Transfer Functions and System Modeling

Contents

The aim of the text is to summarize the properties and characteristics of a dynamic system that has one input signal $u(t)$ and one output signal $y(t)$, both continuous in time. A linear, time-invariant dynamic system is considered.

The term *order of the system* essentially has the same meaning as in the context of differential equations. A differential equation of order n describes a dynamic system of order n. A differential equation of order n is one in which the highest derivative of the unknown is of order n. In the context of the system's transfer function, this means that the characteristic polynomial of the system is of degree n.

It is particularly noted that we are considering a *causal system*, meaning the system's output is a consequence of current and past events. Mathematically, for the transfer function, this means that for the degrees of the polynomials $A(s)$ and $B(s)$, it holds that $n \geq m$, where the characteristic polynomial $A(s)$ is of degree n, the polynomial $B(s)$ is of degree m, and we consider the transfer function in the form

$$G(s) = \frac{B(s)}{A(s)} \tag{1}$$

Moreover, in practice, when mathematically modeling real systems, it often makes sense to talk about systems that do not contain an "energy source" themselves, but are "energy consumers", and are *energetically dissipative*. In such cases, the transfer function's relative degree $n^\star = n - m$ is $n^\star \geq 1$.

1 First Order System

1.1 Transfer Function

When the degree of polynomial $A(s)$ in the transfer function is $n = 1$, we say that the system described by the transfer function is first order. Due to causality, the degree of polynomial $B(s)$ can be equal or less, i.e. $m \leq n$. Thus in general, a first order system is

$$G(s) = \frac{b_1 s + b_0}{a_1 s + a_0} \tag{2}$$

Typically (and often very usefully) $A(s)$ is presented as a monic polynomial, one that has coefficient equal to 1 for the highest power of s. Thus in this case

$$G(s) = \frac{b_1 s + b_0}{s + a_0} \tag{3}$$

Moreover, in practice, in modeling (and in nature) it makes sense in many cases to talk about systems that do not contain an "energy source" themselves, are only "energy consumers", are energetically dissipative. In such case, for the transfer function it holds that its relative degree $n^\star = n - m$ is $n^\star \geq 1$. In this case therefore

$$G(s) = \frac{b_0}{s + a_0} \tag{4}$$

is a typical example of a first order transfer function. Such a transfer function is also called a *positive real transfer function* (if it is a stable system).

For completeness, $B(s) = b_0$ is of degree $m = 0$ and $A(s) = s + a_0$ is of degree $n = 1$. The coefficients of these polynomials are system parameters.

1.2 Differential Equation

To connect to the previous part and also show the conversion of the system from transfer function to differential equation, let us state that

$$G(s) = \frac{Y(s)}{U(s)} \tag{5}$$

where $Y(s)$ is the Laplace transform of the output signal and $U(s)$ is the Laplace transform of the input signal. In this case therefore

$$\begin{aligned}
Y(s) = G(s)U(s) &= \frac{b_0}{s + a_0} U(s) && \text{(6a)}\\
(s + a_0)\, Y(s) &= b_0 U(s) && \text{(6b)}\\
sY(s) + a_0 Y(s) &= b_0 U(s) && \text{(6c)}\\
sY(s) &= -a_0 Y(s) + b_0 U(s) && \text{(6d)}
\end{aligned}$$

and thus the differential equation is

$$\dot{y}(t) = -a_0 y(t) + b_0 u(t) \tag{7}$$

The conversion in the opposite direction, from differential equation to transfer function, is of course standard application of Laplace transformation to equation (7) with zero initial conditions.

1.3 State Space Representation

In the state space, it is necessary to introduce the state vector $x(t) \in \mathbb{R}^n$. In general, the representation of a linear system in the state space is in the form

$$\dot{x}(t) = Ax(t) + bu(t) \tag{8a}$$

$$y(t) = c^\top x(t) \tag{8b}$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}^n$ are the matrix and vectors representing the system parameters.

When determining the vector $x(t)$, it generally involves rewriting a higher-order differential equation into a system of first-order equations. This results in new signals, which are the unknowns in the system of first-order equations and are elements of the state vector $x(t)$. In this case, we have the differential equation (7), which is already a first-order equation. Formally, let's choose

$$x_1(t) = y(t) \tag{9}$$

and thus

$$\dot{x}_1(t) = \dot{y}(t) = -a_0 x_1(t) + b_0 u(t) \tag{10}$$

is essentially a "system" of one differential equation. Formally:

$$\dot{x}_1(t) = -a_0 x_1(t) + b_0 u(t) \tag{11a}$$

$$y(t) = x_1(t) \tag{11b}$$

is the state space representation where $x_1(t)$ is the state variable. For completeness, the state vector in this case is $x(t) = x_1(t)$, the matrix $A = -a_0$, the vector $b = b_0$, and the vector $c = 1$.

1.4 Stability

The term *system stability* typically refers to several different cases concerning the general solution of the differential equation describing the dynamic system. An intuitive term is *BIBO stability* (bounded input, bounded output), where the case is examined when the input signal $u(t)$ is bounded, its maximum value is less than infinity. If the output signal $y(t)$ is also bounded, the system is said to be BIBO stable. Essentially, this examines the forced component of the solution to the non-homogeneous differential equation. The natural component of the solution, dependent on initial conditions, can also be examined and is related to the term *asymptotic stability.*

For a linear system, it holds that the properties of the system from any stability perspective are completely determined by the system poles, i.e., the roots of the characteristic polynomial. A necessary and sufficient condition for the stability of a linear system is that all system poles lie in the left half-plane of the complex plane, i.e., their real parts are negative. If at least one pole lies on the imaginary axis, the system is said to be on the verge of stability. If at least one pole is in the right half-plane, its real part is positive, the system is said to be unstable.

The stability of the system is determined by the roots of the characteristic polynomial $A(s)$. In this case, the transfer function of the first-order system is in the form (4), and thus the characteristic polynomial is

$$A(s) = s + a_0 \tag{12}$$

The root is $s_1 = -a_0$. The system is stable if $a_0 > 0$, unstable if $a_0 < 0$, and if $a_0 = 0$, the system is on the verge of stability.

1.5 Static Gain and Astaticism

When examining the properties of a system, it is often necessary to first understand the so-called static properties of the system. In general, this concerns the steady states of the system. A typical example is a situation where the input signal $u(t)$ is constant, its value does not change over time. The steady value of the input signal is denoted as $u(\infty)$, emphasizing that it is the value at time infinity, which in practice is the time when all transient phenomena are considered finished. The question is whether the value of the output signal $y(t)$ will also settle at some value $y(\infty)$.

At first glance, it is clear that examining the static properties of a system does not make sense for an unstable system.

1.5.1 Static Gain

Consider a system that is not unstable. If none of the system poles are zero, the system is termed static. However, we still have in mind a dynamic system, which in this case is given by the first-order transfer function in the form (4). Collectively, this can be referred to as a *first-order static system*, abbreviated as SS1R.

For such a system, it is possible to determine its static gain. The static gain is the ratio of the output to the input in the steady state.

In the steady state, the signals do not change, meaning their time derivatives are zero. Consider the differential equation (7). In the steady state, $\dot{y}(\infty) = 0$, where ∞ symbolizes the time when the signals are already settled, and thus

$$0 = -a_0 y(\infty) + b_0 u(\infty) \tag{13}$$

The ratio of the output to the input is

$$\frac{y(\infty)}{u(\infty)} = \frac{b_0}{a_0} \tag{14}$$

which is the static gain of the system. This value can be designated as a separate system parameter, e.g., $K = \frac{b_0}{a_0}$.

