Introductory Exercise

Contents

The goal of the introductory exercise is to provide insights into the concept of a *system* from the perspective of the subject Modeling and Control of Systems. The material is designed in such a way that students should be able to engage with it without the need for a prior lecture.

The following two specific examples can be used for an introductory discussion on the following concepts and topics:

- System – has an output and an input (can have only an output. . .)
- Signal. In this subject, a signal is essentially denoted as a function of time, as a time-varying value of a quantity, e.g., $y(t)$.
- Dynamic system
- System whose dynamics do not need to be considered.
- Equation as a tool for the mathematical description of a system.
- Differential equation – mathematical description of a dynamic system.

1 Tasks

1. Answer the questions listed in section 2.1.
2. Formulate the differential equation that describes the process of capacitor discharge (see section 2.2).
3. Determine the units (dimensions) of all parameters and signals (quantities) in the formulated equation.
4. Find the analytical solution of the given differential equation.
5. Draw the graph of the time function, which is the analytical solution of the differential equation. The necessary numerical values of parameters and signals can be arbitrary.
6. Find the numerical solution of the differential equation (using Simulink).

2 Specific Illustrative Examples

2.1 Gain of a Resistor Voltage Divider

Consider a classic resistor voltage divider as shown in the following figure.

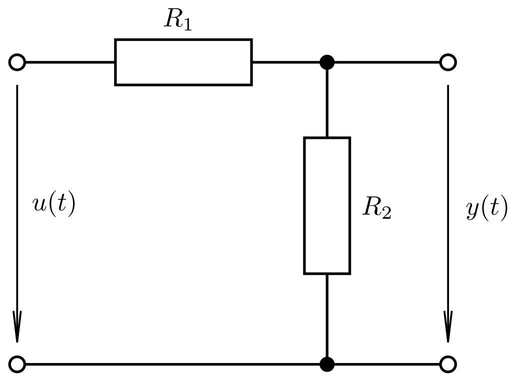


Figure 1: Resistor Voltage Divider

Let the input of the considered system be the voltage denoted as $u(t)$ and the output signal be the voltage $y(t)$.

Questions

- Let the value of the input signal be constant, not changing, and steady. What is the value of the output signal, given that we know the values of resistors R_1 and R_2?
- How would you define the gain of the considered system?
- What is the magnitude of the gain of the considered system?

2.2 Discharging a Capacitor – Mathematical Model of the Process

Consider an RC circuit as shown in Fig. 2.

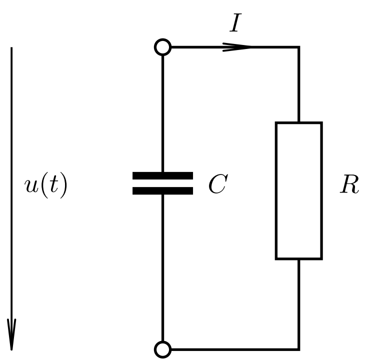


Figure 2: RC Circuit

Let the capacitor C be charged initially at time $t = 0$, with a voltage u_0 across its terminals. In other words, the voltage $u(t)$ at time 0 is u_0, i.e., $u(0) = u_0$.

The capacitor C is connected to a resistor R, and thus the capacitor discharges over time.

2.2.1 Formulating the Differential Equation

Let's formulate the differential equation that describes the process of capacitor discharge.

For a capacitor, the following holds:

$$Q = CU \tag{1}$$

which means that the electric charge Q accumulated in the capacitor is proportional to the voltage across the capacitor terminals U (perhaps overly simplified, but the reader can certainly look up the details). The parameter C represents, as is surely evident, the capacitance of the capacitor.

If the capacitor discharges, the charge changes over time. Therefore, it makes sense to investigate the time course of the charge magnitude. This provides an overall understanding of other quantities related to the capacitor discharge process.

The time change of electric charge is electric current, thus

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -I \tag{2}$$

where I is the electric current and the reason for the negative sign is that the direction of the electric current is marked exactly opposite to the direction of the movement of the negative charge.

