Selected Characteristics of Dynamic Systems

Contents

THE main topic of this text is the *static characteristic* and *step response* of a dynamic system. We consider a system that has one input signal $u(t)$ and one output signal $y(t)$, both continuous in time, and the dynamic system is time-invariant.

1 On the Steady State of the System

When examining the properties of a system, it is often necessary to first understand the so-called static properties of the system. Generally, this concerns the steady states of the system. A typical example is a situation where the input signal $u(t)$ is constant, its value does not change over time. Let us denote the steady value of the input signal as $u(\infty)$, emphasizing that it is the value as if at infinite time, which in practice is the time when all transient processes are considered finished. The question is whether the value of the output signal $y(t)$ will also settle at some value $y(\infty)$.

At first glance, it is clear that the indicated static properties of the system do not make sense to examine for a system that is unstable.

System Stability

The term *system stability* typically refers to several different cases concerning the general solution of the differential equation describing the dynamic system. An intuitive term is *BIBO stability* (bounded input, bounded output), where the case is examined when the input signal $u(t)$ is bounded, its maximum value is less than infinity. If the output signal $y(t)$ is also bounded, we say that the system is BIBO stable. Essentially, the forced component of the solution of the non-homogeneous differential equation is examined. The natural component of the solution, dependent on the initial conditions, can be examined similarly and is related to the term *asymptotic stability*.

For a linear system, it holds that the properties of the system from any stability perspective are completely determined by the poles of the system, i.e., the roots of the characteristic polynomial. A necessary and sufficient condition for the stability of a linear system is that all poles of the system lie in the left half-plane of the complex plane, i.e., their real parts are negative. If at least one pole lies on the imaginary axis, we say that the system is on the verge of stability. If at least one pole is in the right half-plane, its real part is positive, we say that the system is unstable.

Static Gain versus Astatism

Consider a system that is not unstable. From the perspective of the static properties of the system, two basic properties can be distinguished, which are related to the steady state of the system. The first is the *static gain* of the system, and the second is the *astatism* of the system.

If the input signal $u(t)$ is constant, its value is $u(\infty)$, and the output signal $y(t)$ settles at the value $y(\infty)$, then we say that the system is in a steady state. It is possible to determine the static gain of the system, which is the ratio of the steady value of the output signal to the steady value of the input signal.

If the input signal $u(t)$ is constant, its value is $u(\infty)$, and the output signal $y(t)$ does not settle, does not stop changing, does not stop growing, we talk about the astatism of the system, the system is astatic, it does not settle.

For a linear system, if none of the poles of the system is zero, then we attribute the system the property of being static. However, we still have in mind a dynamic system, which is given, for example, by the transfer function of the system. For such a system, it is possible to determine its static gain. The static gain is the ratio of the output to the input in the steady state. If one of the poles of the system is zero, we say that the system is astatic ("contains astatism"). If exactly one pole is zero, we talk about first-order astatism (if two poles, then second-order astatism, etc.). Let us remind that we consider a system that is not unstable. A zero pole means, of course, that its real part is zero. From the perspective of stability, this means that the system is on the verge of stability.

Static Characteristics

In the context of the static properties of the system, it generally makes sense to talk about the static characteristic of the system. The static characteristic is the dependence of the steady values of the output signal of the system on the steady values of the input signal of the system.

It is clear that the static characteristic concerns systems with the attribute of being static, i.e., those that are not astatic.

In the case of linear systems, the static characteristic is a straight line, and without loss of generality, we can consider that it passes through the origin of the coordinate system. The slope of the line is given by the static gain of the system.

2 On Measuring the Static Characteristic

The static characteristic is the dependence of the steady values of the output variable on the steady values of the input variable. The static characteristic, thus characterizes the system only in steady states. It does not contain information about the dynamics of the system.

Simulated System

In the following, we will attempt to outline the measurement of the static characteristic. However, nothing will actually be measured here; the real system will be replaced by a simulated one. The process of obtaining "raw" data, which is necessary for determining the static characteristic, and the process of processing these data will be the same as if it were a real system.

The system that will be the subject of investigation, whose static characteristic we will measure, is a pendulum, as described earlier. Additionally, unknown noise will be added to the output variable of the simulated system for the reader. The reason is to better mimic the situation the reader would be in if dealing with the output variable of a real system.

Recall that we consider a pendulum whose oscillations are damped by viscous friction with a coefficient β [kg m^2 s^{-1}]. The pendulum is shown in Fig. 1, where a mass point with mass m [kg] attached to an arm of negligible mass and length l [m] oscillates, o denotes the axis of rotation perpendicular to the plane in which the pendulum oscillates, the angle between the vertical and the arm of the pendulum is denoted by φ [rad], and the gravitational acceleration is g [m s^{-2}].

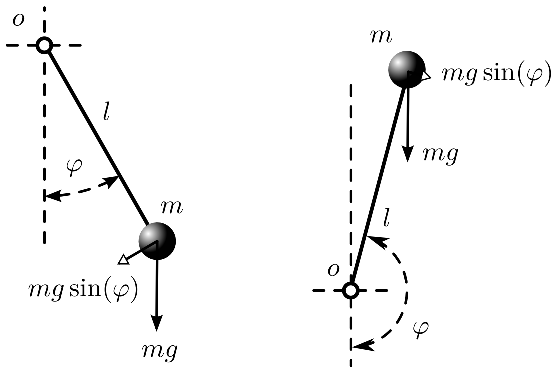


Figure 1: Pendulum

The equation of motion describing the dynamics of the rotational motion of the pendulum can be written as

$$\ddot{\varphi}(t) = -\frac{\beta}{ml^2}\dot{\varphi}(t) - \frac{g}{l}\sin\left(\varphi(t)\right) + \frac{1}{ml^2}u(t) \tag{1}$$

where $u(t)$ [kg m^2 s^{-2}] is the external torque acting on the arm of the pendulum, $\dot{\varphi}(t)$ [rad s^{-1}] is the angular velocity, and $\ddot{\varphi}(t)$ [rad s^{-2}] is the angular acceleration of the pendulum arm. The numerical values of the pendulum parameters are given in Table 1.

Table 1: Pendulum Parameters

Parameter	Value	Units
m	1	kg
l	1	m
g	9.81	m s^{-2}
β	$2\sqrt{g/l}$	kg m^2 s^{-1}

2.1 Data for Determining the Static Characteristic

As already mentioned, the subject of interest is the steady values of the output signal. If we apply an input signal with some constant input value, then wait for a certain time for the output signal to settle, we can then read (measure) the steady value of the output signal. This way, one point of the static characteristic is obtained.

Immediately after, it is possible to change the value of the input signal and again wait for the output to settle.

This procedure, the values of the input signal, and the times during which we "wait" for the output to settle can be expressed in a table as follows. The first column is the time at which the input changes (switches), and the second column is the value (constant) to which it changes (switches).

Table 2: Input Signal Values

time [s]	value [kg m^2 s^{-2}]
0	0.0
10	1.0
30	2.0
20	3.0
40	4.0
50	5.0
60	6.0
70	7.0
80	8.0
95	9.0
110	9.5
135	9.81

From the above, it is also clear that the interval of input values for which we determine the steady values of the output is 0 to 9.81 [kg m^2 s^{-2}]. Other values would have little significance given the specific pendulum considered here.

The input signal as defined by Table 2 can be illustrated as in Fig. 2.

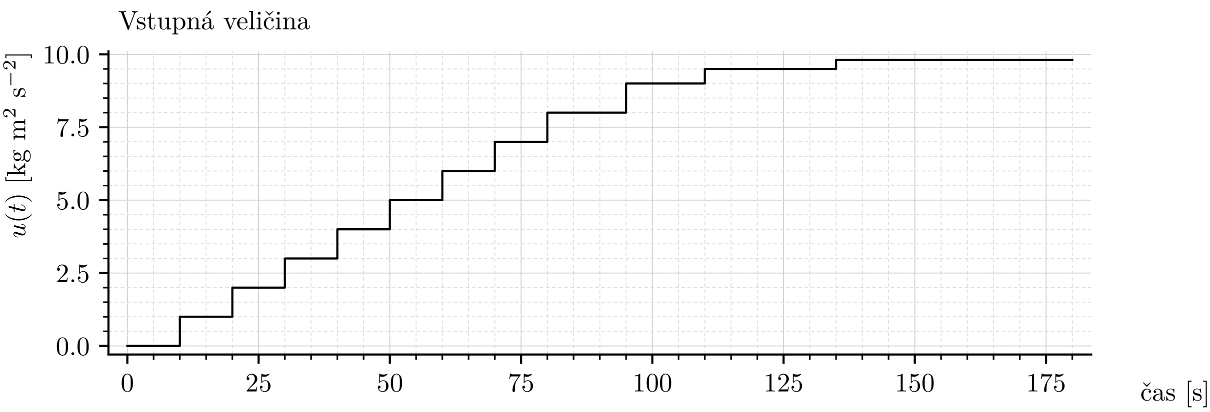


Figure 2: Course of the Input Signal

Now, let's simulate the course of the output variable of the pendulum (pendulum deflection) for the given input signal. The result is in Fig. 3.

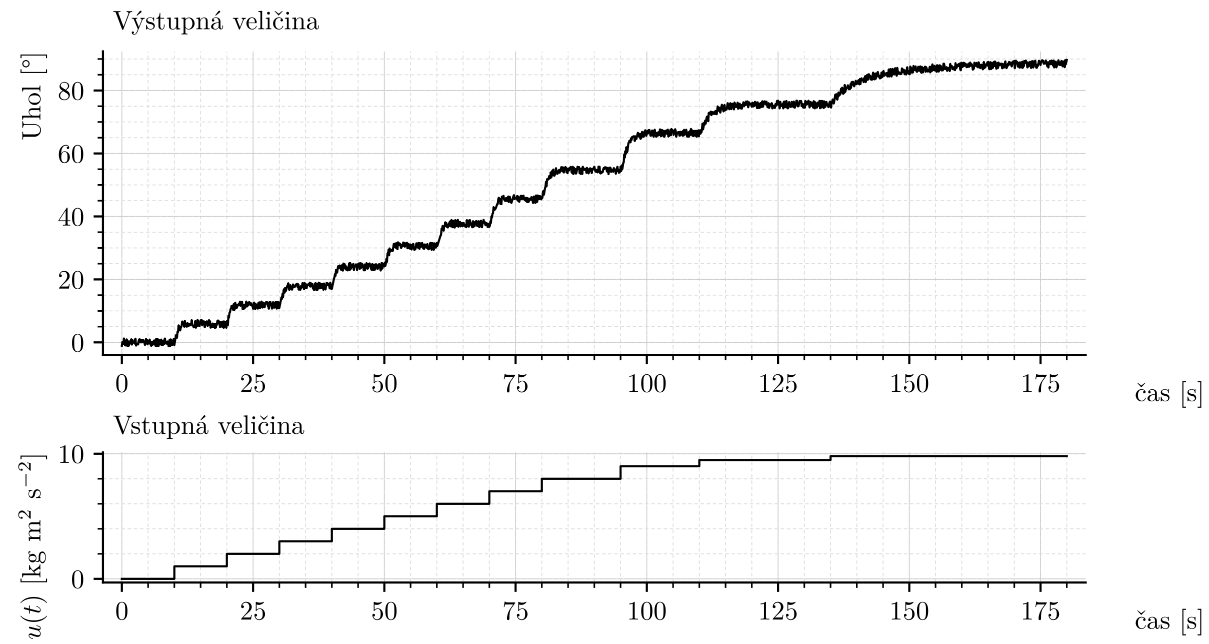


Figure 3: Raw Data. Note: The output variable in Fig. 3 is deliberately "noisy". This mimics the potential noise of the sensor measuring the given variable. The reason is mainly that it better illustrates the individual steps necessary for the general processing of the measured signal, which are presented in the following sections.

