Differential equations

Contents

The goal of this text is to define the concept of a differential equation within the scope of the subject. A differential equation will serve as a mathematical description of a dynamic system. First and foremost, it is necessary to focus on solving the differential equation. It is important to study what a solution to a differential equation is, and then the ways and methods to find the solution to a differential equation.

1 Introduction to Basic Concepts

Let us consider the equation

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + ay(t) = 0 \tag{1}$$

The unknown in this equation is the function of time $y(t)$. The symbol t represents time. The coefficient $a \in \mathbb{R}$ is a real number (it is not a function of time).

The first term on the left-hand side of the equation is the time derivative of the function $y(t)$.

We call this a differential equation because the derivative of the unknown function $y(t)$ appears in the equation.

The derivative of the function $y(t)$ with respect to time is denoted as $\frac{\mathrm{d}y(t)}{\mathrm{d}t}$ or $\dot{y}(t)$. The second derivative would be denoted $\frac{\mathrm{d}^2y(t)}{\mathrm{d}t^2}$ or $\ddot{y}(t)$, and so on. Thus, equation (1) could also be written in the form

$$\dot{y}(t) + ay(t) = 0 \tag{2}$$

Moreover, equation (1) is:

- An ordinary differential equation (ODE).
  A differential equation is ordinary when the unknown function is a function of only one variable (in this case, time t).
- A first-order equation.
  The equation is first-order because the highest derivative of the unknown function is of first order (order of the derivative). The order of the equation is determined by the highest derivative of the unknown function.
- A linear equation.
  We can say that the unknown function $y(t)$ and its derivatives appear only in linear combinations. We refer to such an equation as linear. It is also useful to view the equation in a generalized form, where the highest derivative of the unknown function is on the left-hand side in this case, the first derivative and the right-hand side is treated as a function of the remaining derivatives of the unknown function, in this case, the zeroth derivative:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = f\left(y(t)\right) \tag{3}$$

If this function f is linear, then the equation is linear.

Homogeneous and Non-homogeneous Equation

Equation (1) is homogeneous. A differential equation is homogeneous when only the unknown function of time appears in the equation, with no other functions of time present. From the system's perspective, this means that the system has only an output, i.e., just the output signal $y(t)$. This can be schematically represented as:

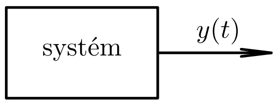

Figure 1: System with output signal (output variable)

An example of a non-homogeneous equation could be the equation

$$\dot{y}(t) + ay(t) = u(t) \tag{4}$$

where $u(t)$ is a function of time. However, it is not the unknown function of time. The only unknown in the equation remains $y(t)$. Equation (4) is non-homogeneous because it contains other functions of time besides the unknown.

From the system's perspective, this means that the system also has an input, the input signal $u(t)$. This can be schematically represented as:

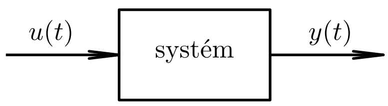

Figure 2: System with one input signal and one output signal.

Initial Condition

The time function $y(t)$ takes on a value at any time t. The specific value, so to speak, "at the beginning of time," formally at time $t = 0$, is called the initial condition. Formally,

$$y(0) = y_0 \tag{5}$$

where y_0 is the specific value of the function $y(t)$ at time $t = 0$.

Higher-Order Equations

An example of a second-order homogeneous differential equation could be

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = 0 \tag{6}$$

where a_1, a_0 are coefficients, real numbers.

An example of a second-order non-homogeneous differential equation could be

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = u(t) \tag{7}$$

where $u(t)$ is a function of time representing the input signal of the system. In general, we can also consider derivatives of the input signal $u(t)$, for example,

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = u(t) + \dot{u}(t) \tag{8}$$

which still remains a second-order differential equation, though non-homogeneous.

In general terms, an ordinary linear non-homogeneous differential equation of n-th order is:

$$a_n \frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1} \frac{\mathrm{d}^{n-1} y(t)}{\mathrm{d}t^{n-1}} + \cdots + a_0 y(t) = b_m \frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} + b_{m-1} \frac{\mathrm{d}^{m-1} u(t)}{\mathrm{d}t^{m-1}} + \cdots + b_0 u(t) \tag{9}$$

The coefficients $a_n, \ldots, a_0$ and $b_m, \ldots, b_0$ are constants. The number n indicates the highest time derivative of the output signal, and the number m indicates the highest time derivative of the input signal. We assume that the equation describes a *causal* dynamic system, so it must hold that $n > m$.