Equation (2) is essentially a differential equation. It contains the time derivative of the quantity – electric charge. However, in this form, the equation cannot be used to obtain the time course of the quantity itself (electric charge). Namely, not only Q is unknown but also I.

Instead of the quantity I, it would be appropriate to have the quantity Q on the right side of equation (2). From Ohm's law, it follows

$$I = \frac{U}{R} \tag{3}$$

The voltage U, which pertains to our problem, is related to the quantity Q, see equation (1). Specifically,

$$U = \frac{Q}{C} \tag{4}$$

Substituting (4) into (3) yields

$$I = \frac{Q}{RC} \tag{5}$$

and subsequently substituting (5) into (2)

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\frac{1}{RC}Q \tag{6}$$

The differential equation (6) contains one unknown. The unknown is the quantity Q. More generally, the unknown is the time course of the quantity. Therefore, we write that we are dealing with the signal (quantity) $Q(t)$. The values of R and C are just fixed values of resistance and capacitance (see Fig. 2). We do not consider them to change over time. Therefore, we do not denote them as signals (functions of time). Thus, we denote the signal as, for example, $Q(t)$ and the constant as, for example, R.

Typically, and for simplicity, the equation (6) is also written in the form

$$\dot{Q}(t) = -\frac{1}{RC}Q(t) \tag{7}$$

where the dot $\dot{}$ denotes the derivative with respect to time, just like the operator $\frac{\mathrm{d}}{\mathrm{d}t}$.

The solution to equation (7) is some time function, some signal, some time course, specifically the time course of the electric charge, which we denote here as $Q(t)$.

To find a unique solution, it is necessary to supplement the task with an initial condition. This is the condition that the sought signal $Q(t)$ must satisfy at the beginning, i.e., at time $t = 0$. Recall that the voltage before discharging is given (known) and has the value u_0. It is therefore clear that the value $Q(0) = Cu_0$ is also known. For simplicity, let's denote it as $Q(0) = Q_0$.

2.2.2 Outline of Solving the Differential Equation by Separation of Variables

We are dealing with the problem in the form

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = -\frac{1}{RC}Q(t) \qquad Q(0) = Q_0 \tag{8}$$

where $Q(t)$ is the unknown time function. The constants (independent of time) R, C, and Q_0 are known. However, there is one more variable in the equation, which is time t. As is known, time just flows. It is a variable because, for example, it is "differentiated with respect to it."

By the way

- What are the units (dimensions) of the term RC in equation (8)?

Let's modify the differential equation (8) so that the same variables are on the same sides. In the form (8), the signal $Q(t)$ is on both sides of the equation. Let it be only on the left side. Similarly, let time t be only on the right side. Thus,

$$\frac{1}{Q(t)}\mathrm{d}Q(t) = -\frac{1}{RC}\mathrm{d}t \tag{9}$$

Notice that now it is possible to integrate both sides of the equation, each according to its own variable, thus

$$\int \frac{1}{Q(t)}\mathrm{d}Q(t) = \int -\frac{1}{RC}\mathrm{d}t \tag{10}$$

The result of the integration is

$$\ln\left(Q(t)\right) + k_1 = -\frac{1}{RC}t + k_2 \tag{11}$$

where k_1 and k_2 are constants resulting from the indefinite integrals (and we also silently considered that $Q(t)$ will not take negative values).

Equation (11) is no longer differential. No quantity in it is derived with respect to time.

Let's express the signal $Q(t)$ from equation (11). By rearranging

$$\ln\left(Q(t)\right) = -\frac{1}{RC}t + k_3 \tag{12}$$

we introduced the constant $k_3 = k_2 - k_1$. Further

$$Q(t) = e^{\left(-\frac{1}{RC}t + k_3\right)} \tag{13a}$$

$$Q(t) = e^{\left(-\frac{1}{RC}t\right)}e^{k_3} \tag{13b}$$

Already at this point, equation (13b) is a prescription that gives the time dependence of the quantity Q. It expresses the signal (time function) $Q(t)$. The time function $Q(t)$ is the solution of the differential equation (9).

