Laplace Transform and Transfer Functions

Contents

Laplace transform allows for efficient handling of linear dynamic systems. It transforms and thereby simplifies operations related to finding solutions to linear differential equations (LDE). Primarily, it simplifies working with the convolution equation (convolution integral) (see section 6).

The use of the Laplace transform in solving differential equations also includes the concept of the *transfer function.* When we talk about transfer functions, we are referring to a tool that allows for analytical work with dynamic systems. However, the aim of this text is not to directly address transfer functions. It is a broader concept, or a separate tool, that is not limited to just solving differential equations.

1 Definition of the Laplace Transform

In broad terms, the definition of the Laplace transform can be summarized as follows.

Consider a time function $f(t)$ (with suitable properties, which we will not specify here). The Laplace transform (LT) transforms or maps this function to another function. Let us denote this other function as $F(s)$. LT is defined by the relation

$$F(s) = \int_0^\infty f(t)e^{-st}\mathrm{d}t \tag{1}$$

where s is a complex variable (a complex number).

We say that it is a transformation from the time domain to the domain of the complex variable s. The variable s is often also called the Laplace operator (the connections will become clear later). Since $s = \sigma + j\omega$ and thus $e^{-(\sigma+j\omega)t}$ is a signal that generally contains a harmonic (oscillatory) component, in this context, we also say that LT is a transformation from the time domain to the frequency domain.

The resulting transformed function $F(s)$ is also called the *image* of the original signal $f(t)$ (or the Laplace image of the signal).

LT is a linear transformation, i.e., if we want to transform the sum of two signals (two time functions) $f(t) + g(t)$ as a whole, it is possible to do so by transforming the signals individually and then summing the transformed functions $F(s) + G(s)$.

2 Laplace Transforms of Signals

Consider a signal $f(t)$. The Laplace transform (L-transform) of this signal is $F(s)$ (according to the definition of LT), and the transformation operation itself is denoted as

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_0^\infty f(t)e^{-st}\mathrm{d}t \tag{2}$$

2.1 Derivative

Let's find the L-transform of the signal $\frac{\mathrm{d}f(t)}{dt}$ (or the signal $\dot{f}(t)$), i.e.,

$$\mathcal{L}\left\{\frac{\mathrm{d}f(t)}{dt}\right\} = \int_0^\infty \frac{\mathrm{d}f(t)}{\mathrm{d}t}e^{-st}\mathrm{d}t \tag{3}$$

This integral can be found using the method of integration by parts, which generally states

$$\int_0^\infty u(t)v'(t)\mathrm{d}t = \left[u(t)v(t)\right]_0^\infty - \int_0^\infty u'(t)v(t)\mathrm{d}t \tag{4}$$

Consider $u(t) = e^{-st}$ and $v(t) = f(t)$, then

$$\begin{aligned} \int_0^\infty \frac{\mathrm{d}f(t)}{\mathrm{d}t}e^{-st}\mathrm{d}t &= \left[e^{-st}f(t)\right]_0^\infty - (-s)\int_0^\infty f(t)e^{-st}\mathrm{d}t \\ &= 0 - f(0) + sF(s) \\ &= sF(s) - f(0) \end{aligned} \tag{5}$$

is the L-transform of the signal $\frac{\mathrm{d}f(t)}{dt}$.

2.2 Integral

Similarly, we could find the transform of the signal $\int_0^t f(\tau)\mathrm{d}\tau$, i.e.,

$$\mathcal{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau\right\} = \int_0^\infty \left(\int_0^t f(\tau)\mathrm{d}\tau\right) e^{-st}\mathrm{d}t \tag{6}$$

Let's find the L-transform by introducing the signal $g(t) = \int_0^t f(\tau)\mathrm{d}\tau$, which means that $\dot{g}(t) = f(t)$. We seek $\mathcal{L}\left\{g(t)\right\} = G(s)$. First, note that

$$\begin{aligned} \mathcal{L}\left\{\dot{g}(t)\right\} =& sG(s) - g(0) \\ sG(s) - g(0) =& F(s) \end{aligned} \tag{7}$$

and we see that $g(0) = \int_0^0 f(\tau)\mathrm{d}\tau = 0$. Thus,

$$sG(s) = F(s) \tag{8a}$$

$$G(s) = \frac{1}{s}F(s) \tag{8b}$$

thereby finding

$$\mathcal{L}\left\{\int_0^t f(\tau)\mathrm{d}\tau\right\} = \frac{1}{s}F(s) \tag{9}$$

