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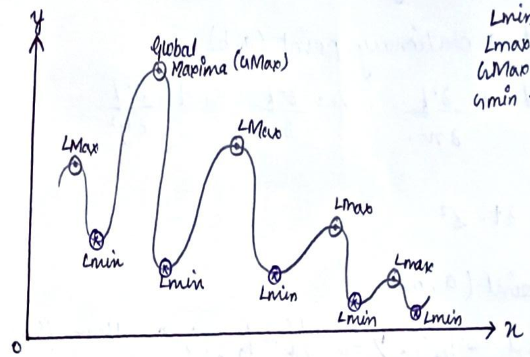
Maxima & Minima



Lmin - Local Minimum
Lmax - Local Maximum
GMax - Global Maximum
Gmin - Global minimum ($-\infty$)

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Maxima:-

* A function $f(x,y)$ is said to have a maximum value at point (a,b) if $f(a,b) > f(a+h, b+k)$ where h & k are small and independent may be +ve or -ve.

Minima:-

* A function $f(x,y)$ is said to have a minimum value at point (a,b) if $f(a,b) < f(a+h, b+k)$ for small & independent value of h & k may be +ve or -ve.

→ A maximum or a minimum value of a function is called its extreme value.

→ **Stationary Point:-**

A point (a,b) is said to be a stationary point by the function $f(x,y)$ if $f(x,y)$ either max or min at that point.

→ **Saddle Point:-**

A point (a,b) is said to be a saddle point of the function $f(x,y)$ is neither max nor min at that point.

→ **Working Rule:-**

For the function $f(x,y)$ is max or min at point (a,b). There are following steps:-

Step 1:- Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

Step 2:- For max or min

$$\frac{\partial f}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} = 0.$$

we find a stationary point (a,b)

Step 3:- Find $r = \frac{\partial^2 f}{\partial x^2}$, $s = \frac{\partial^2 f}{\partial x \partial y}$, $t = \frac{\partial^2 f}{\partial y^2}$

Step 4:- Find $rt - s^2$

Step 5:- At point (a,b)

(i) If $\boxed{[rt-s^2]_{(a,b)} > 0}$, $\boxed{[r]_{(a,b)} > 0}$ then $f(x,y)$ has a minimum value at point (a,b).

(ii) If $\boxed{[rt-s^2]_{(a,b)} > 0}$ $\boxed{[r]_{(a,b)} < 0}$ then $f(x,y)$ has a maximum value at point (a,b).

(iii) If $\boxed{[rt-s^2]_{(a,b)} < 0}$ then the point (a,b) is saddle point at which $f(x,y)$ is neither max nor min.

(iv) If $\boxed{[rt-s^2]_{(a,b)} = 0}$ then further Investigation.

Que Examine for extreme value of $x^2 + y^2 + 6x + 12$.

Soln Consider $f(x,y) = x^2 + y^2 + 6x + 12$. —(1)

$$\frac{\partial f}{\partial x} = 2x + 6 \quad \Big| \quad \frac{\partial f}{\partial y} = 2y$$

for max or min

$$\frac{\partial f}{\partial x} = 0 \quad \Big| \quad \frac{\partial f}{\partial y} = 0$$

$2x + 6 = 0$ | $2y = 0$

$2x = -6$ | $\boxed{y = 0}$

$\boxed{x = -3}$

$\therefore$ $\boxed{\text{Stationary Point } (-3,0)}$

Now, $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}[2x+6] = 2$

$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(2y) = 0$

$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(2y) = 2$

Now, $rt - s^2 = 2 \times 2 - 0^2 = 4$

which is greater than 0

$\therefore$ $\boxed{rt - s^2 = 4}$ & $\boxed{r = 2}$

At point $(-3,0)$

$\therefore$ $[rt - s^2]_{(-3,0)} = 4$ & $[r]_{(-3,0)} = 2$

Hence, $\boxed{rt - s^2 > 0}$ & $\boxed{r > 0}$

$\therefore$ $f(x,y)$ is minimum at point $(-3,0)$

$f_{min} = [x^2 + y^2 + 6x + 12]_{(-3,0)}$

$\Rightarrow 9 - 18 + 12$

$\Rightarrow 3$ Ans

$\boxed{f_{min} = 3}$

Que Test $f(x,y) = x^3 y^2 (6 - x - y)$ for maxima & minima for point not at origin. $f(x,y) = x^3y^2(6-x-y)$

Soln Given $f(x,y) = 6x^3y^2 - x^4y^2 - x^3y^3$ —(1)

$\frac{\partial f}{\partial x} = 18y^2x^2 - 4x^3y^2 - 3x^2y^3$ | $\frac{\partial f}{\partial y} = 12x^3y - 2x^4y - 3y^2x^3$

$\Rightarrow x^2y^2(18 - 4x - 3y)$ —(2) | $= x^3y(12 - 2x - 3y)$ —(3)

for max & min

$$\frac{\partial f}{\partial x} = 0$$

$(x^2y^2)(18-4x-3y)=0$

$x=0$

$y=0$

$18-4x-3y=0 \quad \text{—④}$

$$\frac{\partial f}{\partial y} = 0$$

$x^3y(12-2x-3y)=0$

$x=0$

$y=0$

$12-2x-3y=0 \quad \text{—⑤}$

from —④ & —⑤

$$\begin{aligned} 18-4x-\cancel{3y} &= 0 \\ 12-2x-\cancel{3y} &= 0 \\ - \quad + \quad + & \\ \hline 6-2x &= 0 \end{aligned}$$

$6=2x$

$\boxed{x=3}$

put the value of x in —⑤

$12-2x-3y=0$

$12-6-3y=0$

$6=3y$

$\boxed{y=2}$

→ the stationary points are $(3,2)$

put $\boxed{x=0}$ in —④

$\boxed{y=6}$

$(0,6)$

put $\boxed{y=0}$ in —④

$x=\frac{18}{4}=\frac{9}{2}$

Now, $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$

$= \frac{\partial}{\partial x}\left(x^2y^2(18-4x-3y)\right)$

$x^2y^2(-4) + 2xy^2(18-4x-3y)$

→ Lagranges Method of Undermind Multipliers :-

→ Lagrange's Method does not enable us to find whether there is a maximum or minimum. This fact is determined from the physical consideration of the ~~prople~~ problem.

Let us consider there is a function $f(x,y,z)$ it is maximum or minimum by a ~~given~~ condition $\phi(x,y,z)$ by the Lagrange's Function,

$$F = f(x,y,z) + \lambda\phi(x,y,z) = 0$$

For max or min :-

$$dF = 0$$

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial\phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial\phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial\phi}{\partial z}\right)dz = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} + \lambda\frac{\partial\phi}{\partial x} = 0 \quad \text{—— (1)}$$

$$\Rightarrow \frac{\partial f}{\partial y} + \lambda\frac{\partial\phi}{\partial y} = 0 \quad \text{—— (2)}$$

$$\Rightarrow \frac{\partial f}{\partial z} + \lambda\frac{\partial\phi}{\partial z} = 0 \quad \text{—— (3)}$$

Que Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid (3D ellipse)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Solu Since the ~~ellipsoid~~ parallelopiped is inscribed in the ellipsoid.

Let $2x, 2y, 2z$ be the length, breadth & height of the parallelopiped, so,

Volume $= 2x\,2y\,2z$.

$$\boxed{V = 8xyz}$$

$V = 8xyz$ = (function) = $f(x,y,z)$ say.

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Given, ellipsoid,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\phi(x,y,z) \equiv \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c} - 1 = 0 \quad \text{—— (1)}$$

Now, the Lagrange's Function.

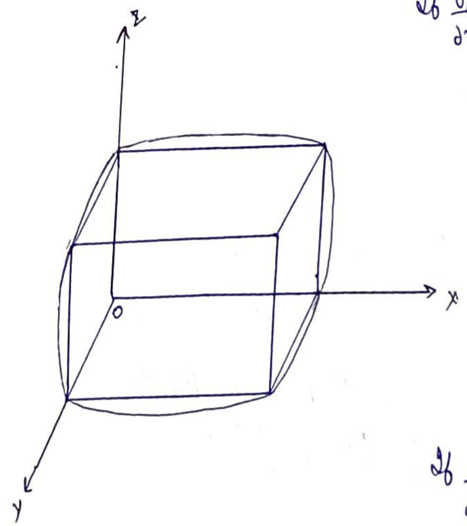
$$F = f + \lambda\phi$$

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c} - 1\right) \quad \text{—— (2)}$$

for max or min.

$$dF = 0$$

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial\phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial\phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial\phi}{\partial z}\right)dz = 0$$



If $\frac{\partial f}{\partial x} + \lambda\frac{\partial\phi}{\partial x} = 0$

$$= 8yz + \lambda\left(\frac{2x}{a^2}\right) = 0 \quad \text{—— (3)}$$

If $\frac{\partial f}{\partial y} + \lambda\frac{\partial\phi}{\partial y} = 0$

$$= 8zx + \lambda\left(\frac{2y}{b^2}\right) = 0 \quad \text{—— (4)}$$

If $\frac{\partial f}{\partial z} + \lambda\frac{\partial\phi}{\partial z} = 0$

$$= 8xy + \lambda\left(\frac{2z}{c^2}\right) = 0 \quad \text{—— (5)}$$

Now multiplying eq-③, ④ & ⑤ by x, y, z respectively then add.

we get,

$$24xyz + 2\lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right) = 0$$

$$= 24xyz + 2\lambda(1) = 0 \quad [\text{using eq.(2)}]$$

$$\Rightarrow 24xyz = -2\lambda$$

$$\boxed{\lambda = -12xyz}$$

From ③

$$8yz + \frac{2x}{a^2}(-12xyz) = 0$$

$$= 8yz\left[1 - \frac{3x^2}{a^2}\right] = 0$$

$$\Rightarrow 1 - \frac{3}{a^2}x^2 = 0$$

$$1 = \frac{3x^2}{a^2}$$

$$x^2 = \frac{a^2}{3}$$

$$\boxed{x = \pm\frac{a}{\sqrt{3}}}$$

Similary $\Rightarrow \boxed{y = \pm\frac{b}{\sqrt{3}}}, \boxed{z = \pm\frac{c}{\sqrt{3}}}$

Largest volume of rectangular parallelopiped.

$$V = 8xyz = 8\left(\pm\frac{a}{\sqrt{3}}\right)\left(\pm\frac{b}{\sqrt{3}}\right)\left(\pm\frac{c}{\sqrt{3}}\right).$$

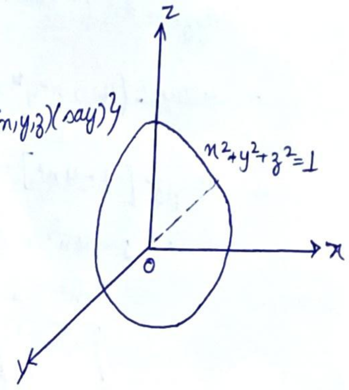
$$= \frac{8}{3\sqrt{3}}(abc) \text{ cubic unit}$$

Ans

Que The temperature 'T' at any point (x, y, z) in space is $T = 400xyz^2$. find the highest temperature at the surface of unit sphere, $x^2+y^2+z^2=1$

Soln Given temperature in space

$$T = 400xyz^2 \{\text{function } f(x,y,z) \text{ (say)}\} \quad \text{—①}$$



given unit sphere

$$x^2+y^2+z^2 = 1$$

$$\therefore \phi(x,y,z) \equiv x^2+y^2+z^2-1 = 0 \quad \text{—②}$$

Now, Lagrange's function.

$$F(xyz) = f(x,y,z) + \lambda\phi(x,y,z)$$

for max or min $dF = 0$

$$\left(\frac{\partial f}{\partial x} + \frac{\lambda\partial\phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \frac{\lambda\partial\phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \frac{\lambda\partial\phi}{\partial z}\right)\cancel{dz} = 0$$

$$\text{If } \frac{\partial f}{\partial x} + \frac{\lambda\partial\phi}{\partial x} = 0 \Rightarrow (400yz^2) + \lambda(2x) \quad \text{—③}$$

$$\text{If } \frac{\partial f}{\partial y} + \frac{\lambda\partial\phi}{\partial y} = 0 \Longrightarrow (400xz^2) + \lambda(2y) \quad \text{—④}$$

$$\text{If } \frac{\partial f}{\partial z} + \frac{\lambda\partial\phi}{\partial y} = 0 \Longrightarrow (800xyz) + \lambda(2z) \quad \text{—⑤}$$

multiply —③, —④, —⑤ by x, y, z respectively & add them we get.

$$16\cancel{4}00xyz + 2\lambda(x^2+y^2+z^2) = 0$$

$$\Rightarrow 1600xyz^2 + 2\lambda(1) = 0 \quad [\text{using eq —②}]$$

$$1600xyz^2 = -2\lambda$$

$$\boxed{\lambda = -800xyz^2}$$

from — 3

$$400yz^2 + 2x(-800xyz^2) = 0$$

$$= 400yz^2(1600x^2yz^2) = 0$$

$$= 400yz^2[1-4x^2] = 0$$

$$\therefore 1-4x^2 = 0$$

$$4x^2 = 1$$

$$\boxed{x = \pm\frac{1}{2}}$$

Similarly $\boxed{y = \pm\frac{1}{2}}$.

from — 5

$$800xyz + 2z(-800xyz^2) = 0$$

$$800xyz(1 - 2z^2) = 0$$

$$\therefore 1-2z^2 = 0$$

$$\boxed{z = \pm\frac{1}{\sqrt{2}}}$$

Maximum Temperature in surface of given unit sphere.

$$T = 400xyz^2$$

$$= 400\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right)^2.$$

$$\Rightarrow 50^\circ \text{ Ans}$$

Que Find the maximum & minimum distance of the point (3,4,12) from the sphere $x^2+y^2+z^2=1$

Soln Consider a pt $Q(x,y,z)$ whose distance is

$$QP = \sqrt{(x-3)^2+(y-4)^2+(z-12)^2}$$

P(3,4,12) Q(x,y,z)

$$(QP)^2 = (x-3)^2+(y-4)^2+(z-12)^2 \quad (1)$$

$$= \text{function } f(x,y,z) \text{ say.}$$

Given condition,

$$\therefore \phi(x,y,z) = x^2+y^2+z^2-1 = 0 \quad (2)$$

The Lagranges function,

$$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$$

$$dF = 0$$

$$\left(\frac{\partial f}{\partial x} + \frac{\lambda\partial\phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \frac{\lambda\partial\phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \frac{\lambda\partial\phi}{\partial z}\right)dz = 0$$

If $\frac{\partial f}{\partial x} + \frac{\lambda\partial\phi}{\partial x} = 0 \Rightarrow 2(x-3) + \lambda \cdot 2x = 0$ — (3)

If $\frac{\partial f}{\partial y} + \frac{\lambda\partial\phi}{\partial y} = 0 \Rightarrow 2(y-4) + \lambda 2y = 0$ — (4)

If $\frac{\partial f}{\partial z} + \frac{\lambda\partial\phi}{\partial z} = 0 \Rightarrow 2(z-12) + \lambda 2z = 0$ — (5)

from — 3

$$2(x-3) + 2x\lambda = 0$$

$$2x - 6 + 2\lambda x = 0$$

$$2x + 2\lambda x = 6$$

$$2x(1+\lambda) = 6$$

$$\boxed{x = \frac{3}{1+\lambda}}$$

from — 4

$$2(y-4) + \lambda 2y = 0$$

$$2y - 8 + 2y\lambda = 0$$

$$2y(1+\lambda) = 8$$

$$y(1+\lambda) = 4$$

$$\boxed{y = \frac{4}{1+\lambda}}$$

from — 5

$$2(z-12) + 2\lambda z = 0 \Rightarrow 2z + 2\lambda z = 24$$

$$z(1+\lambda) = 12 \Rightarrow \boxed{z = \frac{12}{1+\lambda}}.$$

Given cond^n $x^2+y^2+z^2=1$

$$\frac{9}{(1+\lambda)^2}+\frac{16}{(1+\lambda)^2}+\frac{144}{(1+\lambda)^2}=1$$

$$\frac{169}{(1+\lambda)^2}=1$$

$$(1+\lambda)^2=169$$

$$(1+\lambda)=\pm 13$$

$1+\lambda=-13$ $\boxed{\lambda=12}$ | $1+\lambda=-13$ $\boxed{\lambda=-14}$

When $\boxed{\lambda=12}$

$$x=\frac{3}{1+12}=\frac{3}{13}$$

$$y=\frac{4}{1+12}=\frac{4}{13}$$

$$z=\frac{12}{1+12}=\frac{12}{13}$$

$$(QP)=\sqrt{(x-3)^2+(y-4)^2+(z-12)^2}=\sqrt{\left(\frac{3}{13}-3\right)^2+\left(\frac{4}{13}-4\right)^2+\left(\frac{12}{13}-12\right)^2}$$

$$QP=\frac{1}{13}\sqrt{(-36)^2+(-48)^2+(\cancel{156})^2}$$

$$=\cancel{12}$$

when $\boxed{\lambda=-14}$

$$x=\frac{3}{-13},\quad y=\frac{4}{-13},\quad z=\frac{12}{-13}$$

$$QP=\sqrt{\left(\frac{-3}{13}-3\right)^2+\left(\frac{-4}{13}-4\right)^2+\left(\frac{-12}{13}-12\right)^2}$$

$$=$$

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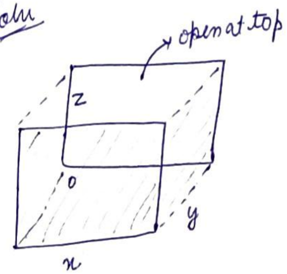
Que In A rectangular box which is open at the top has a capacity of 256 cubic feet determine the dimension of the box such that least material is required for the construction of the box. ~~Here~~. Use Lagranges method of multipliers to obtain the solutions.

Soln



→ let x, y, z be the length, breadth & height of the rectangular box

$\therefore$ given volume of box

$$V=xyz$$

$$256=xyz$$

$\therefore$ cond^n $\Rightarrow$ $\phi(x,y,z)\equiv xyz-256=0$ — (1)

Let S be the surface area of the box.

$$S=xy+2yz+2zx \text{ — (2)}$$

function $f(x,y,z)$ say. ($\because$ open at top)

Lagrange function :-

$$F(x,y,z)=f(x,y,z)+\lambda\phi(x,y,z)$$

for max or min

$$dF=0$$

$$\left(\frac{\partial f}{\partial x}+\frac{\lambda\partial\phi}{\partial x}\right)dx+\left(\frac{\partial f}{\partial y}+\frac{\lambda\partial\phi}{\partial y}\right)dy+\left(\frac{\partial f}{\partial z}+\frac{\lambda\partial\phi}{\partial z}\right)dz=0$$

if $\frac{\partial f}{\partial x}+\frac{\lambda\partial\phi}{\partial x}=0 \Rightarrow (y+2z)+\lambda(yz)=0$ — (3)

if $\frac{\partial f}{\partial y}+\frac{\lambda\partial\phi}{\partial y}=0 \Rightarrow (x+2z)+\lambda(xz)=0$ — (4)

if $\frac{\partial f}{\partial z}+\frac{\lambda\partial\phi}{\partial z}=0 \Rightarrow (2x+2y)+\lambda(xy)=0$ — (5)

eq — (3) mul by x — eq — (4) mul by y, we get

$$xy+2xz+\lambda xyz=0$$

$$\underline{xy+2yz+\lambda xyz=0}$$

$z \neq 0$

$\therefore x - y = 0$

$\boxed{x=y}$

Now; eq (4) mul by y − eq (5) mul by z, we get

$xy + 2\cancel{yz} + \lambda \cancel{xyz} = 0$

$2\cancel{yz} + 2xz + \lambda \cancel{xyz} = 0$

$xy - 2xz = 0$

$x(y-2z) = 0$

$x \neq 0 \quad \therefore y - 2z = 0$

$\boxed{y = 2z}$

Dimension of box for least material

$\Rightarrow \boxed{x = y = 2z}$

Que Find the dimension of rectangular box of maximum capacity whose surface area is given when (a) box is open at the top (b) box is closed.

Soln Let x, y, z be the length, breadth & height of rectangular box then there volume $\boxed{V = xyz}$ this work as a function:

$V = xyz$ [function $f(x,y,z)$ say] — (1)

(a) Box is opened at top :–

$S_1 = xy + 2yz + 2zx$ — (2) [condition] $= \phi(x,y,z)$ (say)

$\therefore$ Lagranges function.

$F(x,y,z) = V + \lambda S_1$

for max or min

$dF = 0$

$$\left(\frac{\partial V}{\partial x} + \frac{\lambda \partial S_1}{\partial x}\right)dx + \left(\frac{\partial V}{\partial y} + \frac{\lambda \partial S_1}{\partial y}\right)dy + \left(\frac{\partial V}{\partial z} + \frac{\lambda \partial S_1}{\partial z}\right)dz = 0$$

if $\frac{\partial V}{\partial x} + \frac{\lambda \partial S_1}{\partial x} = 0 \Rightarrow yz + \lambda(y + 2z) = 0$ — (3)

if $\frac{\partial V}{\partial y} + \frac{\lambda \partial S_1}{\partial y} = 0 \Rightarrow xz + \lambda(x + 2z) = 0$ — (4)

if $\frac{\partial V}{\partial z} + \frac{\lambda \partial S_1}{\partial z} = 0 \Rightarrow xy + \lambda(2y + 2x) = 0$ — (5)

Now eq (3) × x − eq (4) × y, we get.

$\cancel{xyz} + \lambda(xy + 2xz) = 0$

$\cancel{xyz} + \lambda(xy + 2yz) = 0$

$\lambda[\cancel{xy} + 2xz - \cancel{xy} - 2yz] = 0$

$\lambda[2z(x-y)] = 0 \quad \Big| \quad \lambda \neq 0, \; z \neq 0$

$\therefore x - y = 0$

$\boxed{x=y}$

Now eq (4) × y − eq (5) × z

$\cancel{xyz} + \lambda(xy + 2zy) = 0$

$\cancel{xyz} + \lambda(2yz + 2xz) = 0$

$\lambda[xy + \cancel{2zy} - \cancel{2yz} - 2xz] = 0$

$\lambda[x(y - 2z)] = 0$

$y - 2z = 0 \quad \Big| \quad \lambda \neq 0, \; x \neq 0$

$\therefore \boxed{y = 2z}$

Hence $\boxed{x = y = 2z}$

(b) when box is closed at top

Surface Area of box $\rightarrow S_2 = 2xy + 2yz + 2zx$ — (2)

(condition) $= \phi(x,y,z)$

$\therefore$ lagranges function

$F(x,y,z) = V + \lambda S_2$

For max or min

$dF = 0$

$$\left(\frac{\partial V}{\partial x} + \frac{\lambda \partial S_2}{\partial x}\right)dx + \left(\frac{\partial V}{\partial y} + \frac{\lambda \partial S_2}{\partial y}\right)dy + \left(\frac{\partial V}{\partial z} + \frac{\lambda \partial S_2}{\partial z}\right)dz = 0$$

If $\frac{\partial V}{\partial x} + \lambda \frac{\partial S_2}{\partial x} = 0 \Rightarrow yz + \lambda(2y+2z) = 0$ — (3)

If $\frac{\partial V}{\partial y} + \lambda \frac{\partial S_2}{\partial y} = 0 \Rightarrow xz + \lambda(2x+2z) = 0$ — (4)

If $\frac{\partial V}{\partial z} + \lambda \frac{\partial S_2}{\partial z} = 0 \Rightarrow xy + \lambda(2y+2x) = 0$ — (5)

Now eq (3) $\times x$ − eq (4) $\times y$, we get

$$\cancel{xyz} + \lambda(2xy + 2xz) = 0$$
$$\cancel{xyz} + \lambda(2xy + 2yz) = 0$$
$$\lambda[\cancel{2xy} + 2xz - \cancel{2xy} - 2yz] = 0$$
$$\lambda[2z(x-y)] = 0$$

$\therefore x - y = 0$ | $\lambda \neq 0$, $z \neq 0$

$\boxed{y = x}$

Now eq (4) $\times y$ − eq (5) $\times z$, we get

$$\cancel{xyz} + \lambda(2xy + 2yz) = 0$$
$$\cancel{xyz} + \lambda(2yz + 2xz) = 0$$
$$\lambda[2xy + \cancel{2yz} - \cancel{2yz} - 2zx] = 0$$
$$\lambda[2xy - 2xz] = 0$$
$$\lambda x(2y - 2z) = 0$$
$$\therefore 2y - 2z = 0$$

$\boxed{y = z}$

Dimension of closed box $\Rightarrow \boxed{x = y = z}$

Que Find the maximum value of $\cos A \cos B \cos C$ in $\triangle ABC$.

Soln Given function,

$f(A,B,C) = \cos A \cos B \cos C$ — (1)

$\because$ In $\triangle ABC$.

$A + B + C = \pi$

condition

$\phi(A,B,C) = A + B + C - \pi = 0$ — (2)

$dF = f + \lambda \phi$

for max or min

$dF = 0$

$$\left(\frac{\partial f}{\partial A} + \lambda \frac{\partial \phi}{\partial A}\right) dA + \left(\frac{\partial f}{\partial B} + \lambda \frac{\partial \phi}{\partial B}\right) dB + \left(\frac{\partial f}{\partial C} + \lambda \frac{\partial \phi}{\partial C}\right) dC = 0$$

If $\frac{\partial f}{\partial A} + \lambda \frac{\partial \phi}{\partial A} = 0 \Rightarrow -\sin A \cos B \cos C + \lambda = 0$ — (3)

If $\frac{\partial f}{\partial B} + \lambda \frac{\partial \phi}{\partial B} = 0 \Rightarrow -\cos A \sin B \cos C + \lambda = 0$ — (4)

If $\frac{\partial f}{\partial C} + \lambda \frac{\partial \phi}{\partial C} = 0 \Rightarrow -\cos A \cos B \sin C + \lambda = 0$ — (5)

from eq (3) & eq (4)

$$-\sin A \cos B \cos C + \cancel{\lambda} = -\cos A \sin B \cos C + \cancel{\lambda}$$
$$-\sin A \cos B \cos C + \cos A \sin B \cos C = 0$$
$$-\cos C[\sin A \cos B - \cos A \sin B] = 0$$
$$-\cos C(\sin(A-B)) = 0$$

If $\cos C = 0$, $C = \frac{\pi}{2}$ | If $\sin(A-B) = 0$, $A - B = 0$, $\boxed{A = B}$

Similarly from eq (4) & (5)

$A = \frac{\pi}{2}$ | $\boxed{B = C}$

There is no 2 angles will be 90°

$\therefore A = \frac{\pi}{2}$ & $C = \frac{\pi}{2}$ [not Accepted]

$\therefore A = B$ & $B = C$

$\Rightarrow A = B = C$

Maximum value of $\cos A \cos B \cos C$.

$= \cos 60° \cos 60° \cos 60° = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{1}{8}$ Ans.

Que Find minimum value of $\sin x \sin y \sin(x+y)$

Sol^n

$x + y + z = \pi$

$x + y = \pi - z$

$\sin x \sin y \sin(\pi - z)$

$\sin x \sin y \sin z$.

8/2/2023.

Que Divide 24 into 3 parts such that the continued product of 1st, square of 2nd & cube of third is maximum.

Soln Let x, y, z be the 3 parts of 24.

$$\boxed{x+y+z=24}$$

$\therefore$ Condⁿ is $\phi(x,y,z) = x+y+z-24=0$ — (1)

Acc. to given question,

$f(x,y,z) = x \cdot y^2 z^3$ — (2)

The Lagrange's function,

$$F = f + \lambda\phi$$

for max or min

$$\boxed{dF=0}$$

$$\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

If $\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow y^2 z^3 + \lambda \cdot 1 = 0$ — (3)

If $\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 2xyz^3 + \lambda \cdot 1 = 0$ — (4)

If $\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 3xy^2z^2 + \lambda \cdot 1 = 0$ — (5)

eq (3); (4) & (5), mul by x, y, z then adding.

