black card out of 26 black cards can be drawn in ${}^{26}C_1$ ways.]

$$\therefore \; P(A) = \frac{{}^{26}C_1 \times {}^{26}C_1}{{}^{52}C_2} = \frac{26\times26}{26\times51} = \frac{26}{51}$$

(ii) Let B be the event of getting both the cards of the same suit and i.e. two cards of spade or two cards of club or 2 cards of diamond or 2 cards of heart.

$\therefore$ Number of favourable cases for event B $= {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2 + {}^{13}C_2$

$$= 4\times{}^{13}C_2 = 4\times\frac{13\times12}{2!} = 2\times13\times12$$

$$\therefore \; P(B) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{2\times13\times12}{26\times51} = \frac{4}{17}$$

(iii) Let C be the event of getting a jack and a king.

$$\therefore P(C) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{{}^{4}C_1\times{}^{4}C_1}{{}^{52}C_2} = \frac{4\times4}{26\times51} = \frac{8}{663}$$

(iv) Let D be the event of getting one red and one card of club.

$$\therefore P(D) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{{}^{26}C_1\times{}^{13}C_1}{{}^{52}C_2} = \frac{26\times13}{26\times51} = \frac{13}{51}$$

Example 13: If the letters of the word STATISTICS are arranged randomly then find the probability that all the three T's are together.

Solution: Let E be the event that selected word contains 3 T's together.

There are 10 letters in the word STATISTICS. If we consider three T's as a single letter $\boxed{TTT}$, then we have 8 letters i.e.1 '$\boxed{TTT}$'; 3 'S' ; 1 'A'; 2 'I' and 1 'C'

Number of possible arrangements with three T's coming together $= \dfrac{8!}{2!.3!}$

Number of favourable cases for event E $= \dfrac{8!}{2!.3!}$ and

Number of exhaustive cases = Total number of permutations of 10 letters in the word STATISTICS

$$= \frac{10!}{2!.3!.3!}$$

$\left[\because \text{out of 10 letters, 3 are T's, 2 are I's and 3 are S's}\right]$

$$P(A) = \frac{\frac{8!}{2!.3!}}{\frac{10!}{2!.3!.3!}} = \frac{8!.3!}{10!} = \frac{8!\times 6}{10\times 9\times 8!} = \frac{6}{10\times 9} = \frac{1}{15}$$

Example 14: In a lottery, one has to choose six numbers at random out of the numbers from 1 to 30. He/ she will get the prize only if all the six chosen numbers matched with the six numbers already decided by the lottery committee. Find the probability of wining the prize.

Solution: Out of 30 numbers 6 can be drawn in

$$^{30}C_6 = \frac{30\times 29\times 28\times 27\times 26\times 25}{6!} = \frac{30\times 29\times 28\times 27\times 26\times 25}{720} = 593775 \text{ ways}$$

$\therefore$ Number of exhaustive cases = 593775

Out of these 593775 ways, there is only one way to win the prize (i.e. choose those six numbers that are already fixed by committee).

Here, the number of favourable cases is 1.

Hence, $P(\text{wining the prize}) = \dfrac{\text{Favourable cases}}{\text{Exhaustive cases}} = \dfrac{1}{593775}$

Now, you can try the following exercises.

E3) If two coins are tossed then find the probability of getting.

(i) At least one head

(ii) head and tail

(iii) At most one head

E4) If three dice are thrown, then find the probability of getting

(i) triplet
(ii) sum 5
(iii) sum at least 17
(iv) prime number on first die and odd prime number on second and third dice.

E5) Find the probability of getting 53 Mondays in a randomly selected leap year.

1.7 CONCEPT OF ODDS IN FAVOUR OF AND AGAINST THE HAPPENING OF AN EVENT

Let n be the number of exhaustive cases in a random experiment which are mutually exclusive and equally likely as well. Let m out of these n cases are favourable to the happening of an event A (say). Thus, the number of cases against A are $n-m$

Then odds in favour of event A are $m:n-m$ (i.e. m ratio $n-m$) and odds against A are $n-m:m$ (i.e. $n-m$ ratio m)

Example 15: If odds in favour of event A are $3:4$, what is the probability of happening A?

Solution: As odds in favour of A are $3:4$,

$\therefore$ $m=3$ and $n-m=4$ implies that $n=7$. Thus,

Probability of happening A i.e. $P(A)=\frac{m}{n}=\frac{3}{7}$.

Example 16: Find the probability of event A if

(i) Odds in favour of event A are 4 : 3

(ii) Odds against event A are 5 : 8

Solution: (i) We know that if odds in favour of A are $m:n$, then

$$P(A)=\frac{m}{m+n} \Rightarrow P(A)=\frac{4}{4+3}=\frac{4}{7}$$

(ii) Here, $n-m=5$ and $m=8$, therefore, $n=5+8=13$.

Now, as we know that if odds against the happening of an event A are $n-m:n$, then

$$P(A)=\frac{m}{n} \Rightarrow P(A)=\frac{8}{13}$$

Example 17 If $P(A)=\frac{3}{5}$ then find

(i) odds in favour of A; (ii) odds against the happening of event A.

Solution: (i) As $P(A) = \frac{3}{5}$,

$\therefore$ odds in favour of A in this case are $3:5-3 = 3:2$

(ii) We know that if $P(A) = \frac{m}{n}$, then odds against the happening of A are $n-m:m$

$\therefore$ In this case odds against the happening of event A are $5-3 : 3 = 2 : 3$

Now, you can try the following exercises.

E6) The odds that a person speaks the truth are 3 : 2. What is the probability that the person speaks truth?

E7) The odds against Manager X setting the wage dispute with the workers are 8 : 6. What is the probability that the manager settles the dispute?

E8) The probability that a student passes a test is $\frac{2}{3}$. What are the odds against passing the test by the student?

E9) Find the probability of the event A if

(i) Odds in favour of the event $\bar{A}$ are $1:4$ (ii) Odds against the event $\bar{A}$ are $7:2$

1.8 SUMMARY

Let us now summarize the main points which have been covered in this unit.

1) An experiment in which all the possible outcomes are known in advance but we cannot predict as to which of them will occur when we perform the experiment is called **random experiment**. Performing an experiment is called **trial**.
2) Set of all possible outcomes of a random experiment is known as **sample space**. Each outcome of an experiment is visualised as a **sample point** and set of one or more possible outcomes constitutes what is known as **event**. The total number of elements in the sample space is called the number of **exhaustive cases** and number of elements in favour of the event is the number of **favourable cases** for the event.
3) Cases are said to be **mutually exclusive** if the happening of any one of them prevents the happening of all others in a single experiment and if we do not have any reason to expect one in preference to others, then they are said to be **equally likely**.
4) **Classical Probability** of happening of an event is the ratio of number of favourable cases to the number of exhaustive cases, provided they are equally likely, mutually exclusive and finite.
5) **Odds in favour of an event** are the number of favourable cases: number of cases against the event, whereas **Odds against the event** are the number of cases against the event : number of cases favourable to the event.

1.9 SOLUTIONS/ANSWERS

E 1) Let suffices C, D, S, H denote that corresponding card is a club, diamond, spade, heart respectively then sample space of drawing a card can be written as

$$\{1_C, 2_C, 3_C, ..., 9_C, 10_C, J_C, Q_C, K_C, 1_D, 2_D, 3_D, ..., 9_D, 10_D, J_D, Q_D, K_D,$$
$$1_S, 2_S, 3_S, ..., 9_S, 10_S, J_S, Q_S, K_S, 1_H, 2_H, 3_H, ..., 9_H, 10_H, J_H, Q_H, K_H\}$$

E 2) If a die and a coin are tossed simultaneously then sample space is

{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6}

(i) Let A be the event of getting head and prime number, then A = {H2, H3, H5}

(ii) Let B be the event of getting tail and even number, then B = {T2, T4, T6}

(iii) Let C be the event of getting head and multiple of 3, then C = {H3, H6}

E 3) When two coins are tossed simultaneously then sample space is

S = {HH, HT, TH, TT}

(i) Let A be the event of getting at least one head, then

A = {HH, HT, TH} and $P(A) = \frac{3}{4}$

(ii) Let B be the event of getting both head and tail, then

B = {HT, TH} and $P(B) = \frac{2}{4} = \frac{1}{2}$

(iii) Let C be the event of getting at most one head, then

C = {TT, TH, HT} and $P(C) = \frac{3}{4}$

E 4) When 3 dice are thrown, then the sample space is

S = {(1, 1, 1), (1, 1, 2), (1, 1, 3), … , (1, 1, 6),
(1, 2, 1), (1, 2, 2), (1, 2, 3), … , (1, 2, 6),
(1, 3, 1), (1, 3, 2), (1, 3, 3), … , (1, 3, 6),
.
.
.
(6, 6, 1), (6, 6, 2), (6, 6, 3), …, (6, 6, 6)}

Number of elements in the sample space = $6 \times 6 \times 6 = 216$

$$\left[\begin{array}{l}\because \text{We have to fill up 3 positions (.,.,.) and each position can be} \\ \text{filled with 6 options, this can be done in } 6\times6\times6 = 216 \text{ ways}\end{array}\right]$$

(i) Let A be the event of getting triplet i.e. same number on each die.

$\therefore A = \{(1,1,1), (2,2,2), (3,3,3), (4,4,4), (5,5,5), (6,6,6)\}$

and hence P(A) = $\dfrac{6}{216} = \dfrac{1}{36}$

(ii) Let B be the event of getting sum 5

$\therefore B = \{(1,1,3), (1,3,1), (3,1,1), (1,2,2), (2,1,2), (2,2,1)\}$

and P (B) = $\dfrac{6}{216} = \dfrac{1}{36}$

(iii) Let C be the event of getting sum at least17 i.e. sum 17 or 18

$\therefore C = \{(5,6,6), (6,5,6), 6,6,5), (6,6,6)\}$

and hence P (C) = $\dfrac{4}{216} = \dfrac{1}{54}$

(iv) Let D be the event of getting prime number on first die and odd prime number on second and third dice.
i.e. first die can show 2 or 3 or 5 and second, third dice can show 3 or 5

$$\therefore D = \{(2,3,3), (2,3,5), (2,5,3), (2,5,5), (3,3,3), (3,3,5), \\ (3,5,3), (3,5,5), (5,3,3), (5,3,5), (5,5,3), (5,5,5)\}$$

and hence P (D) = $\dfrac{12}{216} = \dfrac{1}{18}$

E5) We know that there are 366 days in a leap year. i.e. $\dfrac{365}{7} = 52\dfrac{2}{7}$ weeks

i.e one leap year = (52 complete weeks + two over days).

These two over days may be

(i) Sunday and Monday

(ii) Monday and Tuesday

(iii) Tuesday and Wednesday

(iv) Wednesday and Thursday

(v) Thursday and Friday

(vi) Friday and Saturday

(vii) Saturday and Sunday

$\therefore$ Number of exhaustive Cases = 7

Let A be the event of getting 53 Mondays

There will be 53 Mondays in a leap year if and only if these two over days are

"Sunday and Monday" or "Monday and Tuesday"

$\therefore$ Number of favourable cases for event A are 2

and $P(A) = \frac{\text{Number of favourable cases}}{\text{Number of exhaustive cases}} = \frac{2}{7}$

E6) Here, the odds in favour of speaking the truth are 3 : 2,

$\therefore$ here m = 3, n – m = 2 and hence n = 5.

Hence, the probability of speaking the truth $= \frac{3}{5}$

E7) As the odds against Manager X setting the wage dispute with the workers are 8 : 6, hence odds in favour of settling the dispute are 6 : 8.

Thus, the probability that the manager settles the dispute $= \frac{6}{14} = \frac{3}{7}$.

E8) As the probability that student pass a test = 2/3,

$\therefore$ the number of favourable cases = 2 and the number of exhaustive cases = 3, and hence the number of cases against passing the test = 3 – 2 = 1.

Thus, odds against passing the test

= the number cases against the event : the number cases favourable to the event

= 1 : 2

E 9) (i) Odds in favour of the event $\overline{A}$ are given as 1 : 4.

We know that if odds in favour of event E are $m:n$ then P(E) $= \frac{m}{m+n}$

$\therefore$ In this case $P(\overline{A}) = \frac{1}{1+4} = \frac{1}{5}$ and $P(A) = 1 - P(\overline{A}) = 1 - \frac{1}{5} = \frac{4}{5}$

(ii) Odds against the happening of the event $\overline{A}$ are given as 7 : 2.

We know that if odds against the happening of an event E are $m:n$, then $P(E) = \frac{n}{m+n}$.

$\therefore$ In this case $P(\overline{A}) = \frac{2}{7+2} = \frac{2}{9}$ and hence

$P(A) = 1 - P(\overline{A}) = 1 - \frac{2}{9} = \frac{7}{9}$.