2.3 Image of the Dirac Impulse

The Dirac impulse is a signal such that (for example)

$$\delta(t) = \begin{cases} 0 & \text{if } t \neq 0 \\ \infty & \text{if } t = 0 \end{cases} \tag{10}$$

where it holds that

$$\int_{-\infty}^{\infty} \delta(\tau)\mathrm{d}\tau = 1 \tag{11}$$

Depending on how we mathematically specify the Dirac impulse $\delta(t)$, the specific methods of applying the LT (calculating the integral) might formally differ, but in any case, it always holds that

$$\mathcal{L}\left\{\delta(t)\right\} = 1 \tag{12}$$

2.4 Image of the Unit Step

For the so-called unit step, it is considered that at time 0, the value of the signal jumps from 0 to 1 (it has the value of "one unit"). Since we are only considering time greater than zero, we can consider that we are looking for the image of the signal $f(t) = 1$, thus

$$\begin{aligned} \mathcal{L}\left\{1\right\} &= \int_0^{\infty} 1e^{-st}\mathrm{d}t \\ &= \left[-\frac{1}{s}e^{-st}\right]_0^{\infty} \\ &= 0 - \frac{1}{s}e^{-s0} \\ &= \frac{1}{s} \end{aligned} \tag{13}$$

2.5 Image of the Exponential Function

Let's find the image of $f(t) = e^{at}$.

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{at}e^{-st}\mathrm{d}t \\ &= \int_0^{\infty} e^{(a-s)t}\mathrm{d}t \\ &= \left[\frac{1}{a-s}e^{(a-s)t}\right]_0^{\infty} \\ &= 0 - \frac{1}{a-s} \\ &= \frac{1}{s-a} \end{aligned} \tag{14}$$

2.6 Image of Time Shift

Consider a signal $f(t)$. A time-shifted signal is $f(t-D)$ (in terms of input-output delay or transport delay). The image of $f(t)$ is $F(s)$. The image of $f(t-D)$ is

$$\int_0^\infty f(t-D)e^{-st}\mathrm{d}t \tag{15}$$

Introduce the substitution $\tau = t - D$, thus $t = \tau + D$ and also $\mathrm{d}t = \mathrm{d}\tau$ since D is constant in time. Then

$$\int_0^\infty f(\tau)e^{-s(\tau+D)}\mathrm{d}\tau = e^{-sD}\int_0^\infty f(\tau)e^{-s\tau}\mathrm{d}\tau \tag{16}$$

and it is clear that

$$e^{-sD}F(s) \tag{17}$$

is the image of the shifted signal $f(t-D)$.

3 Inverse Laplace Transform

At this point, it is appropriate to introduce the inverse of the Laplace transform, known as the inverse Laplace transform. It is denoted as

$$\mathcal{L}^{-1}\left\{F(s)\right\} = f(t) \tag{18}$$

and is formally defined by the relation

$$f(t) = \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}\mathrm{d}s \tag{19}$$

The calculation of the inverse LT is generally not straightforward. In practice, a table of Laplace transforms is used, which lists L-images and their corresponding time-domain signals. The table contains a selection of typical and important signals used in the analysis of dynamic systems.

A complex image of the solution to a differential equation can usually be simplified to reveal individual component images corresponding to typical signals (listed in the table). From these typical time-domain signals, the time-domain function corresponding to the overall solution (in the time domain) can be constructed.

4 Table of Laplace Transforms of Signals

$f(t)$	$\mathcal{L}\{f(t)\}$	Note
$f(t)$	$F(s)$	
$\dot{f}(t)$	$sF(s) - f(0)$	
$\dfrac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}$	$s^n F(s) - s^{(n-1)}f(0) - \cdots - f^{(n-1)}(0)$	
1	$\dfrac{1}{s}$	Step change at time 0
$t^n \ (n = 0, 1, 2, \dots)$	$\dfrac{n!}{s^{n+1}}$	