In equation (13b), the constant e^{k_3} appears. This is a general constant and can take any value. It can be shown, although we will not provide a formal demonstration here, that this constant is determined by the initial condition assigned to the differential equation. In this case, $e^{k_3} = Q_0$.

The sought solution of the differential equation is a time function in the form

$$Q(t) = Q_0\ e^{\left(-\frac{1}{RC}t\right)} \tag{14}$$

The function is graphically shown in Figure 3.

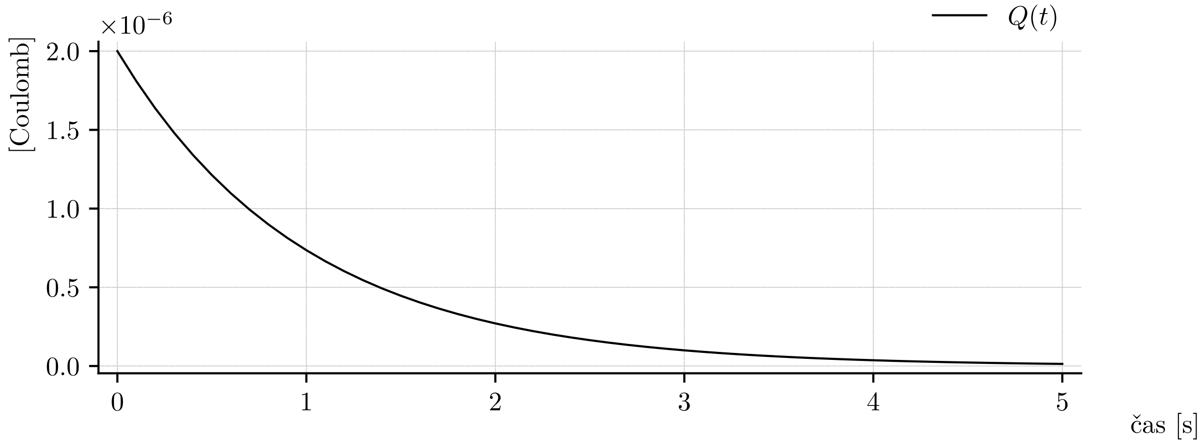


Figure 3: Graph of the function (14) for $R = 10^6$ [Ω], $C = 1$ [μF] and $Q_0 = 2 \cdot 10^{-6}$ [Coulomb] (arbitrary values just as an example)

2.2.3 Voltage on the Capacitor

We have examined the time course of the electric charge during the discharge of the capacitor. The description of the situation at the beginning of section 2.2 indirectly assumes that we will deal with voltage. We already know the mutual relationship, and its more formally precise notation (voltage $u(t)$ as a signal) is

$$u(t) = \frac{1}{C} Q(t) \tag{15}$$

So if we know the course of $Q(t)$, we also know the course of $u(t)$.

The initial condition for the signal $Q(t)$, i.e., the value $Q(0)$, can of course also be determined from the desired (given) initial condition of the signal $u(t)$.

$$Q(0) = C u_0 \tag{16}$$

In the sense of the introduction of section 2.2, consider the following example

$$\begin{aligned} C &= 1 \ [\mu\text{F}] \\ R &= 10^6 \ [\Omega] \\ u_0 &= 5 \ [\text{V}] \end{aligned}$$

For this example, the initial condition for the signal $Q(t)$ is

$$Q(0) = 10^{-6} \cdot 5 = 0.000050 \ [\text{Coulomb}] \tag{17}$$

The resulting voltage course is shown in Fig. 4.