UNIT 2 DIFFERENT APPROACHES TO PROBABILITY THEORY

Structure

2.1 INTRODUCTION

In the previous unit, we have defined the classical probability. There are some restrictions in order to use it such as the outcomes must be equally likely and finite. There are many situations where such conditions are not satisfied and hence classical definition cannot be applied. In such a situation, we need some other approaches to compute the probabilities.

Thus, in this unit, we will discuss different approaches to evaluate the probability of a given situation based on past experience or own experience or based on observed data. Actually classical definition is based on the theoretical assumptions and in this unit, our approach to evaluate the probability of an event is different from theoretical assumptions and will put you in a position to answer those questions related to probability where classical definition does not work. The unit discusses the relative frequency (statistical or empirical probability) and the subjective approaches to probability. These approaches, however, share the same basic axioms which provide us with the unified approach to probability known as axiomatic approach. So, the axiomatic approach will also be discussed in the unit.

Objectives

After studying this unit, you should be able to:

- explain the relative frequency approach and statistical(or empirical) probability;
- discuss subjective approach to probability; and
- discuss axiomatic approach to probability.

2.2 RELATIVE FREQUENCY APPROACH AND STATISTICAL PROBABILITY

Classical definition of probability fails if

i) the possible outcomes of the random experiment are not equally likely or/and

ii) the number of exhaustive cases is infinite.

In such cases, we obtain the probability by observing the data. This approach to probability is called the relative frequency approach and it defines the statistical probability. Before defining the statistical probability, let us consider the following example:

Following table gives a distribution of daily salary of some employees:

Salary per day (In Rs)	**Below 100**	**100-150**	**150-200**	**200 and above**
Employees	**20**	**40**	**50**	**15**

If an individual is selected at random from the above group of employees and we are interested in finding the probability that his/her salary was under Rs. 150, then as the number of employees having salary less than Rs 150 is 20 + 40 = 60 and the total number employees is 20 + 40 + 50 + 15 = 125, therefore the relative frequency that the employee gets salary less than Rs. 150 is

$$\frac{60}{125} = \frac{12}{25}.$$

This relative frequency is nothing but the probability that the individual selected is getting the salary less than Rs. 150.

So, in general, if X is a variable having the values $x_1, x_2, \ldots, x_n$ with frequencies $f_1, f_2, \ldots, f_n,$ respectively. Then

$$\frac{f_1}{\sum_{i=1}^{n} f_i}, \frac{f_2}{\sum_{i=1}^{n} f_i}, \ldots, \frac{f_n}{\sum_{i=1}^{n} f_i}$$

are the relative frequencies of $x_1, x_2, \ldots, x_n$ respectively and hence the probabilities of X taking the values $x_1, x, \ldots, x_n$ respectively.

But, in the above example the probability has been obtained using the similar concept as that of classical probability.

Now, let us consider a situation where a person is administered a sleeping pill and we are interested in finding the probability that the pill puts the person to sleep in 20 minutes. Here, we cannot say that the pill will be equally effective for all persons and hence we cannot apply classical definition here.

To find the required probability in this case, we should either have the past data or in the absence of the past data, we have to undertake an experiment where we administer the pill on a group of persons to check the effect. Let m be the number of persons to whom the pill put to sleep in 20 minutes and n be the total number of persons who were administered the pill.

Then, the relative frequency and hence the probability that a particular person will put to sleep in 20 minutes is $\frac{m}{n}$. But, this measure will serve as probability only if the total number of trials in the experiment is very large.

In the relative frequency approach, as the probability is obtained by repetitive empirical observations, it is known as statistical or empirical probability.

Statistical (or Empirical) Probability

If an event A (say) happens m times in n trials of an experiment which is performed repeatedly under essentially homogeneous and identical conditions (e.g. if we perform an experiment of tossing a coin in a room, then it must be performed in the same room and all other conditions for tossing the coin should also be identical and homogeneous in all the tosses), then the probability of happening A is defined as:

$$P(A) = \lim_{n\to\infty} \frac{m}{n}.$$

As an illustration, we tossed a coin 200 times and observed the number of heads. After each toss, proportion of heads i.e. $\frac{m}{n}$ was obtained, where m is the number of heads and n is the number of tosses as shown in the following table (Table 2.1):

Table 2.1: Table Showing Number of Tosses and Proportion of Heads

n (Number of Tosses)	m (Number of Heads)	Proportion of Heads i.e.P(H)=m/n
1	1	1
2	2	1
3	2	0.666667
4	3	0.75
5	4	0.8
6	4	0.666667
7	4	0.571429
8	5	0.625
9	6	0.666667
10	6	0.6
15	10	0.666667
20	12	0.6
25	14	0.56
30	16	0.533333
35	18	0.514286

40	22	0.55
45	25	0.555556
50	29	0.58
60	33	0.55
70	41	0.585714
80	46	0.575
90	52	0.577778
100	53	0.53
120	66	0.55
140	72	0.514286
160	82	0.5125
180	92	0.511111
200	105	0.525

Then a graph was plotted taking number of tosses (n) on x-axis and proportion of heads $\left(\frac{m}{n}\right)$ on y-axis as shown in Fig. 2.1.

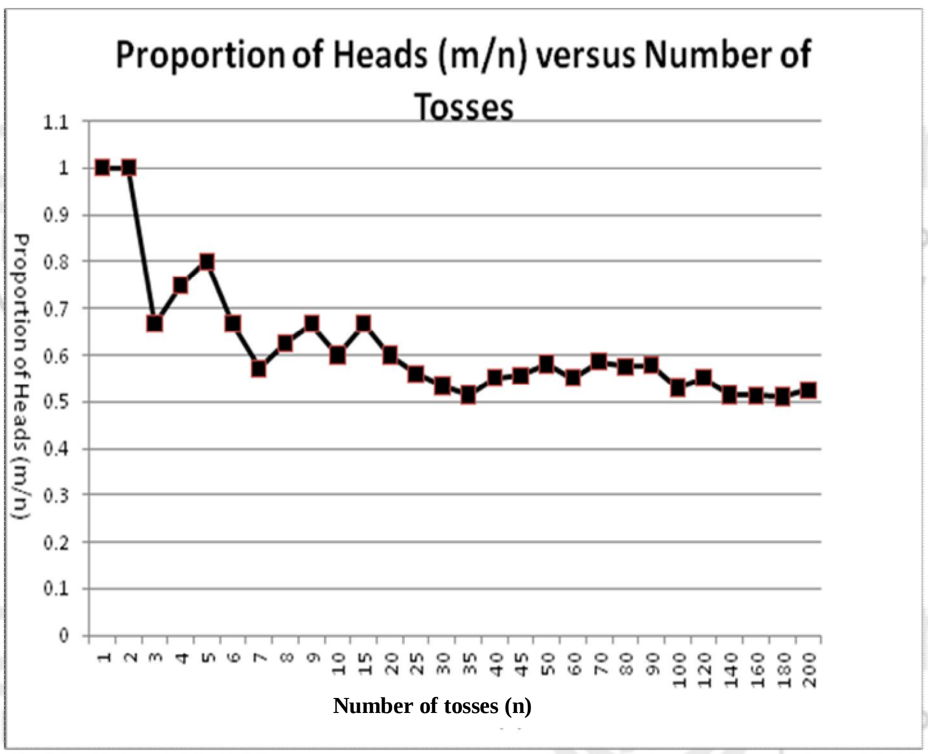


Fig. 2.1: **Proportion of Heads versus Number of Tosses**

The Graph reveals that as we go on increasing n,

$\frac{m}{n}$ tends to $\frac{1}{2}$

i.e. $\lim\limits_{n\to\infty}\frac{m}{n}=\frac{1}{2}$

Hence, by the statistical (or empirical) definition of probability, the probability of getting head is

$$\lim_{n\to\infty}\frac{m}{n}=\frac{1}{2}.$$

Statistical probability has the following limitations:

(i) The experimental condition may get altered if it is repeated a large number of times.

(ii) $\lim\limits_{n\to\infty}\frac{m}{n}$ may not have a unique value, however large n may be.

2.3 PROBLEMS BASED ON RELATIVE FREQUENCY

Example 1: The following data relate to 100 couples

Age of wife / Age of Husband	10-20	20-30	30-40	40-50	50-60
15-25	6	3	0	0	0
25-35	3	16	10	0	0
35-45	0	10	15	7	0
45-55	0	0	7	10	4
55-65	0	0	0	4	5

(i) Find the probability of a couple selected at random has a "age of wife" in the interval 20-50.

(ii) What is the probability that the age of wife is in the interval 20-40 and the age of husband is in the interval 35-45 if a couple selected at random?

Solution: (i) Required probability is given by

$$=\frac{(3+16+10+0+0)+(0+10+15+7+0)+(0+0+7+10+4)}{100}$$

$$=\frac{82}{100}=0.82$$

(ii) Required probabilty $=\frac{10+15}{100}=\frac{25}{100}=0.25$

Example 2: A class has 15 students whose ages are 14, 17, 15, 21, 19, 20, 16, 18, 20, 17, 14, 17, 16, 19 and 20 years respectively. One student is chosen at random and the age of the selected student is recorded. What is the probability that

(i) the age of the selected student is divisible by 3,

(ii) the age of the selected student is more than 16, and

(iii) the selected student is eligible to pole the vote.

Solution:

Age X	Frequency f	Relative frequency
14	2	2/15
15	1	1/15
16	2	2/15
17	3	3/15
18	1	1/15
19	2	2/15
20	3	3/15
21	1	1/15

(i) The age divisible by 3 is 15 or 18 or 21.

$$\therefore \text{ Required Probability} = \frac{1+1+1}{15} = \frac{3}{15} = \frac{1}{5}$$

(ii) Age more than 16 means, age may be 17, 18, 19, 20, 21

$$\therefore \text{ Required Probability} = \frac{3+1+2+3+1}{15} = \frac{10}{15} = \frac{2}{3}$$

(iii) In order to poll the vote, age must be ≥ 18 years. Thus, we are to obtain the probability that the selected student has age 18 or 19 or 20 or 21.

$$\therefore \text{ Required Probability} = \frac{1+2+3+1}{15} = \frac{7}{15}$$

Example 3: A tyre manufacturing company kept a record of the distance covered before a tyre needed to be replaced. The following table shows the results of 2000 cases.

Distance (in km)	Less than 4000	4001-10000	100001-20000	20001-40000	More than 40000
Frequency	20	100	200	1500	180

If a person buys a tyre of this company then find the probability that before the need of its replacement, it has covered

(i) at least a distance of 4001 km.

(ii) at most a distance 20000 km

(iii) more than a distance 20000 km

(iv) a distance between 10000 to 40000

Solution: The record is based on 2000 cases,

$\therefore$ Exhaustive cases in each case = 2000

(i) Out of 2000 cases, the number of cases in which tyre covered at least 4001 km

= 100 + 200 + 1500 + 180 = 1980

$$\therefore \text{Required Probability} = \frac{1980}{2000} = \frac{198}{200} = \frac{99}{100}$$

(ii) Number of cases in which distance covered by tyres of this company is at most 20000 km = 20 + 100 + 200 = 320

$$\therefore \text{Required Probability} = \frac{320}{2000} = \frac{32}{200} = \frac{4}{25}$$

(iii) Number of cases in which tyres of this company covers a distance of more than 20000 = 1500 + 180 = 1680

$$\therefore \text{Required Probability} = \frac{1680}{2000} = \frac{168}{200} = \frac{21}{25}$$

(iv) Number of cases in which tyres of this company covered a distance between 10000 to 40000 = 200 + 1500 = 1700

$$\therefore \text{Required Probability} = \frac{1700}{2000} = \frac{17}{20}$$

Now, you can try the following exercises.

E 1) An insurance company selected 5000 drivers from a city at random in order to find a relationship between age and accidents. The following table shows the results related to these 5000 drivers.

Age of driver (in years)	Accidents in one year				
Class interval	0	1	2	3	4 or more
18-25	600	260	185	90	70
25-40	900	240	160	85	65
40-50	1000	195	150	70	50
50 and above	500	170	120	60	30

If a driver from the city is selected at random, find the probability of the following events:

(i) Age lying between 18-25 and meet 2 accidents

(ii) Age between 25-50 and meet at least 3 accidents

(iii) Age more than 40 years and meet at most one accident

(iv) Having one accident in the year

(v) Having no accident in the year.

E 2) Past experience of 200 consecutive days speaks that weather forecasts of a station is 120 times correct. A day is selected at random of the year, find the probability that

(i) weather forecast on this day is correct

(ii) weather forecast on this day is false

E 3) Throw a die 200 times and find the probability of getting the odd number using statistical definition of probability.