$f(t)$	$\mathcal{L}\{f(t)\}$	Note
$\delta(t)$	1	Dirac impulse
$\delta(t-t_0)$	$1\,e^{-st_0}$	Time delay
e^{at}	$\dfrac{1}{s-a}$	
e^{-at}	$\dfrac{1}{s+a}$	
$\sin(kt)$	$\dfrac{k}{s^2+k^2}$	
$\cos(kt)$	$\dfrac{s}{s^2+k^2}$	
$\sinh(kt)$	$\dfrac{k}{s^2-k^2}$	
$\cosh(kt)$	$\dfrac{s}{s^2-k^2}$	
$\int_0^t f(x)g(t-x)\mathrm{d}x$	$F(s)G(s)$	Convolution integral
$t^n f(t)$	$(-1)^n \dfrac{\mathrm{d}^n F(s)}{\mathrm{d}s^n}$	
te^{at}	$\dfrac{1}{(s-a)^2}$	
$t^n e^{at}$	$\dfrac{n!}{(s-a)^{n+1}}$	
$t\sin kt$	$\dfrac{2ks}{(s^2+k^2)^2}$	
$t\cos kt$	$\dfrac{s^2-k^2}{(s^2+k^2)^2}$	
$t\sinh kt$	$\dfrac{2ks}{(s^2-k^2)^2}$	
$t\cosh kt$	$\dfrac{s^2+k^2}{(s^2-k^2)^2}$	
$e^{at}f(t)$	$F(s-a)$	
$e^{at}\sin kt$	$\dfrac{k}{(s-a)^2+k^2}$	
$e^{at}\cos kt$	$\dfrac{s-a}{(s-a)^2+k^2}$	
$e^{at}\sinh kt$	$\dfrac{k}{(s-a)^2-k^2}$	
$e^{at}\cosh kt$	$\dfrac{s-a}{(s-a)^2-k^2}$	

5 Laplace Image and Original Solution of a Differential Equation

5.1 Example with a Homogeneous Differential Equation

Consider the differential equation

$$\dot{y}(t) - ay(t) = 0 \qquad y(0) = y_0 \tag{20}$$

Apply the LT to the individual signals in this equation.

$$(sY(s) - y(0)) - aY(s) = 0 \tag{21}$$

where $Y(s)$ is the image of the signal $y(t)$. $Y(s)$ is thus the image of the solution of the equation. Express $Y(s)$:

$$\begin{aligned}(s-a)Y(s) - y(0) = 0\\ Y(s) = \frac{1}{(s-a)}y(0)\end{aligned} \tag{22}$$

The question is, if we know the signal in the s-domain (in the Laplace domain), can we determine the original signal in the time domain? Can we find the original solution $y(t)$ using the image of the solution $Y(s)$?

In this case, it is clear from section 2.5 that

$$\mathcal{L}^{-1}\{Y(s)\} = y(t) = e^{at}y(0) \tag{23}$$

where $\mathcal{L}^{-1}\{\}$ represents the inverse LT transformation. It is also clear that (23) is the correct solution to the differential equation (20).

5.2 Example with a Non-Homogeneous Differential Equation

Consider the equation

$$\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = u(t) \qquad y(0) = 3, \dot{y}(0) = -2 \tag{24}$$

where the input signal $u(t) = 12$ (constant over time). Apply the LT

$$\big(s\mathcal{L}\{\dot{y}\} - \dot{y}(0)\big) + 4\left(sY(s) - y(0)\right) + 3Y(s) = U(s) \tag{25a}$$

$$\Big(s\big(sY(s) - y(0)\big) - \dot{y}(0)\Big) + 4sY(s) - 4y(0) + 3Y(s) = U(s) \tag{25b}$$

$$s^2Y(s) - sy(0) - \dot{y}(0) + 4sY(s) - 4y(0) + 3Y(s) = U(s) \tag{25c}$$

$$s^2Y(s) + 4sY(s) + 3Y(s) - sy(0) - \dot{y}(0) - 4y(0) = U(s) \tag{25d}$$

and thus

$$\left(s^2 + 4s + 3\right) Y(s) = sy(0) + \dot{y}(0) + 4y(0) + U(s) \tag{26a}$$

$$Y(s) = \frac{sy(0) + \dot{y}(0) + 4y(0)}{(s^2 + 4s + 3)} + \frac{1}{(s^2 + 4s + 3)}U(s) \tag{26b}$$

We also know the specific form of the image $U(s)$, since $u(t) = 12$, then $U(s) = 12\frac{1}{s}$, thus

$$Y(s) = \frac{sy(0) + \dot{y}(0) + 4y(0)}{(s^2 + 4s + 3)} + \frac{1}{(s^2 + 4s + 3)}12\frac{1}{s} \tag{27}$$

and this is the image of the solution of the differential equation.

Notice that there are two components present

$$Y(s) = \underbrace{\frac{3s + 10}{(s^2 + 4s + 3)}}_{\text{initial condition response}} + \underbrace{\frac{12}{(s^2 + 4s + 3)\,s}}_{\text{input response}} \tag{28}$$

where we have also numerically substituted the values of the initial conditions.

When the image of the solution is in the form (28), it is practically impossible to assign the original time signal to it – there are no obvious typical images of typical signals.

