A new upper bound on Rubik's cube group

Silviu Radu

Lunds Institute of Technology
e-mail: silviu@bredband.net

August 5, 2018

Abstract

In this paper a computational method is introduced for proving that the diameter of the Rubik cube group must be less then or equal to 40 in the quarter turn metric. Recently it has been proven that when restricting the cube to edges the diameter is 18(see [5]). Now given an arbitrarily scrambled cube one can solve the edges in at most 18 moves. When the edges are in place there are 44089920 possibilities for the corners. We prove that each of these states can be solved in at most 22 moves giving a new upper bound of 18+22=40. The program to solve this problem is made in the GAP language and takes about 10 hours to verify on a standard 3Ghz PC. Later some suggestions of further improvements on the diameter are given.

General concepts

In this section we introduce some general notions. The Rubik's cube consists of 26 smaller cubies. We divide those cubies in three sets: the edge cubies ,the corner cubies and the center cubies. The edge cubies are those cubies that only have two colours and the corner cubies are those that have three colours. Finally the center cubies have only one color. So observe that an edge cubie has two faces and a corner cubie three faces.

We will work with GAP. GAP is a computer algebra program and has a lot of special functions that makes handling of sets, permutations and other mathematical structures very easy. A manual with all instructions can be found at [3]. When one wants to work with the cube group in GAP one has to specify its generators as permutations in cycle form. One suggestion of how this can be

done is given in Martin Schoenerts example [4]. Here the same example is used with a different enumeration:

```
25
                            26
                       1
                  8
                       \mathbf{U}
                             2
                 36
                            29
                       4
31
      6
           40
                 27
                       22
                            33
                                                               46
                                  45
                                        17
                                              44
                                                    38
                                                         11
3
     \mathbf{L}
           12
                 15
                       \mathbf{F}
                            18
                                   9
                                        \mathbf{R}
                                              13
                                                    5
                                                          \mathbf{B}
                                                               19
28
     20
           47
                 35
                       23
                            43
                                  37
                                        21
                                             48
                                                   30 24 42
                 32
                            39
                       16
                 14
                       \mathbf{D}
                            10
                       7
                 41
                            34
```

The picture above is a picture of the cube and we have enumerated each corner cubic face and each edge cubic face. U stands for up, L for left, R for right, D for down, F for front, B for back.

We use the same letters for the generators as for the faces. For example the generator U twists the face "up" clockwise. We define the generators by typing in GAP:

```
 \begin{array}{l} U\!:=\!(1,2,4,8)\,(6,11,17,22)\,(25,26,29,36)\,(27,31,38,45)\,(33,40,46,44)\,;\\ L\!:=\!(3,6,12,20)\,(8,15,14,19)\,(25,27,32,42)\,(28,31,40,47)\,(35,41,46,36)\,;\\ F\!:=\!(4,9,16,12)\,(15,22,18,23)\,(27,33,43,35)\,(29,37,32,40)\,(36,45,39,47)\,;\\ R\!:=\!(2,5,10,18)\,(9,17,13,21)\,(26,30,39,33)\,(29,38,34,43)\,(37,45,44,48)\,;\\ B\!:=\!(1,3,7,13)\,(5,11,19,24)\,(25,28,34,44)\,(26,31,41,48)\,(30,38,46,42)\,;\\ D\!:=\!(7,14,16,10)\,(20,23,21,24)\,(28,35,37,30)\,(32,39,34,41)\,(42,47,43,48)\,; \end{array}
```

Looking at these permutations one sees that the edges are numerated from 1 to 24 and the corners from 25 to 48. We generate the cube group by typing in GAP:

```
gap>cube:=Group(U,L,F,R,B,D);
cpermutation group with 6 generators>
```

Now we want to let the cube act on the points 1 to 48. And we want the orbit of this action. This is done by typing in GAP:

```
gap>Orbits(cube,[1..48],OnPoints);
[[1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22,23,24],[25,26,27,28,29,30,31,32,33,34,35,36,37,38,39,40,41,42,43,44,45,46,47,48]]
```

The response to the last command shows that points 1 to 24(edges) are permuted among themselves and the points 25 to 48(corners) are permuted among themselves. This means that each group element of the cube can be uniquely written as the product of a permutation on the set 1 to 24 and a permutation on the set 25 to 48. We will write it as (g_1, g_2) where g_1 is a permutation on the set 1 to 24 and g_2 is a permutation on the set 25 to 48. The edge group is the group with generators U, L, F, B, R, D restricted to 1 to 24 i.e. if we write the generators as $(u_e, u_c) = U, (l_e, l_c) = L, (f_e, f_c) = F, (b_e, b_c) = B, (r_e, r_c) = R, (d_e, d_c) = D$ then the edge group is generated by $u_e, l_e, f_e, b_e, r_e, d_e$. We denote it by R_e . The corner group is defined to be the group with generators u_c , l_c , f_c , b_c , r_c , d_c . The Rubik cube group will be denoted by G_{cube} . Note that neither the edge group or the corner group is a subgroup of the cube. Actually if we for example intersect the corner group with the cube group we get a group two times smaller then corner group. The analogue holds for the edge group. This can be verified in GAP. The intersection of the corner group with the cube group we will call "the group that fixes edges" or E_f and is very important in this text.

The direct product between the corner group and the edge group gives a group twice the size of Rubik cube group and Rubik cube group is a subgroup of index 2 of this direct product. This can be checked with GAP.

Each element in a group can be written as an word where the generators of the group and the inverses of the generators are the letters. The length of an element of the group is defined as the minimal number of letters required to express it. The length of an element g is written L(g) when we use notation L(g) we mean the length of an element in G_{cube} with respect to generators $\{U, L, F, B, R, D\}$. And we write $L_e(g)$ when we mean the length of an element in the group R_e with respect to generators $\{u_e, l_e, f_e, b_e, r_e, d_e\}$.

The Symmetry Group

```
Define M to be the group with generators k_1=(1,24,16,22)(2,19,10,15)(3,21,12,17)(4,11,7,23)(5,14,18,8)(6,13,20,9) (25,48,32,45)(26,41,39,36)(27,44,42,37)(28,43,40,38)(29,31,34,47)(30,35,33,46) and k_2=(1,5,7,19)(2,10,14,8)(3,11,13,24)(4,18,16,15)(6,17,21,20)(9,23,12,22) (25,44,34,28)(26,48,41,31)(27,33,43,35)(29,37,32,40)(30,42,46,38)(36,45,39,47). This group acts on the set T=\{U,L,F,B,R,D,U^{-1},L^{-1},F^{-1},B^{-1},R^{-1},D^{-1}\} by conjugation i.e. if g\in M and x\in T then g^{-1}xg\in T. The action of conjugation by k_1 can be written as a permutation with cycles: (U,B^{-1},D,F^{-1})(L,R^{-1})(F,U^{-1},B,D^{-1})(R,L^{-1}) and the action of k_2:
```

 $(U, R, D, L)(U^{-1}, R^{-1}, D^{-1}, L^{-1}).$

This action can be extended from T to the cube group. It is evident that two M conjugate elements have the same length. More details about this group can be found in [1].

Equivalence relations

There are two important equivalence relations that we will refer to often in this text:

- 1. Two elements g_1, g_2 are equivalent if there $\exists h \in M$ such that $g_1 = h^{-1}g_2h$. We refer to this relation as \sim .
- 2. Two elements g_1, g_2 are equivalent if there $\exists h \in M$ such that $g_1 = h^{-1}g_2h$ or $g_1^{-1} = h^{-1}g_2h$. We refer to this relation as \approx . In both cases two equivalent elements have the same length.

Calculation of the diameter of the Edge Group (R_e)

We will also use that the diameter of the Cayley graph with respect to generators $u_e, l_e, f_e, b_e, d_e, r_e$ defined above is 18. This table is presented below and was found in [5]. The first column gives the number of elements of lengths 0 to 18. The second column gives the number of equivalence classes under relation \sim and the second column the number of equivalence classes under relation \approx .

Pos	ted to Yahoo by 7	Tom Rokicki on S	Jan 2, 2004
Dis	t Positions	Unique wrt M	Unique wrt M+inv
0	1	1	1
1	12	1	1
2	114	5	5
3	1068	25	17
4	9819	215	128
5	89,392	1,886	986
6	807,000	16,902	8,652
7	7,209,384	150,442	75,740
8	63,624,107	1,326,326	665,398
9	552,158,812	11,505,339	5,759,523
10	4,643,963,023	96,755,918	48,408,203
11	36,003,343,336	750,089,528	375,164,394
12	208,075,583,101	4,334,978,635	2,167,999,621
13	441,790,281,226	9,204,132,452	4,603,365,303
14	277,713,627,518	5,785,844,935	2,894,003,596
15	12,144,555,140	253,044,012	126,739,897
16	23,716	750	677
17	30	3	3
18	1	1	1
	980,995,276,80	0 20,437,847,37	76 10,222,192,146

This calculation implies that given an arbitrary element in the cube group we need to multiply it by at most 18 generators (in other words to apply at most 18 moves to it) to take it in E_f (the group that fixes edges).

Mapping the corner group on numbers

It is well known that the corner group is isomorphic to $S_8 \ltimes \mathbb{Z}_3^7$. This means physically that each configuration in the corner group can be described as a permutation of the cubies and an orientation of the cubies. Let $\phi: R_c \mapsto S_8 \ltimes \mathbb{Z}_3^7$ be such a isomorphism. If $g \in R_c$ then $\phi(g)$ can be uniquely written as $\phi(g)_1\phi(g)_2$, $\phi(g)_1 \in S_8$, $\phi(g)_2 \in \mathbb{Z}_3^7$. It is obvious how to find a bijection from Z_3^7 to $[0, 3^7 - 1]$. Call this bijection ψ_1 . There is a simple way to define a bijective map $\psi_2: S_8 \mapsto [0, 8! - 1]$. One can define a natural order relation on permutations in S_8 which means that given two permutations σ_1, σ_2 and $\sigma_1 < \sigma_2$ then either $\sigma_1(1) < \sigma_2(1)$ or if n is the biggest positive integer with the property $\sigma_1(k) = \sigma_2(k), k = 1, ..., n$ then $\sigma_1(n+1) < \sigma_2(n+1)$. We simply define $\psi_2(\sigma) = \sharp \{\tau | \tau < \sigma\}$. Now simply define a bijection $f: R_c \mapsto [1, 88179840]$ as $f(g) = 3^7 \cdot \psi_2(\phi(g)_1) + \psi_1(\phi(g)_2)$.

The main objective of this paper

In this paper we show how to construct explicit expressions in generators of length 22 or less for each element in E_f ("the group that fixes edges"). This group has 44089920 elements, this can be shown by typing in GAP:

```
gap> cube:=Group(U,L,F,B,R,D);
<permutation group with 6 generators>
gap> corners:=Group(uc,lc,fc,bc,rc,dc);
<permutation group with 6 generators>
gap> Size(Intersection(corners,cube));
44089920
```

Since we need at most 18 moves to take an element to E_f and at most 22 moves to take an element of E_f to the identity. This means also that the diameter of the Rubik's cube group is less then or equal to 40.