2.4 SUBJECTIVE APPROACH TO PROBABILITY

In this approach, we try to assess the probability from our own experiences. This approach is applicable in the situations where the events do not occur at all or occur only once or cannot be performed repeatedly under the same conditions. Subjective probability is based on one's judgment, wisdom, intuition and expertise. It is interpreted as a measure of degree of belief or as the quantified judgment of a particular individual. For example, a teacher may express his /her confidence that the probability for a particular student getting first position in a test is 0.99 and that for a particular student getting failed in the test is 0.05. It is based on his personal belief.

You may notice here that since the assessment is purely subjective one, it will vary from person to person, depending on one's perception of the situation and past experience. Even when two persons have the same knowledge about the past, their assessment of probabilities may differ according to their personal prejudices and biases.

2.5 AXIOMATIC APPROACH TO PROBABILITY

All the approaches i.e. classical approach, relative frequency approach (Statistical/Empirical probability) and subjective approach share the same basic axioms. These axioms are fundamental to the probability and provide us with unified approach to probability i.e. axiomatic approach to probability. It defines the probability function as follows:

Let S be a sample space for a random experiment and A be an event which is subset of S, then P(A) is called probability function if it satisfies the following axioms

(i) $P(A)$ is real and $P(A) \geq 0$

(ii) P (S) = 1

(iii) If $A_1, A_2, \ldots$ is any finite or infinite sequence of disjoint events in S, then

$$P(A_1 \text{ or } A_2 \text{ or} \ldots \text{or } A_n) = P(A_1) + P(A_2) + \ldots + P(A_n)$$

Now, let us give some results using probability function. But before taking up these results, we discuss some statements with their meanings in terms of set theory. If A and B are two events, then in terms of set theory, we write

i) 'At least one of the events A or B occurs' as $A \cup B$

ii) 'Both the events A and B occurs' as $A \cap B$

iii) 'Neither A nor B occurs' as $\bar{A} \cap \bar{B}$

iv) 'Event A occurs and B does not occur' as $A \cap \bar{B}$

v) 'Exactly one of the events A or B occurs' as $(\bar{A} \cap B) \cup (A \cap \bar{B})$

vi) 'Not more than one of the events A or B occurs' as

$$\left(A \cap \bar{B}\right) \cup \left(\bar{A} \cap B\right) \cup \left(\bar{A} \cap \bar{B}\right).$$

Similarly, you can write the meanings in terms of set theory for such statement in case of three or more events e.g. in case of three events A, B and C, happening of at least one of the events is written as $A \cup B \cup C$.

2.6 SOME RESULTS USING PROBABILITY FUNCTION

1 Prove that probability of the impossible event is zero

Proof: Let S be the sample space and ϕ be the set of impossible event.

$\therefore\ S \cup \phi = S$

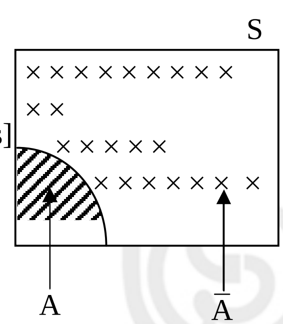
$\Rightarrow P(S \cup \phi) = P(S)$

$\Rightarrow P(S) + P(\phi) = P(S)$ [By axiom (iii)]

$\Rightarrow 1 + P(\phi) = 1$ [By axiom (ii)]

$\Rightarrow P(\phi) = 0$

2 Probability of non-happening of an event A i.e. complementary event $\bar{A}$ of A is given by $P(\bar{A}) = 1 - P(A)$



Proof: If S is the sample space then

$A \cup \bar{A} = S$ [$\because$ A and $\bar{A}$ are mutually disjoint events]

$\Rightarrow P(A \cup \bar{A}) = P(S)$

$\Rightarrow P(A) + P(\bar{A}) = P(S)$ [Using axiom (iii)]

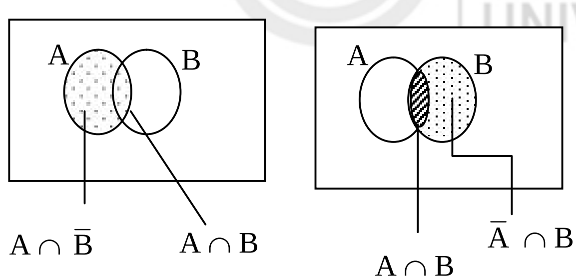
$= 1$ [Using axiom (ii)]

$\Rightarrow P(\bar{A}) = 1 - P(A)$

3. Prove that

(i) $P(A \cap \bar{B}) = P(A) - P(A \cap B)$

(ii) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$



Proof

If S is the sample space and A, B $\subset$ S then

(i) $A = (A \cap \bar{B}) \cup (A \cap B)$

$$\Rightarrow P(A) = P((A \cap \bar{B}) \cup (A \cap B))$$

$$= P(A \cap \bar{B}) + P(A \cap B)$$

[Using axiom (iii) as $A \cap \bar{B}$ and $A \cap B$ are mutually disjoint]

$$\Rightarrow P(A \cap \bar{B}) = P(A) - P(A \cap B)$$

(ii) $B = (A \cap B) \cup (\bar{A} \cap B)$

$$P(B) = P((A \cap B) \cup (\bar{A} \cap B))$$

$$= P(A \cap B) + P(\bar{A} \cap B)$$

[Using axiom (iii) as $A \cap B$ and $\bar{A} \cap B$ are mutually disjoint]

$$\Rightarrow P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

Example 4: A, B and C are three mutually exclusive and exhaustive events associated with a random experiment. Find P(A) given that :

$$P(B) = \frac{3}{4}P(A) \text{ and } P(C) = \frac{1}{3}P(B)$$

Solution:

As A, B and C are mutually exclusive and exhaustive events,

$\therefore A \cup B \cup C = S$

$\Rightarrow P(A \cup B \cup C) = P(S)$

$\Rightarrow P(A) + P(B) + P(C) = 1$ $\left[\begin{array}{l} \because \text{using axiom (iii) as A, B, C} \\ \text{are mutually disjoint events} \end{array}\right]$

$$\Rightarrow P(A) + \frac{3}{4}P(A) + \frac{1}{3}P(B) = 1$$

$$\Rightarrow P(A) + \frac{3}{4}P(A) + \frac{1}{3}\left(\frac{3}{4}P(A)\right) = 1$$

$$\Rightarrow \left(1 + \frac{3}{4} + \frac{1}{4}\right)P(A) = 1$$

$$\Rightarrow 2P(A) = 1$$

$$\Rightarrow P(A) = \frac{1}{2}$$

Examples 5: If two dice are thrown, what is the probability that sum is

a) greater than 9, and

b) neither 10 or 12.

Solution:

a) P[sum > 9] = P[sum = 10 or sum = 11 or sum = 12]

= P[sum =10] + P[sum = 11] + P[sum = 12]

[using axiom (iii)]

$$= \frac{3}{36}+\frac{2}{36}+\frac{1}{36} = \frac{6}{36}=\frac{1}{6}$$

[$\because$ for sum = 10, there are three favourable cases (4, 6), (5, 5) and (6, 4). Similarly for sum =11 and 12, there are two and one favourable cases respectively.]

Let A denotes the event for sum =10 and B denotes the event for sum = 12,

$\therefore$ Required probability = $P\left(\bar{A}\cap\bar{B}\right)=P\left(\overline{A\cup B}\right)$ [Using De- Morgan's law

(see Unit 1 of Course MST-001)]

= 1 –P(A ∪ B)

= 1– [P(A) + P(B)] [Using axiom (iii)]

$$= 1-\left[\frac{3}{36}+\frac{1}{36}\right]=1-\frac{4}{36}=1-\frac{1}{9} = \frac{8}{9}$$

Now, you can try the following exercises.

E4) If A, B and C are any three events, write down the expressions in terms of set theory:

a) only A occurs
b) A and B occur but C does not
c) A, B and C all the three occur
d) at least two occur
e) exactly two do not occur
f) none occurs

E5) Fourteen balls are serially numbered and placed in a bag. Find the probability that a ball is drawn bears a number multiple of 3 or 5.

2.7 SUMMARY

Let us summarize the main topics covered in this unit.

1) When classical definition fails, we obtain the probability by observing the data. This approach to probability is called the **relative frequency approach** and it defines the statistical probability. If an event A (say) happens m times in n trials of an experiment which is performed repeatedly under essentially homogeneous and identical conditions, then the (**Statistical or Empirical**) probability of happening A is defined as

$$P(A) = \lim_{n \to \infty} \frac{m}{n}.$$

2) **Subjective probability** is based on one's judgment, wisdom, intuition and expertise. It is interpreted as a measure of degree of belief or as the quantified judgment of particular individual.

3) If S be a sample space for a random experiment and A be an event which is subset of S, then P(A) is called probability function if it satisfies the following axioms

(i) P(A) is real and $P(A) \geq 0$

(ii) P (S) = 1

(iii) If $A_1, A_2, \ldots$ is any finite or infinite sequence of disjoint events in S, then

$$P(A_1 \text{ or } A_2 \text{ or} \ldots \text{or } A_n) = P(A_1) + P(A_2) + \ldots + P(A_n).$$

This is the **axiomatic approach to the probability**.

2.8 SOLUTIONS/ANSWERS

E 1) Since the information is based on 5000 drivers,

the number of exhaustive cases is = 5000.

Thus,

(i) the required probability = $\frac{185}{5000} = \frac{37}{1000}$

(ii) the required probability = $\frac{85+65+70+50}{5000} = \frac{270}{5000} = \frac{27}{500}$

(iii) the required probability = $\frac{1000+195+500+170}{5000} = \frac{1865}{5000} = \frac{373}{1000}$

(iv) the required probability = $\frac{260+240+195+170}{5000} = \frac{865}{5000} = \frac{173}{1000}$

(v) the required probability = $\frac{600+900+1000+500}{500} = \frac{3000}{5000} = \frac{3}{5}$

E 2) Since the information is based on the record of 200 days, so the number of exhaustive cases in each case = 200.

(i) Number of favourable cases for correct forecast = 120

$\therefore$ the required probability = $\frac{120}{200} = \frac{12}{20} = \frac{3}{5}$

(iii) Number of favourable outcomes for incorrect forecast = 200 – 120

= 80

$\therefore$ the required probability = $\frac{80}{200} = \frac{2}{5}$

E 3) First throw a die 200 times and note the outcomes. Then construct a table for the number of throws and the number of times the odd number turns up as shown in the following format:

Number of Throws(n)	Number of times the odd number turns up (m)	Proportion (m/n)
1 2 3 . . . 200		

Now, plot the graph taking number of throws (n) on x-axis and the proportion ($\frac{m}{n}$) on y-axis in the manner as shown in Fig. 2.1. Then see to which value the proportion ($\frac{m}{n}$) approaches to as n becoming large. This limiting value of $\frac{m}{n}$ is the required probability.

E 4) a) $A \cap \bar{B} \cap \bar{C}$

b) $A \cap B \cap \bar{C}$

c) $A \cap B \cap C$

d) $\left(A \cap B \cap \bar{C}\right) \cup \left(A \cap \bar{B} \cap C\right) \cup \left(\bar{A} \cap B \cap C\right) \cup \left(A \cap B \cap C\right)$

e) $\left(\bar{A} \cap \bar{B} \cap C\right) \cup \left(\bar{A} \cap B \cap \bar{C}\right) \cup \left(A \cap \bar{B} \cap \bar{C}\right)$

f) $\bar{A} \cap \bar{B} \cap \bar{C}$

E 5) Let A be the event that the drawn ball bears a number multiple of 3 and B be the event that it bears a number multiple of 5, then

A = {3, 6, 9, 12} and B = {5, 10}

$$\therefore\ P(A) = \frac{4}{14} = \frac{2}{7} \text{ and } P(B) = \frac{2}{14} = \frac{1}{7}$$

The required probability = P(A or B)

$$= P(A) + P(B)$$

$$= \frac{2}{7} + \frac{1}{7} = \frac{3}{7}$$

[Using axiom (iii) as A and B are mutually disjoint]

UNIT 3 LAWS OF PROBABILITY

Structure

3.1 INTRODUCTION

In the preceding two units of this block, you have studied various approaches to probability, their direct applications and various types of events in terms of set theory. However, in many situations, we may need to find probability of occurrence of more complex events. Now, we are adequately equipped to develop the laws of probability i.e. the law of addition and the law of multiplication which will help to deal with the probability of occurrence of complex events.

Objectives

After completing this unit, you should be able to discuss:

- addition law of probability;
- conditional probability;
- multiplication law of probability;
- independent events;
- probability of happening at least one of the independent events; and
- problems on addition and multiplicative laws of probability.