Let's decompose into partial fractions

$$\frac{3s+10}{(s^2+4s+3)} = \frac{7}{2(s+1)} - \frac{1}{2(s+3)} \tag{29}$$

$$\frac{12}{(s^2+4s+3)\,s} = \frac{4}{s} - \frac{6}{(s+1)} + \frac{2}{(s+3)} \tag{30}$$

and it immediately becomes clear that (29) has the original

$$y_{init}(t) = \frac{7}{2}e^{-t} - \frac{1}{2}e^{-3t} \tag{31}$$

and (30) has the original

$$y_{input}(t) = 4 - 6e^{-t} + 2e^{-3t} \tag{32}$$

The overall solution is

$$\begin{aligned} y(t) &= \frac{7}{2}e^{-t} - \frac{1}{2}e^{-3t} + 4 - 6e^{-t} + 2e^{-3t} \\ &= 4 - \frac{5}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{aligned} \tag{33}$$

6 Connections with the General Solution of Non-Homogeneous Differential Equations

Similarly to finding the solution of a homogeneous differential equation, where the starting point is to assume a solution in the form of an exponential function e^{st}, when finding the solution of a non-homogeneous differential equation, it is possible to examine the assumption that the input signal is in the form of an exponential function e^{st}.

First, let's recall that the solution to the homogeneous differential equation

$$\dot{y}(t) + ay(t) = 0 \qquad y(0) = y_0 \tag{34}$$

is

$$y(t) = e^{-at}y_0 \tag{35}$$

and this is a first-order equation.

Formally, it is also possible to apply the decomposition of a higher-order differential equation into a system of first-order equations in the sense of

$$\dot{x}(t) = ax(t) \qquad x(0) = x_0 \tag{36a}$$

$$y(t) = x(t) \tag{36b}$$

where $a \in \mathbb{R}$ and $x(t)$ is a state variable. For a higher-order differential equation, $x(t)$ would be a vector of state variables and would give a system of equations in the form of

$$\dot{x}(t) = Ax(t) \qquad x(0) = x_0 \tag{37a}$$

$$y(t) = c^{\mathsf{T}}x(t) \tag{37b}$$

where $A \in \mathbb{R}^{n\times n}$ is a matrix, $c \in \mathbb{R}^n$ is a vector, and $x_0 \in \mathbb{R}^n$ is a vector. The solution is

$$y(t) = c^{\mathsf{T}}e^{At}x_0 \tag{38}$$

where we used the object e^{At} which is the so-called matrix exponential function. We will not go into its definition in detail here; the reader is referred to, for example, [1].

It is clearly a generalization of the scalar case (first-order systems) to the vector case (higher-order systems). The definition and subsequent use of the matrix e^{At} is the basis for concepts such as fundamental solutions of the system (differential equation). The matrix e^{At} itself is also referred to as the matrix of fundamental solutions. The "effect" of the matrix e^{At} is given by the matrix A, which can be characterized by its eigenvalues (and eigenvectors). These are then the source of the definition of the characteristic equation as used in finding the analytical solution of the differential equation.

In the case of a non-homogeneous differential equation, the system is given by a set of equations in the form of

$$\dot{x}(t) = Ax(t) + bu(t) \qquad x(0) = x_0 \tag{39a}$$

$$y(t) = c^{\mathsf{T}} x(t) \tag{39b}$$

where $u(t)$ is the input signal, $b \in \mathbb{R}^n$ is a vector. It can be shown that

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} bu(\tau) \mathrm{d}\tau \tag{40}$$

and thus the solution itself (the output signal $y(t)$) is

$$y(t) = c^{\mathsf{T}} x(t) \tag{41a}$$

$$y(t) = c^{\mathsf{T}} e^{At}x(0) + \int_0^t c^{\mathsf{T}} e^{A(t-\tau)} bu(\tau) \mathrm{d}\tau \tag{41b}$$

The first term (on the right side of equation (41b)) is called the *natural component of the solution* (it is caused by the initial conditions) and the second term is called the *forced component of the solution* (it is caused by the input signal).

As we mentioned, the intention is to examine the assumption that the input signal is in the form of an exponential function

$$u(t) = e^{st} \tag{42}$$

where $s = \sigma + j\omega$ (in general). The fact that s is a complex number (complex variable) allows us to consider this special signal actually as a class of signals (of different types). The real part of the variable s determines exponential growth or decay (even if $s = 0$ then the special signal is actually constant) and the imaginary part determines harmonic oscillation of the signal.

We have (40), and thus:

$$x(t) = e^{At}x(0) + \int_0^t \left(e^{A(t-\tau)} b e^{s\tau} \right) \mathrm{d}\tau \tag{43}$$

where when manipulating the expression $\left(e^{A(t-\tau)} b e^{s\tau}\right)$, one must manipulate with regard to the fact that these are matrices and vectors. In any case, after integration, we get

$$x(t) = e^{At}x(0) + e^{At} \left(sI - A\right)^{-1} \left(e^{(sI-A)t} - I \right) b \tag{44}$$

where I is the identity matrix.