Algorithm to express elements of E_f in generators

We need to introduce some notations. Define the following function $p:G_{cube} \mapsto R_e$ by $p((g_1,g_2))=g_1$. We call this map "projection on the first component". Now for a fix $g \in R_e$ define the set $S(n,g)=\{(g,x)\in G_{cube}|L((g,x))\leq n\}$ i.e. the set of all \mathbf{g} in the cube group that can be expressed as a product of at most n generators and $p(\mathbf{g})=g$ for all $\mathbf{g}\in S(n,g)$. One can easily

see that S(n,g) is empty if $L_e(g) > n$. Define $S(k,g) \cdot S(n,h) = \{x \cdot y | x \in S(k,g), y \in S(n,h)\}$. It is obvious that $\bigcup_{g \in R_e} S(10,g) \cdot S(12,g^{-1}) = S(22,id)$. So $\bigcup_{g \in R_e, L_e(g) \le 10} S(10,g) \cdot S(12,g^{-1}) = S(22,id)$.

Following relations are helpful when trying to understand the algorithm we will describe:

- 1. $h^{-1}S(q,n)h = S(h^{-1}qh,n), h \in M$
- $2. \ h^{-1}S(k,g) \cdot S(n,g^{-1})h = S(k,h^{-1}gh) \cdot S(n,h^{-1}g^{-1}h), h \in M.$
- 3. $(S(q,n))^{-1} = S(q^{-1},n)$

In order to keep track on which elements we have found generator expressions for, a "bit vector" is used. A "bit vector" is a vector in which each entry can take only two values 0 or 1. GAP uses "false" for 0 and "true" for 1. Initially all entries in this "bit vector" are set to 1 or "true". A bijection $f: R_c \mapsto [1,88179840]$ was defined on page 6. When generator expressions for a given element $g \in E_f$ have been found entry number f(g) is set to 0 or "false". When we use the expression "element g has been checked of" we mean that entry f(g) in the "bit vector" has been set to "false".

We also need to collect information about how a given element can be expressed in generators. In GAP we use for this a set we call "positions". The algorithm we describe stores only one representative from each equivalence class under relation \approx . When we write "store generator expression" we mean that the generator expression will be saved to the set "positions".

Now we are ready to describe the main step of the algorithm: Define A to be the set $A = \{g \in R_e | L_e(g) \le 10\}$. Fix a $g \in A$ and do following steps:

- 1. Compute S(10,q) and generator expressions for its elements
- 2. Compute $S(12, q^{-1})$ and generator expressions for its elements
- 3. Compute $S(10,g) \cdot S(12,g^{-1})$ and generator expressions for its elements
- 4. For each element elm in $S(10,g) \cdot S(12,g^{-1})$ do following steps:
 - (a) Check if elm has been checked. If yes do nothing if no do the following steps
 - i. Store generator expression of elm
 - ii. Compute the orbit of elm under relation \approx .
 - iii. Check of every element in the orbit.
- 5. Compute $S(10, g^{-1}) \cdot S(12, g)$ and generator expressions for its elements

- 6. For each element elm in $S(10, g^{-1}) \cdot S(12, g)$ do following steps:
 - (a) Check if elm has been checked. If yes do nothing if no do the following steps
 - i. Store generator expression of elm.
 - ii. Compute the orbit of elm under relation \approx .
 - iii. Check of every element in the orbit.

Let O(g) be the orbit of g under relation \approx . Then the above algorithm checks of every element in $\bigcup_{O(g)} S(10,g) \cdot S(12,g^{-1})$.

The algorithm consists of repeating above procedure for enough elements $g \in A$ which are not equivalent under relation \approx . It should be noted that building S(12,g) is the most time consuming step in the algorithm. Once we have S(12,g) it takes no time to build $S(12,h^{-1}gh)$. Thats why we do the above steps for the whole equivalence class.

If we for example run the algorithm with g choused as identity than 3079007 elements are checked of.

Optimal solvers

As mentioned above S(22, id) can be written using following formula:

$$\bigcup_{g \in R_e, L_e(g) \le 10} S(10, g) \cdot S(12, g^{-1}) = S(22, id)$$

In the above algorithm we see how to use it in order to compute each element in E_f as a product of generators. Obviously some elements we compute might have length that is less then 22. To compute those we use the same algorithm with the exception that S(10,g) is replaced by S(8,g). And S(20,id) can be written as:

$$\bigcup_{g \in R_e, L_e(g) \le 8} S(8, g) \cdot S(12, g^{-1}) = S(20, id)$$

Now we have all elements of length less then or equal to 20 in E_f . To get only elements of length exactly 22 we must take the difference between S(22, id) and S(20, id). We continue computing the set S(18, id) using the formula:

$$\bigcup_{g \in R_e, L_e(g) \le 8} S(8, g) \cdot S(10, g^{-1}) = S(18, id)$$

And finally we compute S(16, id), S(14, id), S(12, id), S(10, id) and S(8, id). To compute elements of length n we simply take the difference S(n, g) - S(n - 1, g). The program for this calculation is not given here but we give the result on next page. Observe that 22q means length 22.

Distance	Nr of pos	Unique wrt M	Unique wrt M + inv
0q	1	1	1
2q	0	0	0
4q	0	0	0
6q	0	0	0
8q	240	5	3
10q	288	6	3
12q	8764	197	113
14q	116608	2475	1303
16q	840513	17620	9091
18q	9884342	206424	104523
20q	30623418	639362	323859
22q	2615746	54895	28528
_			
	44089920	920980	467424

We have also computed the length of all elements in the "Group that fixes cubies". This is the set of all elements in the Rubik cube group such that the cubies are in their correct spot but their orientation is arbitrary. This is an abelian normal subgroup of the cube. This can be checked with GAP. The order of this group is 4478976. Here is the table with lengths of each element:

Distance	Nr of pos	Unique wrt M	Unique wrt M + inv
0q	1	1	1
2q	0	0	0
4q	0	0	0
6q	0	0	0
8q	0	0	0
10q	0	0	0
12q	441	11	8
14q	3944	87	52
16q	32110	708	396
18q	456025	9656	5009
20q	2873194	60399	30978
22q	1113236	23652	12424
24q	25	4	4
	4478976	94518	48872

One of the positions of length 24 is the well known "superflip". This position alone generates the center of the Rubik cube group. The other positions up to conjugation with M are:

```
U, B
                          R
                             B2 D
                                   В
                                      R' U' F' B R U2 D' (24q*,21f)
R' F2 R' U
           F
              U
                    B2 R
                          В
                             D2 F
                                            В
                                              L' D' R' U'
                 L
                                  R' B' D
                                                           (24q*, 21f)
              R' D2 B' D' L
                             F L2 U
                                      B' U
                                            F L2 U' B'
                                                           (24q*, 20f)
```

This positions were the only ones not computable in 22q and the expressions in generators were found using Michael Reids solver see [2].

Upperbounds on different subsets

Take an element $g \in R_e$ and L(g) = n (L(g) is the length of g) in R_e . Then for any element $(g,y) \in G_{cube}$ we need at most n moves to make it an element of the form $(id,x) \in G_{cube}$. And an element $(id,x) \in G_{cube}$ needs at most 22 moves applied on it to make it the identity. So $L((g,y)) \leq 22 + n$ in G_{cube} . There is only one element h in R_e with L(h) = 18 in R_e (according to the above diagram). So the only elements in G_{cube} that may need 40 moves are those in the set $S = \{(h,x) \in G_{cube}\}$. If a solver solves each position in S in less then 40 moves then one gets a new upper bound 39. Actually one only needs to solve a position from each equivalence class under relation \approx which gives about 500000 positions needed to be solved. This computation has not been done here. But soon after the author posted generator expressions for elements in E_f this has been done by another author. See [6] for details.

The program

To begin with we are going to create some sets we are going to work with. The sets will be presented in GAP as lists. The first one is **s**. This set is made by typing in gap:

```
gap>s:=[U,L,F,B,R,D,U^-1,L^-1,F^-1,B^-1,R^-1,D^-1];
```

The other sets are:

Some sets

veven the set of all elements in the edge group of length 0, 2, 4, 6. **vodd** the set of all elements in the edge group of length 1, 3, 5. **vevenr** the set of all elements in the Rubik cube group of length 0, 2, 4, 6.

voddr the set of all elements in the Rubik cube group of length 1, 3, 5. The elements of this sets are stored as permutations with disjoint cycles.

vevenc Contains at entry k the same element that **vevenr** contains at entry k but stored as products of generators. Throughout this program we store a product of generators as a vector whose entries are integers between 1 and 12. Integer n means generator s[n] and s was defined above.

voddc is analogue to vevenc but the set vevenr is replaced with voddr.

All the sets containing permutations with cycles are sorted with respect to the natural order relation on permutations which means that given two permutations σ_1, σ_2 and $\sigma_1 < \sigma_2$ then either $\sigma_1(1) < \sigma_2(1)$ or if n is the biggest positive integer with the property $\sigma_1(k) = \sigma_2(k), k = 1, ..., n$ then $\sigma_1(n+1) < \sigma_2(n+1)$. The elements in the set **vevenr** are ordered in following way:

 $(g_1, h_{k_1}), (g_1, h_{k_2}), ..., (g_1, h_{k_m}), (g_2, h_{l_1}), (g_2, h_{l_2}),$ This is because the first component is a permutation on the set 1 to 24 while the second component a permutation on the set 25 to 48. Then the set **veven** is ordered in following way:

 g_1, g_2, \dots Note that we refer to an element of **vevenr** as an element of the form (g, h) although it is stored as a permutation with disjoint cycles.

We use two more vectors $\mathbf{y1}$ and $\mathbf{y2}$ where the first has the same size as **vodd** and the second the same size as **veven**. The vector $\mathbf{y1}$ contains at position k a number that gives the position in **voddr** where we have the first occurrence of an element of the form $(\mathbf{vodd}(k), h)$.

Finally we call the "bit vector" for \mathbf{rcv} which has the same length as R_c (88179840 elements). When the program starts all its entries are set to true.

Some functions

We will predefine some functions in order to make the program code easier:

MakeSetv(n) if n=1 then it returns set [vevenr, voddr] if n=2 returns set [veven, vodd].

MakeSetvc() returns [vevenc, voddc].

MakeSety(n) if n=1 returns set y1 if n=2 returns set y2.

MakeSet(\mathbf{n} , \mathbf{g}) This function will give a set of the form (g, h), $h \in R_c$ and all its elements of length n or less. The returned set is of the form [set1, set2] where the elements of set1 are expressed as the elements of vevenr and the elements of set2 are expressed as products of generators.

PermToNr(g) This is a bijective function $PermToNr : R_c \mapsto [1,88179840]$.

