Introductory Lecture

Contents

The aim of this text is primarily to create an introductory list of terms that the reader needs to work with when studying in the field of cybernetics, robotics, control theory, and so on. In general, these are quite broad terms, but in this text, it is possible to see how they are handled from the perspective of technical cybernetics, the (technical) theory of systems, automation, and control systems.

1 Course: Modeling and Control of Systems

The course *Modeling and Control of Systems* can be divided into two main parts: system modeling and system control. For both parts, it is a primary set of information for the reader whose goal is to study cybernetics, robotics, or further control theory, system identification, etc. (very extensive possibilities).

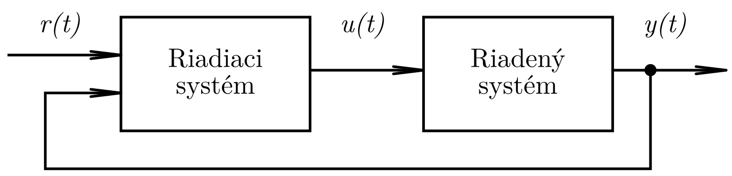

Figure 1: Closed-loop control system. Part of the course Modeling and Control of Systems deals with the controlled system (its mathematical modeling) and part of the course deals with the control system.

Mathematics

For work in this subject (or in these areas), it is practically essential to use certain tools, here meaning knowledge and practical skills mainly (but not only) from mathematics - differential and integral calculus, working with matrices, and algebra. The subject is not focused on these areas. The subject uses them. It is very advantageous if the student has encountered the use of these tools in the past, but it is not necessary. Everything needed will be included in the subject, although often only to a minimal extent, with the understanding that the reader should have a clear direction in which to further utilize these tools (such as mathematics).

Software and Computational Tools

A very related set of tools that need to be used here is computational software, such as MATLAB, GNU Octave, NumPy libraries for Python, and similar libraries for other scripting languages like Julia, R, and so on. Similarly, it applies that within the subject, everything necessary for successful work and further expanding knowledge, experience, and skills will be provided (it is, of course, not possible to cover everything in one subject).

The main goal of the subject is thus to create a space for the first contact with the terms discussed below, but not in general, rather from the perspective of cybernetics, control theory, automation, and so on.

2 Cybernetics

2.1 Cybernetics and Mathematics

From a certain perspective on the history of mathematics as such, it can be concluded that Cybernetics is a relatively young branch of mathematics, which emerged around the mid-20th century [1].

Cybernetics is a science. At first glance, it clearly appears as a technical science. Especially if we consider other fields that have emerged from Cybernetics, such as computer science, robotics, and so on. It would be incomplete to state that Cybernetics is only about technical tasks and problems, about machines and their control, about computers and control systems in general. However, it is precisely computers, automated machines, and technical applications in general that allow the connection of (mathematical) theory with everyday life. Often, it is Cybernetics that finds the path from mathematical theories, including the most abstract ones, to concrete applications in the real world.

Cybernetics can, for example, be divided into theoretical and technical. Just like computer science. It is important to consider that this means connections to other sciences, technical fields, and so on. The results of theoretical Cybernetics can build on the results and tools from mathematics and other theoretical areas, and the results of technical Cybernetics can be called the application of abstract theory to specific technical problems.

Cybernetics and what emerges from it is not the culmination of mathematics, but one of the bridges between mathematics and other fields (both scientific and practical). Mathematics continues to evolve and will continue to evolve. Cybernetics creates a path through which mathematics reaches people in everyday life. Its greatness lies in the fact that it has changed and continues to change the world like hardly any other science before it [1].

2.2 History of Cybernetics

The emergence of Cybernetics is due to the convergence of various scientific disciplines that intersect, overlap, and influence each other, especially in its early stages. These can be divided into groups:

- scientific disciplines on system control and decision-making,
- scientific disciplines on information processing and transmission,

- computer sciences in the narrower sense of automata and algorithms.

A classic definition of Cybernetics comes from Norbert Wiener's work from 1948: *Cybernetics is the science of control and communication in living organisms, human society, and machines* [2].

Norbert Wiener is considered the first author to see the above-mentioned scientific disciplines as a meaningful independent whole, as an independent science, and he names it Cybernetics. His work also implies that for Cybernetics, the matter (and energy) entering (and exiting) the studied process, phenomenon, or object is not essential. What is essential is the structure it forms within them and the laws that govern this structure.