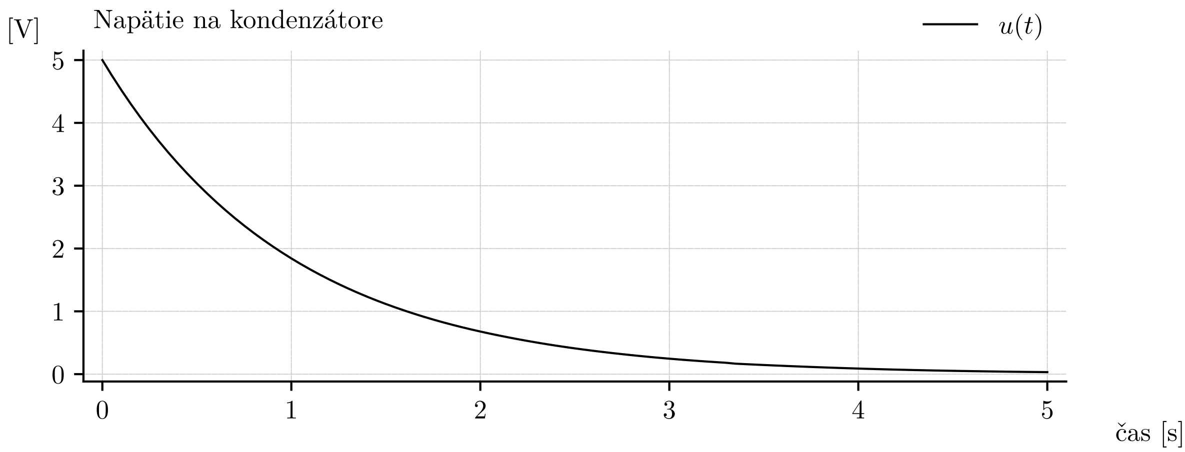


Figure 4: Time course of voltage on the capacitor

Table 1: Examples of different parameters

	C [F]	R [Ω]	u_0 [V]
Example 1	$2 \cdot 10^{-6}$	10^6	5
Example 2	$\frac{1}{2} \cdot 10^{-6}$	10^6	5
Example 3	10^{-6}	$\frac{1}{10} \cdot 10^6$	3

2.2.4 Examples for Different Parameters R and C

For example, let's show the voltage course for different parameters R and C. The examples are summarized in Table 1. The time courses are graphically shown in Fig. 5.

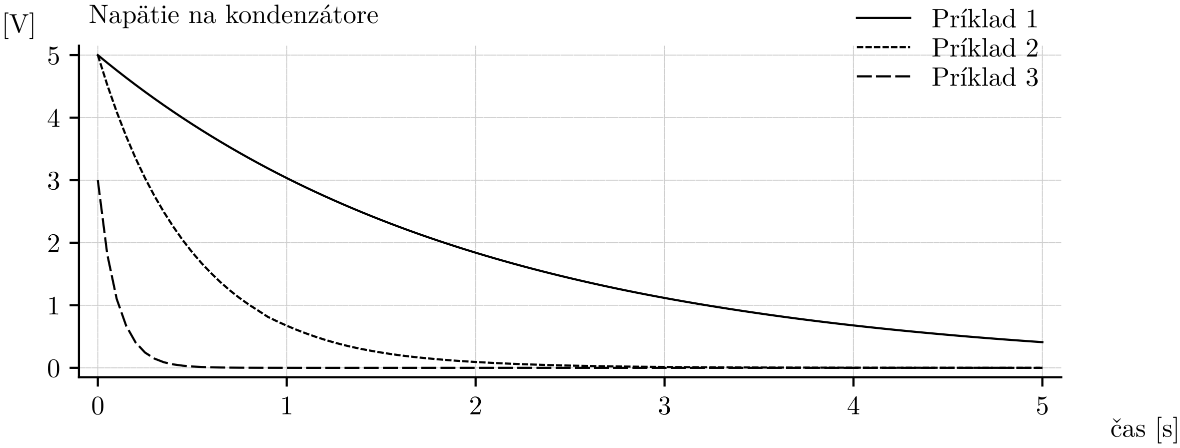


Figure 5: Time course of voltage on the capacitor

3 Additional Notes

3.1 Plotting the Time Function Graph – MATLAB

The goal is to plot the graph of the time function (14), i.e.,

$$Q(t) = Q_0 \; e^{\left(-\frac{1}{RC}t\right)}$$

The example of the resulting graph is shown in Fig. 3.