EqClassRel2(el) Returns the set $\{PermToNr(g)|g\approx el\}$.

WriteRcv(set1) For each $x \in set1$ does rcv(x) = false.

 $\mathbf{Redf}(\mathbf{set1})$.Partitions set set1 into equivalence classes under relation \approx defined above and returns a set containing a representative from each class.

CharToPerm(\mathbf{x}) is a function that converts a text string to a permutation. This was done so that the attached data don't take to much space in this paper. **MultEl**(\mathbf{x}) if for example x := [1, 2, 1, 4, 11, 2] then this function returns s[1] * s[2] * s[1] * s[4] * s[1] * s[2].

s[2] * s[1] * s[4] * s[11] * s[2]. s[2] * s[1] * s[4] * s[11] * s[2]. s[2] * s[1] * s[4] * s[11] * s[2]. s[2] * s[1] * s[4] * s[11] * s[2]. s[2] * s[1] * s[4] * s[11] * s[2]. s[2] * s[1] * s[4] * s[11] * s[2]. s[2] * s[1] * s[4] * s[11] * s[2]. s[2] * s[1] * s[4] * s[11] * s[2]. s[2] * s[4] * s[4] * s[11] * s[2].s[2] * s[4] * s

The Code

Before we describe the code some GAP commandos must be explained. Those can also be found in the manual [3] but we describe them here for completeness. To build a set in GAP one types:

```
gap>set:=[];
To build a set of sets type:
gap>set:=[[],[],[],[]];
The above set contains 4 empty subsets. To store the identity in the first subset type in GAP:
gap>set[1][1]:=();
Now the first set in the set "set" contains the identity.
To build a "bit vector" of size 1000 with all entries set to "true" or 1 just type:
gap>vec:=BlistList([1..1000],[1..1000]);
To build a "bit vector" with all entries false just type:
gap>vec:=BlistList([1..1000],[]);
If we type in GAP:
gap>set:=[1,1];
then we have build a set that contains the same element on two different entries, maybe it is better to call it a list. But if we type:
```

then the set "set2" will contain only element 1. If we instead of union would have taken intersection the same thing should have happened. A union, intersection or difference of two sets will never return a set with doubles. Furthermore after performing those operations the set will even be sorted.

gap>set2:=Union(set,set);

Generators and the Symmetry Group

We define all generators we are going to use and the symmetry group in following file:

```
#Generators for the cube group
U:=(1,2,4,8)(6,11,17,22)(25,26,29,36)(27,31,38,45)(33,40,46,44);
L:=(3,6,12,20)(8,15,14,19)(25,27,32,42)(28,31,40,47)(35,41,46,36);
F := (4,9,16,12) (15,22,18,23) (27,33,43,35) (29,37,32,40) (36,45,39,47);
R := (2,5,10,18) (9,17,13,21) (26,30,39,33) (29,38,34,43) (37,45,44,48);
B := (1,3,7,13)(5,11,19,24)(25,28,34,44)(26,31,41,48)(30,38,46,42);
D := (7,14,16,10)(20,23,21,24)(28,35,37,30)(32,39,34,41)(42,47,43,48);
#Generators for edge group an corner group
ue:=RestrictedPerm(U,[1..24]);uc:=RestrictedPerm(U,[25..48]);
le:=RestrictedPerm(L,[1..24]);lc:=RestrictedPerm(L,[25..48]);
fe:=RestrictedPerm(F,[1..24]);fc:=RestrictedPerm(F,[25..48]);
re:=RestrictedPerm(R,[1..24]);rc:=RestrictedPerm(R,[25..48]);
be:=RestrictedPerm(B,[1..24]);bc:=RestrictedPerm(B,[25..48]);
de:=RestrictedPerm(D,[1..24]);dc:=RestrictedPerm(D,[25..48]);
#Generators for the group of symmetries
k1 := (1,24,16,22)(2,19,10,15)(3,21,12,17)(4,11,7,23)(5,14,18,8)(6,13,20,9)
(25,48,32,45)(26,41,39,36)(27,44,42,37)(28,43,40,38)(29,31,34,47)(30,35,33,46);
k2 := (1,5,7,19)(2,10,14,8)(3,11,13,24)(4,18,16,15)(6,17,21,20)(9,23,12,22)
(25,44,34,28)(26,48,41,31)(27,33,43,35)(29,37,32,40)(30,42,46,38)(36,45,39,47);
#Generators for the group of symmetries restricted to edge resp corners.
k1e:=RestrictedPerm(k1,[1..24]);k2e:=RestrictedPerm(k2,[1..24]);
k1c:=RestrictedPerm(k1, [25..48]); k2c:=RestrictedPerm(k2, [25..48]);
#symmetry group
M:=Group(k1,k2);
#symmetry group restricted to corners
Mc:=Group(k1c,k2c);
#symmetry group restricted to edges
Me:=Group(k1e,k2e);
```

This code will be saved to file gens.txt.

Function MakeSetv

This function builds the sets veven, vodd, vevenr, voddr. The function first selects generators depending on if we want to build veven, vodd or vevenr, voddr. If we want to build vevenr, voddr we type:

```
gap>a:=MakeSetv(1);
```

The function will return a set of sets containing the sets vevenr and voddr. In the above case a[1] is vevenr and a[2] is voddr. Lets describe the algorithm that this program uses:

1. First a set of sets is build containing 6 empty sets and a set containing the identity:

```
v := [[()], [], [], [], [], [], []]
```

- 2. Now take each element v[1] and multiply it be each generator and store the result in the next set i.e. v[2].
- 3. Now take v[2] and multiply each element of it with the generators and store the result in v[3]. Remove doubles from v[3] and elements contained in v[1].
- 4. Now take the next set in v i.e. v[3] multiply each element by generators, store the result in v[4] remove doubles from v[4] and elements contained in v[2].
- 5. Continue doing this procedure until v[6] i.e. v[6] is the last set of which elements will me multiplied by generators and the result stored in v[7]
- 6. Now return the set $\{Union(v[1],v[3],v[5],v[7]),Union(v[2],v[4],v[6])\}$ i.e. $\{veven,vodd\}$.

See next page for the code of above described function:

```
MakeSetv:=function(n)
local veven, vodd, vevenr, voddr, gens, selct, v, k, x, z, y, cnt;
gens:=[[()],[()]];selct:=n;
gens[1]:=[U,L,F,B,R,D,U^-1,L^-1,F^-1,B^-1,R^-1,D^-1];
gens[2]:=[ue,le,fe,be,re,de,ue^-1,le^-1,fe^-1,be^-1,re^-1,de^-1];
v:=[[()],[()],[()],[()],[()],[()];
for y in [1..6] do
   cnt:=1;
   for x in v[y] do
        for z in [1..12] do
           k[z][cnt]:=x*gens[selct][z];
        od;
        cnt:=cnt+1;
   od;
   v[y+1]:=Union(k[1],k[2],k[3],k[4],k[5],k[6],k[7],k[8],k[9],k[10],k[11],k[12]);
   if y>1 then v[y+1]:=Difference(v[y+1],v[y-1]);fi;
if selct=1 then
   vevenr:=Union(v[1],v[3],v[5],v[7]);voddr:=Union(v[2],v[4],v[6]);
   return [vevenr, voddr];
elif selct=2 then
   veven:=Union(v[1],v[3],v[5],v[7]); vodd:=Union(v[2],v[4],v[6]);
   return [veven, vodd];
fi;
end;;
```

This code will be saved to file makesetv.txt.

Function MakeSetvc

This function will build the sets vevenc and voddc. We remind that this sets contain the same number of elements as vevenr and voddr. We have defined a list "s" above as $s = [U, L, F, B, R, D, U^{-1}, L^{-1}, F^{-1}, B^{-1}, R^{-1}, D^{-1}]$. To define it in GAP just type:

```
gap>s:=[U,L,F,B,R,D,U^{-1},L^{-1},F^{-1},B^{-1},R^{-1},D^{-1}];
```

We now want to store generator expressions for elements in vevenr and voddr in sets vevenc and voddc if for instance vevenc contains at position 10 generator expression [1,2,7] then vevenr should contain at position 10 element s[1]*s[2]*s[7]. We will describe an algorithm how to achieve this. But first we need a few preparations. GAP has a function called "Tuples" that we will use. We explain it by giving an example:

```
gap>Tuples([1..5],2);
```

After typing this GAP will return a set containing all possible elements of form (x, y) with $x \in \{1, 2, 3, 4, 5\}, y \in \{1, 2, 3, 4, 5\}$. And we also use a function called PositionSet. We explain it using an example:

```
gap>PositionSet(veven,(1,2,3));
```

If (1,2,3) is contained in veven then this function will return a number n such that veven[n]=(1,2,3). If it is not contained it will just return fail. Now we are ready to describe the algorithm. It is described only for vevenc since voddc is analogue:

- 1. Build a "bit vector" with same size a vevenr with all elements set to "false".
- 2. Build the set of all tuples on elements $\{1,2,3,4,5,6,7,8,9,10,11,12\}$ and length 2. Call this set Tu2.
- 3. Take each element $(x,y) \in Tu2$ and find the position in vevenr of the element s[x] * s[y]. Call this position n. If entry number n is "false" in the "bit vector", set it to "true" and put element (x,y) in vevenc at position n. If entry number n in the "bit vector" is "true" then do nothing.
- 4. Repeat step 2 and 3 for tuples of length 4 and 6 on the set $\{1,2,3,4,5,6,7,8,9,10,11,12\}$.

