Understanding The Rubik's Cube: A Group Theoretic Approach

Pesara Amarasekera Zeyn Jaswal Tony Xu

Introduction

The Rubik's Cube is perhaps the world's most famous puzzle and one that has caught the attention of many a mathematician. The first rigorous mathematical analysis of a Rubik's Cube was done by British-American Mathematician David Singmaster, though solutions were known to a number of mathematicians at the time (like J.H. Conway and Roger Penrose).

In this project, we attempt to understand the inner workings of the Rubik's cube via abstract algebra and group theory.

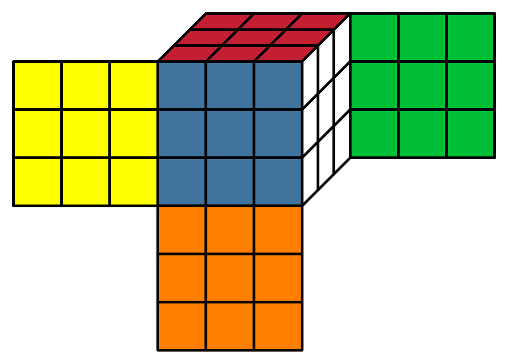

Figure 1. A flat map of the cube

Preliminary Definitions

Semi-direct products:

Let G be a group with identity element e, a subgroup H and a normal subgroup $N \triangleleft G$ such that $N \cap H = \{e\}$. Then the **semi-direct product** of N and H is $G = N \rtimes H$ if and only if for all $g \in G$ there exists a unique $n \in N$ and $h \in H$ such that $g = nh$.

Wreath products:

For X be a finite set, G a group and H a group acting on X, fix $\{x_1, x_2, \ldots, x_t\}$, a labelling of X with $|X| = t$ and let G^t be the direct product of G with itself t times. Then the **wreath product** of G and H is $G \wr H = G \rtimes H$ where H acts on G by its action on X.

Position Vectors:

For any configuration of a Rubik's Cube there exists a corresponding **position vector** which is a 4 tuple (ρ, σ, v, w) where $\rho \in S_8, \sigma \in S_{12}$ describe the permutations of the cubies and $v \in \mathbb{Z}_3^8, w \in \mathbb{Z}_2^{12}$ describe the orientations of the cubies.

Remark: We will adopt Singmaster notation where we label the face with respect to it lying flat on a plane and one is facing the front face. The letters also represent a 90 degree clockwise turn of that face, the inverse of which is a 90 degree anticlockwise turn.

- Let F denote the front face.
- Let L denote the left face.
- Let U denote the upward (top) face.
- Let D denote the downward (bottom) face.
- Let R denote the right face.
- Let B denote the back face.

Conjugates:

For a group G with $g, h \in G$, the **conjugate** of g by h is $g^h = h^{-1}gh$

Commutators:

For a group G with $g, h \in G$ the **commutator** of g and h is $[g, h] = ghg^{-1}h^{-1}$

Relevant Theorems and Remarks

Remark on Orientation:

The orientation number for any facet is determined by comparing the primary orientation with the numbering of the colored tiles. Note that the center facets are unaffected by the moves so they do not require an orientation.

Theorem on Parity:

The cube always has even parity, or an even number of cubies exchanged from the starting position.

Second Fundamental Theorem of Cubology:

An operation of the cube is possible if and only if:

- The total number of edge and corner cycles of even length is even.
- The number of corner cycles twisted right is equal to the number of corner cycles twisted left (up to modulo 3).
- There is an even number of reorienting edge cycles.

Note that these theorems can be proved as part of the **Fundamental theorem of Cubology**

The Fundamental Theorem of Cubology

A move sequence is possible if and only if:

1. The permutation of the corner cubies has the same parity as the permutation of the edge cubies
$$\text{sign}(\rho) = \text{sign}(\sigma)$$
2. The number of corners that are twisted clockwise is equal to the number that are twisted counterclockwise modulo 3
$$v_1 + v_2 + \cdots + v_8 \equiv 0 \mod 3$$
3. The number of flipped edges is even
$$w_1 + w_2 + \cdots + w_{12} \equiv 0 \mod 2$$

Proof of the Fundamental Theorem of Cubology (Summarized)

The Rubik's cube consists of 54 facets and the entire set of arrangements of the Rubik's Cube are a subset of all possible permutations of these 54 facets denoted by the Rubik's Cube group i.e. $LRC_3 = \langle F, L, U, D, R, B \rangle \subset S_{54}$.

