Actuarial Statistics

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$$P[x < X < x+n] = S(x) - S(x+n)$$

Hence we get,

$$_nd_x = E[_n\mathscr{D}_x] = l_0[S(x) - S(x+n)]$$
$$nd_x = l_x - l_{x+n}$$

when n=1,we omit the prefixes on $_n\mathscr{D}_x$ and $_nd_x$
$_1\mathscr{D}_x$ is equivalent to $\mathscr{D}_x$ and $_1d_x$ is equivalent to d_x

0.1 Relation between u_x and l_x

we have,

$$l_x = l_0S(x)$$
$$log(l_x) = log(l_0) + log(S(x))$$

note that as S(x) is differentiable function of x and hence l_x is also differentiable function of x

$$\frac{-d}{dx}log(l_x) = \frac{-d}{dx}log(S(x)$$
$$\frac{-1}{l_x}\frac{d}{dx}l_x = \frac{-1}{S(x)}\frac{d}{dx}S(x) = \frac{-S'(x)}{S(x)} = u_x$$

$$\frac{-d}{dx}l_x = u_xl_x$$
$$-dl_x = l_xu_xd_x$$

the factor l_xu_x can be interpreted as the expected density of deaths in the age interval (x,x+d_x)

$$l_x = l_0S(x) = l_0e^{-\int_0^x u_ydy}$$
$$l_{x+n} = l_{xn}p_x$$
$$l_{x+n} = l_xe^{-\int_x^{x+n} u_ydy}$$

Also as,

$$\frac{-d}{dy}l_y = u_yl_y$$
$$\int_x^{x+n}\left(\frac{-d}{dy}l_y\right)dy = \int_x^{x+n} l_yu_ydy$$
$$l_x - l_{x+n} = \int_x^{x+n} l_yu_ydy$$
$$_nd_x = l_x - l_{x+n} = \int_x^{x+n} l_yu_ydy$$

for convenience of reference,we call this concept of l_0 newborns,each with survival function S(x) a random survivorship group
In the following we have an example of life table.

Table 1: Life Table for the total population:US(1979-81)

Age Interval	${}_tq_x$	l_x	${}_td_x$	${}_t\mathscr{L}_x$	T_x	E(T(x))
days						
0-1	0.00463	100,000	463	273	7387758	73.88
1-7	0.00246	99537	245	1635	7387485	74.22
7-28	0.00139	99292	138	5708	7385850	74.38
28-365	0.00418	99154	414	91357	7380142	74.43
years						
0-1	0.01260	100000	1260	98973	7387758	73.88
1-2	0.00093	98740	92	98694	7288785	73.82
2-3	0.00065	98648	64	98617	7190091	72.89
3-4	0.00050	98584	49	98560	7091474	71.93
: :	: :	: :	: :	: :	: :	: :
108-109	0.35453	51	18	42	115	2.24
109-110	0.35988	33	12	27	73	2.2

Table 2: Following table provides values of q_x from some National Life tables

x (years)	Australian 2000-02		English 1990-92		US 2002-04	
	Male	Female	Male	Female	Male	Female
0	0.00567	0.00466	0.00814	0.00632	0.00764	0.00627
1	0.00044	0.00043	0.00062	0.00055	0.00053	0.00042
2	0.00031	0.00019	0.00038	0.00030	0.00037	0.00028
10	0.00013	0.00008	0.00018	0.00013	0.00018	0.00013
20	0.00096	0.00036	0.00084	0.00031	0.00139	0.00045
40	0.00159	0.00088	0.00172	0.00107	0.00266	0.00149
90	0.15934	0.12579	0.20465	0.15550	0.16805	0.13328
100	0.24479	0.23863	0.38705	0.32489	-	-