It is also a convention to generally consider the input as "unitary," simply that $u(\infty) = 1$, and thus it is written $y(\infty) = \frac{b_0}{a_0}$, but it still means the static gain of the system.

We reach the same conclusion if we consider a constant, steady signal at the input, generally $u(t) = 1$. This is a unit step, and thus $U(s) = \frac{1}{s}$. Then

$$Y(s) = \frac{b_0}{s + a_0} \frac{1}{s} \tag{15}$$

The final value of this signal transform ($Y(s)$ is the transform of $y(t)$) is the value at which the system output potentially settles. Using the final value theorem:

$$y(\infty) = \lim_{s \to 0} s \left(\frac{b_0}{s + a_0} \frac{1}{s} \right) \tag{16a}$$

$$y(\infty) = \lim_{s \to 0} \left(\frac{b_0}{s + a_0} \right) \tag{16b}$$

$$y(\infty) = \frac{b_0}{a_0} \tag{16c}$$

1.5.2 Astaticism

If one of the poles of a system is zero, we say the system is astatic ("contains astaticism"). If exactly one pole is zero, we refer to it as first-order astaticism (if there are two zero poles, then second-order astaticism, etc.). Recall that we are considering a system that is not unstable. A zero pole means, of course, that its real part is zero. This implies that the system is on the boundary of stability. In this case, we can refer to it as a *first-order astatic system*, abbreviated as AS1R.

In this case, we have only one pole, which is zero when $a_0 = 0$. In such a case, it is not possible to determine the value of $y(\infty)$. If we consider an input signal $u(t) = 1$, the output variable $y(t)$ will increase indefinitely, without stabilizing. This is evident from the differential equation (7) when $a_0 = 0$:

$$\dot{y}(t) = b_0 u(t) \tag{17}$$

It is clear that the change in the signal $y(t)$, represented by $\dot{y}(t)$, will be zero only if $u(t)$ is a zero signal; otherwise, $y(t)$ will generally change.

For $a_0 = 0$, and without loss of generality, if we set $b_0 = 1$, we have

$$G(s) = \frac{1}{s} \tag{18}$$

which is the transfer function of an integrator. The integrator is a first-order system with first-order astaticism.

1.6 Static Characteristic

In the context of the static properties of a system, it is generally meaningful to discuss the static characteristic of the system. The static characteristic is the relationship between the steady-state values of the system's output signal and the steady-state values of the input signal.

It is evident that the static characteristic applies to systems with a static attribute, meaning those that are not astatic.

In the case of linear systems, the static characteristic is a line, and without loss of generality, we can assume it passes through the origin of the coordinate system. The slope of the line is determined by the system's static gain; if we use the notation above, the slope of the static characteristic of a linear system is $K = \frac{b_0}{a_0}$.

1.7 Impulse Response

The impulse response is the system's response to a Dirac impulse.

The Dirac impulse is an impulse with a unit area and infinitesimal width. In other words, it is an impulse that is zero for $t \neq 0$ and has a unit area for $t = 0$. The Laplace transform of the Dirac impulse is $U(s) = 1$.

Since we have a mathematical description of the system, we can analytically find the impulse response. The transfer function of a first-order system is (4). The Laplace transform of the input signal is $U(s) = 1$. The Laplace transform of the output signal then becomes

$$Y(s) = G(s)U(s) = \frac{b_0}{s + a_0} \cdot 1 \tag{19a}$$

$$Y(s) = \frac{b_0}{s + a_0} \tag{19b}$$

$$Y(s) = b_0 \frac{1}{s + a_0} \tag{19c}$$

The inverse transform of this result is

$$y(t) = b_0 e^{-a_0 t} \tag{20}$$

which is the time-domain function that analytically expresses the impulse response of the system.

It is evident that for the impulse response (ICH), it is possible to distinguish qualitatively different cases based on the system's single pole. The system pole is $s_1 = -a_0$.

In the context mentioned above, the following cases can be distinguished: a first-order static system (SS1R), a first-order astatic system (AS1R), and an unstable system.

1.7.1 ICH SS1R

The time function (20) will be the impulse response of a first-order static system if $a_0 > 0$. Let us choose $a_0 = 1$ and, for example, $b_0 = 1$. The graph of the resulting time function is shown in the following figure.

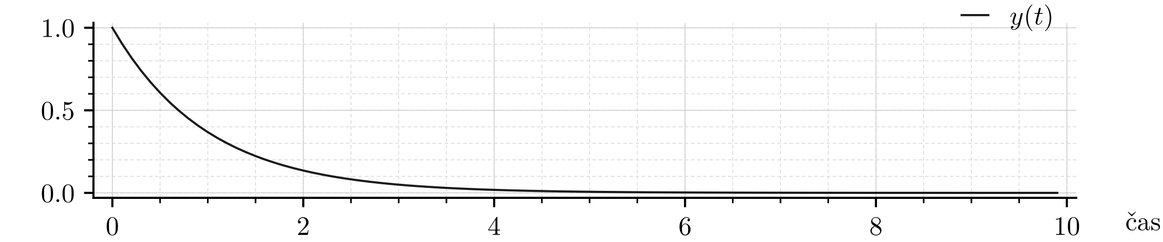


Figure 1: Impulse response of a first-order static system for $a_0 = 1$ and $b_0 = 1$

1.7.2 ICH AS1R

The time function (20) will be the impulse response of a first-order astatic system if $a_0 = 0$. The graph of the resulting time function is shown in the following figure.

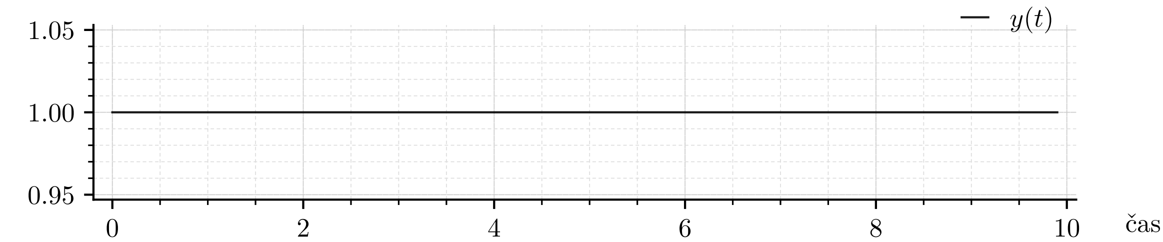


Figure 2: Impulse response of a first-order static system for $a_0 = 0$ and $b_0 = 1$

1.7.3 ICH of an unstable first-order system

For completeness, let us also include the case where $a_0 < 0$, i.e., the system is unstable. Let us choose $a_0 = -1$. The graph of the resulting time function is shown in the following figure.