2 Schematic Representation of a Differential Equation (Dynamic System)

2.1 Homogeneous First-Order Differential Equation

Let us consider the differential equation in the form:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -a y(t) \qquad y(0) = y_0 \tag{10}$$

where $y(t)$ is the unknown function of time, and we seek this time function as the solution to the equation. The parameter $a \in \mathbb{R}$ is a given real number. The value y_0 in the initial condition is also specified.

This is a homogeneous differential equation because it contains no terms other than those involving the unknown $y(t)$. If we move the terms containing the unknown to the left side, we have:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} + a y(t) = 0 \tag{11}$$

where zero is on the right side. Thus, the equation is homogeneous.

From the system perspective, $y(t)$ is the output variable (output signal), and the number a is a system parameter. The initial condition $y(0) = y_0$ indicates that y_0 is the value of the signal $y(t)$ at time $t = 0$.

In this case, there is no input variable of the system (input signal). As we mentioned, from the viewpoint of the differential equation or in terms of the concept of a dynamic system, we are dealing with a homogeneous differential equation. It contains only the unknown itself (output variable, output signal).

An important fact is that this equation is called a first-order differential equation because the highest derivative of the unknown is of the first order.

The differential equation, or from another perspective, the dynamic system in the form (10), can be schematically represented using basic functional blocks, the building elements, among which are signals (time-varying quantities).

The resulting scheme is as follows:

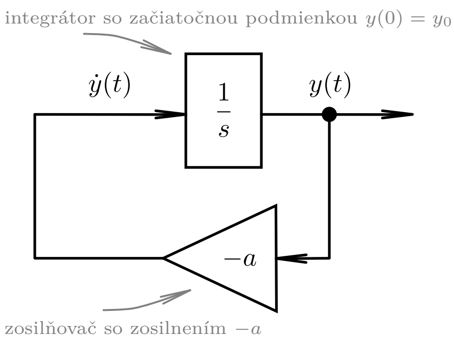

Figure 3: Schematic of the dynamic system corresponding to equation (10).

2.2 Elements of a Block Diagram

A dynamic system, or the differential equation that describes it, can also be represented graphically using a *block diagram*. Such a diagram may even allow for a reverse process, meaning that the diagram specifies the differential equation. The resulting block diagram also facilitates further analysis of the dynamic system.

Typically, the following elements and blocks are used within such a diagram.

Signal

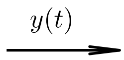

Figure 4: Signal in a block diagram.

A signal is represented by a line with an arrow indicating the direction of information flow. The label of the signal is provided alongside the line.

Amplifier (Gain)

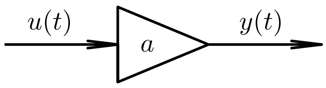

Figure 5: Gain in a block diagram.

This is a block that generally amplifies the input signal to this block. Of course, it may also represent "attenuation." In other words, this block multiplies the value of the input signal $u(t)$ by the parameter a, resulting in the output signal $y(t)$. Mathematically, this can be expressed as

$$y(t) = a\,u(t) \tag{12}$$

The parameter a can take any value; it may be less than 1 (attenuation) or negative.

Summator

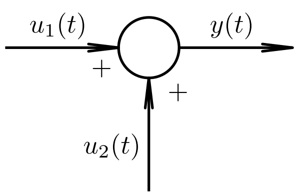

Figure 6: Summator in a block diagram.

The summator performs the addition of two or more signal values. It can also perform subtraction. An example of the mathematical representation is:

$$y(t) = u_1(t) + u_2(t) \tag{13}$$

Integrator

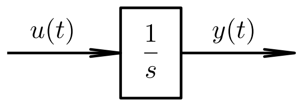

Figure 7: Integrator in a block diagram.

The integrator represents the time integration of the input signal in terms of the mathematical representation:

$$y(t) = \int u(t)\,\mathrm{d}t \qquad y(0) = y_0 \tag{14}$$

where y_0 is the initial value of the signal $y(t)$, that is, the initial condition.

Differentiator (Derivative)

For completeness, we also include the block that performs the time differentiation of the signal. It is the opposite of the integrator. However, this block is used less frequently in practice. The reason is that implementing time differentiation is practically

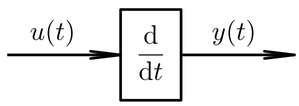

Figure 8: Time differentiation in a block diagram.

2.3 Homogeneous First-Order Differential Equation: Example of Block Diagram Construction

Consider the dynamic system described by the differential equation in the form:

$$\dot{y}(t) + ay(t) = 0 \qquad y(0) = y_0 \tag{15}$$

The procedure for constructing the block diagram of the dynamic system can be outlined as follows.