The code for this function is given on next page.

```
s := [U,L,F,B,R,D,U^{-1},L^{-1},F^{-1},B^{-1},R^{-1},D^{-1}];
MakeSetvc:=function()
local binv,binv2,vevenc,voddc,x,set,z,elm,g,pos;
binv:=BlistList([1..Size(vevenr)],[]);
binv2:=BlistList([1..Size(voddr)],[]);
vevenc:=[];
voddc:=[];
for x in [1..6] do
if x=1 or x=3 or x=5 then
set:=Tuples([1..12],x);
for z in set do
elm:=();
for g in [1..Size(z)] do
            elm:=elm*s[z[g]];
od;
pos:=PositionSet(voddr,elm);
if binv2[pos]=false then voddc[pos]:=z;binv2[pos]:=true;fi;
fi;
if x=2 or x=4 or x=6 then
set:=Tuples([1..12],x);
for z in set do
elm:=();
for g in [1..Size(z)] do
elm:=elm*s[z[g]];
pos:=PositionSet(vevenr,elm);
if binv[pos]=false then vevenc[pos]:=z;binv[pos]:=true;fi;
od;
fi;
od;
vevenc[1]:=[];
return[vevenc, voddc];
end;;
```

Function MakeSety

As described before vector y1 has the same size as vodd and vector y2 the same size as veven. We only describe how to build y1 since y2 is analogue. Remember that if an element $g \in R_e$ has position n in veven then y[n] contains a number m that gives the position in vevenr of the first element who's projection on R_e is g. By projection we mean the map $(g, x) \mapsto g$. We use following algorithm to build y2:

- 1. Build a vector "vtemp" the same size as vevenr that contains at position n the projection of vevenr[n] on R_e .
- 2. Do following for each $g \in veven$.
 - (a) Find position of g in veven call it "pos"
 - (b) Find the position of the first occurrence of g in vtemp call it "pos2".
 - (c) Set y2[pos]=pos2

The code is described below:

```
MakeSety:=function(n) local vtemp,v,vr,x,y,cnt,g;
vtemp:=[()];cnt:=1;
if n=1 then v:=vodd;vr:=voddr;
elif n=2 then v:=veven;vr:=vevenr;
fi;
x:=[1..Size(vr)];y:=[1..Size(v)];
for g in x do
    vtemp[g]:=RestrictedPerm(vr[g],[1..24]);
od;
for g in v do
    y[PositionSorted(v,g)]:=PositionSorted(vtemp,g);
od;
return y; end;;
```

This code will be saved to file makesety.txt

Function PermToNr

Here is the program described in the section "How to map a permutation on a number". It takes a $g \in R_c$ and maps it on $S_8 \ltimes \mathbb{Z}_3^7$. Lets describe how to map a permutation of S_8 on numbers [1,8!] in more detail. Given a $\sigma \in S_8$ we want to determine all elements that are less then σ with respect to the order relation

that was defined on page 11. This is a simple combinatorial problem and we give an example of how to determine the number of elements less then a given permutation and then generalize this to an arbitrary one. Take for example $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 3 & 4 & 2 & 1 & 8 & 7 \end{pmatrix}$. We divide the set P of permutations less than σ in following disjoint subsets:

- 1. The subset with $\{\tau(1) < 5, \tau \in P\}$. Then $\tau(1)$ can be any element in the set $\{1, 2, 3, 4\}$ so this set has size $7! \cdot 4$
- 2. The subset with $\{\tau(1) = 5, \tau \in P\}$. Then $\tau(2)$ can be any element in the set $\{1, 2, 3, 4\}$ so the size of this set is $6! \cdot 4$.
- 3. The subset with $\{\tau(1) = 5, \tau(2) = 6, \tau \in P\}$ then $\tau(3)$ can be any element in the set $\{1, 2\}$ and the size of this set is $5! \cdot 2$.
- 4. The subset with $\{\tau(1) = 5, \tau(2) = 6, \tau(3) = 3, \tau \in P\}$ then $\tau(4)$ can be any element in the set $\{2, 1\}$. The size of this subset is $4! \cdot 2$.
- 5. The subset with $\{\tau(1) = 5, \tau(2) = 6, \tau(3) = 3, \tau(4) = 4\}$ then $\tau(5), \tau \in P$ is in the set $\{1\}$. The size of this set is 3!.
- 6. The subset with $\{\tau(1) = 5, \tau(2) = 6, \tau(3) = 3, \tau(4) = 4, \tau(5) = 2, \tau \in P\}$ and $\tau(6)$ can be any element in the set $\{\}$. The size of this subset is 0.
- 7. The subset with $\{\tau(1) = 5, \tau(2) = 6, \tau(3) = 3, \tau(4) = 4, \tau(5) = 2, \tau(6) = 1, \tau \in P\}$ then $\tau(7)$ can be any element in the set $\{7\}$. The size of this subset is 1!.

Since all of this subsets are disjoint and their union is P it follows that the size of P is $4 \cdot 7! + 4 \cdot 6! + 2 \cdot 5! + 2 \cdot 4! + 3! + 0 + 1!$. This can be easily generalized for an arbitrary σ . Let $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) & \sigma(6) & \sigma(7) & \sigma(8) \end{pmatrix}$ then the number of permutations less than σ is

$$\sum_{k=1}^{7} (7 - k + 1)! \cdot (\sigma(k) - 1) \cdot \sharp \{ \sigma(1) < \sigma(k) \mid l \in \{1, ..., k - 1\} \}$$

The code for bijection $f: R_c \mapsto [1,88179840]$ can be seen on next page.

```
PermToNr:=function(g) local m1,m2,m3,m4,m5,m6,m7,m8,y,w,nr,k1,k2;
y := [1..48];
v[25] := 1; v[31] := 2; v[46] := 3; v[26] := 1; v[38] := 2; v[44] := 3; v[27] := 1; v[36] := 2; v[40] := 3;
y[28] := 1; y[41] := 2; y[42] := 3; y[29] := 1; y[33] := 2; y[45] := 3; y[30] := 1; y[34] := 2; y[48] := 3;
y[32] := 1; y[35] := 2; y[47] := 3; y[37] := 1; y[39] := 2; y[43] := 3; w := [1..48];
w[25]:=1;w[31]:=1;w[46]:=1;w[26]:=2;w[38]:=2;w[44]:=2;w[27]:=3;w[36]:=3;w[40]:=3;
w[28] := 4; w[41] := 4; w[42] := 4; w[29] := 5; w[33] := 5; w[45] := 5; w[30] := 6; w[34] := 6; w[48] := 6;
w[32] := 7; w[35] := 7; w[47] := 7; w[37] := 8; w[39] := 8; w[43] := 8;
m1:=w[25^g];m2:=w[26^g];m3:=w[27^g];m4:=w[28^g];m5:=w[29^g];m6:=w[30^g];m7:=w[32^g];
m8:=w[37^g]; if m1<m2 then m2:=m2-1; fi; if m1<m3 then m3:=m3-1; fi;
if m2 < m3 then m3:=m3-1; fi; if m1 < m4 then m4:=m4-1; fi; if m2 < m4
then m4:=m4-1; fi; if m3 < m4 then m4:=m4-1; fi; if m1 < m5 then
m5:=m5-1; fi; if m2<m5 then m5:=m5-1; fi; if m3<m5 then m5:=m5-1;
fi; if m4 < m5 then m5 := m5 - 1; fi; if m1 < m6 then m6 := m6 - 1; fi; if m2 < m6
then m6:=m6-1; fi; if m3<m6 then m6:=m6-1; fi; if m4<m6 then
m6:=m6-1; fi; if m5<m6 then m6:=m6-1; fi; if m1<m7 then m7:=m7-1;
fi; if m2 < m7 then m7:=m7-1; fi; if m3 < m7 then m7:=m7-1; fi; if m4 < m7
then m7:=m7-1; fi; if m5 < m7 then m7:=m7-1; fi; if m6 < m7 then
m7:=m7-1; fi;
k1:=(m1-1)*7*6*5*4*3*2+(m2-1)*6*5*4*3*2+(m3-1)*5*4*3*2+(m4-1)*4*3*2+
(m5-1)*3*2+(m6-1)*2+m7-1;
k2 := (y[25^g]-1)*3^0+(y[26^g]-1)*3^1+(y[27^g]-1)*3^2+(y[28^g]-1)*3^3+
(y[29^g]-1)*3^4+(y[30 ^g]-1)*3^5+(y[32^g]-1)*3^6; nr:=2187*k1+k2+1;
return nr; end;;
This code will be saved to file permtonr.txt.
Nobody is expected to follow this code. That is why a program is given to verify
that this function is a bijection and it is all that is needed for the proof:
corners:=Group(uc,lc,fc,bc,rc,dc); rcv:=BlistList([1..88179840],[]);
for x in corners do
rcv[PermToNr(x)]:=true;
```

If the program returns true then it is a surjection and consequently a bijection.

Print(rcv=BlistList([1..88179840],[1..88179840]),"\n");

The function MakeSet(n,g)

This function builds S(10, g) and S(12, g). To build S(10, g) just type:

```
gap>MakeSet(10,g);
```

We explain the algorithm for S(12, g)

- 1. Build an empty set S
- 2. For each element h in veven do
 - (a) Build the set $S_h = \{(h, x) \in vevenr\}$ and the set $S_{h^{-1}g} = \{(h^{-1}g, x) \in vevenr\}$.
 - (b) Build the set $\{g_1\cdot g_2\mid g_1\in S_h,g_2\in S_{h^{-1}g}\}$ and put each element from this set in S.
- 3. Return the set S (or the set S = S(12, g)

The implementation of this function as code can be seen on next page.

```
MakeSet:=function(n,g) local set,y,veo,veor,a,b,z,x,kl,j,k,l,m,t,set2,set3,nre,veoc;
if n=10 then
y:=y1;veo:=vodd;veor:=voddr;veoc:=voddc;
elif n=12 then
y:=y2;veo:=veven;veor:=vevenr;veoc:=vevenc;
set:=[];t:=[1..Size(veo)];kl:=Size(veo);set2:=[];set3:=[];nre:=0;
for x in t do
 if x=kl then
    a:=Size(veor)+1-y[x];
 else
    a:=y[x+1]-y[x];
 fi;
 z:=PositionSet(veo,veo[x]^-1*g);
   if z<>fail then
      if z=Size(veo) then b:=Size(veor)+1-y[z];
      else b:=y[z+1]-y[z];
     fi;
 1:=[1..a];m:=[1..b];
     for j in m do
        for k in 1 do
          if PositionSet(set,veor[y[x]+k-1]*veor[y[z]+j-1])=fail then nre:=nre+1;
          AddSet(set, veor[y[x]+k-1]*veor[y[z]+j-1]);
          set2[nre]:=veor[y[x]+k-1]*veor[y[z]+j-1];
          set3[nre]:=Concatenation(veoc[y[x]+k-1],veoc[y[z]+j-1]);
          fi;
       od;
    od;
    fi;
od;
return [set2,set3];end;;
```

This code will be saved to file makeset.txt

The function EqClassRel2(el)

This function returns a set that contains the image under the bijection $f: R_c \mapsto [1,88179840]$ of every element equivalent to el under relation \approx . The algorithm is evident and need not to be explained.

```
EqClassRel2:=function(g) local h,set;
set:=[];
for h in Mc do
   AddSet(set,PermToNr(h^-1*g*h));
   AddSet(set,PermToNr(h^-1*g^-1*h));
od;
return set; end;;
```

This code will be saved to file eqclassrel2.txt

The function WriteRcv(set)

This function has as input a set of indices or numbers and is called by command:

```
gap>WriteRcv(set);
```

The command above takes every $n \in set$ and sets entry n in the "bit vector" to "false" or 0.

```
WriteRcv:=function(set)
local x;
for x in set do
  rcv[x]:=false;
od;
end;;
```

This code will be saved to file writercv.txt

The function Redf(set)