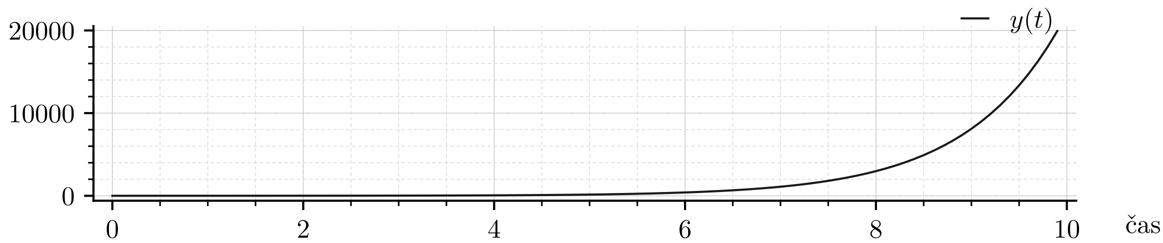


Figure 3: Impulse response of a first-order static system for $a_0 = -1$ and $b_0 = 1$

1.7.4 Python script for plotting impulse response graphs

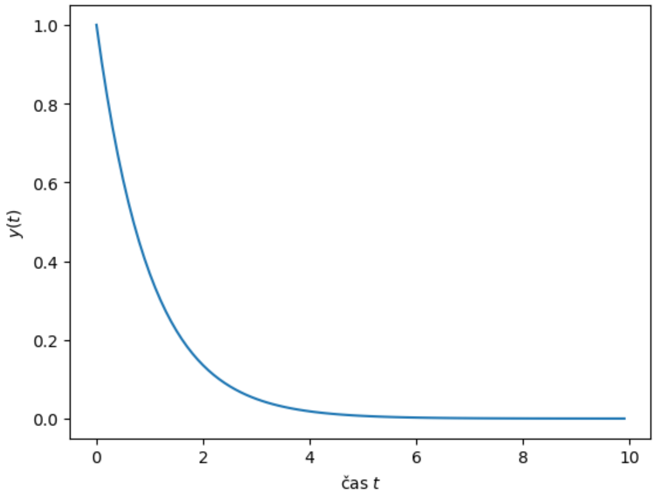
This section presents a script in the Python programming language for plotting the above-mentioned impulse response graphs. The script is presented in the form of a Jupyter notebook, and the individual cells of the notebook are displayed below.

Code Listing 1: Súbor MRS10_ICH1R.ipynb cell:02

```python
import numpy as np
import matplotlib.pyplot as plt

# Parametre
b_0 = 1
a_0 = 1

# Súradnice bodov na x-ovej osi
plotData_x = np.arange(0, 10, 0.1)

# Výpočet hodnôt na y-ovej osi v zmysle danej časovej funkcie
plotData_y = b_0 * np.exp(-a_0 * plotData_x)
```

Code Listing 2: Súbor MRS10_ICH1R.ipynb cell:03

```python
# Kreslenie grafu
plt.plot(plotData_x, plotData_y)
plt.xlabel('čas $t$')
plt.ylabel('$y(t)$')
plt.show()
```



Code Listing 3: Súbor MRS10_ICH1R.ipynb cell:04

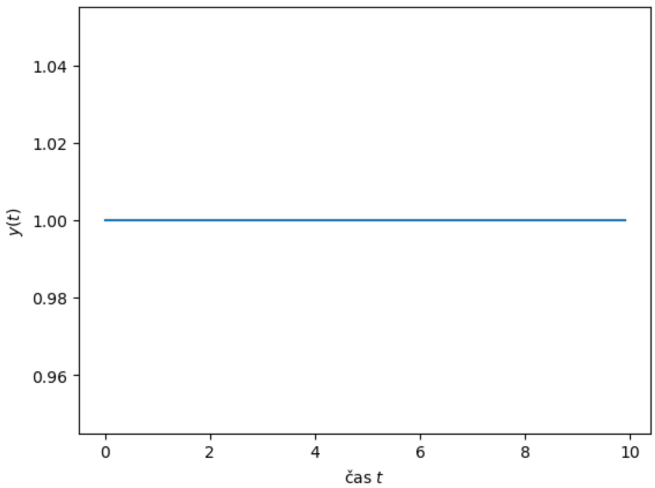
```python
# Obrázok pre hlavný text
figName = 'ICH_SS1R'
figNameNum = 0
exec(open('./figjobs/MRS10_figJob_01.py', encoding='utf-8').read())
```

Code Listing 4: Súbor MRS10_ICH1R.ipynb cell:05

```python
# Zmeňme hodnotu parametra a_0
a_0 = 0

# Výpočet hodnôt na y-ovej osi v zmysle danej časovej funkcie
plotData_y = b_0 * np.exp(-a_0 * plotData_x)
```

Code Listing 5: Súbor MRS10_ICH1R.ipynb cell:06

```
# Kreslenie grafu
plt.plot(plotData_x, plotData_y)
plt.xlabel('čas $t$')
plt.ylabel('$y(t)$')
plt.show()
```



Code Listing 6: Súbor MRS10_ICH1R.ipynb cell:07

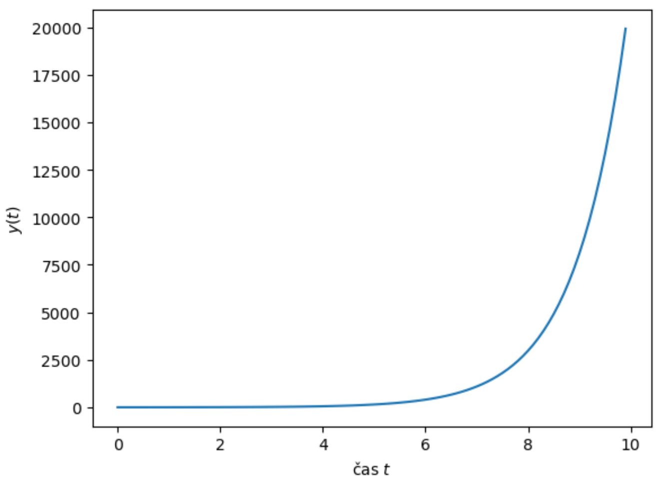
```
# Obrázok pre hlavný text
figName = 'ICH_AS1R'
figNameNum = 0
exec(open('./figjobs/MRS10_figJob_01.py', encoding='utf-8').read())
```

Code Listing 7: Súbor MRS10_ICH1R.ipynb cell:08

```
# Zmeňme hodnotu parametra a_0
a_0 = -1

# Výpočet hodnôt na y-ovej osi v zmysle danej časovej funkcie
plotData_y = b_0 * np.exp(-a_0 * plotData_x)
```

Code Listing 8: Súbor MRS10_ICH1R.ipynb cell:09

```
# Kreslenie grafu
plt.plot(plotData_x, plotData_y)
plt.xlabel('čas $t$')
plt.ylabel('$y(t)$')
plt.show()
```



Code Listing 9: Súbor `MRS10_ICH1R.ipynb cell:10`

```
# Obrázok pre hlavný text
figName = 'ICH_unstable1R'
figNameNum = 0
exec(open('./figjobs/MRS10_figJob_01.py', encoding='utf-8').read())
```

1.7.5 MATLAB: Control System Toolbox

Using the *Control System Toolbox* in MATLAB, the impulse response can be obtained with the command `impulse()`. Of course, it is first necessary to define the system whose impulse response is of interest, which can be done in this toolbox directly in the form of a transfer function using the command `tf()`. Thus:

```
G = tf([1], [1, 1])
impulse(G)
```

The `impulse()` command directly plots the graph as well.