3.2 ADDITION LAW

Addition Theorem on Probability for Two Events

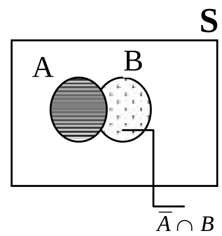
Statement

Let S be the sample space of a random experiment and events A and $B \subseteq S$ then $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof: From the Venn diagram, we have

$$A \cup B = A \cup \left(\bar{A} \cap B\right)$$

$$\therefore\ P\left(A \cup B\right) = P\left(A\right) + P\left(\bar{A} \cap B\right) \qquad \left[\begin{array}{l}\text{By axiom (iii)} \because\ A \text{ and } \bar{A} \cap B \\ \text{are mutually disjoint}\end{array}\right]$$



$$= P(A) + P(B) - P\left(A \cap B\right) \left[\begin{array}{l}\text{Refer to result 3 of Sec. 2.6} \\ \text{of Unit 2 of this Course}\end{array}\right]$$

Hence proved

Corollary: If events A and B are mutually exclusive events, then

$P(A \cup B) = P(A) + P(B)$.

[This is known as the addition theorem for mutually exclusive events]

Proof: For any two events A and B, we know that

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Now, if the events A and B are mutually exclusive then

$A \cap B = \phi$

Also, we know that probability of impossible event is zero i.e.

$P(A \cap B) = P(\phi) = 0$.

Hence,

$P(A \cup B) = P(A) + P(B)$

Similarly, for three non-mutually exclusive events A, B and C, we have

$$P\left(A \cup B \cup C\right) = P\left(A\right) + P\left(B\right) + P\left(C\right) - P\left(A \cap B\right) - P\left(A \cap C\right) - P\left(B \cap C\right) + P\left(A \cap B \cap C\right)$$

and for three mutually exclusive events A, B and C, we have

$P\left(A \cup B \cup C\right) = P\left(A\right) + P\left(B\right) + P\left(C\right)$.

The result can similarly be extended for more than 3 events.

Applications of Addition Theorem of Probability

Example 1: From a pack of 52 playing cards, one card is drawn at random. What is the probability that it is a jack of spade or queen of heart?

Solution: Let A and B be the events of drawing a jack of spade and queen of heart, respectively.

$\therefore$ P(A) $= \frac{1}{52}$ and P(B) $= \frac{1}{52}$ $\left[\begin{array}{l}\because \text{there is one card each of jack} \\ \text{of spade and queen of heart}\end{array}\right]$

Here, a card cannot be both the jack of spade and the queen of heart, hence A and B are mutually exclusive,

$\therefore$ applying the addition theorem for mutually exclusive events,

the required probability = $P(A \cup B) = P(A) + P(B)$

$$= \frac{1}{52} + \frac{1}{52} = \frac{2}{52} = \frac{1}{26}.$$

Example 2: 25 lottery tickets are marked with first 25 numerals. A ticket is drawn at random. Find the probability that it is a multiple of 5 or 7.

Solution: Let A be the event that the drawn ticket bears a number multiple of 5 and B be the event that it bears a number multiple of 7.

Therefore,

A = {5, 10, 15, 20, 25}

B = {7, 14, 21}

Here, as $A \cap B = \phi$,

$\therefore$ A and B are mutually exclusive, and hence,

$$P(A \cup B) = P(A) + P(B) = \frac{5}{25} + \frac{3}{25} = \frac{8}{25}$$

Example 3: Find the probability of getting either a number multiple of 3 or a prime number when a fair die is thrown.

Solution: When a die is thrown, then the sample space is

S = {1, 2, 3, 4, 5, 6}

Let A be the event of getting a number multiple of 3 and B be the event of getting a prime number,

$\therefore$ A = {3, 6}, B = {2, 3, 5}, $A \cap B$ = {3}

Here as $A \cap B$ is not empty set,

$\therefore$ A and B are non-mutually exclusive and hence,

the required probability = $P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{2}{6} + \frac{3}{6} - \frac{1}{6}$$

$$= \frac{2+3-1}{6} = \frac{4}{6} = \frac{2}{3}.$$

Example 4: There are 40 pages in a book. A page is opened at random. Find the probability that the number of this opened page is a multiple of 3 or 5.

Solution Let A be the event that the number of the opened page is a multiple of 3 and B be the event that it is a multiple of 5.

$\therefore$ A = {3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39},

B = {5, 10, 15, 20, 25, 30, 35, 40}, and

$A \cap B = \{15, 30\}$

As A and B are non-mutually exclusive,

$\therefore$ the required probability = $P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{13}{40} + \frac{8}{40} - \frac{2}{40} = \frac{19}{40}.$$

Example 5: A Card is drawn from a pack of 52 playing cards, find the proabability that the drawn card is an ace or a red colour card.

Solution: Let A be the event that the drawn card is a card of ace and B be the event that it is red colour card.

Now as there are four cards of ace and 26 red colour cards in a pack of 52 playing cards. Also, 2 cards in the pack are ace cards of red colour.

$$\therefore P(A) = \frac{4}{52},\ P(B) = \frac{26}{52}\ \text{, and}\ P(A \cap B) = \frac{2}{52}$$

$\therefore$ the required probability = $P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B)$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}.$$

Now, you can try the following exercises.

E1) A card is drawn from a pack of 52 playing cards. Find the probability that it is either a king or a red card.

E2) Two dice are thrown together. Find the probability that the sum of the numbers turned up is either 6 or 8.

3.3 CONDITIONAL PROBABILITY AND MULTIPLICATIVE LAW

Conditional Probability

We have discussed earlier that P(A) represents the probability of happening event A for which the number of exhaustive cases is the number of elements in the sample space S. P(A) dealt earlier was the unconditional probability. Here, we are going to deal with conditional probability.

Let us start with taking the following example:

Suppose a card is drawn at random from a pack of 52 playing cards. Let A be the event of drawing a black colour face card. Then $A = \{J_s, Q_s, K_s, J_c, Q_c, K_c\}$ and hence

P(A) = 6/52 = 3/26.

Let B be the event of drawing a card of spade i.e.

$B = \{1_s, 2_s, 3_s, 4_s, 5_s, 6_s, 7_s, 8_s, 9_s, 10_s, J_s, Q_s, K_s\}$.

If after a card is drawn from the pack of cards, we are given the information that card of spade has been drawn i.e., B has happened, then the probability of event A will no more be $\frac{3}{26}$, because here in this case, we have the information that the card drawn is of spade (i.e. from amongst 13 cards) and hence there are 13 exhaustive cases and not 52. From amongst these 13 cards of spade, there are 3 black colour face cards and hence probability of having black colour face card given that it is a card of spade i.e. P(A|B) = 3/13, which is the conditional probability of A given that B has already happened.

Note: Here, the symbol '|' used in P(A|B) should be read as 'given' and not 'upon'. P(A|B) is the conditional probability of happening A given that B has already happened i.e. here A happens depending on the condition of B.

So, the conditional probability P(A|B) is also the probability of happening A but here the information is given that the event B has already happened. P(A|B) refers to the sample space B and not S.

Remark 1: P(A|B) is meaningful only when $P(B) \neq 0$ i.e. when the event B is not an impossible event.

Multiplication Law of Probability

Statement: For two events A and B,

$$P(A \cap B) = P(A)\,P(B|A),\quad P(A) > 0 \qquad \ldots (1)$$

$$= P(B)\,P(A|B),\quad P(B) > 0, \qquad \ldots (2)$$

where P(B|A) is the conditional probability of B given that A has already happened and P(A|B) is the conditional probability of A given that B has already happened.

Proof: Let n be the number of exhaustive cases corresponding to the sample space S and m_1, m_2, m_3 be the number of favourable cases for events A, B and $A \cap B$ respectively.

$$\therefore\ P(A) = \frac{m_1}{n},\ P(B) = \frac{m_2}{n},\ P(A \cap B) = \frac{m_3}{n}$$

Now, as B|A represents the event of happening B given that A has already happened and hence it refers to the sample space A ($\because$ we have with us the information that A has already happened) and thus the number of exhaustive cases for B|A is m_1 (i.e. the number of cases favourable to "A relative to sample space S"). The number of cases favourable to B|A is the number of those elements of B which are in A i.e. the number of favourable cases to B|A is the number of favourable cases to "$A \cap B$ relative to S". So, the number of favourable cases to B|A is m_3

$$\therefore\ P(B\,|\,A) = \frac{m_3}{m_1}$$

$$= \frac{m_3/n}{m_1/n} \qquad \left[\begin{array}{l}\text{Dividing the numerator}\\ \text{and denominator by n}\end{array}\right]$$

$$= \frac{P(A \cap B)}{P(A)},\ P(A) \neq 0$$

$\Rightarrow P(A \cap B) = P(A)\ P(B|A),\ P(A) \neq 0$

Similarly, you can prove yourself that

$P(A \cap B) = P(B)\ P(A|B),\ P(B) \neq 0.$

This result can be extended for 3 or more events e.g. for three events A, B and C, we have

$P(A \cap B \cap C) = P(A)\ P(B|A)\ P(C|A \cap B),$

where $P(C|A \cap B)$ represents the probability of happening C given that A and B both have already happened.

3.4 INDEPENDENT EVENTS

Before defining the independent events, let us again consider the concept of conditional probability taking the following example:

Suppose, we draw a card from a pack of 52 playing cards, then probability of drawing a card of spade is 13/52. Now, if we do not replace the card back and draw the next card. Then, the probability of drawing the second card 'a card of spade' if it being given that the first card was spade would be 12/51 and it is the conditional probability. Now, if the first card had been replaced back then this conditional probability would have been 13/52. So, if sampling is done without replacement, the probability of second draw and that of subsequent draws made following the same way is affected but if it is done with replacement, then the probability of second draw and subsequent draws made following the same way remains unaltered.

So, if in the above example, if the next draw is made with replacement, then the happening or non-happening of any draw is not affected by the preceding draws. Let us now define independent events.

Independent Events

Events are said to be independent if happening or non-happening of any one event is not affected by the happening or non-happening of other events. For example, if a coin is tossed certain number of times, then happening of head in any trial is not affected by any other trial i.e. all the trials are independent.

Two events A and B are independent if and only if $P(B|A) = P(B)$ i.e. there is no relevance of giving any information. Here, if A has already happened, even then it does not alter the probability of B. e.g. Let A be the event of getting head in the 4^{th} toss of a coin and B be the event of getting head in the 5^{th} toss of the coin. Then the probability of getting head in the 5^{th} toss is $\frac{1}{2}$, irrespective of the case whether we know or don't know the outcome of 4^{th} toss, i.e. $P(B|A) = P(B)$.

Multiplicative Law for Independent Events:

If A and B are independent events, then

$P(A \cap B) = P(A)\ P(B).$

This is because if A and B are independent then $P(B|A) = P(A)$ and hence the equation (1) discussed in Sec. 3.3 of this unit becomes $P(A \cap B) = P(A)\,P(B)$.

Similarly, if A, B and C are three independent events, then

$P(A \cap B \cap C) = P(A)\,P(B)\,P(C)$.

The result can be extended for more than three events also.

Remark 2: Mutually exclusive events can never be independent.

Proof: Let A and B be two mutually exclusive events with positive probabilities (i.e. $P(A) > 0$, and $P(B) > 0$)

$\therefore\ P(A \cap B) = 0\ [\because A \cap B = \phi]$

Also, by multiplication law of probability, we have

$P(A \cap B) = P(A)\,P(B|A),\ P(A) \neq 0.$

$\therefore\ 0 = P(A)\,P(B|A).$

Now as $P(A) \neq 0,\ \therefore\ P(B|A) = 0$

But $P(B) \neq 0 \quad [\because P(B) > 0]$

$\therefore\ P(B|A) \neq P(B)$,

Hence A and B are not independent.

Result: If events A and B are independent then prove that

(i) A and $\overline{B}$ are independent

(ii) $\overline{A}$ and B are independent

(iii) $\overline{A}$ and $\overline{B}$ are independent

Proof:

(i) We know that

$$\begin{aligned} P(A \cap \overline{B}) &= P(A) - P(A \cap B) && \text{[Already proved in Unit 2 of this course]} \\ &= P(A) - P(A)P(B) && [\because \text{events A and B are independent.}] \\ &= P(A)(1 - P(B)) \\ &= P(A)P(\overline{B}) \end{aligned}$$

$\Rightarrow$ Events A and $\overline{B}$ are independent

(ii) We know that

$$\begin{aligned} P(\overline{A} \cap B) &= P(B) - P(A \cap B) && \text{[Already proved in Unit 2 of this course]} \\ &= P(B) - P(A)P(B) && [\because \text{ events A and B are independent.}] \\ &= P(B)(1 - P(A)) \\ &= P(B)P(\overline{A}) \\ &= P(\overline{A})P(B) \end{aligned}$$

$\Rightarrow$ Events $\bar{A}$ and B are independent

(iii) $P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$ [By De-Morgan's law]

$= 1 - P(A \cup B)$ $\left[\because P(E) + P(\bar{E}) = 1\right]$

$= 1 - [P(A) + P(B) - P(A \cap B)]$ [Using addition law on probability]

$= 1 - [P(A) + P(B) - P(A)P(B)]$ $[\because$ events A and B are independent. $]$

$= 1 - P(A) - P(B) + P(A)P(B)$

$= (1 - P(A)) - P(B)(1 - P(A))$

$= (1 - P(A))(1 - P(B))$

$= P(\bar{A})P(\bar{B})$

$\Rightarrow$ Events $\bar{A}$ and $\bar{B}$ are independent

Now let us take up some examples on conditional probability, multiplicative law and independent events:

Example 6: A die is rolled. If the outcome is a number greater than 3, what is the probability that it is a prime number?

