Exercise two and three

Contents

THE aim of the exercises is to cover topics related to the schematic representation of a dynamic system given by a differential equation (or a system of differential equations), topics related to the decomposition of a higher-order differential equation into a system of first-order equations, and obtaining numerical solutions using Simulink and MATLAB (or other software).

1 Exercise two

1.1 Task 1

Consider a dynamic system given by the differential equation in the form

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_0 u(t) \qquad y(0) = y_0 \qquad \dot{y}(0) = z_0 \tag{1}$$

where a_0, a_1, b_0 are constants and $u(t)$ is a known input signal.

- Schematically represent the dynamic system given by equation (1).

1.2 Task 2

- Rewrite the differential equation (1) as a system of first-order differential equations.
  - How many first-order equations will this result in?

1.3 Task 3

Consider the mathematical model of a direct current (DC) motor with constant external excitation.

Basic information about the motor: Nominal power 39 [kW], nominal voltage 520 [V], and nominal current 89 [A]. Nominal motor speed: 1113 [rpm] and nominal torque: 337 [Nm]. Thus, it is a relatively powerful, large motor.

To create a mathematical model of the motor, with the aim of describing the dynamic processes during start-up and operation, it is necessary to consider that this is a case with constant external excitation, which simplifies the description of the electromagnetic part of the system. At the same time, information about so-called iron losses is available in this case, resulting in the possibility of considering the so-called motor voltage constant $C_{u\omega}$ [Vs] and motor torque constant C_{uM} [Nm/A]. In this case (without providing further details), we have

$$C_{u\omega} = 3.903 \text{ [Vs]} \tag{2a}$$
$$C_{uM} = 3.787 \text{ [Nm/A]} \tag{2b}$$

The electromagnetic subsystem of the motor (which in this case is essentially just the rotor winding) can be described by the differential equation in the form

$$u(t) = Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + u_i(t) \tag{3}$$

where $u(t)$ [V] is the voltage on the rotor winding of the motor, $i(t)$ [A] is the current through the rotor winding, R [Ω] is the electrical resistance of the winding, L [H] is the inductance of the winding, and u_i is the back electromotive force, which is essentially the result of the changing magnetic field relative to the winding. Here, the motor voltage constant is utilized when $u_i(t) = C_{u\omega}\omega(t)$, where $\omega(t)$ [rad/s] is the angular velocity of the motor. Thus, equation (3) can be written in the form

$$u(t) = Ri(t) + L\frac{\mathrm{d}i(t)}{\mathrm{d}t} + C_{u\omega}\omega(t) \tag{4}$$

The numerical values of the parameters in this case are: $R = 0.737$ [Ω] and $L = 0.00905$ [H].

The mechanical subsystem of the motor can be described by the well-known equation

$$M_m(t) = J_m\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} \tag{5}$$

where $M_m(t)$ [Nm] is the torque produced by the motor, J_m [kg m^2] is the moment of inertia, and $\omega(t)$ [rad/s] is the angular velocity of the motor. The torque produced by the motor can be determined using the motor torque constant by the relation $M_m(t) = C_{uM}i(t)$. The attentive reader will notice that equation (5) does not consider the mechanical load of the motor (load torque), which can be easily added.

In this case, we can also write equation (5) in the form

$$C_{uM}i(t) = J_m\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} \tag{6}$$

The numerical values of the parameters in this case are: $J_m = 0.5$ [kg m^2].

- Formally adjust the differential equations describing the given dynamic system into a form suitable for analysis from the perspective of numerical simulation in Simulink.
- Simulate the start of the motor (mechanically unloaded) at a constant supply voltage of 520 [V].

1.4 Task 4

Consider a dynamic system given by the differential equation in the form

$$\dot{y}(t) + a_0y(t) = b_0u(t) \qquad y(0) = y_0 \tag{7}$$

where a_0, a_1, b_0 are constants and $u(t) = 1$ is the input signal.