2.2.1 Norbert Wiener

- Year 1948, book *Cybernetics or Control and Communication in the Animal and the Machine*
  The book is considered the birth of Cybernetics as a science.
  It builds on journal publications by three authors: Norbert Wiener, who is also the author of the mentioned book, Arturo Rosenblueth, a Mexican physician and psychologist, and Julian Bigelow, an electrical engineer and mathematician, Wiener's assistant.

The Word Cybernetics

The ancient Greek term *κυβερνητικός* (kubernētikos), meaning good at steering or good at governing, can be found in Plato's works *The Republic* and *Alcibiades*. Essentially, it refers to a helmsman, but in the context of governing or managing people.

The word *cybernétique* (French word) was also used by André-Marie Ampère to denote the sciences of governance or management in his classification of human knowledge.

3 System, Signal, and Dynamics

The theory of any real phenomenon is based on a concept called a model. Without introducing any restrictions, a model can be represented by mathematical relationships, and these mathematical relationships are called a system.

However, we understand the meaning of the word system more broadly. The diversity of reality is not only given by the diversity of individual elements that compose it. It is significantly more contributed to by the variety of interactions between these elements. The intensity of this interaction varies: sometimes intense, other times weak, or none at all.

System and Signal

We use the word system (also referred to as a plant) to denote a certain number of elements whose mutual interaction is relatively intense and can be assigned some meaning (sense). Interactions with the environment, on the other hand, are significantly weaker.

The relationship of the system with the environment is expressed through connections referred to as *inputs and outputs*.

In the context of technical relations, these inputs and outputs are quantities such as temperature, electrical voltage, water level, and so on. These generally change over time; their values change over time. Collectively, we refer to them as signals, input and output signals. It is equally meaningful to talk about input and output quantities.

From a mathematical perspective, when we talk about a signal, it (almost always) refers to a function of time. Let us denote time as t. For example, let us denote the output of the system, the output quantity of the system, by the symbol y. The notation $y(t)$ means that it is the output of the system (the output quantity of the system), which is a function of time t. In general, it changes over time. We say that $y(t)$ is a signal.

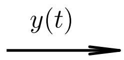

Figure 2: Signal in a block diagram. Often, it is useful to represent systems and signals graphically, schematically within so-called block diagrams. A signal is represented by a line with an arrow indicating the direction of information transfer. The signal designation is given next to the line.

We will understand the input of the system as a signal that is the external cause of changes in the system. If these are intentional (desired) changes, we generally speak of control signals, more specifically in the context of system control, we speak of control actions. Undesired changes are the result of the influence of input signals referred to as disturbances.

It makes sense to consider systems that do not have any defined input (e.g., signal generators). Such systems are also called free, autonomous systems.

On the other hand, we understand the output of the system as a signal that we observe (measure) in the system, or a signal through which the system affects its surroundings.

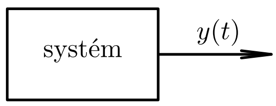

Figure 3: System with an output signal (with an output quantity)

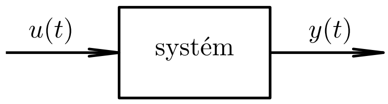

Figure 4: System with one input signal and one output signal. SISO system.

A system with one input and one output is called a single-dimensional system, abbreviated as a SISO system (Single Input - Single Output). It is represented by a block (Fig. 4). The input signal $u(t)$ and the output signal $y(t)$ are indicated by arrows.

A system with multiple input and output variables is called a multi-dimensional, or MIMO (Multi Input - Multi Output) system. Defining input and output variables may not be a trivial and straightforward problem.

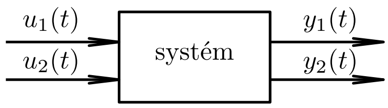

Figure 5: System with multiple input and output signals. MIMO system.

Dynamic System

For example, from a physical perspective, it often makes sense to emphasize that an essential property of a system is its variability over time. What exactly varies over time can be quite diverse. We attribute the term dynamic to such a system.