1.7.6 MATLAB: Simulink

In Simulink, for example, the Dirac impulse approximation can be implemented using the `Step` block with the following settings:

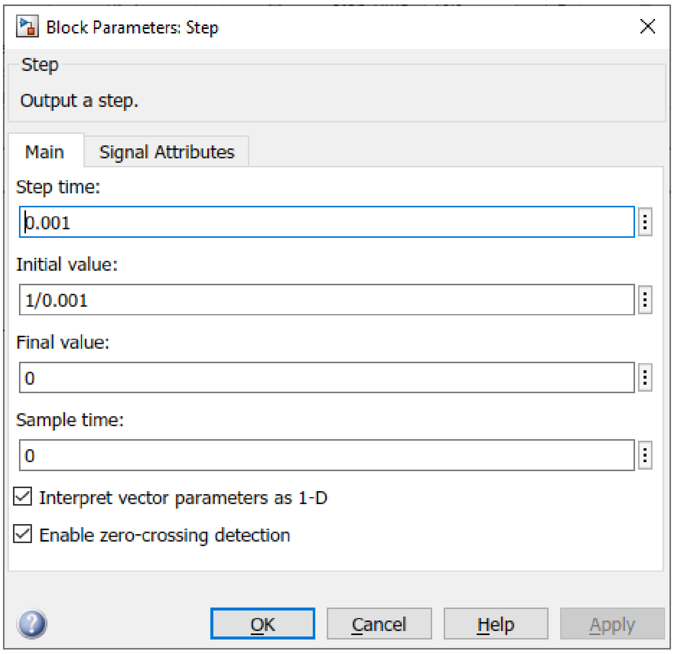


Figure 4: Settings of the `Step` block.

The block is part of the schematic:

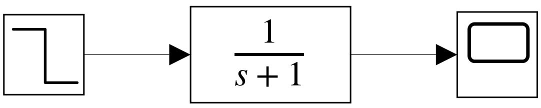


Figure 5: Simulation schematic for ICH SS1R

1.8 Step Response

The step response is the system's reaction to a unit step input.

A unit step is a signal that is zero for $t < 0$ and has a magnitude of one for $t \geq 0$. It represents a sudden change at time $t = 0$. The Laplace transform of the unit step is $U(s) = \frac{1}{s}$.

Since we have a mathematical description of the system, the step response can be determined analytically. The transfer function of a first-order system is given in (4). The Laplace transform of the input signal is $U(s) = \frac{1}{s}$. The Laplace transform of the output signal is then

$$Y(s) = G(s)U(s) = \frac{b_0}{s + a_0}\frac{1}{s} \tag{21a}$$

$$Y(s) = \frac{b_0}{s(s + a_0)}. \tag{21b}$$

To find the inverse transform of this expression, it is convenient to rewrite it as a partial fraction expansion:

$$\frac{b_0}{s(s + a_0)} = \frac{A}{s} + \frac{B}{s + a_0} \tag{22a}$$

$$b_0 = A(s + a_0) + Bs. \tag{22b}$$

Here, A and B are unknown coefficients. This equation holds for any value of s. For $s = 0$, we obtain

$$b_0 = Aa_0 \tag{23a}$$

$$A = \frac{b_0}{a_0}. \tag{23b}$$

For $s = -a_0$, we obtain

$$b_0 = B(-a_0) \tag{24a}$$

$$B = -\frac{b_0}{a_0}. \tag{24b}$$

Thus, the Laplace transform of the output signal becomes

$$Y(s) = \frac{b_0}{a_0}\frac{1}{s} - \frac{b_0}{a_0}\frac{1}{s + a_0}, \tag{25}$$

and its inverse transform is

$$y(t) = \frac{b_0}{a_0} - \frac{b_0}{a_0}e^{-a_0 t} \tag{26a}$$

$$y(t) = \frac{b_0}{a_0}\left(1 - e^{-a_0 t}\right). \tag{26b}$$

This time-domain function is the analytical expression for the system's step response. In this derivation, we assumed $a_0 \neq 0$.

If $a_0 = 0$, the Laplace transform of the output signal becomes

$$Y(s) = \frac{b_0}{s^2} \tag{27a}$$

$$Y(s) = b_0\frac{1}{s^2}. \tag{27b}$$

The inverse transform of this is

$$y(t) = b_0 t, \tag{28}$$

which is the time-domain function representing the system's step response when $a_0 = 0$.

It is clear that for the step response (PCH), different qualitative cases can be identified based on the single pole of the system. The system's pole is $s_1 = -a_0$.

Based on the above, we can distinguish the following cases: first-order static system (SS1R), first-order astatic system (AS1R), and unstable system.

1.8.1 Step Response of a Static First-Order System (PCH SS1R)

The time-domain function (26b) will be the step response of a first-order static system if $a_0 > 0$. Let us choose $a_0 = 1$ and $b_0 = 1$. The graph of the resulting time function is shown below.

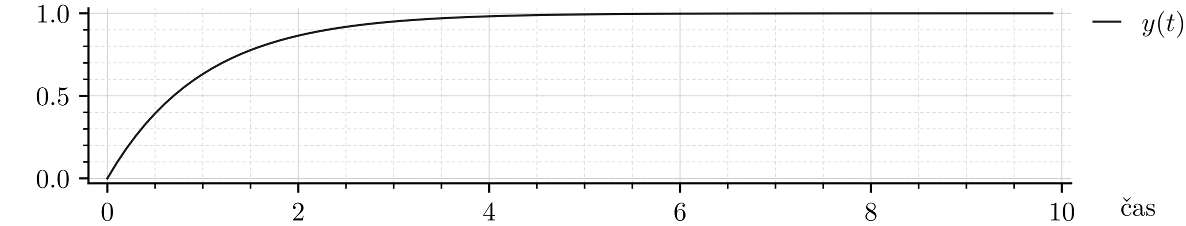


Figure 6: Step response of a first-order static system for $a_0 = 1$ and $b_0 = 1$

1.8.2 Step Response of an Astatic First-Order System (PCH AS1R)

The time-domain function (28) will be the step response of a first-order astatic system if $a_0 = 0$. The graph of the resulting time function is shown below.

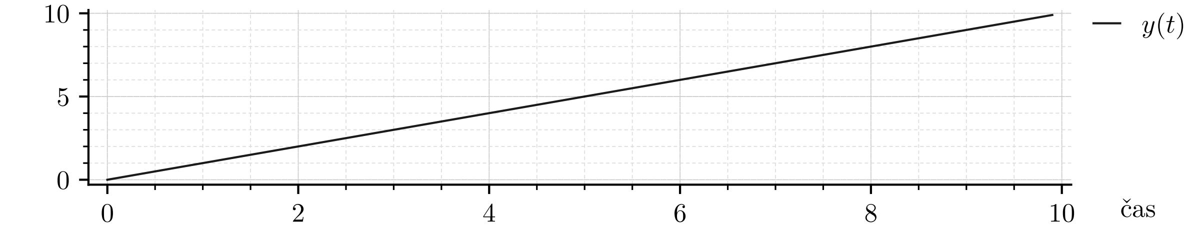


Figure 7: Step response of a first-order astatic system for $a_0 = 0$ and $b_0 = 1$

1.8.3 Step Response of an Unstable First-Order System

For completeness, let us include the case where $a_0 < 0$, i.e., the system is unstable. Let us choose $a_0 = -1$. The graph of the resulting time function is shown below.