As mentioned earlier in the text the algorithm finds generator expressions for elements in E_f and each time it does this we need to give it a $g \in R_e$. It was also mentioned that if given the algorithm a $g \in R_e$ that is equivalent under relation \approx with another $h \in R_e$ that has been before, then this $h \in R_e$ will not yield any new generator expressions. Redf(set) returns a set containing only one representative from each equivalence class under relation \approx . The algorithm is simple and can be seen directly by looking at the code.

```
Redf:=function(set) local x,y,set2,setemp,vec;
vec:=BlistList([1..Size(set)],[1..Size(set)]);set2:=[];
for x in [1..Size(set)] do
    setemp:=[];
    if vec[x] then
        for y in Me do
            vec[PositionSet(set,y^-1*set[x]*y)]:=false;
            vec[PositionSet(set,y^-1*set[x]^-1*y)]:=false;
        AddSet(setemp,y^-1*set[x]*y);
        AddSet(setemp,y^-1*set[x]^-1*y);
        od;
        if IsEmpty(Intersection(setemp,set2)) then AddSet(set2,set[x]); fi;
        fi;
od;
return set2; end;;
```

This code will be saved to file redf.txt

The function CharToPerm(x)

Part of the $g \in R_e$ we are giving as input to the algorithm will come from the set Redf(veven). With this set we find generator expression for 43984820 elements in E_f . To find expressions for the rest of 105100 we need some more $g \in R_e$ that has been extracted from a large list and are given at the end of this paper. This list took about 2 days to find on 20 computers. It consisted of representatives under relation \approx of elements of the set $\{g \in R_e, L(g) = 8\}$ and very few from $\{g \in R_e, L(g) = 10\}$. From the last set only 17. Those were found using a solver found at [2]. In order to compress the space the elements have been well coded and in order to make them permutations in R_e we need the function CharToPerm(x). If we want to convert an element of this list to a permutation we type in GAP:

```
gap>CharToPerm("acekrihownvs");
(4,6,5,21,23,15,7,22,8,13,10,16,12,24)(14,20)
```

Since this is just something technical it needs no further explanation.

```
CharToPerm:=function(g)
local s,v,w,m,k;
v := [1..24];
v[1] := 11; v[2] := 17; v[3] := 19; v[4] := 22; v[5] := 13; v[6] := 8; v[7] := 24; v[8] := 6; v[9] := 18;
v[10] := 21; v[11] := 1; v[12] := 15; v[13] := 5; v[14] := 20; v[15] := 12; v[16] := 23; v[17] := 2;
v[18] := 9; v[19] := 3; v[20] := 14; v[21] := 10; v[22] := 4; v[23] := 16; v[24] := 7;
w:=[1..256];k:=InputTextString("abcdefghijklmnopqrstuvwx");
w[ReadByte(k)] := 1; w[ReadByte(k)] := 11; w[ReadByte(k)] := 2; w[ReadByte(k)] := 17;
w[ReadByte(k)]:=3;w[ReadByte(k)]:=19;w[ReadByte(k)]:=4;w[ReadByte(k)]:=22;
w[ReadByte(k)]:=5;w[ReadByte(k)]:=13;w[ReadByte(k)]:=6;w[ReadByte(k)]:=8;
w[ReadByte(k)]:=7;w[ReadByte(k)]:=24;w[ReadByte(k)]:=9;w[ReadByte(k)]:=18;
 w [ReadByte(k)] := 10; w [ReadByte(k)] := 21; w [ReadByte(k)] := 12; w [ReadByte(k)] := 15; 
w [ReadByte(k)] :=14; w [ReadByte(k)] :=20; w [ReadByte(k)] :=16; w [ReadByte(k)] :=23;
m:=InputTextString(g);s:=[1..24];
s[1] := w[ReadByte(m)]; s[v[1]] := v[s[1]]; s[2] := w[ReadByte(m)]; s[v[2]] := v[s[2]];
s[3] := w[ReadByte(m)]; s[v[3]] := v[s[3]]; s[4] := w[ReadByte(m)]; s[v[4]] := v[s[4]];
s[5] := w[ReadByte(m)]; s[v[5]] := v[s[5]]; s[6] := w[ReadByte(m)]; s[v[6]] := v[s[6]];
s[7] := w[ReadByte(m)]; s[v[7]] := v[s[7]]; s[9] := w[ReadByte(m)]; s[v[9]] := v[s[9]];
s[10] := w[ReadByte(m)]; s[v[10]] := v[s[10]]; s[12] := w[ReadByte(m)]; s[v[12]] := v[s[12]];
s[14]:=w[ReadByte(m)];s[v[14]]:=v[s[14]];s[16]:=w[ReadByte(m)];s[v[16]]:=v[s[16]];
return PermList(s);end;;
```

This code will be saved to file chartoperm.txt

The function MultEl(el)

We write elements in generator form as a vector. For example [1,3,2] means the element UFL = s[1] * s[2] * s[3] and s was defined above. When we want a conversion from the form [1,...,8] to a permutation with cycles we use the function MultEl. Its code can be seen below:

```
MultEl:=function(el)
local x,an;
an:=();
for x in [1..Size(el)] do
an:=an*s[el[x]];
od;
return an;
end;;
```

The function Invg(el)

This function takes an element in generator form as defined above and maps it to its inverse also given in generator form.

```
Invg:=function(g)
local set,x,y;
set:=[];y:=[];
y[1]:=7;y[2]:=8;y[3]:=9;y[4]:=10;y[5]:=11;y[6]:=12;
y[7]:=1;y[8]:=2;y[9]:=3;y[10]:=4;y[11]:=5;y[12]:=6;
for x in [1..Size(g)] do
set[Size(g)+1-x]:=y[g[x]];
od;
return(set);end;;
```

The main program

The algorithm for the program was explained in the beginning and here we give the implementation as GAP code:

```
sh:=0;cnt2:=1;positions:=[];
for x in sett do
    set1:=MakeSet(10,x);
    set2:=MakeSet(12,x^-1);
   cnt:=0;
   for y in [1..Size(set1[1])] do
        for z in [1..Size(set2[1])] do
        g:=set1[1][y]*set2[1][z];
        if rcv[PermToNr(g)] then
             positions[cnt2]:=Concatenation(set1[2][y],set2[2][z]);cnt2:=cnt2+1;
             set3:=EqClassRel2(g);
             WriteRcv(set3);
             cnt:=cnt+Size(set3);
        fi;
        g:=set1[1][y]^-1*set2[1][z]^-1;
        if rcv[PermToNr(g)] then
             positions[cnt2]:=
             Concatenation(Invg(set1[2][y]),Invg(set2[2][z]));cnt2:=cnt2+1;
             set3:=EqClassRel2(g);
             WriteRcv(set3);
             cnt:=cnt+Size(set3);
        fi;
        od;
    od;
    sh:=sh+cnt;
    Print("Positions checked:",sh," ","Positions left:",88179840/2-sh,"\n");
od;
```

This code will be saved to file mainprg2.txt

Running the program and output

To run the program we make use of all above functions and sets. To read those in memory and start the program the following code is used:

```
Read("makesetv.txt"); #This file contains function MakeSetv
a:=MakeSetv(1):
vevenr:=a[1];
voddr:=a[2];
a:=MakeSetv(2);
veven:=a[1];
vodd:=a[2];
Read("makesetvc.txt");
a:=MakeSetvc();
vevenc:=a[1];
voddc:=a[2];
Read("makesety.txt"); #This file contains function MakeSety
v1:=MakeSety(1);
y2:=MakeSety(2);
Read("makeset.txt"); #This file contains function MakeSet
Read("redf.txt");#This file contains function Redf
rcv:=BlistList([1..88179840],[1..88179840]);
Read("permtonr.txt");# This file contains function PermToNr
Read("eqclassrel2.txt"); #This file contains function EqClassRel2
Read("writercv.txt"); #This file contains function WriteRcv
sett:=Redf(veven);
Read("datav.txt"); #This file contains the data included in the paper
Read("chartoperm.txt"); #This file contains function CharToPerm
for x in datav do
AddSet(sett,CharToPerm(x));
Read("inverseingens.txt"); #This file contains function Invg
Read("mainprg2.txt"); #This file contains the Main program
SaveWorkspace("test2");
quit;
```

This code is put in a file called start.txt. And all the files(including datav.txt at the end of the paper) in the \gap4r4\bin catalog. The program is run under unix with command:

```
gap.sh -o 1000M start.txt
```

The program will output:

```
Positions checked:3079007 Positions left:41010913
Positions checked:18604155 Positions left:25485765
Positions checked:19391883 Positions left:24698037
Positions checked:22124037 Positions left:21965883
.....

And a couple of hours later...

Positions checked:44089776 Positions left:144
Positions checked:44089824 Positions left:96
Positions checked:44089872 Positions left:48
Positions checked:44089920 Positions left:0
gap>
```

What does this program do?

As told in the beginning we need to compute each element in the group $G_{cube} \cap R_c = E_f$ in at most 22 moves. This program does this. When the program is finished the set "positions" contains generator expressions of each element in E_f . An element of "positions" has the form [1,2,8,7]. Here 1 means U, 2 means F more general n means s[n]. Actually not all elements of E_f are contained in positions but only representatives from each equivalence class under relation \approx . To make a final check that confirms that this set contains a representative from each equivalence class under relation \approx the following program can be run:

```
Read("multel.txt");
corners:=Group(uc,lc,fc,bc,rc,dc);
rcv:=BlistList([1..Size(corners)],[1..Size(corners)]);
#rcv is a binary vector with all its entries set to true and size 88179840
for x in positions do
set:=EqClassRel2(MultEl(x));
WriteRcv(set);
od;
cnt:=0;
#the program below counts the number of entries in rcv that are true
for x in rcv do
if x then cnt:=cnt+1;fi;
od;
Print(cnt,"\n");
```

This program takes each element in "positions" converts it to a permutation with disjoint cycles (this is the form which GAP can handle best). Then build

all elements in the equivalence class under relation \approx and checks them of in the bit vector. After this procedure is done the program counts the number of entries in the bit vector that has been set to "false" and prints them out. If it returns 44089920 then every element in E_f has been checked of and we are finished. Note that this final check actually proves the whole content of this paper is correct. Actually this verification has already been done by Bruce Norskog soon after i published the results of this work on the internet.