Practically, we could always find a perspective from which we could label a system as dynamic. Essentially, everything is a dynamic system. However, in the modeling and control of systems, the opposite perspective is very significant, i.e., recognizing a system where it makes sense to neglect its potential dynamic properties.

Better terms in this context are *inertia and non-inertia*.

3.1 Inertial and Non-inertial Systems

When we talk about a dynamic system, we usually mean an *inertial* system.

A *non-inertial* system is one whose output depends only on the instantaneous value of the input. Past input values do not affect the current output value. An example of a non-inertial system can be a resistive voltage divider.

An *inertial* system can be thought of as a system "with memory." Its current output depends on the instantaneous value of the input as well as the input values in the past. Such a system is *causal*.

Incidentally, it is theoretically possible to talk about non-causal systems. In such a case, the output also depends on future input values. The words *inertia* and *causality* are recommended to be perceived here mainly from a physical point of view.

An example of an inertial causal system can be an electrical capacitor. The current electric charge $Q(t)$ on the capacitor is given by the entire history of the electric current $I(t)$ through the capacitor. Formally, the charge at time t_0 will be

$$Q(t_0) = \int_{-\infty}^{t_0} I(t)\mathrm{d}t \tag{1}$$

4 System State and Initial Conditions

The previous subsection leads to questions such as how far into the past do the input values still affect the current output? Is it perhaps necessary to know the complete past of the system? This introduces the concept of the *system state*.

The mentioned charge Q on the capacitor has a certain value at time t_0. Let's denote it as Q_0. The capacitor as a system is in such a state that its quantity has the value Q_0. This is exactly at time t_0.

If we came to this system at time t_0, we would be able to determine that it is in some state, in this case expressible by the value Q_0. If we wanted to change the value of the charge on the capacitor, i.e., the state it is in, we would need to act on the input of this system for some time (apply voltage to the capacitor terminals).

We know the state of the system at the beginning of our action on the input, and subsequently, at the end of the action, the system will generally be in a different state than at the beginning. The system is in some state at every moment.

Unlike the fairly intuitive state of the system in the case of a capacitor, determining which quantities indicate the state of the system and whether we can measure them is a separate question.

4.1 Example with a Mechanical System

Consider the translational motion of a body with mass m, on which a force $u(t)$ acts. In general, a time-varying force is the input of the system. The body moves along a straight line. The position of this body, i.e., its distance from a defined zero point, is the output variable of the system. Let us denote it as $y(t)$.

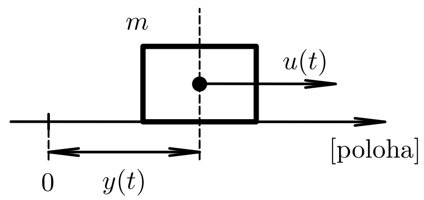

Figure 6: Translational motion of a body with mass m.

The equation of motion in this case is[1]

$$m\ddot{y}(t) = u(t) \tag{2}$$

[1]Newton's $F = ma$ where a is acceleration, which is the second derivative of position with respect to time, i.e., $\ddot{y}$.

It describes the given situation as a dynamic system. It relates the input and output of the system. It is a second-order differential equation.

But how do we determine the state of the system?

What do we need to know to determine the output of the system from some initial time t_0 onwards, i.e., over the interval $\langle t_0, t \rangle$?

The position, the output of the system, is given by the initial position at time t_0, denoted as y_0, and further by the integral of the velocity of the body $\dot{y}(t)$ over the given interval.

$$y(t) = y_0 + \int_{t_0}^{t} \dot{y}(\tau)\mathrm{d}\tau \tag{3}$$

where, of course, $\tau \in \langle t_0, t \rangle$.

But what determines the course of the velocity $\dot{y}(t)$? Over the interval $\langle t_0, t \rangle$, it must evidently hold that

$$\dot{y}(t) = z_0 + \int_{t_0}^{t} \ddot{y}(\tau)\mathrm{d}\tau \tag{4}$$

where z_0 is the initial velocity of the body at time t_0 and $\ddot{y}(t)$ is the time course of the body's acceleration.

Do we know the time course of the acceleration over the interval $\langle t_0, t \rangle$? Yes. Directly thanks to the differential equation describing the system itself. From equation (2), it directly follows that

$$\dot{y}(t) = \frac{1}{m} \int_{t_0}^{t} u(\tau)\mathrm{d}\tau \tag{5}$$

We generally know the time course of the input signal $u(t)$. We control the input variable.