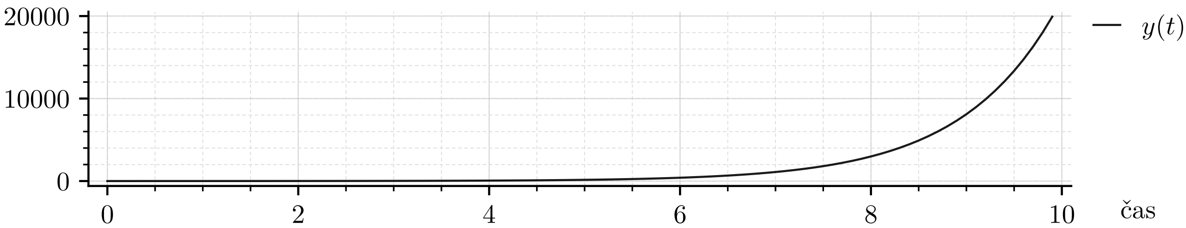


Figure 8: Step response of a first-order static system for $a_0 = -1$ and $b_0 = 1$

1.8.4 Python Script for Plotting Step Response Graphs

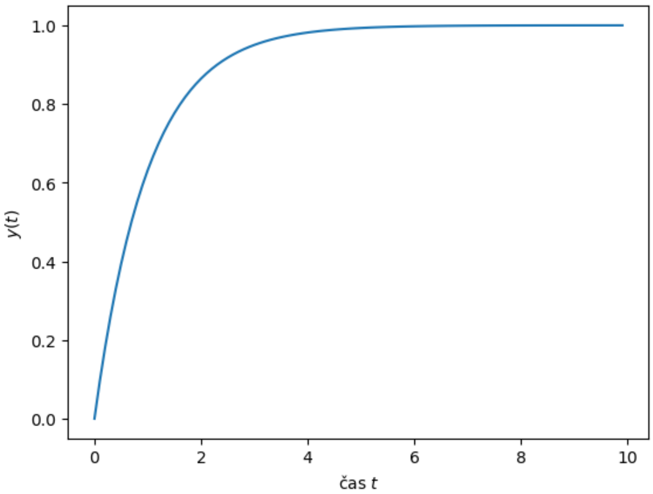
This section presents a script in the Python programming language to plot the above-mentioned step response graphs. The script is provided in the form of a Jupyter notebook, and the notebook cells are shown below.

Code Listing 10: Súbor MRS10_PCH1R.ipynb cell:02

```
import numpy as np
import matplotlib.pyplot as plt

# Parametre
b_0 = 1
a_0 = 1

# Súradnice bodov na x-ovej osi
plotData_x = np.arange(0, 10, 0.1)

# Výpočet hodnôt na y-ovej osi v zmysle danej časovej funkcie
plotData_y = (b_0/a_0) * (1 - np.exp(-a_0 * plotData_x))
```

Code Listing 11: Súbor MRS10_PCH1R.ipynb cell:03

```
# Kreslenie grafu
plt.plot(plotData_x, plotData_y)
plt.xlabel('čas $t$')
plt.ylabel('$y(t)$')
plt.show()
```



Code Listing 12: Súbor MRS10_PCH1R.ipynb cell:04

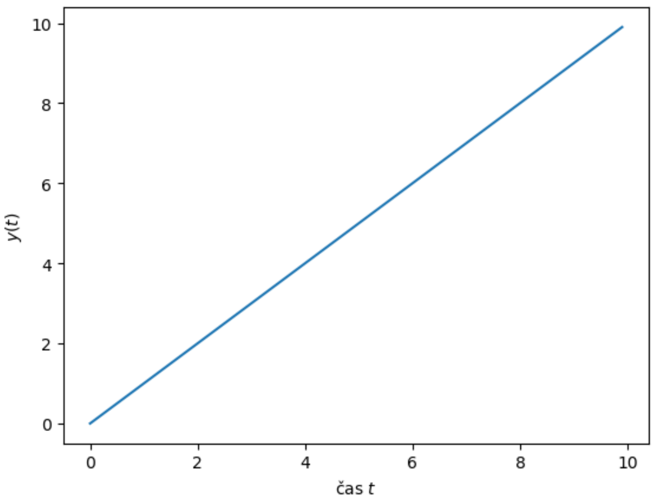
```
# Obrázok pre hlavný text
figName = 'PCH_SS1R'
figNameNum = 0
exec(open('./figjobs/MRS10_figJob_01.py', encoding='utf-8').read())
```

Code Listing 13: Súbor MRS10_PCH1R.ipynb cell:05

```
# Zmeňme hodnotu parametra a_0
a_0 = 0

# Výpočet hodnôt na y-ovej osi v zmysle danej časovej funkcie
plotData_y = b_0 * plotData_x
```

Code Listing 14: Súbor MRS10_PCH1R.ipynb cell:06

```
# Kreslenie grafu
plt.plot(plotData_x, plotData_y)
plt.xlabel('čas $t$')
plt.ylabel('$y(t)$')
plt.show()
```



Code Listing 15: Súbor MRS10_PCH1R.ipynb cell:07

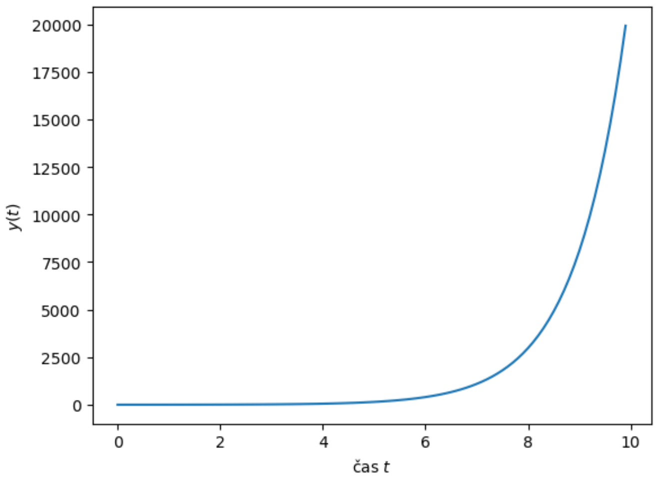
```
# Obrázok pre hlavný text
figName = 'PCH_AS1R'
figNameNum = 0
exec(open('./figjobs/MRS10_figJob_01.py', encoding='utf-8').read())
```

Code Listing 16: Súbor MRS10_PCH1R.ipynb cell:08

```
# Zmeňme hodnotu parametra a_0
a_0 = -1

# Výpočet hodnôt na y-ovej osi v zmysle danej časovej funkcie
plotData_y = (b_0/a_0) * (1 - np.exp(-a_0 * plotData_x))
```

Code Listing 17: Súbor MRS10_PCH1R.ipynb cell:09

```
# Kreslenie grafu
plt.plot(plotData_x, plotData_y)
plt.xlabel('čas $t$')
plt.ylabel('$y(t)$')
plt.show()
```



Code Listing 18: Súbor MRS10_PCH1R.ipynb cell:10

```
# Obrázok pre hlavný text
figName = 'PCH_unstable1R'
figNameNum = 0
exec(open('./figjobs/MRS10_figJob_01.py', encoding='utf-8').read())
```

1.8.5 MATLAB: Control System Toolbox

Using the *Control System Toolbox* in MATLAB, the step response can be obtained with the `step()` command. First, it is necessary to define the system whose step response is of interest. This can be done directly in the form of a transfer function using the `tf()` command. For example:

```
G = tf([1], [1, 1])
step(G)
```

The `step()` command handles time simulation settings (e.g., appropriate ODE solver settings) and directly plots the graph.

