Compsci 240, S2021: HW2 Solutions

[10 points] Problem 1**.** An ant starts from the origin of a 2-D grid, at the point (0,0). Then each time it moves either i) right by one unit distance, e.g., from (0,0) to (1,0), ii) up by one unit distance, e.g., from (0,0) to (0,1), or iii) stay still, with probabilities 0.3, 0.2, and 0.5, respectively. After 10 moves, what is the probability that the ant ends up at point (2,5)?
Solution: The number of combinations is calculated as using the partition formula as $\frac{n}{n_1!n_2!n_3!} = \frac{10!}{2!*5!*3!}$, where n_1 denotes the number of times the ant moves right, n_2 denotes the number of times the ant moves up, and n_3 denotes the number of times the ant stays still. The probability of each exact move is $p_1^{n_1}p_2^{n_2}p_3^{n_3} = 0.3^2 * 0.2^5 * 0.5^3$, where p_1 denotes the probability the ant moves right, p_2 denotes the probability the ant moves up, and p_3 denotes the probability the ant stays still. So the final probability is $\frac{10!}{2!*5!*3!} * 0.3^2 * 0.2^5 * 0.5^3 = 0.009072$.

[3 + 5 + 2 = 10 points] Problem 2**.**

(a) In how many ways, can you arrange the letters in the word “massachusetts”?
(b) Let’s introduce an additional constraint: a vowel letter has to be the first letter in your arrangement. How many ways can you arrange these letters now?
(c) In this word, letters “a” and “t” appears twice, and the letter “s” appears four times. How many ways can you arrange these letters so that the occurrences of these letters are symmetric w.r.t. the middle position (7th letter among the 13 total)? For example, “mcsstauatsshe” and “thusasesasmct” are arrangements that fit this requirement.

Solution:

(a) Let’s look at the number of occurrences of each letter. $m:1$, $s:4$, $a:2$, $c:1$, $h:1$, $u:1$, $e:1$, $t:2$. There are a total of 13 occurrences. So this means I have 13 position indices; I need to put them into buckets of these sizes. Therefore, this is a partition problem, so the total number of ways is
$$\frac{13!}{4!2!2!1!1!1!1!1!} = 64864800.$$
(b) Now we need to look at cases for each possibility of the first letter, and then sum over all cases (first is A, first is U, and first is E). The total number of ways is
$$\frac{12!}{4!2!1!1!1!1!1!1!} + \frac{12!}{4!2!2!1!1!1!1!} + \frac{12!}{4!2!2!1!1!1!1!} = 9979200 + 2 \cdot 4989600 = 19958400.$$
(c) There are many correct lines of reasoning for this part, here’s one. Let’s look at how you can construct such a word. Look at the first 6 positions: we know they are made of 1 “a”, 1 “t”, 2 “s”s, and 2 other letters that occur only once. So the number of ways you pick a position for “a” among these 6 is 6. Let it be position k. Then, the corresponding symmetric position in the last 6 positions, i.e., $14-k$, will also be “a”. Then we pick out a position for “t”, there are 5 such possibilities. Now, for the remaining 4 positions, we need to pick two for “s”; the number of ways you can do that is $\binom{4}{2} = 6$. The number of remaining positions is now 5 (2 from the first 6, 2 from the last 6, and the middle one). The number of ways you can assign the single-occurrence letters, “m,c,h,u,e” to these 5 positions is 5!. Therefore, the total number of such arrangements is given by $6 \cdot 5 \cdot 6 \cdot 5! = 21600$.

[4 + 6 = 10 points] Problem 3**.** Let X be a discrete random variable, and let $Y = |X|$.

(a) Assume that the PMF of X is

$$p_X(x) = \begin{cases} Kx^2 & \text{if } x = -3, -2, -1, 0, 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

where K is a suitable constant. Determine the value of K.

(b) For the PMF of X given in part (a) calculate the PMF of Y, i.e., write down what $p_Y(y)$ is.

Solution:

(a) We must have $\sum_{x=-3}^{3} p_X(x) = 1$, so $\sum_{x=-3}^{3} Kx^2 = 1$, so

$$K = \frac{1}{\sum_{x=-3}^{3} x^2} = \frac{1}{28}.$$

(b) Using the formula $p_Y(y) = \sum_{\{x:|x|=y\}} p_X(x)$, we obtain

$$p_Y(y) = \begin{cases} 2Ky^2 & \text{if } y = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{y^2}{14} & \text{if } y = 1, 2, 3 \\ 0 & \text{otherwise.} \end{cases}$$

Note that we have used the notation $x : |x| = y$ to make it look cleaner.

[10 points] Problem 4**.** Let X be a discrete random variable that is uniformly distributed over the set of integers in the range $[a, b]$, where a and b are integers with $a < 0 < b$. Find the PMF of the random variables $\max(0, X)$ and $\min(0, X)$.

Solution: Let $Y = \max(0, X)$. By using the formula

$$p_Y(y) = \sum_{\{x|\max(0,x)=y\}} p_X(x)$$

we have

$$p_Y(y) = \begin{cases} 0 & \text{if } y < 0 \text{ or } b < y \\ \frac{1-a}{b-a+1} & \text{if } y = 0 \\ \frac{1}{b-a+1} & \text{if } 0 < y \leq b \end{cases}$$

Let $Y = \min(0, X)$. Similarly, we have

$$p_Y(y) = \begin{cases} 0 & \text{if } 0 < y \text{ or } y < a \\ \frac{1+b}{b-a+1} & \text{if } y = 0 \\ \frac{1}{b-a+1} & \text{if } a \leq y < 0 \end{cases}$$

[10 points] Problem 5**.** You are taking a multiple choice test with 10 questions, with 4 different options per question. For each question, there's only one correct answer. You haven't studied for the test and you decide to choose the answers at random (each option equally likely). What is the probability that you get at least 7 of them right?

Solution: Each correct has probability 1/4, and each incorrect has probability 3/4. The probability of getting at least 7 of them correct is equal to the probability that they get at most 3 incorrect. So total probability is calculated as the probability of getting zero incorrect plus the probability of getting one incorrect plus the probability of getting two incorrect plus the probability of getting three incorrect. Therefore, using the PMF of binomial random variables with $p = 3/4$, we have

$$\binom{10}{0}*(1/4)^10*(3/4)^0+\binom{10}{1}*(1/4)^9*(3/4)^1+\binom{10}{2}*(1/4)^8*(3/4)^2+\binom{10}{3}*(1/4)^7*(3/4)^3 = 0.0035.$$