To determine the output of the system at time t, we needed to know the following: the initial position, the initial velocity, and the time course of the input variable from the beginning to time t.

The state of the system at the beginning, but not only at the beginning, is given by two variables: position and velocity. Their values can be considered the state of the system. These variables can be referred to as the state variables of the system.

The state of the system changes if the input signal acts for some time.

4.2 State Vector

In general, there are several state variables. At least as many as the order of the differential equation describing the dynamic system. Thus, we speak of the order of the system. In the previous example, the system is of the second order, $n = 2$.

It is practical to arrange the state variables into a state vector. In the previous example, the state vector consists of two state variables. Let us denote the state vector as $x(t)$, and formally its dimension is given as $x(t) \in \mathbb{R}^n$.

$$x(t) = \begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} \tag{6}$$

where we have listed the state variables from the previous example. Usually, state variables are denoted by separate symbols, typically

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{7}$$

The elements of the state vector $x(t)$ can take on various values; in other words, the vector $x(t)$ is in some space, which we refer to as the *state space*.

5 Mathematical Description of the System

5.1 Example of a Non-inertial System

Consider a classic resistive voltage divider as shown in the following figure.

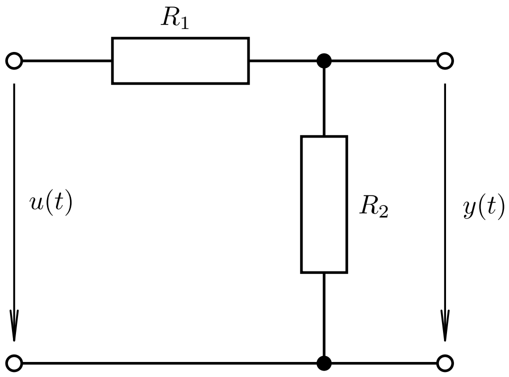

Figure 7: Resistive Voltage Divider

It is evident that we can talk about a system here. We can define an output signal and an input signal. Let the input of the system be the electrical voltage denoted as $u(t)$ and the output signal be the voltage $y(t)$.

Let the input be a constant voltage, the value of the input signal $u(t)$ does not change, the input is steady. As a result, the electric current through resistors R_1 and R_2 is constant, i.e., $I(t) = u(t)/(R_1 + R_2)$. The voltage across resistor R_2 is then $y(t) = R_2 I(t) = R_2 u(t)/(R_1 + R_2)$. The relationship between the output and input of the system is described by the equation

$$y(t) = \frac{R_2}{R_1 + R_2} u(t) \tag{8}$$

This equation is the mathematical model of the system.

If we consider a time-varying signal $u(t)$ at the input, this equation would still hold. It would still be useful from the perspective of mathematical modeling of the system.

Equation (8) is an algebraic equation. It is not a differential equation. The unknown in the equation is the output signal $y(t)$, and the derivative of the unknown does not appear in the equation. In such a case, we talk about a non-inertial system. In this case, it is not practical to consider the system as dynamic. The moment the value of $u(t)$ is given, the value of $y(t)$ is also given at the same moment. Sometimes we refer to such a system as static in the sense that it does not make sense to consider the dynamics of the system, and thus the system is static.

5.1.1 System Gain

However, it is always important to consider the steady state of the system, i.e., the situation when the input $u(t)$ and the output $y(t)$ do not change their values. The steady state can be considered a property of the system. Either it is possible for the system to be in a steady state, or it is not.

If it is possible for the system to be in a steady state, we say that the system is static. If it is not possible, we talk about an astatic system, i.e., one where even if the input is steady, the output never stabilizes.

The values of the input and output in the steady state are denoted, for example, as $y(\infty)$ and $u(\infty)$, where ∞ denotes time at infinity, which in practice is the time required for the practical stabilization of the values of the given signals.