2.2 Data Processing

From the raw data, it is necessary to obtain individual points of the static characteristic. This primarily means being able to read the steady value of the output signal (from the raw data). For illustration, let's focus on the measured data from the tenth second to the twentieth second, i.e., for the interval during which the input value was $u = 1$ [kg m^2 s^{-2}]. This part of the data is plotted in Fig. 4.

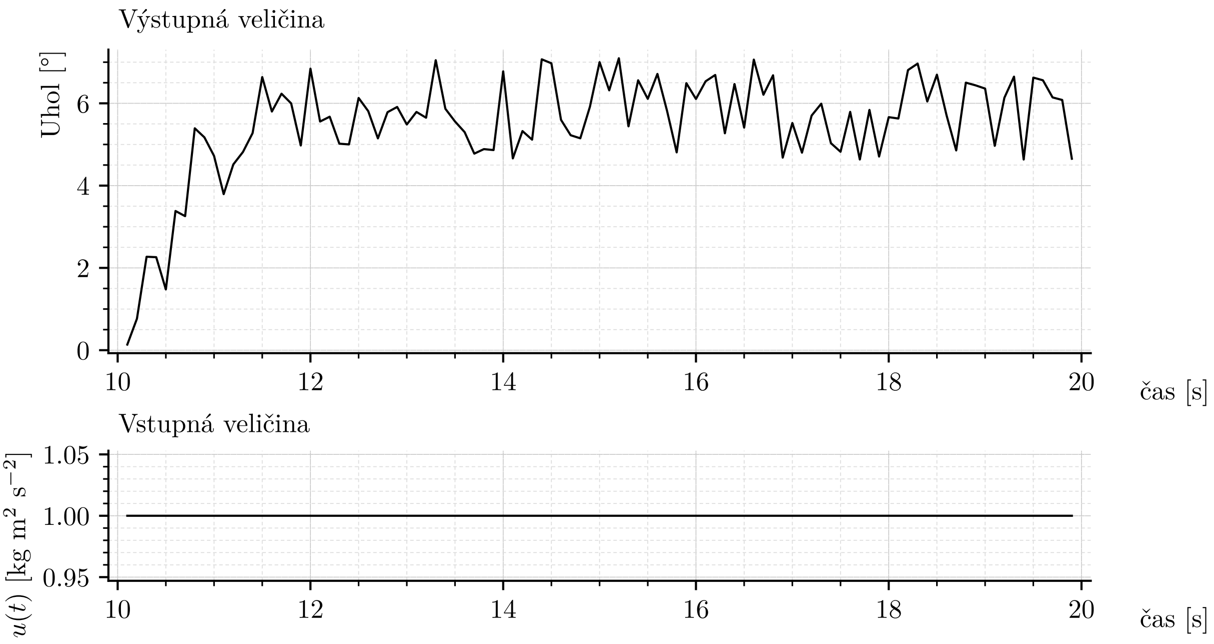


Figure 4: Raw Data - detail

From Fig. 4, it is clear that in the last third of the interval, the output value can already be considered steady. Let's highlight this part of the data - see Fig. 5.

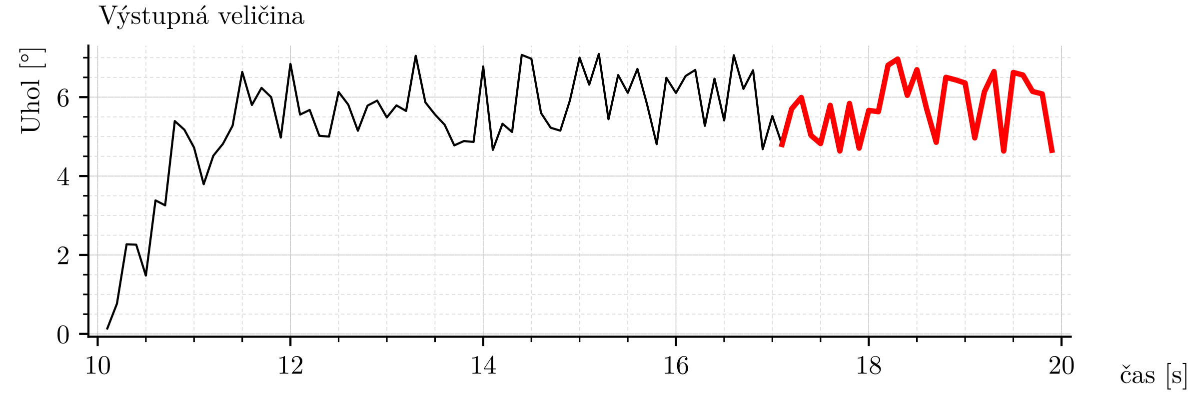


Figure 5: Raw Data - detail

The question is how to determine a single value - the steady value - from the highlighted section. A natural choice is to take the average of the selected (highlighted) part of the data. The average is highlighted in Fig. 6

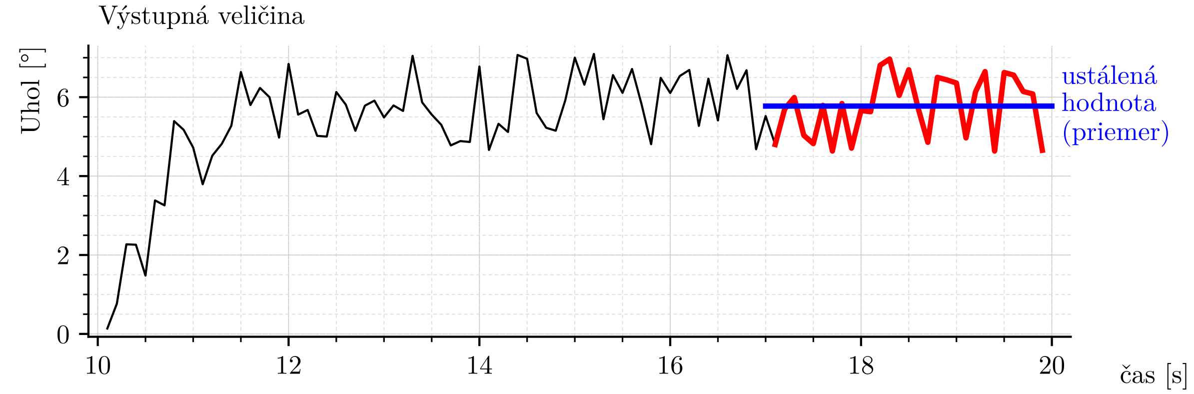


Figure 6: Raw Data - detail

Of course, the same procedure can be applied to all points of the static characteristic. All steady values read from the "raw data" are shown in Fig. 7.

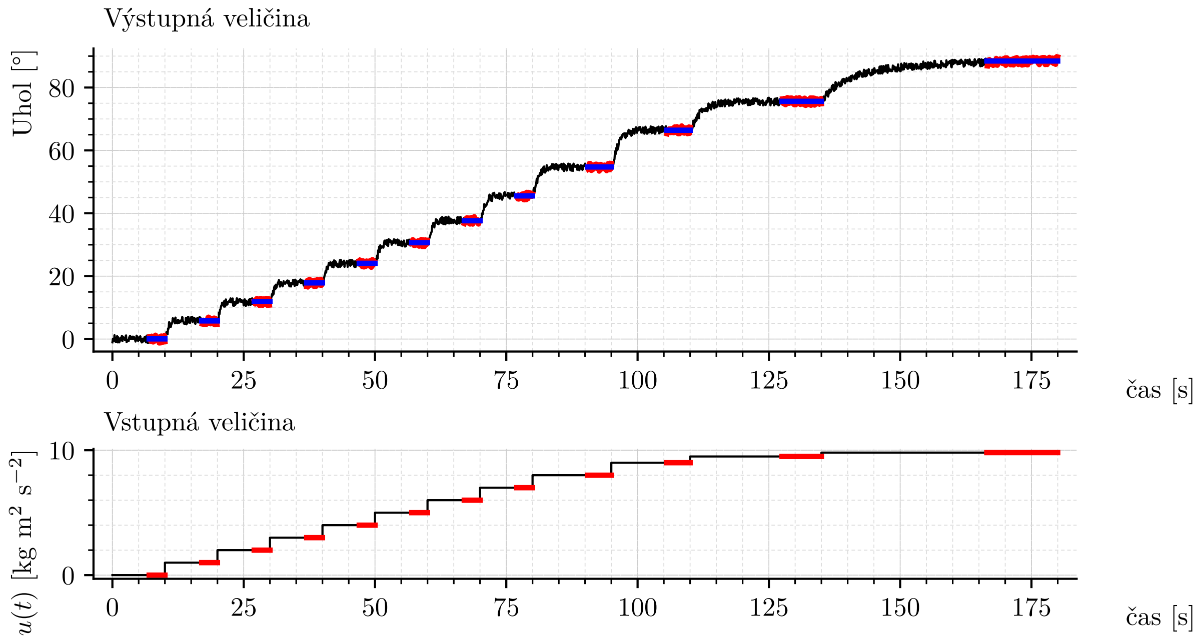


Figure 7: Raw Data

The read values are then listed in Table 3. The static characteristic is graphically represented in Fig. 8.

Table 3: Static Characteristic

input [kg m^2 s^{-2}]	output [°]
0.0	0.00691
1.0	5.7
2.0	11.8
3.0	17.7
4.0	24.3
5.0	30.6
6.0	37.6
7.0	45.2
8.0	54.5
9.0	66.5
9.5	75.6
9.81	88.7

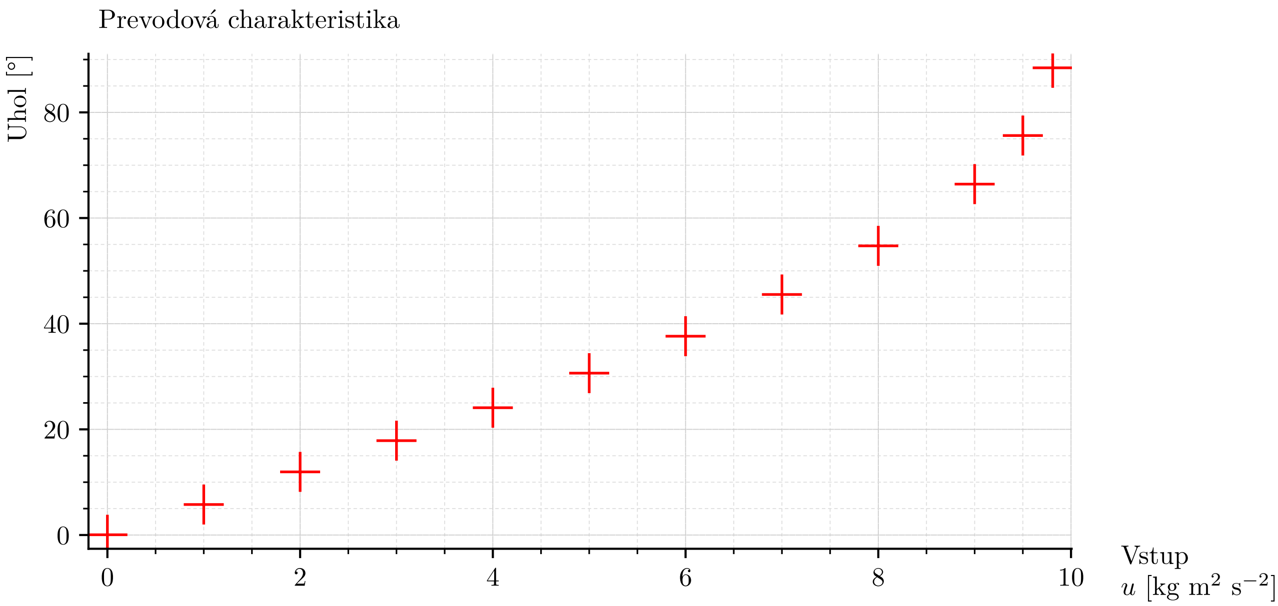


Figure 8: Static Characteristic

2.3 Supplementary Text: On Approximating the Static Characteristic

Let's have a measured static characteristic, the data is in Table 3. The static characteristic is graphically represented in Fig. 8.