$6xy^2z^3 + \lambda(x+y+z) = 0$

$6xy^2z^3 + \lambda 24 = 0$

$$\boxed{\lambda = -\frac{1}{4}xy^2z^3}$$

put value of λ in (3), (4) & (5)

from (3)

$$y^2z^3 - \frac{xy^2z^3}{4} = 0 \Rightarrow y^2z^3\left(1 - \frac{x}{4}\right) = 0$$

$$1 - \frac{x}{4} = 0 \Rightarrow 1 = \frac{x}{4} \quad \boxed{x=4}$$

from (4)

$$2xyz^3 - \frac{xy^2z^3}{4} = 0$$

$$xyz^3\left[2 - \frac{y}{4}\right] = 0$$

$$2 - \frac{y}{4} = 0$$

$$\frac{y}{4} = 2$$

$$\boxed{y=8}$$

Since $x+y+z=24$

$4+8+z=24$

$12+z=24$

$$\boxed{z=12}$$

$\therefore$ the three parts of 24 are

$$\boxed{x=4,\ y=8,\ z=12}$$

Que Find the minimum value of $x^2+y^2+z^2$ given that $ax+by+cz=p$

Soln Given function $f(x,y,z) = x^2+y^2+z^2$ — (1)

given condⁿ $\phi(x,y,z) = ax+by+cz-p=0$ — (2)

The Lagrange function

$F(x,y,z) = f(x,y,z) + \lambda\phi(x,y,z)$

for max or min

$dF=0$

$$\Rightarrow \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right)dx + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right)dy + \left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right)dz = 0$$

If $\left(\frac{\partial f}{\partial x} + \lambda\frac{\partial \phi}{\partial x}\right) = 0 \Rightarrow 2x + \lambda a = 0$ — (3)

If $\left(\frac{\partial f}{\partial y} + \lambda\frac{\partial \phi}{\partial y}\right) = 0 \Rightarrow 2y + \lambda b = 0$ — (4)

If $\left(\frac{\partial f}{\partial z} + \lambda\frac{\partial \phi}{\partial z}\right) = 0 \Rightarrow 2z + \lambda c = 0$ — (5)

eq (3), (4) & (5) mul by x, y, z & then add.

$2[x^2+y^2+z^2] + \lambda[ax+ \ldots]= \ldots$

$2[x^2+y^2+z^2] + \ldots$

$2f + \lambda p = 0$

$2f = -\lambda p$

$\boxed{\lambda = \frac{-2f}{p}}$

from ③

$$2x + \left(\frac{-2f}{p}\right) a = 0$$

$$\boxed{x = \frac{f}{p} a}$$

Similarly, from ④ $\boxed{y = \frac{f}{p} b}$

from ⑤ $\boxed{z = \frac{f}{p} c}$

Since $f = x^2 + y^2 + z^2$

$$f = \frac{f^2}{p^2} a^2 + \frac{f^2}{p^2} b^2 + \frac{f^2}{p^2} c^2$$

$$1 = \left(\frac{a^2}{p^2} + \frac{b^2}{p^2} + c^2\right) f$$

$$1 = \left(\frac{a^2 + b^2 + c^2}{p^2}\right) f$$

$$f = \frac{p^2}{a^2 + b^2 + c}$$

$$\boxed{f_{min}(x,y,z) = \frac{p^2}{a^2 + b^2 + c}}$$ Ans.

Que Find the extreme value of $x^3 + y^3 - 3axy$.

Soln Given $f(x,y) = x^3 + y^3 - 3axy$

$$\frac{\partial f}{\partial x} = 3x^2 - 3ay$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax$$

for max or min.

$$\frac{\partial f}{\partial x} = 0 \quad \& \quad \frac{\partial f}{\partial y} = 0$$

$3x^2 - 3ay = 0$ & $3y^2 - 3ax = 0$

$\boxed{x^2 - ay = 0}$ — ① $\boxed{y^2 - ax = 0}$ — ②

from ①

$\boxed{y = \frac{x^2}{a}}$ use it in — ②

$$\left(\frac{x^2}{a}\right)^2 - ax = 0$$

$$\frac{x^4}{a^2} - a^3 x = 0 \Rightarrow \frac{x(x^3 - a^3)}{a^2} = 0$$

$\Rightarrow x(x^3 - a^3) = 0$

$x(x-a)(x^2 + ax + a^2) = 0$

$\boxed{x = 0}$

$x - a = 0$

$\boxed{x = a}$

but, $x^2 + ax + a^2 = 0$

$\rightarrow x$ is imaginary value (not accepted)

$\therefore \boxed{x = 0 \ \& \ x = a}$

$\because \boxed{y = \frac{x^2}{a^2}}$

$x = 0$, $y = 0$

$x = a$, $y = a$.

$\boxed{\text{Stationary Pt} = (0,0),\ (a,a)}$

current → Tensor.

8/2/2023.

Unit-4

Vector Differentiation

Quantity

Scalar

$\phi(x,y,z)=C$

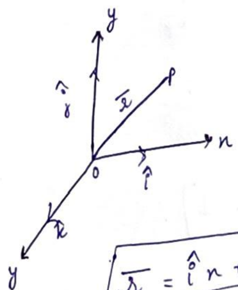
Eg- Time, ~~[illegible]~~, distance, Mass, temperature etc.

Vector.

$V = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$

Eg:- Velocity, workdone, momentum etc.

$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$



$\therefore \underbrace{|\vec{r}|}_{\text{vector}} = \underbrace{r}_{\text{scalar}}$

$$\boxed{\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z.}$$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$r = \sqrt{x^2+y^2+z^2}$

$$\boxed{r^2 = x^2+y^2+z}$$

15/2/2023

→ Scalar Point Function :-

Let 'R' be a region of space. At this point of which a scalar. $\phi = \phi(x,y,z)$ is given, than ϕ is called a scalar point function and R is called scalar field.

The temperature distribution in a medium, the distribution of atmospheric pressure in space are example of Scalar Point function.

Now, $r = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}(3x^2 - 3ay) = 6x$.

$$s = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}(3y^2 - 3ax) = -3a$$

$$t = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}(3y^2 - 3ax) = 6y.$$

$$rt - s^2 \Rightarrow 6x \times 6y - (-3a)^2$$

$$\boxed{36xy - 9a^2}$$

Now $\boxed{rt - s^2 = 36xy - 9a^2}$ & $\boxed{r = 6a}$

(i) At point (0,0) :-

$$[rt - s^2]_{(0,0)} = -9a^2 = -ve < 0$$

pt (0,0) is saddle point at this pt. function neither max nor min.

(ii) At point (a,a)

$$[rt - s^2]_{(a,a)} = 36a \cdot a \cdot - 9a^2 = 27a^2 = +ve$$

$$\boxed{rt - s^2 > 0}$$

and $[r]_{(a,a)} = 6a$.

case 1 :-

If $a > 0$

$[rt - s^2]_{(a,a)} > 0$, $r > 0$

$f(x,y)$ has a minimum value at (a,a) when a is +ve

case 2 :-

If $a < 0$

$[rt - s^2]_{(a-a)} > 0$, $[r]_{(a,a)} < 0$

$f(x,y)$ has a maximum value at (a,a) when a is -ve.

$$f(max) = (-a)^3 + (-a^3) - 3(-a)(-a)(-a)$$

$$= -a^3 + 3a^3 - a^3$$

$$\boxed{f(max) = +a^3}$$

Find the extreme ...

(i) $f(x,y) = x^3 y^2 (6 - x - y)$

imp (ii) $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$

Unit-4

8/2/2023

Vector Differentiation

Quantity

Scalar

$\phi(x,y,z) = c$

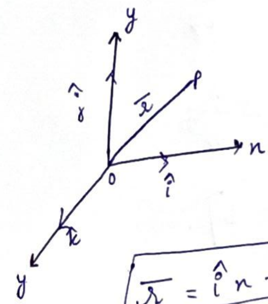
Eg.- Time, ~~amount~~; distance, Mass, temperature etc.

Vector.

$V = V_1 \hat{i} + V_2 \hat{j} + V_3 \hat{k}$

Eg:- Velocity, workdone, momentum etc.

$\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$



$\therefore |\bar{r}| = r$

vector scalar.

$$\boxed{\bar{r} = \hat{i}x + \hat{j}y + \hat{k}z.}$$

$$|\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r = \sqrt{x^2+y^2+z^2}$$

$$\boxed{r^2 = x^2+y^2+z}$$

15/2/2023

Scalar Point Function :-

Let 'R' be a region of space. At this point of which a scalar. $\phi = \phi(x,y,z)$ is given, than ϕ is called a scalar point function and R is called scalar field.

The temperature distribution in a medium, the distribution of atomospheric pressure in space are example of Scalar Point function.

15/2/2023

→ Vector Point Function :-

Let R be a region of space at each point of which a vector $\vec{V} = \vec{V}(x, y, z)$ is given, then $\vec{V}$ is called Vector point function and R is called as vector field.

The velocity of moving fluid at any instant, gravitational force are eg. of vector point functions.

→ Differential Operation (∇) Del

Differential operator is denoted by sample ∇ read as Del

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

or

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\left[\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1\right].$$

$$\left[\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0\right]$$

→ Gradient of a Scalar Field :-

Let $\phi(x, y, z)$ be a scalar field (a scalar) then its gradient is denoted by grad ϕ or $\nabla\phi$ and defined as

$$\text{grad}\,\phi \text{ or } \nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\phi$$

$$= \frac{\hat{i}\,\partial\phi}{\partial x} + \frac{\hat{j}\,\partial\phi}{\partial y} + \frac{\hat{k}\,\partial\phi}{\partial z}$$

= Vector.

→ Divergence of a Vector :-

Let $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector then its divergence is denoted by div $\vec{F}$ or $\nabla\cdot\vec{F}$ and defined as,

$$\text{div}\,\vec{F} \text{ or } \nabla\cdot\vec{F} = \left(\frac{\hat{i}\,\partial}{\partial x} + \frac{\hat{j}\,\partial}{\partial y} + \frac{\hat{k}\,\partial}{\partial z}\right)\cdot\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3 = \text{Scalar}.$$

→ Curl of a Vector :-

Let $\vec{P} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ be a vector then its curl is denoted by curl $\vec{F}$ or $\nabla\times\vec{P}$ and defined as

$$\text{curl}\,\vec{P} \text{ or } \nabla\times\vec{P} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \frac{\hat{k}\,\partial}{\partial z}\right)\times\left(F_1\hat{i} + F_2\hat{j} + F_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

= Vector.

→ grad $\phi = \nabla\phi$ = Vector

→ div $\vec{F} = \nabla\cdot\vec{F}$ = Scalar

→ curl $\vec{F} = \nabla\times\vec{F}$ = Vector.

Que If $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$ Find grad r^n where $|\vec{r}| = r$. Also find $\nabla\left(\frac{1}{r}\right)$

Soln Given $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$\therefore |\vec{r}| = \sqrt{x^2+y^2+z^2}$

But $|\vec{r}| = r$

or $r = \sqrt{x^2+y^2+z^2}$

on squaring both side

$\boxed{r^2 = x^2+y^2+z^2}$ —— (1)

Differentiate —(1) partially wrt x both side.

$\frac{\partial}{\partial x} r^2 = \frac{\partial}{\partial x}(x^2+y^2+z^2)$

$2r\frac{\partial r}{\partial x} = 2x + 0 + 0$

$\therefore \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}$

Similarly $\boxed{\frac{\partial r}{\partial y} = \frac{y}{r}}$ and $\boxed{\frac{\partial r}{\partial z} = \frac{z}{r}}$

Now, grad $r^n = \nabla r^n$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) r^n$

$= \hat{i}\left(\frac{\partial r^n}{\partial x}\right) + \hat{j}\left(\frac{\partial r^n}{\partial y}\right) + \hat{k}\left(\frac{\partial r^n}{\partial z}\right)$

$= \hat{i}\left(n r^{n-1}\frac{\partial r}{\partial x}\right) + \hat{j}\left(n r^{n-1}\frac{\partial r}{\partial y}\right) + \hat{k}\left(n r^{n-1}\frac{\partial r}{\partial z}\right)$

$= n r^{n-1}\left[\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right]$

$= n r^{n-1}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right]$

$= n r^{n-1} \cdot \frac{1}{r}\left(\hat{i}x + \hat{j}y + \hat{k}z\right)$

$= n \cdot r^{n-2}\vec{r}$

$\therefore \boxed{\text{grad } r^n = n r^{n-2}\vec{r}}$

where $r = |\vec{r}|$

Now, $\nabla\left(\frac{1}{r}\right) = \nabla(r^{-1})$

$= -1(r^{-1-2})\vec{r}$

$= -r^{-3}\vec{r}$

$\boxed{\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}}$ Ans.

Que Find grad $\log|\vec{r}|$ where $|\vec{r}| = r$ and $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$.

Soln Given $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$|\vec{r}| = \sqrt{x^2+y^2+z^2}$

But $|\vec{r}| = r$

$\therefore r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$ —— (1)

$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$

Now, grad $\log|\vec{r}|$

$= \nabla(\log r) \qquad (\because |\vec{r}| = r)$

$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\log r)$

$$= \hat{i}\frac{\partial}{\partial x}\log r + \hat{j}\frac{\partial(\log r)}{\partial y} + \hat{k}\frac{\partial}{\partial z}(\log r)$$

$$= \hat{i}\left(\frac{1}{r}\cdot\frac{\partial r}{\partial x}\right) + \hat{j}\left(\frac{1}{r}\frac{\partial r}{\partial y}\right) + \hat{k}\left(\frac{1}{r}\frac{\partial r}{\partial z}\right)$$

$$= \frac{1}{r}\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right]$$

$$= \frac{1}{r^2}(\hat{i}x + \hat{j}y + \hat{k}z) = \frac{1}{r^2}\vec{r} \quad \text{Ans}$$

Que ☆ If $u = x+y+z$
$v = x^2+y^2+z^2$
and $w = yz + zx + xy$.
prove that grad u, grad v and grad w are coplanar vectors

Soln If grad u, grad v and grad w are coplanar vectors
then we have to show that,

$$[\text{grad}\,u \cdot \text{grad}\,v\ \text{grad}\,w] = 0$$

$$\text{grad}\,u = \nabla u = \hat{i}\frac{\partial u}{\partial x} + \hat{j}\frac{\partial u}{\partial y} + \hat{k}\frac{\partial u}{\partial z}$$

$$= \hat{i}(1) + \hat{j}(1) + \hat{k}(1)$$

$$\boxed{\text{grad}\,u = \hat{i}+\hat{j}+\hat{k}}$$

$$\text{grad}\,v = \nabla v = \hat{i}\frac{\partial v}{\partial x} + \hat{j}\frac{\partial v}{\partial y} + \hat{k}\frac{\partial v}{\partial z}$$

$$= \hat{i}(2x) + \hat{j}(2y) + \hat{k}(2z)$$

$$\boxed{\text{grad}\,v = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}}$$

$$\text{grad}\,w = \nabla w = \hat{i}\frac{\partial w}{\partial x} + \hat{j}\frac{\partial w}{\partial y} + \hat{k}\frac{\partial w}{\partial z}$$

$$\boxed{\text{grad}\,w = \hat{i}(y+z) + \hat{j}(x+z) + \hat{k}(x+y)}$$

LHS $[\text{grad}\,u\ \text{grad}\,v\ \text{grad}\,w]$

$$= (\text{grad}\,u)\cdot(\text{grad}\,v \times \text{grad}\,w)$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ (y+z) & (x+z) & (x+y) \end{vmatrix}$$

$$C_1 \to C_1 - C_2, \quad C_2 \to C_2 - C_3$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & 2z \\ -(x-y) & -(y-z) & (x+y) \end{vmatrix}$$

$$= \cancel{1[2(x-y)(-y+z) + (x-y)(2(y-z))]}$$

$$= 2(x-y)(y-z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ -1 & -1 & x+y \end{vmatrix}$$

$$2(x-y)(y-z) \times 0 \quad [C_1 = C_2]$$

$$\Rightarrow 0 = \text{RHS}$$

hence grad u, grad v and grad w are coplanar Vectors.

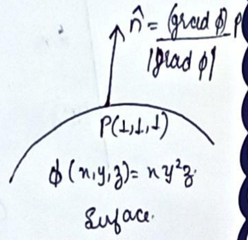
16/2/2023

Que Find outward unit normal vector on the given surface. $\phi(x,y,z) = xy^2z$ at pt $(1,1,1)$. Greatest rate of increase of ϕ

Soln Given surface $\phi(x,y,z) = xy^2z$.

$$\text{grad}\,\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}[y^2z] + \hat{j}[2xyz] + \hat{k}[xy^2]$$



$$(\text{grad}\,\phi)_{P(1,1,1)} = \hat{i} + 2\hat{j} + \hat{k}$$

$$|\text{grad}\,\phi| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$$

$\therefore$ outward unit normal vector on given surface ϕ at point $(1,1,1)$ is

$$\hat{n} = \frac{(\text{grad}\,\phi)_P}{|\text{grad}\,\phi|} = \frac{\hat{i}+2\hat{j}+\hat{k}}{\sqrt{6}} \text{ Ans}$$

Greatest Maximum rate of increase of $\phi(x,y,z) = |(\text{grad}\,\phi)_P|$

$$= \sqrt{6} \text{ Ans.}$$

~~Def~~ → **Directional Derivative of a Scalar field :-**

The Directional Derivative of a Scalar field $\phi(x,y,z)$ at point $P(x,y,z)$ in direction of given vector $\vec{a}$ is

$$= (\text{grad}\,\phi)_{P(x,y,z)} \cdot \hat{a}$$

$$= (\text{grad}\,\phi)_{P(x,y,z)} \cdot \frac{\vec{a}}{|a|}$$

Que Find the directional derivative in direction of $\frac{1}{r}$ in direction of $\vec{r} = \hat{i}x + \hat{j}y + kz$. where $|\vec{r}| = r$.

Soln Given $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

or $r = \sqrt{x^2+y^2+z^2}$

or $r^2 = x^2+y^2+z^2$.

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

D. Deri. of $\frac{1}{r}$ in direction of $\bar{r}$

$$= \left(\text{grad}\frac{1}{r}\right) \cdot \hat{r}$$

$$= (\text{grad}\, r^{-1}) \cdot \frac{\bar{r}}{|\bar{r}|} \qquad \left[\because \hat{r} = \frac{\bar{r}}{|\bar{r}|}\right]$$

$$= \left\{(-1)\, r^{-1-2}\, \bar{r}\right\} \cdot \frac{\bar{r}}{r} \qquad \left[\begin{array}{l} \text{grad}\, r^n = n\, r^{n-2}\, \bar{r} \\ \therefore |\bar{r}| = r \end{array}\right)$$

$$= -\frac{1}{r^3} \cdot \frac{1}{r}(\bar{r}\cdot\bar{r})$$

$$= \frac{-1}{r^4} \cdot r^2 \qquad [\bar{r}\cdot\bar{r} = |\bar{r}|^2 = r^2]$$

$$= \frac{-1}{r^2} \text{ Ans.}$$

Que Find the directional derivative of $\phi(x,y,z) = x^2yz + yxz^2$ at pt $(1,-2,1)$ in the direction of vector $2i - j - 2k$.

Soln Given Scalar field. $\phi(x,y,z) = x^2yz + yxz^2$

$$\therefore \text{grad}\,\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\left[2xyz + yz^2\right] + \hat{j}\left[x^2z + xz^2\right] + \hat{k}\left[x^2y + 2xyz\right]$$

$$\therefore (\text{grad}\,\phi)_{P(1,-2,1)} = \hat{i}(2\cdot1\cdot-2 + (-2)\cdot1^2) + \hat{j}\,[1^2\cdot1 + 1\cdot1^2]$$

$$k = [1^2(-2) + 2(1)(-2)1]$$

$$= -6\hat{i} + 2\hat{j} - 6\hat{k}$$

Now Directional Derivative of ϕ at Pt (P) in direction of given vector $\vec{a} = 2\hat{i} - \hat{j} - 2\hat{k}$.

$$= (\text{grad}\,\phi)_{P(1,-2,1)} \cdot \hat{a}$$

$$= (-6\hat{i} + 2\hat{j} - 6k) \cdot \frac{\vec{a}}{|a|}$$

$$= (-6\hat{i} + 2\hat{j} - 6k) \cdot \frac{(2\hat{i} - \hat{j} - 2k)}{\sqrt{2^2 + (-1)^2 + (-2)^2}} = \frac{(-6)(2) + 2(-1) + (-6)(-2)}{3}$$

$$\Rightarrow \frac{-12 - 2 + 12}{3} = \frac{-2}{3} \simeq \frac{2}{3} \text{ (ignoring '-' sign)}$$

Que Find the Directional Derivative of $\phi = x^2 - y^2 + 2z^2$ at pt $(1,2,3)$ in direction of line PQ where Q is pt $(5,0,4)$

Soln Given scalar field,

$$\phi = x^2 - y^2 + 2z.$$

$$\therefore \text{grad}\,\phi = \nabla\phi$$

$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + \frac{k\,\partial\phi}{\partial z}$$

$$= i\{2x\} + j\{-2y\} + k\{4z\}$$

$$\therefore (\text{grad}\,\phi)_{(1,2,3)} = 2\hat{i} - 4\hat{j} + 12\hat{k}.$$

Now $\overline{PQ}$ obtained by joining Pt $P = (1,2,3)$ and $Q = (5,0,4)$

$$\overline{PQ} = (5-1)\hat{i} + (0-2)\hat{j} + (4-3)\hat{k}$$

$$\overrightarrow{PQ} = 4\hat{i} - 2\hat{j} + \hat{k}$$

$$\hat{PQ} = \frac{\overline{PQ}}{|\overline{PQ}|} = \frac{4\hat{i} - 2\hat{j} + k}{\sqrt{4^2 + (2^2) + 1^2}} = \frac{4\hat{i} - 2j + k}{\sqrt{21}}$$

$\therefore$ Directional Derivative of ϕ at $P(1,2,3)$ in direction of vector PQ is

$$(\text{grad}\,\phi)_{P(1,2,3)} \cdot \hat{PQ}$$

$$= (2\hat{i} - 4\hat{j} - 12\hat{k}) \left(\frac{4\hat{i} - 2\hat{j} + k}{\sqrt{21}}\right) = \frac{2(4) + (-4)(-2) + 12(1)}{\sqrt{21}}$$

$$= \frac{8 + 8 + 12}{\sqrt{21}} = \frac{28}{\sqrt{21}} \text{ Ans}$$

Que Find the Directional Derivative of $\phi = 5x^2y - 5y^2z + \frac{5}{2}z^2x$ at point $(1,1,1)$ in the direction of line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

Soln $\text{grad}\,\phi = \nabla\phi = \frac{i\,\partial\phi}{\partial x} + \frac{j\,\partial\phi}{\partial y} + \frac{k\,\partial\phi}{\partial z^2}$

$$= i\left(10xy + \frac{5}{2}z^2\right) + j(5x^2 - 10yz) + k(5xz)$$

$$(\text{grad}\,\phi)_{(1,1,1)} = \frac{25\hat{i}}{2} - 5\hat{j} + 5\hat{k}$$

Now, given line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

$\therefore$ The vector $\hat{a}$ about given line.

$$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}$$

Now, DD of ϕ at point $(1,1,1)$ in direction of given vector a along given line.

$$= (\text{grad}\,\phi)_{P(1,1,1)} \cdot \hat{a}$$

$$= \left(\frac{25\hat{i}}{2} - 5\hat{j} + 5\hat{k}\right) \cdot \frac{\bar{a}}{|\bar{a}|} = \left(\frac{25}{2}i + 5j + 5k\right) \cdot \frac{(2\hat{i} + 2\hat{j} + \hat{k})}{\sqrt{(2)^2 + (-2)^2 + (1)^2}}$$

$$= \frac{\left(\frac{25}{2}\right)(2) + (-5)(-2) + 5\times 1}{3}$$

$$= \frac{25 + 10 + 5}{3} = \frac{40}{3} \text{ Ans}$$

Que If the D.D of $\phi = ax^2y + by^2 + cz^2x$ at pt $(1,1,1)$ has maximum magnitude 15 in direction of parallel line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$ find a, b & c.

Soln Given scalar field

$$\phi = ax^2y + by^2z + cz^2x.$$

$\therefore$ grad $\phi = \nabla\phi$

$$= \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}$$

$$= \hat{i}(2axy + cz^2) + \hat{j}(ax^2 + 2by) + \hat{k}(2czx + y^2b)$$

$$(\text{grad}\,\phi)_{(1,1,1)} = (2a+c)\hat{i} + (a+2b)\hat{j} + (2c+b)\hat{k}$$

given line $\frac{x-1}{2} = \frac{y-3}{2} = \frac{z}{1}$

$\vec{a} \parallel$ to line $= 2\hat{i} - 2\hat{j} + \hat{k}$

According to que, Max magnitude = 15

$$\therefore 15 = \sqrt{(2a+c)^2 + (a+2b)^2 + (b+2c)^2}$$

on squaring both side

$$225 = (2a+c)^2 + (a+2b)^2 + (b+2c)^2 \quad \text{—(1)}$$

$$\therefore \frac{2a+c}{2} = \frac{a+2b}{-2} = \frac{b+2c}{1} \quad (\parallel)$$ parallel

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~~maximum~~ Since the magnitude is maximum in the direction $\parallel$ to line $\frac{x-1}{2} = \frac{y-3}{-2} = \frac{z}{1}$

If $\frac{2a+c}{2} = \frac{a+2b}{-2}$

$2a+c = -(a+2b)$

$2a+c+2b+a=0$

$3a+2b+c=0$ —(2)

If $\frac{2b+a}{-2} = \frac{2c+b}{1}$

$2b+a = -4c-2b$

$2b+a+4c+2b=0$

$a+4b+4c=0$ —(3)

Taking cross product in eq —(2) & —(3)

$3a+2b+c=0$

$a+4b+4c=0$

$$\frac{a}{\substack{2\ 1\\4\ 4}} = \frac{b}{\substack{1\ 3\\4\ 1}} = \frac{c}{\substack{3\ 2\\1\ 4}} \Rightarrow \frac{a}{8-4} = \frac{b}{1-12} = \frac{c}{12-2}$$

$$\frac{a}{4} = \frac{b}{-11} = \frac{c}{10} = k$$

$\boxed{a=4k}$ $\boxed{b=-11k}$ $\boxed{c=10k}$

$\because$ we have $225 = [(2a+c)^2 + (2b+a)^2 + (2c+b)^2]$

$225 = [(8k+10k)^2 + (-22k+4k)^2 + (20k-11k)^2]$

$225 = 324k^2 + 324k^2 + 81k^2$

$225 = 729k^2$

$$k^2 = \frac{225}{729} = \frac{\pm 15}{27} = \frac{\pm 5}{9}$$

$$\boxed{k = \frac{\pm 5}{9}}$$

$\therefore a = 4\left(\frac{\pm 5}{9}\right) = \frac{\pm 20}{9}$ Ans

$b = -11\left(\frac{\pm 5}{9}\right) = \frac{\pm 55}{9}$ Ans

$c = 10k = 10\left(\frac{\pm 5}{9}\right) = \frac{\pm 50}{9}$ Ans

VVI

Que Find div. $\bar{r}$ and curl $\bar{r}$ if $\bar{r} = \hat{i}x + y\hat{j} + \hat{k}z$.

Soln $\because \vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$

$\therefore |\vec{r}| = \sqrt{x^2+y^2+z^2} \Rightarrow r = \sqrt{x^2+y^2+z^2} \quad (\because |\bar{r}| = r)$

$r^2 = x^2+y^2+z^2$ —(1)

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}.$$

Now, $\operatorname{div}\bar{r} = \nabla\cdot\vec{r}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(\hat{i}x + \hat{j}y + \hat{k}z\right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}$$

$$= 1+1+1$$

$$= 3$$

$$\boxed{\operatorname{div}\bar{r} = 3}$$

grad $\vec{r}$ not possible.

$\operatorname{curl}\bar{r} = \nabla\times\vec{r}$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\times\left(\hat{i}x + \hat{j}y + \hat{k}z\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x & z \end{vmatrix} + k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x & y \end{vmatrix}$$

$$= i\left\{\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right\} - j\left\{\frac{\partial z}{\partial x} - \frac{\partial x}{\partial z}\right\} + k\left\{\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right\}$$

$$= i(0-0) - j(0-0) + k(0-0)$$

$$= 0$$

$$\boxed{\operatorname{curl}\bar{r} = 0}$$

Remark :- If $\operatorname{div}\vec{F} = 0 \rightarrow F$ is called Solenoidal.

curl $\rightarrow$ rotation.