The equation (15) is of the first order, and the unknown is the time function $y(t)$. Let's rewrite the equation (15) so that the left side contains only the highest derivative of the unknown, in this case, the signal $\dot{y}(t)$. Thus, we have:

$$\dot{y}(t) = -ay(t) \tag{16}$$

The equation in this form serves as the basis for constructing the block diagram. It is clear that the signal $\dot{y}(t)$ exists. In other words, what we definitely have available is the signal $\dot{y}(t)$. If this signal did not exist, then the equation (16) would be nonsensical. Therefore, in the diagram, we have the signal $\dot{y}(t)$.

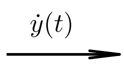

Figure 9: Block diagram of the equation (15), step one.

The equation in the form (16) also directly shows that the signal $\dot{y}(t)$ is equivalent to the expression $-ay(t)$. Can we construct a block diagram for this expression? It is obviously a gain block with a gain of $-a$, which has the signal $y(t)$ as its input. Let's add to the block diagram:

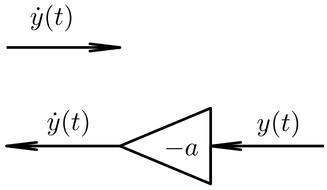

Figure 10: Block diagram of the equation (15), step two.

Since literally $\dot{y}(t) = -ay(t)$, we have:

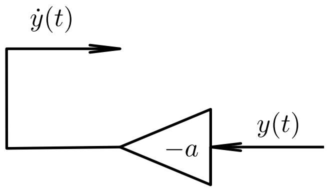

Figure 11: Block diagram of the equation (15), step three.

It is important to remember that the signal $\dot{y}(t)$ essentially exists; it is available. However, the actual signal $y(t)$ is not available. It needs to be created from what is already available. It is clear that the signal $y(t)$ can be obtained by integrating $\dot{y}(t)$, thus:

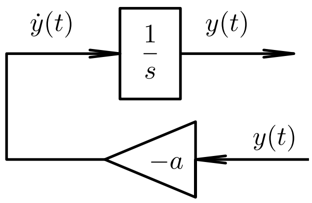

Figure 12: Block diagram of the equation (15), step four.

The integrator must have the initial condition $y(0) = y_0$ (according to (15)).

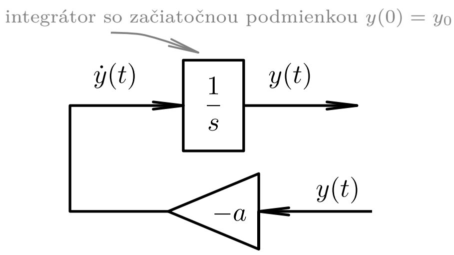

Figure 13

Finally, we arrive at:

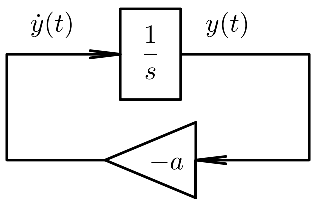

Figure 14: Block diagram of the equation (15).

This is the block diagram of the dynamic system corresponding to the differential equation (15).

2.4 Non-Homogeneous First-Order Differential Equation: Example of Block Diagram Construction

Consider the dynamic system described by the differential equation in the form:

$$\dot{y}(t) + ay(t) = bu(t) \qquad y(0) = y_0 \tag{17}$$

where a and b are constants and $u(t)$ is a known input signal.

Let's rewrite the equation (17) so that the left side contains only the highest derivative of the unknown, which is the signal $\dot{y}(t)$:

$$\dot{y}(t) = -ay(t) + bu(t) \tag{18}$$

Initially, we have the signal $\dot{y}(t)$ available, so we start our block diagram construction with this signal:

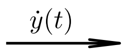

Figure 15: Block diagram of the equation (17), step one.

The signal $\dot{y}(t)$ is the sum of two other signals.

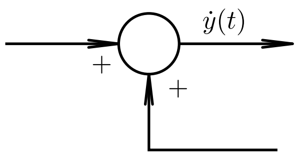

Figure 16: Block diagram of the equation (17), step two.

The first signal is obtained by amplifying the signal $y(t)$ through a gain block with a parameter of $-a$.

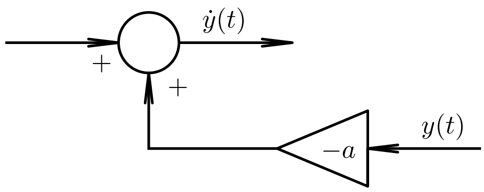

Figure 17: Block diagram of the equation (17), step three.

The second signal is obtained by amplifying the known (available) signal $u(t)$ through a gain block with a parameter of b.