The Actuarial present value for this insurance is,

$$(DA)|_{x:\overline{n}|} = \sum_{k=0}^{n-1}(n-k)\,v^{k+1}\,{}_kp_x q_{x+k}$$

consider,

$$(DA)|_{x:\overline{n}|} = \sum_{k=0}^{n-1}(n-k)\,v^{k+1}\,{}_kp_x q_{x+k}$$

$$= \sum_{k=0}^{n-1}(n-k)\,{}_{k|}A_{|x:\overline{1}|}$$

as by defination of actuarial present value of a unit benefit in an m-year deferred n year term insurance

$$_{m|}A^{1}_{x:\overline{n}|} = \sum_{k=m}^{m+n-1} v^{k+1}\, {}_{k}p_x q_{x+k}$$

with m=k and n=1 , we get $_{k|}A^{1}_{x:\overline{1}|} = v^{k+1}\, {}_{k}p_x q_{x+k}$ so we have,

$$(DA)^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} (n-k)\, {}_{k|}A^{1}_{x:\overline{1}|} = \sum_{k=0}^{n-1} (n-1)\, {}_{k|}A_x$$

Also in above expression if we substitute,

$$n-k = \sum_{j=0}^{n-k-1} (1)$$

we get,

$$(DA)^{1}_{x:\overline{n}|} = \sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} (1)\, v^{k+1}\, {}_{k}p_x q_{x+k}$$

By interchanging the order of summation we obtain ,

$$\sum_{k=0}^{n-1} \sum_{j=0}^{n-k-1} (1)\, v^{k+1}\, {}_{k}p_x q_{x+k} = \sum_{j=0}^{n-1} A^{1}_{x:\overline{n-j}|} = A^{1}_{x:\overline{n-j}|}$$

This is Actuarial present value of $(n-j)$ year term insurance when benefit of 1 unit is payable at the end of death year.
Hence we obtain,one more expression for $(DA)^{1}_{x:\overline{n}|}$ as,

$$(DA)^{1}_{x:\overline{n}|} = \sum_{j=0}^{n-1} A^{1}_{x:\overline{n-j}|}$$

In the following we give the summary of different insurances under which benefit amount is payable at the end of year of death.we also provide the summary of the recursion relations for their actuarial present values(NSPs)

Table 3: Summary of insurances payable at the end of year of death

(1)	(2)	(3)	(4)	(5)
Insurance Name	benefit function b_{k+1}	Discount function v_{k+1}	present value function z_{k+1}	APV
whole life	1	v^{k+1}	v^{k+1}	A_x (*)
n year term	1 k=0,1,...,n-1 0 k=n,n+1,...	v^{k+1}	v^{k+1} k=0,1,...,n-1 0 k=n,n+1,...	$A_{\|x:\overline{n}\|}$ (*)
n year endowment	1	v^{k+1} k=0,1,...,n-1 v^n k=n,n+1,...	v^{k+1} k=0,1,...,n-1 v^n k=n,n+1,...	$A_{x:\overline{n}\|}$ (*)
m year deferred n year term	1 k=m...m+n-1 0 k=0,...m-1 k=m+n,m+n+1,...	v^{k+1}	v^{k+1} k=m...m+n-1 0 k=0,...m-1 k=m+n,...	${}_{m\|n}A_x$ (*)
n year term increasing annuity	k+1 k=0,1,...n-1 0 k=n,n+1,...	v^{k+1}	(k+1) v^{k+1} k=0,1,...n-1 0 k=n,n+1,...	$(IA)\|_{x:\overline{n}\|}$
n year term decreasing annuity	n-k k=0,1,...n-1 0 k=n,n+1,...	v^{k+1}	(n-k) v^{k+1} k=0,1,...n-1 0 k=n,n+1,..	$(DA)\|_{x:\overline{n}\|}$
whole life increasing annuity	k+1 k=0,1,...	v^{k+1}	(k+1) v^{k+1}, k=0,1,...	$(IA)_x$

(*):Rule of moments holds,thus, v(Z)=2A-A^2