Solution: The sample space of the experiment is

S = {1, 2, 3, 4, 5, 6}

Let A be event that the outcome is a number greater than 3 and B be the event that it is a prime number.

$\therefore$ A = {4, 5, 6}, B = {2, 3, 5} and hence $A \cap B$ = {5}.

$\Rightarrow$ P(A) = 3/6, P(B) = 3/6, $P(A \cap B)$ = 1/6.

Now, the required probability = P(B|A)

$$= \frac{P(A \cap B)}{P(A)} \qquad \left[\begin{array}{l}\text{Refer the multiplication} \\ \text{law given by (1) in} \\ \text{Sec. 3.3 of the Unit}\end{array}\right]$$

$$= \frac{1/6}{3/6} = \frac{1}{3}$$

Example 7: A couple has 2 children. What is the probability that both the children are boys, if it is known that?

(i) younger child is a boy

(ii) older child is a boy

(iii) at least one of them is boy

Solution: Let B_i, G_i denote that i^{th} birth is of boy and girl respectively, i =1, 2.

Then for a couple having two children, the sample space is

$S=\{B_1B_2,\ B_1G_2, G_1B_2, G_1G_2\}$

Let A be the event that both children are boys then

$A = \{B_1B_2\}$

(i) Let B be the event of getting younger child as boy i.e.

$B=\{B_1B_2, G_1B_2\}$. Hence $A \cap B = \{B_1B_2\}$

$$\therefore \text{required probability } P(A\mid B) = \frac{P(A\cap B)}{P(B)} = \frac{1/4}{2/4} = \frac{1}{2}$$

(ii) Let C be the event of getting older child as boy, then C = $\{B_1B_2,\ B_1G_2\}$

and hence $A \cap C = \{B_1B_2\}$.

$$\therefore \text{ required probability } P(A\mid C) = \frac{P(A\cap C)}{P(C)} = \frac{1/4}{2/4} = \frac{1}{2}.$$

(iii) Let D be the event of getting at least one of the children as boy, then

$D=\{B_1B_2,\ B_1G_2,\ G_1B_2\}$ and hence

$A \cap D = \{B_1B_2\}$.

$$\therefore \text{required probability } P(A\mid D) = \frac{P(A\cap D)}{P(D)} = \frac{1/4}{3/4} = \frac{1}{3}$$

Example 8: An urn contains 4 red and 7 blue balls. Two balls are drawn one by one without replacement. Find the probability of getting 2 red balls.

Solution: Let A be the event that first ball drawn is red and B be the event that the second ball drawn is red.

$$\therefore P(A) = 4/11 \text{ and } P(B|A) = 3/10 \begin{bmatrix} \because \text{it is given that one red ball} \\ \text{has already been drawn} \end{bmatrix}$$

$\therefore$ The required probability = P(A and B)

$$= P(A)\,P(B|A)$$

$$= \left(\frac{4}{11}\right)\left(\frac{3}{10}\right) = \frac{6}{55}$$

Example 9: Three cards are drawn one by one without replacement from a well shuffled pack of 52 playing cards. What is the probability that first card is jack, second is queen and the third is again a jack.

Solution: Define the following events

E_1 be the event of getting a jack in the first draw,

E_2 be the event of getting a queen in second draw, and

E_3 be the event of getting a jack in third draw,

$\therefore$ Required probability = $P(E_1 \cap E_2 \cap E_3)$

$$= P(E_1)P(E_2|E_1)P(E_3 | E_1 \cap E_2)$$

$$\left[\begin{array}{l}\because \text{ cards are drawn without replacement and} \\ \text{hence the events are not independent}\end{array}\right]$$

$$= \frac{4}{52} \times \frac{4}{51} \times \frac{3}{50} = \frac{1}{13} \times \frac{2}{17} \times \frac{1}{25} = \frac{2}{5525}.$$

Example 10: (i) If A and B are independent events with

$P(A \cup B) = 0.8$ and $P(B) = 0.4$ then find P(A).

(ii) If A and B are independent events with

$P(A) = 0.2$, $P(B) = 0.5$ then find $P(A \cup B)$.

(iii) If A and B are independent events and

$P(A) = 0.4$ and $P(B) = 0.3$, then find $P(A|B)$ and $P(B|A)$.

(iv) If A and B are independent events with $P(A) = 0.4$ and $P(B) = 0.2$, then find

$$P(\overline{A} \cap B),\ P(A \cap \overline{B}),\ P(\overline{A} \cap \overline{B})$$

Solution:

(i) We are given

$$P(A \cup B) = 0.8,\ P(B) = 0.4.$$

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{[By Addition theorem of probability]}$$

$$= P(A) + P(B) - P(A)P(B) \quad [\because \text{ events A and B are independent}]$$

$$\Rightarrow\ 0.8 = P(A) + 0.4 - 0.4P(A)$$

$$\Rightarrow\ 0.4 = (1 - 0.4)\, P(B)$$

$$= 0.6\, P(B)$$

$$\Rightarrow P(B) = \frac{0.4}{0.6} = \frac{4}{6} = \frac{2}{3}.$$

(ii) We are given that $P(A) = 0.2$, $P(B) = 0.5$.

We know that

$$P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad \text{[Addition theorem of probability]}$$

$$= P(A) + P(B) - P(A)P(B) \quad [\because \text{ events A and B are independent}]$$

$$= 0.2 + 0.5 - 0.2 \times 0.5$$

$$= 0.7 - 0.10 = 0.6$$

(iii) We are given that $P(A) = 0.4$, $P(B) = 0.3$.

Now, as A and B are independent events,

$\therefore P(A \cap B) = P(A)P(B)$

$= 0.4 \times 0.3 = 0.12$

And hence from conditional probability, we have

$$P(A\,|\,B) = \frac{P(A \cap B)}{P(B)} = \frac{0.12}{0.3} = \frac{12}{30} = \frac{2}{5} = 0.4,$$

and $P(B\,|\,A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{0.12}{0.4} = \dfrac{12}{40} = \dfrac{3}{10} = 0.3.$

(iv) We are given that P(A) = 0.4, P(B) = 0.2.

We know that if two events A and B are independent then

$\overline{A}$ and B; A and $\overline{B}$; $\overline{A}$ and $\overline{B}$ are also independent events.

$\therefore P(\overline{A} \cap B) = P(\overline{A})P(B)$ [Using the concept of independent events]

$= (1 - P(A))(P(B))$

$= (1 - 0.4)\,(0.2)$

$= (0.6)\,(0.2)$

$= 0.12$

$P(A \cap \overline{B}) = P(A)\,P(\overline{B})$ [$\because$ A and $\overline{B}$ are independent]

$= (0.4)\,(1 - 0.2) = 0.32$

$$P\left(\overline{A} \cap \overline{B}\right) = P\left(\overline{A}\right)P\left(\overline{B}\right) = (1-0.4)(1-0.2) = (0.6)(0.8) = 0.48.$$

Example 11: Three unbiased coins are tossed simultaneously. In which of the following cases are the events A and B independent?

(i) A be the event of getting exactly one head

B be the event of getting exactly one tail

(ii) A be the event that first coin shows head

B be the event that third coin shows tail

(iii) A be the event that shows exactly two tails

B be the event that third coin shows head

Solution: When three unbiased coins are tossed simultaneously, then the sample space is given by

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

(i) A = {HTT, THT, TTH}

B = {HHT, HTH, THH}

$A \cap B = \{\} = \phi$

$$\therefore P(A) = \frac{3}{8}, P(B) = \frac{3}{8}, P(A \cap B) = \frac{0}{8} = 0$$

Hence, P(A) P(B) = $\dfrac{3}{8} \times \dfrac{3}{8} = \dfrac{9}{64}$

$\Rightarrow P(A \cap B) \neq P(A)P(B)$

$\Rightarrow$ Events A and B are not independent.

(ii) A = {HHH, HHT, HTH, HTT}

B = {HHT, HTT, THT, TTT}

$A \cap B = \{HHT, HTT\}$

$\therefore P(A) = \frac{4}{8} = \frac{1}{2}, P(B) = \frac{4}{8} = \frac{1}{2}, P(A \cap B) = \frac{2}{8} = \frac{1}{4}$

$P(A)P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(A \cap B)$

$\Rightarrow$ Events A and B are independent.

(iii) A= {HTT, THT, TTH}

B = {HHH, HTH, THH, TTH}

$A \cap B = \{TTH\}$

$P(A) = \frac{3}{8}, P(B) = \frac{4}{8} = \frac{1}{2}, P(A \cap B) = \frac{1}{8}.$

Hence, $P(A)P(B) = \frac{3}{8} \times \frac{1}{2} = \frac{3}{16} \neq P(A \cap B)$

$\Rightarrow$ Events A and B are not independent.

Example 12: Two cards are drawn from a pack of cards in succession with replacement of first card. Find the probability that both are the cards of 'heart'.

Solution: Let A be the event that the first card drawn is a heart card and B be the event that second card is a heart card.

As the cards are drawn with replacement,

$\therefore$ A and B are independent and hence the required probability

$$= P(A \cap B) = P(A)P(B) = \left(\frac{13}{52}\right)\left(\frac{13}{52}\right) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}.$$

Example 13: A class consists of 10 boys and 40 girls. 5 of the students are rich and 15 students are brilliant. Find the probability of selecting a brilliant rich boy.

Solution: Let A be the event that the selected student is brilliant, B be the event that he/she is rich and C be the event that the student is boy.

$\therefore P(A) = \frac{15}{50}, P(B) = \frac{5}{50}, P(C) = \frac{10}{50}$ and hence

the required probability = $P(A \cap B \cap C)$

$$= P(A)P(B)P(C) \quad [\because A, B \text{ and } C \text{ are independent}]$$

$$= \left(\frac{15}{50}\right)\left(\frac{5}{50}\right)\left(\frac{10}{50}\right) = \left(\frac{3}{10}\right)\left(\frac{1}{10}\right)\left(\frac{1}{5}\right) = \frac{3}{500}$$

Here are some exercises for you.

E3) A card is drawn from a well-shuffled pack of cards. If the card drawn is a face card, what is the probability that it is a king?

E4) Two cards are drawn one by one without replacement from a well shuffled pack of 52 cards. What is the probability that both the cards are red?

E5) A bag contains 10 good and 4 defective items, two items are drawn one by one without replacement. What is the probability that first drawn item is defective and the second one is good?

E6) The odds in favour of passing driving test by a person X are $3:5$ and odds in favour of passing the same test by another person Y are $3:2$. What is the probability that both will pass the test?

3.5 PROBABILITY OF HAPPENING AT LEAST ONE OF THE INDEPENDENT EVENTS

If A and B be two independent events, then probability of happening at least one of the events is

$$P(A \cup B) = 1 - P(\overline{A \cup B})$$

$$= 1 - P(\bar{A} \cap \bar{B}) \qquad \text{[By DeMorgan's Law]}$$

$$= 1 - P(\bar{A})P(\bar{B}) \qquad \left[\because \text{ A and B and hence } \bar{A} \text{ and } \bar{B} \text{ are independent.}\right]$$

Similarly if we have n independent event $A_1, A_2, \ldots, A_n$, then probability of happening at least one of the events is

$$P(A_1 \cup A_2 \cup \ldots \cup A_n) = 1 - \left[P(\bar{A}_1)P(\bar{A}_2) \ldots P(\bar{A}_n)\right]$$

I.e. probability of happening at least one of the independent events

= 1 – probability of happening none of the events.

Example 14: A person is known to hit the target in 4 out of 5 shots whereas another person is known to hit 2 out of 3 shots. Find the probability that the target being hit when they both try.