If $\operatorname{curl}\vec{F} = 0 \rightarrow \vec{F}$ is irrotational

divergence = flow of liquid in pipe

divergence grad = slope.

$\rightarrow$ Some Important Formulae :-

1.) $\operatorname{div}(\phi\bar{F}) = (\operatorname{grad}\phi)\cdot\bar{F} + \phi\operatorname{div}\bar{F}$

2) $\operatorname{curl}(\phi\bar{F}) = (\operatorname{grad}\phi)\times\bar{F} + \phi\operatorname{curl}\vec{F}$

3) $\operatorname{div}(\bar{a}\times\bar{b}) = \bar{b}\cdot\operatorname{curl}\bar{a} - \bar{a}\cdot\operatorname{curl}\bar{b}$

VMI 2023 Que Prove that $\operatorname{div}(\operatorname{grad} r^n) = n(n+1)r^{n-2}$ where $\bar{r} = \hat{i}x + \hat{j}y + \hat{k}z$

Soln $\operatorname{div}(\operatorname{grad} r^n)$

$= \operatorname{div}\left[n r^{n-2}\bar{r}\right]$ $\quad$ ($\because \operatorname{grad} r^n = n r^{n-2}\bar{r}$ where $r = |\bar{r}|$)

$= n\operatorname{div}\left[\underset{\text{Scalar}}{r^{n-2}}\ \underset{\text{Vector.}}{\bar{r}}\right]$

We know that, $\operatorname{div}(\phi\bar{F}) = (\operatorname{grad}\phi)\cdot\bar{F} + \phi\operatorname{div}\bar{F}$

$\therefore \operatorname{div}[\operatorname{grad} r^n] = n[(\operatorname{grad} r^{n-2})\cdot\bar{r} + r^{n-2}\operatorname{div}\vec{r}]$

$= n\left[\{(n-2)r^{(n-2)-2}\bar{r}\}\cdot\bar{r} + r^{n-2}\cdot 3\right]$

[$\because$ using $\operatorname{grad} r^n = n r^{n-2}\bar{r}$, $\operatorname{div}\bar{r} = 3$]

$= n\left[n-2\ r^{n-4}(\bar{r}\cdot\bar{r}) + 3r^{n-2}\right]$

$= n\left[n-2\ r^{n-4}\cdot r^2 + 3r^{n-2}\right]$ $\quad$ [$\because \vec{r}\cdot\vec{r} = |\bar{r}|^2 = r^2$)

$= n\left[(n-2)r^{n-2} + 3r^{n-2}\right]$

$= n\left[(n-2)+3\right]r^{n-2}$

$= n(n+1)r$

hence proved

Scalar Potential :-

The scalar Potential of a scalar field $\phi(x,y,z)$ is

$$d\phi = \bar{F} \cdot d\bar{r}$$

$$\boxed{\phi = \int \bar{F} \cdot d\bar{r} + C}$$

where C is constant of integration

Que Find the scalar potential of F for $\bar{A} = y\hat{i} + 2xy\hat{j} - z^2\hat{k}$

Que The scalar Potential ~~to~~ $\phi(x,y,z)$ for a vector field $\bar{F}$ is

$$d\phi = \bar{F} \cdot d\bar{r}$$

$$\phi = \int \bar{F} \cdot d\bar{r} + c$$

Let ϕ is scalar potential of $\vec{A}$

$$d\phi = \bar{A} \cdot d\bar{r}$$

$$\phi = \int \vec{A} \cdot d\vec{r}$$

$$\phi = \int (y\hat{i} + 2xy\hat{j} - z^2 k) \cdot (\hat{i}\,dx + \hat{j}dy + \hat{k}dz)$$

$$\because \vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$$

$$d\vec{r} = \hat{i}\,dx + \hat{j}dy + \hat{k}\,dz$$

$$= \int y\,dx + 2xy\,dy - z^2 dz$$

$$y\int dx + 0 - \int z^2 dz$$

$$= yx - \frac{z^3}{3} + C$$

This is required scalar Potential.

Remark $\int (y+z)\,dx + \int (z+x)\,dy + \int (x+y)\,dz$

$(y+z)\int dx + z\int dy + 0 + 0$

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VIMP

Que A fluid motion is given by vector $\vec{V} = (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}$ show that the motion is irrotional. and find its velocity Potential (. ~~For~~ is the motion possible for an incompressible fluid.

Soln If the ~~mo~~ velocity of motion is irrotional then prove that curl vector $\vec{V} = 0$.

Now curl $\vec{V} = \nabla \times \vec{V}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \times \left[(y+z)i + (z+x)j + (x+y)k\right]$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (y+z) & (z+x) & (x+y) \end{vmatrix}$$

$$= i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z+x) & x+y \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ (y+z) & (x+y) \end{vmatrix} + k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ (y+z) & (z+x) \end{vmatrix}$$

$$\Rightarrow i\left[\frac{\partial}{\partial y}(x+y) - \frac{\partial}{\partial z}(x+z)\right] - j\left[\frac{\partial}{\partial x}(x+y) - \frac{\partial}{\partial z}(y+z)\right] + k\left[\frac{\partial}{\partial x}(z+x) - \frac{\partial}{\partial y}(y+z)\right]$$

$$\Rightarrow i(0-0) - j[0-0] + k(0-0)$$

$$\Rightarrow 0$$

$\because$ curl $\vec{V} = 0$

$\Rightarrow \vec{V}$ is irrational.

Velocity Potential :-

Let ϕ be velocity potential for $\vec{V}$

$$d\phi = \vec{V} \cdot d\vec{r}$$

$$\phi = \int \vec{V} \cdot d\vec{r}$$

$$\phi = \int \{(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\} \cdot \{\hat{i}dx + \hat{j}dy + \hat{k}dz\}$$

$$\phi = \int (y+z)\,dx + (z+x)\,dy + (x+y)\,dz$$

$$= y+z \int dx + z\int dy + 0\int dz$$

$$\Rightarrow (y+z)x + zy + 0 + C$$

$\Rightarrow \quad xy + zx + zy + c$ where c is constant of integration.

→ To show that the fluid motion is possible for an incompressible fluid, we will show that divergence $= 0$.

$$\text{div}\,\vec{V} = 0$$

Now, $\text{div}\,\vec{V} = \nabla \cdot \vec{V}$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \{(y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\}$$

$$= \frac{\partial}{\partial x}(y+z) + \frac{\partial}{\partial y}(z+x) + \frac{\partial}{\partial z}(x+y)$$

$$= 0 + 0 + 0$$

$\therefore \text{div}\,\vec{V} = 0$

$\therefore$ motion is possible for an incompressible fluid!!

Que If $\vec{f} = 2mx\hat{i} + 3y\hat{j} + 2z\hat{k}$ find the value of m. If $\vec{f}$ is solenoidal.

Solⁿ Given vector $\vec{f}$ is solenoidal,

$\therefore \text{div}\,\vec{F} = 0$

$\nabla \cdot \vec{F} = 0$

$$\left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left(2mx\hat{i} + 3y\hat{j} + 2z\hat{k}\right) = 0$$

$$\frac{\partial(2mx)}{\partial x} + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(2z) = 0$$

$$2m + 3 + 2 = 0$$

$$2m = -5$$

$$\boxed{m = \frac{-5}{2}} \text{ Ans}$$

VMI Que Show that the vector field $\vec{F} = \frac{\vec{r}}{|\vec{r}|^3}$ is irrotational as well as solenoidal. find its scalar potential.

Solⁿ (i) If vector $\vec{f}$ is irrotational then we have to show that $\text{curl}\,\vec{f} = 0$

Now, $\text{curl}\,\vec{f} = \text{curl}\left(\frac{\vec{r}}{|\vec{r}|^3}\right)$

$$= \text{curl}\left(\frac{\vec{r}}{r^3}\right) \quad [\because |\vec{r}| = r]$$

$$= \text{curl}\left(\underbrace{\frac{1}{r^3}}_{\text{scalar}} \cdot \underbrace{\vec{r}}_{\text{vector}}\right)$$

We know that,

$$[\text{curl}(\phi \cdot \vec{A}) = (\text{grad}\,\phi) \times \vec{A} + \phi\,\text{curl}\,\vec{A}]$$

$$= \left(\left(\text{grad}\frac{1}{r^3}\right) \times \vec{r} + \frac{1}{r^3}(\text{curl}\,\vec{r}\right)$$

$= (\text{grad } r^{-3}) \times \vec{r} + \frac{1}{r^3}(0) \quad [\because \text{curl } \vec{r} = 0]$

$= (-3 r^{-3-2} \cdot \vec{r}) \times \vec{r} \quad [\because \text{grad } r^n = n r^{n-2} \vec{r}]$

$= \left(\frac{-3}{r^5} \cdot \vec{r}\right) \times \vec{r}$

$\Rightarrow \frac{-3}{r^5}(\vec{r} \times \vec{r}) \quad [\because \vec{r} \times \vec{r} = 0]$

$\text{curl } \vec{V} \Rightarrow 0$

$\rightarrow F$ is irrational

$\rightarrow$ If vector f is solenoidal then we have to prove that, $\text{div } \vec{f} = 0$

Now, $\text{div } \vec{F} = \text{div}\left(\frac{\vec{r}}{|\vec{r}|^3}\right)$

$= \text{div}\left(\frac{\vec{r}}{r^3}\right) \quad [\because |\vec{r}| = r]$

$= \text{div}\left(\underbrace{\frac{1}{r^3}}_{\text{scalar}} \underbrace{\vec{r}}_{\text{vector}}\right)$

We know that $\text{div}(\phi \vec{A}) = (\text{grad }\phi) \cdot \vec{A} + \phi(\text{div } \vec{A})$

$\text{div } \vec{F} = (\text{grad}(r^{-3})) \cdot \vec{r} + \frac{1}{r^3}(\text{div } \vec{r})$

$= (-3 r^{-3-2} \vec{r}) \cdot \vec{r} + \frac{1}{r^3}(3) \quad [\because \text{div } \vec{r} = 3]$

$\Rightarrow \frac{-3}{r^5}(\vec{r} \cdot \vec{r}) = \frac{3}{r^3}$

$= \frac{-3}{r^5} \times r^2 + \frac{3}{r^3} \quad (\because \vec{r} \cdot \vec{r} = |\vec{r}|^2 = r^2)$

$\text{div } \vec{V} = 0$

$\Rightarrow F$ is solenoidal

$\rightarrow$ **Scalar Potential :-**

Let us consider ϕ is scalar potential of given vector.

$d\phi = \vec{F} \cdot d\vec{r}$

on integration,

$\int d\phi = \int \vec{F} \cdot d\vec{r}$

$\phi = \int \frac{\vec{r}}{|\vec{r}|^3} \cdot d\vec{r}$

$\phi = \int \frac{(x\hat{i} + y\hat{j} + z\hat{k})}{(\sqrt{x^2+y^2+z^2})^3} \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$= \int \frac{x\,dx + y\,dy + z\,dz}{(x^2+y^2+z^2)^{3/2}}$

Let $x^2+y^2+z^2 = t$

on diff. both side as Total

$d(x^2+y^2+z^2) = dt$

$2x\,dx + 2y\,dy + 2z\,dz = dt$

$2(x\,dx + y\,dy + z\,dz) = dt$

$\therefore x\,dx + y\,dy + z\,dz = \frac{dt}{2}$

$\Rightarrow \int \frac{dt/2}{(t^{3/2})}$

$= \frac{1}{2}\int t^{-3/2}\,dt$

$= \frac{1}{2}\left[\frac{t^{-3/2+1}}{\frac{-3}{2}+1}\right] + C \quad \Rightarrow \frac{1}{2}\left(\frac{t^{\frac{-1}{2}}}{-1/2}\right) + C$

$\Rightarrow -\frac{1}{\sqrt{t}} + C$

$$= -\frac{1}{\sqrt{x^2+y^2+z^2}} + c$$

$$= \boxed{-\frac{1}{r} + c} \text{ Ans}$$

This is required Scalar Potential.

~~Que~~ $\cancel{\text{If } \vec{f} = r^2\vec{r}}$.

Que Show that $\vec{f} = r^2 . \vec{r}$ is conservative & find its Scalar Potential.

Que If vector $\vec{f}$ is conservation then curl vector $\vec{f}$ is zero.

$$\text{curl} \vec{f} = \text{curl}(\underbrace{r^2}_{\text{scalar}} . \underbrace{\vec{r}}_{\text{vector}}) \quad (\because |\vec{r}| = \sqrt{x^2+y^2+z^2} = r)$$

We know that $\text{curl}(\phi\vec{A}) = (\text{grad}\,\phi) \times \vec{A} + \phi(\text{curl}\,\vec{A})$

$$\therefore \text{curl}\,\vec{f} = (\text{grad}\,r^2) \times \vec{r} + r^2 \text{curl}(\vec{r}) \quad [\because \text{curl}\,\vec{r} = 0]$$

$$\text{curl}\,\vec{f} = (\text{grad}\,r^2) \times \vec{r}$$

$$= (2 r^{2-2} \vec{r}) \times \vec{r} \qquad [\because \vec{r} \times \vec{r} = \vec{0}]$$

$$\Rightarrow 2 \times 0$$

$$\Rightarrow 0$$

Hence $\vec{f}$ is conservative.

→ Scalar Potential.

Let us consider ϕ is scalar potential of given vector

$$d\phi = \vec{F} . \vec{dr}$$

on integration

$$\phi = \int \vec{f} . \vec{dr}$$

$$= \int (r^2 \vec{r}) . dr$$

$$= \int (x^2+y^2+z^2)(x\hat{i} + y\hat{j} + z\hat{k}) . (\hat{i}dx + \hat{j}dy + \hat{k}dz)$$

$$= \int (x^2+y^2+z^2)(x\,dx + y\,dy + z\,dz)$$

Let $x^2+y^2+z^2 = t$

$2x\,dx + 2y\,dy + 2z\,dz = dt$

$x\,dx + y\,dy + z\,dz = \frac{dt}{2}$

$$\int t \frac{(dt)}{2}$$

$$\Rightarrow \frac{1}{2}\int t\,dt$$

$$\frac{1}{2}\frac{t^2}{2} + c \Rightarrow \frac{x^2+y^2+z^2}{4} + c$$

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Que find divergence & curl of $\vec{V} = (xyz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - yz)k$. at point $(1,-1,1)$ also find its scalar potential.

Soln. $\text{curl}\,\vec{V} = \nabla \times \vec{V}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & (3x^2y) & (xz^2 - y^2z) \end{vmatrix}$$

$$= i\begin{vmatrix} \partial/\partial y & \partial/\partial z \\ 3x^2y & (xz^2-y^2z) \end{vmatrix} - j\begin{vmatrix} \partial/\partial x & \partial/\partial z \\ xyz & (xz^2-y^2z) \end{vmatrix} + k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xyz & 3x^2y \end{vmatrix}$$

$$= \hat{i}\left\{\frac{\partial}{\partial y}(xz^2 - y^2z) - \frac{\partial}{\partial z}(3x^2y)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(xz^2 - y^2z) - \frac{\partial}{\partial z}(xyz)\right\} + k\left\{\frac{\partial}{\partial x}(3x^2y) - \frac{\partial}{\partial y}(xyz)\right\}$$

$$\Rightarrow i(0 - 2yz) - 0\} - j\{(z^2 - 0) - (xy \cdot 1)\} + k\{6xy - xz\}$$

$$\Rightarrow (-2yz)\hat{i} - (z^2 - xy)\hat{j} + (6xy - xz)\hat{k} \text{ Ans.}$$

$(\text{curl}\,\vec{V})_{(1,1,1)} = 2\hat{i} - 2\hat{j} - 7\hat{k}$ Ans.

Now, divergence $\vec{V}$,

$$\text{div}\,\vec{V} = \nabla \cdot \vec{V}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \left[xyz\,\hat{i} + 3x^2y\,\hat{j} + (xz^2 - y^2z)\,\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(xyz) + \frac{\partial}{\partial y}(3x^2y) + \frac{\partial}{\partial z}(xz^2 - y^2z)$$

$$= yz + 3x^2 + (2xz - y^2)$$

$$(\text{div}\,\vec{V})_{(1,-1,1)} = -1 + 3 + (2-1)$$

$$= 3 \text{ Ans}$$

Scalar Potential:- Let ϕ be scalar potential of given vector $V(\vec{V})$

$\because d\phi = \vec{V} \cdot d\vec{r}$

on integration.

$$\phi = \int \vec{V} \cdot d\vec{r}$$

$$= \int (x, yz)\hat{i} + (3x^2y)\hat{j} + (xz^2 - y^2z)\hat{k} \} \cdot \{\hat{i}dx + \hat{j}dy + \hat{k}dz\}$$

$$= (xyz)dx + \overset{=0}{(3x^2y)}dy + (xz^2 - y^2z)dz$$

$$= yz\frac{x^2}{2} + 0 + 0 + c$$

$$(\phi)_{(1,-1,1)} = \frac{-1}{2} + c \text{ Ans.}$$

Que Prove that $f(r)\vec{r}$ is irrational

Soln To show that $f(r)\vec{r}$ is irrational, we will prove that $\text{curl}(f(r)\vec{r}) = 0$

Now, $\text{curl}[f(r)\ \vec{r}]$.

scalar vector.

$$= [\text{grad}\,f(r)] \times \vec{r} + f(r)\,\text{curl}\,\vec{r}$$

$$= \left[\hat{i}\frac{\partial}{\partial x}f(r) + \hat{j}\frac{\partial}{\partial y}f(r) + \hat{k}\frac{\partial}{\partial z}f(r)\right] \times \vec{r} + f(\vec{r})\,\vec{0} \quad [\because \text{curl}\,\vec{r} = 0]$$

$$= \left[\hat{i} f'(r)\frac{\partial r}{\partial x} + \hat{j} f'(r)\frac{\partial r}{\partial y} + \hat{k} f'(r)\frac{\partial r}{\partial z}\right] \times \vec{r}$$

$$\Rightarrow f'(r)\left[\hat{i}\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right] \times \vec{r}$$

$$\Rightarrow \left[\frac{f'(r)}{r}(\hat{i}x + \hat{j}y + \hat{k}z)\right] \times \vec{r}$$

$$\Rightarrow \left[\frac{f'(r)}{r}\vec{r}\right] \times \vec{r}$$

$$\Rightarrow \frac{f'(r)}{r}(\vec{r} \times \vec{r}) = \frac{f(r)}{r} = 0 \text{ Ans.}$$

Hence $f(r)\vec{r}$ is irrational.

Que Find the value of n for which r^n vector r is solenoidal, where $\vec{r} = \hat{i}x + \hat{j}y + \hat{k}z$.

Ans If r^n is solenoidal.

then $\text{div}(r^n\,\vec{r}) = 0$

$$(\text{grad}\,r^n) \cdot \vec{r} + r^n\,\text{div}\,\vec{r} = 0$$

$$(nr^{n-2}\vec{r}) \cdot \vec{r} + r^n(3) = 0 \quad \{\because \text{div}\,\vec{r} = 3\}$$

$$n r^{n-2}(\vec{r} \cdot \vec{r}) + 3r^n = 0$$

$$n r^{n-2} r^2 + 3r^n = 0 \quad (\because \vec{r} \cdot \vec{r} = |\vec{r}|^2 = r^2).$$

$n\,r^{n} \cancel{-2r} + 3r^{n} = 0$

$(n+3)\,r^{n} = 0$

$r^{n} \neq 0 \quad \therefore n+3=0$

$\boxed{n=-3}$ Ans

Que Show ~~that~~ . If $\vec{V} = x\hat{i} + y\hat{j} + z\hat{k} / \sqrt{x^2+y^2+z^2}$

find $\mathrm{div}(u\vec{r})$ if $u = x^2 + y^2 + z^2$ where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Soln

$\mathrm{div}(u\vec{V})$

$\mathrm{div}\left(r^{2}\,\frac{\bar{r}}{r}\right)$

$\mathrm{div}(r\,\vec{r})$

$(\mathrm{grad}\,r)\cdot\vec{r} + r\,\mathrm{div}\,\vec{r}$

$= \{(\cancel{1})\,r^{1-2}\,\vec{r}\}\cdot\vec{r} + r(3)$

$= \frac{1}{r}(\vec{r}\cdot\vec{r}) + 3r$

$= \frac{1}{r}\,r^{2} + 3r$

$r + 3r = 4r$

$= 4\sqrt{x^2+y^2+z^2}$ Ans.

Que ★ Show that $\mathrm{curl}(\mathrm{grad}\,\phi) = 0$ & $\mathrm{div}(\mathrm{curl}\,\vec{V}) = 0$

Soln $\mathrm{grad}\,\phi = \frac{i\,\partial\phi}{\partial x} + \frac{j\,\partial\phi}{\partial y} + \frac{k\,\partial\phi}{\partial z}$

$\mathrm{div}\,\vec{V} = \nabla\cdot\vec{V}$

$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$

$= \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$ Ans

curl $\mathrm{grad}\,\phi = 0$

Proof:- $\mathrm{grad}\,\phi = \nabla\phi$

$= \left(\frac{i\,\partial}{\partial x} + \frac{j\,\partial}{\partial y} + \frac{k\,\partial}{\partial z}\right)\phi$

$\mathrm{grad}\,\phi = \frac{i\,\partial\phi}{\partial x} + \frac{j\,\partial\phi}{\partial y} + \frac{k\,\partial\phi}{\partial z}$

LHS curl grad ϕ

$= \nabla \times \mathrm{grad}\,\phi$

$= \left(\frac{i\,\partial}{\partial x} + \frac{j\,\partial}{\partial y} + \frac{k\,\partial}{\partial z}\right) \times \left(\frac{i\,\partial\phi}{\partial x} + \frac{j\,\partial\phi}{\partial y} + \frac{k\,\partial\phi}{\partial z}\right)$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix}$$

$$\Rightarrow i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial y} & \frac{\partial\phi}{\partial z} \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial z} \end{vmatrix} + k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ \frac{\partial\phi}{\partial x} & \frac{\partial\phi}{\partial y} \end{vmatrix}$$

$$i\left\{\frac{\partial^2\phi}{\partial y\partial z} - \frac{\partial^2\phi}{\partial y\partial z}\right\} - j\left\{\frac{\partial^2\phi}{\partial x\partial z} - \frac{\partial^2\phi}{\partial x\partial z}\right\} + k\left\{\frac{\partial^2\phi}{\partial x\partial y} - \frac{\partial^2\phi}{\partial x\partial y}\right\}$$

$= 0$

Hence proved

Beta & Gamma Function:-

Gamma Function:-

If n is a +ve no. i.e. $n>0$ then definite integral $\boxed{\int_0^\infty e^{-x} x^{n-1} dx}$ is a function of n called Gamma function and denoted by Γn (read as Gamma n)

$$\boxed{\Gamma n = \int_0^\infty e^{-x} . x^{n-1} dx \quad , \quad n>0}$$

$\Gamma 1$:-

$$\Gamma 1 = \int_0^\infty e^{-x} . x^{1-1} dx$$

$$= \int_0^\infty e^{-x} . x^0 dx$$

$$= \int_0^\infty e^{-x} \, 1 \, dx$$

$$= \left[\frac{e^{-x}}{-1}\right]_0^\infty = -\left[e^{-\infty} - e^0\right] = \left[\frac{1}{e^\infty} - \frac{1}{e^0}\right] = \left[\frac{1}{\infty} - \frac{1}{1}\right]$$

$$= -[0-1] = 1$$

$$\boxed{\Gamma n = 1}$$

Remark:- $\Gamma n+1 = n\Gamma n$ if n is fraction (+ve)

① $\Gamma \frac{7}{2} = \frac{5}{2}\Gamma\frac{5}{2}$

$= \frac{5}{2} \cdot \frac{3}{2}\Gamma\frac{3}{2}$

$= \frac{5}{2} \cdot \frac{3}{2} \frac{1}{2}\Gamma\frac{1}{2}$

$\Gamma \frac{9}{2} = \frac{7}{2} \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2}$

$\Gamma \frac{7}{3} = \frac{4}{3} \cdot \frac{1}{3}\Gamma\frac{1}{3}$

② $\Gamma n+1 = \lfloor n$

$\Gamma 7 = \lfloor 6 = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$$\boxed{\Gamma n+1 = n\Gamma n = \lfloor n}$$

$\Gamma \frac{9}{2} = \lfloor 7/2$ X

Ay prove $\boxed{\Gamma \frac{1}{2} = \sqrt{\pi}}$

① $\int_0^\infty e^{-x} x^{n-1} dx = \Gamma n$

② $\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma n}{k^n}$

③ $\int_0^1 \log\left(\frac{1}{x}\right)^{n-1} dx = \Gamma n$

→ Beta Function :-

If $m>0$ & $n>0$ - then the definite integral,

$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ [denoted by $\beta(m,n)$] is called Beta function of function m & n..

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx}$$

Beta function (Another definition)

$$\boxed{\therefore\ \beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx}$$

Beta function is symmetrical

$$\boxed{\beta(m,n) = \beta(n,m)}$$

Relation b/w beta & gamma :-

$$\boxed{\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma(m+n)}}$$

Some Important Formulae :-

(1) $$\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\ \Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}$$

(2) $$\Gamma n\ \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

(3) Duplication Formulae.

$$\Gamma n\ \Gamma\left(n+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{n-1}}\,\Gamma 2n$$

(4) $$\Gamma\frac{1}{n}\ \Gamma\frac{2}{n}\ \Gamma\frac{3}{n} \cdots \Gamma\frac{n-1}{n} = \frac{(2\pi)^{n-1/2}}{\sqrt{n}}$$

* **Que** Prove that $\int_0^\infty \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$.

Soln $I = \int_0^\infty \frac{x^c}{c^x}\,dx$

$$= \int_0^\infty c^{-x} x^c\,dx \qquad \left[\because a^b = e^{b(\log a)}\right]$$

$$= \int_0^\infty e^{-x(\log c)}\ x^{(c+1)-1}\,dx$$

$$I = \int_0^\infty e^{-(\log c)x} \cdot x^{(c+1)-1}\,dx.$$

we know that,

$$I = \int_0^\infty e^{-x} x^{n-1}\,dx = \Gamma n$$

$$\int_0^\infty e^{-kx} x^{n-1}\,dx = \frac{\Gamma n}{k^n}$$

$$I = \frac{\Gamma(c+1)}{(\log c)^{c+1}} = RHS$$

Que Find $\Gamma\left(\frac{-1}{2}\right)$

Soln

We know,

$$\Gamma n \cdot \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

putting $n = \frac{-1}{2}$

$$\Gamma\left(\frac{-1}{2}\right) \Gamma\left(1-\left(\frac{-1}{2}\right)\right) = \frac{\pi}{\sin\left(\frac{-\pi}{2}\right)} \qquad \left(\because \sin(-\theta) = -\sin\theta\right)$$

$$\Rightarrow \Gamma\left(\frac{-1}{2}\right) \Gamma\left(\frac{3}{2}\right) = \frac{\pi}{-\sin\left(\frac{\pi}{2}\right)}$$

$$\Gamma\left(\frac{-1}{2}\right)\Gamma\left(\frac{3}{2}\right) = \frac{+\pi}{(-1)} \Rightarrow \Gamma\left(\frac{-1}{2}\right) = \frac{-\pi}{\Gamma\left(\frac{3}{2}\right)}$$

$$= \frac{-\pi}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{-2\pi}{\sqrt{\pi}} \qquad \left(\because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}\right)$$

$$\Gamma\left(\frac{-1}{2}\right) = -2\sqrt{\pi} \text{ Ans}$$

Que Find $\Gamma\left(\frac{-3}{2}\right)$

Soln We know,

$$\Gamma n \cdot \Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

putting $n = \frac{-3}{2}$

$$\Gamma\left(\frac{-3}{2}\right) \Gamma\left(1-\left(\frac{-3}{2}\right)\right) = \frac{\pi}{\sin\left(\frac{-3}{2}\pi\right)} \qquad \left(\because \sin(-\theta) = -\sin\theta\right)$$

$$\Gamma\left(\frac{-3}{2}\right)\Gamma\left(\frac{5}{2}\right) = \frac{\pi}{-\sin\frac{3}{2}\pi} \Rightarrow \Gamma\left(\frac{-3}{2}\right)\Gamma\left(\frac{5}{2}\right) = \frac{\pi}{-\sin\left(\pi+\frac{\pi}{2}\right)}$$

$$\Rightarrow -\frac{\pi}{\left(-\sin\frac{\pi}{2}\right)} \qquad \left[\sin(\pi+\theta) = -\sin\theta\right]$$

$$\Gamma\left(\frac{-3}{2}\right)\Gamma\left(\frac{5}{2}\right) = \frac{\pi}{1} \Rightarrow \Gamma\left(\frac{-3}{2}\right) = \frac{\pi}{\Gamma\left(\frac{5}{2}\right)} \Rightarrow \frac{\pi}{\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}$$

$$= \frac{4}{3}\frac{\pi}{\sqrt{\pi}} = \frac{4\sqrt{\pi}}{3} \text{ Ans}$$

Que Evaluate $\displaystyle\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$

Soln

$$I = \int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{(x^8 - x^{14})}{(1+x)^{24}}\,dx$$

$$= \int_0^\infty \frac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \frac{x^{14}}{(1+x)^{24}}\,dx$$

We know,

$$\beta(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$I = \int_0^\infty \frac{x^{9-1}}{(1+x)^{15+9}}\,dx - \int_0^\infty \frac{x^{15-1}}{(1+x)^{9+15}}\,dx$$

$= \beta(15,9) - \cancel{\beta(9,15)} - \beta(15,9) - \beta(9,15)$

$= 0 \quad (\because \beta(m,n) = \beta(n,m))$

Que Prove that,

$$\beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$$

Soln RHS :-

$\beta(m+1,n) + \beta(m,n+1)$

$$= \frac{\Gamma(m+1)\Gamma n}{\Gamma(m+1+n)} + \frac{\Gamma m\,\Gamma(n+1)}{\Gamma(m+n+1)} \quad \left(\because \beta(p,q) = \frac{\Gamma p\,\Gamma q}{\Gamma(p+q)}\right)$$

$$= \frac{m\Gamma m\cdot\Gamma n}{(m+n)\Gamma(m+n)} + \frac{\Gamma m\; n\Gamma n}{(m+n)\Gamma(m+n)}$$

$$= \frac{\Gamma m\,\Gamma n}{\cancel{(m+n)}\Gamma(m+n)}\cancel{(m+n)} = \frac{\Gamma m\cdot\Gamma n}{\Gamma(m+n)} = \beta(m,n) = \text{RHS.}$$

Hence proved.