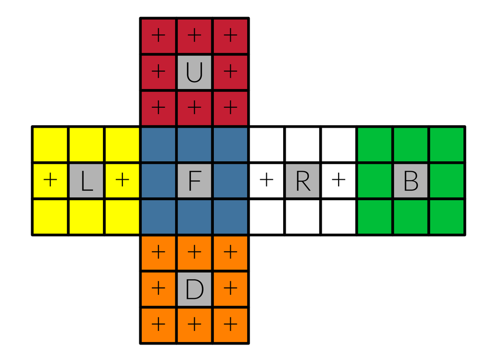

Figure 2. The primary orientation of the Cubies

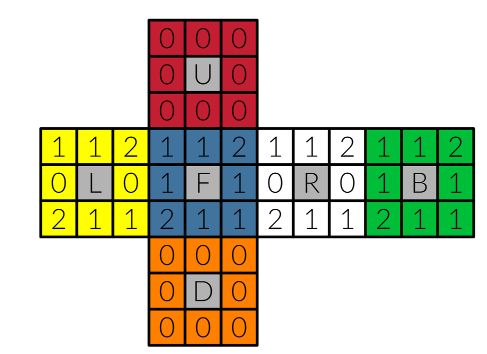

Figure 3. The numbering of the coloured tiles

($\Longrightarrow$) **We show that if X is a legal configuration then the three conditions hold.**

Essentially, the six position vectors (F, L, U, D, R, B) are perserved under these conditions.

$$F \to ((3,7,8,4),(3,7,11,8),(0,0,1,2,0,0,2,1),(0,0,0,0,0,0,0,0,0,0,0,0))$$
$$L \to ((1,4,8,5),(4,8,12,5),(2,0,0,1,1,0,0,2),(0,0,0,1,1,0,0,1,0,0,0,1))$$
$$U \to ((1,2,3,4),(1,2,3,4),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0,0,0,0,0))$$
$$D \to ((5,8,7,6),(9,12,11,10),(0,0,0,0,0,0,0,0),(0,0,0,0,0,0,0,0,0,0,0,0))$$
$$R \to ((2,6,7,3),(2,6,10,7),(0,1,2,0,0,2,1,0),(0,1,0,0,0,1,1,0,0,1,0,0))$$
$$B \to ((1,5,6,2),(1,5,9,6),(1,2,0,0,2,1,0,0),(0,0,0,0,0,0,0,0,0,0,0,0))$$

Notice that each permutation is a 4-cycle which is odd with sign -1, the sum of the component of each corner orientation vectors is either 0 or 6, both divisible by 3 and the sum of the components of each edge orientation vector is either 0 or 4, both divisible by 2.

($\Longleftarrow$) **We show that if the 3 conditions hold then X is a solvable configuration.**

For a legal configuration X with position vector (ρ, σ, v, w) assume that $\text{sign}(\rho) = \text{sign}(\sigma)$. Notice that the corner cubies and the edge cubies have even parity so the corner cubies can be returned to their originial position by performing 3-cycles on them. Call this configuration X' with position vector (ρ', σ', v', w'). Since X satisfies the three conditions X' also does with $\rho' = \epsilon, \sigma' = \epsilon$ because the solved state of the cube has position vector $(\epsilon, \epsilon, 0, 0)$.

We can solve all edge cubies since we have moves to flip any pair of edges. Now twist any clockwise or counterclockwise pairs to their original position. Notice that all leftover corner twists occur in triplets, either all clockwise or all counterclockwise and are solvable with the corner twisting move.

Constructing the Legal Rubik's Cube Group

The Illegal Rubik's Cube group is:

$$IRC_3 = (\mathbb{Z}_2^{12} \wr S_{12}) \times (\mathbb{Z}_3^8 \wr S_8)$$
$$|IRC_3| = 2^{12} \times 12! \times 3^8 \times 8! = 519024039293878272000$$

- The corner facets are represented by cyclic group of 3 elements $\mathbb{Z}_3$ and since there are 8 copies we denote it by $\mathbb{Z}_3^8$.
- The possible arrangements of the corner cubes are represented by S_8 (since we are permuting those), and postions of all the corner facets are represented by $\mathbb{Z}_3^8 \wr S_8$.
- Using a similar argument, the position of all the edge facets on the Rubik's Cube are represented by the group $\mathbb{Z}_2^{12} \wr S_{12}$.