Final comments

It might seem that we were lucky that all elements in E_f had length 22 or less but the truth is that we had a backup plan if this would be the case. We find generator expressions for each element g in the Rubik cube group in two steps. First g is multiplied by $b \in G_{cube}$ from the left and then $bg \in E_f$ and afterwards bg is multiplied by a which takes it to the identity.

$$abg = id \Rightarrow g = b^{-1}a^{-1} \Rightarrow g^{-1} = ab$$

Assume that $a \in E_f$ has length 24. Assume that the diameter of the Rubik cube group is 24. Compute the set

 $X_1 = \{aU^{-1}, aL^{-1}, aF^{-1}, aB^{-1}, aR^{-1}, aD^{-1}, aU, aL, aF, aB, aR, aD\}$. Then each element of this set has length 23 or less. Then $a = ex_1$ where $e_1 \in X_1, L(e_1) \leq 23$ and x_1 is a generator. Then $b = x_2c$ where x_2 is some generator and c_1 is an element of length 17 or less. When writing $g^{-1} = ab = e_1x_1x_2c_1$ we can choose $x_1 = x_2^{-1}$ and cancellation occurs giving that g^{-1} has length 40 or less. It is well known that there exists a position of length 26 so even this procedure might fail. So assume that aU has length 25 in the list above. Then compute the set

 $X_2 = \{aUU^{-1}, aUL^{-1}, aUF^{-1}, aUB^{-1}, aUR^{-1}, aUD^{-1}, aUU, aUL, aUF, aUF, aUB, aUR, aUD\} \text{ and assume that all elements of this set have length 24.}$ Then $g^{-1} = ab = e_1x_1x_2c_1$. If $x_2 \neq U$ we solve this problem as before. If $x_2 = U$ then $g^{-1} = ab = e_2x_1U^{-1}Ux_3c_2$. And c_2 has length 16 or less. Again

we can choose $x_1 = x_3^{-1}$ and the length of g^{-1} is less than or equal to 40. Finally some elements in the last set might have length 26 in which case we must carry one last step of this. Obviously if there were too many elements in E_f having length 24 this computation could take years. I must remark though that the elements of length 24 are very rare and very few are known. In 1995-1998 the only known element of length 24 was the "superflip" and today about 150 others are known to the best of my knowledge.

Conclusions

In this paper we have constructed a program that returns generator expressions of length 22 or less for each element in the subgroup of the cube group that fixes the edges. As discussed in the abstract this means we have given a upper bound of 40 for the cube group.

Acknowledgements

I want to thank my advisor Gert Almkvist for his help and suggestions and my second advisor Victor Ufnarovski for taking the time to look through this work in detail. I further want to my father, my wife and my mother for their help and support. Thanks to Tomas Rokicki for publishing his result for the edges analysis and Bruce Norskog who verified the output of this program. I also thank Michael Reid for sharing his optimal solver. Finally thanks to the people who contributed to GAP development, this program makes it very easy to test mathematical ideas.

