School of Engineering and Applied Science (SEAS), Ahmedabad University

BTech(ICT) Semester IV :Probability and Random Processes(MAT202)

Homework Assignment-3

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1. Given :

$$f_{X|M=0}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{x^2}{2\sigma^2})$$

and

$$f_{X|M=1}(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{(x-m)^2}{2\sigma^2})$$

Since messages sent are equiprobable , we have Pr(M=0)=$\frac{1}{2}$ and Pr(M=1)=$\frac{1}{2}$

a) To find Pr(M=0 | X= x) :

Given $\sigma^2 = 1$

$$Pr(M=0|X=x) = \frac{Pr(X=x|M=0) \times Pr(M=0)}{Pr(X=x)}$$

$$= \frac{Pr(X=x|M=0) \times Pr(M=0)}{Pr(X=x|M=0) \times Pr(M=0) + Pr(X=x|M=1) \times Pr(M=1)}$$

$$= \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times \frac{1}{2}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} \times \frac{1}{2} + \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-1)^2}{2}} \times \frac{1}{2}} \tag{1}$$

$$= \frac{e^{\frac{-x^2}{2}}}{e^{-\frac{x^2}{2}} + e^{-\frac{(x-1)^2}{2}}}$$

$$= \frac{1}{1 + e^{\frac{2x-1}{2}}}$$

For $\sigma^2 = 5$

$$Pr(M=1|X=x) = \frac{Pr(X=x|M=0)\times Pr(M=0)}{Pr(X=x)}$$

$$= \frac{Pr(X=x|M=0)\times Pr(M=0)}{Pr(X=x|M=0)\times Pr(M=0) + Pr(X=x|M=1)\times Pr(M=1)}$$

$$= \frac{\frac{1}{\sqrt{10\pi}}exp(-\frac{x^2}{10})\times\frac{1}{2}}{\frac{1}{\sqrt{10\pi}}exp(-\frac{x^2}{10})\times\frac{1}{2} + \frac{1}{\sqrt{10\pi}}exp(-\frac{(x-1)^2}{10})\times\frac{1}{2}} \tag{2}$$

$$= \frac{e^{\frac{-x^2}{10}}}{e^{-\frac{x^2}{10}} + e^{-\frac{(x-1)^2}{10}}}$$

$$= \frac{1}{1+e^{\frac{2x-1}{10}}}$$

b) Now given, Pr(M=0)=$\frac{1}{4}$ and Pr(M=1)=$\frac{3}{4}$

$$Pr(M=0|X=x) = \frac{Pr(X=x|M=0)\times Pr(M=0)}{Pr(X=x)}$$

$$= \frac{Pr(X=x|M=0)\times Pr(M=0)}{Pr(X=x|M=0)\times Pr(M=0) + Pr(X=x|M=1)\times Pr(M=1)}$$

$$= \frac{\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\times\frac{1}{4}}{\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\times\frac{1}{4} + \frac{1}{\sqrt{2\pi}}e^{-\frac{(x-1)^2}{2}}\times\frac{3}{4}} \tag{3}$$

$$= \frac{e^{\frac{-x^2}{2}}\times\frac{1}{4}}{e^{-\frac{x^2}{2}}\times\frac{1}{4} + e^{-\frac{(x-1)^2}{2}}\times\frac{3}{4}}$$

$$= \frac{1}{1+3e^{\frac{2x-1}{2}}}$$

For σ^2=5,

$$Pr(M=0|X=x) = \frac{Pr(X=x|M=0) \times Pr(M=0)}{Pr(X=x)}$$

$$= \frac{Pr(X=x|M=0) \times Pr(M=0)}{Pr(X=x|M=0) \times Pr(M=0) + Pr(X=x|M=1) \times Pr(M=1)}$$

$$= \frac{\frac{1}{\sqrt{10\pi}} e^{-\frac{x^2}{10}} \times \frac{1}{4}}{\frac{1}{\sqrt{10\pi}} e^{-\frac{x^2}{10}} \times \frac{1}{4} + \frac{1}{\sqrt{10\pi}} e^{-\frac{(x-1)^2}{10}} \times \frac{3}{4}} \tag{4}$$