The points of the static characteristic correspond to a certain property of a real system (a real existing system). They correspond to the dependence of the system's output on the system's input, of course in a steady state. However, only a few points are measured. In these points, the given property of the system is known. But what if it were necessary to know the given property outside the measured points? That is, outside the input values for which the static characteristic was measured.

For these purposes, it is advantageous to use a model. A model of the real property of the system. The measured dependence (static characteristic) corresponds to the property of the system. By approximating this dependence, it is possible to obtain a model.

In this case, let the model be a mathematical relationship, a functional dependence, a certain prescription. If the value at the input of the model is the same as the value at the input of the real system, then the value at the output of the model should be "approximately the same" as the real value. This should apply to all measured points of the static characteristic. That is, the model should approximately match the real data. If this applies to the measured points, then it presumably applies to other points as well. It applies to any value at the input of the model: that the output of the model approximately matches the real output of the system.

2.3.1 Model

Such a generally described model can be specified, for example, as follows: Let the model be a polynomial function

$$\hat{y} = \Theta_3 u^3 + \Theta_2 u^2 + \Theta_1 u + \Theta_0 \tag{2}$$

where the "input of the model" is u and the "output of the model" is $\hat{y}$. The parameters of the model are the coefficients (numbers) Θ_3, Θ_2, Θ_1, and Θ_0.

By the way, this is a linear model. The parameters of the model are in a linear relationship with the "signals" of the model (with the inputs of the model).

2.3.2 Simple Search for Polynomial Coefficients

In this section, MATLAB will be used to find the parameters (coefficients) of the polynomial function (2). For these purposes, let us have a variable `transferChar`, whose first column contains the values of the input variable and the second column contains the values of the output variable. Thus, Fig. 8 would be plotted in MATLAB as follows:

```
figure(1);
plot(transferChar(:,1), transferChar(:,2), '+r');
```

2.3.3 Function polyfit

The `polyfit` function is generally used to find the coefficients of a polynomial (polynomial function) of a given degree so that the polynomial function approximates the given data (e.g., the measured x-y dependence). The criterion for finding the coefficients is the minimization of the squares (quadratic) deviations between the measured value and its approximation. More precisely, the minimization of the sum of the squares of the deviations.

The use of the `polyfit` function in this specific case would be as follows:

```
polyCoef = polyfit(transferChar(:,1),transferChar(:,2), 3)
```

and the variable `polyCoef` contains the values of the polynomial coefficients. The equation (2) with the found coefficients is:

$$\hat{y} = 0.1105u^3 - 1.1071u^2 + 8.8873u - 1.146 \tag{3}$$

2.3.4 Function polyval

To calculate the values (outputs) $\hat{y}$ for the desired inputs u, the `polyval` function can be used. So if we want to calculate the approximation $\hat{y}$ according to the model (3) for each input for which the output value was measured, we just need to call:

```
y_hat = polyval(polyCoef, transferChar(:,1));
```

A figure showing both the measured data and the model output (3) can be plotted as follows:

```
figure(2);
hold on;
plot(transferChar(:,1), transferChar(:,2), '+r')
plot(transferChar(:,1), y_hat, '.b')
```

The figure would be similar to Fig. 9

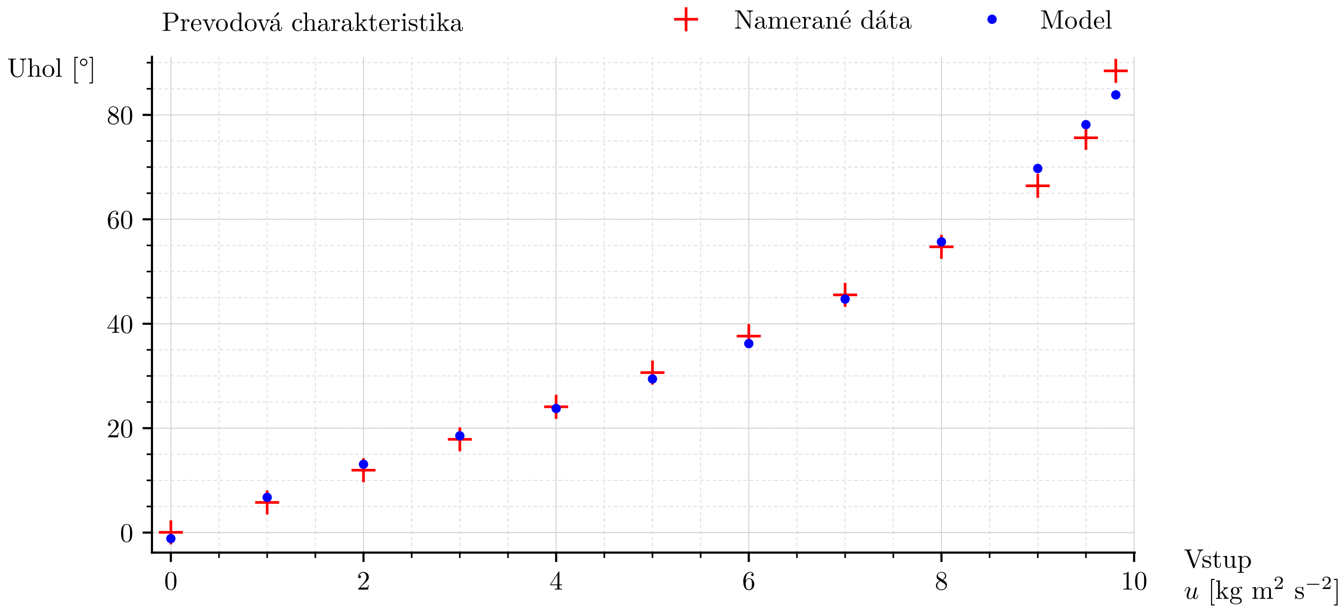


Figure 9: Static Characteristic

2.3.5 Using the Model

The model, of course, allows calculating the approximation of the real output for any input value - not just for the input values for which the actual system values were measured. Let's calculate the model outputs for these input values (given by the vector):

```
u_other = [0:0.1:9.81];
```

So we call the `polyval` function.

```
y_other_1 = polyval(polyCoef, u_other);
```

Let's plot the figure (see Fig. 10)

```
figure(3);
hold on;
plot(transferChar(:,1), transferChar(:,2), '+r')
plot(u_other, y_other_1, '.b')
```

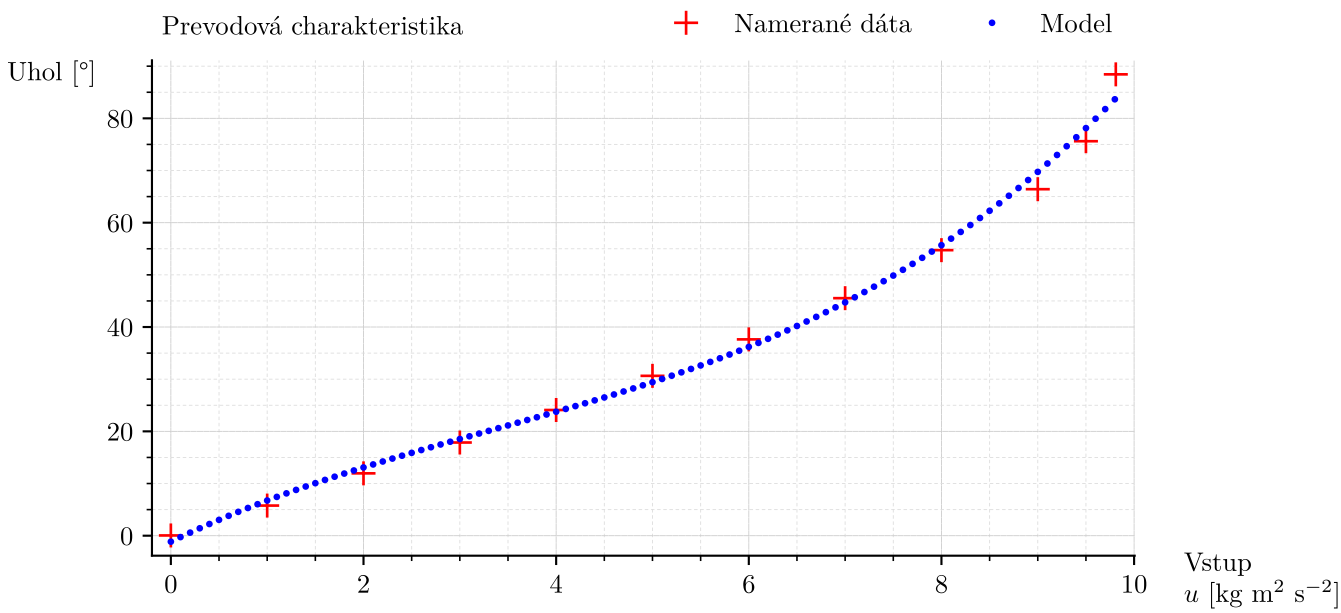


Figure 10: Static Characteristic

3 On Measuring the Step Response

The following text aims to inspire the reader to gain a better understanding of how to conduct meaningful measurements of step responses (even though in this case, the reader will only deal with simulations). In this case, the task is not only to obtain the step response. It is also necessary to analyze the problem in terms of the given real (simulated) system (whose properties we are investigating). Therefore, it is first necessary to explain the terms used in describing a dynamic system.

Recall that at this point, we have certain information about the system under investigation. It is the static characteristic - see Fig. 8.

The static characteristic itself describes the so-called static properties of the system. The properties of the system in a steady state. Specifically, it is possible to determine the static gain of the system from the static characteristic.

The static characteristic in Fig. 8 shows that the gain of the system at lower input signal values is different from the gain at higher input signal values. Let's use the fact that in this case, we have an available model of the static characteristic (it is not necessary to have such a model). The model is a polynomial function, specifically:

$$\hat{y} = 0.1105u^3 - 1.1071u^2 + 8.8873u - 1.146 \tag{4}$$

Let's use this model to calculate the outputs (estimated outputs) of the system at these input signal values:

```
u_other = [0:0.1:9.81];
```

We get Fig. 11.