The overall solution, in other words, the output signal of the system, is then

$$\begin{aligned}
y(t) &= c^{\mathsf{T}} e^{At}x(0) + c^{\mathsf{T}} e^{At} \left(sI - A\right)^{-1} \left(e^{(sI-A)t} - I \right) b \\
&= c^{\mathsf{T}} e^{At}x(0) + c^{\mathsf{T}} e^{At} \left(sI - A\right)^{-1} \left(e^{st} e^{-At} - I \right) b \\
&= c^{\mathsf{T}} e^{At}x(0) + c^{\mathsf{T}} e^{At} \left(sI - A\right)^{-1} \left(e^{st} e^{-At} b - b \right) \\
&= c^{\mathsf{T}} e^{At}x(0) + \left(c^{\mathsf{T}} e^{At} \left(sI - A\right)^{-1} e^{st} e^{-At} b - c^{\mathsf{T}} e^{At} \left(sI - A\right)^{-1} b \right) \\
&= c^{\mathsf{T}} e^{At}x(0) + \left(c^{\mathsf{T}} \left(sI - A\right)^{-1} e^{st} b - c^{\mathsf{T}} e^{At} \left(sI - A\right)^{-1} b \right)
\end{aligned} \tag{45}$$

At this point, it is possible to state:

$$y(t) = \underbrace{c^\mathsf{T} e^{At} x(0)}_{\text{initial condition response}} + \underbrace{\left(c^\mathsf{T}\left(sI - A\right)^{-1} e^{st} b - c^\mathsf{T} e^{At}\left(sI - A\right)^{-1} b\right)}_{\text{input response}} \tag{46}$$

and at the same time:

$$\begin{aligned} y(t) &= c^\mathsf{T} e^{At}\left(x(0) - \left(sI - A\right)^{-1} b\right) + \left(c^\mathsf{T}\left(sI - A\right)^{-1} b e^{st}\right) \\ &= \underbrace{c^\mathsf{T} e^{At}\left(x(0) - \left(sI - A\right)^{-1} b\right)}_{\text{transient response}} + \underbrace{\left(c^\mathsf{T}\left(sI - A\right)^{-1} b\right) e^{st}}_{\text{pure exponential response}} \end{aligned} \tag{47}$$

The effect of the special signal e^{st} on the overall solution is thus determined by the expression $c^\mathsf{T}\left(sI - A\right)^{-1} b$. Formally,

$$G(s) = c^\mathsf{T}\left(sI - A\right)^{-1} b \tag{48}$$

is called the transfer function of the system.

The above is based on the fact expressed by the general solution (40), which is the solution of a system of first-order differential equations in the form of (39). The original higher-order differential equation for this case is in the form of

$$\frac{\mathrm{d}^n y(t)}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{(n-1)} y(t)}{\mathrm{d}t^{(n-1)}} + \cdots + a_0 y(t) = b_m \frac{\mathrm{d}^m u(t)}{\mathrm{d}t^m} + b_{m-1}\frac{\mathrm{d}^{m-1} u(t)}{\mathrm{d}t^{m-1}} + \cdots + b_0 u(t) \tag{49}$$

Then if we consider $u(t) = e^{st}$ at the input and at the same time know that the solution of the system is also some exponential signal, which can generally be expressed as $y(t) = y_0 e^{st}$ (where y_0 mainly distinguishes $y(t)$ from $u(t)$). If we substitute $y(t)$ and $u(t)$ into (49), we see that

$$\begin{aligned} \frac{\mathrm{d}^n y_0 e^{st}}{\mathrm{d}t^n} + a_{n-1}\frac{\mathrm{d}^{(n-1)} y_0 e^{st}}{\mathrm{d}t^{(n-1)}} + \cdots + a_0 y_0 e^{st} &= b_m \frac{\mathrm{d}^m e^{st}}{\mathrm{d}t^m} + b_{m-1}\frac{\mathrm{d}^{m-1} e^{st}}{\mathrm{d}t^{m-1}} + \cdots + b_0 e^{st} \\ y_0 e^{st} s^n + a_{n-1} y_0 e^{st} s^{(n-1)} + \cdots + a_0 y_0 e^{st} &= b_m e^{st} s^m + b_{m-1} e^{st} s^{m-1} + \cdots + b_0 e^{st} \\ \left(s^n + a_{n-1} s^{(n-1)} + \cdots + a_0\right) y_0 e^{st} &= \left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0\right) e^{st} \\ y_0 e^{st} &= \frac{\left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0\right)}{\left(s^n + a_{n-1} s^{(n-1)} + \cdots + a_0\right)} e^{st} \end{aligned} \tag{50}$$

and thus we can say that the solution of the system dependent on the special signal e^{st} is

$$y(t) = \frac{\left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0\right)}{\left(s^n + a_{n-1} s^{(n-1)} + \cdots + a_0\right)} e^{st} \tag{51}$$