$$= \frac{e^{\frac{-x^2}{10}} \times \frac{1}{4}}{e^{-\frac{x^2}{10}} \times \frac{1}{4} + e^{-\frac{(x-1)^2}{10}} \times \frac{3}{4}}$$

$$= \frac{1}{1+3e^{\frac{2x-1}{10}}}$$

2. Assuming the two messages are equiprobable : Pr(M=1)=Pr(M=0)= $\frac{1}{2}$
   From the above question, we found Pr(M=0 | X= x)=$\frac{1}{1+e^{\frac{2x-1}{2}}}$
   It is a decreasing function.

   To find x , we equate it to 0.9

$$\frac{1}{1+e^{\frac{2x-1}{2}}} \geq \frac{9}{10}$$
$$1+e^{\frac{2x-1}{2}} \geq \frac{10}{9}$$
$$e^{\frac{2x-1}{2}} \geq \frac{1}{9}$$
$$x \leq \frac{2ln(\frac{1}{9})+1}{2}$$
$$\leq -1.67$$

   $\therefore$ Pr(M=0 | X= x) $\geq$ 0.9 for x ranging between $-\infty$ to -1.697

$$\frac{1}{1+e^{\frac{-2x+1}{2}}} \geq \frac{9}{10}$$
$$1+e^{\frac{-2x+1}{2}} \geq \frac{10}{9}$$
$$e^{\frac{-2x+1}{2}} \geq \frac{1}{9}$$
$$x \leq -\frac{2ln(\frac{1}{9})-1}{2}$$
$$\leq -1.67$$

   $\therefore$ Pr(M=1 | X= x) $\geq$ 0.9 for x ranging between $-\infty$ to 2.698

Decide not to decide Pr(M = 0 | X = x) > 0.9 and Pr(M = 1 | X = x)> 0.9 Range will be the intersection of previous both ranges i.e. (-1.6972,2.6972)

b) The probability that the receiver erases a symbol is given by :

$$\begin{aligned}
Pr(erased) =& Pr(erased|M=0)Pr(M=0) + Pr(erased|M=1)Pr(M=1)\\
=& \frac{Pr(erased|M=0) + Pr(erased|M=1)}{2}\\
=& \frac{Pr(-1.6972 < x < 2.6972|M=0) + Pr(-1.6972 < x < 2.6972|M=1)}{2}\\
=& \frac{[Q(-1.6972) - Q(2.6972)] + [Q(-1.6972-1) - Q(2.6972-1)]}{2}\\
=& \frac{2 - 2Q(1.6972) - 2Q(2.6972)}{2}\\
=& 1 - Q(1.6972) - Q(2.6972)\\
\approx& 0.95197
\end{aligned}$$

* *using properties of Q-function* $= Q\left(\frac{x-\mu}{\sigma}\right)$ *and Q-table*

c) The probability that the receiver makes an error is given by;

$$\begin{aligned}
Pr(error) =& Pr(error|M=0)Pr(M=0) + Pr(error|M=1)Pr(M=1)\\
=& \frac{Pr(error|M=0) + Pr(error|M=1)}{2}\\
=& \frac{Pr(x \geq 2.6972|M=0) + Pr(x \leq -1.6972|M=1)}{2}\\
=& \frac{[Q(2.6972)] + [1 - Q(-1.6972-1)]}{2}\\
=& \frac{2Q(2.6972)}{2}\\
=& Q(2.6972)\\
\approx& 0.003467
\end{aligned}$$