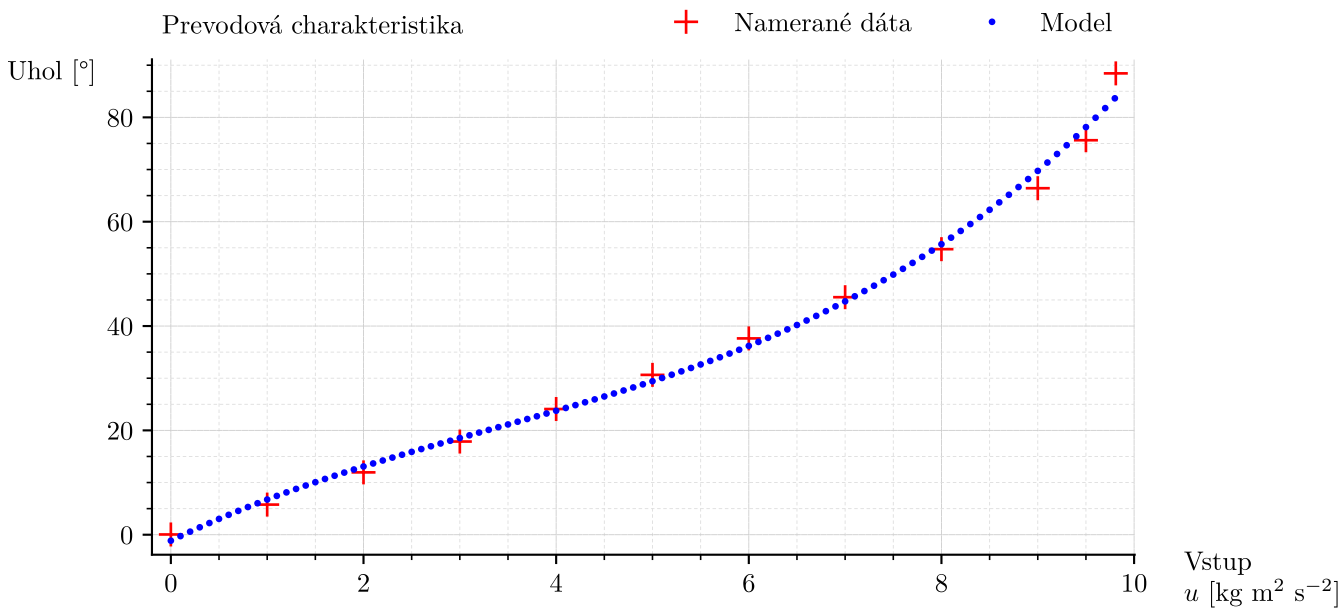


Figure 11

3.1 Operating Point

The main task in this text is to obtain the step responses of the subject system (laboratory system) at different operating points. The operating points should be chosen with regard to the static characteristic of the system. First of all, what is an *operating point*?

An operating point is defined by the steady value of the input signal, to which (uniquely) corresponds the steady value of the output signal. The pair of values, the value at the input and the value at the output, forms the operating point.

If the steady value of the input signal is given, then it is possible to find the corresponding steady value of the output signal using the static characteristic.

The term operating point also relates to the term *vicinity of the operating point.* In the vicinity of the operating point, the properties of the system are relatively the same as in the operating point. From the perspective of the static properties of the system, this means that the slope of the static characteristic does not change significantly in the vicinity of the operating point. In other words, the static gain of the system does not change. Similarly, the dynamic properties of the system are relatively unchanged in the vicinity of the operating point - the time constants of the system do not change.

In two different operating points, a real system may have, for example, different static gains, i.e., static properties. The static gain of the system in the operating point can be determined based on the static characteristic. It is given by the slope of the static characteristic in the vicinity of the operating point.

Any difference in static properties in different operating points, however, does not say anything about any difference in the dynamic properties of the system. Dynamic properties can be evaluated based on the step response.

A *step response* is the system's response to a unit step.

The term *unit step* refers to a step change in the signal (input), and the magnitude of this change is unitary. It is unitary in the sense that any magnitude of the step change can be expressed as a multiple of the unit step change. Naturally, it is assumed that the unit change is such that it does not cause the system to go beyond the vicinity of the operating point.

3.2 Choice of Operating Points

The operating points should be chosen with regard to the static characteristic of the system. How should this be considered? As already mentioned, from the static characteristic, it is clear that a certain property of the system is different at low values of the input signal and different at high values. This property is the static gain. Other than the so-called static properties of the system, it is not possible to deduce anything else from the static characteristic. The question then is whether other properties of the system are different at different values of the input signal.

Therefore, let's choose two operating points - one to represent a low value of the input signal and the other a high value. The chosen operating points are listed in Table 4.

Table 4: Chosen Operating Points

OP	value	units
1.	4	[kg m^2 s^{-2}]
2.	9.5	[kg m^2 s^{-2}]

Based on the measured points of the static characteristic, we could assign output values to the chosen steady input values:
1.OP: $y = 23.76$ [°]
2.OP: $y = 78.13$ [°]

However, let's work with the approximation of the static characteristic, i.e., its model. The model will allow us to obtain information that was not actually measured.

For $u = 4$ [kg m^2 s^{-2}], according to the model of the static characteristic, the steady output value is:

$$\hat{y}_{OP1} = 0.1105\ u_{OP1}^3 - 1.1071\ u_{OP1}^2 + 8.8873\ u_{OP1} - 1.146 \tag{5}$$

where if $u_{OP1} = 4$ [kg m^2 s^{-2}], then $\hat{y}_{OP1} = 23.73$ [°]. Similarly, for $u = 9.5$ [kg m^2 s^{-2}], according to the model of the static characteristic, the steady output value is $\hat{y}_{OP2} = 78.06$ [°]. Let's illustrate the operating points - see Fig. 12.

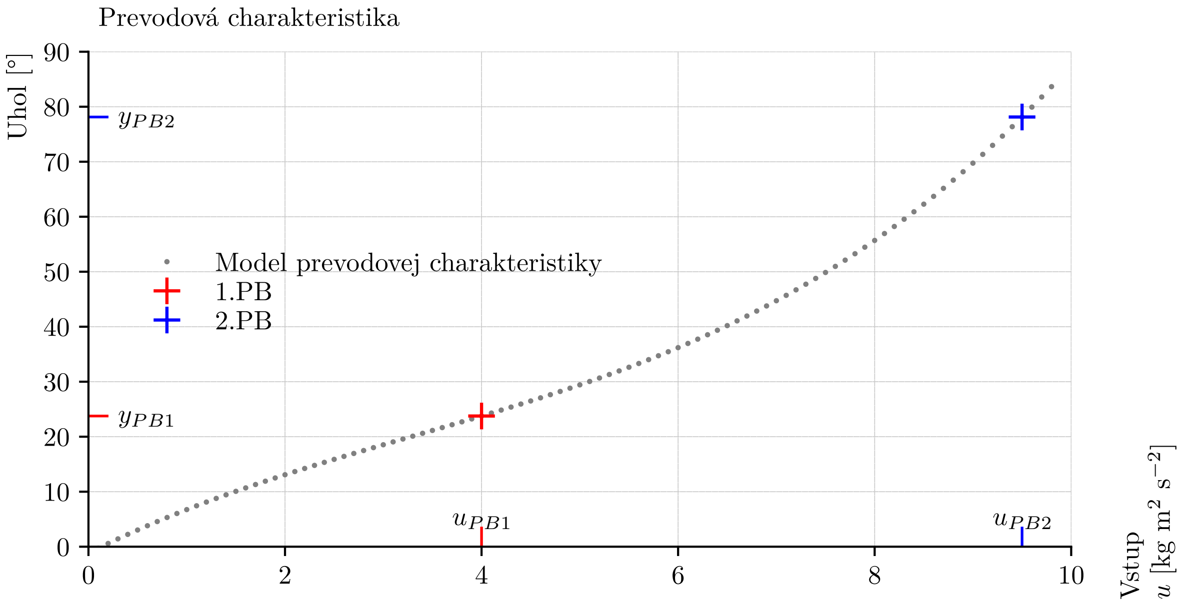


Figure 12

Furthermore, it is, of course, necessary to appropriately choose the vicinity of the operating point (for each operating point). Essentially, it is necessary to choose the operating point and the corresponding vicinity of the operating point simultaneously. Here, we have separated it for better clarity.

Recall that in the vicinity of the operating point, the properties of the system are expected to be relatively unchanged. Based on the static characteristic, the static properties of the system can be assessed. Based on this, for the 1st operating point (OP1), choose the vicinity $u = 4 \pm 0.8$ [kg m^2 s^{-2}]. For OP2, choose $u = 9.5 \pm 0.25$ [kg m^2 s^{-2}]. Let's illustrate the operating points and their vicinities - see Fig. 13. In Fig. 13, the boundaries of the vicinity of the operating point are also marked, e.g., u_{OP1_l} as the lower boundary of the vicinity of the operating point and u_{OP1_h} as the upper boundary. The corresponding values of the output variable, values y_{OP1_l} and y_{OP1_h}, are also marked. Similarly for the second operating point.

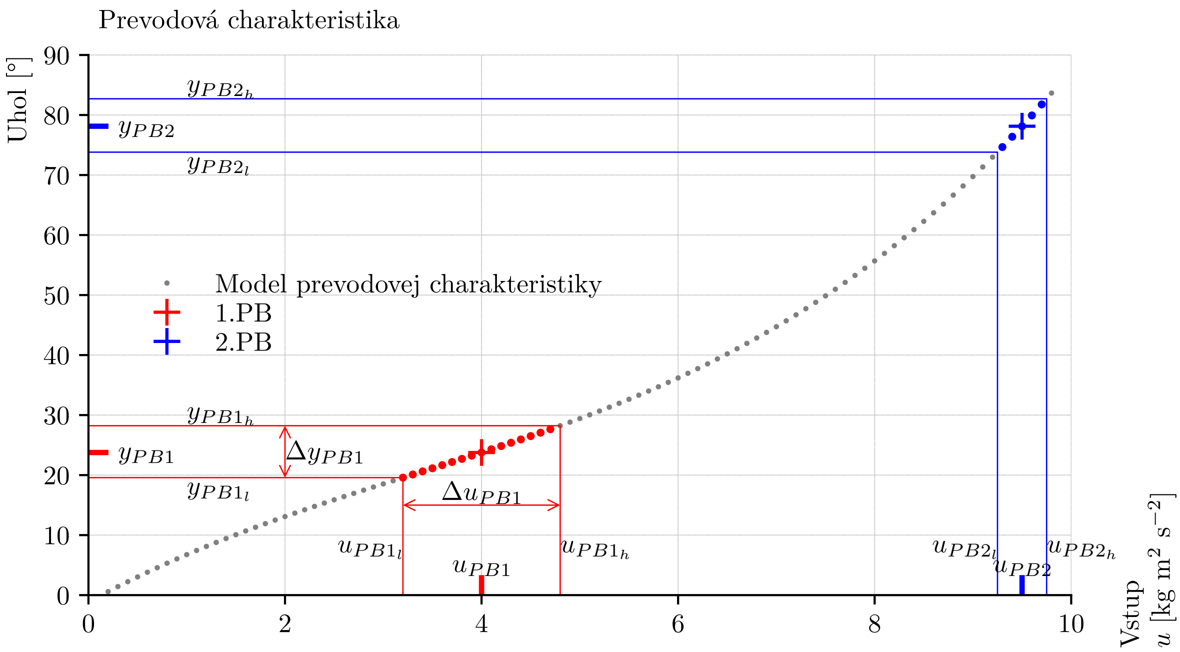
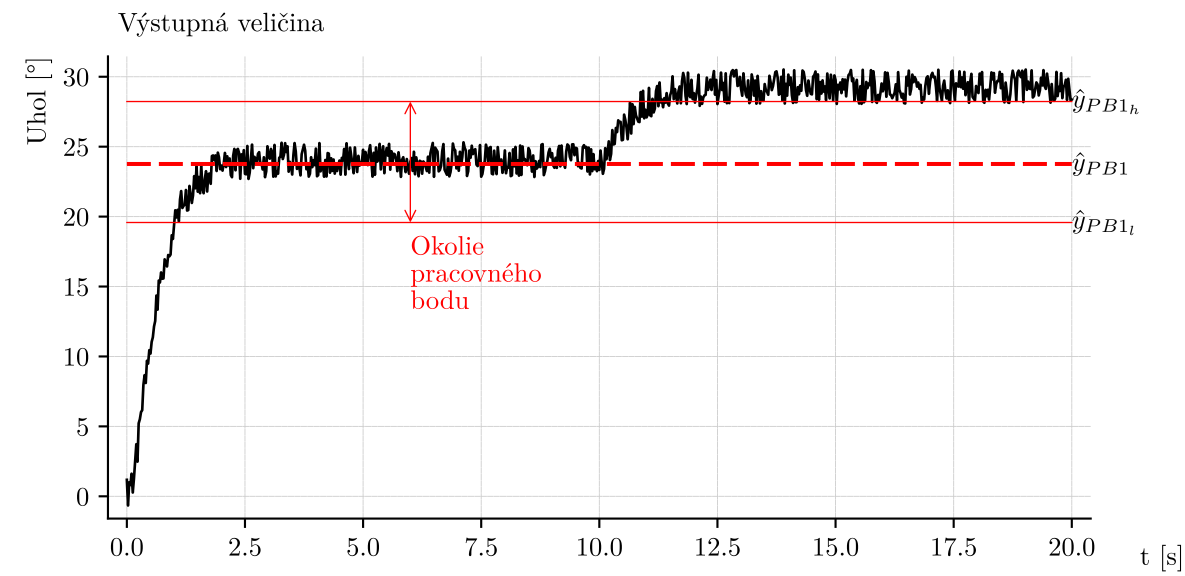