Let's denote

$$B(s) = \left(b_m s^m + b_{m-1} s^{m-1} + \cdots + b_0\right) \tag{52a}$$

$$A(s) = \left(s^n + a_{n-1} s^{(n-1)} + \cdots + a_0\right) \tag{52b}$$

and the expression

$$G(s) = \frac{B(s)}{A(s)} \tag{53}$$

expresses the transfer function of the system.

7 Transfer Function

The transfer function is a tool for the mathematical modeling of linear time-invariant dynamic systems.

Primarily, dynamic systems are described by differential equations. If these equations are linear, we say that the system they describe is linear. If the coefficients in the differential equation are not functions of time, we say that the system is time-invariant.

In general, we seek the solution of a differential equation. In the context of dynamic systems, the solution to the differential equation is a function of time. From the system's perspective, this function is referred to as the output signal of the system. The solution is influenced by several factors. In general, the solution is determined by the differential equation itself, its order, and the values of its coefficients. Specific solutions are then determined by the initial conditions and the input signal of the system.

From the system's perspective, we refer to the order of the differential equation and its coefficients as the system's parameters. Of course, it also makes sense to talk about the system's initial conditions. Ultimately, we are interested in the influence of the input signal on the output signal of the system, and the concept of the transfer function is related to the mathematical modeling of this influence. Figuratively, we talk about the transfer from the input to the output of the system.

7.1 Definition of the Transfer Function Using Laplace Transformation

The transfer function is defined as the ratio of the Laplace transform of the system's output signal to the Laplace transform of the input signal under the assumption of zero initial conditions for the system.

The Laplace transform applies to linear time-invariant systems. Let us consider such a system, where the input signal is denoted by $u(t)$ and the output signal by $y(t)$. In terms of the Laplace transform, the Laplace image of the input signal is $U(s)$, and the Laplace image of the output signal is $Y(s)$, with these images determined under zero initial conditions for the system.

Let's illustrate this with an example. A linear time-invariant system can be described by the differential equation in the form

$$a_1\dot{y}(t) + a_0 y(t) = b_0 u(t) \tag{54}$$

where $y(t)$ and $u(t)$ are, of course, the output and input signals. The coefficients $a_1, a_0, b_0 \in \mathbb{R}$ are constants. Let's apply the Laplace transform to the elements of this differential equation.

$$\begin{aligned} a_1\mathcal{L}\left\{\dot{y}(t)\right\} + a_0\mathcal{L}\left\{y(t)\right\} &= b_0\mathcal{L}\left\{u(t)\right\} \\ a_1 sY(s) - a_1 y(0) + a_0 Y(s) &= b_0 U(s) \end{aligned} \tag{55}$$

With zero initial conditions, we have

$$a_1 sY(s) + a_0 Y(s) = b_0 U(s) \tag{56}$$

The transfer function is defined as the ratio $Y(s)/U(s)$, so

$$Y(s)\left(a_1 s + a_0\right) = b_0 U(s) \tag{57a}$$

$$\frac{Y(s)}{U(s)} = \frac{b_0}{a_1 s + a_0} \tag{57b}$$

The transfer function as an independent entity is often denoted separately, for instance, as $G(s)$, so in this case, we have

$$G(s) = \frac{b_0}{a_1 s + a_0} \tag{58}$$

and in general

$$G(s) = \frac{Y(s)}{U(s)} \tag{59}$$

From another perspective, it also makes sense to denote the polynomials in the numerator and denominator of the transfer function separately. The polynomial in the numerator is typically denoted as $B(s)$, and the polynomial in the denominator is denoted as $A(s)$. In this case,

$$B(s) = b_0 \qquad A(s) = a_1 s + a_0 \tag{60}$$

and in general,

$$G(s) = \frac{B(s)}{A(s)} \tag{61}$$

7.2 Related Concepts

The transfer function describes a linear time-invariant dynamic system, assuming the system's initial conditions are zero. A given dynamic system can be described either by a differential equation or by a transfer function, and these two descriptions are equivalent.

Consider a general transfer function in the form

$$G(s) = \frac{B(s)}{A(s)} \tag{62}$$

where $A(s)$ and $B(s)$ are polynomials with the Laplace operator s as their independent variable (where s is a complex number).