- Choose the values of the system parameters a_0, b_0 with $a_0 > 0$. Choose the value of the initial condition y_0.
- Using the `ode45()` function in MATLAB, perform a numerical simulation of the system and plot the signal $y(t)$ and the signal $\dot{y}(t)$.

1.5 Bonus Tasks

- In section 1.1, consider the input signal $u(t)$ as a constant signal and choose its constant value. Also, choose the values of the initial conditions y_0 and z_0. Lastly, choose the values of the constants a_0, a_1, and b_0. Using the constructed system scheme, perform a numerical simulation of the system in Simulink.
- Schematically represent the resulting system of differential equations from section 1.2.
- In section 1.4, experiment with the values of the parameters a_0, b_0, and the initial condition y_0 and observe how these values affect the course of the signal $y(t)$.

2 Exercise three

2.1 Task 1

- Find the analytical solution of the differential equation using the characteristic equation method.

$$\ddot{y}(t) + 6\dot{y}(t) + 5y(t) = 0 \qquad y(0) = 4 \qquad \dot{y}(0) = 3$$

2.2 Task 2

2.2.1 Pendulum – creating a numerical simulation (simulation scheme)

Consider a pendulum whose oscillations are damped by viscous friction with a coefficient β [kg m^2 s^{-1}]. The pendulum is shown in Fig. 1, where a mass point with mass m [kg] attached to an arm with negligible mass and length l [m] oscillates, o denotes the axis of rotation perpendicular to the plane in which the pendulum oscillates, the angle between the vertical and the pendulum arm is denoted by φ [rad], and the gravitational acceleration is g [m s^{-2}].

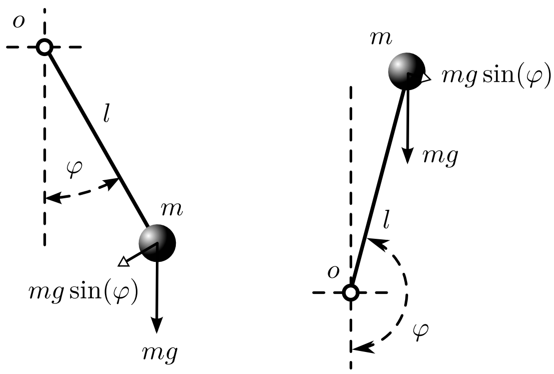

Figure 1: Pendulum

The equation of motion describing the dynamics of the rotational motion of the pendulum is in the form

$$ml^2\ddot{\varphi}(t) + \beta\dot{\varphi}(t) + mgl\sin\varphi(t) = u(t) \tag{8a}$$

$$ml^2\ddot{\varphi}(t) = -\beta\dot{\varphi}(t) - mgl\sin\varphi(t) + u(t) \tag{8b}$$

where $u(t)$ [kg m^2 s^{-2}] is the external torque acting on the pendulum arm, $\dot{\varphi}(t)$ [rad s^{-1}] is the angular velocity, and $\ddot{\varphi}(t)$ [rad s^{-2}] is the angular acceleration of the pendulum arm. The numerical values of the pendulum parameters are given in Table 1.

Table 1: Pendulum Parameters

Parameter	Value	Units
m	1	kg
l	1	m
g	9.81	m s^{-2}
β	$2 \cdot 0.5 \cdot \sqrt{g/l}$	kg m^2 s^{-1}

- Create a numerical (computer) simulation of the time course of the pendulum deflection (pendulum as a nonlinear dynamic system).
  Alternatives:
  - Schematic representation for implementation in MATLAB-Simulink environment.
  - Implementation using a general ODE solver (without Simulink).

2.2.2 Pendulum – Simulation of Various Scenarios

- Simulate the deflection of the pendulum.
  Scenarios:

a) Initial state: $\varphi = 0.25$ [rad], $\dot{\varphi} = 0$ [rad/s]. Input signal $u(t) = 0$ [kg m^2 s^{-2}].
b) Initial state: $\varphi = 0.1$ [rad], $\dot{\varphi} = 1$ [rad/s]. Input signal $u(t) = 0$ [kg m^2 s^{-2}].
c) Initial state: $\varphi = 0$ [rad], $\dot{\varphi} = 0$ [rad/s]. Input signal $u(t) = 3$ [kg m^2 s^{-2}].
d) Initial state: $\varphi = 0$ [rad], $\dot{\varphi} = 0$ [rad/s]. Input signal $u(t) = 9.81$ [kg m^2 s^{-2}].
e) Initial state: $\varphi = 0$ [rad], $\dot{\varphi} = 0$ [rad/s]. Input signal $u(t) = 9.82$ [kg m^2 s^{-2}].