3. PDF of a Rayleigh RV is given as :

$$f(r,\sigma) = \frac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}$$

R1=3/2=1.5 ft
R2=0.5/2=0.25 ft
$\sigma^2 = 4$
a)Probability of Mr Hood hitting the target: Pr(R $\leq$ R1) $= \int_0^{R1} f_R(r)dr$

$$= \int_0^{R1} \frac{r}{\sigma^2}e^{-\frac{r^2}{2\sigma^2}}$$

Replacing r^2 by x and 2rdr by dx, we get :

$$\int_0^{2.25} \frac{1}{2\sigma^2}e^{-\frac{x}{2\sigma^2}}dx$$

$$Pr(R \leq R1) = 1 - e^{-\frac{R1}{2\sigma^2}}$$

$$= 1 - e^{-\frac{R1^2}{8}}$$

Limits will change from 1.5 to 2.25 while substitution

$$= 0.24$$

b)Probability of hitting bulls eye :
Pr(R $\leq$ R2) = $\int_0^{R2} f_R(r)dr$

$$= \int_0^{R2} \frac{r}{\sigma^2} e^{-\frac{r^2}{2\sigma^2}}$$

Replacing r^2 by x, we get :

$$\int_0^{0.0625} \frac{1}{2\sigma^2} e^{-\frac{x}{2\sigma^2}} dx$$

$$Pr(R \leq R2) = 1 - e^{-\frac{R2^2}{2\sigma^2}}$$

$$= 1 - e^{-\frac{R2^2}{8}}$$

Limits will change from 0.25 to 0.0625

$$= 0.007$$

c)Probability that Mr Hood hit bulls eye given he hit the target :

$$= \frac{\text{Probability of hitting bulls eye} \cap \text{Probability of hitting target}}{\text{Probability of hitting target}}$$

Pr(Hitting bulls-eye) $\subset$ Pr(hitting target)

$$= \frac{\text{Probability of hitting bulls eye}}{\text{Probability of hitting target}}$$

$$= \frac{0.007}{0.245} = 0.0317$$

4. PDF of Gaussian random variable :

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{(-\frac{(x-m)^2}{2\sigma^2})}$$

Odd central moments are given y $E[(X-\mu_X)^k]$ where k= 2x-1 and x $\in$ N and $\mu_x = E[X]$

$$\therefore E[(X-\mu_X)^k] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} (x-\mu)^k e^{(-\frac{(x-m)^2}{2\sigma^2})} dx$$

Substituting , $\frac{x-\mu}{\sigma} = t$, we get ,

$$E[(X-\mu_X)^k] = \frac{\sigma^k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t^k e^{\frac{-t^2}{2}} dt$$

When k is odd, the function becomes odd over $-\infty$ to ∞ and total area becomes zero. When k is even,

$$I_k = \int_{-\infty}^{\infty} t^k e^{-\frac{t^2}{2}} dt$$

Integrating by parts, x = t^{k-1} and dy = $te^{-\frac{t^2}{2}} dt$

$$I_k = (k-1) \int_{-\infty}^{\infty} t^{k-2} e^{-\frac{t^2}{2}}$$

$$= (k-1) I_{k-2}$$

Following the above steps, it is observed that ,

$$I_k = (k-1)(k-3)(k-5)...3.1.I_0$$

$$I_0 = \sqrt{2\pi}$$

For k $\geq$ 2,

$$E[(X-\mu_X)^k] = \sigma^k \frac{I_k}{I_0}$$

$$E[(X-\mu_X)^k] = \frac{\sigma^2 k!}{(k/2)! 2^{k/2}}$$

5. Given PDF a gaussian random variable :

$$f_X(x) = ce^{-(2x^2+3x+1)}$$

To bring it into gaussian form, we will convert the quadratic equation into whole square form:

$2x^2 + 3x + 1$

$= 2(x^2 + \frac{3}{2}x + \frac{1}{2})$

$= 2(x^2 + 2.\frac{3}{4}x + \frac{1}{2})$

$= 2(x^2 + 2.\frac{3}{4}x + \frac{9}{16} - \frac{9}{16} + \frac{1}{2})$

$= 2((x + \frac{3}{4})^2 - (\frac{1}{4})^2$

Our PDF, after multiplying and dividing 2 in the numerator and denominator,will now look like, :

$$ce^{\frac{-4((x+\frac{3}{4})^2-(\frac{1}{4})^2)}{2}}$$
$$= c.e^{\frac{1}{8}}e^{-\frac{(x+\frac{3}{4})^2}{2.\frac{1}{4}}}$$