The polynomial $A(s)$ has degree n, and the polynomial $B(s)$ has degree m.

Real-world dynamic processes/systems are, of course, causal[1], meaning the output is a consequence of the present and past events. If the transfer function describes a causal system, then the degrees of the polynomials $A(s)$ and $B(s)$ must satisfy $n \geq m$.

The polynomial $A(s)$ is called the characteristic polynomial of the transfer function. The term "characteristic equation" or "characteristic polynomial" is also used in the context of analytical methods for solving linear differential equations. These are equivalent concepts—the characteristic polynomial of the transfer function is the same as the characteristic polynomial of a linear differential equation.

The degree of the polynomial $A(s)$, denoted by n, is called the order of the transfer function. This corresponds to the order of the dynamic system (the highest degree of the derivative of the unknown variable in the differential equation).

The roots of the polynomial $A(s)$ are called the poles of the transfer function. Equivalently, we can refer to the poles of the linear dynamic system. Since they are the roots of the characteristic polynomial, the system's poles are directly related to the fundamental solutions of the differential equation. The fundamental solutions are determined by the system's poles. Another term for the fundamental solutions is the modes of the dynamic system.

In terms of system stability, we say that the system is stable if all of its poles lie in the left half of the complex plane. In other words, the system is stable if the real parts of all the poles are negative. By stability, we are referring to the stability of the equilibrium state of the linear dynamic system.

The roots of the polynomial $B(s)$ are called the zeros of the transfer function (or zeros of the linear dynamic system).

The zeros of the system are primarily related to the input signal. The broader interpretation of the transfer function, as we know, involves examining the impact of an exponential input signal $u(t) = e^{st}$ (where s is a complex number) on the system output. Simply put, the zeros "nullify" corresponding input exponential signals, meaning they are not transferred to the output. The position of a zero in the complex plane determines the signal e^{st} that is nullified and does not affect the output (it does not influence the output variable). Further discussion on this topic is beyond the scope of this text, and the reader is referred to relevant literature, such as [1].

In connection with the transfer function, expressed in the form

$$G(s) = \frac{Y(s)}{U(s)} \tag{63}$$

it is useful to apply the *final value theorem*. If we have the Laplace transform of the solution to a differential equation, i.e., the Laplace image of the system's output signal $Y(s)$, the final value theorem states that the final value of the output signal $y(t)$ denoted by $y(\infty)$ is given by

$$y(\infty) = \lim_{s \to 0} s\, Y(s) \tag{64}$$

[1]Non-causality is more of a mathematical/abstract concept.

For example, if we know the transfer function (58) and the system's input is a unit step function, whose Laplace image is $U(s) = 1/s$, then the Laplace image of the output signal is

$$Y(s) = G(s)U(s) = \left(\frac{b_0}{a_1 s + a_0}\right)\frac{1}{s} \tag{65}$$

The final value of this signal will be

$$y(\infty) = \lim_{s\to 0} s\ Y(s) = \lim_{s\to 0} s\left(\frac{b_0}{a_1 s + a_0}\right)\frac{1}{s} \tag{66a}$$

$$= \lim_{s\to 0}\left(\frac{b_0}{a_1 s + a_0}\right) = \frac{b_0}{a_0} \tag{66b}$$

8 Transfer Function Algebra

The transfer function is a tool for the mathematical modeling of linear time-invariant dynamic systems. A transfer function can also be viewed as a single block in a block diagram, as shown below:

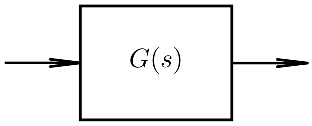

Figure 1: Transfer function as a single block in a block diagram

Manipulating these blocks is one of the applications of transfer function algebra. In this context, we need to consider three basic situations: series connection of blocks, parallel connection of blocks, and feedback connection of blocks.

8.1 Series Connection of Blocks

Consider a system composed of a cascade combination of two subsystems. The transfer functions of the subsystems are $G_1(s)$ and $G_2(s)$. The input to the first subsystem is also the input to the overall system. The output of the first subsystem is the input to the second subsystem. The output of the second subsystem is also the output of the overall system. This is a series connection of subsystems.

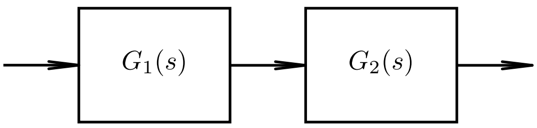

Figure 2: Series connection of blocks

We are looking for the transfer function of the overall system, denoted as $G(s)$. For a series connection of subsystems, the following holds:

$$G(s) = G_1(s)\ G_2(s) \tag{67}$$

Thus, the resulting transfer function is obtained by multiplying the transfer functions of the subsystems.