3 Notes on the Tasks

Schematic Representation of the Differential Equation

For the schematic representation of the differential equation (1), it is advantageous to rewrite this equation so that only the highest derivative of the unknown, i.e., the signal $\ddot{y}(t)$, is on the left side. Thus,

$$\ddot{y}(t) = -a_1\dot{y}(t) - a_0 y(t) + b_0 u(t) \tag{9}$$

Initially, we have the signal $\ddot{y}(t)$ available, so

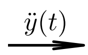

Figure 2: Block diagram of equation (1), step one.

The signal $\ddot{y}(t)$ is essentially the sum of three other signals.

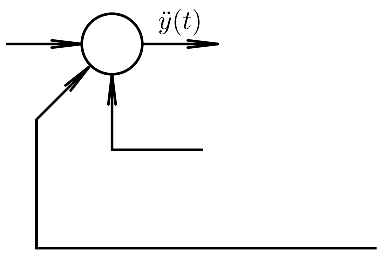

Figure 3: Block diagram of equation (1), step two.

The first signal is obtained by amplifying the signal $\dot{y}(t)$ with an amplifier of gain a_1. The signal $\dot{y}(t)$ can be obtained by integrating the signal $\ddot{y}(t)$.

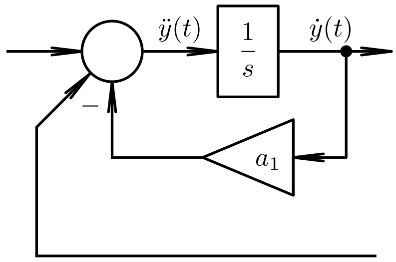

Figure 4: Block diagram of equation (1), step three.

The second signal is obtained by amplifying the signal $y(t)$ with an amplifier of gain a_0. The signal $y(t)$ can be obtained by integrating the signal $\dot{y}(t)$.

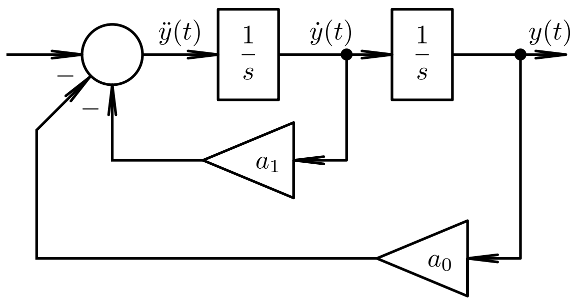

Figure 5: Block diagram of equation (1), step four.

The third signal is obtained by amplifying the known (available) signal $u(t)$ with an amplifier of gain b_0.

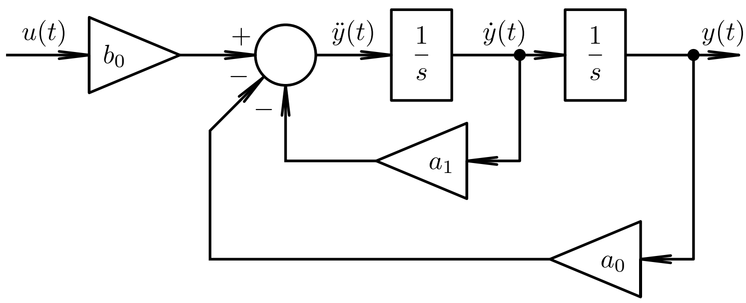

Figure 6: Block diagram of equation (1).

The respective integrators in the final diagram must have initial conditions $y(0) = y_0$ and $\dot{y}(0) = z_0$ (according to (1)).