If it is possible for the system to be in a steady state, this property is described by the term *static gain of the system* or simply *system gain*. The system gain is the ratio between the steady-state value of the system's output signal and the steady-state value of the system's input signal. Let us denote it as K. Thus,

$$K = \frac{y(\infty)}{u(\infty)} \tag{9}$$

5.2 Example of a Dynamic System

Consider an RC circuit as shown in Fig. 8. Let the capacitor C be charged initially at time $t = 0$ with a voltage value of u_0. In other words, the voltage $u(t)$ at time 0 is u_0, i.e., $u(0) = u_0$. The capacitor C is connected to a resistor R, and therefore, the capacitor discharges over time.

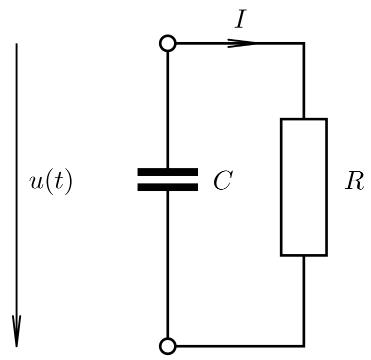

Figure 8: RC circuit

This is a system that has only an output signal. The output is the voltage across the capacitor C, in this case denoted as $u(t)$. From the physical nature of the system, the voltage across the capacitor can be described by a differential equation in the form

$$\dot{Q}(t) = -\frac{1}{RC}Q(t) \tag{10}$$

where $Q(t)$ is the electric charge on the capacitor, and for the voltage across the capacitor, it holds that $u(t) = Q(t)/C$.

The mathematical model of the system is thus a differential equation. The unknown in the equation in this case is $Q(t)$. If we know $Q(t)$, we also know the output of the system.

Since the mathematical description of the system is a differential equation, meaning the equation involves derivatives of the unknown, we refer to it as a dynamic system. In other words, it is an inertial system, in this case, the inertia is evident from the nature of the capacitor, which cannot be discharged instantly.

The unknown of the differential equation is a function, in this case, a function of time. The unknown of the differential equation thus corresponds to the signal of the system, principally the output of the system. We can study the properties of the system, its dynamics, by finding the solution to the differential equation.

5.2.1 Solution of the Differential Equation

We are dealing with a differential equation in the form

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = -\frac{1}{RC}Q(t) \qquad Q(0) = Q_0 \tag{11}$$

where $Q(t)$ is the unknown time function. The independent variable in this function is time t, and the dependent variable is "Q". The constants (not functions of time) R, C, and also Q_0 are known.

Let's rearrange the differential equation (11) so that the same variables ($Q(t)$ and t) are on the same sides. In the form (11), the signal $Q(t)$ is on both sides of the equation. Let it be only on the left side. Similarly, let time t be only on the right side. Thus

$$\frac{1}{Q(t)}\mathrm{d}Q(t) = -\frac{1}{RC}\mathrm{d}t \tag{12}$$

Notice that now it is possible to integrate both sides of the equation, each according to its own variable, thus

$$\int \frac{1}{Q(t)}\mathrm{d}Q(t) = \int -\frac{1}{RC}\mathrm{d}t \tag{13}$$

The result of the integration is

$$\ln\left(Q(t)\right) + k_1 = -\frac{1}{RC}t + k_2 \tag{14}$$

where k_1 and k_2 are constants resulting from indefinite integrals (and also $Q(t) > 0 \; \forall \; t$).

Equation (14) is no longer differential. No quantity in it is derived with respect to time. Let's express the signal $Q(t)$ from equation (14). By rearranging

$$\ln\left(Q(t)\right) = -\frac{1}{RC}t + k_3 \tag{15}$$

we introduced the constant $k_3 = k_2 - k_1$. Further

$$Q(t) = e^{\left(-\frac{1}{RC}t + k_3\right)} \tag{16a}$$

$$Q(t) = e^{\left(-\frac{1}{RC}t\right)} e^{k_3} \tag{16b}$$

It can be shown that $e^{k_3} = Q_0$.

The sought solution of the differential equation is the time function in the form

$$Q(t) = Q_0 \; e^{\left(-\frac{1}{RC}t\right)} \tag{17}$$

6 Questions and Tasks

1. Explain in your own words the term *Cybernetics* (what is Cybernetics?).
2. Explain the term *system gain* (or *static system gain*).
3. What is the ratio between the steady-state value of the system's output signal and the steady-state value of the system's input signal called?
4. Explain the difference between a non-inertial and an inertial system.
5. What are the *initial conditions* of a dynamic system?

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