Que Evaluate $\int_0^\infty e^{-\sqrt{x}}\, x^{1/4}\, dx$.

Soln Let $I = \int_0^\infty e^{-\sqrt{x}}\, x^{1/4}\, dx$ —— (1)

$\left(\text{we know } \Gamma n = \int_0^\infty e^{-x} x^{n-1} dx\right)$

Suppose $\sqrt{x} = t$ —— (2)

or $x^{1/2} = t$

$\boxed{x = t^2}$

On differentiating w.r.t. t

$dx/dt = 2t$

$\boxed{dx = 2t\,dt}$

when $x=0$, $t = \sqrt{0} = 0$

when $x=\infty$, $t = \sqrt{\infty} = \infty$

$$I = \int_0^\infty e^{-t}\,(t^2)^{1/4}\, 2t\, dt.$$

$$= 2\int_0^\infty e^{-t}\, t^{1/2+1}\, dt$$

$$= 2\int_0^\infty e^{-t}\, t^{(3/2+1)-1}\, dt$$

$$= 2\int_0^\infty e^{-t}\, t^{5/2-1}\, dt.$$

$$= 2\Gamma\frac{5}{2} = \cancel{2}\cdot\frac{3}{2}\cdot\frac{1}{\cancel{2}}\Gamma\frac{1}{2} = \frac{3}{2}\sqrt{\pi} \text{ Ans}$$

Que Evaluate $\int_0^1 x^5(1-x^3)^{10}\, dx$.

Soln We know,

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx.$$

Let $I = \int_0^1 x^5(1-x^3)^{10}\, dx$ —— (1)

Suppose $x^3 = t$ —— (2)

$x = t^{1/3}$

on diff.

$dx = \frac{1}{3}t^{1/3-1}\, dt.$

when $x=0$, $t = (0)^3 = 0$

when $x=1$, $t = 1^3 = 1$

$$I = \int_0^1 (t^{1/3})^5 (1-t)^{10}$$

$$= \frac{1}{3}\int_0^1 t^{5/3 + 1/3 - 1}(1-t)^{10} dt$$

$$= \frac{1}{3}\int_0^1 t^{2-1}(1-t)^{11-1} dt$$

$m = 2$, $n = 11$

$$= \frac{1}{3}\beta(2, 11)$$

$$= \frac{1}{3}\frac{\sqrt{2}\,\sqrt{11}}{\sqrt{2+11}} \Rightarrow \frac{1}{3}\frac{\lfloor 1 \, \lfloor 10}{\lfloor 12} \Rightarrow \frac{1}{3}\frac{1 \cdot 1 \cdot \cancel{\lfloor 10}}{12 \times 11 \, \cancel{\lfloor 10}}$$

$$\Rightarrow \frac{1}{3 \times 12 \times 11} = \frac{1}{396} \text{ Ans.}$$

21/2/2023

☆ **Que** Evaluate $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$

Soln Let $I = \int_0^1 \frac{1}{\sqrt{-\log x}} dx$

$$= \int_0^1 \frac{1}{\sqrt{\log x^{-1}}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{\log \frac{1}{x}}} dx = \int_0^1 \frac{1}{(\log \frac{1}{x})^{1/2}} dx$$

$$= \int_0^1 \left(\log \frac{1}{x}\right)^{-1/2} dx$$

We know $\boxed{\int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx = \sqrt{n}}$

$$I = \int_0^1 \left(\log \frac{1}{x}\right)^{\left(-\frac{1}{2}+1\right)-1} dx$$

$$= \int_0^1 \left(\log \frac{1}{x}\right)^{1/2 - 1} dx$$

$$= \sqrt{\tfrac{1}{2}} = \sqrt{\pi} \text{ (Hence proved)}$$

Que Evaluate $\int_0^2 x(8-x^3)^{1/3} dx$.

Soln Let $I = \int_0^2 x(8-x^3)^{1/3} dx$

$$= \int_0^2 x\,(8)^{1/3}\left(1 - \frac{x}{8}\right)^{1/3} dx$$

$$= \int_0^2 x\,(2^3)^{1/3}\left[1 - \left(\frac{x}{2}\right)^3\right]^{1/3} dx$$

$$= 2\int_0^2 x\left[1 - \left(\frac{x}{2}\right)^3\right]^{1/3} dx \quad \text{—— (1)}$$

we know $\left[\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx\right]$

Suppose $\left(\frac{x}{2}\right)^3 \ t$ —— (2)

$$\frac{x}{2} \ t^{1/3}$$

$$x = 2t^{1/3}$$

on differentiating.

$$dx = 2\cdot\frac{1}{3} t^{1/3-1}\, dt$$

Now limit :-

when $x = 0,\ t = 0$

$x = 2,\ t = 1$

$$I = 2\int_0^1 2t^{1/3}(1-t)^{1/3}\,\frac{2}{3}\,t^{1/3-1}\,dt.$$

$$= \frac{8}{3}\int_0^1 t^{1/3+1/3-1}(1-t)^{1/3}\,dt$$

$$= \frac{8}{3}\int_0^1 t^{2/3-1}(1-t)^{(1/3+1)-1}\,dt$$

$$= \frac{8}{3}\int_0^1 t^{2/3-1}(1-t)^{4/3-1}\,dt.$$

$$= \frac{8}{3}\,\beta\left(\frac{2}{3},\frac{4}{3}\right)$$

$$= \frac{8}{3}\,\frac{\Gamma\frac{2}{3}\,\Gamma\frac{4}{3}}{\Gamma\left(\frac{2}{3}+\frac{4}{3}\right)} \Rightarrow \frac{\frac{8}{3}\,\Gamma\frac{2}{3}\,\frac{1}{3}\Gamma\frac{1}{3}}{\Gamma\frac{6}{3} \to \Gamma 2}$$

$$= \frac{\frac{8}{9}\,\Gamma\frac{1}{3}\,\Gamma\frac{2}{3}}{1!} \Rightarrow \frac{8}{9}\,\Gamma\frac{1}{3}\cdot\Gamma\frac{2}{3} = I$$

We know that,

$$\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

put $n = \frac{1}{3}$

$$\Gamma\frac{1}{3}\,\Gamma\left(1-\frac{1}{3}\right) = \frac{\pi}{\sin\left(\frac{1}{3}\pi\right)}$$

$$\Gamma\frac{1}{3}\,\Gamma\frac{2}{3} = \frac{\pi}{\frac{\sqrt{3}}{2}}$$

$$\Gamma\frac{1}{3}\,\Gamma\frac{2}{3} = \frac{2\pi}{\sqrt{3}}$$

$$I = \frac{8}{9}\,\frac{2\pi}{\sqrt{3}} \Rightarrow \frac{16\pi}{9\sqrt{3}}$$ **Ans.**

Que Evaluate $\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}$

we know $\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

put $n = \frac{1}{4}$

$$\Gamma\frac{1}{4}\,\Gamma\left(1-\frac{1}{4}\right) = \frac{\pi}{\sin\left(\frac{1}{4}\pi\right)} \Rightarrow \Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)}$$

$$\boxed{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4} = \sqrt{2}\,\pi}$$ **Ans.**

Que Evaluate $\int_0^2 (8-x^3)^{-1/3}\,dx.$

Soln Let $I = \int_0^2 (8-x^3)^{-1/3}\,dx.$

$$= \int_0^2 (8)^{-1/3}\left(1-\frac{x^3}{8}\right)^{-1/3} dx.$$

$$= \int_0^2 (2^3)^{-\frac{1}{3}} \left[1-\left(\frac{x}{2}\right)^3\right]^{-1/3} dx.$$

$$= \frac{1}{2}\int_0^2 \left[1-\left(\frac{x}{2}\right)^3\right]^{-1/3} dx \quad \text{—(1)}$$

Suppose $\left(\frac{x}{2}\right)^3 = t$ —(2)

$$\frac{x}{2} = t^{1/3}$$

$$x = 2t^{1/3}$$

$$dx = \frac{2}{3} t^{1/3-1} dt.$$

Limit :- when $x=0,\ t=0$

when $x=2,\ t=1$

from —(1)

$$I = \frac{1}{\cancel{2}}\int_0^1 (1-t)^{-1/3} \frac{\cancel{2}}{3} t^{1/3-1} dt$$

$$I = \frac{1}{3}\int_0^1 t^{1/3-1} (1-t)^{(-1/3+1)-1} dt$$

$$= \frac{1}{3}\int_0^1 t^{1/3-1} (1-t)^{2/3-1} dt$$

$$= \frac{1}{3}\beta\left(\frac{1}{3},\frac{2}{3}\right)$$

$$= \frac{\frac{1}{3}\Gamma\frac{1}{3}\,\Gamma\frac{2}{3}}{\Gamma\left(\frac{1}{3}+\frac{2}{3}\right)} \Rightarrow \frac{\frac{1}{3}\Gamma\frac{1}{3}\,\Gamma\frac{2}{3}}{\Gamma 1} = \frac{1}{3}\Gamma\frac{1}{3}\,\Gamma\left(1-\frac{1}{3}\right)$$

$$= \frac{1}{3}\frac{\pi}{\sin\left(\frac{1}{3}\pi\right)}$$

$$= \frac{1}{3}\frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{3\sqrt{3}} \text{ Ans}$$

Que $\displaystyle\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx \times \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{4\sqrt{2}}.$

Soln $B = I_1 \times I_2$ —(1)

where $I_1 = \displaystyle\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx$, $I_2 = \displaystyle\int_0^\infty x^2 e^{-x^4} dx$

Now, $I_1 = \displaystyle\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx.$

$\because$ we know that $\Gamma n = \displaystyle\int_0^\infty e^{-x} x^{n-1} dx$

$$I_1 = \int_0^\infty e^{-x^2} x^{-1/2} dx \quad \text{—(2)}$$

Suppose $x^2 = t$ —(3)

$x = t^{1/2} \Rightarrow dx = \frac{1}{2} t^{1/2-1} dt.$

Now limit when $x=0,\ t=0$

when $x=\infty,\ t=\infty$

$$I_1 = \int_0^\infty e^{-t}\left(t^{\frac{1}{2}}\right)^{-1/2} \frac{1}{2} t^{1/2-1} dt$$

$$= \frac{1}{2}\int_0^\infty e^{-t}\, t^{-1/4+\frac{1}{2}-1} dt$$

$$I_1 = \frac{1}{2}\int_0^\infty e^{t}\, t^{1/4-1} dt$$

$$\boxed{I_1 = \frac{1}{2}\Gamma\frac{1}{4}}$$

Now, $I_2 = \int_0^\infty x^2 e^{-x^4} dx.$

Suppose $x^4 = t$

$x = t^{1/4}$

$dx = \frac{1}{4} t^{1/4-1} dt$

<u>Now limit</u> :- when $x = 0,\ t = 0$
when $x = \infty,\ t = \infty$

$$I_2 = \int_0^\infty (t^{1/4})^2 e^{-t} \frac{1}{4} t^{1/4-1} dt$$

$$= \frac{1}{4}\int_0^\infty t^{1/2+\frac{1}{4}-1} e^{-t} dt$$

$$= \frac{1}{4}\int_0^\infty e^{-t} t^{\frac{3}{4}-1} dt.$$

$$\boxed{I_2 = \frac{1}{4}\Gamma\frac{3}{4}}$$

$$I = I_1 \times I_2$$

$$I = \frac{1}{2}\Gamma\frac{1}{4} \times \frac{1}{4}\Gamma\frac{3}{4}$$

$$= \frac{1}{8}\Gamma\frac{1}{4}\Gamma\frac{3}{4} \Rightarrow \frac{1}{8}\Gamma\frac{1}{4}\Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{8}\left(\frac{\pi}{\sin(\frac{1}{4}\pi)}\right). \qquad \left[\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$I = 18\sqrt{2}\ \pi$$

$$= \frac{1}{4\sqrt{2}}\sqrt{2}\pi \Rightarrow \frac{1}{4\sqrt{2}}\pi \quad \text{Hence proved}$$

<u>Que</u> Prove that $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi$

<u>Soln</u> $I = I_1 \times I_2$ —(1)

where $I_1 = \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}}$, $I_2 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$

Now $I_1 = \int_0^{\pi/2} \frac{1}{\sqrt{\sin\theta}} d\theta$

$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cdot \cos^0\theta\, d\theta$$

we know, $\left(\int_0^{\pi/2} \sin^m\theta \cdot \cos^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2} \cdot \Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}\right)$

$$I_1 = \frac{\Gamma\frac{-\frac{1}{2}+1}{2}\ \Gamma\frac{0+1}{2}}{2\Gamma\frac{-\frac{1}{2}+0+2}{2}} \Rightarrow \frac{\Gamma\frac{1}{4}\Gamma\frac{1}{2}}{2\Gamma\frac{3}{4}}$$

$$I_2 = \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta$$

$$= \int_0^{\pi/2} \sin^{1/2}\theta\, \cos^0\theta\, d\theta$$

$m = \frac{1}{2},\ n = 0$

$$I_2 = \frac{\Gamma\frac{\frac{1}{2}+1}{2}\ \Gamma\frac{0+1}{2}}{2\Gamma\frac{\frac{1}{2}+0+2}{2}} = \frac{\Gamma\frac{3}{4}\ \Gamma\frac{1}{2}}{2\Gamma\frac{5}{4}}$$

from — (1)

$I = I_1 \times I_2$

$$I = \frac{\Gamma\frac{1}{4}\Gamma\frac{1}{2}}{2\,\Gamma\frac{3}{4}} \times \frac{\Gamma\frac{3}{4}\Gamma\frac{1}{2}}{2\,\Gamma\frac{5}{4}} = \frac{\frac{1}{4}\Gamma\frac{1}{4}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{5}{4}}$$

$$= \frac{\cancel{\frac{1}{4}}\cancel{\Gamma\frac{1}{4}}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\cancel{\frac{1}{4}}\cancel{\Gamma\frac{1}{4}}} \Rightarrow \Gamma\frac{1}{2}\,\Gamma\frac{1}{2} = \sqrt{\pi}\times\sqrt{\pi} = \pi \text{ Ans}$$

Que Show that $\int_0^3 \frac{dx}{\sqrt{3x-x^2}} = \pi$.

Soln Let $I = \int_0^3 \frac{dx}{\sqrt{3x-x^2}}$

$\because \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$$I = \int_0^3 \frac{1}{\sqrt{3x\left(1-\frac{x^2}{3x}\right)}}\,dx$$

$$= \int_0^3 \frac{1}{\sqrt{3}\sqrt{x}\sqrt{1-\frac{x}{3}}}\,dx$$

$$= \frac{1}{\sqrt{3}}\int_0^3 x^{-1/2}\cdot\left(1-x/3\right)^{-1/2}\,dx \quad \text{— (1)}$$

Suppose $\frac{x}{3} = t$

$x = 3t$

$dx = 3dt$

Now limit when $x=0,\ t=0$

$x=3,\ t=1$

$$I = \frac{1}{\sqrt{3}}\int_0^1 (3t)^{-1/2}(1-t)^{-1/2}\,3dt$$

$$= \frac{1}{\sqrt{3}}\,3\int_0^1 (3)^{-1/2}\,t^{-1/2}\,(1-t)^{-1/2}\,dt$$

$$= \frac{3}{\sqrt{3}}\cdot\frac{1}{\sqrt{3}}\int_0^1 t^{(-1/2+1)-1}\,(1-t)^{(-1/2+1)-1}\,dt$$

$$= \int_0^1 t^{1/2-1}\,(1-t)^{1/2-1}\,dt$$

$$= \beta\left(\tfrac{1}{2},\tfrac{1}{2}\right)$$

$$= \frac{\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} \qquad \left[\because \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right.$$

$$= \frac{\sqrt{\pi}\,\sqrt{\pi}}{\Gamma 1} = \frac{\pi}{1} = \pi \text{ Ans}$$

Que Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^n}}$ — (1)

Soln $\int_0^1 (1-x^n)^{-1/2}\,dx$

Let $x^n = t$ — (2)

$x = t^{1/n}$

$dx = \frac{1}{n}\,t^{1/n-1}\,dt$

Limit → when $x=0,\ t=0$

$x=1,\ t=(1)^n=1$

from eq — (1)

$$I = \int_0^1 (1-t)^{-1/2}\,\frac{1}{n}\,t^{1/n-1}\,dt$$

$$= \frac{1}{n}\int_0^1 t^{1/n-1}(1-t)^{-1/2}\,dt$$

$$= \frac{1}{n}\int_0^1 t^{1/n-1}(1-t)^{(-1/2+1)-1}\,dt.$$

$$I = \frac{1}{n}\int_0^1 t^{1/n-1}(1-t)^{1/2-1}\,dt.$$

$$m=\frac{1}{n},\quad n=\frac{1}{2}$$

$$= \frac{1}{n}\,\beta\left(\frac{1}{n},\frac{1}{2}\right) \text{ Ans.}$$

Que Prove that $\int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta = \int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta = \frac{\pi}{\sqrt{2}}$

Soln Let $I = \int_0^{\pi/2}\sqrt{\tan\theta}\,d\theta$

using property of definite integral.

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$$

$$I = \int_0^{\pi/2}\sqrt{\tan(\pi/2-\theta)}\,d\theta$$

$$= \int_0^{\pi/2}\sqrt{\cot\theta}\,d\theta \quad \text{— (i) part proved}$$

$$I = \int_0^{\pi/2}\sqrt{\frac{\cos\theta}{\sin\theta}}\,d\theta$$

$$= \int_0^{\pi/2}\sin^{-1/2}\theta\,\cos^{1/2}\theta\,d\theta$$

$$\left[\text{using } \int_0^{\pi/2}\sin^m\theta\cos^n\theta\,d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}\right]$$

$$\therefore\ m=-\frac{1}{2},\ n=\frac{1}{2}$$

$$I = \frac{\Gamma\frac{-\frac{1}{2}+1}{2}\,\Gamma\frac{\frac{1}{2}+1}{2}}{2\,\Gamma\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}} \Rightarrow \frac{\Gamma\frac{1}{4}\,\Gamma\frac{3}{4}}{2\Gamma 1}$$

$$\Rightarrow \frac{1}{2}\Gamma\frac{1}{4}\Gamma\frac{3}{4} \quad (\because \Gamma 1 = 1)$$

$$= \frac{1}{2}\Gamma\frac{1}{4}\Gamma\left(1-\frac{1}{4}\right) \Rightarrow \frac{1}{2}\frac{\pi}{(\sin\frac{1}{4}\pi)} \quad \left[\Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right]$$

$$= \frac{1}{2}\frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)} \Rightarrow \frac{1}{2}(\sqrt{2}\,\pi) = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}\cdot\sqrt{2}\,\pi$$

$$\Rightarrow \frac{\pi}{\sqrt{2}} \text{ Ans}$$

Que Find $\Gamma .1\ \Gamma .2\ \Gamma .3\ \Gamma .4 \cdots \Gamma .9$

Soln $\Gamma .1\ \Gamma .2\ \Gamma .3\ \Gamma .4 \cdots \Gamma .9$

$$= \Gamma\frac{1}{10}\Gamma\frac{2}{10}\Gamma\frac{3}{10}\cdots\Gamma\frac{9}{10}$$

$$= \Gamma\frac{1}{10}\Gamma\frac{2}{10}\Gamma\frac{3}{10}\cdots\Gamma\frac{10-1}{10}$$

we know $\Gamma\frac{1}{n}\Gamma\frac{2}{n}\Gamma\frac{3}{n}\cdots\Gamma\frac{n-1}{n} = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$

$$\Gamma .1\ \Gamma .2\ \Gamma .3\cdots\Gamma .9 = \frac{(2\pi)^{\frac{10-1}{2}}}{\sqrt{10}}$$

$$= \frac{(2\pi)^{9/2}}{\sqrt{10}} \text{ Ans.}$$

☆ **Que** Prove that $\int_0^{\infty}(x-a)^m(b-x)^n\,dx = (b-a)^{m+n+1}\cdot\beta(m+1,n+1)$

Soln Let $I = \int_0^{\infty}(x-a)^m(b-x)^n\,dx$ — (1)

put $x = a+(b-a)z$.

$\because (x-a) = (b-a)z$

$(b-x) = b-[a-(b-a)z]$

$= b-a+(b-a)z$

$(b-x) = (b-a)(1+z)$

$\because x = a+(b+a)z$ — (2)

$dx = 0+(b-a)dz$

$dx = (b-a)dz$

when $x=0$, $0 = a+(b-a)z$

Soln $\int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx$, $\int_0^\infty \frac{1}{1+x^4}\,dx$

Soln $I = \int_0^\infty \frac{1}{1+x^4}\,dx$ — (1)

$\because \beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$

Let $x^4 = t$ — (2)

$x = t^{1/4}$

$dx = \frac{1}{4} t^{1/4-1}\,dt$

Now Limit, when $x=0$, $t=0$

$x=\infty$, $t=\infty$

$I = \int_0^\infty \frac{1}{(1+t)} \frac{1}{4} t^{1/4-1}\,dt.$

$= \frac{1}{4}\int_0^\infty \frac{t^{1/4-1}}{(1+t)^{3/4+\frac{1}{4}}}\,dt$

$= \frac{1}{4}\beta\left(\frac{3}{4},\frac{1}{4}\right)$

$= \frac{\frac{1}{4}\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{\Gamma\left(\frac{3}{4}+\frac{1}{4}\right)}$ $\left(\because \beta(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}\right.$

$= \frac{\frac{1}{4}\Gamma\frac{1}{4}\Gamma 1-\frac{1}{4}}{\Gamma 1} = \frac{1}{4}\frac{\pi}{(\sin\frac{1}{4}\pi)} = \frac{1}{4}\frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)} = \frac{1}{4}\sqrt{2}\pi$

$= \frac{\pi}{2\sqrt{2}}$ Ans

Que. evaluate $\int_0^1 \frac{1}{\sqrt{1+x^4}}\, dx.$

Soln Let $I = \int_0^1 \frac{1}{\sqrt{1+x^4}}\, dx$

put $x^2 = \tan\theta$

$x = \tan^{1/2}\theta$

$dx = \left(\frac{1}{2}\tan^{(1/2-1)}\theta\right) \cdot \sec^2\theta\, d\theta.$

when $x = 0,\ 0 = \tan\theta \Rightarrow \tan\theta = \tan 0^\circ \Rightarrow \theta = 0^\circ$

$x = 1,\ 1 = \tan\theta \Rightarrow \tan\theta = \tan\frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{4}$

$$I = \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \frac{1}{2}(\tan^{(1/2-1)}\theta) \cdot \sec^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\cancel{\sec\theta}} \cdot \frac{1}{\sqrt{\tan\theta}} \cdot \sec^{\cancel{2}}\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{\sec\theta}{\sqrt{\tan\theta}}\, d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/4} \frac{\sqrt{\cos\theta}}{\sqrt{\sin\theta}} \cdot \frac{1}{\cos\theta}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{\sin\theta}\sqrt{\cos\theta}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\sqrt{\frac{2}{2}\sin\theta\cos\theta}}\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{1}{\frac{1}{\sqrt{2}}\sqrt{\sin 2\theta}}\, d\theta$$

$$I = \frac{\sqrt{2}}{2}\int_0^{\pi/4} \frac{1}{\sqrt{\sin 2\theta}}\, d\theta$$

Let $2\theta = t$

$2\,d\theta = dt$

$d\theta = \frac{dt}{2}$

Now limit $\theta = 0,\ t = 0$

$\theta = \frac{\pi}{4},\ t = \pi/2.$

$$I = \frac{1}{\sqrt{2}}\int_0^{\pi/2} \frac{1}{\sqrt{\sin t}} \cdot \frac{dt}{2}$$

$$\Rightarrow \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2} t \cdot \cos^0 t\, dt.$$

$$\left[\text{using } \int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\,\Gamma\frac{n+1}{2}}{2\,\Gamma\frac{m+n+2}{2}}\right]$$

$$= \frac{1}{2\sqrt{2}} \frac{\Gamma\frac{-\frac{1}{2}+1}{2}\,\Gamma\frac{0+1}{2}}{2\,\Gamma\frac{-\frac{1}{2}+0+2}{2}} = \frac{1}{2\sqrt{2}} \cdot \frac{\frac{1}{2}\Gamma\frac{1}{4}\,\Gamma\frac{1}{2}}{\Gamma\frac{3}{4}}$$

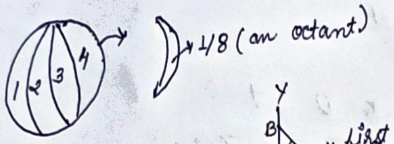
$$I = \frac{1}{4\sqrt{2}}\sqrt{\pi}\,\frac{\Gamma\frac{1}{4}}{\Gamma\frac{3}{4}}\ \text{Ans.}$$

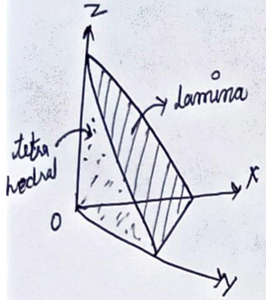
$$= \because \Gamma\tfrac{1}{4}\,\Gamma\tfrac{3}{4} = \frac{\pi}{\sin\frac{\pi}{4}} = \sqrt{2}\pi$$

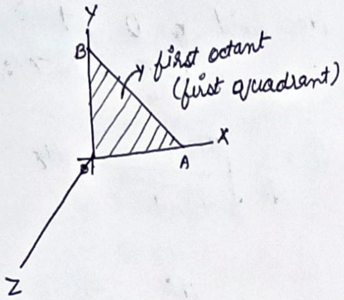
22/2/2023

Dirichlets Theorem :-

An Octant $= \frac{1}{8}$ part of a Solid.







Statement :- To the find the volume of an octant, $x \geq 0, y \geq 0, z \geq 0$ and $x+y+z \leq 1$

$$\boxed{\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}}$$

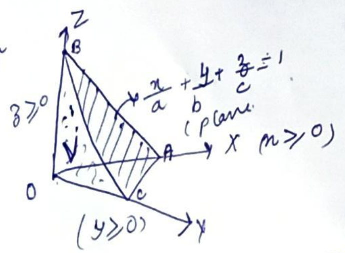
This is called Dirichlets Theorem.