Decomposition into a System of First-Order Differential Equations

In general, any higher-order differential equation can be decomposed (rewritten, transformed) into a system of first-order equations. The number of these equations is at least n, where n is the order of the original differential equation. For the equation (1), the order of the equation is 2, so the system will have at least 2 equations.

For these two new equations, it is necessary to consider two quantities that will take the place of the unknown in the new differential equations. Let's denote them as $x_1(t)$ and $x_2(t)$.

First, let's choose

$$x_1(t) = y(t) \tag{10}$$

This means

$$\dot{x}_1(t) = \dot{y}(t) \tag{11}$$

which, however, is not in the form we are looking for. The original quantity $y(t)$ appears on the right side.

Therefore, the second choice should be

$$x_2(t) = \dot{y}(t) \tag{12}$$

because then we can write the first differential equation in the form

$$\dot{x}_1(t) = x_2(t) \tag{13}$$

It remains to construct the second differential equation.

Since we chose (12), it is clear that

$$\dot{x}_2(t) = \ddot{y}(t) \tag{14}$$

The question is $\ddot{y}(t) = ?$ The answer is the original second-order differential equation. Let's rewrite (1) in the form

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_0 u(t) \tag{15}$$

$$\ddot{y}(t) = -a_1\dot{y}(t) - a_0 y(t) + b_0 u(t) \tag{16}$$

This means that

$$\dot{x}_2(t) = -a_1\dot{y}(t) - a_0 y(t) + b_0 u(t) \tag{17}$$

which, however, is still not the desired form of the second sought differential equation. On the right side of equation (17), only the new quantities $x_1(t)$ and $x_2(t)$ can appear, not the original quantity $y(t)$. However, it is sufficient to notice the previously chosen (10) and (12). Then we can write

$$\dot{x}_2(t) = -a_1 x_2(t) - a_0 x_1(t) + b_0 u(t) \tag{18}$$

which is the second sought first-order differential equation.

We have transformed the second-order differential equation (1) into a system of first-order differential equations

$$\dot{x}_1(t) = x_2(t) \tag{19}$$

$$\dot{x}_2(t) = -a_1 x_2(t) - a_0 x_1(t) + b_0 u(t) \tag{20}$$

Differential Equations of a DC Motor

In terms of the physical nature of a DC motor with constant external excitation, the differential equations describing the electromagnetic and mechanical subsystems of the motor are in the form (4) and (6). From the perspective of setting up a numerical simulation, it is advantageous to adjust these equations into a form that highlights the individual signals in terms of whether they are inputs or outputs, and thus the overall structure of the dynamic system.

Moreover, it can be stated (here without providing details) that an ODE solver in general (i.e., a tool performing numerical simulation) works with a function that realizes the functional relationship between the time derivatives of quantities and their (non-derivative) values. Therefore, it is useful to have the differential equations (first-order) in a form where only the time derivatives are on the left side.

Equations (4) and (6) can be written in the form

$$\frac{\mathrm{d}i(t)}{\mathrm{d}t} = -\frac{R}{L}i(t) - \frac{C_{u\omega}}{L}\omega(t) + \frac{1}{L}u(t) \tag{21}$$

$$\frac{\mathrm{d}\omega(t)}{\mathrm{d}t} = \frac{C_{uM}}{J_m}i(t) \tag{22}$$

If we wanted to further highlight the relationship between the rate of change of signals (time derivative), their instantaneous values, and other signals, we could use a matrix notation, in this case:

$$\begin{bmatrix} \dot{i}(t) \\ \dot{\omega}(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{C_{u\omega}}{L} \\ \frac{C_{uM}}{J_m} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ \omega(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} u(t) \tag{23}$$

Sample Simulation of a DC Motor

A sample result of the simulation of the motor start (mechanically unloaded) at a constant supply voltage of 520 [V] is shown in the following figure:

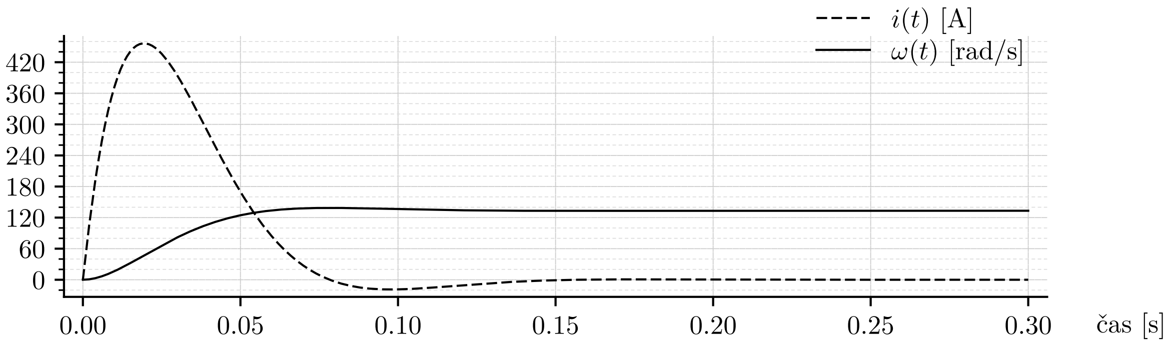

Figure 7: Graphical representation of the numerical simulation result.

Numerical Simulation of the Pendulum

The differential equation describing the pendulum (8) can be rewritten as a system of first-order equations in the form

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{\beta}{ml^2} x_2(t) - \frac{g}{l} \sin\left(x_1(t)\right) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u(t) \tag{24}$$

where the state of the pendulum consists of two quantities: the angle of the pendulum arm φ and the angular velocity of the pendulum arm $\dot{\varphi}$. The state vector therefore has two elements $x^\mathsf{T} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$, where $x_1 = \varphi$ and $x_2 = \dot{\varphi}$.

Let's create a function that implements the system of differential equations (24), but consider that the input signal $u(t)$ is zero. In other words, let's not consider the input signal at all. In other words, the external torque is zero, $u(t) = 0$, and therefore we can write

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{\beta}{ml^2} x_2 - \frac{g}{l} \sin(x_1) \end{bmatrix} \tag{25}$$

This is an autonomous nonlinear time-invariant second-order system. Its behavior depends only on the initial state at the beginning of the considered time.

The function that implements the given system can be as follows:

Entire file PravaStr.m

```
function dotx = PravaStr(t,x)

global m l g beta

dotx1 =  x(2);
dotx2 = - (beta/m*l^2)*x(2)  -  (g/l)*sin(x(1));

dotx = [dotx1; dotx2];

end
```

Let's create a "main script" where we set everything necessary and where we will call the ODE solver. First, let the global variables (in this case, the pendulum parameters) be:

Part of the file hlSkript.m

```
global m l g beta

m = 1; %kg
l = 1; %m
g = 9.81; %m/s^2
beta = 2*0.5*sqrt(g/l); %kgm^2/s
```

Define the time vector, which will determine for which time points the ODE solver will return the numerical solution:

Part of the file hlSkript.m

```
timeVect = 0:0.1:5;
```

Call the ODE solver, while choosing the initial conditions - the initial state of the pendulum. Let the initial state be $x_1(0) = 0.25$ [rad] and $x_2(0) = 0$ [rad/s].

Part of the file hlSkript.m

```
[t,x] = ode45(@(t,x) PravaStr(t,x), timeVect, [0.25; 0]);
```

The variable `x` now contains two columns - the first column is the first state variable and the second column is the second state variable. To plot the calculated solution:

Part of the file hlSkript.m

```
figure(1)
plot(t,x)
```

The resulting numerical solution is graphically shown in Fig. 8.

