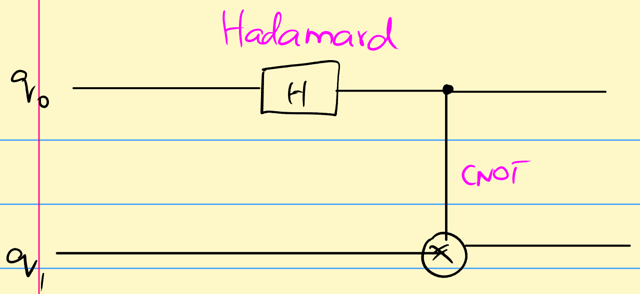
08.06.2024



Let $q_0 = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

If $q_1 = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$; Initial state $\Rightarrow$ $|\psi_i\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|11\rangle$

$CNOT|\psi_i\rangle = \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle = |\psi_f\rangle = |\psi^+\rangle$ $\Rightarrow$ Bell state

||ly if $q_1 = |0\rangle$, $|\psi_i\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle$; $CNOT|\psi_i\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle = |\psi_f\rangle = \underset{\text{Bell state}}{|\phi^+\rangle}$

Applying $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = NOT$ on q_0 and $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ on q_1 is equivalent to applying $X \otimes \mathbb{1}$ on $|\psi_f\rangle$.

Let $|\psi_f\rangle = |\phi^+\rangle$,

We know $E[X] = \sum_{x \in \Omega} x \, P[X = x] = \int_{-\infty}^{\infty} x \, \overbrace{f(x)}^{f \text{ is PDF}} dx$

For quantum vector,

$$|\phi\rangle = \sum_{(a,b) \in |a\rangle \otimes |b\rangle} \alpha_{ab} |ab\rangle \; ; \quad P\big[|\phi\rangle = |cd\rangle\big] = |\alpha_{cd}|^2$$

$$\langle \phi^+ | A \otimes \mathbb{1} | \phi^+ \rangle = \left(\frac{1}{\sqrt{2}} \langle 00| + \frac{1}{\sqrt{2}} \langle 11| \right) A \otimes \mathbb{1} \left(\frac{1}{\sqrt{2}} |00\rangle + \frac{1}{\sqrt{2}} |11\rangle \right)$$

$$= \frac{1}{2} \Big[\langle 00|A|00\rangle + \langle 00|A|11\rangle + \langle 11|A|00\rangle + \langle 11|A|11\rangle \Big]$$

$\langle 00|A|00\rangle = \langle 0|A|0\rangle \langle 0|\mathbb{1}|0\rangle = \langle 0|A|0\rangle$; $\langle 11|A|00\rangle = 0$

$\langle 00|A|11\rangle = \langle 0|A|1\rangle \underbrace{\langle 0|\mathbb{1}|1\rangle}_{0} = 0$; $\langle 11|A|11\rangle = \langle 1|A|1\rangle$

If $A = X$,

$$\langle 0|X|0\rangle = (1\ 0)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = (1\ 0)\begin{pmatrix}0\\1\end{pmatrix} = 0 = \langle 1|X|1\rangle$$

If $A = Z$,

$$\langle 0|Z|0\rangle = (1\ 0)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = (1\ 0)\begin{pmatrix}1\\0\end{pmatrix} = 1$$

$$\langle 1|Z|1\rangle = (0\ 1)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = (0\ 1)\begin{pmatrix}0\\-1\end{pmatrix} = -1$$

$\therefore$ $\langle 0|Z|0\rangle + \langle 1|Z|1\rangle = 0$

If we apply X on both,

$$E = \langle \phi^+|X\otimes X|\phi^+\rangle = \frac{1}{2}\left(\langle 00| + \langle 11|\right)\Big| X\otimes X \Big|\left(|00\rangle + |11\rangle\right)$$

$$= \frac{1}{2}\left[\langle 00|XX|00\rangle + \langle 00|XX|11\rangle + \langle 11|XX|00\rangle + \langle 11|XX|11\rangle\right]$$

$$\Rightarrow \frac{1}{2}\left[\underbrace{\langle 0|X|0\rangle}_{0}\underbrace{\langle 0|X|0\rangle}_{0} + \langle 0|X|1\rangle\langle 0|X|1\rangle + \langle 1|X|0\rangle\langle 1|X|0\rangle + \underbrace{\langle 1|X|1\rangle}_{0}\underbrace{\langle 1|X|1\rangle}_{0}\right]$$

$$= \frac{1}{2}(1+1) = 1$$

$|||^{\text{ly}}$ $\langle \phi^+|Z\otimes Z|\phi^+\rangle = 1$