By this theorem we find volume of a solid in octant i.e first quadrant.

where, l, m, n are +ve no.

Que find the volume of tetrahedral bounded by the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and coordinate planes.

Soln



Volume of tetrahedral OABCO bonded by coordinate planes & given surfaces is

$$V = \iiint dx\, dy\, dz$$

Given surface $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

Consider, $\frac{x}{a} = u$ | $\frac{y}{b} = v$ | $\frac{z}{c} = w$

$x = au$ | $y = bv$ | $z = wc$

$dx = a\,du$ | $dy = b\,dv$ | $dz = c\,dw$

$$V = \iiint a\,du\; b\,du\; c\,dw$$

$$= abc \iiint du\, dv\, dw$$

By Dirichlet's theorem,

coordinate plane $\begin{cases} x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{cases}$ | $x+y+z \geq 1$ is surface

$$\therefore \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}$$

$$V = abc \iiint u^{1-1} v^{1-1} w^{1-1}\, du\, dv\, dw$$

Subject to condition $u \geq 0, v \geq 0, w \geq 0, u+v+w=1$

$$= \frac{abc\, \Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)} = \frac{abc\, \Gamma 1\, \Gamma 1\, \Gamma 1}{\Gamma(1+1+1+1)} = \frac{abc \cdot 1 \cdot 1 \cdot 1}{\Gamma 4}$$

$$= \frac{abc}{\lfloor 3} = \frac{abc}{6}\ \text{Ans.}$$

Que Find the volume of solid ~~ssr~~ surrounded by solid

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$$

Soln Volume of solid of an octant , $V = \iiint dx\,dy\,dz$.

By Dirichlet's theorem

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

Subject to condition

$x \geq 0,\ y \geq 0,\ z \geq 0$ and $\boxed{x+y+z \leq 1}$

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$$

$\left(\frac{x}{a}\right)^{2/3} = u$	$\left(\frac{y}{b}\right)^{2/3} = v$	$\left(\frac{z}{c}\right)^{2/3} = w$
$\frac{x}{a} = u^{3/2}$	$\frac{y}{b} = v^{3/2}$	$\frac{z}{c} = w^{3/2}$
$x = a u^{3/2}$	$y = b \cdot v^{3/2}$	$z = c w^{3/2}$
$dx = a\frac{3}{2} u^{\frac{3}{2}-1} du$	$dy = b\frac{3}{2} v^{3/2-1}$	$dz = c\frac{3}{2} w^{3/2-1}\, dw$.

$$V = \iiint \frac{3}{2}a\,\frac{3}{2}b\,\frac{3}{2}c\; u^{3/2-1}\, v^{3/2-1}\, w^{3/2-1}\, du\,dv\,dw.$$

$$= \frac{27}{8} abc \iiint u^{3/2-1}\, v^{3/2-1}\, w^{3/2-1}\, du\,dv\,dw$$

Subject to cond^n → $u \geq 0,\ v \geq 0,\ w \geq 0$ & $u+v+w \leq 1$

$$= \frac{27}{8} abc \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

$$\Rightarrow \frac{27}{8} abc \frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma\left(\frac{3}{2}+\frac{3}{2}+\frac{3}{2}+1\right)}$$

$$= \frac{27}{8} abc \frac{\frac{1}{2}\Gamma\frac{1}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2} \cdot \frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{11}{2}} \Rightarrow \frac{\frac{27}{8} abc \frac{1}{8} \Gamma\frac{1}{2}\,\Gamma\frac{1}{2}\,\Gamma\frac{1}{2}}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}$$

$$V = \frac{27}{8 \times 8} abc \frac{\sqrt{\pi}\cdot\sqrt{\pi}\; 32}{9 \times 7 \times 5 \times 3 \times 1}$$

$$= \frac{abc}{70}\pi$$

This is volume of solid of in an octant.

∴ volume of solid surrounded by given surface

$$V = 8 \times \frac{abc}{70}\pi = \frac{4abc}{35}\pi \text{ Ans}$$

Que Find the mass of an octant of an ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$

The density at any pt $\rho = kxyz$.

Soln The mass of solid of an octant.

$m = \rho \times V$ (density × volume)

$= \rho \iiint dx\,dy\,dz.$

$= \iiint \rho \cdot dx\,dy\,dz.$

mass $= \iiint kxyz\, dx\,dy\,dz.$

given surface $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$

Let

$\left(\frac{x}{a}\right)^2 = u$	$\left(\frac{y}{b}\right)^2 = v$	$\left(\frac{z}{c}\right)^2 = w$
$\frac{x}{a} = u^{1/2}$	$y = b v^{1/2}$	$w = c\, w^{1/2}$
$x = a u^{1/2}$	$du = b\frac{1}{2} v^{1/2-1} dv$	$dw = c\frac{1}{2} v^{1/2-1} dw$
$dx = a\frac{1}{2} u^{1/2-1} du$		

Que Find the volume of ...

mass $= k \iiint a u^{1/2} \cdot b v^{1/2}\, c w^{1/2} \frac{a}{2} u^{1/2-1} du\, \frac{b}{2} v^{1/2-1} \frac{c}{2} w^{1/2} dw$

$= \frac{k\, a^2 b^2 c^2}{8} \iiint u^{1-1} v^{1-1} w^{1-1}\, du\, dv\, dw.$

Subject to condⁿ $= u \geq 0, v \geq 0, w \geq 0$ & $u+v+w \leq 1$

By Dirichlet theorem,

$\iiint u^{l-1} v^{m-1} w^{n-1}\, du\,dv\,dw = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}$

$= \frac{k a^2 b^2 c^2}{8} \frac{\Gamma 1\, \Gamma 1\, \Gamma 1}{\Gamma(1+1+1+1)} = \frac{k a^2 b^2 c^2}{8} \frac{1 \cdot 1 \cdot 1}{\Gamma 4}$

$\Rightarrow \frac{k a^2 b^2 c^2}{8 \lfloor 3} = \frac{k a^2 b^2 c^2}{48}$ Ans

Que Evaluate $\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz$ where, x, y, z are all positive but limited by the condⁿ, $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r \leq 1$

Soln Given condⁿ $\rightarrow \left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r$

Suppose.

$\left(\frac{x}{a}\right)^p = u$	$\left(\frac{y}{b}\right)^q = v$	$\left(\frac{z}{c}\right)^r = w$
$\frac{x}{a} = u^{1/p}$	$y = b v^{1/q}$	$z = c w^{1/r}$
$x = a u^{1/p}$	$dy = \frac{b}{q} v^{1/q-1} dv$	$dz = \frac{c}{r} w^{1/r-1} dw$
$dx = \frac{a}{p} u^{1/p-1} du$		

$I = \iiint (a u^{1/p})^{l-1} (b v^{1/q})^{m-1} (c w^{1/r})^{n-1} \frac{a}{p} u^{1/p-1} du.$

$\frac{b}{q} v^{1/q-1} dv.\, \frac{c}{r} w^{1/r-1} dw.$

$= \frac{a^l b^m c^n}{pqr} \iiint u^{\frac{l-1}{p}+\frac{1}{p}-1} v^{\frac{m-1}{q}+\frac{1}{q}-1} w^{\frac{n-1}{r}+\frac{1}{r}-1}\, du\,dv\,dw$

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$= \frac{a^l b^m c^n}{pqr} \iiint u^{\frac{l}{p}-1} v^{\frac{m}{q}-1} w^{\frac{n}{r}-1}\, du\,dv\,dw$

Subject to condⁿ $\rightarrow u \geq 0, v \geq 0, w \geq 0$ & $u+v+w \leq 1$

By Dirichlet's theorem,

$\frac{a^l b^m c^n}{pqr} \iiint x \quad \frac{a^l b^m c^n}{pqr} \frac{\Gamma \frac{l}{p}\, \Gamma \frac{m}{q}\, \Gamma \frac{n}{r}}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)}$ Ans.

Que Find the volume of the solid bounded by coordinate planes & surface $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1.$

Soln The volume of solid bounded by coordinate plane & given surface $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1$

is given by $V = \iiint dx\,dy\,dz.$

Subject to condition.

$x \geq 0, y \geq 0, z \geq 0$ & $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1.$

By Dirichlet integral theorem,

we know that,

$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\,dy\,dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n+1)}$

where $x \geq 0, y \geq 0, z \geq 0,$ & $x+y+z \leq 1.$

Suppose

$\sqrt{\frac{x}{a}} = u$	$\sqrt{\frac{y}{b}} = v$	$\sqrt{\frac{z}{c}} = w$
$\left(\frac{x}{a}\right)^{1/2} = u$	$y = b v^2$	$z = c w^2$
$\frac{x}{a} = u^2$	$dy = 2bv^{2-1} dv$	$dz = 2cw^{2-1} dw.$
$x = a u^2$	$= 2bv\,dv$	$2c\,w\,dw$
$dx = 2au^{2-1} du$		
$= 2au\,du$		

$V = \iiint 2au\, 2bv\, 2cw\, du\, dv\, dw.$

$8abc \iiint u\ v\ w\ du\, dv\, dw.$

$V = 8abc \iiint u^{2-1} v^{2-1} w^{2-1}\, du\, dv\, dw.$

where $u \geq 0,\ v \geq 0,\ w \geq 0$ & $u+v+w = 1.$

$$V = \frac{8abc\, \Gamma 2\, \Gamma 2\, \Gamma 2}{\Gamma 2+2+2+1}$$

$$V = \frac{8abc\, \Gamma 2\, \Gamma 2\, \Gamma 2}{\Gamma 7}$$

$$V = \frac{\cancel{8}abc\, \cancel{1}.\cancel{1}.\cancel{1}}{\cancel{6}.6.5.\cancel{4}.3.\cancel{2}.1} \Rightarrow \frac{8abc}{\cancel{720}} = \frac{4abc}{105}\ \text{Ans}$$

$$\Rightarrow \frac{abc}{90}\ \text{Ans}$$

→ Liouville's Extension of Dirichlet's theorem

Statement :- If the variable x, y, z are all +ve i.e $x \geq 0$, $y \geq 0$, $z \geq 0$ such that $h_1 < x+y+z < h_2$ then the triple integral of $f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}$

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma l+m+n} \times$$

$$\int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du, \text{ where } u = x+y+z.$$

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Que Show that $\iiint \frac{dx\, dy\, dz}{(x+y+z+1)^3} = \frac{1}{2} \log 2 - \frac{5}{16}$, the integral being taken through out the volume bounded by the planes $x=0,\ y=0,\ \&\ z=0$ & $x+y+z=1$.

Not related to each other.

Soln Let $I = \iiint \frac{1}{(x+y+z+1)^2}\, dx\, dy\, dz.$

Subject to condn,

$\boxed{x=0,\ y=0,\ z=0}$ & $\boxed{x+y+z=1}$

here, $h_1 = 0,\ h_2 = 1$

$\therefore\ 0 \leq x+y+z \leq 1$

By Louville's theorem, we know that.

$$\iiint f(x+y+z)\, x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma l+m+n} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du$$

where $u = x+y+z.$

and $x \geq 0,\ y \geq 0,\ z \geq 0$ & $x+y+z \leq 1.$

$$\therefore I = \iiint \frac{1}{(x+y+z+1)^3}\, x^{1-1} y^{1-1} z^{1-1}\, dx\, dy\, dz.$$

$$= \frac{\Gamma 1\, \Gamma 1\, \Gamma 1}{\Gamma 1+1+1} \int_0^1 \cancel{f}\frac{1}{(x+y+z+1)^3} \cdot \left[\frac{1}{(x+y+z+1)^3}\right]^2 du.$$

$$= \frac{1}{\Gamma 2} \int_0^1 \frac{1}{(x+y+z+1)^3} \cdot \left[\frac{1}{(x+y+z+1)^{+3}}\right]^2 du.$$

$$= \frac{1}{2} \int_0^1 \frac{1}{(u+1)^3}\, u^2\, du.$$

$$= \frac{1}{2}\int_0^1 \frac{u^2}{(1+u)^3}\, du$$

put $(1+u) = t$

$u = t-1$

$\boxed{du = dt}$

Now limit, when $u=0$, $t=+1$

$u=1$, $t=2$

$$I = \frac{1}{2}\int_1^2 \frac{(t-1)^2}{(1+t-1)^3}\, dt$$

$$= \frac{1}{2}\int_1^2 \frac{(t-1)^2}{(t)^3}\, dt$$

$$= \frac{1}{2}\int_1^2 \frac{t^2+1-2t}{t^3}\, dt$$

$$= \frac{1}{2}\int_1^2 \left(\frac{1}{t} + t^{-3} - 2t^{-2}\right) dt.$$

$$= \frac{1}{2}\int_1^2 \frac{1}{t}\, dt + \int_1^2 t^{-3} dt - 2\int_1^2 t^{-2} dt$$

$$= \frac{1}{2}\left[\log t \Big|_1^2 + \left(\frac{t^{-2}}{-2}\right) + \left(\frac{-2t^{-1}}{-1}\right)\Big|_1 \right]^2$$

$$= \frac{1}{2}\left[\log t - \frac{t^{-2}}{2} + 2t^{-1}\right]_1^2$$

$$= \frac{1}{2}\left[\log t - \frac{1}{2t^2} + \frac{2}{t}\right]_1^2$$

$$\frac{1}{2}\left[\left(\log 2 - \frac{1}{2(2)^2} + \frac{2}{2}\right) - \left(\underset{0}{\log 1} - \frac{1}{2} + 2\right)\right]$$

$$= \frac{1}{2}\left[\log 2 + \frac{1}{8} + 1 + \frac{1}{2} - 2\right]$$

$$= \frac{1}{2}\left[\log 2 - \frac{1}{8} + \frac{1}{2} - 1\right]$$

$$= \frac{1}{2}\left[\log 2 - \left(\frac{1}{8} - \frac{1}{2} + 1\right)\right]$$

$$= \frac{1}{2}\left[\log 2 - \left(\frac{1-4+8}{8}\right)\right]$$

$$= \frac{1}{2}\left[\log 2 - \frac{5}{8}\right] \text{ Ans.}$$

Que Evaluate $\iiint xyz \sin(x+y+z)\, dx\, dy\, dz$. the integral being extended to all +ve values of the variables. subject to cond^n $x+y+z \le \pi/2$.

Soln Given . $0 \le x+y+z \le \pi/2$

$h_1 = 0, \quad h_2 = \frac{\pi}{2}$

$$I = \iiint \sin(x+y+z)\, x^{2-1} y^{2-1} z^{2-1}\, dx\, dy\, dz.$$

Subject to cond^n, $x \ge 0$, $y \ge 0$, $z \ge 0$ & $x+y+z \le \pi/2$

By Liouville's theorem,

$$I = \frac{\Gamma l\, \Gamma m\, \Gamma n}{\Gamma(l+m+n)} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du.$$

$$= \frac{\Gamma 2\, \Gamma 2\, \Gamma 2}{\Gamma 6} \int_0^{\pi/2} \sin u\, (u)^5\, du$$

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$$\frac{\lfloor 1 \lfloor 1 \lfloor 1}{\lfloor 5} \int_0^{\pi/2} \underset{I}{u^5} \underset{II}{\sin u}\, du$$

$$= \frac{1}{120} \int \Big[u^5(-\cos u) - (5u^4)(-\sin u) + 20u^3(\cos u) - (60u^2)(\sin u) + 120u(-\cos u) - (120)(\sin) + 0 \Big]_0^{\pi/2}$$

$$= \frac{1}{120} \Big[-u^5\cos u + 5u^4 \sin u + 20u^3 \cos u - 60u^2 \sin u - 120u\cos u + 120 \sin u \Big]_0^{\pi/2}$$

$$= \frac{1}{120} \Big[0 + 5\left(\frac{\pi}{2}\right)^4 + 0 - \frac{60}{1}\left(\frac{\pi}{2}\right)^2 - 0 + 120 - 0 \Big]$$

$$= \frac{1}{120} \Big[\frac{5}{16}\pi^4 - 15\pi^2 + 120 \Big] \text{ Ans.}$$

Que Find the value of $\iiint \log(x+y+z)\, dx\, dy\, dz$. The integral extending for all +ve & 0 values of x, y, z. Subject to condⁿ $x+y+z < 1$

Soln Acc. to given question the integral is extending for all +ve and 0 values of variable such that $x+y+z < 1$

$$0 \leq x+y+z < 1$$
$$h_1 = 0 \qquad h_2 = 1$$

Given integral can be written as:-

$$I = \iiint \log(x+y+z)\, x^{1-1} y^{1-1} z^{1-1}\, dx\, dy\, dz.$$

By Liouvilles theorem.

$$I = \frac{\sqrt{l}\sqrt{m}\sqrt{n}}{\sqrt{l+m+n}} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du.$$

$$I = \frac{\sqrt{1}\sqrt{1}\sqrt{1}}{\sqrt{1+1+1}} \int_0^1 \log(u)\, u^{1+1+1-1}\, du$$

$$= \frac{1 \cdot 1 \cdot 1}{\lfloor 3} \int_0^1 \underset{I}{\log u}\ \underset{II}{u^2}\, du.$$

$$\Rightarrow \frac{1}{2} \Big[\left\{ (\log u) \int u^2 du \right\}_0^1 - \int_0^1 \left\{ \left(\frac{d}{du} \log u \right) \int u^2 du \right\} du \Big]$$

$$\Rightarrow \frac{1}{2} \Big[\left\{ (\log u) \cdot \frac{u^3}{3} \right\}_0^1 - \int_0^1 \left(\frac{1}{u} \cdot \frac{u^3}{3} \right) du \Big]$$

$$= \frac{1}{2} \Big[\left((\log 1)\frac{1}{3} - 0 \right) - \frac{1}{3} \int_0^1 u^2 du \Big]$$

$$= \frac{1}{2} \Big[-\frac{1}{3} \left(\frac{u^3}{3} \right) \Big]_0^1 = \frac{-1}{18}(1-0) = \frac{-1}{18} \text{ Ans.}$$

Que Evaluate $\iiint \frac{dx\, dy\, dz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$ for all +ve values of the variables for which the expression is real.

Que Acc. to given question, expression is real when $a^2 - x^2 - y^2 - z^2 > 0$

$$a^2 - (x^2 + y^2 + z^2) > 0$$
$$a^2 > x^2 + y^2 + z^2$$
$$x^2 + y^2 + z^2 < a^2$$
$$\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2} < 1$$

Suppose $\frac{x^2}{a^2} = u \qquad \left(\frac{x}{a}\right)^2 = u.$

$\frac{x}{a} = u^{1/2} \Rightarrow x = au^{1/2}$

$$\boxed{dx = \frac{a}{2} u^{1/2 - 1} du}$$

$\frac{y^2}{a^2} = v$

$\left(\frac{y}{a}\right)^2 = v$

$\left(\frac{y}{a}\right) = v^{1/2}$

$y = a v^{1/2}$

$$\boxed{dy = \frac{a}{2} v^{1/2 - 1} dv}$$

$\frac{z^2}{a^2} = w$

$\left(\frac{z}{a}\right)^2 = w$

$\left(\frac{z}{a}\right) = w^{1/2}$

$z = a w^{1/2}$

$$\boxed{dz = \frac{a}{2} w^{1/2 - 1} dw}$$

Now, the given integral can be written as,

$$I = \iiint \frac{1}{\sqrt{a^2\left[1 - \left(\frac{x^2}{a^2} + \frac{y^2}{a^2} + \frac{z^2}{a^2}\right)\right]}} dx\, dy\, dz.$$

$$= \frac{1}{a} \iiint \frac{1}{\sqrt{1 - (u+v+w)}} a/2\, u^{1/2 - 1} du \cdot \frac{a}{2} v^{1/2 - 1} dv \cdot \frac{a}{2} w^{1/2 - 1} dw$$

$$= \frac{1}{a} \cdot \frac{a^3}{8} \iiint \frac{1}{\sqrt{1 - (u+v+w)}} u^{1/2 - 1} v^{1/2 - 1} w^{1/2 - 1} du\, dv\, dw.$$

This is Liouville's integral.

Double Integral

$$\iint_R f(x,y)\,dx\,dy$$

If we integrate w.r.t x, y will be constant & if we integrate w.r.t y, x will be constant.

Que Evaluate $\displaystyle\int_{y=1}^{\log 8}\int_{x=0}^{\log y} e^{x+y}\,dx\,dy$.

Soln $\displaystyle\int_{y=1}^{\log y}\left(\int_{x=0}^{\log y} e^{x}\cdot e^{y}\,dx\right)dy$

$$= \int_{y=1}^{\log 8} e^{y}\left(\int_{x=0}^{\log y} e^{x}\,dx\right)dy$$

$$= \int_{y=1}^{\log 8} e^{y}\,(e^{x})_0^{\log y}\,dy$$

$$= \int_{y=1}^{\log 8} e^{y}\left[e^{\log y} - e^{0}\right]dy$$

$$= \int_{y=1}^{\log 8} e^{y}(y-1)\,dy$$

$$= \int_{y=1}^{\log 8} \underset{I}{(y-1)}\,\underset{II}{e^{y}}\,dy$$

$$= \left[(y-1)(e^{y}) - 1(e^{y}) + 0\right]_{y=1}^{\log 8}$$

$$= \left[(y-1-1)\,e^{y}\right]_{y=1}^{\log 8}$$

$$= \left[(y-2)e^{y}\right]_{y=1}^{\log 8}$$

$$\left[\log 8 - 2\, e^{\log 8} - (1-2)\,e\right] \Rightarrow \left[(\log 8 - 2)8 + e\right] \text{Ans.}$$

Que Evaluate $\displaystyle\int_0^1\int_0^1 \frac{1}{\sqrt{(1-x^2)(1-y^2)}}\,dx\,dy$

Soln Given integral can be written as,

$$I = \int_{y=0}^{1}\int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}\sqrt{1-y^2}}\,dx\,dy.$$

$$I = \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}}\left(\int_{x=0}^{1} \frac{1}{\sqrt{1-x^2}}\,dx\right)dy.$$

$$= \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}}\,(\sin^{-1}x)_0^1\,dy.$$

$$= \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}}\,(\sin^{-1}1 - \sin^{-1}0)\,dy.$$

$$= \int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}}\left(\frac{\pi}{2} - 0\right)dy$$

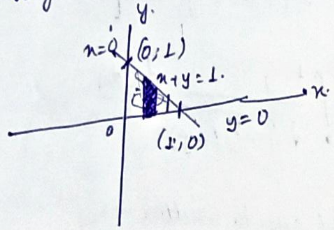
$$= \frac{\pi}{2}\int_{y=0}^{1} \frac{1}{\sqrt{1-y^2}}\,dy.$$

$$= \frac{\pi}{2}\,(\sin^{-1}y)_0^1 \Rightarrow \frac{\pi}{2}\left[\sin^{-1}1 - \sin^{-1}0\right]$$

$= \frac{\pi}{9}\left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{4}$ Ans

Que Evaluate $\iint xy\, dx\, dy$ over the region in the +ve quadrant for which $x+y \le 1$

Soln Given region:- $x \geq 0,\ y \geq 0$ & $x+y+z \le 1$



$y \longrightarrow y = 0$ to $y = 1-x$

$x \longrightarrow x = 0$ to $x = 1$

$$I = \int_{x=0}^{1}\left(\int_{y=0}^{1-x} xy\, dy\right) dx$$

$$= \int_{x=0}^{1} x\left(\int_{y=0}^{1-x} y\, dy\right) dx$$

$$= \int_{x=0}^{1} \left(\frac{y^2}{2}\right)_{y=0}^{1-x} dx$$

$= \cancel{\frac{1}{2}\int_{x=0}^{1} x(1-0)\, dx = \frac{1}{2}\int_{x=0}^{1} x\, dx}$

$$= \frac{1}{2}\int_{x=0}^{1} \frac{x}{2}\left[(1-x)^2 - 0\right] dx \Rightarrow \frac{1}{2}\int_{x=0}^{1} x(1+x^2-2x)\, dx$$

$$= \frac{1}{2}\int_{x=0}^{1} (x + x^3 - 2x^2)\, dx \Rightarrow \frac{1}{2}\left[\frac{x^2}{2} + \frac{x^4}{4} - \frac{2x^3}{3}\right]_0^1$$

$$\Rightarrow \frac{1}{2}\left[\left(\frac{1}{2} + \frac{1}{4} - \frac{2}{3}\right) - 0\right] \Rightarrow \frac{1}{2}\left[\frac{6+3-8}{12}\right] = \frac{1}{24} \text{ Ans}$$

Que Evaluate $\iint xy\, dx\, dy$, A is the domain bounded by x axis, ordinate $x = 2a$ & $x^2 = 4ay$.

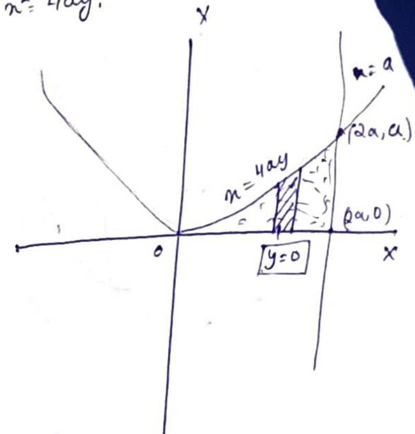
Soln Given region,

x axis $\rightarrow \boxed{y=0}$

ordinate $\boxed{x = 2a}$

Parabola $\boxed{x^2 = 4ay}$

$x = 2a$, $y = a$: $(2a, a)$ $y = 0$, $x = 0$: $(0,0)$



$y \longrightarrow y = 0$ & $y = \frac{x^2}{4a}$

$x \longrightarrow x = 0$ to $x = 2a$

$$= \int_{x=0}^{2a}\left(\int_{y=0}^{\frac{x^2}{4a}} y\, dy\right) dx \Rightarrow \int_{x=0}^{2a} x\left(\frac{y^2}{2}\right)_{y=0}^{\frac{x^2}{4a}} dx$$

$$\int_{x=0}^{2a} \frac{x}{2}\left(\left(\frac{x^2}{4a}\right)^2 - 0\right) dx$$

$$\Rightarrow \int_{x=0}^{2a} \frac{x}{2} \cdot \frac{x^4}{16a^2}\, dx$$

$$= \frac{1}{32a^2}\int_0^{2a} x^5\,dx = \frac{1}{32a^2}\left(\frac{x^6}{6}\right)_0^{2a}$$

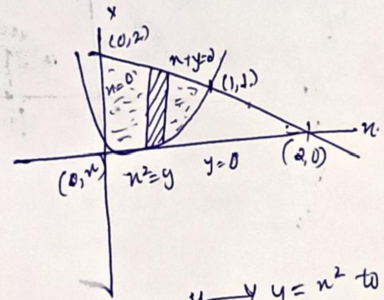
$$= \frac{1}{32a^2}\cdot\frac{1}{6}\left[(2a)^6 - 0\right]$$

$$= \frac{1}{32a^2\times 6}\cdot 64a^6 = \frac{a^4}{3}\ \text{Ans}$$

Que Evaluate $\iint y\,dx\,dy$, area bounded by $x=0$, $y=x^2$, $x+y=2$

Soln Given region,

$\boxed{x=0}$ $\boxed{y=x^2}$ $\boxed{x+y=2}$



$y \longrightarrow y = x^2$ to $y = 2-x$.
$x \longrightarrow x = 0$ to $x = 1$.

$$\therefore \iint y\,dx\,dy = \int_{x=0}^{1}\left(\int_{y=x^2}^{2-x} y\,dy\right)dx.$$

$$= \int_{x=0}^{1}\left(\frac{y^2}{2}\right)_{y=x^2}^{2-x} dx$$

$$= \int_{x=0}^{1}\frac{1}{2}\left[(2-x)^2 - (x^2)^2\right]dx.$$

$$= \frac{1}{2}\int_0^1 (4 + x^2 - 4x - x^4)\,dx$$

$$= \frac{1}{2}\left[4x + \frac{x^3}{3} - \frac{4x^2}{2} - \frac{x^5}{5}\right]_0^1$$

$$= \frac{1}{2}\left[\left(4+\frac{1}{3} - 2 - \frac{1}{5}\right) - 0\right]$$

$$= \frac{1}{2}\left[2 + \frac{1}{3} - \frac{1}{5}\right] = \frac{1}{2}\left[\frac{30+5-3}{15}\right] = \frac{16}{15}\ \text{Ans}$$

Que Evaluate $\iint xy\,dx\,dy$ when R is region bounded by $y = x^2$ & $x = y^2$.