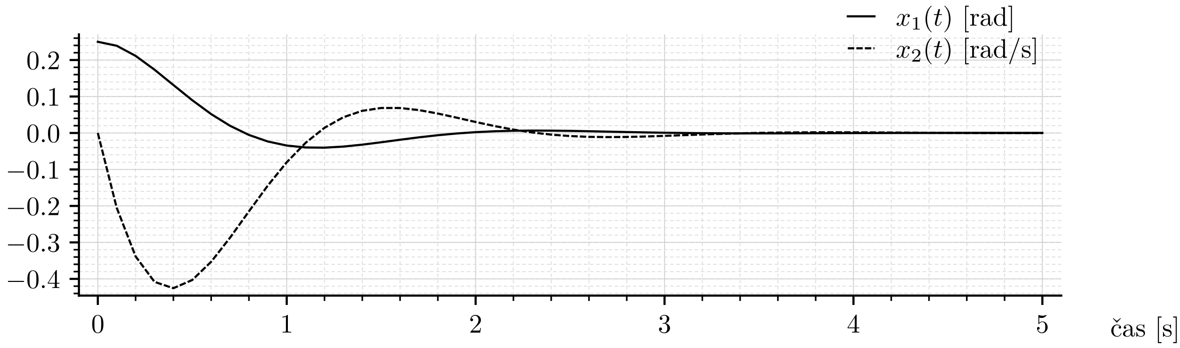

Figure 8: Graphical representation of the numerical solution.

In Fig. 8, however, it is just a basic representation. It might be more meaningful, for example, to plot only the pendulum position (deflection) separately and additionally not in radians but in degrees – see Fig. 9. For such a plot, you can add to the main script:

Part of the file hlSkript.m

```
    figure(2)
    plot(t,x(:,1)*180/pi)
```

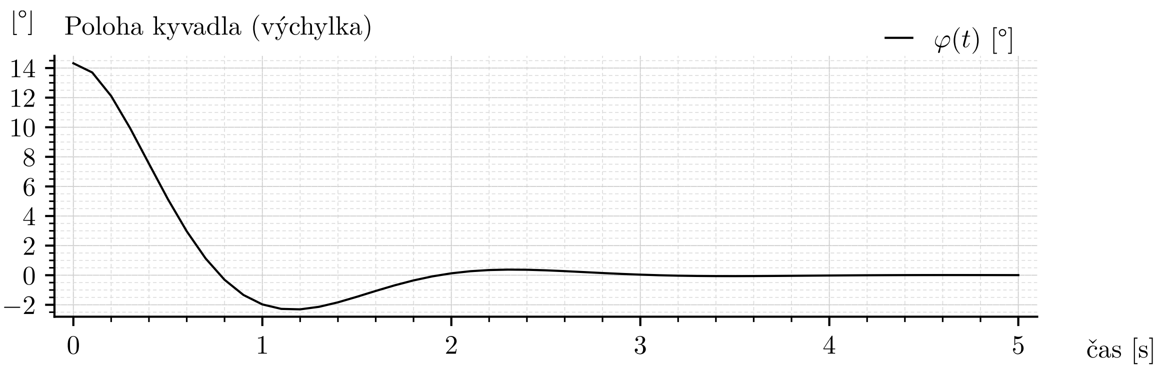

Figure 9: Graphical representation of the pendulum position.

Let the initial conditions (initial state) be: $x_1(0) = 0.1$ [rad] and $x_2(0) = 1$ [rad/s]. Meanwhile, let the input $u(t)$ still be zero. The simulation result is shown in Fig. 10.