Figure 13

Furthermore, in this case, it is necessary to consider the magnitude of the step change that will be used as the unit step. Given the circumstances, there is no reason why the unit magnitude should not be the value defining the vicinity of the operating point. This satisfies the requirement that the unit step does not cause the system to go beyond the vicinity of the operating point (it will be on the edge, but not beyond). Therefore, for OP1, let the unit magnitude of the step change be equal to $u_{s1} = 0.8$ [kg m^2 s^{-2}], and for OP2, let the unit magnitude of the step be equal to $u_{s2} = 0.25$ [kg m^2 s^{-2}].

3.3 Conducting the Step Response Measurement

To perform a unit step (step change of the system's input signal with unit magnitude) in the vicinity of the operating point, it is first necessary to bring the system to the operating point. If the value of the input signal is u_{OP}, and we leave it for some time, then based on the static characteristic, we expect the system's output to settle at the value y_{OP}. The system will be at the operating point. Then it is possible to step increase the value of the input signal by the value u_s. This will realize a unit step in the vicinity of the operating point. The simulation of this is shown in Fig. 14.



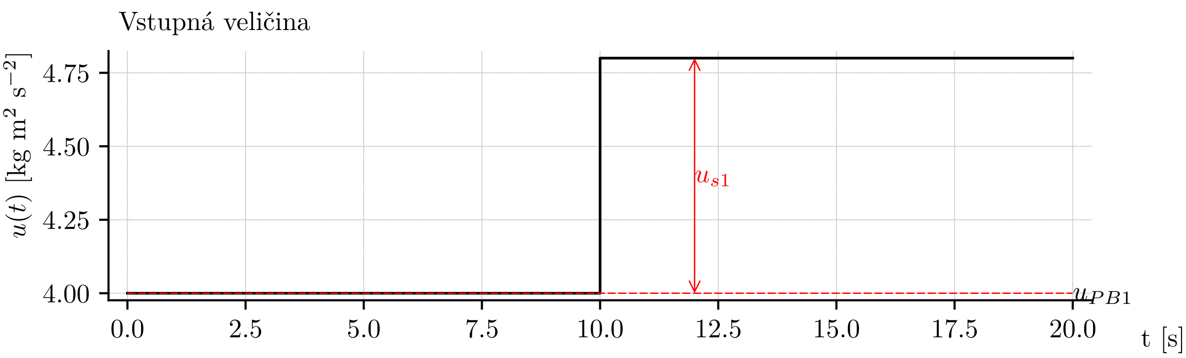


Figure 14

The magnitude of the step change of the input signal is marked as u_{s1} in Fig. 14. In this case, given the chosen vicinity of the operating point, $u_{s1} = 0.8$.

The unit step occurred at time $t = 10$ [s]. Before this time, the system was reaching the operating point. From time $t = 10$ [s] until the output variable of the system settles again, the transient process occurs, which is the step response (since the input was a unit step).

3.4 Processing the Measured Data

Before the unit step, we expected the output variable to settle at the value y_{PB}. According to the model of the static characteristic, for this operating point, the value is $\hat{y}_{PB1} = 23.73$ [°].

The average value of the output variable during the 5 seconds before the unit step is 24.04 [°]. The deviation of the average value, around which the system settled in the operating point, from the expected value according to the model of the static characteristic is approximately 1%. This is, of course, an acceptable deviation. Therefore, we can consider the value $\hat{y}_{PB1}$ according to the model of the static characteristic as the value at which the output variable was settled before the step change of the input signal.

After the transient process ends, according to the model of the static characteristic, the output variable is expected to settle at the value $\hat{y}_{PB1_h} = 28.19$ [°].

From Fig. 14, it is clear that the output variable actually settles at a slightly higher value. More precisely, if we consider the time interval after the unit step during which the output variable is already settled, let it be the interval from 2.5 to 5 seconds after the unit step, then the average value of the output variable in this interval is 29.2 [°].

If at the value y_{PB1} the difference between the expected (according to the model of the static characteristic) and the measured value was almost negligible, at the value y_{PB1_h} it is not so clear. It is not clear that the difference is negligible. This problem, however, is related to the measurement of the static characteristic and the subsequent choice of the model of the static characteristic, which we decided to use here for estimating the expected values y_{PB1} and y_{PB1_h}. If we decided to use the given model of the static characteristic, then we must account for any deviations that are clearly not errors.

This is to say that despite the fact that regarding the expected values, after the unit step, the output variable left the expected vicinity of the operating point. However, this is only the expected, estimated vicinity (based on the model of the static characteristic). Deviations from the "real" values are acceptable, and thus we can continue without the need to reconsider the choice of the vicinity of the operating point.

3.4.1 "Cutting Out" the Step Response

Let's select from the data in Fig. 14 only the part that corresponds to the step response, i.e., the data from time 10 to the time of settling (let it be time 15). The result is in Fig. 15.

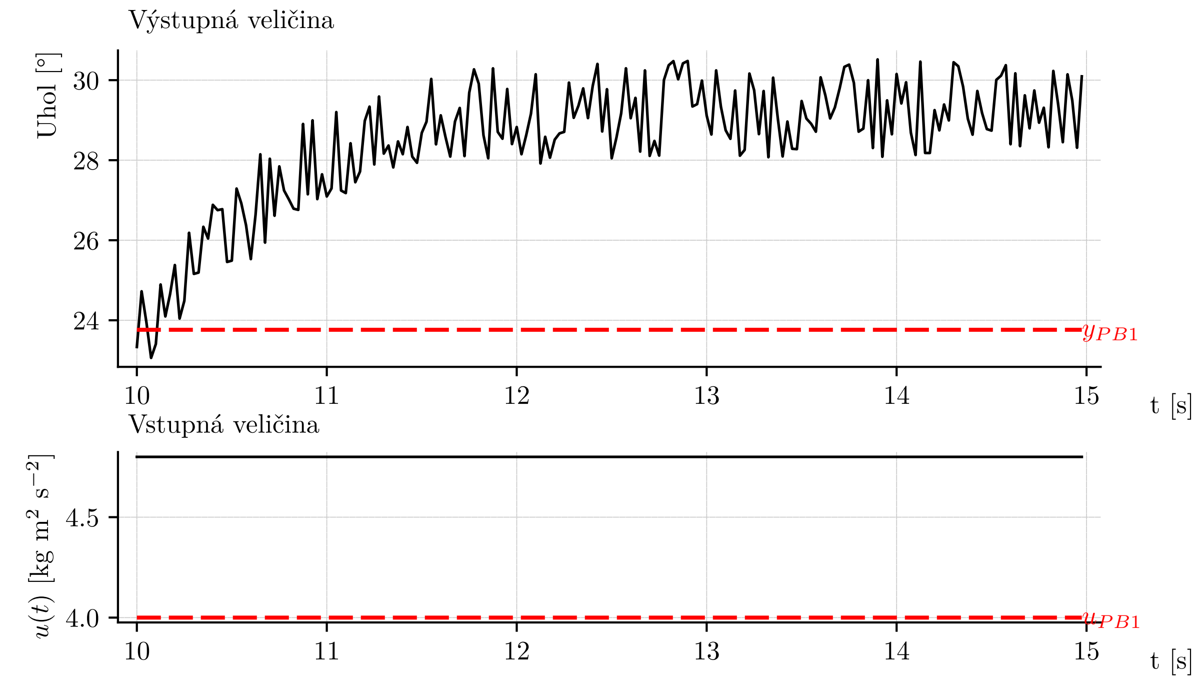


Figure 15

3.4.2 "Shifting" the Step Response

For further work with the step response, it is usually advantageous to shift the measured data so that the beginning of the step response is at the point (0,0), meaning that the step response starts at time 0 and the value of the output variable at the beginning is also zero (at least philosophically).

Specifically: from the obtained course of the output variable, it is necessary to subtract the value y_{PB}, because this way the beginning shifts in the y-axis direction to zero (philosophically... now the noise might spoil it). Similarly, the course of the input variable needs to be shifted in the direction of the axis by the value u_{PB}. Of course, from the time vector, it is necessary to subtract the time at which the unit step occurred. The result is in Fig. 16.

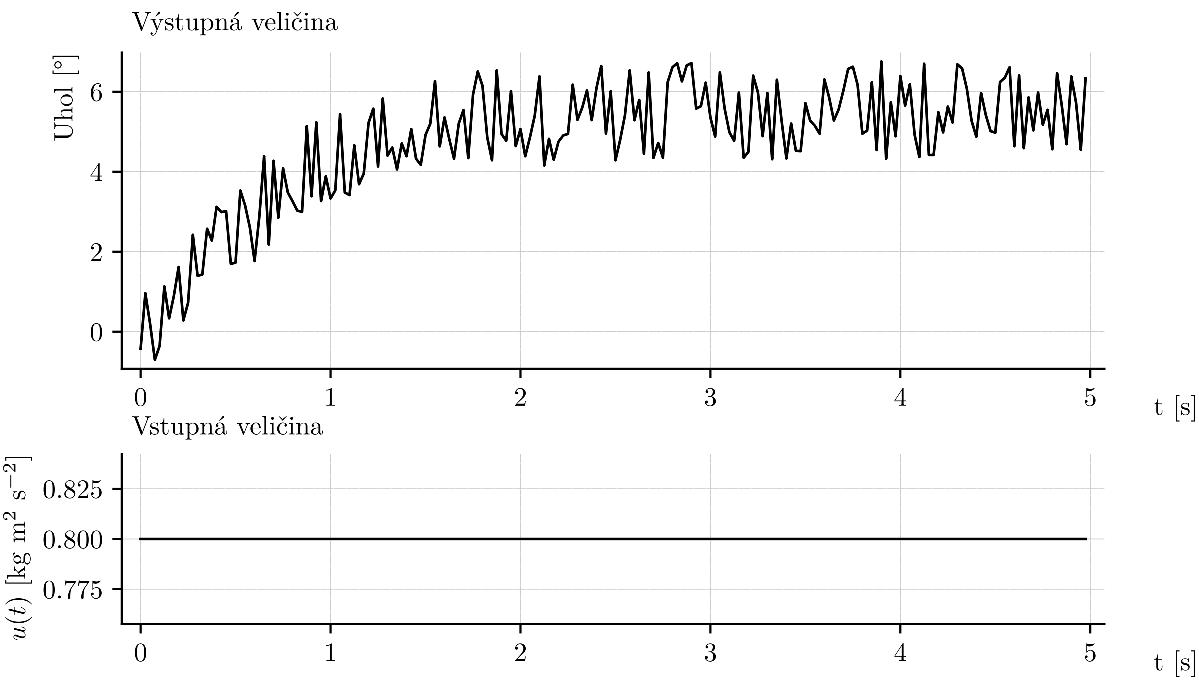


Figure 16

3.5 Notes on Reading Values from the Step Response Graph

3.5.1 Measured Step Response

From the previous, the measured and processed step response (SR) of the subject system is available. It is the step response in the first operating point. It is shown in Fig. 16.