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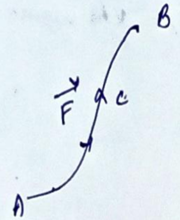
→ Line Integral :-

Any integral which is to be evaluated along a curve (a line) is called line Integral.



The line integral is denoted by $\boxed{\int_c \vec{F} \cdot \vec{dr}}$

If path is closed the line integral is denoted by.

$\boxed{\oint_c F \cdot dr}$, $\oint_c$ is called circulation.

Workdone :- By a force $\vec{F}$, whose displacement is $\vec{AB}$.

$$\boxed{W = \int_A^B \vec{F} \cdot \vec{dr}}$$

Que Suppose $\vec{F} = x^3\hat{i} + y\hat{j} + z\hat{k}$ is the force field, find workdone by vector F along the line $(1,2,3)$ to $(3,5,4)$

Soln Since Workdone $= W = \int_A^B \vec{F} \cdot dr$.

$$\int_{(1,2,3)}^{(3,5,4)} (x^3\hat{i} + y\hat{j} + zk) \cdot (\hat{i}dx + \hat{j}dy \cdot kdz)$$

$$= \int_{(1,2,3)}^{(3,5,4)} x^3 dx + y dy + z dz$$

$$= \int_{x=1}^{3} x^3 dx + \int_{y=2}^{5} y\, dy + \int_{z=3}^{4} z\, dz.$$

$$= \left(\frac{x^4}{4}\right)_1^3 + \left(\frac{y^2}{2}\right)_2^5 + \left(\frac{z^2}{2}\right)^4$$

$$= \left(\frac{81-1}{4}\right) + \left(\frac{25-4}{2}\right) + \left(\frac{16-9}{2}\right)$$

$$= \cancel{2}0 + \frac{21}{2} + \frac{\cancel{20}7}{2} \Rightarrow \frac{\cancel{101}}{4} \Rightarrow \frac{4\cancel{2}0+21+7}{2} \Rightarrow \frac{\cancel{708}}{\cancel{2}} = \cancel{354}$$

$$\frac{68}{2} = 34 \text{ Ans.}$$

Que If $\vec{F} = 3xy\hat{i} - y^2\hat{j}$. Evaluate $\int_c \vec{F} \cdot \vec{dr}$ where C is the arc of the parabola $y = 2x^2$ from $(0,0)$ to $(1,2)$

Soln $\int_c \vec{F} \cdot \vec{dr} = \int_c (3xy\hat{i} - y^2\hat{j}) \cdot (\hat{i}dx + \hat{j}dy + kdz)$

$$= \int_c (3xy)dx - y^2 dy.$$

Here c is parabola, $y = 2x^2$

$dy = 4x\, dx$

and $x \longrightarrow x = 0$ to $x = 1$

$$\because \int_C \vec{F} \cdot \vec{dr} = \int_{(0,0)}^{(1,2)} 3xy\, dx - y^2 dy.$$

$$= \int_{x=0}^{1} 3x \cdot 2x^2 dx - (2x^2)^2 4x dx.$$

$$= \int_{x=0}^{1} (6x^3 - 16x^5)\, dx.$$

$$= \left(\frac{6x^4}{4} - 16 \cdot \frac{x^6}{6} \right)_0^1$$

$$= \frac{6}{4} \; \frac{-16}{6}$$

$$\Rightarrow \frac{3}{2} - \frac{8}{3} \Rightarrow \frac{9-16}{6} \Rightarrow \frac{-7}{6} \text{ Ans.}$$

Que A vector field is given by $\vec{F}$ $(\sin y)\hat{i} + x(1 + x\cos y)\hat{j}$. Evaluate the line integral over the circular path. Given by $x^2 + y^2 = a^2, z = 0$

Que The line integral over circular path $\cdot \oint_C \vec{F} \cdot \vec{dr}$

$$= \oint \{ (\sin y)\hat{i} + x(1+\cos y)\hat{j} \} \cdot (\hat{i}dx + \hat{j}dy) \quad , z=0$$

$$= \oint_C \sin y \, dx + x(1+\cos y)\, dy.$$

$$= \oint_C \sin y \, dx + x\,dy + x\cos y\, dy.$$

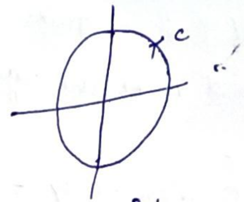
$$= \oint_C (\sin y\, dx + x\cos y\, dy) + x\,dy.$$

$$= \oint_C d(x \sin y) + \oint_C x\,dy.$$

Since, here C is circle $x^2 + y^2 = a^2$

put $y = a\sin t$
$x = a\cos t$

$t \longrightarrow t = 0$ to 2π.



$$= \int_{t=0}^{2\pi} \{ d \cdot a\cos t \sin(a \sin t) \} + \int_0^{2\pi} a\cos t \cdot (+a\cos t)\, dt.$$

$$= \left[a\cos t, \sin(a\cos t) \right]_0^{2\pi} + a^2 \int_0^{2\pi} \cos^2 t \, dt$$

$$= +a^2 \int_0^{2\pi} \left(\frac{1+\cos 2t}{2} \right) dt$$

$$= \frac{+a^2}{2} \left[t + \frac{\sin 2t}{2} \right]_0^{2\pi}$$

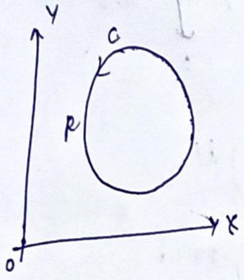
$$= \frac{-a^2}{2}(2\pi - 0) = +a^2\pi \text{ Ans.}$$

→ **Green's Theorem :-** on the plane region i.e on x, y → 2D

Statement :- If C is a regular closed curve in xy plane and R be the region bounded by C.

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$$

where M & N are continuously differentiable functions.



Que A vector field, $\vec{F} = \sin y\,\hat{i} + x(1+\cos y)\,\hat{j}$ evaluate $\oint_C \vec{F}\cdot d\vec{r}$ where C is the circular path given by $x^2 + y^2 = a^2$

Soln $\oint_C \vec{F}\cdot d\vec{r} = \oint_C \{(\sin y)\hat{i} + x(1+\cos y)\hat{j}\} \cdot \{\hat{i}\,dx + \hat{j}\,dy + k\,dz\}$

$= \oint_C \sin y\,dx + x(1+\cos y)\,dy)$ $(z=0)$

By Green theorem, we know that

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = \oint_C \underbrace{\sin y}_{M}\,dx + \underbrace{x(1+\cos y)}_{N}\,dy.$$

$$= \iint_R \left[\left(\frac{\partial\, x(1+\cos y)}{\partial x} \right) - \left(\frac{\partial}{\partial y}(\sin y) \right) \right] dx\,dy.$$

$$= \iint_R (1+\cos y) - \cos y]\,dx\,dy$$

$$= \iint_R dx\,dy = \text{Area of region } R.$$

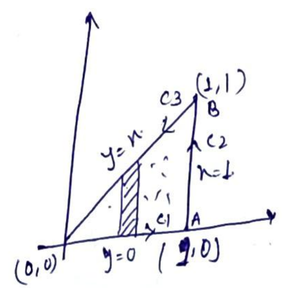
$= \text{area of Circle}$ $(x^2+y^2=a^2)$

$= \pi\,(\text{radius})^2$

$= \pi a^2$ Ans

Que Verify Green Theorem, for closed path $\oint_C x^2y\,dx + x^2\,dy$ where C is the boundary, described counter clockwise of the Δ with vertices, $(0,0)$, $(1,0)$, $(1,1)$.

Soln Given curve C is a boundary of a Δ with vertices $(0,0)$, $(1,0)$, $(1,1)$



$y \longrightarrow y=0$ to $y=x$.

$x \longrightarrow x=0$ to $x=1$.

By Green Theorem, We Know that, closed path C, $\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy$

$$\oint_C \underbrace{x^2y}_{M}\,dx + \underbrace{x^2}_{N}dy = \iint_R \left[\frac{\partial (x^2)}{\partial x} - \frac{\partial}{\partial y}(x^2y)\right] dx\,dy$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} (2x - x^2)\,dx\,dy$$

$$= \int_{x=0}^{1} (2x - x^2) \left[\int_{y=0}^{x} dy\right] dx$$

$$= \int_0^1 (2x - x^2)\,(y)_0^x\,dx$$

$$= \int_0^1 (2x - x^2)(x - 0)\,dx$$

$$= \int_0^1 (2x^2 - x^3)\,dx$$

$$= \left(2\cdot\frac{x^3}{3} - \frac{x^4}{4}\right)_0^1$$

$$= \frac{2}{3} - \frac{1}{4} = \frac{8-3}{12} = \frac{5}{12}$$

LHS :- $\oint_C x^2y\,dx + x^2\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3}$

Now (i) Path C_1

Now, (i) Path C_1 : $y = 0$, $dy = 0$, $x \longrightarrow x = 0$ to $x = 1$

$$\int_{C_1} x^2y\,dx + x^2 dy = \int_{x=0}^{1} (x^2\cdot 0\,dx + x^2\cdot 0) = 0$$

(ii) Path C_2 : $x = 1 \Longrightarrow dx = 0$, $y \longrightarrow y = 0$ to $y = 1$

$$\int_{C_2} x^2y\,dx + x^2 dy = \int_{y=0}^{1} (1\cdot y\cdot 0 + 1\,dy) = (y_0^1) = 1.$$

(iii) Path C_3 :- $y = x$; $dy = dx$; $x \longrightarrow x = 1$ to $x = 0$.

$$\int_{C_3} x^2y\,dx + x^2 dy = \int_{x=1}^{0} x^2\cdot x\,dx + x^2\,dx.$$

$$\Rightarrow \int_1^0 x^3\,dx + x^2\,dx.$$

$$= \left(\frac{x^4}{4} + \frac{x^3}{3}\right)_1^0 = 0 - \left(\frac{1}{4} - \frac{1}{3}\right) \Rightarrow \frac{-7}{12}.$$

$\therefore$ Now $\oint_C x^2y\,dx + x^2 dy \Rightarrow 0 + 1 + \left(-\frac{7}{12}\right)$

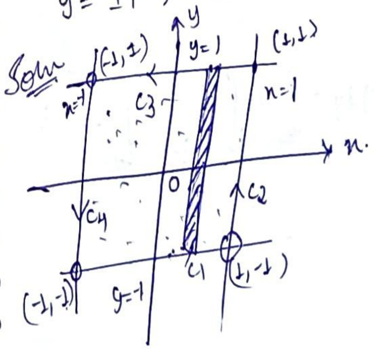
$$= \frac{5}{12}$$

LHS = RHS

Hence Green theorem is verified.

Que Verify green theorem to Evaluate $\oint_C (x^2 + xy)dx + (x^2 + y^2)dy$ where C is the square formed by the line $y = \pm 1$, $x = \pm 1$.

Soln



$y \longrightarrow y = -1$ to $y = 1$

$x \longrightarrow x = -1$ to $x = 1$

By Green theorem we know that,

$$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy.$$

$$\therefore \oint_C (x^2+xy)\,dx + (x^2+y^2\,dy)$$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(x^2+xy)\right] dx\,dy.$$

$$= \iint_R \left[(2x+0) + (0+x \cdot 1)\right] dx\,dy.$$

$$= \iint_R x\, dx\, dy.$$

$$\int_{x=-1}^{1}\int_{y=-1}^{1} x\,dx\,dy.$$

$$\Rightarrow \underbrace{\left(\int_{x=-1}^{1} x\,dx\right)}_{0}\left(\int_{y=-1}^{1} dy\right) \left[\text{using } \int_{-a}^{a} f(x)\,dx = 0 \text{ if } f(x) \text{ is odd function}\right]$$

$$= 0\left(\int_{y=-1}^{1} dy\right) = 0$$

→ **Verification :-**

$$\oint_C (x^2+xy)\,dx + (x^2+y^2)\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \quad \text{—(1)}$$

C_1 : $y=-1$	C_2 : $x=1$	C_3 : $y=1$	C_4 : $x=-1$
$dy = 0$	$dx = 0$	$\therefore dy = 0$	$dx = 0$
$x \to x=-1$ to $x=1$	$\therefore y \to y=-1$ to $y=1$	$\therefore x \to x=1$ to $x=-1$	$\therefore y \to y=1$ to $y=-1$
$\int_{C_1} (x^2+xy)dx + \underbrace{(x^2+y^2)dy}_{0}$	$\int_{C_2} \underbrace{(x^2+xy)dx}_{0} + (x^2+y^2)dy$	$\int_{C_3} (x^2+xy)dx + \underbrace{(x^2+y^2)dy}_{0}$	$\int_{C_4} \underbrace{(x^2+xy)dx}_{0} + (x^2+y^2)dy$
$= \int_{x=-1}^{1} [x^2+x(-1)]dx$	$= \int_{y=-1}^{1} (1+y^2)\,dy$	$= \int_{x=1}^{-1} (x^2+x \cdot 1)dx$	$\int_{y=1}^{-1} (1+y^2)\,dy$
$= \int_{-1}^{1} (x^2 - x)\,dx$	$= \left(y + \frac{y^3}{3}\right)_{-1}^{+1}$	$= \left(\frac{x^3}{3} + \frac{x^2}{2}\right)_1^{-1}$	$= \left(y + \frac{y^3}{3}\right)_1^{-1}$
$= \left(\frac{x^3}{3} - \frac{x^2}{2}\right)_{-1}^{1}$	$= \left(1+\frac{1}{3}\right) - \left(-1-\frac{1}{3}\right)$	$= \left(-\frac{1}{3}+\frac{1}{2}\right) - \left(\frac{1}{3}+\frac{1}{2}\right)$	$\Rightarrow \left(-1-\frac{1}{3}\right) - \left(1+\frac{1}{3}\right)$
$= \left(\frac{1}{3}-\frac{1}{2}\right) - \left(-\frac{1}{3}-\frac{1}{2}\right)$	$= 2 + \frac{2}{3} = \frac{8}{3}$	$= -\frac{2}{3}$	$= -2-\frac{2}{3} = -\frac{8}{3}$
$\Rightarrow \frac{2}{3}$			

from —(1)

$$\oint_C (x^2+xy)\,dx + (x^2+y^2\,dy)\,dy = \left(\frac{2}{3}\right) + \left(\frac{8}{3}\right) + \left(\frac{-2}{3}\right) + \left(\frac{-8}{3}\right)$$
$$= 0$$

RHS = LHS

$\therefore$ Green's theorem is verified.

→ **Area By Green Theorem :-**

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx.$$

Que Using green theorem find the area of region in first quadrant bounded by curve $y = x$, $y = \frac{1}{x}$, $y = \frac{x}{4}$.

Soln Given curve,

$y = x$ —(1) (Straight line)

$y = 1/x$ —(2) (Rectangular Hyperbola)

$y = \frac{x}{4}$ — ③ (Straight line)

Point of intersection of ① & ②

$x = \frac{1}{x}$

$x^2 = 1$

$x = \pm 1$

$x = 1, \ x = -1$

$\therefore y = x$

$y = 1, -1$

pt $(1,1)$ $(-1,-1)$

Pt of intersection of ② & ③

$\frac{1}{x} = \frac{x}{4}$

$x^2 = 4$

$x = \pm 2$

$x = 2, \ x = -2$

$\therefore y = \frac{1}{x}$

$y = \frac{1}{2}, -\frac{1}{2}$

pt $(2, \frac{1}{2})$ $(-2, -\frac{1}{2})$

point of intersection of ① & ③

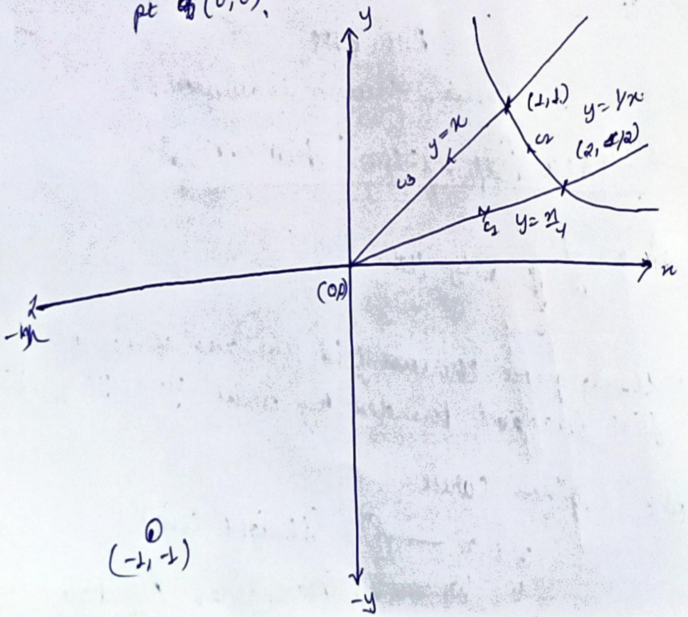
$x = \frac{x}{4}$ → $4x - x = 0$, $3x = 0$, $x = 0$

~~No point.~~

$\therefore y = x$

$y = 0$

pt of $(0,0)$.



$A = \frac{1}{2} \oint_C x\,dy - y\,dx$

$= \frac{1}{2}\left[\int_{c_1} + \int_{c_2} + \int_{c_3}\right]$ — ①

① **Along C_1:-**

$y = \frac{x}{4} \Rightarrow dy = \frac{dx}{4}$, $x \to x = 0$ to $x = 2$

$\int_{c_1} x\,dy - y\,dx = \int_{x=0}^{2} x \cdot \frac{dx}{4} - \frac{x}{4} dx$

$= 0$

② **Along C_2:-**

$y = \frac{1}{x} \Rightarrow dy = -\frac{1}{x^2} dx$, $x \to x = 2$ to $x = 1$

$\int_{c_2} x\,dy - y\,dx = \int_{x=2}^{1} x\left(-\frac{1}{x^2}\right)dx - \frac{1}{x} dx$

$= \int_{x=2}^{1} -\frac{2}{x}\, dx$

$= -2(\log x)_2^1 \Rightarrow -2[\log 1 - \log 2]$, with $\log 1 = 0$

$\Rightarrow 2\log 2$

③ **Along C_3:-**

$y = x, \ dy = dx$, $x \to x = 1$ to $x = 0$

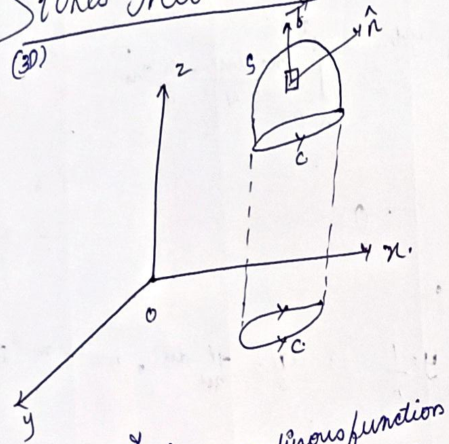
$\int_{c_3} x\,dy - y\,dx = \int_{x=1}^{0} x\,dx - x\,dx$

$= 0$

Required Area $= \frac{1}{2}\left[\oint_C x\,dy - y\,dx\right] = \frac{1}{2}\left[0 + 2\log 2 + 0\right]$

$= \log 2$ Ans.

Stokes' Theorem :- (3D)



Statement :- If $\vec{F}$ be a continous function & S is a given surface enclosed by curve C. then

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\vec{F}\cdot\hat{n}\,ds$$

where $\hat{n}$ is outward unit normal vector surface S.

Remarks :- dS = Projection of Surface S in x, y plane

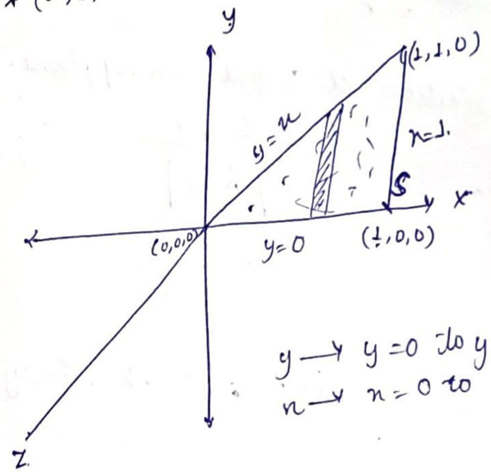
$$= \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

Projection of Surface S in yz plane $= \dfrac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$

Projection of Surface S in zx plane $= \dfrac{dx\,dz}{|\hat{n}\cdot\hat{j}|}$

Que Evaluate $\oint_C \vec{F}\cdot d\vec{r}$, $F = y^2 i + x^2 j - (x+z)k$ and C is the boundary of the $\triangle$ with vertices $(0,0,0)$, $(1,0,0)$ & $(1,1,0)$

Soln



$y \to y=0$ to $y=x$
$x \to x=0$ to $x=1$.

By Stokes Theorem, we know that,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,ds$$

$$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix}$$

$$= i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -(x+z) \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ y^2 & -(x+z) \end{vmatrix} + k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ y^2 & x^2 \end{vmatrix}$$

$$= i\left\{\frac{\partial}{\partial y}(-(x+z)) - \frac{\partial x^2}{\partial z}\right\} - j\left\{\frac{\partial}{\partial x}(-x-z) - \frac{\partial y^2}{\partial z}\right\}$$

$$+ k\left\{\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}y^2\right\}$$

$= \hat{i}(0-0) - \hat{j}(-1-0) + k(2x \ldots) = \hat{j} + 2(x-y)k$

Here $\hat{n}$ = outward unit normal vector on surface S in xy plane. $= \hat{k}$

ds = projection of surface S in xy plane.

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|}$$

$$= \frac{dx\,dy}{1}$$

$$\oint \vec{F}\cdot d\vec{r} = \iint \{\hat{j} + 2(x-y)\hat{k}\}\cdot\hat{k}\ .\ dx\,dy.$$

$$= \iint_S 2(x-y)\,dx\,dy$$

$$\Rightarrow 2\int_{x=0}^{1}\int_{y=0}^{x} (x-y)\,dx\,dy.$$

$$= 2\int_{x=0}^{1} \left(xy - \frac{y^2}{2}\right)_0^x dx$$

$$= 2\int_{x=0}^{1} \left(x^2 - \frac{x^2}{2}\right) dx$$

$$= \cancel{2}\int_{x=0}^{1} \frac{x^2}{\cancel{2}}\,dx$$

$$= \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}\ \text{Ans}$$

2/03/2023.

Ques Verify Stokes theorem for $\vec{F} = x^2\hat{i} + xy\hat{j}$ integration round the square in plane $z=0$ & bounded by line $x=0, y=0, x=a, y=a$.

Soln By Stokes theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds \quad \text{--- (1)}$$

RHS: $\text{Curl}\,\vec{F} = \nabla\times\vec{F}$

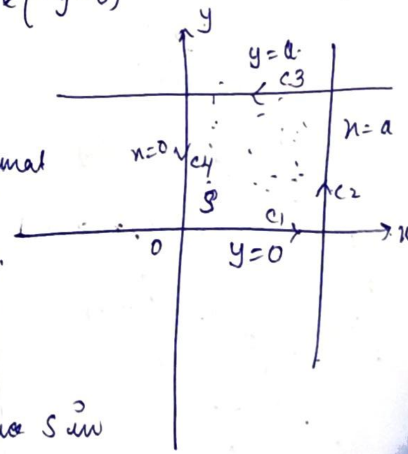
$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & -xy & 0 \end{vmatrix}$$

$$= i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -xy & 0 \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ x^2 & 0 \end{vmatrix} + k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ x^2 & -xy \end{vmatrix}$$

$$= i\left[\frac{\partial}{\partial y}\cdot 0 - \frac{\partial}{\partial z}(-xy)\right] - j\left[\frac{\partial\cdot 0}{\partial x} - \frac{\partial x^2}{\partial z}\right] + k\left[\frac{\partial}{\partial x}(-xy) - \frac{\partial x^2}{\partial y}\right]$$

$$= i(0-0) - j(0-0) + k(-y-0)$$

$$\text{curl}\,\vec{F} = -y\hat{k}.$$



Now $\hat{n}$ = outward unit normal vector on surface S in xy plane i.e along z axis,

$$\boxed{\hat{n} = \hat{k}}$$

ds = projection of surface S in xy plane.

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} = \frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} = \frac{dx\,dy}{1}$$

$$\boxed{ds = dx\,dy}$$

$$\therefore \iint_S \text{curl } \vec{F}\cdot\hat{n}\,ds = \iint_S -y\underbrace{\hat{k}\cdot\hat{k}}_{1}\cdot dx\,dy$$

$$\Rightarrow -\iint_{x=0,\,y=0}^{a\quad a} y\,dx\,dy \Rightarrow -\int_{x=0}^{a} dx \cdot \int_{y=0}^{a} y\,dy$$

$$= -(x)_0^a \left(\frac{y^2}{2}\right)_0^a$$

$$= -a\cdot\frac{a^2}{2} = \frac{-a^3}{2} \text{ Ans}$$

LHS $= \oint \vec{F}\cdot d\vec{r} = \oint (x^2\hat{i} - xy\hat{j})(\hat{i}dx + \hat{j}dy + \hat{k}dz)$

$$= \oint_C x^2dx - xy\,dy = \int_{C_1} + \int_{C_2} + \int_{C_3} + \int_{C_4} \quad \text{—(1)}$$

(i) Along C_1 :

$$\int_{C_1} x^2dx - xy\,dy$$

C_1 : $y=0$, $dy=0$; $x \to x=0$ to $x=a$

$$\int_{C_1} x^2dx - \underset{0}{xy\,dy} = \int_0^a x^2dx$$

$$= \left(\frac{x^3}{3}\right)_0^a$$

$$= \frac{a^3}{3}$$

(ii) Along C_2 :-

$$\int_{C_2} x^2dx - xy\,dy.$$

C_2 : $x = a \to dx = 0$

$y \to y = 0$ to $y = a$.

$$\int_{C_2} \underset{0}{x^2dx} - xy\,dy$$

$$= \int_{y=0}^{a} -ay\,dy$$

$$= -a\left(\frac{y^2}{2}\right)_0^a = -\frac{a^3}{2}.$$

Along C_3 :-

$$\int_{C_3} x^2dx - xy\,dy.$$

C_3 : $y = a$, $dy = 0$

$x \to x=0$ to $x=0$

$$\int_{C_3} x^2dx - \underset{0}{xy\,dy}$$

$$= \int_{x=a}^{0} x^2dx$$

$$= \left(\frac{x^3}{3}\right)_a^0$$

$$= \frac{-a^3}{3}$$

Along C_4 :-

$$\int_{C_4} x^2dx - xy\,dy$$

$C_4 = x=0$, $dx=0$

$y \to y=a$ to $y=0$

$= 0$

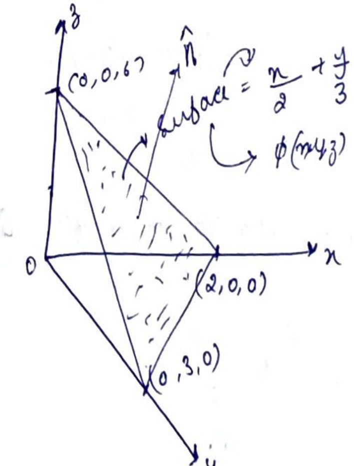
$$\therefore \oint \vec{F}\cdot d\vec{r} = \cancel{\frac{a^3}{3}} - \frac{a^3}{2} + \cancel{\left(\frac{-a^3}{3}\right)} + 0$$

$$= \frac{-a^3}{2} \text{ (RHS.)}$$

$\therefore$ Stokes theorem is verified.

Que Use stokes theorem to evaluate $\oint_C (x+2y)dx + (x-z)dy + (y-z)dz$, with vertices $(2,0,0)$, $(0,3,0)$, $(0,0,6)$ oriented Anti clockwise.