Solution: Let A be the event that first person hits the target and B be the event that second person hits the target.

$$\therefore P(A) = \frac{4}{5}, \quad P(B) = \frac{2}{3}$$

Now, as both the persons try independently,

$\therefore$ the required probability = probability that the target is hit

= probability that at least one of the persons hits the target

$= P(A \cup B)$

$= 1 - P(\bar{A})P(\bar{B})$

$$= 1 - \left(1 - \frac{4}{5}\right)\left(1 - \frac{2}{3}\right)$$

$$= 1 - \left(\frac{1}{5}\right)\left(\frac{1}{3}\right) = 1 - \frac{1}{15} = \frac{14}{15}.$$

E 7) A problem in statistics is given to three students A, B and C whose chances of solving it are 0.3, 0.5 and 0.6 respectively. What is the probability that the problem is solved?

3.6 PROBLEMS USING BOTH ADDITION AND MULTIPLICATIVE LAWS

Here we give some examples which are based on both the addition and multiplication laws.

Example15: Husband and wife appear in an interview for two vacancies for the same post. The probabilities of husband's and wife's selections are $\frac{2}{5}$ and $\frac{1}{5}$ respectively. Find the probability that

(i) Exactly one of them is selected

(ii) At least one of them is selected

(iii) None is selected.

Solution: Let H be the event that husband is selected and W be the event that wife is selected. Then,

$$P(H) = \frac{2}{5},\ P(W) = \frac{1}{5}$$

$$\therefore P(\overline{H}) = 1 - \frac{2}{5} = \frac{3}{5},\ P(\overline{W}) = 1 - \frac{1}{5} = \frac{4}{5}$$

(i) The required probability $= P\left[(H \cap \overline{W}) \cup (\overline{H} \cap W)\right]$

$$= P(H \cap \overline{W}) + P(\overline{H} \cap W)$$

$\left[\begin{array}{l}\text{By Addition theorem for}\\ \text{mutually exclusive events}\end{array}\right]$

$$= P(H)P(\overline{W}) + P(\overline{H})P(W)$$

$\left[\begin{array}{l}\because \text{selection of husband and}\\ \text{wife are independent}\end{array}\right]$

$$= \frac{2}{5} \times \frac{4}{5} + \frac{3}{5} \times \frac{1}{5} = \frac{8}{25} + \frac{3}{25} = \frac{11}{25}.$$

(ii) The required probability $= P(H \cup W)$

$$= 1 - P\left(\overline{H}\right)P\left(\overline{W}\right)$$

$$\left[\begin{array}{l}\because \text{H and W are independent and hence} \\ \overline{\text{H}} \text{ and } \overline{\text{W}} \text{ are independent events}\end{array}\right]$$

$$= 1 - \frac{3}{5} \times \frac{4}{5} = 1 - \frac{12}{25} = \frac{13}{25}.$$

(iii) The required probability = $P(\overline{H} \cap \overline{W})$

$$= P(\overline{H})P(\overline{W}) \left[\because \overline{H} \text{ and } \overline{W} \text{ are independent}\right]$$

$$= \frac{3}{5} \times \frac{4}{5} = \frac{12}{25}$$

Example 16: A person X speaks the truth in 80% cases and another person Y speaks the truth in 90% cases. Find the probability that they contradict each other in stating the same fact.

Solution: Let A, B be the events that person X and person Y speak truth respectively, then

$$P(A) = \frac{80}{100} = 0.8, \; P(B) = \frac{90}{100} = 0.9.$$

$$\therefore P(\overline{A}) = 1 - 0.8 = 0.2, \; P(\overline{B}) = 1 - 0.9 = 0.1.$$

Thus, the required probability $= P\left[\left(A \cap \overline{B}\right) \cup \left(\overline{A} \cap B\right)\right]$

$$= P(A \cap \overline{B}) + P(\overline{A} \cap B) \left[\begin{array}{l}\text{By addition law for mutually} \\ \text{exclusive events}\end{array}\right]$$

$$= P(A)P(\overline{B}) + P(\overline{A})P(B) \left[\begin{array}{l}\text{By multiplication law for} \\ \text{independent events}\end{array}\right]$$

$$= 0.8 \times 0.1 + 0.2 \times 0.9$$

$$= 0.08 + 0.1 = 0.26 = 26\%.$$

Here is an exercise for you.

E 8) Two cards are drawn from a pack of cards in succession presuming that drawn cards are replaced. What is the probability that both drawn cards are of the same suit?

3.7 SUMMARY

Let us summarize the main points covered in this unit:

1) For two non-mutually exclusive events A and B, the **addition law** of probability is given by $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. If these events are mutually exclusive then, $P(A \cup B) = P(A) + P(B)$.

2) **Conditional probability** for happening of an event say B given that an event A has already happened is given by

$P(B|A) = \dfrac{P(A \cap B)}{P(A)}$, P(A) > 0. The **Multiplicative law** of probability for any two events is stated as $P(A \cap B) = P(A)\,P(B|A)$.

3) Events are said to be **independent** if happening or non-happening of any one event is not affected by the happening or non-happening of other events. Two events A and B are independent if and only if $P(B|A) = P(B)$ and hence **multiplicative law for two independent events** is given by $P(A \cap B) = P(A)\,P(B)$.

4) If events are independent then their complements are also independent. **Probability of happening at least one of the independent events** can be obtained on subtracting from 1 the probability of happening none of the events.

3.8 SOLUTIONS/ANSWERS

E1) Let A be the event of getting a card of king and B be the event of getting a red card.

$\therefore$ the required probability $= P(A \cup B)$

$$= P(A) + P(B) - P(A \cap B) \quad \text{[By addition theorem]}$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52} \quad \left[\begin{array}{l}\because \text{there are two cards which} \\ \text{are both king as well as red}\end{array}\right]$$

$$= \frac{4}{52} + \frac{26}{52} - \frac{2}{52}$$

$$= \frac{4 + 26 - 2}{52} = \frac{28}{52} = \frac{7}{13}$$

E2) Here, the number of exhaustive cases = $6 \times 6 = 36$.

Let A be the event that the sum is 6 and B be the event that the sum is 8.

$\therefore$ A = {(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)}, and

B = {(2, 6), (3, 5), (4, 4), (5, 3), (6, 2)}.

Here, as $A \cap B = \phi$,

$\therefore$ A and B are mutually exclusive and hence

the required probability = $P(A \cup B) = P(A) + P(B)$

$$= \frac{5}{36} + \frac{5}{36} = \frac{10}{36} = \frac{5}{18}$$

E 3) Let A be the event that the card drawn is face card and B be the event that it is a king.

$\therefore$ $A = \{J_s, Q_s, K_s, J_h, Q_h, K_h, J_d, Q_d, K_d, J_c, Q_c, K_c\}$

$B = \{K_s, K_h, K_c, K_d\}$

$\Rightarrow A \cap B = \{K_s, K_h, K_c, K_d\}$

$\Rightarrow P(A) = \frac{12}{52},\ P(B) = \frac{4}{52}$, and $P(A \cap B) = \frac{4}{52}$.

$\therefore$ the required probability $= P(B \mid A) = \frac{P(A \cap B)}{P(A)} = \frac{4/52}{12/52} = \frac{4}{12} = \frac{1}{3}$.

E 4) Let A be the event that first card drawn is red and B be the event that second card is red.

$\therefore P(A) = \frac{26}{52}$ and $P(B|A) = \frac{25}{51}$.

Thus, the desired probability = $P(A \cap B)$

$$= P(A)\,P(B|A) \quad \begin{bmatrix}\text{By multiplication}\\ \text{theorem}\end{bmatrix}$$

$$= \left(\frac{26}{52}\right)\left(\frac{25}{51}\right) = \frac{25}{102}.$$

E 5) Let E_1 be the event of getting a defective item in first draw and E_2 be the event of getting a good item in the second draw.

$\therefore$ the required probability $= P(E_1 \cap E_2)$

$$= P(E_1)P(E_2 \mid E_1) \quad \begin{bmatrix}\text{By multiplication}\\ \text{theorem}\end{bmatrix}$$

$$= \frac{4}{14} \times \frac{10}{13} = \frac{20}{91}.$$

E 6) Let A be the event that person X passes the test and B be the event that person Y passes the test.

$\therefore P(A) = \frac{3}{3+5} = \frac{3}{8}$

and $P(B) = \frac{3}{3+2} = \frac{3}{5}$

Now, as both the person take the test independently,

$\therefore$ the required probability $= P(A \cap B)$

$$= P(A)P(B) = \left(\frac{3}{8}\right)\left(\frac{3}{5}\right) = \frac{9}{40}$$

E 7) Let E_1, E_2 and E_3 be the events that the students A, B and C solves the problem respectively.

$\therefore P(E_1) = 0.3$, $P(E_2) = 0.5$ and $P(E_3) = 0.6$.

Now, as the students try independently to solve the problem,

$\therefore$ the probability that the problem will be solved

= Probability that at least one of the students solves the problem

$= P(E_1 \cup E_2 \cup E_3)$

$= 1 - P\left(\bar{E}_1\right)P\left(\bar{E}_2\right)P\left(\bar{E}_3\right)$

$= 1 - (1 - 0.3)\,(1 - 0.5)\,(1 - 0.6)$

$= 1 - (0.7)\,(0.5)\,(0.4)$

$= 1 - 0.140 = 0.86.$

E 8) Let S_1, C_1, H_1 and D_1 be the events that first card is of spade, club, heart and diamond respectively; and let S_2, C_2, H_2 and D_2 be the events that second card is of spade, club, heart and diamond respectively.

Thus, the required probability

$= P[(S_1 \cap S_2) \text{ or } (C_1 \cap C_2) \text{ or } (H_1 \cap H_2) \text{ or } (D_1 \cap D_2)]$

$= P(S_1 \cap S_2) + P(C_1 \cap C_2) + P(H_1 \cap H_2) + P(D_1 \cap D_2)$

$= P(S_1)P(S_2) + P(C_1)P(C_2) + P(H_1)P(H_2) + P(D_1)P(D_2)$

$\left[\begin{array}{l} \because \text{cards are drawn with replacement and} \\ \text{hence the draws are independent} \end{array}\right]$

$$= \frac{13}{52} \times \frac{13}{52} + \frac{13}{52} \times \frac{13}{52} + \frac{13}{52} \times \frac{13}{52} + \frac{13}{52} \times \frac{13}{52}$$

$$= 4 \times \frac{13}{52} \times \frac{13}{52}$$

$$= \frac{1}{4}$$

UNIT 4 BAYES' THEOREM

Structure

4.1 INTRODUCTION

In the first three units of this block, we have seen as to how probability of different types of events is calculated. We have also discussed the operation of events and evaluation of their probabilities by using addition and multiplication laws. Still, there are situations whose probability evaluations require use of more results.

Probabilities obtained in the earlier units are per-revised or priori probabilities. However, these probabilities can be revised on the basis of some new related information. The revised probabilities are obtained using Bayes' theorem for which knowledge of total probability is also required. So, the present unit first discusses the law of total probability, its applications and then Bayes' theorem and its applications.

Objectives

After completing this unit, you should be able to:

- explain law of total probability;
- know as to how and when to apply the law of total probability;
- learn Bayes' theorem; and
- learn as to how and when to apply Bayes' theorem.

4.2 LAW OF TOTAL PROBABILITY

There are experiments which are conducted in two stages for completion. Such experiments are termed as two-stage experiments. At the first stage, the experiment involves selection of one of the given numbers of possible mutually exclusive events. At the second stage, the experiment involves happening of an event which is a sub-set of at least one of the events of first stage.

As an illustration for a two-stage experiment, let us consider the following example:

Suppose there are two urns – Urn I and Urn II. Suppose Urn I contains 4 white, 6 blue and Urn II contains 4 white, 5 blue balls. One of the urns is selected at random and a ball is drawn. Here, the first stage is the selection of one of the urns and second stage is the drawing of a ball of particular colour.