Further, information about the operating point in which the SR was measured is available. The value of the input signal in the operating point is $u = 4$ [kg m^2 s^{-2}] and the vicinity of the operating point is considered to be $u = 4 \pm 0.8$ [kg m^2 s^{-2}].

A model of the static characteristic is available, and thus it is possible to estimate the value of the output variable in the operating point, i.e.,

$$\hat{y}_{PB1} = 0.1105\ u_{PB1}^3 - 1.1071\ u_{PB1}^2 + 8.8873\ u_{PB1} - 1.146 \tag{6}$$

where $u_{PB1} = 4$ [kg m^2 s^{-2}] and thus $\hat{y}_{PB1} = 23.73$ [°]. Similarly, it is possible to calculate the value of the output signal for, let's call it, the upper boundary of the vicinity of the operating point, meaning for the input value $u_{PB1_h} = 4 + 0.8$ [kg m^2 s^{-2}]. This corresponds to the value $\hat{y}_{PB1_h} = 28.19$ [°].

Since the step response in Fig. 16 is shifted to zero, i.e., the actual values are subtracted by the values in the operating point, let's make this adjustment for the calculated values as well, i.e.,

$$\Delta u_{PB1} = u_{PB1_h} - u_{PB1} = 0.8 \tag{7}$$

$$\Delta \hat{y}_{PB1} = \hat{y}_{PB1_h} - \hat{y}_{PB1} = 4.52 \tag{8}$$

3.5.2 Static Gain K

Let's determine the static gain of the system in the vicinity of the considered operating point. We need the value at which the output variable settled after the transient process. From the TCH graph, let's assume that the output variable is already settled after time $t = 3$ [s] (let's assume this for now). The average value of the output variable after this time is $\Delta y = 5.52$ [°].

Thus, after performing a unit step in the vicinity of the operating point, the output variable changed by Δy [°]. The change at the input Δu was, of course, exactly unitary (because it was a unit step). In this case, the unit step has the magnitude of the vicinity of the operating point $\Delta u = 0.8$ [kg m^2 s^{-2}].

The static gain of the system, based on the step response, denoted as K, is $K = \frac{\Delta y}{\Delta u}$, numerically

$$K = 6.9[°/(\text{kg m}^2\ \text{s}^{-2})] \tag{9}$$

This can also be illustrated in the graph - see Fig. 17

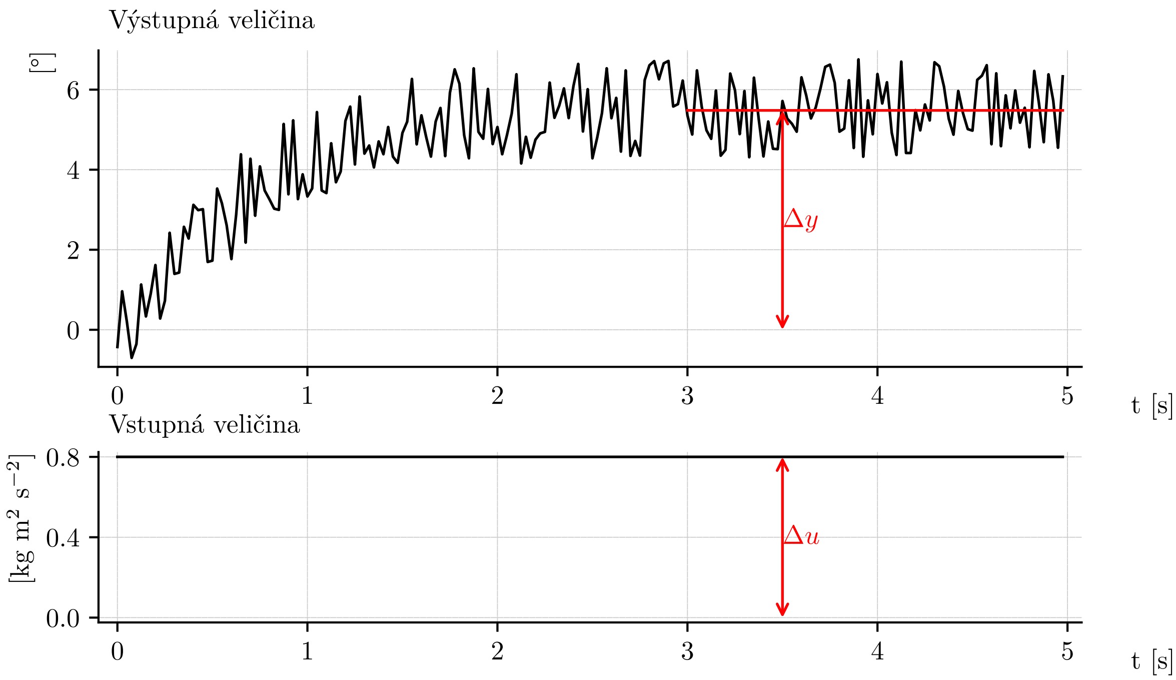


Figure 17

The static gain of the system can, of course, also be determined using the static characteristic. In fact, everything necessary is already available.

By the way, if we were not lazy, we would find the tangent in the operating point, and its slope would be the static gain. That would be the formally correct procedure.

However, we are lazy, so: we look for the slope of the static characteristic in the vicinity of the operating point. Practically, let the slope be given by the operating point and the point bounding the vicinity of the operating point from above. Formally, slope $= \frac{\Delta y}{\Delta u}$ where $\Delta y = \hat{y}_{PB_h} - \hat{y}_{PB}$ and $\Delta u = u_{PB_h} - u_{PB}$. This is, of course, the same as derived from using the step response above. Here, however, the numerical values are not read from the step response but from the model of the static characteristic. Specific numbers:

$$\text{slope} = \frac{\hat{y}_{PB_h} - \hat{y}_{PB}}{u_{PB_h} - u_{PB}} = \frac{4.52}{0.8} = 5.65 \tag{10}$$

The deviation from the static gain determined from the step response is -1.25 [°], i.e., 18.10 [%] (this is, of course, also due to the fact that we are using the model of the static characteristic, as the specific necessary values within the measured static characteristic are not available).

3.5.3 Time Constant T for a First-Order Linear Dynamic System

Next, it is possible to find a model that should capture the dynamics (dynamic properties) of the real system. Let the model be a linear dynamic system.

A qualified estimate based on the graphical representation of the subject step response leads to the possibility that the system model could be a first-order dynamic system. This can be written in the form of a transfer function

$$G(s) = \frac{y(s)}{u(s)} = \frac{K}{Ts+1} \tag{11}$$

where K can be interpreted as the static gain of the system and T is the time constant.

The time constant can be found based on the step response. It is the time from the beginning of the step response (from the time of the unit step) at which the output variable reached approximately 63

Why exactly 63

100

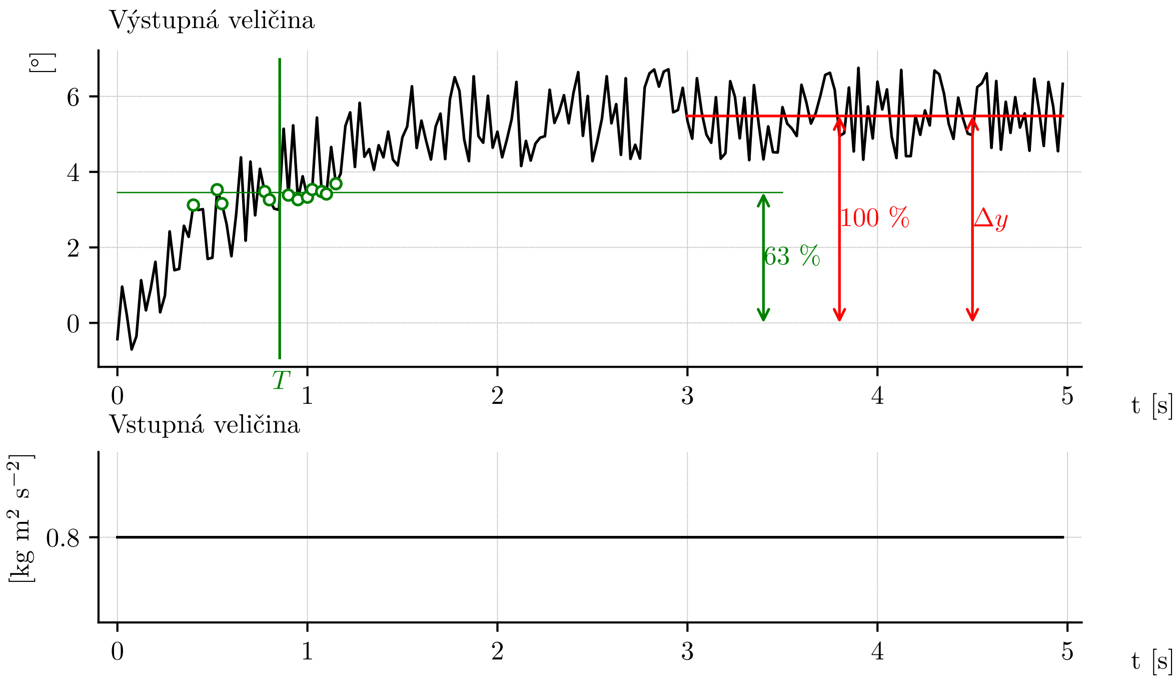


Figure 18

The value of T can now be found "by eye", literally using the TCH graph, or "by eye", but slightly differently - for example: Find the values of the output variable that are in the band (let's call it "by eye") $\pm$?% around the value Δy_{63}. More precisely, find the times of the samples that are in that band. The found points in the "by eye" band around the value Δy_{63} are marked as small green circles in Fig. 18. The average of the found times is

$$T = 0.81[\mathrm{s}] \tag{12}$$

And this value can be quite well "by eye" read as the time constant. All of the above is drawn in Fig. 18.

3.5.4 Verification of the Identified Dynamic Model

In the previous section, based on the step response, the parameters of a linear dynamic system that should be the model of the real system were determined. This model can be expressed in the form of a transfer function

$$\frac{y(s)}{u(s)} = \frac{K}{Ts+1} \tag{13}$$

For verification of the model, a graphical comparison of the step response of the model and the actual step response can be used. To obtain the TCH of the model, use numerical simulation. The given transfer function corresponds to the differential equation in the form

$$T\dot{y}(t) + y(t) = Ku(t) \tag{14}$$

$$T\dot{y}(t) = -y(t) + Ku(t) \tag{15}$$

$$\dot{y}(t) = -\frac{1}{T}y(t) + \frac{K}{T}u(t) \tag{16}$$

Choose the input signal to be the same as the magnitude Δu. This will ensure the corresponding magnitude of the unit step used in the numerical simulation to obtain the TCH.