1.8.6 MATLAB: Simulink

Simulink directly allows working with transfer functions, automatically converting them into state-space representations and performing numerical simulations. For this case, the Simulink diagram would appear as follows:

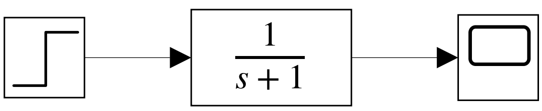


Figure 9: Simulation diagram for PCH SS1R

In the `Step` block, the step is configured to occur at time 0, changing from a value of 0 to 1.

2 Zero-Order System

The degree of the polynomial $A(s)$ can also be $n = 0$. In this case, we refer to a zero-order system. The transfer function for this scenario (considering both causality and positive realness) is

$$G(s) = \frac{b_0}{a_0}. \tag{29}$$

In this case, discussing system dynamics is essentially meaningless. Such a system generally functions as an amplifier, whose static gain is

$$\frac{y(\infty)}{u(\infty)} = \frac{b_0}{a_0}. \tag{30}$$

This system exhibits only static properties (static gain – the slope of the static characteristic). Discussions of dynamic properties, such as astatism, stability, and step response, are irrelevant in this context.

3 Second-Order System

3.1 Transfer Function

If the degree of the polynomial $A(s)$ is $n = 2$, we say that the system is of second order. To ensure causality and positive realness, we consider $m < n$, and thus, in general:

$$G(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}, \tag{31}$$

where $A(s)$ is given, without loss of generality, as a monic polynomial.

Similarly, second-order transfer functions can take the following forms:

$$G(s) = \frac{b_0}{s^2 + a_1 s + a_0}, \tag{32a}$$

$$G(s) = \frac{b_1 s}{s^2 + a_1 s + a_0}. \tag{32b}$$

3.2 Differential Equation

Let the system be described by a transfer function

$$G(s) = \frac{Y(s)}{U(s)}, \tag{33}$$

where $Y(s)$ is the Laplace transform of the output signal and $U(s)$ is the Laplace transform of the input signal. If we aim to rewrite it as a differential equation, then:

$$Y(s) = G(s)U(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} U(s), \tag{34a}$$

$$(s^2 + a_1 s + a_0)Y(s) = (b_1 s + b_0)U(s), \tag{34b}$$

$$s^2 Y(s) + a_1 s Y(s) + a_0 Y(s) = b_1 s U(s) + b_0 U(s), \tag{34c}$$

$$s^2 Y(s) = -a_1 s Y(s) - a_0 Y(s) + b_1 s U(s) + b_0 U(s), \tag{34d}$$

and thus the corresponding differential equation is:

$$\ddot{y}(t) = -a_1 \dot{y}(t) - a_0 y(t) + b_1 \dot{u}(t) + b_0 u(t). \tag{35}$$

Conversely, transforming the differential equation back into the transfer function involves the standard application of the Laplace transform to (35) under zero initial conditions.

3.3 State-Space Representation

In the state space, we introduce a state vector $x(t) \in \mathbb{R}^n$. In general, the state-space representation of a linear system is written as:

$$\dot{x}(t) = Ax(t) + bu(t), \tag{36a}$$

$$y(t) = c^{\mathsf{T}} x(t), \tag{36b}$$

where $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}^n$ are system parameters represented as a matrix and vectors.

Choosing the state vector $x(t)$ involves rewriting the higher-order differential equation as a system of first-order equations. This introduces new variables that become the elements of the state vector $x(t)$.

3.3.1 Example Procedure for Choosing State Variables

The transformation from a transfer function to a state-space representation is not unique and depends on the choice of state variables (state space). Here, we demonstrate a choice of state variables that results in a so-called controllable canonical form.

The transfer function of the system under consideration is given by:

$$\frac{Y(s)}{U(s)} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0}. \tag{37}$$

The question is how to transform this transfer function into a state-space representation and how to choose state variables.

When the numerator contains only a constant (i.e., the system has no zeros), the choice of state variables is relatively intuitive. Let us rewrite the transfer function (37) as two transfer functions in series:

$$\frac{Z(s)}{U(s)} = \frac{1}{s^2 + a_1 s + a_0}, \tag{38}$$

$$\frac{Y(s)}{Z(s)} = b_1 s + b_0, \tag{39}$$

where $Z(s)$ represents an auxiliary variable corresponding to $z(t)$. Clearly, the following relationship holds:

$$\frac{Y(s)}{U(s)} = \frac{Y(s)}{Z(s)} \frac{Z(s)}{U(s)}, \tag{40}$$

or explicitly:

$$\frac{Y(s)}{U(s)} = (b_1 s + b_0) \frac{1}{s^2 + a_1 s + a_0}. \tag{41}$$

By the way, the transfer function (39) is mathematically valid, but its numerator is of degree 1, and its denominator is of degree 0, meaning it represents a non-causal system. Therefore, the transfer function (39) alone is not suitable for modeling a real physical system.

The first transfer function (38) can be rewritten as a second-order differential equation:

$$\ddot{z}(t) + a_1 \dot{z}(t) + a_0 z(t) = u(t). \tag{42}$$

This can be converted into a system of first-order differential equations by choosing state variables. For example, let:

$$x_1(t) = z(t), \tag{43}$$

where $x_1(t)$ is the first state variable. Then:

$$\dot{x}_1(t) = \dot{z}(t). \tag{44}$$

Let the second state variable be:

$$x_2(t) = \dot{z}(t), \tag{45}$$

so that:

$$\dot{x}_2(t) = \ddot{z}(t). \tag{46}$$

At this point, we can write:

$$\dot{x}_1(t) = x_2(t). \tag{47}$$

This is the first differential equation! It involves only the newly introduced state variables ($x_1(t)$ and $x_2(t)$). The second differential equation is simply (46). However, can we express $\ddot{z}(t)$ solely in terms of the new state variables? Yes. From (42), it is evident that:

$$\ddot{z}(t) = -a_1\dot{z}(t) - a_0 z(t) + u(t) = -a_1 x_2(t) - a_0 x_1(t) + u(t). \tag{48}$$

Thus, (46) becomes:

$$\dot{x}_2(t) = -a_1 x_2(t) - a_0 x_1(t) + u(t). \tag{49}$$

This is the second differential equation.
Both equations together:

$$\dot{x}_1(t) = x_2(t), \tag{50}$$

$$\dot{x}_2(t) = -a_1 x_2(t) - a_0 x_1(t) + u(t). \tag{51}$$

In matrix form:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t). \tag{52}$$

Returning to the transfer function (39), it can be expressed as a differential equation:

$$y(t) = b_1\dot{z}(t) + b_0 z(t). \tag{53}$$

However, we have already chosen $z(t) = x_1(t)$ and $\dot{z}(t) = x_2(t)$. Thus, (53) becomes:

$$y(t) = b_1 x_2(t) + b_0 x_1(t), \tag{54}$$

or in matrix form:

$$y(t) = \begin{bmatrix} b_0 & b_1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \tag{55}$$

This is a system of the form:

$$\dot{x}(t) = Ax(t) + bu(t), \tag{56a}$$

$$y(t) = c^{\mathsf{T}} x(t), \tag{56b}$$

where:

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad c = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}. \tag{57}$$

3.3.2 Subsequent Examples of Direct Description of a System in State Space

We see that if we have a transfer function in the form

$$\frac{Y(s)}{U(s)} = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \tag{58}$$

then the state-space representation of the system is given as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{59a}$$

$$y(t) = \begin{bmatrix} b_0 & b_1 \end{bmatrix} x(t) \tag{59b}$$

where, of course, $x(t) \in \mathbb{R}^2$ is the state vector. Similarly, if we have a transfer function in the form

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^2 + a_1 s + a_0} \tag{60}$$

then the state-space representation of the system is given as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{61a}$$

$$y(t) = \begin{bmatrix} b_0 & 0 \end{bmatrix} x(t) \tag{61b}$$

which can also be written in the form

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ b_0 \end{bmatrix} u(t) \tag{62a}$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \tag{62b}$$

because we only changed where the coefficient b_0 multiplies the corresponding signal. It does not matter whether it is at the input or the output.