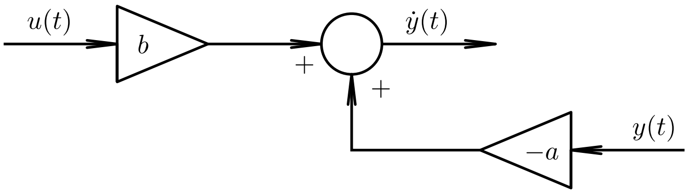

Figure 18: Block diagram of the equation (17), step four.

However, in this step, the signal $y(t)$ is not available. It needs to be created from what is already available. The signal $y(t)$ can be obtained by integrating the signal $\dot{y}(t)$.

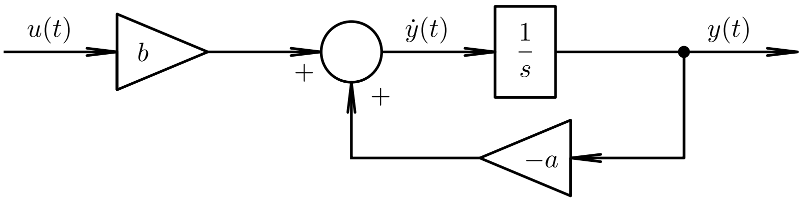

Figure 19: Block diagram of the equation (17).

The integrator must have the initial condition $y(0) = y_0$ (according to (17)).

3 Decomposition into a System of First-Order Differential Equations

In general, any higher-order differential equation can be decomposed (rewritten, transformed) into a system of first-order equations. The number of these equations is at least n.

As an example, consider the differential equation in the form

$$a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = b_0u(t) \tag{19}$$

The task is to decompose this second-order differential equation into two first-order differential equations.

For these two new equations, we need to consider two variables that will replace the unknown in the new differential equations. Let us denote them by $x_1(t)$ and $x_2(t)$.

We are looking for two differential equations; in other words, we seek

$$\begin{aligned} \dot{x}_1(t) &= \quad ? \\ \dot{x}_2(t) &= \quad ? \end{aligned}$$

where the right-hand side must contain only terms that involve the new variables $x_1(t)$ and $x_2(t)$ and do not include the original variable $y(t)$. By the way, the variable $u(t)$, i.e., the input signal of the system, is known in terms of solving the differential equation. The unknown is the output variable $y(t)$. Therefore, the signal $u(t)$ can appear unchanged in the newly sought differential equations. This can be achieved through the following procedure.

First, *let's choose*

$$x_1(t) = y(t) \tag{20}$$

This means

$$\dot{x}_1(t) = \dot{y}(t) \tag{21}$$

which, however, is not in the form we are looking for. The original variable $y(t)$ appears on the right-hand side.

For the second choice, let

$$x_2(t) = \dot{y}(t) \tag{22}$$

because then we can write the first differential equation in the form

$$\dot{x}_1(t) = x_2(t) \tag{23}$$

It remains to construct the second differential equation.

Since we chose (22), it is clear that

$$\dot{x}_2(t) = \ddot{y}(t) \tag{24}$$

The question is $\ddot{y}(t) = ?$ The answer is the original second-order differential equation. Let's rearrange (19) into the form

$$\ddot{y}(t) + \frac{a_1}{a_2}\dot{y}(t) + \frac{a_0}{a_2}y(t) = \frac{b_0}{a_2}u(t) \tag{25}$$

$$\ddot{y}(t) = -\frac{a_1}{a_2}\dot{y}(t) - \frac{a_0}{a_2}y(t) + \frac{b_0}{a_2}u(t) \tag{26}$$

This means that

$$\dot{x}_2(t) = -\frac{a_1}{a_2}\dot{y}(t) - \frac{a_0}{a_2}y(t) + \frac{b_0}{a_2}u(t) \tag{27}$$

which is still not in the desired form of the second sought differential equation. On the right side of equation (27), only the new variables $x_1(t)$ and $x_2(t)$ can appear, not the original variable $y(t)$. However, it is sufficient to notice the previously chosen (20) and (22). We can then write

$$\dot{x}_2(t) = -\frac{a_1}{a_2}x_2(t) - \frac{a_0}{a_2}x_1(t) + \frac{b_0}{a_2}u(t) \tag{28}$$

which is the second sought first-order differential equation.

We have transformed the second-order differential equation

$$a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0y(t) = b_0u(t) \tag{29}$$

into a system of first-order differential equations

$$\dot{x}_1(t) = x_2(t) \tag{30}$$

$$\dot{x}_2(t) = -\frac{a_1}{a_2}x_2(t) - \frac{a_0}{a_2}x_1(t) + \frac{b_0}{a_2}u(t) \tag{31}$$

4 Analytical Solution of Differential Equations

The solution to the differential equation is a function of time $y(t)$. If we find such a function of time $y(t)$ that, when substituted into the differential equation, satisfies the equation, then we have found the solution to the differential equation.