The Legal Rubik's Cube group is a subgroup of the Illegal Rubik's Cube Group because there are configurations of the cube that cannot be realized by only using the rotaions.

$$LRC_3 \approx (\mathbb{Z}_3^7 \wr S_8) \times (\mathbb{Z}_2^{10} \wr S_{12}) \subset (\mathbb{Z}_2^{12} \wr S_{12}) \times (\mathbb{Z}_3^8 \wr S_8) \approx IRC_3$$
$$|LRC_3| = 2^{10} \times 12! \times 3^7 \times 8! = 43252003274489856000$$

Unsolvable Cubes

The following cubes are unsolvable.

1. **Exactly one edge cubie is flipped:** A singular edge cubie flipped is not possible since flipped edges occur in pairs.

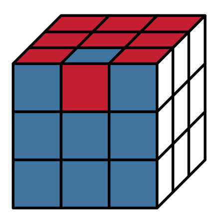

Figure 4. One edge cubie is flipped

2. **Corner twist:** A singular twisted corner is an impossible case since the number of corners twisted clockwise should be the same as those twisted counter clockwise modulo 3.

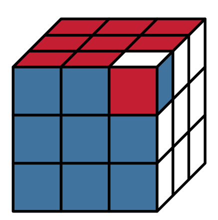

Figure 5. Corner twist

3. **Swapped pieces:** For both edge and corner pieces, exactly two adjacent or opposite pieces cannot be swapped.

Figure 6. Swapped pieces

4. **Two Corners twisted in the same direction** Exactly two corner cubies cannot be twisted in the same direction.

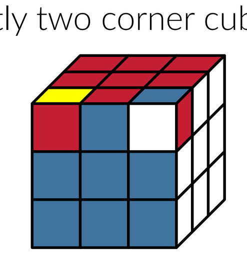

Figure 7. Two corner cubies twisted clockwise

How does this relate to MATH 4GR3?

When analyzing permutations, certain moves sequences are more effective than others like moves done by commutators.

If y and h are moves that fail to commute with each other by "a little bit", then $[y, h]$ will be close to the identity (i.e. the solved state).

For a permutation $\alpha \in LRC_3$ define $\text{fix}(\alpha)$ to be the set of all cubies that are not moved and the set of cubies moved as the support of α (denoted by $\text{supp}(\alpha)$):

$$\text{supp}(\alpha) = \overline{\text{fix}(\alpha)} = \{m \in \text{cubies} : \alpha(m) \neq m\}$$

Commutators provide us with a method for creating puzzles moves that affect a small number of pieces and Conjugation is a process for modifying the moves of the commutators to produce new moves that have similar structure.

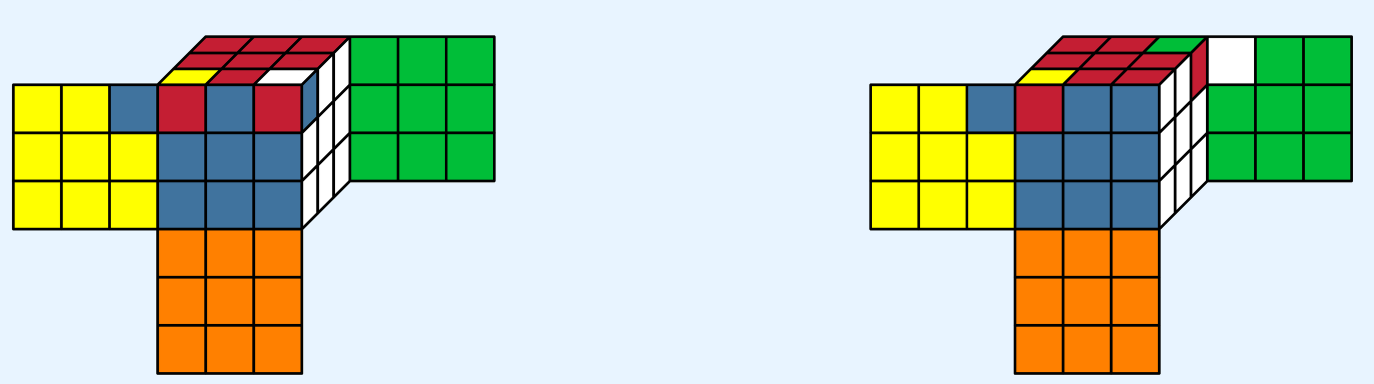

Figure 8. $x = [LD^2L^{-1}F^{-1}D^2F, U]$ twists adjacent corners, it can be conjugated as R^x to twist opposite corners

References

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