8.2 Parallel Connection of Blocks

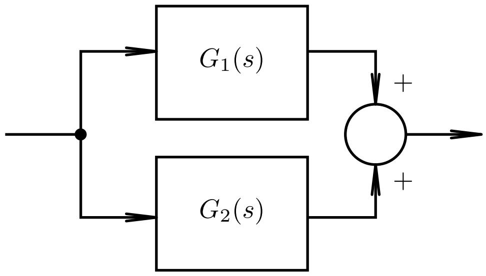

Figure 3: Parallel connection of blocks

In the case of parallel connection of subsystems with transfer functions $G_1(s)$ and $G_2(s)$, the output of the overall system is simply the sum of the outputs of the subsystems. For the overall system's transfer function $G(s)$, we have:

$$G(s) = G_1(s) + G_2(s) \tag{68}$$

8.3 Feedback Connection of Blocks

The feedback connection of blocks is illustrated in Fig. 4. For better clarity, the input of the overall system is denoted as u, and the output as y. The output signal y serves as the input to the feedback subsystem $G_2(s)$. This feedback is subtracted from the input signal u (representing negative feedback), resulting in the error signal e, which is fed into the subsystem $G_1(s)$.

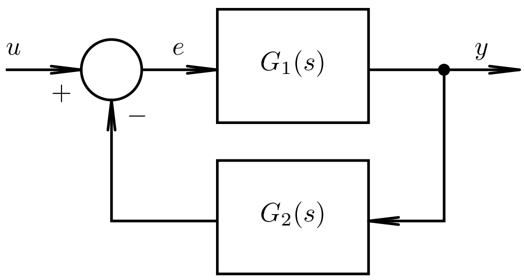

Figure 4: Feedback connection of blocks

Without going into details and assumptions, we can express the error signal as:

$$e = u - G_2(s)y \tag{69}$$

and then:

$$\begin{aligned} y &= G_1(s)e && \text{(70a)} \\ y &= G_1(s)\left(u - G_2(s)y\right) && \text{(70b)} \\ \left(1 + G_1(s)G_2(s)\right) y &= G_1(s)u && \text{(70c)} \\ y &= \frac{G_1(s)}{(1 + G_1(s)G_2(s))} u && \text{(70d)} \end{aligned}$$

Thus, the transfer function of the overall system $G(s)$ is given by:

$$G(s) = \frac{G_1(s)}{(1 + G_1(s)G_2(s))} \tag{71}$$

9 Questions and Tasks

1. Write the equation that defines the Laplace transform.
2. Write the Laplace transform of the derivative of a time function $\frac{\mathrm{d}f(t)}{dt}$.
3. Write the Laplace transform of the unit step function.
4. Write the Laplace transform of the Dirac impulse.
5. Find the analytical solution of the differential equation using the Laplace transform:

$$\dot{y}(t) + a_0 y(t) = b_0 u(t) \qquad y(0) = y_0 \qquad a_0, b_0, y_0 \in \mathbb{R} \qquad u(t) = 1$$

6. Find the analytical solution of the differential equation using the Laplace transform:

$$\dot{y}(t) + a_0 y(t) = b_0 u(t) \qquad y(0) = y_0 \qquad a_0, b_0, y_0 \in \mathbb{R} \qquad u(t) = \delta(t)$$

7. Find the analytical solution of the differential equation using the Laplace transform:

$$\ddot{y}(t) + (a+b)\dot{y}(t) + aby(t) = 0 \qquad y(0) = y_0,\ \dot{y}(0) = z_0 \qquad a \in \mathbb{R},\ b \in \mathbb{R},\ y_0 \in \mathbb{R},\ z_0 \in \mathbb{R}$$

8. Find the analytical solution of the differential equation using the Laplace transform:

$$\ddot{y}(t) + 4\dot{y}(t) + 3y(t) = u(t) \qquad y(0) = 3,\ \dot{y}(0) = -2 \qquad u(t) = 1$$

A table of Laplace transforms is available:

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$\frac{\mathrm{d}^n f(t)}{\mathrm{d}t^n}$	$s^n F(s) - s^{(n-1)} f(0) - \cdots - f^{(n-1)}(0)$
e^{at}	$\frac{1}{s-a}$
1	$\frac{1}{s}$
$\delta(t)$	1

References

[1] Karl Johan Åström and Richard M. Murray. *Feedback Systems: An Introduction for Scientists and Engineers.* Princeton University Press, Jan. 2020. ISBN: 978-0-691-13576-2. URL: `https://fbswiki.org/wiki/index.php/Main_Page`.