In a common figure, plot the measured TCH and the TCH of the system model - see Fig. 19

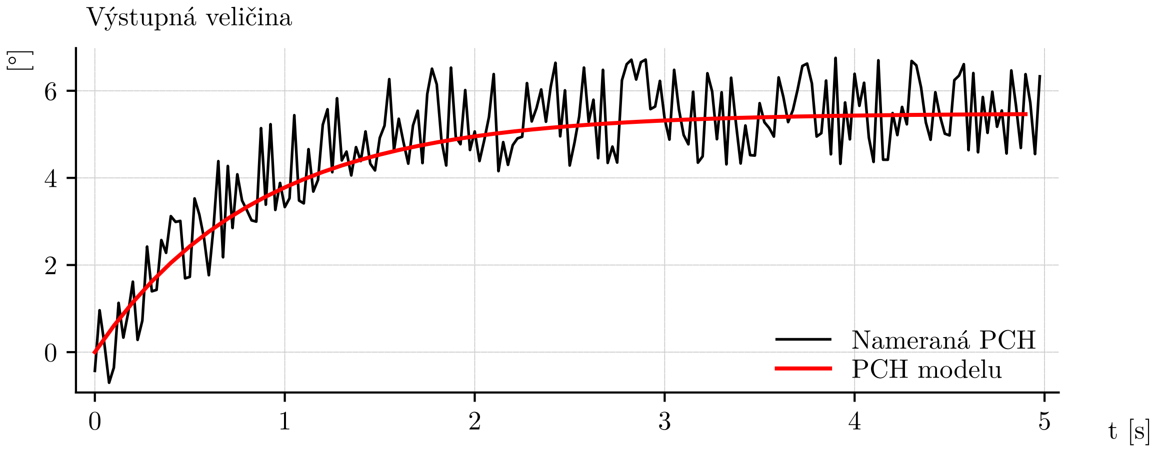


Figure 19

With this (at least for our needs), the model can be considered verified - it means that the given model is capable of capturing the properties of the real system and that it is possible to find the model parameters based on the available information (step response).

4 Supplementary Text: On the Stability of a Dynamic System

Recall the mathematical model of the pendulum, which we dealt with in previous topics. The equation of motion describing the dynamics of the rotational motion of the pendulum is in the form

$$ml^2\ddot{\varphi} = -\beta\dot{\varphi} - mgl\sin\varphi + u \tag{17}$$

where the pendulum's oscillations are damped by viscous friction with a coefficient β [kg m^2 s^{-1}], a mass point with mass m [kg] attached to an arm of negligible mass and length l [m] oscillates around the axis of rotation, and the angle from the vertical is denoted by φ [rad]. g [m s^{-2}] is the gravitational acceleration.

Furthermore, we considered that the state of the pendulum consists of two quantities: the angle of the pendulum arm φ and the angular velocity of the pendulum arm $\dot{\varphi}$. The state vector therefore has two elements $x^\mathsf{T} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$, where $x_1 = \varphi$ and $x_2 = \dot{\varphi}$. The pendulum model in state space is in the form

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{\beta}{ml^2}x_2 - \frac{g}{l}\sin(x_1) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{ml^2} \end{bmatrix} u \tag{18a}$$

$$\varphi = x_1 \tag{18b}$$

This is a nonlinear time-invariant second-order system.

In this section, we will consider the given dynamic system, but without input, in other words, the external torque is zero, $u(t) = 0$. Then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{\beta}{ml^2}x_2 - \frac{g}{l}\sin(x_1) \end{bmatrix} \tag{19a}$$

$$\varphi = x_1 \tag{19b}$$

4.1 Vector Field, Phase Portrait, Equilibrium

The qualitative behavior of a nonlinear dynamic system is important for understanding key concepts of Lyapunov's theory of system stability. For analysis, a certain class of systems called planar dynamic systems is important. These systems have two state variables $x \in \mathbb{R}^2$, which allows the state space to be represented in a plane with a coordinate system (x_1, x_2). Moreover, the results of qualitative analysis generally apply and can be used for higher-order systems as well. Therefore, these systems are important from the perspective of analysis. The pendulum model belongs to this class of systems.

An advantageous way to understand the behavior of a dynamic system with state $x \in \mathbb{R}^2$ is to draw the *phase portrait of the system*. We start by introducing the concept of a *vector field*. For a system of ordinary differential equations written compactly in a vector equation (like equation (19a)) in the form

$$\dot{x} = F(x) \tag{20}$$

the right-hand side of the equation defines the velocity $F(x) \in \mathbb{R}^n$ at each $x \in \mathbb{R}^n$. This velocity indicates how x changes and can be represented by a vector.

In a planar dynamic system, each state corresponds to a point in the plane, and $F(x)$ is a velocity vector representing the magnitude and direction of the change (velocity) of that state. These vectors can be plotted on a grid of points in the plane to obtain a visual representation of the system's dynamics, as shown in Fig. 20.

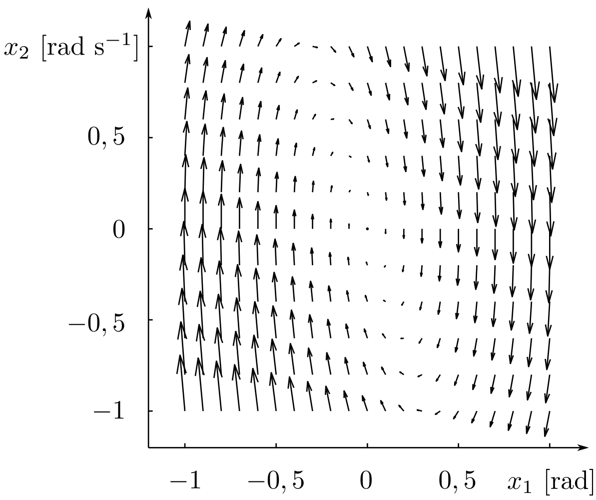


Figure 20: Vector Field Representing Pendulum Dynamics (image created in MATLAB, see text)

Vector Field

The vector field in Fig. 20 was generated in Matlab using the following code:

Code to generate Fig. 20

```
m = 1; %kg
l = 1; %m
g = 9.81; %m/s^2
beta = 2*0.5*sqrt(g/l); %kgm^2/s
[x1, x2] = meshgrid(-1:.1:1, -1:.2:1);
x1dot = x2;
x2dot = -(beta/m*l^2).*x2 - (g/l).*sin(x1);
quiver(x1,x2,x1dot,x2dot,1.5);
axis([-1.2 1.2 -1.2 1.2])
axis equal
```

Points where the velocity vector is zero are particularly interesting because they define the system's equilibrium points: if an autonomous system starts in such a state, it will remain in that state indefinitely.

Phase Portrait

The phase portrait (also called a phase diagram) consists of *streamlines* drawn according to the vector field. In other words, for a given set of initial conditions, we plot the solutions to the differential equation in the plane and indicate the direction of motion in the state space with an arrow. This corresponds to following the *vector field arrows* at each point in the state space and drawing the resulting trajectory. After plotting several trajectories for different initial conditions, we obtain a phase portrait like the one in Fig. 21.

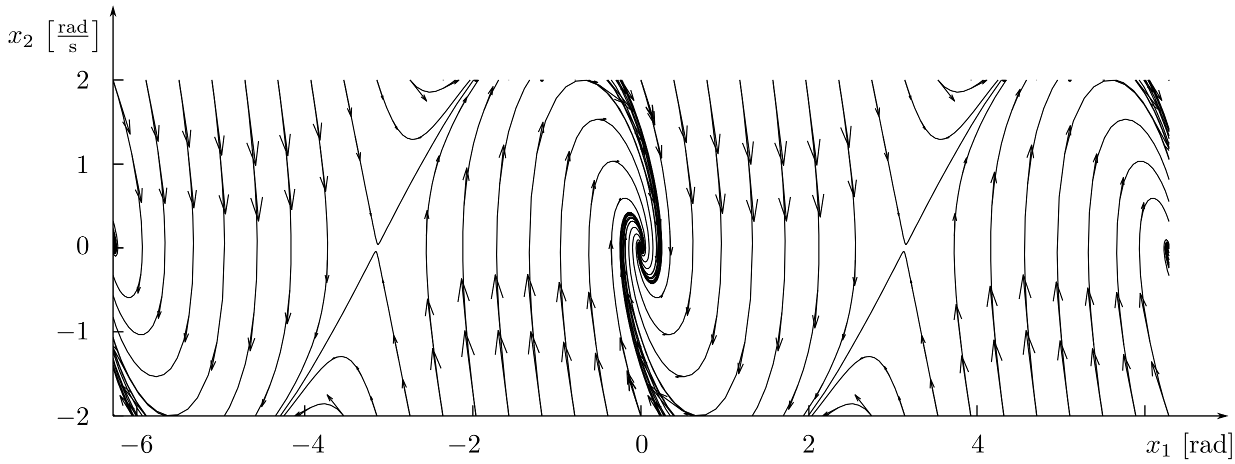


Figure 21: Phase portrait of the pendulum (image generated in Matlab, see text for source code)

The source code in MATLAB to generate this image is as follows:

Code to generate Fig. 21

```
global m l g beta
m = 1; %kg
l = 1; %m
g = 9.81; %m/s^2
beta = 2*0.5*sqrt(g/l); %kgm^2/s

for angular_velocity = -2:4:2
  for angle = -360:22.5:360
    [t,x]=ode45(@RightHandSide,[0 5],[angle*pi/180 angular_velocity])
    ;
    hold on
    state = x;
    x = x(1:5:end-70,:);
    x1dot = x(:,2);
    x2dot=-(beta/m*l^2)*x(:,2)-(g/l)*sin(x(:,1));
    quiver(x(:,1),x(:,2),x1dot,x2dot,0.5,'k')
    plot(state(:,1),state(:,2),'k');
    hold off
  end
end

axis equal
axis([-2*pi 2*pi -2 2])
```

where the function `RightHandSide` is defined as

```
function dotx = RightHandSide(t,x)
  global m l g beta
  dotx(1)=x(2);
  dotx(2)=-(beta/m*l^2)*x(2)-(g/l)*sin(x(1));
  dotx=dotx';
end
```

The phase portrait is a tool that allows one to assess the overall dynamics of a system by plotting several solutions in the system's state space (plane). For example, it is possible to see whether all trajectories converge to a single point over time or if the system exhibits more complex behavior. However, the phase portrait does not provide information about the magnitude of the rate of state change (though this can be inferred from the length of the vectors in the system's vector field).

Equilibrium

The equilibrium of a dynamic system is a point in the state space that represents the system's steady-state conditions. It is a stationary point where the velocity vector of the system's trajectory is zero, as previously mentioned.

We say that the state x_e is an equilibrium of the dynamic system

$$\dot{x} = F(x)$$

if $F(x_e) = 0$. If an autonomous system starts with the initial condition $x(0) = x_e$, it will remain in this state, and the solution is $x(t) = x_e$ for all $t > 0$, assuming the initial time is $t_0 = 0$.

Stationary points (equilibria) are among the most important features of a dynamic system because they define conditions where the system's operating parameters remain unchanged. A system can have zero, one, or more stationary points.

The stationary points of the pendulum system are

$$x_e = \begin{bmatrix} \pm n\pi \\ 0 \end{bmatrix} \tag{21}$$

where $n = 0, 1, 2, \ldots$. For even n, these represent states where the pendulum hangs downward, and for odd n, the pendulum is in an inverted position. The phase portrait in Fig. 21 is drawn for $-2\pi \leq x_1 \leq 2\pi$, so the diagram shows five stationary points.