Data vector

This should be placed in a file called datav.txt:

```
{\tt datav} := [\ "aceglpmwqius",\ "acegpkwsujqm",\ "acejhtmoqluw",\ "acejpkwmuqsg",\ "acekrihownvs",\ "acek
"acelismhqvpw", "acelismwohuq", "acelismwqhou", "aceliwqgonvt", "aceliwqugtmo", "acelqwmspgui", "acepikm-pacelismhqvpw", "acelismwohuq", "acepikm-pacelismhqvpw", "acelismwohuq", "acepikm-pacelismhqvpw", "acepikm-pacelismwohuq", "acepikm-pacelismhqvpw", "acepikm-pacelismwohuq", "acepikm-pacelis
  {\it sqhuw",\ "acerikmtohwv",\ "acesikwmqhpu",\ "acexigqkosmv",\ "acfgkojqmvxs",\ "acfkihmoqvtw",\ "acfunilqshow",\ "acexigqkosmv",\ "acexigqk
    acgtikquwnoe", "achfikmoqwvt", "achliewuonqt", "achmikqxosfv", "achpiemsqkuw", "achpikfqmsvw", "achrvkmo-
jsfx", "achseuqowkni", "achsikqownfv", "achtikuqonwf", "achuislmfxoq", "achvikoqfswn", "achvismoqekw", "achviswlonqe", "achv
  "achxikmvprfs", "achxikufmsor", "acjfrnlxosvh", "acjgnrloewsu", "acjgnrlwesou", "acjgrnewlsou", "acjtlnhqevow",
    "acktieqowhvn", "ackxiegqmsov", "ackxiequosgn", "aclgniewprus", "aclgniuoexrs", "aclgwieqmsup", "aclheipqmvws"
    acloniesqgwu", "aclonieswguq", "aclonisewgqu", "aclsvhmqjeow", "aclxrimohsfv", "acmgeujwoktq", "acmgkfjw-
prus", "acmgkfxqjsup", "acmgpkjwequs", "acmgsljwqvof", "acmoglqfwvjs", "acmokfjsqgwu", "acmokfjwqvtg", "acmgpkjwequs", "acmg
teojwqkug", "acmxvgjoqetk", "acnfikhsqpwu", "acnfikhwqspu", "acnfikpxqsvh", "acnfikqhwspu", "acnfikqxovth",
    acnkihvoqetw", "acolihmsweuq", "acoqihmewsul", "acoxikmvqsge", "acplietqmgux", "acplismwqveh", "acptieuqmkwg"
  "acpvikmqesxh", "acqqfnjwlsou", "acqokfmsjgwu", "acqsikuwmgoe", "acruikomxsge", "acsuikomgeqw", "acsvri-
hownke", \ "acsxikfoqngv", \ "actfikmqvhow", \ "actfikmuqgxp", \ "actfikmjwqhou", \ "actlvieqmhow", \ "actpieuqmkgw", \ "actfikmjwqhou", \ "actf
  "acugfnjwoslq", "acugkfojwsmq", "acuokfmswgjq", "acuokfsmjgwq", "acvfikmswphq", "acvfqjmkpsxh", "acvgpetk-
mjwq", "acvlhjmoqftw", "acvmietoqkxh", "acvmikqfosxh", "acvnkfxqjtgo", "acvriktomewh", "acvsilmoqexh", "acvwikm-reference acvoided by the control of the c
rosge", "acwtieqgokmv", "acwuiemoqslh", "acwuiksqmego", "acxfikhmupqs", "acxfikoqmsvh", "acxlisumfhoq", "adegmki-
  wpqus", "adegqkmupitx", "adeohkmwqius", "adesukhomiwq", "adeuokqsximg", "adeuqkmgtiow", "adewqomglius", "adeunkqsximg", "adeunkqsyimg", "ade
  "adexikmsqoug", "adfvslnohiwq", "adgfqkmvxitp", "adhuisrwpnle", "adkgmsfwpivq", "adkgqmewpius", "adkgws-
  fqmipv", "adknqgpuwifs", "adksowhfuiqn", "adktiomrhvfw", "admfkuhxrisp", "adpgikxeunqs", "adpgslmwrifv", "advardigent advardigent advard
```

```
"adpgslwfuirn", "adptkgnwqifv", "adrxsnlohifv", "adsvowqlhime", "advlmerwpgti", "advsiemkqoxh", "adxkmeuqpgti",
"aegoidmlqsuw", "aehtikmcquwo", "aehtikmvqwdo", "aekgismvqwdo", "aelgniwcuqso", "aesoiwmvqclg", "aesuqk-
mgpwdi", "aevdikmhqwto", "aexoivtgmqlc", "afckiumoqgtw", "afkoismvqwgc", "afkrcsmxipug", "aflcqhpuwims",
 aflcqhupmiws", "afqckhjuwpms", "agclnpewqjus", "agcuikmoqswe", "agdsiemoqluw", "agdsikfqmxov", "agecltqow,
imu", "agecqkmpxius", "agecskmoqiuw", "ageisdmoqluw", "agejckmwqpus", "agejktmoqcuw", "agekciurmpws", "agelcmwqpus", "ageisdmoqluw", "ageisd
{\it cjqosumw",~"agelicjurmspw",~"agelicpmrixus",~"agelipquwcms",~"ageliqmwocut",~"agelismoqvdx",~"agenckuoitwq",}
 "agenckuqitow", "agenikqxocvt", "agenikuwqcot", "agenksrdpiuw", "ageoikmdqtuw", "ageoikmrxsuc", "ageoikm-
sqwuc", "ageoikmsqxud", "ageoikmsxcur", "ageoitmlqcuw", "ageonkqcjsuw", "agepltmricuw", "ageqckuniswp", "age-pltmricuw", "ageqckuniswp", "age-pltmricuw", "ageqckuniswp", "age-pltmricuw", "ageqckuniswp", "age-pltmricuw", "age-
qxkmcosuj", "agesckuomxri", "agesikmcqwuo", "agesikmrxcuo", "agesldmoqiuw", "agesnkuwocjq", "agesqkmwcoui",
 "agesqkpuwidm", "agesqkuomidx", "agesrkjomcuw", "agesrkjwmcou", "agesvkjwocmq", "agesvkojmcqw", "agesxk-
 moucqj", "ageuckiosxmr", "ageuckqoixmt", "ageuikmwqsoc", "agevismoqlxd", "agewikmsoquc", "agewikmsqvpc",
 " agewikmvqsdo", " agewikqucsmo", " agewikucmsoq", " agewikuqmtdo", " agewmkuordsi", " agewpkmjcsuq", " agexikm-
cpsuq", "agexikmcqpus", "agfkvqxdmitp", "agflniquoctw", "agfsnivkqcpw", "agjsfvmwoclq", "agknwsjoqcfv", "agkncpusture, "agflniquoctw", "agfsnivkqcpw", "agjsfvmwoclq", "agknwsjoqcfv", "agknwsjoqcfv", "agfsnivkqcpw", "agfs
smoivfx", "agksiqfomcwv", "agksiwmqecvp", "agksrwmfocjv", "agktipmwqcfv", "agkuiwtoqcne", "agkvpsfncqwi",
 agkvqsmoxifd", "agkvsfjoqcnw", "agkvwsmpqidf", "agkwiomcqsfv", "agkwiomrtcfv", "agkwiomsqfdu", "agkwismc-
qvfo", "agkxcjmoqsfv", "aglcrtmoweju", "aglfixuomctq", "aglovieqmscw", "agloximcquse", "aglrniewpcus", "aglsjweo-
 qcmu", "aglwvieomqsc", "agmctkjoqeuw", "agmldpjwqeus", "agmodejvqktw", "agmsikqoecuw", "agmsikqwecou",
 agmsikuwoceq", "agmwkfjcqsuo", "agnfiwquoctl", "agnrsljowceu", "agodikmvqswe", "agplismwqcfv", "agpsikfwqcmu",
  agqsikeomcuw", "agqsikewmcou", "agscikmoqeuw", "agsjrkeowcmu", "agswnielqcuo", "agtdikmwprfv", "agtovqedg
mikx", "aguerisowcml", "aguliemoqcxs", "agusikewocmq", "agusikfdpnwq", "aguwkfjqmsoc", "agvcelmoqjtw", "agvcelmoq
qitw", "agvcfpmiqktw", "agvfikmcqspx", "agvfkmjoqctw", "agvkiemcqsxo", "agvkiemwqods", "agvrcemoiltw", "agvr
moitxk", "agvsipmxqcle", "agvwiemcqkto", "agwsipmlqcue", "agwvieqtocmk", "agxkcetnupqi", "ahecitmrkpuw",
 "aheodirsxnul", "ahesqkmowdui", "ahnuieqcowlt", "aidkmersowhv", "aiegckqwnpus", "aiegquptwcmk", "aiehmutq-
pldx", "aiekqumdpsxh", "aielqspdwnvh", "aielqspudhmw", "aienckwguqsp", "aienpkwcuqsg", "aieomkugqwsc", "aier-
{\tt ckmsgwup","aierpkmscwug","aieskhmdpruw","aiesmkuwqgoc","aiespkmdgquw","aievqkpdwhms","aiexckmgqusp",}
"aikgcsmqvpfx", "aikhscqoxnfu", "ailgncesqxup", "aimgekupqsdw", "aimgslpfdvrw", "aiswceukmroh", "aiurewd-
kmptg", "aiurkfdgmwsp", "aivgcewktnqp", "aivrcehkmwsp", "aixfmkptdhuq", "ajchkfosxqnv", "ajegpkuwcmsq",
"ajegxkqtcpmu"," ajeovkmwcqsg"," ajesqkpdhmuw"," ajewtkucmqgo"," ajkgxsmrcpfv"," ajkvmsfdhqwp"," ajndwepthumum ajewtkucmqgo"," ajevtkucmqqo"," ajevtkucmqqo", ajevtkucmqqo"
{\tt kvqtg", "ajogmewcuqtk", "ajsgpvqlcwme", "akecqhiwmpus", "akecwhirosmu", "akehctmoiquw", "akeoiguqmwsc", "akencymiyathur "akehctmoiquw", "akencymiyathur "akencymiyathur "akencymiyathur", "akencymiyathur "akencymiyathur", "akencymiyathur", "akencymiyathur "akencymiyathur", "aken
 "akeoiqncwshu", "akerchmsixup", "akeritmwphuc", "akeschmriouw", "akesiwugmcoq", "akhpcsmwiqfv", "akhtqmew-
pjdv", "akmchfjrosuw", "akmuetjcqwph", "akqhcemsiwup", "akvcmetiqpwh", "akvncetgiqwp", "akvscemriphw",
"akvsiehqmcow", "akvtweqppimd", "akvwceitgpmr", "akxoivtemqhc", "akxqcemgipus", "alcgifmwqous", "alcoifug-line akvtweqppimd", "akvwceitgpmr", "akvoivtemqhc", "akvqcemgipus", "alcgifmwqous", "alcoifug-line akvtweqppimd", "akvwceitgpmr", "akvoivtemqhc", "akvqcemgipus", "alcgifmwqous", "alcoifug-line akvtweqppimd", "akvwceitgpmr", "akvoivtemqhc", "akvqcemgipus", "alcgifmwqous", "alcgifmwqous", "alcoifug-line akvtweqppimd", "akvwceitgpmr", "akvoivtemqhc", "akvqcemgipus", "alcgifmwqous", "alcoifug-line akvtweqppimd", "akvwceitgpmr", "akvwc
mosw", "alcsrfmowgiu", "alcsvfmojgwq", "alcsvfmojgow", "alegctirwpmu", "alegmtudpjwg", "alegptmjgcuw", "ales-
almsfdjogguw", "alpscfmriguw", "alsgivmwqoec", "alsgrieowcmu", "alsgvieomcwq", "alvntfjogcxh", "alvsdemo-
qgxi", "amcgikouwqes", "amegiksqcuwo", "amegikuqcwso", "ameoikwruchs", "amepqkwguisd", "amesikgocquw",
 "amesikocwuqg", "amesikoqugwc", "ameuhkscjqwo", "amlgipewqcus", "amrfikuhscwp", "amswpvceqjlg", "amugk-
fwcjqso", "andtgvlqipfx", "anegikqcwups", "anigkfquwpds", "anksiwqhdufo", "ankwisfgqcuo", "anlgriewpcus", "anlpchuqiews",
"anlygqepxsdi", "anpuiglcqsfw", "anykjcewqpth", "anywcepqitgk", "aoclrfwjugms", "aocuikqswgme", "aoecikquwtmh",
 aoecikuqmtwh", "aoecktmwqguj", "aoednskwqjuh", "aoedqkmvhitw", "aoedukwirgsm", "aoegdmjwqkus", "aoegikm-
rdvxs", "aoegikqsucmw", "aoegikqtvcmw", "aoegikquwntc", "aoegiktqucwn", "aoegikuxmctq", "aoegikwqmctv",
 "aoegisukmcwq", "aoegltmwqiuc", "aoegnkjqucws", "aoegskmqjcux", "aoehskmdqiuw", "aoejmkdwqgus", "aoejrk-
suwcgm", "aoelisdqmhwv", "aoelismdqhxu", "aoelituqmcwg", "aoelnrcuwgjs", "aoelqwmvhitc", "aoenkgjvqctw",
"aoenskwjvcqh", "aoerckmwigut", "aoerckmwjgus", "aoerckwmuhis", "aoerclmwigus", "aoerjkdwngus", "aoerskmx-
icug",\ "aoesihmwqkud",\ "aoesihquwcml",\ "aoesikmcquwg",\ "aoesukwrhcim",\ "aoetikmgqcwv",\ "aoetikqhwcmu",\ "aoetikqhwcmu
 "aoevhkjwqcns"," aofgimuwqctk"," aofgksquwcni"," aoflsdjwqgnv"," aofriktwmcuh"," aogcrnlwfius"," aogmvilqecws",\\
```

```
"aogrldewnius", "aohckfjwqnus", "aohclsqfwimv", "aohlivmwqcfs", "aohlnevwqjds", "aohlvemqjcws", "aohlweiuqcms",
 "aohtifmwqcul", "aohtikfdmvrw", "aohwikmuqcfs", "aoignsquwckf", "aoigrkewncus", "aoilnevwqchs", "aojgfnlsxruc",
   aojgkvcqmews", "aojglrwfuctn", "aojgnveqlcws", "aojkngewqcus", "aojtndkwqgfv", "aojwfnqcguls", "aojwgvsqm-
 cke", "aokdqshfwimv", "aokgcsirvxmf", "aokgismvwcfq", "aokgiswufcmr", "aokgixmuqcfs", "aokgmsqwicfv", "aok-
  grsjwmcfv", "aokgvsjfqcmx", "aokgvsqfucjn", "aokjdemwqgus", "aoknigfwqcus", "aokrsgjfwcmv", "aoksivmwqcfg", "aokgvsjfqcmx", "aokgvsjfqcmx",
 "aokuismhqcfw", "aokvishfrcmw", "aokwiemgqctv", "aokwismgqufd", "aolcniewqtuh", "aolgnvjwqces", "aolgwieqn-tuber ("aokuismbqcfw", "aokvishfrcmw", "aokwiemgqctv", "aokwismgqufd", "aolcniewqtuh", "aolcniewq