If we are interested in finding the probability of the event of second stage, then it is obtained using law of total probability, which is stated and proved as under:

Law of Total Probability

Statement: Let S be the sample space and $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events with $P(E_i) \neq 0$; i = 1, 2, …, n. Let A be any event which is a sub-set of $E_1 \cup E_2 \cup \ldots \cup E_n$ (i.e. at least one of the events $E_1, E_2, \ldots, E_n$) with $P(A) > 0$, then $P(A) = P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + \ldots + P(E_n)\, P(A|E_n)$

$$= \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)$$

Proof: As A is a sub-set of $E_1 \cup E_2 \cup \ldots \cup E_n$

$\therefore\ A = A \cap (E_1 \cup E_2 \cup \ldots \cup E_n)$ [$\because$ if A is sub-set of B, then $A = A \cap B$] A

$\Rightarrow A = (A \cap E_1) \cup (A \cap E_2) \cup \ldots \cup (A \cap E_n)$ [Distributive property of set theory]

$= (E_1 \cap A) \cup (E_2 \cap A) \cup \ldots \cup (E_n \cap A)$

The above expression can be understood theoretically/logically also as explained under :

A happens in any of the following mutually exclusive ways:

(E_1 happens and then A happens) or (E_2 happens and then A happens) or (E_3 happens and then A happens) or … or (E_n happens and then A happens).
Now, as meanings of 'and' and 'or' in set theory are '$\cap$' and '$\cup$' respectively

$\therefore\ A = [(E_1 \cap A) \cup (E_2 \cap A) \cup (E_3 \cap A) \cup \ldots \cup (E_n \cap A)]$

$\Rightarrow P(A) = P[(E_1 \cap A) \cup (E_2 \cap A) \cup \ldots \cup (E_n \cap A)]$

$= P(E_1 \cap A) + P(E_2 \cap A) + \ldots + P(E_n \cap A)$

[$\because$ $E_1, E_2, \ldots, E_n$ and hence $E_1 \cap A, E_2 \cap A, \ldots, E_n \cap A$ are mutually exclusive]

$= P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + \ldots + (E_n)\, P(A|E_n)$

[Using multiplication theorem for dependent events]

$$= \sum_{i=1}^{n} P(E_i)\, P(A \mid E_i)$$

Hence proved

4.3 APPLICATIONS OF LAW OF TOTAL PROBABILITY

Here, in this section, we are going to take up various situations through examples, where the law of total probability is applicable.

Example 1: There are two bags. First bag contains 5 red, 6 white balls and the second bag contains 3 red, 4 white balls. One bag is selected at random and a ball is drawn from it. What is the probability that it is (i) red, (ii) white.

Solution: Let E_1 be the event that first bag is selected and E_2 be the event that second bag is selected.

$$\therefore\ P(E_1) = P(E_2) = \frac{1}{2}.$$

(i) Let R be the event of getting a red ball from the selected bag.

$$\therefore P(R \mid E_1) = \frac{5}{11},\ \text{and}\ P(R \mid E_2) = \frac{3}{7}.$$

Thus, the required probability is given by

$$P(R) = P(E_1)P(R \mid E_1) + P(E_2)P(R \mid E_2)$$

$$= \frac{1}{2} \times \frac{5}{11} + \frac{1}{2} \times \frac{3}{7}$$

$$= \frac{5}{22} + \frac{3}{14} = \frac{35+33}{154} = \frac{68}{154} = \frac{34}{77}$$

(ii) Let W be the event of getting a white ball from the selected bag.

$$\therefore P(W \mid E_1) = \frac{6}{11},\ \text{and}\ P(W \mid E_2) = \frac{4}{7}.$$

Thus, the required probability is given by

$$P(W) = P(E_1)P(W \mid E_1) + P(E_2)P(W \mid E_2)$$

$$= \frac{1}{2} \times \frac{6}{11} + \frac{1}{2} \times \frac{4}{7} = \frac{3}{11} + \frac{2}{7} = \frac{21+22}{77} = \frac{43}{77}.$$

Example 2: A factory produces certain type of output by 3 machines. The respective daily production figures are-machine X : 3000 units, machine Y: 2500 units and machine Z: 4500 units. Past experience shows that 1% of the output produced by machine X is defective. The corresponding fractions of defectives for the other two machines are 1.2 and 2 percent respectively. An item is drawn from the day's production. What is the probability that it is defective?

Solution: Let E_1, E_2 and E_3 be the events that the drawn item is produced by machine X, machine Y and machine Z, respectively. Let A be the event that the drawn item is defective.

As the total number of units produced by all the machines is

3000 + 2500 + 4500 = 10000,

$$\therefore\ P(E_1) = \frac{3000}{10000} = \frac{3}{10},\ P(E_2) = \frac{2500}{10000} = \frac{1}{4},\ P(E_3) = \frac{4500}{10000} = \frac{9}{20}.$$

$$P(A\mid E_1) = \frac{1}{100} = 0.01,\ P(A\mid E_2) = \frac{1.2}{100} = 0.012,\ P(A\mid E_3) = \frac{2}{100} = 0.02.$$

Thus, the required probability = Probability that the drawn item is defective

$= P(A)$

$= P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2) + P(E_3)\, P(A|E_3)$

$$= \frac{3}{10}\times 0.01 + \frac{1}{4}\times 0.012 + \frac{9}{20}\times 0.02$$

$$= \frac{3}{1000} + \frac{3}{1000} + \frac{9}{1000}$$

$$= \frac{15}{1000} = 0.015.$$

Example 3: There are two coins-one unbiased and the other two- headed, otherwise they are identical. One of the coins is taken at random without seeing it and tossed. What is the probability of getting head?

Solution: Let E_1 and E_2 be the events of selecting the unbiased coin and the two-headed coin respectively. Let A be the event of getting head on the tossed coin.

$\therefore\ P(E_1) = \frac{1}{2},\ P(E_2) = \frac{1}{2}$ [$\because$ selection of each of the coin is equally likely]

$P(A\mid E_1) = \frac{1}{2}$ [$\because$ if it is unbiased coin, then head and tail are equally likely]

$P(A|E_2) = 1$ [$\because$ if it is two-headed coin, then getting the head is certain]

Thus, the required probability = P(A)

$= P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2)$

$$= \frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times 1$$

$$= \frac{1}{4} + \frac{1}{2} = \frac{3}{4}.$$

Example 4: The probabilities of selection of 3 persons for the post of a principal in a newly started colleage are in the ratio $4:3:2$. The probabilities that they will introduce co-education in the college are 0.2, 0.3 and 0.5, respectively. Find the probability that co-education is introduced in the college.

Solution: Let E_1, E_2, E_3 be the events of selection of first, second and third person for the post of a principal respectively. Let A be the event that co-education is introduced.

$$\therefore\ P(E_1) = \frac{4}{9},\ P(E_2) = \frac{3}{9},\ P(E_3) = \frac{2}{9}$$

$$P(A\mid E_1) = 0.2,\ P(A\mid E_2) = 0.3,\ P(A\mid E_3) = 0.5.$$

Thus, the required probability = P(A)

$$= P(E_1)P(A|E_1) + P(E_2)P(A|E_2) + P(E_3)\,P(A|E_3)$$

$$= \frac{4}{9}\times 0.2 + \frac{3}{9}\times 0.3 + \frac{2}{9}\times 0.5$$

$$= \frac{0.8}{9} + \frac{0.9}{9} + \frac{1}{9} = \frac{2.7}{9} = 0.3$$

Now, you can try the following exercises.

E1) A person gets a construction job and agrees to undertake it. The completion of the job in time depends on whether there happens to be strike or not in the company. There are 40% chances that there will be a strike. Probability that job is completed in time is 30% if the strike takes place and is 70% if the strike does not take place. What is the probability that the job will be completed in time?

E2) What is the probability that a year selected at random will contains 53 Sundays?

E3) There are two bags, first bag contains 3 red, 5 black balls and the second bag contains 4 red, 5 black balls. One ball is drawn from the first bag and is put into the second bag without noticing its colour. Then two balls are drawn from the second bag. What is the probability that balls are of opposite colours?

4.4 BAYES' THEOREM

In Sec. 4.2 of this unit, we have discussed that if we are interested in finding the probability of the event of second stage, then it is obtained using law of total probability. But if the happening of the event of second stage is given to us and on this basis we find the probability of the events of first stage, then the probability of an event of first stage is the revised (or posterior) probabilities and is obtained using an important theorem known as Bayes' theorem given by Thomas Bayes (died in 1761, at the age of 59), a British Mathematician, published after his death in 1763. This theorem is also known as 'Inverse probability theorem', because here moving from first stage to second stage, we again find the probabilities (revised) of the events of first stage i.e. we move inversely. Thus, using this theorem, probabilities can be revised on the basis of having some related new information.

As an illustration, let us consider the same example as taken in Sec. 4.2 of this unit. In this example, if we are given that the drawn ball is of particular colour and it is asked to find, on this basis, the probability that Urn I or Urn II was selected, then these are the revised (posterior) probabilities and are obtained using Bayes' theorem, which is stated and proved as under:

Statement: Let S be the sample space and $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events with $P(E_i) \neq 0$; i = 1, 2, .., n. Let A be any event which is a sub-set of $E_1 \cup E_2 \cup \ldots \cup E_n$ (i.e. at least one of the events $E_1, E_2, \ldots, E_n$) with $P(A) > 0$ [Notice that up to this line the statement is same as that of law of total probability], then

$$P(E_i \mid A) = \frac{P(E_i)P(A \mid E_i)}{P(A)}, i = 1, 2, ..., n$$

where $P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2) + \ldots + P(E_n)\,P(A|E_n)$.

Proof: First you have to prove that

$P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2) + \ldots + P(E_n)\,P(A|E_n)$.

which is nothing but the law of total probability that has already been proved in Sec. 4.2. After proving this, proceed as under:

$$P(E_i \mid A) = \frac{P(E_i \cap A)}{P(A)} \qquad \left[\begin{array}{l}\text{Applying the formula of conditional} \\ \text{probability i.e. } P(A \mid B) = \dfrac{P(A \cap B)}{P(B)}\end{array}\right]$$

$$= \frac{P(E_i)P(A \mid E_i)}{P(A)} \qquad \left[\begin{array}{l}\text{Applying multiplication theorem for dependent} \\ \text{events i.e. } P(A \cap B) = P(A)P(B \mid A)\end{array}\right]$$

4.5 APPLICATIONS OF BAYES' THEOREM

Example 5: Let us consider the problem given in Example 1 of Sec 4.3 of this unit after replacing the question asked therein (i.e. the last sentence) by the following question:

If it is found to be red, what is the probability of?

i) selecting the first bag

ii) selecting the second bag

Solution: First, we have to give the solution exactly as given for Example 1 of Sec. 4.3 of this unit. After that, we are to proceed as follows:

i) Probability of selecting the first bag given that the ball drawn is red

$$= P(E_1 \mid R)$$

$$= \frac{P(E_1)P(R \mid E_1)}{P(R)} \qquad \left[\begin{array}{l}\text{Applying Bayes'} \\ \text{theorem}\end{array}\right]$$

$$= \frac{\frac{1}{2} \times \frac{5}{11}}{\frac{34}{77}} = \frac{5}{22} \times \frac{77}{34} = \frac{35}{68}$$

ii) Probability of selecting the second bag given that the ball drawn is red

$$P(E_2|R) = \frac{P(E_2)P(R \mid E_2)}{P(R)} = \frac{\frac{1}{2} \times \frac{3}{7}}{\frac{34}{77}} = \frac{3}{14} \times \frac{77}{34} = \frac{33}{68}$$

Example 6: Consider the problem given in Example 2 of Sec. 4.3 of this unit after replacing the question asked therein (i.e. the last sentence) by the following question:

If the drawn item is found to be defective, what is the probability that it has been produced by machine Y?

Solution: Proceed exactly in the manner the Example 2 of Sec. 4.3 of this unit has been solved and then as under:

Probability that the drawn item has been produced by machine Y given that it is defective

$$= P(E_2 \mid A)$$

$$= \frac{P(E_2)P(A \mid E_2)}{P(A)} \quad \begin{bmatrix} \text{Applying Bayes'} \\ \text{theorem} \end{bmatrix}$$

$$= \frac{\frac{1}{4} \times 0.012}{0.015} = \frac{0.003}{0.015} = \frac{1}{5}.$$

Example 7: Consider the problem given in Example 3 of Sec. 4.3 of this unit after replacing the question asked therein (i.e. the last sentence) by the following question:

If head turns up, what is the probability that

i) it is the two-headed coin

ii) it is the unbiased coin.

Solution: First give the solution of Example 3 of Sec. 4.3 of this unit and then proceed as under:

Now, the probability that the tossed coin is two-headed given that head turned up

$$= P(E_2 \mid A)$$

$$= \frac{P(E_2)P(A \mid E_2)}{P(A)} \quad \begin{bmatrix} \text{Applying Bayes'} \\ \text{theorem} \end{bmatrix}$$

$$= \frac{\frac{1}{2} \times 1}{\frac{3}{4}} = \frac{1}{2} \times \frac{4}{3} = \frac{2}{3}.$$

ii) The probability that the tossed coin is unbiased given that head turned up

$$= P(E_1 \mid A)$$

$$= \frac{P(E_1)P(A \mid E_1)}{P(A)} \quad \begin{bmatrix} \text{Applying Bayes'} \\ \text{theorem} \end{bmatrix}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{3}{4}} = \frac{1}{4} \times \frac{4}{3} = \frac{1}{3}$$

Example 8: Consider the statement given in Example 4 of Sec. 4.3 of this unit after replacing the question asked therein (i.e. the last sentence) by the following question:

If the co-education is introduced by the candidate selected for the post of principal, what is the probability that first candidate was selected.