4.1 Homogeneous First-Order Differential Equation

Consider the differential equation in the form:

$$\frac{\mathrm{d}y(t)}{\mathrm{d}t} = -ay(t) \qquad y(0) = y_0 \tag{32}$$

where $y(t)$ is the unknown function of time we seek as the solution to the equation, and $a \in \mathbb{R}$ (it is a real number) is a given parameter. The value y_0 in the initial condition is also given.

4.1.1 Outline of the Method of Separation of Variables

Let's rearrange the differential equation (32) so that the same variables are on the same sides. In the form (32), the signal $y(t)$ appears on both sides of the equation. Let it appear only on the left side. Likewise, let time t appear only on the right side. Thus,

$$\frac{1}{y(t)}\mathrm{d}y(t) = -a\mathrm{d}t \tag{33}$$

Notice that it is now possible to integrate both sides of the equation, each with respect to its own variable, that is,

$$\int \frac{1}{y(t)}\mathrm{d}y(t) = \int -a\mathrm{d}t \tag{34}$$

The result of the integration is

$$\ln\left(y(t)\right) + k_1 = -at + k_2 \tag{35}$$

where k_1 and k_2 are constants resulting from the indefinite integrals (and we have silently assumed that $y(t)$ will not take on negative values).

Equation (35) is no longer a differential equation. No quantity in it is derived with respect to time. Let's express the signal $y(t)$ from equation (35). By rearranging,

$$\ln\left(y(t)\right) = -at + k_3 \tag{36}$$

we have introduced the constant $k_3 = k_2 - k_1$. Further,

$$y(t) = e^{(-at+k_3)} \tag{37a}$$

$$y(t) = e^{(-at)}e^{k_3} \tag{37b}$$

At this point, equation (37b) is a formula that specifies the time dependence of the quantity y. It expresses the signal (time function) $y(t)$. The time function $y(t)$ is the solution to the differential equation (33).

In equation (37b), the constant e^{k_3} is a general constant and can take any value. It can be shown, but we will refrain from providing a formal proof here, that this constant is determined by the initial condition assigned to the differential equation. In this case, it holds that $e^{k_3} = y_0$.

The sought solution of the differential equation is the time function in the form

$$y(t) = y_0 \; e^{(-at)} \tag{38}$$

4.1.2 Outline of the Method Using *Characteristic Equation*

For perhaps the first contact with the term *characteristic equation*, let's provide a brief outline of the essence of this concept.

For the differential equation under consideration, recall

$$\dot{y}(t) + ay(t) = 0 \tag{39}$$

we are essentially looking for a solution in the form

$$y(t) = e^{st} \tag{40}$$

where s is generally a complex number. Then it follows that

$$\dot{y}(t) = s\, e^{st} \tag{41}$$

Substituting this into the equation itself, we can write

$$s\, e^{st} + a\, e^{st} = 0 \tag{42}$$

$$e^{st}\left(s + a\right) = 0 \tag{43}$$

Since $e^{st} \neq 0$ (if it is to be a non-trivial solution), it must hold that

$$(s + a) = 0 \tag{44}$$

This is the *characteristic equation*.

It is evident that we need to find the root of the polynomial, which we will call the characteristic polynomial, $s + a$. The root of the polynomial (the solution to the characteristic equation) in this case is $s_1 = -a$. Thus, we have found a solution, which is called the fundamental solution (let's denote it as $y_f(t)$), in this case:

$$y_f(t) = e^{(-at)} \tag{45}$$

This solution corresponds to the given root of the characteristic equation. However, it does not take into account all possible solutions, which for the homogeneous differential equation are determined by the initial condition. The general solution in this case should take the form

$$y(t) = c\, e^{(-at)} \tag{46}$$

where c is a constant, which allows us to clearly determine the value of the general solution at time $t = 0$ given by the initial condition. The specific solution is thus

$$y(t) = y_0 \; e^{(-at)} \tag{47}$$

Note: In the case of a homogeneous equation of higher order, it would be possible to show that the general solution is always given as a linear combination of fundamental solutions.

4.2 Method of Characteristic Equation

Under this title, we will understand a certain method for finding the general solution of a homogeneous differential equation within this subject. The method is based on the analysis of the problem, which leads to the establishment of a certain structure of solutions for the homogeneous differential equation. Typically, the given problem is examined for the case of a second-order differential equation. The following is processed mainly according to the teaching material [1].

Let us emphasize the attribute *homogeneous* in the sense that the subject of this text is the *general solution* of the differential equation. Such a solution can be further specified in terms of initial (or boundary) conditions.