From the equation, $\mu = -\frac{3}{4}$ and $\sigma = \frac{1}{2}$
For the coefficient :

$$\int_{-\infty}^{\infty} f_X(x) = 1$$
$$= c.e^{\frac{1}{8}} \int_{-\infty}^{\infty} e^{-2(x+\frac{3}{4})^2} dx = 1$$
$$= c.e^{\frac{1}{8}} \sqrt{\frac{\pi}{2}} = 1$$
$$\therefore c = \sqrt{\frac{2}{\pi}} e^{-\frac{1}{8}}$$

6. For a Gaussian Random Variable,

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$$

$c_s = E[(\frac{x-\mu}{\sigma})^3]$ and $c_k = E[(\frac{x-\mu}{\sigma})^4]$

let, $\frac{x-\mu}{\sigma} = y$

$$E[(Y)^3] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^3 exp(-\frac{y^2}{2\sigma^2})dy$$

Evaluates to Zero as the given integral is odd function. Hence,

$$c_s = 0$$

$$E[(Y)^4] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^4 exp(-\frac{y^2}{2\sigma^2})dy$$

$$E[(Y)^4] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^3 \ y \ exp(-\frac{y^2}{2\sigma^2})dy$$

Evaluating the Integral using by parts we get

$$E[(Y)^4] = \frac{3}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y^2 exp(-\frac{y^2}{2\sigma^2})dy$$

$$= 3\sigma^2$$

7. Given

$$U = Xcos\theta - Ysin\theta$$
$$V = Xsin\theta + Ycos\theta$$

Here, X and Y are independaent, $\mu = 0$, $\sigma = 1$, Gaussian Random Variable Joint PDF of U and V is :

$$f_{U,V}(u, v) = \frac{f_{X,Y}}{\left| J \begin{pmatrix} u & v \\ x & y \end{pmatrix} \right|}$$

In the above equation ,

$$\left| J \begin{pmatrix} u & v \\ x & y \end{pmatrix} \right| \text{is the Jacobian matrix}$$

Now we need to find a function for X and Y in terms of U and V

Multiplying equation 1 with $cos\theta$ on both sides,

$$Ucos\theta = Xcos^2\theta - Ysin\theta cos\theta ...(3)$$

Multiplying $sin\theta$ on both sides of equation 2,

$$Vsin\theta = Xsin^2\theta + Ysin\theta cos\theta ...(4)$$

Adding eq 3 and 4,

$$X = Ucos\theta + Vsin\theta$$

Subtracting eq 3 and 4,

$$Y = Vcos\theta - Usin\theta$$

$$\left| J \begin{pmatrix} u & v \\ x & y \end{pmatrix} \right| = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{vmatrix}$$
$$= sin^2\theta + cos^2\theta = 1$$

Hence, the joint PDF of U and V is equal to joint PDF of X and Y.

8. Given pdf of the random variable :

$$f_X(x) = ce^{-2x}u(x)$$

Applying the property of PDF :

$$\int_{-\infty}^{\infty} f_X(x)dx = 1$$

$$c\int_{0}^{\infty} e^{-2x}dx = 1$$

$$-\frac{c}{2}[e^{-2x}]_0^{\infty} = 1$$

$$-\frac{c}{2}[-1] = 1$$

$$\therefore c = 2$$

b)

$$Pr(X > 2) = \int_{2}^{\infty} f_X(x)dx$$

$$\int_{2}^{\infty} 2e^{-2x}dx$$

$$= e^{-4}$$

c)

$$Pr(X < 3) = \int_{0}^{3} 2e^{-2x}dx$$

$$= -[e^{-2x}]_0^3$$

$$= 1 - e^{-6}$$

d)

$$Pr(X < 3|X > 2) = \frac{Pr(2 < X < 3)}{Pr(X > 2)}$$

$$Pr(2 < X < 3) = e^{-4} - e^{-6}$$

$$\frac{e^{-4} - e^{-6}}{e^{-4}}$$

$$= 1 - e^{-2}$$

9. We have a Random variable X which has a uniform distribution over the interval (-a,a) for some positive constant a. Its PDF will be,

$$\begin{aligned} f_X(x) &= \frac{1}{b-a} \\ &= \frac{1}{a-(-a)} \\ &= \frac{1}{2a} \end{aligned}$$