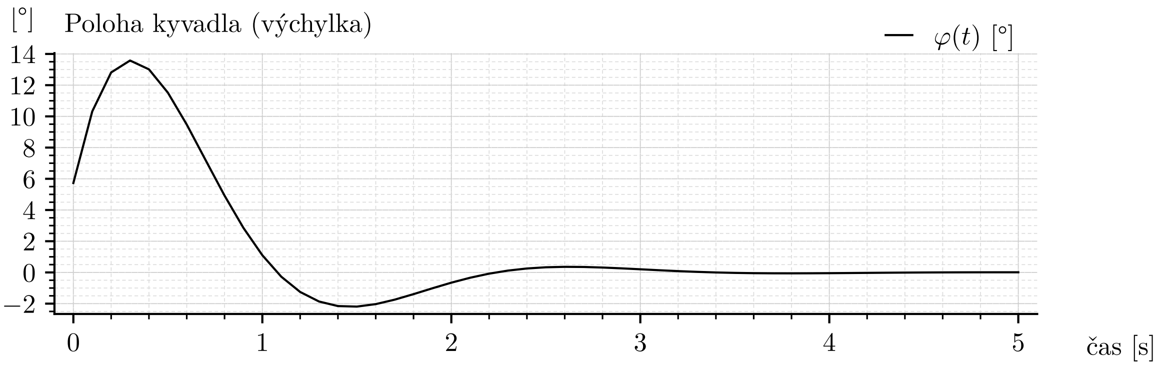

Figure 10: Graphical representation of the pendulum position.

Let's modify the original function and script in MATLAB so that it is possible to simulate a non-zero input signal $u(t)$.

The function that implements the system of differential equations (24) with the input signal $u(t)$:

Entire file PravaStr_u.m

```
function dotx = PravaStr_u(t,x, u)

global m l g beta

dotx1 =  x(2);
dotx2 = - (beta/m*l^2)*x(2) - (g/l)*sin(x(1)) + (1/m*l^2) * u;

dotx = [dotx1; dotx2];

end
```

Let's create a "main script" where we set everything necessary and where we will call the ODE solver:

File hlSkript_u.m

```
global m l g beta

m = 1; %kg
l = 1; %m
g = 9.81; %m/s^2
beta = 2*0.5*sqrt(g/l); %kgm^2/s

u = 3

[t,x] = ode45(@(t,x) PravaStr_u(t,x,u), [0 10], [0; 0]);

figure(3)
plot(t,x(:,1)*180/pi)
```

Let's simulate the case when, for example, $u(t) = 3$ [kg m^2 s^{-2}] (note: for better clarity, let's consider the initial conditions to be zero). The simulation result is shown in Fig. 11.

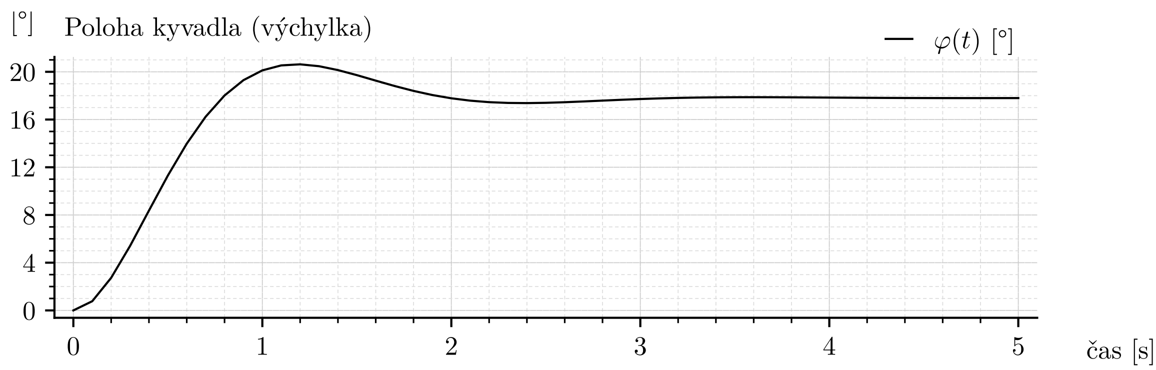

Figure 11: Graphical representation of the pendulum position.

An interesting case is when $u(t) = 9.81$ [kg m^2 s^{-2}]. The simulation result is shown in Fig. 12.