Solution: First give the solution of Example 4 of Sec. 4.3 of this unit and then proceed as under:

The required probability $= P(E_1 \mid A)$

$$= \frac{P(E_1)P(A \mid E_1)}{P(A)} \quad \begin{bmatrix}\text{Applying Bayes'}\\ \text{theorem}\end{bmatrix}$$

$$= \frac{\frac{4}{9}\times 0.2}{0.3} = \frac{4}{9}\times\frac{0.2}{0.3} = \frac{8}{27}.$$

Now, you can try the following exercises.

E4) A bag contains 4 red and 5 white balls. Another bag contains 2 red and 3 white balls. A ball is drawn from the first bag and is transferred to the second bag. A ball is then drawn from the second bag and is found to be red, what is the probability that red ball was transferred from first to second bag?

E5) An insurance company insured 1000 scooter drivers, 3000 car drivers and 6000 truck drivers. The probabilities that scooter, car and truck drivers meet an accident are 0.02, 0.04, 0.25 respectively. One of the insured persons meets with an accident. What is the probability that he is a (i) car driver

(ii) truck driver

E6) By examining the chest X-ray, the probability that T.B is detected when a person is actually suffering from T.B. is 0.99. The probability that the doctor diagnoses incorrectly that a person has T.B. on the basis of X-ray is 0.002. In a certain city, one in 1000 persons suffers from T.B. A person is selected at random and is diagnosed to have T.B., what is the chance that he actually has T.B.?

E7) A person speaks truth 3 out of 4 times. A die is thrown. She reports that there is five. What is the chance there was five?

4.6 SUMMARY

In this unit, we have covered the following points:

1) There are experiments which are conducted in two stages for completion. Such experiments are termed as two-stage experiments. At the first stage, the experiment involves selection of one of the given number of possible mutually exclusive events. At the second stage, the experiment involves happening of an event which is a sub-set of at least one of the events of first stage. If we are interested in finding the probability of the event of second stage, then it is obtained using **law of total probability**, but if the

happening of the event of the second stage is given to us and on this basis we find the probability of an event of the first stage, then this probability of the event of the first stage is the revised (or posterior) probability and is obtained by using **Bayes' theorem.**

2) **Law of total probability** Let S be the sample space and $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events with $P(E_i) \neq 0$; i = 1, 2, …, n. Let A be any event which is a sub-set of $E_1 \cup E_2 \cup \ldots \cup E_n$ (i.e. at least one of the events $E_1, E_2, \ldots, E_n$) with P(A) > 0, then

$$P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2) + \ldots + P(E_n)\,P(A|E_n)$$

$$= \sum_{i=1}^{n} P(E_i)\,P(A \mid E_i).$$

3) **Bayes' theorem** is an extension of law of total probability and is stated as:

Let S be the sample space. Let $E_1, E_2, \ldots, E_n$ be n mutually exclusive and exhaustive events with $P(E_i) \neq 0$; i = 1, 2, …, n. Let A be any event which is a sub-set of $E_1 \cup E_2 \cup \ldots \cup E_n$ (i.e. at least one of the events $E_1, E_2, \ldots, E_n$) with P(A) > 0 [Notice that up to this line the statement is as same as that of law of total probability], then

$$P(E_i \mid A) = \frac{P(E_i)\,P(A \mid E_i)}{P(A)},\ i = 1, 2, \ldots, n$$

where $P(A) = P(E_1)\,P(A|E_1) + P(E_2)\,P(A|E_2) + \ldots + P(E_n)\,P(A|E_n)$.

4.7 SOLUTIONS/ANSWERS

E1) Let E_1 be the event that strike takes place and E_2 be the event that strike does not take place.

Let A be the event that the job will be completed in time.

$$\therefore P(E_1) = 40\% = \frac{40}{100},\ P(E_2) = 1 - P(E_1) = 1 - \frac{40}{100} = \frac{60}{100},$$

$$P(A \mid E_1) = 30\% = \frac{30}{100},\ \ P(A \mid E_2) = 70\% = \frac{70}{100}$$

Thus, the required probability is

$$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$$

$$= \frac{40}{100} \times \frac{30}{100} + \frac{60}{100} \times \frac{70}{100}$$

$$= \frac{1200}{10000} + \frac{4200}{10000} = \frac{12}{100} + \frac{42}{100} = \frac{54}{100} = 54\%$$

E2) Such probability in case of a leap year and in case of a non-leap year has already been obtained in Unit 1 of this course. But here we do not know as to whether it is a leap year or non-leap year, hence we will have to do it using law of total probability considering both the possibilities i.e. we shall proceed as follows:

Let E_1 be the event that the year selected is a leap year and E_2 be the event that it is a non-leap year. Let A be the event that the selected year contains 53 Sundays.

We know that out of 4 consecutive years, 1 is a leap year and 3 are non-leap years, therefore,

$$P(E_1) = \frac{1}{4},\ P(E_2) = \frac{3}{4}.$$

A leap years consists of 366 days, i.e. 52 complete weeks and 2 over days. These 2 over days may be "Sunday and Monday", "Monday and Tuesday", "Tuesday and Wednesday" ," Wednesday and Thursday", "Thursday and Friday", " Friday and Saturday" and " Saturday and Sunday". Out of these 7 cases, Sunday is included in 2 cases and hence the probability that a leap year will consist of 53 Sundays is

$$P(A \mid E_1) = \frac{2}{7}.$$

A non-leap year consists of 365 days, i.e. 52 complete weeks and 1 over day. This over day may be one of the seven days of the week and hence the probability that a non-leap year will consist of 53 sundays is

$$P(A \mid E_2) = \frac{1}{7}.$$

$\therefore$ By the total law of probability, the probability that a year selected at random will consists of 53 Sundays is

$$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2)$$

$$= \frac{1}{4} \times \frac{2}{7} + \frac{3}{4} \times \frac{1}{7} = \frac{2}{28} + \frac{3}{28} = \frac{5}{28}.$$

E3) We are given that first of all one ball is transferred from bag I to bag II and then two balls are drawn from bag II. This job can be done in two mutually exclusive ways

(a) A red ball is transferred from bag I to bag II and then two balls are drawn from bag II

(b) A black ball is transferred from bag I to bag II and then two balls are drawn from bag II

We define the following events:

E_1 be the event of getting a red ball from bag I

E_2 be the event of getting a black ball from bag I

A be the event of getting two balls of opposite colours from the bag II

$$\therefore P(E_1) = \frac{{}^{3}C_1}{{}^{8}C_1} = \frac{3}{8}$$

$$P(E_2) = \frac{{}^{5}C_1}{{}^{8}C_1} = \frac{5}{8}$$

$$P(A|E_1) = \frac{{}^5C_1 \times {}^5C_1}{{}^{10}C_2} = \frac{5 \times 5}{45} = \frac{5}{9}.$$

$$P(A|E_2) = \frac{{}^4C_1 \times {}^6C_1}{{}^{10}C_2} = \frac{4 \times 6}{45} = \frac{8}{15}$$

Hence, the required probability is given by

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2)$$

$$= \frac{3}{8} \times \frac{5}{9} + \frac{5}{8} \times \frac{8}{15}$$

$$= \frac{5}{24} + \frac{1}{3} = \frac{5+8}{24} = \frac{13}{24}$$

E4) Let E_1 be the event that a red ball is drawn from the first bag and E_2 be the event that the drawn ball from the first bag is white. Let A be the event of drawing a red ball from the second bag after transferring the ball drawn from first bag into it.

$$\therefore \ P(E_1) = \frac{4}{9}, P(E_2) = \frac{5}{9}$$

$$P(A|E_1) = \frac{3}{6}, P(A|E_2) = \frac{2}{6}$$

$\therefore$ By law of total probability,

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2)$$

$$= \frac{4}{9} \times \frac{3}{6} + \frac{5}{9} \times \frac{2}{6} = \frac{11}{27}.$$

Thus, the required probability = $P(E_1|A)$

$$= \frac{P(E_1)P(A|E_1)}{P(A)} \quad \left[\begin{array}{l}\text{Applying Bayes'} \\ \text{theorem}\end{array}\right]$$

$$= \frac{\frac{4}{9} \times \frac{3}{6}}{\frac{11}{27}} = \frac{6}{11}$$

E5) Total number of drivers = 1000 + 3000 + 6000 = 10000.

Let E_1, E_2, E_3 be the events that the selected insured person is a scooter driver, car driver, truck driver respectively. Let A be the event that a driver meets with an accident.

$$\therefore \ P(E_1) = \frac{1000}{10000} = \frac{1}{10}, \ P(E_2) = \frac{2000}{10000} = \frac{1}{5}, \ P(E_3) = \frac{6000}{10000} = \frac{6}{10} = \frac{3}{5}.$$

$$P(A|E_1) = 0.02 = \frac{2}{100}, \ P(A|E_2) = 0.04 = \frac{4}{100}, \ P(A|E_3) = 0.25 = \frac{25}{100}$$

$\therefore$ By total probability theorem, we have

$$P(A) = P(E_1)P(A \mid E_1) + P(E_2)P(A \mid E_2) + P(E_3)P(A \mid E_3)$$

$$= \frac{1}{10} \times \frac{2}{100} + \frac{1}{5} \times \frac{4}{100} + \frac{3}{5} \times \frac{25}{100}$$

$$= \frac{1}{500} + \frac{4}{500} + \frac{75}{500} = \frac{80}{500} = \frac{4}{25}$$

i) The required probability $= P(E_2 \mid A)$

$$= \frac{P(E_2)P(A \mid E_2)}{P(A)} \quad \begin{bmatrix}\text{Applying Bayes'} \\ \text{theorem}\end{bmatrix}$$

$$= \frac{\frac{1}{5} \times \frac{4}{100}}{\frac{4}{25}} = \frac{4}{500} \times \frac{25}{4} = \frac{1}{20}.$$

ii) The required probability $= P(E_3 \mid A)$

$$= \frac{P(E_3)P(A \mid E_3)}{P(A)} \quad \begin{bmatrix}\text{Applying Bayes'} \\ \text{theorem}\end{bmatrix}$$

$$= \frac{\frac{3}{5} \times \frac{25}{100}}{\frac{4}{25}} = \frac{75}{500} \times \frac{25}{4} = \frac{75}{80} = \frac{15}{16}.$$

E6) Let E_1 be the event that a person selected from the population of the city is suffering from the T.B. and E_2 be the event that he/she is not suffering from the T.B. Let A be the event that the selected person is diagnosed to have T.B. Therefore, according to given,

$$P(E_1) = \frac{1}{1000} = 0.001, \quad P(E_2) = 1 - 0.001 = 0.999 = \frac{999}{1000},$$

$$P(A \mid E_1) = 0.99 = \frac{99}{100}, P(A \mid E_2) = 0.002 = \frac{2}{1000}.$$

$\therefore$ By total probabilities theorem, we have

$$P(A) = P(E_1)\, P(A|E_1) + P(E_2)\, P(A|E_2)$$

$$= \frac{1}{1000} \times \frac{99}{100} + \frac{999}{1000} \times \frac{2}{1000}$$

$$= \frac{99}{100000} + \frac{1998}{1000000} = \frac{990 + 1998}{1000000} = \frac{2988}{1000000}$$

Thus, the required probability = $P(E_1|A)$

$$= \frac{P(E_1)P(A|E_1)}{P(A)} \quad \left[\begin{array}{l}\text{Applying Bayes'}\\ \text{theorem}\end{array}\right]$$

$$= \frac{\frac{1}{1000} \times \frac{99}{100}}{\frac{2988}{1000000}} = \frac{990}{2988} = \frac{55}{166}.$$

E7) Let E_1 be the event that the person speaks truth, E_2 be the event that she tells a lie and A be the event that she reports a five.

$$\therefore \; P(E_1) = \frac{3}{4}, \; P(E_2) = \frac{1}{4}, \; P(A|E_1) = \frac{1}{6}, \; P(A|E_2) = \frac{5}{6}.$$

By law of total probability, we have

$$P(A) = P(E_1)P(A|E_1) + P(E_2)P(A|E_2)$$

$$= \frac{3}{4} \times \frac{1}{6} + \frac{1}{4} \times \frac{5}{6}$$

$$= \frac{3}{24} + \frac{5}{24} = \frac{8}{24} = \frac{1}{3}.$$

Thus, the required probability = $P(E_1|A)$

$$= \frac{P(E_1)P(A|E_1)}{P(A)} \quad \left[\begin{array}{l}\text{Applying Bayes'}\\ \text{theorem}\end{array}\right]$$

$$= \frac{\frac{3}{4} \times \frac{1}{6}}{\frac{1}{3}} = \frac{3}{4} \times \frac{1}{6} \times \frac{3}{1} = \frac{3}{8}.$$