For completeness, if we have a transfer function in the form

$$\frac{Y(s)}{U(s)} = \frac{b_1 s}{s^2 + a_1 s + a_0} \tag{63}$$

then the state-space representation of the system is given as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{64a}$$

$$y(t) = \begin{bmatrix} 0 & b_1 \end{bmatrix} x(t) \tag{64b}$$

Another example could be a transfer function in the form

$$\frac{Y(s)}{U(s)} = \frac{b_0}{s^2 + a_0} \tag{65}$$

and the state-space representation would be

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ 0 & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{66a}$$

$$y(t) = \begin{bmatrix} b_0 & 0 \end{bmatrix} x(t) \tag{66b}$$

3.3.3 From State-Space Representation to Transfer Function

Let us consider a system given in the state-space form as

$$\dot{x}(t) = Ax(t) + bu(t) \tag{67a}$$

$$y(t) = c^{\mathsf{T}} x(t) \tag{67b}$$

where the state vector $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}^n$ are the matrix and vectors that represent the system parameters.

Equation (67a) is essentially a vector differential equation, meaning the unknown is a vector $x(t)$ containing signals (state variables).

Applying the Laplace transform to equations (67), assuming zero initial conditions (since we are deriving the transfer function), we can write

$$sIX(s) = AX(s) + bU(s) \tag{68a}$$

$$Y(s) = c^{\mathsf{T}} X(s) \tag{68b}$$

where I is the identity matrix of the same size as A, and s is the Laplace operator. The term sI is a matrix with Laplace operators along its diagonal. $X(s)$ is a vector containing the Laplace transforms of the state variables.

The transfer function is the ratio of the output to the input in the Laplace domain. Starting from equation (68a), we express the ratio of $X(s)$ to $U(s)$. Writing

$$sIX(s) = AX(s) + bU(s) \tag{69}$$
$$sIX(s) - AX(s) = bU(s) \tag{70}$$

the dimensions of the matrices and vectors are preserved. Then

$$(sI - A)X(s) = bU(s) \tag{71}$$

where $(sI - A)$ is a matrix. To isolate $X(s)$, we need to multiply the entire equation from the left by the inverse of $(sI - A)$, i.e., $(sI - A)^{-1}$:

$$(sI - A)^{-1}(sI - A)X(s) = (sI - A)^{-1}bU(s) \tag{72}$$
$$X(s) = (sI - A)^{-1}bU(s) \tag{73}$$

Now substituting $X(s)$ into equation (68b), we have

$$Y(s) = c^{\mathsf{T}}X(s) \tag{74}$$
$$Y(s) = c^{\mathsf{T}}(sI - A)^{-1}bU(s) \tag{75}$$

The ratio $Y(s)/U(s)$ is then

$$\frac{Y(s)}{U(s)} = c^{\mathsf{T}}(sI - A)^{-1}b \tag{76}$$

and the transfer function is

$$G(s) = c^{\mathsf{T}}(sI - A)^{-1}b \tag{77}$$

Consider a specific case where the system is given in the state-space form as

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \tag{78a}$$
$$y(t) = \begin{bmatrix} b_0 & b_1 \end{bmatrix} x(t) \tag{78b}$$

Here, $A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $c = \begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$. Let us compute the matrix $(sI - A)$:

$$(sI - A) = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} = \begin{bmatrix} s & -1 \\ a_0 & s + a_1 \end{bmatrix} \tag{79}$$

Its inverse is

$$\begin{aligned} (sI - A)^{-1} &= \begin{bmatrix} s & -1 \\ a_0 & s + a_1 \end{bmatrix}^{-1} = \frac{1}{(s + a_1)s - (-a_0)} \begin{bmatrix} s + a_1 & 1 \\ -a_0 & s \end{bmatrix} \\ &= \frac{1}{s^2 + a_1 s + a_0} \begin{bmatrix} s + a_1 & 1 \\ -a_0 & s \end{bmatrix} \\ &= \begin{bmatrix} \frac{s+a_1}{s^2+a_1 s+a_0} & \frac{1}{s^2+a_1 s+a_0} \\ \frac{-a_0}{s^2+a_1 s+a_0} & \frac{s}{s^2+a_1 s+a_0} \end{bmatrix} \end{aligned} \tag{80}$$

Multiplying from the right by the vector b gives

$$(sI - A)^{-1}b = \begin{bmatrix} \frac{s+a_1}{s^2+a_1 s+a_0} & \frac{1}{s^2+a_1 s+a_0} \\ \frac{-a_0}{s^2+a_1 s+a_0} & \frac{s}{s^2+a_1 s+a_0} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2+a_1 s+a_0} \\ \frac{s}{s^2+a_1 s+a_0} \end{bmatrix} \tag{81}$$

and subsequently from the left by c^{T}:

$$c^{\mathsf{T}}(sI - A)^{-1}b = \begin{bmatrix} b_0 & b_1 \end{bmatrix} \begin{bmatrix} \frac{1}{s^2+a_1 s+a_0} \\ \frac{s}{s^2+a_1 s+a_0} \end{bmatrix} = b_0 \frac{1}{s^2 + a_1 s + a_0} + b_1 \frac{s}{s^2 + a_1 s + a_0} \tag{82}$$

After simplification,

$$c^{\mathsf{T}}(sI - A)^{-1}b = \frac{b_0 + b_1 s}{s^2 + a_1 s + a_0} \tag{83}$$

is the transfer function for this specific case.