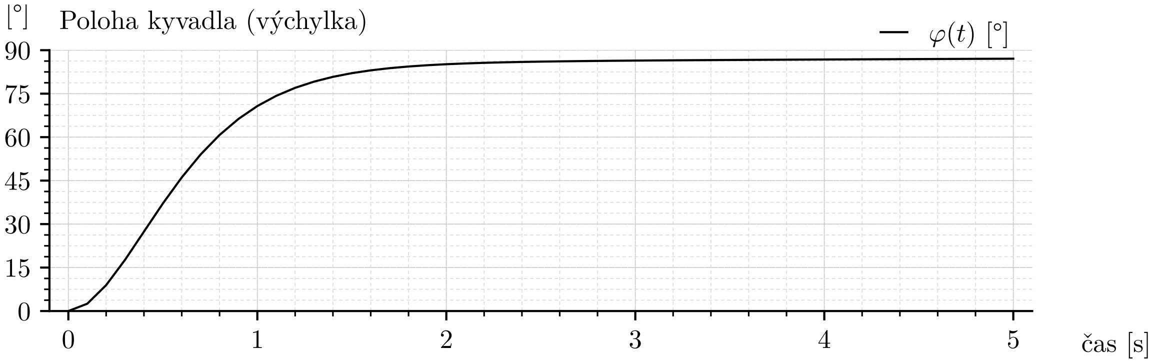

Figure 12: Graphical representation of the pendulum position.

However, it will be better shown if we extend the time vector (simulation time) – see Fig. 13. It is clear that the pendulum is approaching the value of 90 degrees.

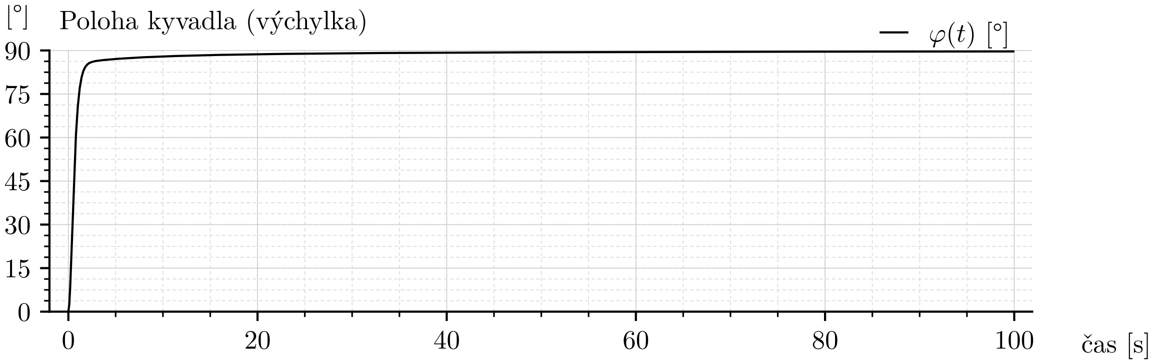

Figure 13: Graphical representation of the pendulum position.

What happens if $u = 9.82$ [kg m^2 s^{-2}]? (Fig. 14)

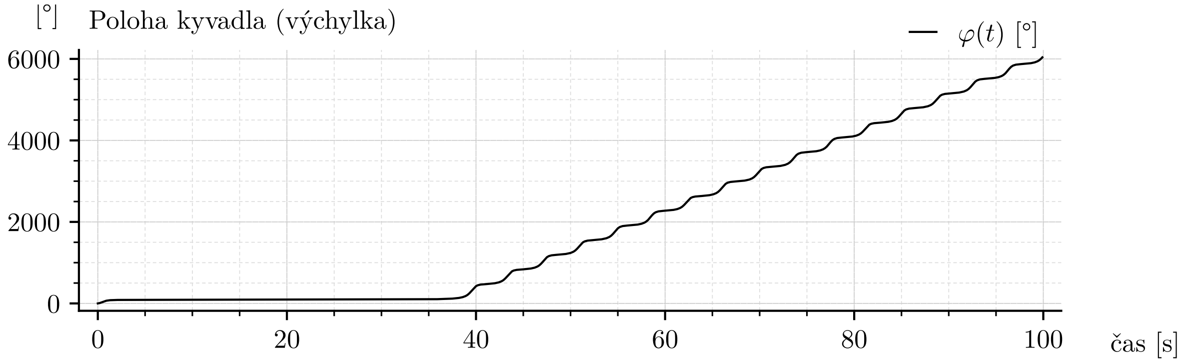

Figure 14: Graphical representation of the pendulum position.