Consider the differential equation in the form

$$\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = 0 \tag{48}$$

where $y(t)$ is the unknown time function. We seek this function to be a solution to the equation (48), that is, such that when substituting $y(t)$ and its derivatives into the equation (48), the equation holds. For completeness, $a_1 \in \mathbb{R}$ and $a_2 \in \mathbb{R}$ are constant coefficients.

The general solution of the differential equation (48) is found by first seeking two linearly independent solutions. These solutions are called *fundamental solutions*. Their linear combination is then the general solution of the differential equation (48).

4.2.1 Fundamental Solutions

Let $y_1(t)$ and $y_2(t)$ be two solutions of the differential equation (48). Then any linear combination $c_1y_1(t) + c_2y_2(t)$, $c_1, c_2 \in \mathbb{R}$ is also a solution of the differential equation (48) (this follows from the linearity of differentiation).

Specifically, if $c_1 = c_2 = 0$, we obtain the solution $y(t) = 0$. This is called the zero solution or *trivial solution*.

Through further analysis, it can be shown that if the two solutions $y_1(t)$ and $y_2(t)$ are linearly independent, then their linear combination can be used to establish the general solution of the differential equation [1].

Having one solution ("some solution") and then attempting to generalize it is essentially not possible. We need at least two solutions (clearly not "just any") and we need them to be linearly independent.

The concept of linear independence of functions is related to the term *Wronskian* or Wronskian determinant.

Two linearly independent solutions of a differential equation are called *fundamental solutions* or *basis of solutions* of the differential equation (see also [2]). It can be shown that the solutions of a differential equation form a vector space and that it always has a basis [2].

The general solution $y(t)$ is called a linear combination of two fundamental solutions $y_{f1}(t)$ and $y_{f2}(t)$, that is, $y(t) = c_1 y_{f1}(t) + c_2 y_{f2}(t)$, where $c_1, c_2 \in \mathbb{R}$ are arbitrary constants.

We do not know a general method by which we could always find fundamental solutions. In practice, various procedures are used, which are usually based on trying different forms of solutions. For linear differential equations with constant coefficients, it is possible to find solutions in the form of exponential functions. In the case of the differential equation (48), it is possible to seek a solution in the form $y(t) = e^{st}$, where $s \in \mathbb{C}$.

4.2.2 General Solution of Homogeneous Differential Equation

Consider the differential equation in the form

$$\ddot{y}(t) + a_1 \dot{y}(t) + a_0 y(t) = 0 \tag{49}$$

where $a_1 \in \mathbb{R}$ and $a_2 \in \mathbb{R}$.

We seek the solution of the equation (49) in the form

$$y(t) = e^{st} \tag{50}$$

where $s \in \mathbb{C}$. Then

$$\dot{y}(t) = se^{st} \tag{51}$$

$$\ddot{y}(t) = s^2 e^{st} \tag{52}$$

Substituting into the equation (49) gives us

$$s^2 e^{st} + a_1 s e^{st} + a_0 e^{st} = 0 \tag{53a}$$

$$e^{st}(s^2 + a_1 s + a_0) = 0 \tag{53b}$$

Since $e^{st} \neq 0$, because we are not looking for a trivial solution, for finding the solution, it must hold that

$$s^2 + a_1 s + a_0 = 0 \tag{54}$$

Characteristic Polynomial and Roots of the Characteristic Polynomial

The equation (54) is called the *characteristic equation* of the differential equation (49), and its solutions s_1 and s_2 are the *characteristic numbers* of the differential equation (49). In other words, the polynomial $s^2 + a_1 s + a_0$ is called the *characteristic polynomial* of the differential equation (49).

The function e^{st} is a solution of the differential equation (49) if s is a solution of the algebraic equation (54). In other words, e^{st} is a solution of the differential equation (49) if s is a root of the characteristic polynomial of the differential equation.

In general, we can determine as many distinct functions e^{st}, i.e., solutions of the differential equation, as there are roots of the characteristic polynomial. In the case of a second-order differential equation, there are two roots. They can be:

- two distinct real roots,
- one double real root,
- two complex conjugate roots.

These cases lead to different types of solutions of the differential equation.