3.3.4 From State-Space Description to a Differential Equation

Consider a system given in the state-space form as:

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a_0 & -a_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u(t) \tag{84}$$

$$y(t) = \begin{bmatrix}b_0 & b_1\end{bmatrix}\begin{bmatrix}x_1(t)\\ x_2(t)\end{bmatrix} \tag{85}$$

The goal is to rewrite this system of differential equations into a single higher-order differential equation. In this case, the output signal $y(t)$ will serve as the unknown variable in the higher-order differential equation. The system of equations is as follows:

$$\dot{x}_1(t) = x_2(t) \tag{86}$$

$$\dot{x}_2(t) = -a_1x_2(t) - a_0x_1(t) + u(t) \tag{87}$$

$$y(t) = b_0x_1(t) + b_1x_2(t) \tag{88}$$

For example, differentiate equation (87) and substitute $\dot{x}_1(t)$ from the first equation (86):

$$\ddot{x}_2(t) = -a_1\dot{x}_2(t) - a_0\dot{x}_1(t) + \dot{u}(t) \tag{89}$$

$$\ddot{x}_2(t) = -a_1\dot{x}_2(t) - a_0x_2(t) + \dot{u}(t) \tag{90}$$

From equation (88), it follows that

$$x_2(t) = \frac{1}{b_1}y(t) + \frac{b_0}{b_1}x_1(t) \tag{91}$$

$$\dot{x}_2(t) = \frac{1}{b_1}\dot{y}(t) + \frac{b_0}{b_1}\dot{x}_1(t) = \frac{1}{b_1}\dot{y}(t) + \frac{b_0}{b_1}x_2(t) \tag{92}$$

and

$$\ddot{x}_2(t) = \frac{1}{b_1}\ddot{y}(t) + \frac{b_0}{b_1}\dot{x}_2(t) \tag{93a}$$

$$= \frac{1}{b_1}\ddot{y}(t) + \frac{b_0}{b_1}\left(-a_1x_2(t) - a_0x_1(t) + u(t)\right) \tag{93b}$$

$$= \frac{1}{b_1}\ddot{y}(t) - \frac{b_0a_1}{b_1}x_2(t) - \frac{b_0a_0}{b_1}x_1(t) + \frac{b_0}{b_1}u(t) \tag{93c}$$

Results (91), (92), and (93) can be substituted into (90), thus:

$$\begin{aligned}&\frac{1}{b_1}\ddot{y}(t) - \frac{b_0a_1}{b_1}x_2(t) - \frac{b_0a_0}{b_1}x_1(t) + \frac{b_0}{b_1}u(t)\\ &= -a_0\left(\frac{1}{b_1}y(t) + \frac{b_0}{b_1}x_1(t)\right) - a_1\left(\frac{1}{b_1}\dot{y}(t) + \frac{b_0}{b_1}x_2(t)\right) + \dot{u}(t)\end{aligned} \tag{94}$$

$$\begin{aligned}&\frac{1}{b_1}\ddot{y}(t) - \frac{b_0a_1}{b_1}x_2(t) - \frac{b_0a_0}{b_1}x_1(t) + \frac{b_0}{b_1}u(t)\\ &= -\frac{a_0}{b_1}y(t) - \frac{a_0b_0}{b_1}x_1(t) - \frac{a_1}{b_1}\dot{y}(t) - \frac{a_1b_0}{b_1}x_2(t) + \dot{u}(t)\end{aligned} \tag{95}$$

and thus:

$$\frac{1}{b_1}\ddot{y}(t) + \frac{b_0}{b_1}u(t) = -\frac{a_0}{b_1}y(t) - \frac{a_1}{b_1}\dot{y}(t) + \dot{u}(t) \tag{96}$$

$$\ddot{y}(t) + b_0u(t) = -a_0y(t) - a_1\dot{y}(t) + b_1\dot{u}(t) \tag{97}$$

$$\ddot{y}(t) = -a_1\dot{y}(t) - a_0y(t) + b_1\dot{u}(t) + b_0u(t) \tag{98}$$

3.4 Stability

The stability of the system is determined by the roots of the characteristic polynomial $A(s)$, in this case:

$$A(s) = s^2 + a_1s + a_0 \tag{99}$$

This polynomial has two roots, which can be:

- two distinct real numbers (the imaginary part of the root is zero),
- a single real number, which is a double root,
- or two complex numbers, which are conjugate pairs.

In all cases, the system is stable if and only if the real parts of the poles are negative (in the left half-plane of the complex plane).

If at least one root lies on the imaginary axis (the real part of the root is zero), the system is said to be on the boundary of stability.

If at least one root has a positive real part, the system is unstable.

3.5 Static Gain and Astaticism

3.5.1 Static Gain

Consider a system that is not unstable, and none of its poles are zero. Such a system can be called static, as the output stabilizes when the input is steady. In this case, we have a second-order system, referred to as a *static second-order system*, abbreviated SS2R.

For such a system, it is possible to determine its static gain. The static gain is the ratio of output to input in steady-state.

In the steady-state, signals do not change, meaning their time derivatives are zero. Consider the differential equation (35). In steady-state, $\dot{y}(\infty) = 0$, where ∞ symbolizes the time at which the signals have stabilized. Similarly, $\ddot{y}(\infty) = 0$ and $\dot{u}(\infty) = 0$, and thus:

$$0 = -a_0 y(\infty) + b_0 u(\infty) \tag{100}$$

The ratio of output to input is:

$$\frac{y(\infty)}{u(\infty)} = \frac{b_0}{a_0} \tag{101}$$

which is the static gain of the system. This value can be designated as an independent parameter of the system, e.g., $K = \frac{b_0}{a_0}$.

It is conventional to assume that the input is "unitary," i.e., $u(\infty) = 1$, and thus it is written as $y(\infty) = \frac{b_0}{a_0}$. The same conclusion is reached if we consider a constant steady signal at the input, $u(t) = 1$. For a unit step input, $U(s) = \frac{1}{s}$. Then:

$$Y(s) = \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \frac{1}{s} \tag{102}$$

The final value of this Laplace-transformed signal ($Y(s)$ is the Laplace transform of $y(t)$) is the potential steady-state value of the system output. Using the final value theorem:

$$\begin{aligned} y(\infty) &= \lim_{s \to 0} sY(s) && \text{(103a)} \\ &= \lim_{s \to 0} s \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} \frac{1}{s} && \text{(103b)} \\ &= \lim_{s \to 0} \frac{b_1 s + b_0}{s^2 + a_1 s + a_0} && \text{(103c)} \\ &= \frac{b_0}{a_0} && \text{(103d)} \end{aligned}$$

3.5.2 Astaticism

If one of the system's poles is zero, the system is said to be astatistic (i.e., it contains an astaticity). If exactly one pole is zero, it is called first-order astaticity. If two poles are zero, it is referred to as second-order astaticity, and so on.

In this case, we have a second-order system, which we call a *second-order astatic system*, abbreviated AS2R.

Here, the system has two poles (generally, two complex numbers). Denote the poles by p_1 and p_2. If one of them is zero, $p_1 = 0$, then:

$$A(s) = (s - p_1)(s - p_2) = (s - 0)(s - p_2) = s(s - p_2) \tag{104}$$

The transfer function of a second-order system with first-order astaticity could therefore take the form:

$$G(s) = \frac{b_1 s + b_0}{s(s - p_2)} \tag{105a}$$

$$G(s) = \frac{b_0}{s(s - p_2)} \tag{105b}$$

Note that if $G(s) = \frac{b_1 s}{s(s-p_2)}$, then $G(s) = \frac{b_1}{s-p_2}$, and thus it is no longer a second-order system[1].

If both poles are zero, i.e., $A(s) = s^2$, then the transfer function of a second-order system with second-order astaticity could take the form:

$$G(s) = \frac{b_1 s + b_0}{s^2} \tag{106a}$$

$$G(s) = \frac{b_0}{s^2} \tag{106b}$$

Incidentally, the transfer function (106b) represents a double integrator.

3.6 Static Characteristic

For linear systems, the static characteristic is a straight line, and without loss of generality, it can be assumed to pass through the origin of the coordinate system. The slope of this line is given by the system's static gain. From the discussion above, the slope of the static characteristic for a linear system is $K = \frac{b_0}{a_0}$.

[1] In general, dividing polynomials must be handled carefully from a mathematical standpoint, as it is not always valid to simply cancel terms.