Now,

$$\begin{aligned} E[X] &= \int_{-a}^{a} x\left(\frac{1}{2a}\right) dx \\ &= \frac{1}{2a}[0] \\ &= 0 \end{aligned}$$

To find the variance we will calculate $E[X^2]$,

$$\begin{aligned} E[X^2] &= \int_{-a}^{a} x^2\left(\frac{1}{2a}\right) dx \\ &= \frac{1}{6a}x^3\Big|_{-a}^{a} \\ \sigma^2 &= \frac{1}{3}a^2 \end{aligned}$$

For coefficient of skewness

$$\begin{aligned} E[X^3] &= \int_{-a}^{a} x^3\left(\frac{1}{2a}\right) dx \\ &= 0 \end{aligned}$$

$$\begin{aligned} c_s &= \frac{E[X^3]}{\sigma^3} \\ &= 0 \end{aligned}$$

Now for coefficient of kurtosis

$$\begin{aligned} E[X^4] &= \int_{-a}^{a} \frac{x^4}{2a} dx \\ &= \frac{1}{10a}x^5\Big|_{-a}^{a} \\ &= \frac{1}{5}a^4 \end{aligned}$$

$$\begin{aligned} c_s &= \frac{E[X^4]}{\sigma^4} \\ &= \frac{\frac{1}{5}a^4}{\left(\frac{1}{3}a^2\right)^2} \\ &= \frac{9}{5} \end{aligned}$$

10. X is a continuous random variable uniform on $[1, 10]$. Hence we know the PDF and CDF of X.

The PDF of X:

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} = \dfrac{1}{9} & 1 \leq x \leq 10 \\ \\ 0 & otherwise \end{cases}$$

The CDF of X:

$$F_X(x) = \begin{cases} 0 & x \leq 1 \\ \\ \dfrac{x-a}{b-a} = \dfrac{x-1}{9} & 1 \leq x \leq 10 \\ \\ 1 & x \geq 10 \end{cases}$$

To find the PDF of $\sqrt{X}$, we will find its CDF and differentiate it.
Let, $Y = \sqrt{X}$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(\sqrt{X} \leq y) \\ &= P(X \leq y^2) \\ &= F_X(y^2) \end{aligned}$$

Differentiating both side,

$$\begin{aligned} f_Y(y) &= 2.y.f_X(y^2) \\ &= \frac{2y}{9} \end{aligned}$$

The PDF of Y is

$$f_Y(y) = \begin{cases} \dfrac{2y}{9} & 1 \leq x \leq \sqrt{10} \\ \\ 0 & otherwise \end{cases}$$

To find the PDF of $\text{-}\ln X$, we will find its CDF and differentiate it.
Let, $Y = \text{-}\ln X$

$$\begin{aligned} F_Y(y) &= P(Y \leq y) \\ &= P(\text{-}\ln X \leq y) \\ &= P(X \geq e^{\text{-}y}) \\ &= 1 - P(X \leq e^{\text{-}y}) \\ &= 1 - F_X(e^{\text{-}y}) \end{aligned}$$

Differentiating both side,

$$\begin{aligned} f_Y(y) &= e^{\text{-}y} f_X(e^{\text{-}y}) \\ &= \frac{e^{\text{-}y}}{9} \end{aligned}$$

The PDF of Y is

$$f_Y(y) = \begin{cases} \dfrac{e^{-y}}{9} & -\ln 10 \leq x \leq 0 \\ \\ 0 & otherwise \end{cases}$$