4.2 Stability of Dynamic Systems in General

Recall that we are dealing with an autonomous system (homogeneous differential equation) in the form

$$\dot{x} = F(x) \tag{22}$$

and let us also recall what we mean by a solution of the system, or simply a solution. We say that $x(t)$ is a *solution* to the differential equation (22) over the time interval from $t_0 \in \mathbb{R}$ to $t_f \in \mathbb{R}$ if

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = F\big(x(t)\big) \qquad \text{for } t_0 < t < t_f \tag{23}$$

This differential equation may have many solutions, but we are usually interested in initial value problems, where $x(t)$ is specified at the initial time t_0, and the task is to find the solution for the entire future time $t > t_0$. In this case, $x(t)$ is the solution to the differential equation (22) with the initial state $x_0 \in \mathbb{R}^n$ at time t_0 if

$$x(t_0) = x_0 \quad \text{and} \quad \frac{\mathrm{d}x(t)}{\mathrm{d}t} = F\big(x(t)\big) \quad \text{for } t_0 < t < t_f \tag{24}$$

We commonly encounter differential equations for which there exists a unique solution, often for all time $t > t_0$, meaning $t_f = \infty$. It is also common for the function F to be time-independent, allowing us to assume $t_0 = 0$.

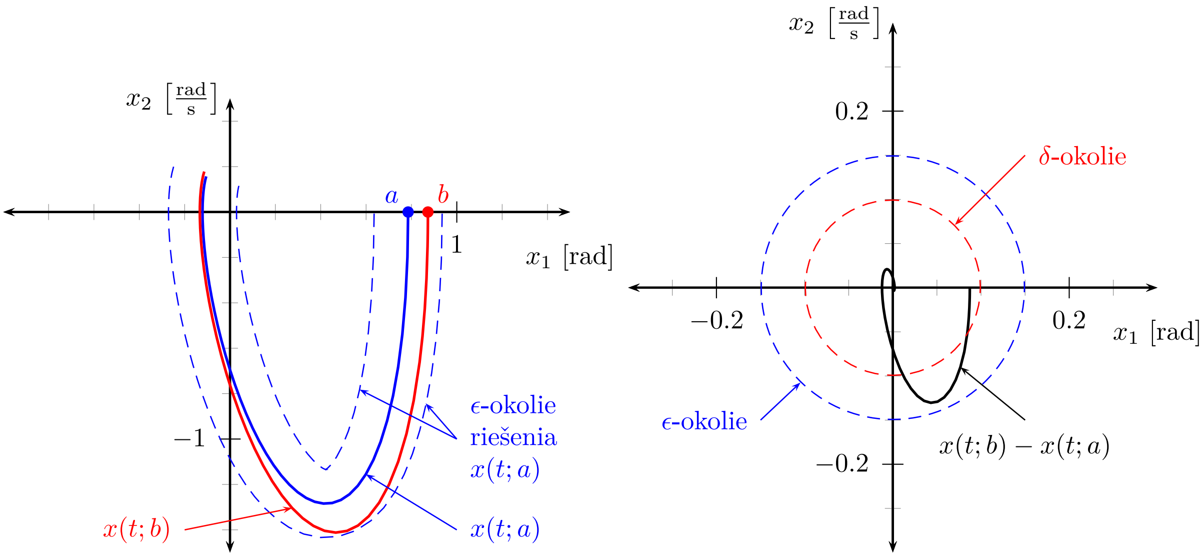
Stability of the solution determines whether other solutions nearby remain close, converge to it, or diverge. Below, we provide some informal and formal definitions of stability:

Let $x(t; a)$ be the solution of the differential equation with the initial state a. This solution is stable if other solutions that start near a remain close to $x(t; a)$. Formally, we say that the solution $x(t; a)$ is stable if for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|b - a\| < \delta \;\Rightarrow\; \|x(t; b) - x(t; a)\| < \epsilon,\ \forall\, t > 0 \tag{25}$$

Note that this does not imply that $x(t; b)$ converges to $x(t; a)$, only that it remains in its vicinity. Moreover, the value of δ may depend on ϵ, meaning if we want to stay very close to a solution, we must start very close to it. This form of stability is called *Lyapunov stability.*

We illustrate the condition (25) using the solution of the pendulum differential equation (19a). Let the initial time be $t_0 = 0$ [s] and the final time be $t_f = 1.4$ [s], with the pendulum's initial angle at $\varphi = 45°$ and initial velocity at zero. The initial state in the state space is $a = \begin{bmatrix} 0.7854 & 0 \end{bmatrix}^{\mathsf{T}}$. The corresponding solution $x(t; a)$ is shown in the state space in Fig. 22a, where the initial state a is also indicated. We will not investigate all $\epsilon > 0$, but only one. For example, for $\epsilon = 0.15$, we search for



(a) Trajektórie riešení systému v stavovom priestore pre rôzne začiatočné stavy a a b v časovom intervale $t_0 < t < t_f$

(b) Trajektória rozdielu dvoch riešení daných začiatočnými stavmi a a b v časovom intervale $t_0 < t < \infty$

Figure 22: Illustrative example for the definition of system solution stability

(b)

$\delta > 0$ that satisfies condition (25). Such a δ exists, as the solution $x(t;b)$, starting from state $b = \begin{bmatrix} 0.8727 & 0 \end{bmatrix}^\mathsf{T}$, satisfies $\|x(t;b) - x(t;a)\| < \epsilon$, as seen in Fig. 22a and Fig. 22b, where the time is extended to infinity. We found, for example, $\delta = 0.1$, since

$$\begin{aligned} \|b - a\| &= \sqrt{(0.8721 - 0.7854)^2 + (0 - 0)^2} = \\ &= 0.0873 < 0.1 \end{aligned} \tag{26}$$

as is evident from Fig. 22b. However, this does not tell us much about the stability of the solution $x(t;a)$, as we have not verified if condition (25) holds for all $\epsilon > 0$.

If a solution is stable in the sense of Lyapunov, but the trajectories of nearby solutions do not converge to it, we say that the solution is *neutrally stable*.

A solution $x(t;a)$ is *asymptotically stable* if it is stable in the sense of Lyapunov and, in addition, $x(t;b) \to x(t;a)$ as time $t \to \infty$, where the initial state b is sufficiently close to the state a.

A very important special case is when the considered solution satisfies $x(t;a) = x_e$. In this case, we do not refer to the stability of the solution but instead to the *stability of the equilibrium point*. An example of an asymptotically stable equilibrium point is the set of points

$$x_{e_{-2}} = \begin{bmatrix} -2\pi \\ 0 \end{bmatrix}, \quad x_{e_0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad x_{e_2} = \begin{bmatrix} 2\pi \\ 0 \end{bmatrix}$$

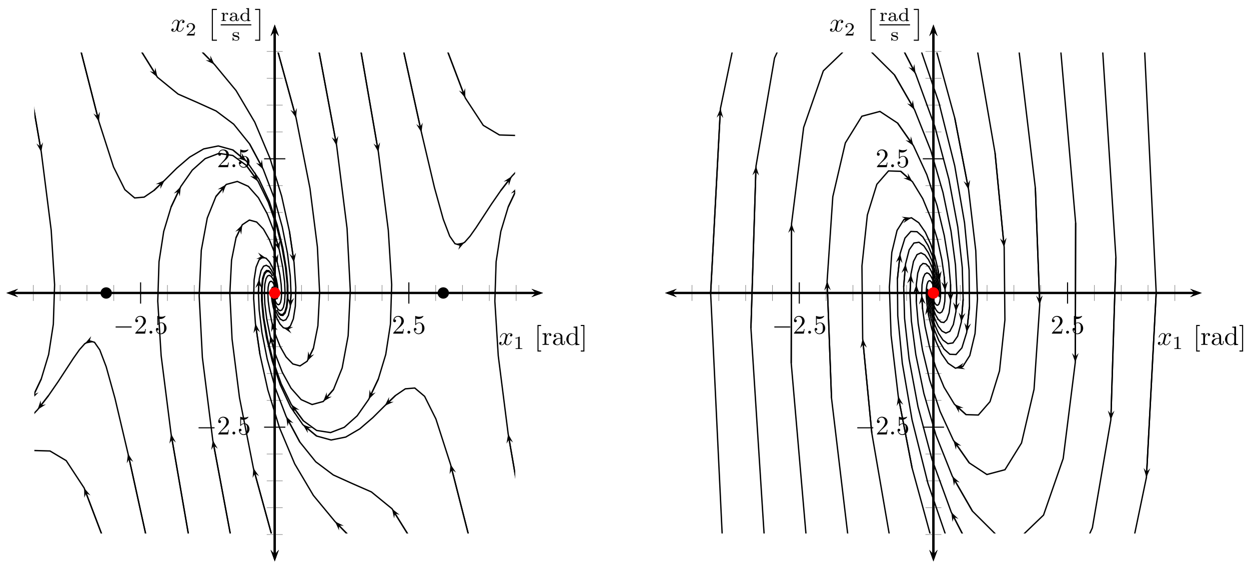
as shown in Fig. 21, where we see that if we start near an asymptotically stable equilibrium point, we approach it as time increases.

A solution $x(t;a)$ is *unstable* if it is not stable. More specifically, we say that a solution $x(t;a)$ is unstable if for any given $\epsilon > 0$, there does not exist a $\delta > 0$ such that if $\|b - a\| < \delta$ then $\|x(t;b) - x(t;a)\| < \epsilon$, $\forall\, t > 0$. An example of an unstable equilibrium point is the set of points

$$x_{e_{-1}} = \begin{bmatrix} -\pi \\ 0 \end{bmatrix}, \quad \text{and} \quad x_{e_1} = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

as shown in Fig. 21.

The previous definitions do not take into account the region over which they apply. It is more precise to define a solution as *locally stable* (or *locally asymptotically stable*) if it is stable for all initial conditions $x \in B_r(a)$, where $B_r(a) = \{x : \|x - a\| < r\}$ is a region with radius $r > 0$ around the point a. A solution is *globally stable* if it is stable for all $r > 0$.



(a) Phase portrait of the nonlinear pendulum model

(b) Phase portrait of the linear pendulum model

Figure 23: Comparison of phase portraits of the nonlinear system and its linearized approximation

4.3 Stability of Linear Dynamical Systems

A linear dynamical system has the form

$$\dot{x} = Ax, \quad x(0) = x_0 \tag{27}$$

where $A \in \mathbb{R}^{n\times n}$ is a square matrix. The origin of the state space is always a stationary point of the linear system, and the stability of this stationary point can be determined using the eigenvalues of the matrix A.

The eigenvalues $\lambda(A)$ are the roots of the *characteristic polynomial* of the system $\det(sI - A)$, where $s \in \mathbb{C}$ is a complex variable and I is the identity matrix. A specific eigenvalue (the i-th eigenvalue) is denoted as λ_i, where $\lambda_i \in \lambda(A)$.

For a linear system, the stability of the stationary point (as a very important special case among all solutions) depends only on the matrix A, meaning that stability is a property of the system. Therefore, for a linear system, we talk about the stability of the system instead of the stability of a specific solution or equilibrium.

The stability of a linear system can be summarized in one sentence:
The system

$$\dot{x} = Ax$$

is asymptotically stable if and only if the real parts of all eigenvalues of the matrix A are negative, and the system is unstable if at least one eigenvalue of the matrix A has a positive real part.