A minimal code for MATLAB could look like this:

Code Listing 1: File MRS01_plotexample.m

```
% Parametre
R = 10^6;
C = 10^-6;
Q_0 = 2*10^-6;


% Súradnice bodov na x osi
plotData_x = 0:0.1:5;

% Výpočet hodnôt na y osi v zmysle danej časovej funkcie
plotData_y = Q_0 * exp( ( -1.0/(R*C)) * plotData_x );

% Kreslenie grafu
plot(plotData_x,plotData_y)
xlabel('čas [sec]')
ylabel('Q [Coulomb]')
```

3.2 Plotting the Time Function Graph – Python

The goal is to plot the graph of the time function (14), i.e.,

$$Q(t) = Q_0 \; e^{\left(-\frac{1}{RC}t\right)}$$

An example of the resulting graph is shown in Fig. 3.

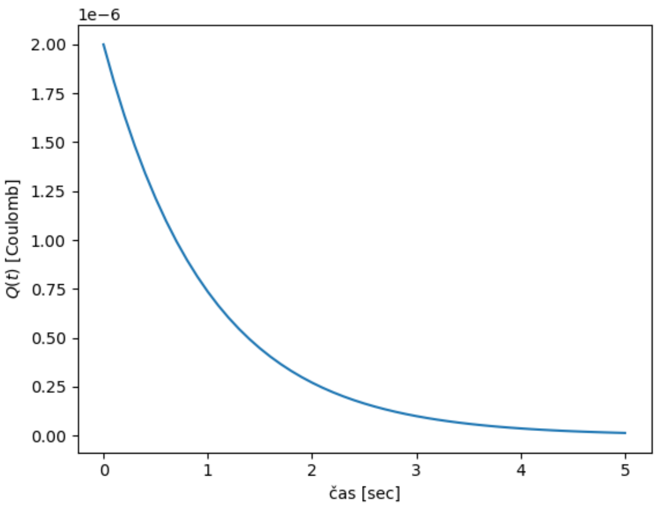
A minimal code for Python using the NumPy and Pyplot modules could look like this, as cells from a Jupyter notebook:

Code Listing 2: Súbor `MRS01_plotexample.ipynb cell:01`

```
import numpy as np
import matplotlib.pyplot as plt
```

Code Listing 3: Súbor `MRS01_plotexample.ipynb cell:02`

```
# Parametre
R = 10**6
C = 10**-6
Q_0 = 2*10**-6

# Súradnice bodov na x osi
plotData_x = np.arange(0,5.1,0.1)

# Výpočet hodnôt na y osi v zmysle danej časovej funkcie
plotData_y = Q_0 * np.exp( (-1.0/(R*C)) * plotData_x )

plt.plot(plotData_x , plotData_y)
plt.xlabel('čas [sec]')
plt.ylabel('$Q(t)$ [Coulomb]')
plt.show ()
```



If the reader is encountering Python for numerical computations for the first time, the following links might be useful:

Python (installed as a package distribution...)

For general use of Python on Windows, especially for "scientific computations," it is recommended to use the Anaconda distribution, as mentioned here: `https://www.scipy.org/install.html`, Anaconda distribution: `https://www.anaconda.com/download/`

Unless explicitly stated otherwise, Python version 3 is used here.

Jupyter

In this context, it is also worth pointing out `https://jupyter.org/`. IPython as well as Jupyter notebook are part of the Anaconda distribution.

3.3 Numerical Simulation – Simulink

As mentioned, the differential equation (8) describes a dynamic system. Recall

$$\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = -\frac{1}{RC}Q(t) \qquad Q(0) = Q_0 \tag{18}$$

where $Q(t)$ is the unknown time function. The constants (independent of time) R, C, and Q_0 are known. The task is to find the time course of the quantity $Q(t)$. To find the solution of the differential equation. In the previous section, we sought the solution analytically, resulting in the time function $Q(t)$, whose graph we then plotted.

In this section, we will seek the numerical solution of the differential equation (18) using Simulink. The result will be the time course of the quantity $Q(t)$. These will be numerical values assigned to time data. The result can then also be plotted as the dependence of $Q(t)$ on time t.