4.2.3 Case: Two Distinct Real Roots

Let the characteristic polynomial have two distinct real roots s_1 and s_2. Then the general solution of the differential equation (49) is

$$y(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} \tag{55}$$

where we consider the fundamental solutions to be

$$y_{f1}(t) = e^{s_1 t} \tag{56a}$$
$$y_{f2}(t) = e^{s_2 t} \tag{56b}$$

Their Wronskian (Wronskian determinant) [1, 2] is

$$\begin{aligned} W(t) &= \begin{vmatrix} e^{s_1 t} & e^{s_2 t} \\ s_1 e^{s_1 t} & s_2 e^{s_2 t} \end{vmatrix} \\ &= e^{s_1 t} s_2 e^{s_2 t} - e^{s_2 t} s_1 e^{s_1 t} \\ &= e^{(s_1+s_2)t}(s_2 - s_1) \neq 0 \end{aligned} \tag{57}$$

which shows that they are linearly independent and confirms that the general solution of the differential equation (49) in this case is

$$y(t) = c_1 e^{s_1 t} + c_2 e^{s_2 t} \tag{58}$$

4.2.4 Case: One Double Real Root

Let the characteristic polynomial have the root s. The solution of the differential equation (49) is $y_{f1} = e^{st}$. However, to find the general solution, we also need a second fundamental solution. It can be shown that $y_{f2} = te^{st}$ is also a solution of the differential equation (49). Thus, we have two fundamental solutions in the form

$$y_{f1} = e^{st} \tag{59a}$$
$$y_{f2} = te^{st} \tag{59b}$$

It can be shown that they are linearly independent.

The general solution of the differential equation (49) in this case is in the form

$$y(t) = c_1 e^{st} + c_2 t e^{st} \tag{60}$$

4.2.5 Case: Two Complex Conjugate Roots

Let the characteristic polynomial have two complex conjugate roots

$$s_1 = a + jb \tag{61a}$$
$$s_2 = a - jb \tag{61b}$$

where $a, b \in \mathbb{R}$ and j is the imaginary unit.

In this case, the functions $e^{(a+jb)t}$ and $e^{(a-jb)t}$ are not real functions. Utilizing Euler's formula, we have

$$\begin{aligned} e^{s_1 t} = e^{at} e^{jbt} &= e^{at}\left(\cos(bt) + j\sin(bt)\right) \\ &= e^{at}\cos(bt) + je^{at}\sin(bt) \end{aligned} \tag{62a}$$
$$\begin{aligned} e^{s_2 t} = e^{at} e^{-jbt} &= e^{at}\left(\cos(bt) - j\sin(bt)\right) \\ &= e^{at}\cos(bt) - je^{at}\sin(bt) \end{aligned} \tag{62b}$$

These functions are thus linear combinations of the functions

$$y_{f1}(t) = e^{at}\cos(bt) \tag{63a}$$
$$y_{f2}(t) = e^{at}\sin(bt) \tag{63b}$$

It can be shown that $y_{f1}(t)$ and $y_{f2}(t)$ are solutions of the differential equation (49) and that they are linearly independent. The general solution of the differential equation (49) in this case is in the form

$$y(t) = c_1 e^{at}\cos(bt) + c_2 e^{at}\sin(bt) \tag{64}$$

5 Examples

5.1 Example 1

Find the analytical solution of the differential equation. Use the characteristic equation method.

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = 0 \qquad y(0) = 3 \qquad \dot{y}(0) = -2$$

The first step is to establish the characteristic equation. In this case,

$$s^2 + 3s + 2 = 0 \tag{65}$$

In the second step, to determine the fundamental solutions, we look for the roots of the characteristic equation. The solutions of the characteristic equation are

$$s_1 = -1 \tag{66a}$$
$$s_2 = -2 \tag{66b}$$

The corresponding fundamental solutions are

$$y_{f1}(t) = e^{-t} \tag{67a}$$
$$y_{f2}(t) = e^{-2t} \tag{67b}$$

The third step is to determine the general solution of the differential equation. It is a linear combination of the fundamental solutions. Thus,

$$y(t) = c_1 e^{-t} + c_2 e^{-2t} \tag{68}$$

where $c_1, c_2 \in \mathbb{R}$ are constants.

In the fourth step, it is possible to establish a specific solution based on the initial conditions. At time $t = 0$, the general solution takes the form

$$y(0) = c_1\, e^{(-1)\cdot 0} + c_2\, e^{(-2)\cdot 0} = c_1 + c_2 \tag{69}$$

Thus, we have utilized the information about the initial value $y(0) = 3$. The second initial condition pertains to the derivative of the solution. The derivative of the general solution is

$$\dot{y}(t) = -c_1 e^{-t} - 2c_2 e^{-2t} \tag{70}$$

At time $t = 0$, the derivative of the general solution takes the form

$$\dot{y}(0) = -c_1 - 2c_2 \tag{71}$$

From this, we derive a system of two equations with two unknown constants c_1 and c_2:

$$c_1 + c_2 = 3 \tag{72a}$$
$$-c_1 - 2c_2 = -2 \tag{72b}$$

It follows that $c_2 = 3 - c_1$, and thus

$$-c_1 - 2(3 - c_1) = -2 \tag{73a}$$
$$-c_1 - 6 + 2c_1 = -2 \tag{73b}$$
$$c_1 = 4 \tag{73c}$$