Soln



surface $= \frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

$\phi(x,y,z) = 3x + 2y + z - 6 = 0$

By Stokes theorem we know that,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds$$

Here $\vec{F} = (x+2y)\hat{i} + (x-z)\hat{j} + (y-z)\hat{k}$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C (x+2y)\,dx + (x-z)\,dy + (y-z)\,dz.$$

$$= \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds. \quad \text{— (1)}$$

→ Now $\operatorname{curl}\vec{F} = \nabla\times\vec{F}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x+2y) & (x-z) & (y-z) \end{vmatrix}$$

$$= i\begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x-z) & (y-z) \end{vmatrix} - j\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ (x+2y) & (y-z) \end{vmatrix} + k\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ (x+2y) & (x-z) \end{vmatrix}.$$

$$= i\left[\frac{\partial}{\partial y}(y-z) - \frac{\partial}{\partial z}(x-z)\right] - j\left[\frac{\partial}{\partial x}(y-z) - \frac{\partial}{\partial z}(x+2y)\right] + k\left[\frac{\partial}{\partial x}(x-z) - \frac{\partial}{\partial y}(x+2y)\right]$$

$$= i[(1-0)-(0-1)] - i[0-0] + k[(1-0)-(0+2)]$$

$$= 2\hat{i} - 0\hat{j} - \hat{k}$$

→ $\hat{n}$ = outward Normal vector on surface S

$$\boxed{\hat{n} = \frac{\operatorname{grad}\phi}{|\operatorname{grad}\phi|}}$$

$$\operatorname{grad}\phi = \nabla\phi = \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= \hat{i}\frac{\partial}{\partial x}(3x+2y+z-6) + \hat{j}\frac{\partial}{\partial y}(3x+2y+z-6) + k\frac{\partial}{\partial z}(3x+2y+z-6)$$

$$= 3\hat{i} + 2\hat{j} + \hat{k}$$

$$|\operatorname{grad}\phi| = \sqrt{(3)^2+(2)^2+(1)^2} = \sqrt{9+4+1} = \sqrt{14}$$

$$\hat{n} = \frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$$ here

$$\because \oint \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,ds$$

$$= \iint_S (2\hat{i}-k)\cdot\left(\frac{3\hat{i}+2\hat{j}+k}{\sqrt{14}}\right)\cdot\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$$

$$= \frac{1}{\sqrt{14}}\iint_S (6-1)\frac{dx\,dy}{|\hat{k}\cdot\hat{k}|} \Rightarrow \frac{1}{\sqrt{14}}5\iint_S dx\,dy$$

$$= \frac{5}{\sqrt{14}} \text{ Area of Base of triangle.}$$

$$= \frac{5}{\sqrt{14}}\cdot\frac{1}{2} \times \text{Base} \times \text{height}$$

$$= \frac{5}{\sqrt{14}}\cdot\frac{1}{2}\times 2\times 3 \Rightarrow \boxed{\frac{15}{\sqrt{14}}} \text{ Ans.}$$

→ Gauss Divergence Theorem :- (GDT)

By GDT we find volume integral with surface integral.

Statement :- If $\vec{F}$ is taken around a closed surface S enclosed by volume V then,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv.$$

where $\hat{n}$ is outward unit normal vector on surface S.

Que State Gauss Divergence Theorem. Evaluate $\iint_S \vec{F}\cdot\hat{n}\,ds$ where S is surface of sphere $x^2+y^2+z^2=16$ & $\vec{F} = 3x\hat{i}+4y\hat{j}+5z\hat{k}$.

Soln By GDT, we know that,

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dv$$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\hat{k}\right)\cdot\left(3x\hat{i}+4y\hat{j}+5z\hat{k}\right)$$

$$= \frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(4y) + \frac{\partial}{\partial z}(5z)$$

$$\Rightarrow 3+4+5 = \underline{12}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V 12\,dV$$

$$= 12\iiint_V dx\,dy\,dz \Rightarrow 12\ \text{Volume of sphere.}$$

$$= 12 \times \frac{4}{3}\pi r^3 \Rightarrow 16\pi(4)^3 = 64\times16\pi$$

$$\Rightarrow 1024\pi \ \underline{\text{Ans.}}$$

Que $\iint_S \vec{F}\cdot\hat{n}\,ds$ where $\vec{F} = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ and S is the surface of the cube bounded by $x=0, x=1, y=0, y=1, z=0, z=1$.

Soln By GDT, we know that.

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \operatorname{div}\vec{F}\,dV$$

$$\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$$

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}k\right)\cdot\left(4xz\hat{i} - y^2\hat{j} + yz\hat{k}\right)$$

$$= \frac{d(4xz)}{dx} + \frac{\partial}{\partial y}(-y^2) + \frac{\partial(yz)}{\partial z}$$

$$= 4z - 2y + y.$$

$$\Rightarrow 4z - y.$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V (4z-y)\,dv.$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z-y)\,dx\,dy\,dz.$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\left(\frac{4z^2}{2} - yz\right)_{z=0}^{1} dy\,dx$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} (2-y)\,dy\,dx$$

$$\Rightarrow \int_{x=0}^{1}\left(2y - \frac{y^2}{2}\right)_0^1 dx \Rightarrow \int_{x=0}^{1}\left(2-\frac{1}{2}\right)dx.$$

$$= \frac{3}{2}(x)_0^1 = \frac{3}{2} \text{ Ans.}$$

2/8/2023

→ Problems in Volume :-

Que 1 An triangular prism is formed by planes whose eq are $ay = bx$, $y=0$ & $x=a$, Find the volume of the prism bounded by plane $z=0$ & surface $z = c + xy$.

Soln Required volume of the prism b/w the given plane & surface.

$$V = \iiint_{x=0,\,y=0,\,z=0}^{a,\ \frac{bx}{a},\ c+xy} dx\,dy\,dz.$$

$$= \int_{x=0}^{a}\int_{y=0}^{\frac{bx}{a}}\int_{z=0}^{c+xy} dz\,dy\,dx.$$

$$= \int_{x=0}^{a}\int_{y=0}^{\frac{bx}{a}} [z]_{z=0}^{c+xy}\,dy\,dx$$

$$= \int_{x=0}^{a}\int_{y=0}^{\frac{b}{a}x} (c+xy)\,dy\,dx.$$

$$= \int_{x=0}^{a}\left(cy + \frac{cy^2}{2}\right)_{y=0}^{\frac{b}{a}x} dx.$$

$$= \int_{x=0}^{a}\left(c\cdot\frac{b}{a}x + \frac{c}{2}\cdot\frac{b^2}{a^2}x^2\right)dx.$$

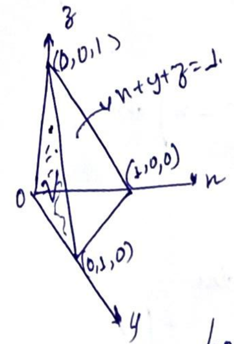
$$\Rightarrow \left[c\cdot\frac{b}{a}\cdot\frac{x^2}{2} + \frac{b^2c}{2a^2}\cdot\frac{x^3}{3}\right]_{x=0}^{a}$$

$$= \frac{cb}{a}\cdot\frac{a^2}{2} + \frac{b^2c}{2a^2}\cdot\frac{a^3}{3}$$

$$= \frac{abc}{2} + \frac{ab^2c}{6} \Rightarrow \frac{abc}{2}\left[1+\frac{b}{3}\right] \text{ Ans}$$

Que Find the volume of solid bounded by coordinate plane $x+y+z=1$

Soln



volume of solid bounded by coordinate plane & surface

$$V = \iiint_V dx\,dy\,dz.$$

lower limit ; $x=0$, $y=0$, $z=0$

upper limit of x: $y=0$, $z=0 \Rightarrow \boxed{x=1}$

upper limit of y: $z=0 \Rightarrow \boxed{y=1-x}$

upper limit of z: $\boxed{z=1-x-y}$

$$V = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} dx\,dy\,dz.$$

$$V = \int_{x=0}^{1} \int_{y=0}^{1-x} [z]_0^{1-x-y} dy\,dz.$$

$$= \int_{x=0}^{1} \left[\int_{y=0}^{1-x} (1-x-y)\,dy \right] dx.$$

$$= \int_{x=0}^{1} \left(y - xy - \frac{y^2}{2} \right)_{y=0}^{1-x} dx.$$

$$= \int_{x=0}^{1} \left[1-x - x(1-x) - \frac{1}{2}(1-x)^2 \right] dx.$$

$$= \int_{x=0}^{1} \left[1-x-x+x^2 - \frac{1}{2}(1+x^2-2x) \right] dx$$

$$V = \int_{x=0}^{1} 1 - 2x + x^2 - \frac{1}{2} - \frac{x^2}{2} - x)\, dx$$

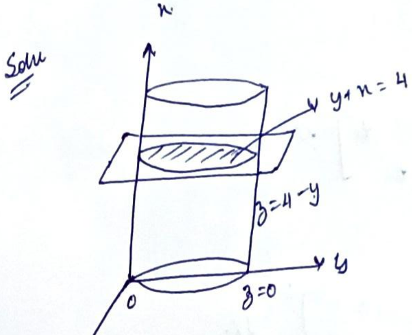
$$\Rightarrow \int_{x=0}^{1} \left(\frac{1}{2} - x + \frac{x^3}{2} \right) dx.$$

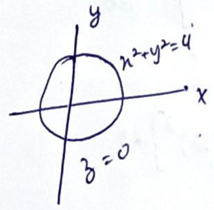
$$\Rightarrow \left(\frac{1}{2}x - \frac{x^2}{2} + \frac{x^3}{6} \right)_0^1$$

$$= \left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) = \frac{1}{6}\ \text{Ans.}$$

☆ **Ques** Find the volume bounded by the cylinder $x^2+y^2=4$, cut off by the ~~and~~ plane $y+z=4$, $z=0$.

Soln





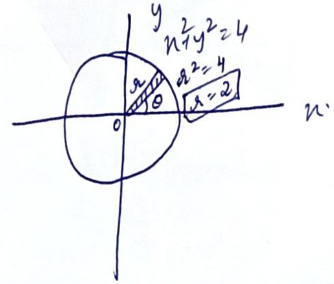
Now the required volume of the cylinder cut off by the plane,

$$V = \iiint dx\,dy\,dz.$$

$$\iint_R \left[\int_{z=0}^{4-y} dz \right] dy\,dx \Rightarrow \iint_R [z]_{z=0}^{4-y} dy\,dx$$

$$\Rightarrow \iint_R (4-y)\,dy\,dx.$$

The base of the cylinder is a circle in xy plane.
i.e $x^2+y^2=4$



$x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$

$r \longrightarrow r = 0$ to $r = 2$.

$\theta \longrightarrow \theta = 0$ to 2π

$$V = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} (4 - r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\int_{r=0}^{r=2} (4r - r^2\sin\theta)\, dr \right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{4r^2}{2} - \sin\theta \cdot \frac{r^3}{3} \right]_{r=0}^{2} d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{4}{2}(2)^2 - \frac{\sin\theta}{3}(2)^3 \right] d\theta$$

$$= \int_{\theta=0}^{2\pi} \left(8 - \underbrace{\frac{8}{3}\sin\theta}_{0} \right) d\theta$$

$$= 8(\theta)_0^{2\pi} = 8(2\pi - 0)$$

$$= 16\pi \text{ Ans.}$$

Some Important Problem in change of order of Integration.

Que Evaluate the given integral by changing into polar form.

$$\int_0^2 \int_0^{\sqrt{2x - x^2}} (x^2 + y^2)\, dy\, dx.$$

Soln Given integration can be written as

$$I = \int_{x=0}^{2} \int_{y=0}^{\sqrt{2x - x^2}} (x^2 + y^2)\, dy\, dx.$$

$x = 0,\ x = 2,\ y = 0,\ y = \sqrt{2x - x^2}$ — $x = 0,\ y = 0,\ (0,0)$; $x = 2,\ y = 0,\ (2,0)$

$$y^2 = 2x - x^2$$

$$x^2 + y^2 = 2x$$

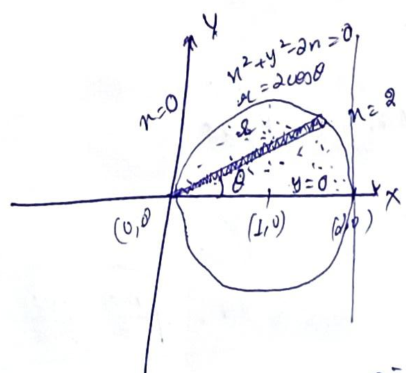
$$x^2 + y^2 - 2x = 0$$

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

$2g = -2$, $g = -1$ | $2f = 0$, $f = 0$ | $c = 0$

centre $= (-g, -f) = (1, 0)$

radius $= \sqrt{g^2 + f^2 - c} = 1$



$\theta \longrightarrow \theta = 0$ to $\theta = \frac{\pi}{2}$

$r \longrightarrow r = 0$ to $r = 2\cos\theta$.

Given Integration can be written as.

$$= \int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$$

$y = r\sin\theta$
$x = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$
$x^2+y^2 = r^2$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{2\cos\theta} r^2 (r\,dr\,d\theta)$$

$$= \int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{2\cos\theta} r^3\, dr \right] d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left(\frac{r^4}{4} \right)_{r=0}^{2\cos\theta} d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} (2\cos\theta)^4\, d\theta$$

$$= \frac{1}{4} \int_{\theta=0}^{\pi/2} 16\cos^4\theta\, d\theta$$

$$= 4 \int_{\theta=0}^{\pi/2} \sin^0\theta \cos^4\theta\, d\theta$$

$$= \int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{\Gamma\frac{m+1}{2}\, \Gamma\frac{n+1}{2}}{2\Gamma\frac{m+n+2}{2}}$$

$$V = \frac{4\,\Gamma\frac{0+1}{2}\, \Gamma\frac{4+1}{2}}{2\Gamma\frac{0+4+2}{2}} = \frac{4\,\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{2\Gamma 3}$$

$$= \frac{\cancel{2}\,\Gamma\frac{1}{2}\ \frac{3}{2}\cdot\frac{1}{2}\,\Gamma\frac{1}{2}}{\cancel{2}} \Rightarrow \frac{3}{\cancel{4}}\sqrt{\pi}\sqrt{\pi} = \frac{3\pi}{4}$$

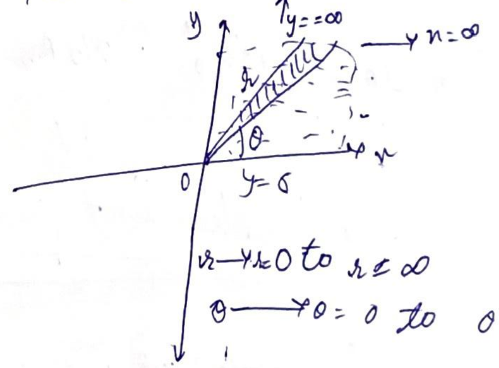
Que Evaluate By change of order of Integration.

$$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

Soln

$$I = \int_{y=0}^\infty \int_{x=0}^\infty e^{-(x^2+y^2)}\, dx\, dy.$$

Here $y=0$, $y=\infty$, $x=0$, $x=\infty$



$r \to r=0$ to $r=\infty$

$\theta \to \theta=0$ to $\theta=\frac{\pi}{2}$

$$I = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

$x = r\cos\theta$
$y = r\sin\theta$
$dx\,dy = r\,dr\,d\theta$
$x^2+y^2 = r^2.$

Suppose $x^2 = t$ —(1)

& differentiating

$2x\,dx = dt$

$dx = \frac{dt}{2x}$

New limit :-

when $x=0$, $t=0$
when $x=\infty$, $t=\infty$

$$\Rightarrow I = \int_{\theta=0}^{\pi/2}\left[\int_{t=0}^{\infty} e^{-t}\,\cancel{x}\,\frac{dt}{2\cancel{x}}\right]d\theta$$

$$= \int_{\theta=0}^{\pi/2}\left(\int_{t=0}^{\infty}\left(e^{-t}\,t^{1-1}\,dt\right)d\theta\right.$$

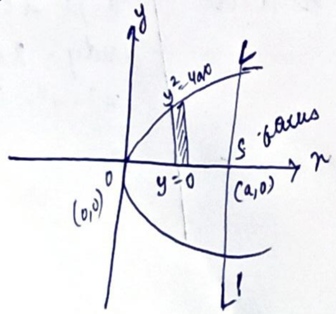
$$= \int_{\theta=0}^{\pi/2} \pi\,d\theta = 1\cdot(\theta)_0^{\pi} = \pi/2 \text{ Ans}$$

Some Problem in Area

Que Find the area ~~of~~ of parabola $y^2 = 4ax$ about its Latus Rectum.

Required area $= \iint_R dx\,dy$

Solⁿ



$$= \int_{x=0}^{a}\int_{y=0}^{\sqrt{4ax}} dy\,dx$$

$$= \int_{x=0}^{a} (y)_{y=0}^{\sqrt{4ax}}\,dx \Rightarrow \int_{x=0}^{a}\sqrt{4ax}\,dx$$

$$= \sqrt{4a}\int_{x=0}^{a} x^{1/2}\,dx$$

$$= 2\sqrt{a}\left(\frac{x^{3/2}}{3/2}\right)_{x=0}^{a}$$

$$= 2\sqrt{a}\cdot\frac{2}{3}\cdot a\sqrt{a} = \frac{4}{3}a^2 \text{ Ans}$$

Que Determine the area bounded by the curve $y = x^2$ & $x + y = 2$.

Solⁿ

$y = x^2$ —(1)

$x + y = 2$ —(2)

pt of intersection

put $y = x^2$ from —(1) in —(2)

$x^2 + x = 2$

$x^2 + x - 2 = 0$

$x^2 + 2x - x - 2 = 0$

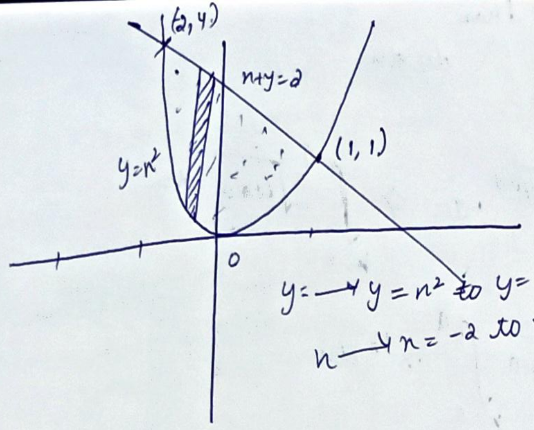
$x(x+2) - 1(x+2) = 0$

$(x-1)(x+2) = 0$

$x = 1,\ x = -2$

$\therefore y = x^2$

$(1,1)$ $x=1$, $y=1$ $\quad$ $x=-2$, $y=4$ $(-2,4)$



$y: \longrightarrow y = x^2$ to $y = 2-x$.

$x \longrightarrow x = -2$ to $x = 1$

Required Area:-

$$A = \int_{x=-2}^{1}\int_{y=x^2}^{2-x} dy\, dx$$

Que Determine Area bounded by $xy = 2$, $4y = x^2$, $y = 4$.

Que Given curve $xy = 2$ — (1)

$4y = x^2$ — (2)

$y = 4$ — (3)

$(4,4)$; $(\frac{1}{2}, 4)$

Point of intersection of (1) & (2)

$4\left(\frac{2}{x}\right) = x^2$

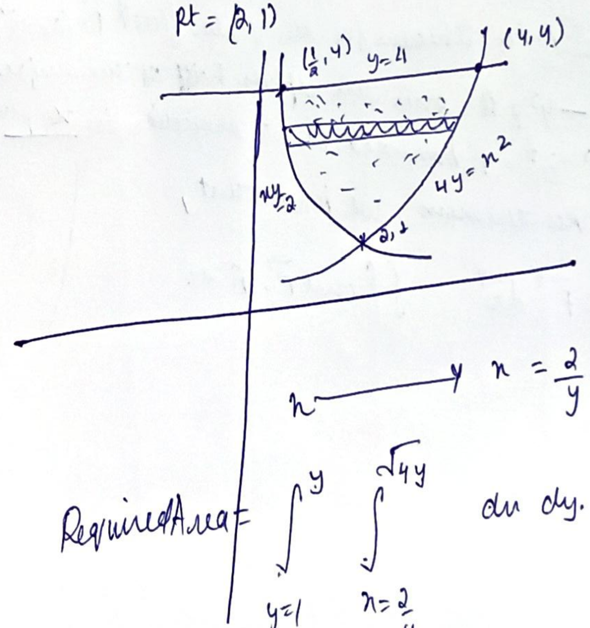
$x^3 = 8$

$x = 2$

$xy = 2$

$y = 1$

Pt = (2, 1)



$x \longrightarrow x = \frac{2}{y}$ to $x = \sqrt{4y}$.

$$\text{Required Area} = \int_{y=1}^{y}\int_{x=\frac{2}{y}}^{\sqrt{4y}} dx\, dy.$$

$$= \int_{y=1}^{4} (x)_{x=\frac{2}{y}}^{\sqrt{4y}}\, dy \Rightarrow \int_{y=1}^{4}\left(\sqrt{4y} - \frac{2}{y}\right) dy.$$

$$= \left[\sqrt{4}\cdot\frac{y^{3/2}}{3/2} - 2\log y\right]_1^4$$

$$= \left[2\cdot\frac{2}{3}(4)^{3/2} - 2\log 4\right] - \left[2\cdot\frac{2}{3}\cdot 1 - 2\underbrace{\log 1}_{0}\right]$$

$$= \frac{4}{3}\cdot 8 - 2\log 4 - \frac{4}{3} = \frac{28}{3} - 2\log 4 \quad \underline{\text{Ans}}.$$

→ Stokes Theorem :-

Que Verify Stokes Theorem for the vector field $\vec{F} = (2x-y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ over the upper half of the surface $x^2+y^2+z^2=1$ bounded by its projection in $\underline{xy \text{ plane}}$.

Soln By Stokes Theorem we know that,

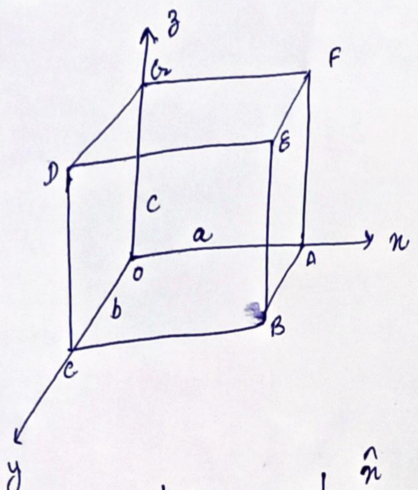
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot\hat{n}\,ds.$$

Que Use Stoke's theorem, evaluate $\oint \vec{F} \cdot d\vec{r}$ where, $\vec{F} = y^2\hat{i} + xy\hat{j} + xz\hat{k}$ and C is the boundary of hemisphere $x^2+y^2+z^2=9$ oriented in the direction.

Que Verify Gauss theorem, $\vec{F} = (x^2 - yz)\hat{i} + (y^2 - zx)\hat{j} + (z^2 - xy)\hat{k}$

Ans taken over the rectangular parallelopiped. $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$

Soln



Face	Plane	$\hat{n}$	ds	limit
1) OABC S_1	XY, $z=0$	$-\hat{k}$	dx dy	$x \to 0$ to a, $y \to 0$ to b
2.) DEFG S_2	XY, $z=c$	$+\hat{k}$	dx dy.	$x \to 0$ to a, $y \to 0$ to b
3.) OAFG S_3	ZX, $y=0$	$-\hat{j}$	dz dx	$x \to 0$ to a, $z \to 0$ to c
4) CBED S_4	ZX, $y=b$	$\hat{j}$	dz dx	$x \to 0$ to a, $z \to 0$ to c
5) OCDG S_5	YZ, $x=0$	$-\hat{i}$	dy dz.	$y \to 0$ to b, $z \to 0$ to c
6) ABEF S_6	YZ, $x=a$	$+\hat{i}$	dy dz	$y \to 0$ to b, $z \to 0$ to c.

By Gauss Divergence Theorem, We know that,

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iiint_V \operatorname{div}\vec{F}\cdot dv.$$

RHS: $\iiint_V \operatorname{div}\vec{F}\cdot dv$

Now :- $\operatorname{div}\vec{F} = \nabla\cdot\vec{F}$

$$= \left(\frac{i\partial}{\partial x} + \frac{j\partial}{\partial y} + \frac{k\partial}{\partial z}\right)\cdot\left[(x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}\right]$$

$$= \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$= 2x + 2y + 2z.$$

$$\therefore \iiint_V \operatorname{div}\vec{F}\cdot dv = \int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c}(2x+2y+2z)\, dz\, dy\, dx$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left(xz + yz + \frac{z^2}{2}\right)_{z=0}^{c} dy\, dx.$$

$$= 2\int_{x=0}^{a}\int_{y=0}^{b}\left(xc + yc + \frac{c^2}{2}\right) dy\, dx.$$

$$= 2\int_{x=0}^{a}\left(xcy + \frac{cy^2}{2} + \frac{c^2}{2}\cdot y\right)_{y=0}^{2} dx$$

$$\Rightarrow 2\int_{x=0}^{a}\left(xcb + \frac{c}{2}b^2 + \frac{c^2 b}{2}\right) dx$$

$$= \left[\frac{cbx^2}{2} + \frac{c}{2}b^2x + \frac{c^2bx}{2}\right]_{x=0}^{a}$$

$$= cba^2 + cb^2a + c^2ba$$

$$= abc(a+b+c)$$

LHS :- $\iint_S \vec{F}\cdot\hat{n}\, ds$

$$= \iint_S \left[(x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}\right]\cdot\hat{n}\, ds$$

$$= \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6} \quad \text{--- (1)}$$

Face 1 :-

$$\iint_{S_1} \left[(x^2-y\underset{0}{z})\hat{i} + (y^2-\underset{0}{z}x)\hat{j} + (\underset{0}{z^2}-xy)\hat{k}\right]\cdot\hat{n}\, ds$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} (x^2\hat{i} + y^2\hat{j} - xy\hat{k})\cdot(-\hat{k})\, dx\, dy$$

$$= \int_{x=0}^{a}\int_{y=0}^{b} xy\, dx\, dy$$

$$= \int_{x=0}^{a} x\,dx \int_{y=b}^{b} y\, dy \Rightarrow \left(\frac{x^2}{2}\right)_0^a \cdot \left(\frac{y^2}{2}\right)_0^b$$

$$= \frac{a^2b^2}{4}$$

Face 2 :- $\iint_{S_2} \vec{F}\cdot\hat{n}\, ds = \int_{x=0}^{a}\int_{y=0}^{b} (z^2-xy)\, dx\, dy.$

$\cancel{\Rightarrow \iint_{S_2} \vec{F}\cdot\hat{n}\, ds}$

$$\Rightarrow \int_{x=0}^{a}\int_{y=0}^{b} (c^2-xy)\, dy\, dx \qquad (\because z=c)$$

$$= \int_{x=0}^{a} \left(c^2y - \frac{xy^2}{2}\right)_{y=0}^{b} dx$$

$$\Rightarrow \int_{x=0}^{a} \left(c^2b - \frac{xb^2}{2}\right) dx$$

$$\Rightarrow \left(c^2bx - \frac{b^2}{2}\frac{x^2}{2}\right)_0^a = abc^2 - \frac{a^2b^2}{4}$$

Face 3 :- $\iint_{S_3} \vec{F}\cdot\hat{n}\, ds = \int_{x=0}^{a}\int_{z=0}^{c} -(\underset{0}{y^2}-zx)\, dz\, dx$

$$\Rightarrow \int_{x=0}^{a}\int_{z=0}^{c} zx\, dz\, dx = \frac{a^2c^2}{4}$$

Face 4 :- $\iint_{S_4} \vec{F}\cdot\hat{n}\, ds = \int_{x=0}^{a}\int_{z=0}^{c} (y^2 - zx)\, dz\, dx.$

$$\Rightarrow \int_{x=0}^{a}\int_{z=0}^{c} +(b^2 - zx)\, dz\, dx = (b^2ac) - \frac{a^2c^2}{4}$$

Face 5 :- $\iint_{S_5} \vec{F}\cdot\hat{n}\, ds = \int_{y=0}^{b}\int_{z=0}^{c} -(\underset{0}{x^2}-yz)\, dy\, dz.$

$$= \int_{y=0}^{b}\int_{z=0}^{c} yz\, dy\, dz = \frac{b^2c^2}{4}$$

(6) For S_6:-

$$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_{y=0}^{b}\int_{z=0}^{c} (a^2 - yz)\,dy\,dz$$

$$= a^2bc - \frac{b^2c^2}{4}$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = c^2ab - \cancel{\frac{a^2b^2}{4}} + b^2ac - \cancel{\frac{a^2c^2}{4}} + a^2bc - \cancel{\frac{b^2c^2}{4}}$$

$$+ \cancel{\frac{a^2b^2}{4}} + \cancel{\frac{b^2c^2}{4}} + \cancel{\frac{c^2a^2}{4}}$$

$$\Rightarrow abc\,(a+b+c)$$

$\Rightarrow$ LHS = RHS

Gauss divergence theorem Verified

Tracing of curve in cartesian form $f(x,y)$

Working Rule :- [Symmetry]

1. If all power of x is even the curve is symmetrical about y-axis and vice versa.
2. If all power x & y both are even then curve is symmetrical about both axis.
3. If all power of y is even the curve is symmetrical about x axis.
4. Replacing x by y and y by x if no change the curve is symmetrical about line $y=x$
5. Replacing x by $-x$ & and y by $-y$ if no change then curve is symmetrical in opp. quadrant. Replacing x by $-y$ & y by $-x$ no change curve is symmetrical about $y=x$.