In Simulink, it is necessary to define the equation (18) in the form of a schematic representation of the dynamic system (see also [KUT007]). This includes setting the initial conditions of the system (initial conditions in integrators).

Attention should also be paid to the required simulation time length, i.e., the length of the time interval over which the numerical solution of the differential equation is required. The signal $Q(t)$ can be displayed using the Scope block.

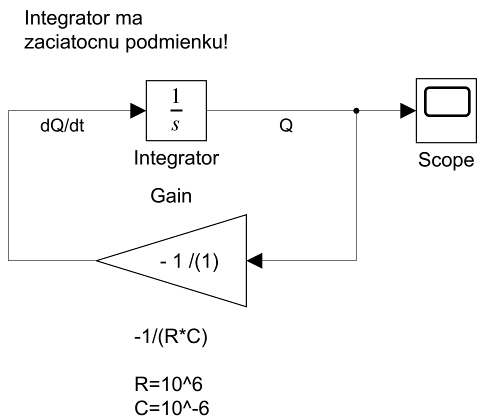


Figure 6: Simulation scheme corresponding to the equation (18)

3.4 Numerical Simulation – ODE Solver (MATLAB)

For the numerical computation of the solution using the `ode45` procedure, it is necessary to write the system (equation) as a function that the `ode45` procedure will use. In this case:

```
function dQ = fundif(t,x);
R = 10^6;
C = 10^(-6);
Q = x;
dQ = -(1/(R*C)) * Q;
```

It is necessary to create a separate file **`fundif.m`**, which will contain the function as shown here.

By the way, we will not provide details about the ODE solver here in the text. The goal is just to familiarize the reader with the possibilities of obtaining a numerical solution. How it "works" will be briefly commented on later.

The actual use of the `ode45` procedure is performed with the following commands (say in a script in another m-file):

```
Q_0 = 2 * 10^(-6);
[t,y] = ode45('fundif',[0 5],[Q_0]);
plot(t,y)
```

The plot is left to the reader...

3.5 Numerical Simulation – Python, SciPy.integrate Library

An example of using the ODE Solver from the SciPy.integrate library can be found in the `jupyter` notebook `PY/MRS01_ODEsolver.ipynb`.

Code Listing 4: Súbor MRS01_ODEsolver.ipynb cell:01

```python
# Import potrebných modulov
import numpy as np
import matplotlib.pyplot as plt
from scipy.integrate import odeint
```

Code Listing 5: Súbor MRS01_ODEsolver.ipynb cell:02

```python
# Definovanie funkcie, ktorá realizuje predmetnú diferenciálnu
    rovnicu

def fcn_difRovnica_01(x, t, param):

    R, C = param
    Q = x
    dotQ = (-1.0/(R*C)) * Q

    return dotQ
```