Then,

$$c_2 = 3 - c_1 \tag{74a}$$
$$c_2 = 3 - 4 \tag{74b}$$
$$c_2 = -1 \tag{74c}$$

We have found the function $y(t)$, which is the solution to the differential equation for the specific initial conditions:

$$y(t) = 4e^{-t} - e^{-2t} \tag{75}$$

5.2 Example 2

Find the analytical solution of the differential equation. Use the characteristic equation method.

$$\ddot{y}(t) + (a+b)\dot{y}(t) + aby(t) = 0 \qquad y(0) = y_0 \qquad \dot{y}(0) = z_0 \qquad a, b \in \mathbb{R}$$

The first step is to establish the characteristic equation. In this case,

$$s^2 + (a+b)s + ab = 0 \tag{76}$$

In the second step, to determine the fundamental solutions, we look for the roots of the characteristic equation. In general,

$$s_{1,2} = \frac{-(a+b) \pm \sqrt{(a+b)^2 - 4ab}}{2} \tag{77}$$

but in this case, we can also see that

$$s^2 + (a+b)s + ab = (s+a)(s+b) \tag{78}$$

The solutions of the characteristic equation are thus

$$s_1 = -a \tag{79a}$$
$$s_2 = -b \tag{79b}$$

The corresponding fundamental solutions are

$$y_{f1}(t) = e^{-at} \tag{80a}$$
$$y_{f2}(t) = e^{-bt} \tag{80b}$$

The third step is to determine the general solution of the differential equation. It is a linear combination of the fundamental solutions. Thus,

$$y(t) = c_1 e^{-at} + c_2 e^{-bt} \tag{81}$$

where $c_1, c_2 \in \mathbb{R}$ are constants.

In the fourth step, it is possible to establish a specific solution based on the initial conditions. At time $t = 0$, the general solution takes the form

$$y(0) = c_1\, e^{(-a)0} + c_2\, e^{(-b)0} = c_1 + c_2 \tag{82}$$

The derivative of the general solution is

$$\dot{y}(t) = -ac_1 e^{-at} - bc_2 e^{-bt} \tag{83}$$

At time $t = 0$, the derivative of the general solution takes the form

$$\dot{y}(0) = -ac_1 - bc_2 \tag{84}$$

From this, we derive a system of two equations with two unknown constants c_1 and c_2:

$$c_1 + c_2 = y_0 \tag{85a}$$
$$-ac_1 - bc_2 = z_0 \tag{85b}$$

Substituting $c_1 = y_0 - c_2$ into the second equation gives:

$$-a(y_0 - c_2) - bc_2 = z_0 \tag{86a}$$
$$-ay_0 + ac_2 - bc_2 = z_0 \tag{86b}$$
$$c_2(a-b) = z_0 + ay_0 \tag{86c}$$
$$c_2 = \frac{z_0 + ay_0}{a-b} \tag{86d}$$

Then,

$$c_1 = y_0 - c_2 \tag{87a}$$

$$c_1 = y_0 - \frac{z_0 + ay_0}{a - b} \tag{87b}$$

$$c_1 = \frac{y_0(a - b) - z_0 - ay_0}{a - b} \tag{87c}$$

$$c_1 = \frac{y_0 a - y_0 b - z_0 - ay_0}{a - b} \tag{87d}$$

$$c_1 = \frac{-y_0 b - z_0}{a - b} \tag{87e}$$

The specific solution to the problem is thus

$$y(t) = \frac{-y_0 b - z_0}{a - b} e^{-at} + \frac{z_0 + ay_0}{a - b} e^{-bt} \tag{88}$$

6 Questions and Exercises

1. What is the solution of an ordinary differential equation (in general)?
2. Explain the difference between a homogeneous and a non-homogeneous ordinary differential equation.
3. Provide an example of a homogeneous ordinary differential equation.
4. Provide an example of a non-homogeneous ordinary differential equation.
5. Explain the term *analytical solution* of an ordinary differential equation.
6. Find the analytical solution of the differential equation

$$\dot{y}(t) + ay(t) = 0 \qquad y(0) = y_0 \qquad a \in \mathbb{R},\ y_0 \in \mathbb{R}$$

7. Find the analytical solution of a differential equation (one will be given).
8. From a higher-order differential equation, construct a system of first-order differential equations (a specific example will be provided).
9. Rewrite the following second-order differential equation as a system of first-order differential equations.

$$a_2\ddot{y}(t) + a_1\dot{y}(t) + a_0 y(t) = b_0 u(t) \qquad a_2, a_1, a_0, b_0 \in \mathbb{R}$$

References

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[2] Jaromír Kuben. *Obyčejné diferenciální rovnice*. 1995.