Que — Trace the Curve $y^2(2a-x)=x^3$.

Sol^n Given curve. $\quad 2ay^2 - xy^2 = x^3$

$$y^2(2a-x) = x^3 \quad \text{——} (1)$$

(1) **Symmetry :-** Here, all powers of y are even so, curve is symmetrical about $\boxed{x \text{ axis}}$.

(2) **Tangent at origin :-**

put $x=0$ & $y=0$

we see that, LHS = RHS. it means that curve is passing through origin

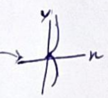
$\because$ To find eq of tangent at origin, equating to 0 the lowest degree terms.

$$2ay^2 = 0 \Rightarrow y^2 = 0$$

$$\boxed{y^2=0,\ y=0}$$

i.e same tangent.

$\therefore$ shape of curve at origin is Cusp.



(3) **Point of Intersection :-**

put $x=0$

$y^2(2a-0)=0$

$\boxed{y=0}$

$(0,0)$ is pt of intersection.

put $y=0$, $x=0$

$\Rightarrow (0,0)$ is pt of intersection.

(4) **Asymptote :-** If Asymptote ∥ to x axis, then equating to 0 the coefficient of highest power of x.

here, higher power of x is x^3 whose coefficient is $1=0$ (Not possible)

It means that No Asymptote ∥ to x axis.

If Asymptote ∥ to y axis, then equating to 0 the coefficient of highest power of y.

here, higher power of y is y^2 whose coefficient is

$$2a-x=0$$

$\Rightarrow \boxed{x=2a}$ is Asymptote ∥ to y axis.

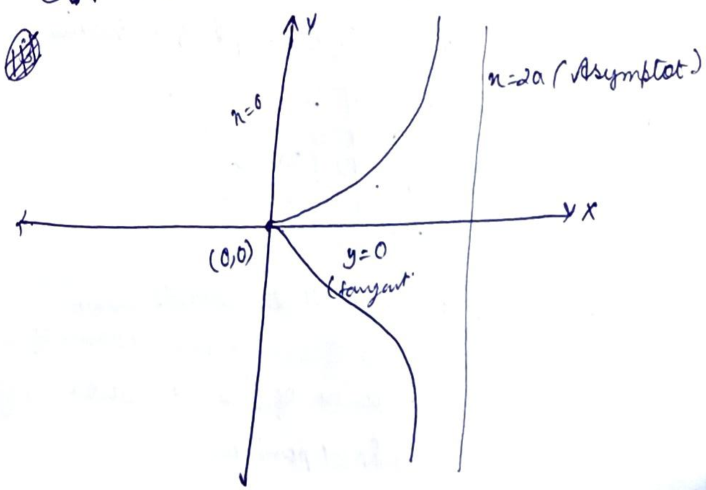
(5) **Region :-** $y^2(2a-x) = x^3$

$$y^2 = \frac{x^3}{2a-x}$$

$$y = \pm\sqrt{\frac{x^3}{2a-x}}$$

(1) If $x > 2a$, y is imaginary (does not exist)

(2) If $x < 0$, y is imaginary (does not exist)



③ Curve lies b/w $x=0$ & $x=2a$.

Que 2. Trace the curve. $y^2(a+x) = x^2(3a-x)$

1) Symmetry $ay^2+xy^2-3ax^2+x^4=0$

Here, all powers of y are even so curve is symmetrical about x axis

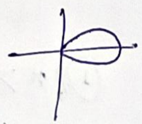
② Tangent at origin :-

Since the curve passes through origin

∴ to find eq. of tangent at origin equating to 0 the lowest degree terms.

$$ay^2-3ax^2=0$$
$$y^2=3x^2$$
$$y=\pm\sqrt{3}x$$
$$y=\sqrt{3}x \;\&\; y=-\sqrt{3}x$$

There are 2 different tangent. hence Shape of curve at origin is Node



③ Point of Intersection :-

→ put $x=0$, $y=0 \Rightarrow (0,0)$ is pt of intersection.

→ put $y=0$; $x^3(3a-x)=0 \Rightarrow x=0, 3a$.

pt of intersection $\boxed{(0,0)\ (3a,0)}$

There are pt of intersection

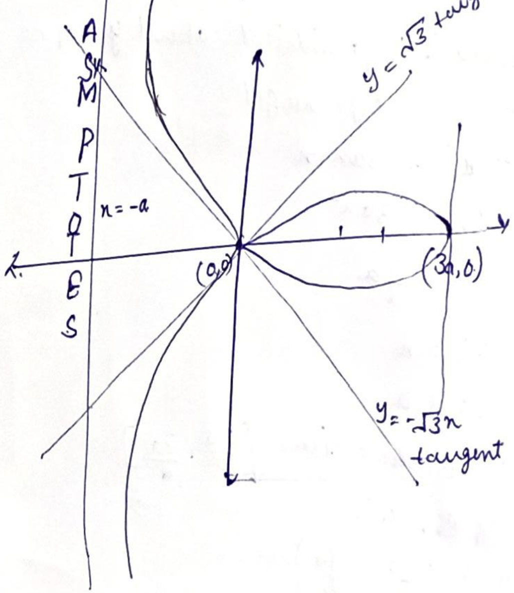
④ Asymptot :- If asymptot ∥ to x axis then equating to 0, the highest power of x here highest power of x is x^3 whose coefficient $1=0$ (Not possible)

→ No asymptot ∥ to x axis

→ If asymptot ∥ to y axis then equating to 0, the highest power of y here highest power of y is y^2 whose coefficient is $a+x=0$

$\boxed{x=-a}$ is Asymptot ∥ to y axis.

⑤ Region :- $y=\pm\sqrt{\dfrac{x^2(3a-x)}{a+x}}$



The curve lies b/w. $x=-a$ & $x=3a$.

Que Trace the curve $x^3+y^3=3axy$ (Folium)

① Symmetry :- Replacing y by x & x by y, there is no change so curve is symmetrical about line $y=x$

② Tangent at origin :- Since the curve passing through origin eqn of tangent at origin equating to 0 the lowest degree term., $3axy=0$

$\boxed{x=0,\ y=0}$ distinct tangent.

then the shape is a node.

(3) Point of intersection $x=0 \longrightarrow y=0$

$(0,0)$ pt of intersection

$y=0,\ x \Longrightarrow 0$

$(0,0)$ pt of intersection.

Since this curve is symmetrical about $y=x$, so put $y=x$ in given eq we get.

$$x^3 + x^3 = 3ax \cdot x$$

$$2x^3 = 3ax^2$$

$$x = \frac{3}{2}a.$$

$$\therefore y = x$$

$$y = \frac{3}{2}a$$

pt of Intersection $\left[\frac{3a}{2}, \frac{3a}{2}\right]$

$\therefore$ Total pt of Intersection.

$(0,0)\ \left(\frac{3a}{2}, \frac{3a}{2}\right)$

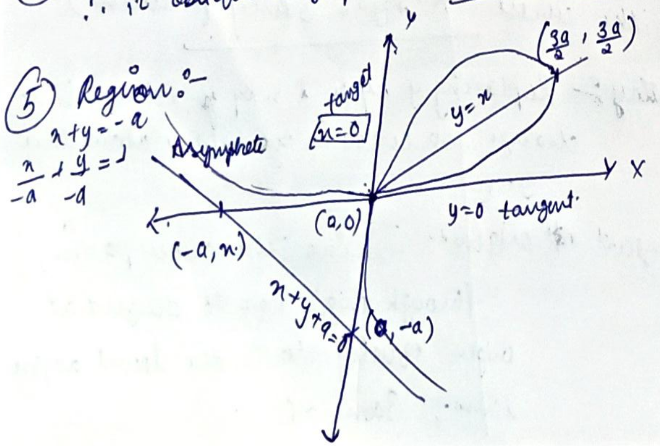
(4) There is not asymptote $\parallel$ to x & y axis

$\therefore$ it oblique asymptote is $\boxed{x+y+a=0}$

(5) Region:-

$x+y=-a$

$\frac{x}{-a}+\frac{y}{-a}=1$



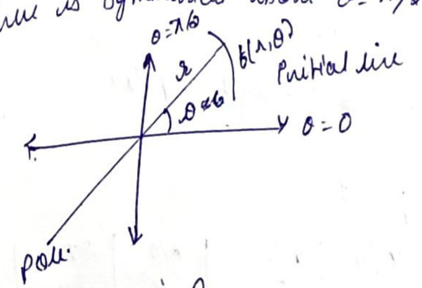
Tracing Curve in Polar Form:- $f(r,\theta)$

→ Symmetry :-

(a) Replacing $\theta=-\theta$, No change, the curve is symmetrical about $\theta=0°$ (x axis)

(b) If power of r is even, curve is symmetrical on pole (origin)

(c) Replacing r by $-r$ & θ by $-\theta$, No change then curve is symmetrical about $\theta=\pi/2$.



(2) Tangent at Pole :-

put $r=0$, find some values of θ will be tangent at pole.

(3) Find some special points (r,θ)

Que Trace the curve $r = a(1+\cos\theta)$ (Cardioid)

Soln ① Symmetry :-

① Replacing $\theta = -\theta$, No change so curve is symmetrical about initial line. i.e $\theta = 0°$ (x axis)

② **Tangent at Pole :-**

Put $r = 0$, we get $0 = a(1+\cos\theta)$

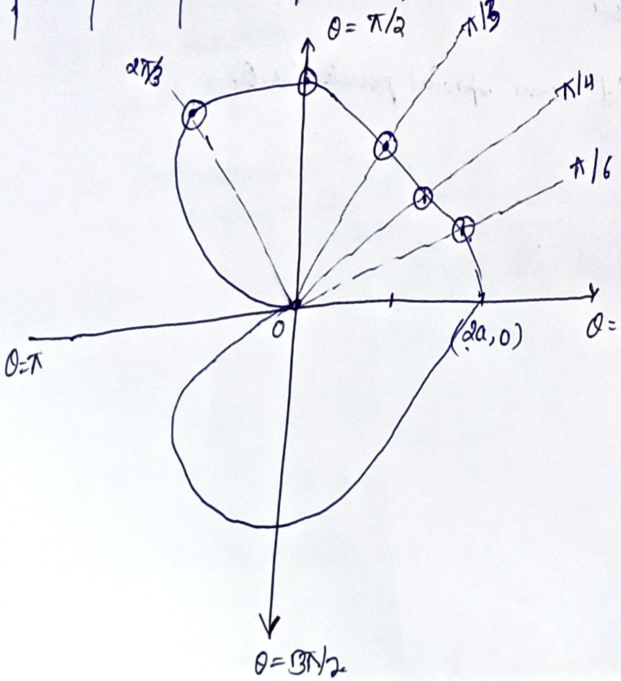
$(1+\cos\theta) = 0$

$\cos\theta = -1 = \cos\pi$

$\boxed{\theta = \pi}$ is tangent at pole.

③ Some special point

θ	$0°$	$(30°)$ $\pi/6$	$\pi/4$ $(45°)$	$\pi/3$ $(60°)$	$\pi/2$ $(90°)$	$2\pi/3$ $(120°)$	π $(180°)$
r	$2a$	$a\left(1+\frac{\sqrt{3}}{2}\right)$	$a\left(1+\frac{1}{\sqrt{2}}\right)$	$\frac{3}{2}a$	a	$\frac{-a}{2}$	0



Que Trace the Curve $r^2 = a^2\cos 2\theta$ (Lemniscate)

① Symmetry :- Replacing $\theta = -\theta$ (No change), curve is symmetrical about initial line.

(ii) here power of r is even, so curve is symmetrical about pole, ~~Replacing~~

Replacing $r \to -r$ & $\theta \to -\theta$, No change, curve is symmetrical about $\theta = \pi/2$

② **Tangent at Pole :-**

put $r = 0$

$a^2\cos 2\theta = 0$

~~$\cos\theta$~~ $= \cos(\pm\pi/2)$

$2\theta = \pm\frac{\pi}{2}$

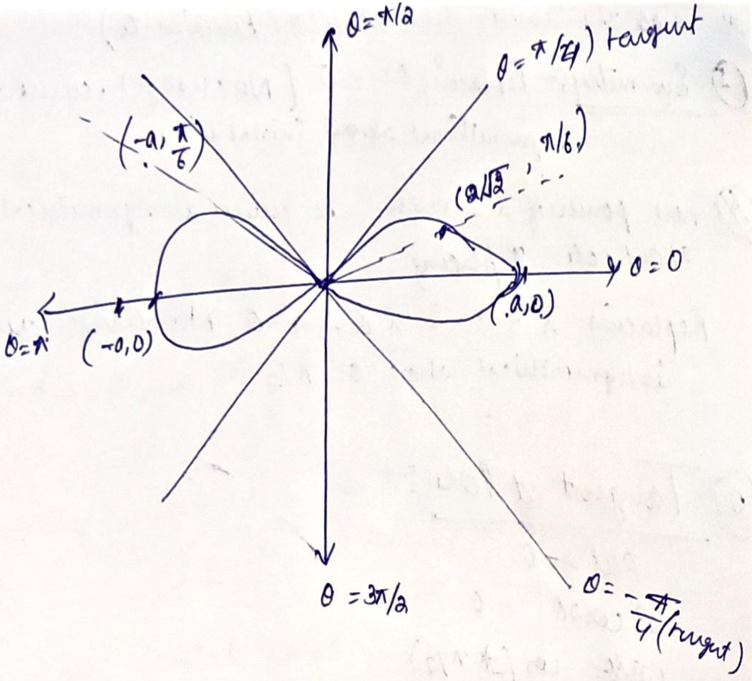
$\boxed{\theta = \pm\pi/4}$

③ ~~Some Special point~~

$\theta = \frac{\pi}{4}$, $\theta = -\frac{\pi}{4}$, there are two distinct tangents so shape of curve at pole will be Node.

③ Some Special point

θ	0	$\theta = \pi/6$	$\theta = \pi/4$
r	$\pm a$	$r = \frac{\pm a}{\sqrt{2}}$	0



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Que Trace the curve $R = a\sin 2\theta$.

The given curve of the type $R = a\sin n\theta$.

$\therefore$ Here $n=2$ i.e even so $2n = 2\times 2 = 4$ loops will be formed

Symmetry:- Replacing $r = -r$, $\theta = -\theta$, No change so curve is symmetrical about $\theta = \pi/2$

Tangent at pole:- Put $r=0$, we get

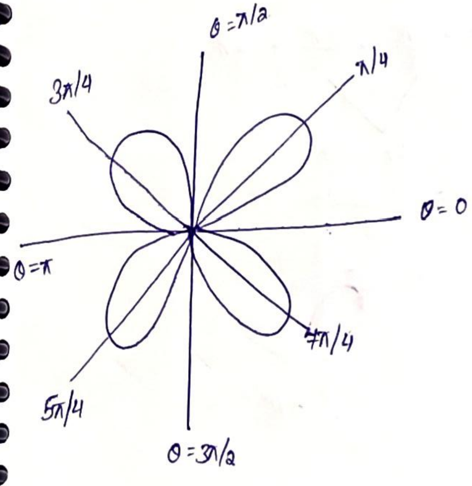
$$0 = a\sin 2\theta$$
$$\sin 2\theta = 0 = \sin n\pi$$
$$2\theta = n\pi$$
$$\theta = \frac{n\pi}{2} \quad (n = 0, 1, 2, ---)$$

$$\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$$

$$\frac{0+\pi/2}{2} = \frac{\pi}{4} \qquad \frac{\frac{\pi}{2}+\pi}{2} = \frac{3\pi}{4} \qquad \frac{\frac{3\pi}{2}+\pi}{2} \Rightarrow \frac{5\pi}{4} \qquad \frac{\frac{3\pi}{2}+2\pi}{2} = \frac{7\pi}{4}$$

Some special point

θ	0	$\pi/4$ ✓	$\pi/2$	$3\pi/4$ ✓	π	$5\pi/4$ ✓	$3\pi/2$	$7\pi/4$ ✓	2π
r	0	$+a$	0	$-a$	0	$+a$	0	a	0



Que Trace the curve $r = a\sin 3\theta$

The given curve of the type $r = a\sin n\theta$, here $n=3$ i.e odd, so there are n loops i.e 3 loops will be formed.

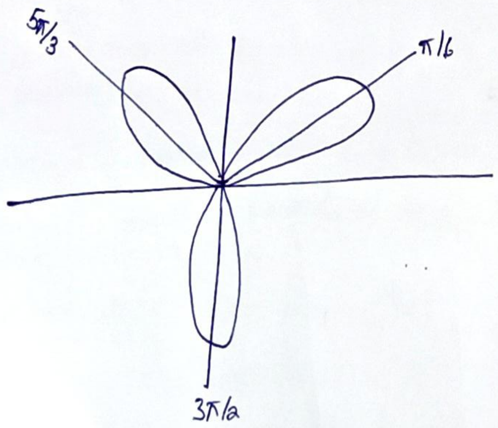
Symmetry = Replacing $r = -r$, $\theta = -\theta$, No change, so curve is symmetrical about $\theta = \pi/2$

Tangent at pole :- put $r=0$, we get $0 = a\sin 3\theta$

$$\sin 3\theta = 0 = \sin n\pi$$
$$3\theta = n\pi$$
$$\theta = \frac{n\pi}{3} \quad (n = 0, 1, 2 \ldots)$$

$$\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi.$$

$$\frac{0+\frac{\pi}{3}}{2} = \frac{\pi}{6} \qquad \frac{\frac{\pi}{3}+\frac{2\pi}{3}}{2} \Rightarrow \frac{5\pi}{6} \qquad \frac{\frac{4\pi}{3}+\frac{5\pi}{3}}{2} \Rightarrow \frac{3\pi}{2}$$



Some Special point

θ	0	$\pi/6$	$\pi/3$	$2\pi/3$	$5\pi/6$	π	$4\pi/3$	$3\pi/2$	$5\pi/3$	2π
r	0	$+a$	0	0	$-a$	0	0	$+a$	0	0

Que Find the value of $\Gamma\frac{1}{2}$

Solu We know that $\Gamma n \cdot \Gamma(1-n) = \frac{\pi}{\sin n\pi}$

put $n = \frac{1}{2}$

$$\Gamma\frac{1}{2}\,\Gamma\left(1-\frac{1}{2}\right) = \frac{\pi}{\sin\frac{1}{2}\pi}$$

$$\Gamma\frac{1}{2}\,\Gamma\frac{1}{2} = \frac{\pi}{1}$$

$$\left(\Gamma\frac{1}{2}\right)^2 = \pi$$

$$\boxed{\Gamma\frac{1}{2} = \sqrt{\pi}} \text{ Ans}$$

Que Prove that $\beta(m,m) = 2^{1-2m}\beta\left(m, \frac{1}{2}\right)$

Proof $\beta\left(m, \frac{1}{2}\right) = \dfrac{\Gamma m\,\Gamma\frac{1}{2}}{\Gamma\left(m+\frac{1}{2}\right)}$ — (1) $\quad \because \beta(p,q) = \dfrac{\Gamma p\,\Gamma q}{\Gamma(p+q)}$

By Lagendu's Dulplication formulae.

$$\Gamma m\,\Gamma\left(\frac{m+1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\cdot \Gamma 2m$$

$$\Gamma\left(m+\tfrac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2m)}{\Gamma(m)\cdot 2^{2m-1}} \quad \text{--- (2)}$$

put $\Gamma\left(m+\tfrac{1}{2}\right)$ from (2) in eq (1)

$$\beta\left(m,\tfrac{1}{2}\right) = \frac{\Gamma(m)\,\Gamma\left(\tfrac{1}{2}\right)}{\left(\frac{\sqrt{\pi}\cdot\Gamma(2m)}{\Gamma(m)\cdot 2^{2m-1}}\right)}$$

$$= \frac{\Gamma(m)\,\cancel{\Gamma\left(\tfrac{1}{2}\right)}\cdot\Gamma(m)\cdot 2^{2m-1}}{\cancel{\sqrt{\pi}}\cdot\Gamma(2m)}$$

$$= \frac{\Gamma(m)\cdot\Gamma(m)}{\Gamma(m+m)}\,2^{2m-1}$$

$$\Rightarrow \beta\left(m,\tfrac{1}{2}\right) = \beta(m,m)\,2^{2m-1}$$

$$\frac{\beta\left(m,\tfrac{1}{2}\right)}{2^{2m-1}} = \beta(m,m)$$

$$\boxed{\beta(m,m) = 2^{1-2m}\cdot\beta\left(m,\tfrac{1}{2}\right)}$$

Hence proved

Que Find Relation b/w beta & Gamma function.

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Soln we know that

$$\Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}\,dx.$$

or $$\int_0^\infty e^{-kx}\cdot x^{n-1}\,dx = \frac{\Gamma(n)}{k^n}$$

or $$\Gamma(n) = k^n\int_0^\infty e^{-kx}\,x^{n-1}\,dx.$$

$$\Gamma(n) = \int_0^\infty e^{-kx}\,k^n\,x^{n-1}\,dx.$$

multiplying both side by $e^{-km}k^{m-1}$

$$e^{-k}\,k^{m-1}\cdot\Gamma(n) = \int_0^\infty e^{-k}\,k^{m-1}\,e^{-kx}\,k^n\,x^{n-1}\,dx.$$

Now integrate both side ~~b/w~~ 0 to ∞ w.r.t. k. we have

$$\left(\int_0^\infty e^{-x}\,k^{m-1}\,dk\right)\Gamma(n) = \int_0^\infty\int_0^\infty\left(e^{-k}\cdot k^{m-1}\,e^{-kx}\,k^n\,x^{n-1}\,dx\right)dk$$

$$= \int_0^\infty x^{n-1}\left(\int_0^\infty e^{-k(x+1)}\,k^{m+n-1}\,dx\right)dx.$$

$$\Gamma(m)\,\Gamma(n) = \int_0^\infty x^{n-1}\,\frac{\Gamma(m+n)}{(x+1)^{m+n}}\,dx.$$

$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = \int_0^\infty \frac{x^{n-1}}{(x+1)^{m+n}} \, dx$$

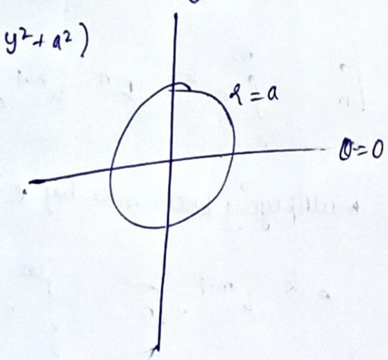
$$\frac{\Gamma m \, \Gamma n}{\Gamma m+n} = \beta(m,n) \Rightarrow \boxed{\beta(m,n) = \frac{\Gamma m \cdot \Gamma n}{\Gamma m+n}}$$

Area of Planar Curves:

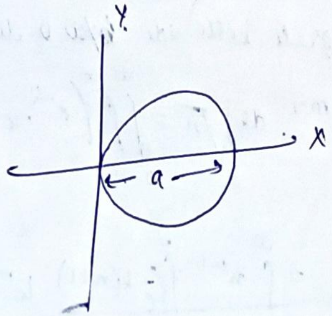
$x = r\cos\theta$; $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$

$x^2+y^2 = a^2 \Rightarrow r^2 = a^2 \Rightarrow \boxed{r=a}$ circle

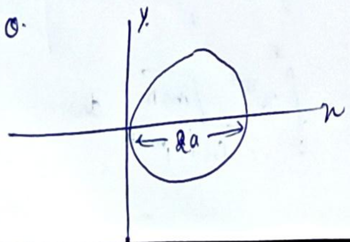
$r=a$ $(x^2+y^2=a^2)$



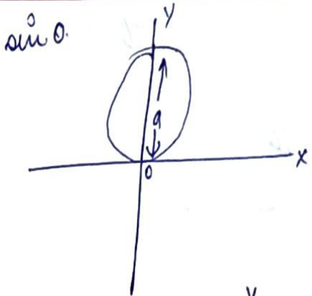
$r = a\cos\theta$.



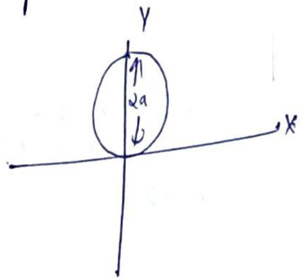
$r = 2a\cos\theta$.



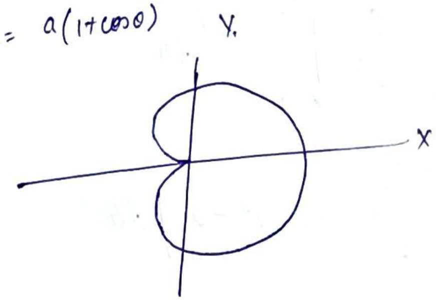
$r = a\sin\theta$.



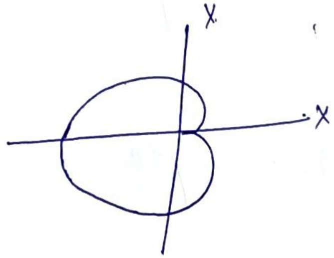
$r = 2a\sin\theta$.



$r = a(1+\cos\theta)$

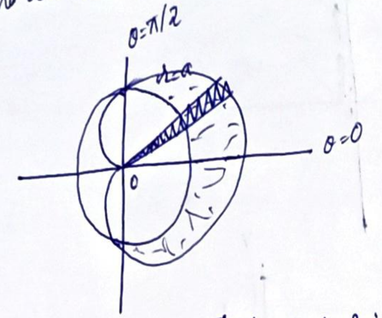


$r = a(1-\cos\theta)$



Que Find the area outside the circle $r = a$, $r = a(1+\cos\theta)$

Soln



Required area outside circle & inside cardioid

$= 2(\text{Area of upper half})$

$$= 2\int_{\theta=0}^{\pi/2}\int_{r=a}^{(1+\cos\theta)a} r\,dr\,d\theta$$

$$= 2\int_{\theta=0}^{\pi/2}\left[\frac{r^2}{2}\right]_{r=a}^{a(1+\cos\theta)} d\theta$$

$$= \frac{2}{2}\int_{\theta=0}^{\pi/2}\left[a^2(1+\cos\theta)^2 - a^2\right] d\theta$$

$$= a^2\int_{\theta=0}^{\pi/2}\left[\cancel{1}+\cos^2\theta + 2\cos\theta - \cancel{1}\right] d\theta$$

$$= a^2\int_{\theta=0}^{\pi/2}\left[\frac{1+\cos 2\theta}{2} + 2\cos\theta\right] d\theta$$

$$= a^2\int_{\theta=0}^{\pi/2}\left(\frac{1+\cos 2\theta + 4\cos\theta}{2}\right) d\theta$$

$$= \frac{a^2}{2}\left[\theta + \underbrace{\frac{\sin 2\theta}{2}}_{0} + \frac{4\sin\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[\left(\frac{\pi}{2} + 0 + 2\cdot 1\right) - (0+0+0)\right]$$

$$= \frac{a^2}{2}\left(\frac{\pi}{2} + 2\right) = \frac{a^2}{4}(\pi + 4)\ \text{Ans}$$