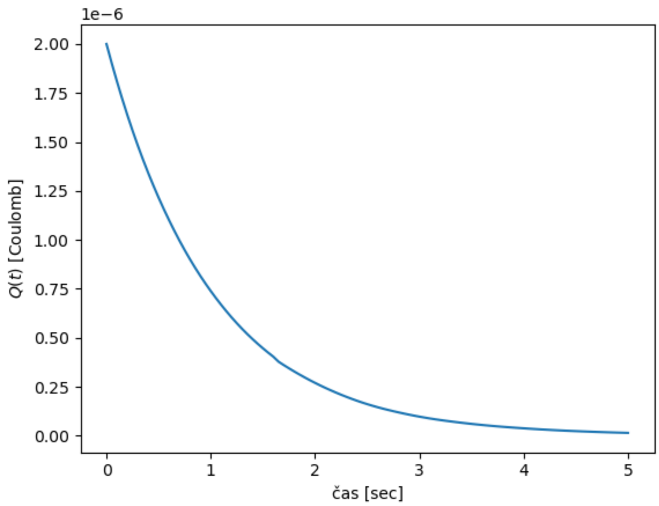
Code Listing 6: Súbor MRS01_ODEsolver.ipynb cell:03

```python
# Použitie ODE solvera (odeint imporotvaný z knižnice scipy)

# Príprava parametrov a začiatočných podmienok
param_C = 10**-6
param_R = 10**6
param = [param_R, param_C]          # zoznam parametrov

Q_0 = 2*10**-6                      # Začiatočná podmienka

# Časový vektor, pre ktorý požadujeme riešenie
sim_t_start = 0
sim_t_final = 5
sim_T_s = 0.05
timeVect = np.arange(sim_t_start, sim_t_final+sim_T_s, sim_T_s)

# Volanie ODE solvera
odeOut = odeint(fcn_difRovnica_01,  # volaná funkcia (dif. rovnica)
                Q_0,                # začiatočná podmienka
                timeVect,           # časový vektor
                args=(param,)       # argumenty volanej funkcie
                )

print(odeOut[:,0]) # Výpis výsledkov simulácie - numerické riešenie
    ...
```

```
[2.00000000e-06 1.90242604e-06 1.80961688e-06 1.72134793e-06
 1.63737823e-06 1.55765170e-06 1.48195836e-06 1.41008245e-06
 1.34180819e-06 1.27691980e-06 1.21520151e-06 1.15643755e-06
 1.10041214e-06 1.04690950e-06 9.95852513e-07 9.47340655e-07
 9.01247200e-07 8.57438923e-07 8.15782598e-07 7.76145000e-07
 7.38392904e-07 7.02393084e-07 6.68012315e-07 6.35129586e-07
 6.03824871e-07 5.74086386e-07 5.45838175e-07 5.19004282e-07
 4.93508750e-07 4.69275623e-07 4.46228943e-07 4.24292755e-07
 4.03391101e-07 3.77739286e-07 3.60602172e-07 3.44032035e-07
 3.28028877e-07 3.12592697e-07 2.97723495e-07 2.83421271e-07
 2.69686025e-07 2.56517758e-07 2.43914212e-07 2.31827496e-07
 2.20238343e-07 2.09146754e-07 1.98552727e-07 1.88456263e-07
 1.78857363e-07 1.69756026e-07 1.61152252e-07 1.53046041e-07
 1.45437393e-07 1.38326308e-07 1.31649509e-07 1.25254891e-07
 1.19135788e-07 1.13292200e-07 1.07724127e-07 1.02431569e-07
 9.74145263e-08 9.26729984e-08 8.82069856e-08 8.40164877e-08
 8.01015049e-08 7.64573621e-08 7.29848135e-08 6.96439350e-08
 6.64347268e-08 6.33571888e-08 6.04113210e-08 5.75971234e-08
 5.49145960e-08 5.23637388e-08 4.99445518e-08 4.76570350e-08
 4.55011884e-08 4.34600932e-08 4.14930814e-08 3.95983705e-08
 3.77759605e-08 3.60258514e-08 3.43480432e-08 3.27425359e-08
 3.12093294e-08 2.97484239e-08 2.83598192e-08 2.70435154e-08
 2.57986173e-08 2.46061198e-08 2.34583775e-08 2.23553905e-08
 2.12971589e-08 2.02836825e-08 1.93149614e-08 1.83909957e-08
 1.75117852e-08 1.66773301e-08 1.58876302e-08 1.51426857e-08
 1.44373720e-08]
```

Code Listing 7: Súbor MRS01_ODEsolver.ipynb cell:04

```
# Grafické zobrazenie výsledkov simulácie

plt.plot(timeVect , odeOut)
plt.xlabel('čas [sec]')
plt.ylabel('$Q(t)$ [Coulomb]')
plt.show()
```



3.6 MATLAB Online Training Suite

For the mentioned topics, the MATLAB Online Training Suite can be recommended, where the basic courses are:

- MATLAB Onramp
  https://matlabacademy.mathworks.com/details/matlab-onramp/gettingstarted
- Simulink Onramp
  https://matlabacademy.mathworks.com/details/simulink-onramp/simulink

Directly related to this text, perhaps:

- Solving Ordinary Differential Equations with MATLAB
  https://matlabacademy.mathworks.com/details/solving-ordinary-differential-equations-with-matlab/odes