 {\it cus",~"} a olhism wqc fv",~" a olmeisw qcgv",~" a olsnihurcew",~" a oltiwhurcfn",~" a oluiqsw ncge",~" a omchvlqejws",~" a omchvlqejw
{\tt ckfhwqius","aomckfjwqtuh","aomedkgwqjus","aomedkjwqtuh","aomgikeuwcqs","aomgxtjlqcue","aomidfhwqlus",}
 "aomligrwecus", "aomtfljwqgdv", "aomtilcwqgfv", "aonfikdsqhwu", "aonfikqxhcvt", "aonfwkqtucih", "aonlisqfwgdu",
   "a on reiuwhtk", "a on uiewckgqt", "a on uisfwqcgk", "a on xieqkvctg", "a oqcehjuwkms", "a oqgntwlucje", "a oqhniexx-
{\tt ckv",\ "aoqhskjfwcmv",\ "aoquismlwche",\ "aosgrvmlwcje",\ "aosgrvmqjcle",\ "aotfikmdwhuq",\ "aotliemdqhuw",\ "aotliemdqh
 tungkwqcie", "aougfrljmcws", "aougriwlfcsn", "aoulcfnqjgws", "aounkfdqihws", "aoutigqlwcme", "aovcgemwqitk", "aounkfdqihws", "aoutigqlwcme", "aovcgemwqitk", "aounkfdqihws", "aoutigqlwcme", "aoutigqlwcme",
"a ovciem wqshk"," a ovcikmhqsxf"," a ovfikmhqcwt"," a ovfikm wqtgd"," a ovgemjdqktw"," a ovliem tqcxg"," a ovrceitwhmk"," a ovciem wqshk"," a ovciem wqshk"," a ovcikmhqsxf"," a ovciem wqshk"," a ovcikmhqsxf"," a ovcikmhqsxf", a ovcikmhqsxf, a 
 "aovsikmdqgxf", "aovtikmgqcfx", "aowckhmeujqs", "aowgnielucqs", "aowjkfmuqgsc", "aowliemsqcgy", "aowuieqt-
   glmc", "aoxcikfqmtuh", "aoxgniesqcuk", "aoxrceuimhtk", "apegikqwsnuc", "apegmkuqxisc", "apejmkugxqsd", "apels-
dmriguw", "apevckqxihms", "apknisfvqcwg", "apvnkfdgiwqt", "aqcwikmfsogv", "aqegcomixlus", "aqegikmdsowv", "apevckqxihms", "a
 "aqegvkmcjwso", "aqelismdxouh", "aqencksuwpig", "aqeoikmgcsuw", "aqeoikutmwdg", "aqesvkmwjgoc", "aqevikxs-respectively and the statement of 
 modh", "aqewikucmosg", "aqewvkmgjcso", "aqjsfnuwlgoc", "aqkgosmiwufd", "aqkncgutixfo", "aqlwniuoesgc", "aqnf-
 pkcjwsvh",\ "aqtvkfjgmwdp",\ "arcgienuwptk",\ "aregxkumdpis",\ "areldinuwphs",\ "arflgsmwipdv",\ "arhojckusxfn",\ "areldinuwphs",\ "artlgsmwipdv",\ "artlgsmw
   arlgcinuwpes", "artocyhimxle", "arvwceipgtmk", "asedikupmqwg", "asegikwpudnr", "asegipulmqwc", "asegixqul-
cmo", "asegvkmqjwco", "asekigmoqcuw", "asekiumgqwdp", "aselcwmgjpqu", "aselihmoqcuw", "asenckigqxuo", "aseoikuqgmwc",
 "aseoikuxmcgr", "aseqikmogcuw", "aseqipulmcwg", "aseqpkujhmwd", "aserckmviwog", "aseuikqcwomg", "asevikmo-
 qchw", "asewcqmlipug", "asewikmogquc", "ashxikfoqncu", "askoigfwqunc", "aslcngeiqwuo", "aslcrimweugc", "aslqvimwgceo", "aslqvimwgceo", "aslcrimweugc", "aslqvimwgceo", "aslcrimweugc", "aslc
 "aslrniegxouc", "aslwipmgqeuc", "aslwnieohcur", "asnuiepwqcgk", "asqofnlugwjc", "asuwikfqmcpg", "asvkiemgqwpc",
   "asvwiemkqodg", "aswvioqlgcme", "asxhikmcqofv", "ategikmvqowc", "atelcuqhiomx", "ateliwmchruo", "ateqhk-
   mociuw", "aterxkigcvmo", "atngceikuxqp", "atnuweqkpdjh", "atpgikudmqfw", "atvgpeikcxmq", "atvwiemkqodh",
     "auegckqswnip", "auegikmxqotc", "auegiknswrdp", "auejpkhwdqms", "aueoikqgtxmc", "aueoqkmsxdgi", "aueoskigqwmc",
"auesikmwqocg", "auesrkiwhomc", "aufmriskwodh", "auhtwkinqofc", "aulsniqogcew", "avcfikmsqwph", "avegckm-
siwor", "avegikmcqspw", "avegikmsqodw", "avegikmmqsdw", "avegikqdswmo", "aveligmoqsdx", "aveoikmcqsxg", "avegikmcqsxg", "ave
   "avesikmwqodg", "avesqkmdpwjg", "avkocgfmixqt", "avkpismhqwfc", "avkwismohcfr", "avkwmsfdhioq", "avwsiemgqcpk",
 "awdjlhmsqofv", "awedismkqouh", "awedqgpukims", "awedqkphvims", "awegikmdqosv", "awegikmuqodt", "awegikom-
 suqc",\ "awegikqscumo",\ "awegmkurpcsi",\ "awegqkmipvtd",\ "awegskqupdmi",\ "awejckosnqug",\ "awekisuqmodh",\ "awegnkurpcsi",\ "awegnkurpcsi
 "awelivmhqods", "awelpdmsirug", "awenckvsiroh", "aweoihmsqluc", "awerckusmioh", "awesikuomgqc", "aweuik-
   moqsgc", "aweuikmsqcgo", "awgjpvmscqle", "awjgfvmsqolc", "awjrcesmuphk", "awkoqsmduifg", "awkoqsmvhifd",
     awmfikhvqods", "awmoekjdqgus", "awmokfjgqusc", "awmuktrepdgi", "awpfikmuqcth", "awrfikdusomh", "awse-
 hvmlqcjo", "awsmikgeqouc", "awsoeqjmuchk", "awvoremtkcjg", "awvsikfqmodh", "axegikusmqdo", "axeoqkmsg-negative awvoremtkcjg", "avvsikfqmodh", "axeqikusmqdo", "axeoqkmsg-negative awvoremtkcjg", "avvsikfqmodh", "axeqikusmqdo", "axeoqkmsg-negative awvoremtkcjg", "avvsikfqmodh", "axeqikusmqdo", "axeqiku
dui", "axhtikqfucpn", "axksiqocfumg", "axphckifsvqn", "axqgemitupdk", "bdmuqtlxjoeg", "bfmsokjhqcuw", "bkeogjmd-
\mathbf{sxuq}", "\mathbf{bwehvqkpjdnt"}, "\mathbf{caljnpegqsuw"}, "\mathbf{camlxfroigus"}, "\mathbf{cbsgivlqmpwe"}, "\mathbf{cbvfqjmwplgs"}, "\mathbf{cealnigousqw"}, "\mathbf{cellibroigus"}, "\mathbf{camlxfroigus"}, "\mathbf{chvfqjmwplgs"}, "\mathbf{chvfqjmwplgs"}, "\mathbf{chvfqjmwplgs}, "\mathbf{chvfqjmwplgs"}, "\mathbf{chvfqjmwplgs"}, "\mathbf{chvfqjmwplgs}, "\mathbf{chvfqjmwplgs"}, "\mathbf{chvfqjmwplgs}, "\mathbf{ch
 mvqwps", "ceilnpawqgus", "cesagjuwmqlo", "ceslibmguwqo", "cfsulranwoig", "cfsuqjmbphxk", "cgalfiuoswmq",
   "cgaliqewosun", "cgalpiuemsqw", "cgalqiemuswp", "cgaltiqouemw", "cgaltiuomeqw", "cgameiwoqslv", "cgateiuq-
 know", "cgauniswoelq", "cgaurimwseol", "cgbxkimoqsfv", "cgeiqpmbxlus", "cgelibmvprtw", "cgelibqtowmv", "cgelib-
 vqmsox", \ "cgelibwusqom", \ "cgelmjpwbsuq", \ "cgelniuwosaq", \ "cgelpbujmsqw", \ "cgelqbouismw", \ "cgelwbiornus", \ "cgelpbujmsqw", \
 "cgesqamokiuw", "cgetibmlovwq", "cgfbniqosuxk", "cgfxblniqsov", "cgfxikmoqsbv", "cgjlbfqwmsou", "cgjlfsaoqxnv",
 "cgkasjqowvfn", "cgktibvmfqow", "cgmakpjwqeus", "cgmesbjoqlxu", "cgmilfrotauw", "cgmpbljrtwfv", "cgmxufao-
 {\tt qkis", "cgnlibfowrvs", "cgnlibfqusow", "cgofibqmxsvk", "cgpilnfqavxs", "cgplibufmsqw", "c
 saeuqmixk", "cgpxikfqmsub", "cgrfibwuokmt", "cgsfibmuqkow", "cgsubfjwomlq", "cgsuibqfomkw", "cgtbniuoeqxk", "cgrfibwuokmt", "cgrfibwuokmt",
```

```
"cgulibfnqsow", "cgvbiemoqlxs", "cgvbfrojnxs", "cgvnafrokixt", "cgvsiembqokw", "cgvtibwoqkfm", "cgvxniqboetl",
 "cgwlbfqjosmu", "cgwlibeqosum", "cgwlibqeusom", "cgwlifmoqvta", "cgwtblqfjvom", "cgxfnbutokjq", "cgxtibmo-
 qlfv", "chelibpuwnrs", "chesikunrwob", "chvnqfptwbjk", "chxlqemtpiub", "cimalfrwpgus", "cixkemugpqtb", "ck-
 afnisgquwo", "ckafsiomywgq", "ckahniuseqwo", "ckgbniseqwpy", "ckwhibmoqyte", "ckxpieuqmath", "claoyimgsweq",
 "clebiomgqsuw", "clergbmoisux", "clewgbmrisup", "clewibmogsuq", "clmgbfusjwoq", "clmsbfjwqgou", "clqajfhu-
osmw", "clqsbfmwjgou", "cluwbfjqmsgo", "cmalriseguwo", "cmulbfjqgwso", "cnebiqkpvgws", "coafnlrwigus", "coaf
 sjmwqguk", "coaheikmuxqs", "coalnwiuqges", "coalriesugmw", "cobgilqfwsmv", "coegikuamqws", "coegmkuaqiws",
  coehibmuqxtk", "coehibukrnws", "coehqbkuwims", "coeiahmwlqus", "coelrbmsvxjg", "coelsbujmgqx", "coergaiuwlms",
"coernbtjygwk", "coesibmyqlhw", "coetbgjwqlnv", "coetkbxqjgnv", "coglbtjqmuwe", "cohtbljqmuex", "cohuibfqmwtk",
  "coifrbvwngtk", "cojhenaqkvws", "cokfibnqsgwv", "cokithewnrbv", "cokjnaquwges", "cokrvbmxjgfs", "coktibfhuxnq",
 "coktibfqmxug", "coljnberhsuw", "comhbljfwtuq", "comlfubqjgws", "conbrixutgle", "confaktwqiuh", "confibtkquwg", "confibtkquw
  confibuqwksh", "conwibqfugtk", "coqlbfwjmgsu", "coqlfnsjwgau", "coqlibesmgwu", "coqlnbvwjgfs", "coqlviweagms",
"coqtbfjukgmw"," coqwbtjufgln"," corlibwmfvtg"," corlibwusneg"," cortliawfgnv"," cosfibqmwguk"," cosnbfjuqgwk"," cortliawfgnv"," cosfibqmwguk"," cosnbfjuqgwk"," cortliawfgnv"," cortliawfgnv," cortliawfgnv," cortliawfgnv," cortliawfgnv," cortliawfgnv," cortliawfgnv, cortliaw
  cosqvnweaglj", "cosubkmwqfgi", "coulbfsjqgmw", "coulibwmqges", "covfibhwnktq", "covfmbhwrkti", "covfnbkwqgjt",
  covgnaewqski", "covlrbjwegns", "covwisfqmglb", "coxfibumtgqk", "coxknbteugqi", "coxtiulqmgbe", "cqaoxiemugsl",
  cqaxnikovgfs", "cqehmbwpuisl", "cqelibuspgwn", "cqewihusmola", "cqjbenxskoug", "cqmlsbvxjofg", "cqsoibml-
  wgue", "csagniuoeqwl", "csahnqelpwui", "csalniouwqge", "csaqnikohwfv", "csawjkegqnuo", "csfbnipgqwuk", "csnhibe-
owkru", "csqlubigwemo", "csvfrbmxgkjo", "csvmikqgwfao", "ctbrgoixlnue", "ctelawmiqoug", "ctvgweikqbmp", "cuakrip-
 swgfn", "cueagsmpirxl", "cuelibmgxqto", "cukgibqnosfx", "cunfaoiswrhk", "cvbrgsmliwfp", "cvksibmoqwfh", "cvk-
tfiqgmwbo", "cympbtlqjwge", "cwehbujsqkno", "cwelmbusjgor", "cwelqbumiosg", "cwevnbikqotg", "cwjlnyeqaosg",
 "cwkhbsjqmoev", "cwluiomsqhae", "cwpakimsqhfv", "cwqpbljfsvmg", "cwvhifqbkoms", "cwvnbfjrpgtk", "cwvnofrsjlhb",
 "dbegikuqmows", "ebmgksrwpiud", "ecmlsajoqguw", "ecpnbkwjusrh", "ecpnbkxjusqh", "ecsxbknqjvgp", "edmkqb-
 hwpius", "edphknasxiqv", "egloibmcqwsu", "egsaidwmouql", "egsvcojrbnlw", "egvkbmitrpdw", "ehpkqbjsxdnv", "elpkqbjsxdnv", "el
 "ejulmbrpgdws", "ejusbkgocmwq", "ejusbkndhqwp", "eknscqjhbovx", "ekvgbmcqwtoj", "elaoidugmqsw", "enkobgqvswjc",
  eogcxkumjaqs", "eokgcsjrbxnv", "eomgjkdwqaus", "epksbwduihrn", "epvgbmtkrdwi", "erhucsblwpjm", "erxgb-
 vtidplm", "esuwbkmogcjq", "euigbkxsqomc", "euinbkqgswdp", "euinbkqtogxc", "evajpdwnglqs", "ewctmibvkqoh", "euinbkqtogxc", "evajpdwnglqs", "ewctmibvkqoh", "e
  ewmockjhbrus", "fcagsqjolnuw", "fhspbmdviwlr", "fhvcqjwmtapk", "fonaidqkwshu", "gcajntuqelow", "gcalitmo-
 {\tt qeuw","gclantuoejwq","gdeaqjmswlup","gemxdaqvoljt","gfurbdjomksx","gipkecmqbtwv","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanqdewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus",",gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddewpjus","gkanddew
can qpuwies", "gkeaspquximc", "glcaxiouesmq", "glebmurdpsxi", "glmrdftojbux", "glnfidqbwspu", "glpxvcbimsfq", "glebmurdpsxi", "glmrdftojbux", "glmrdftojbux", "glpxvcbimsfq", "glebmurdpsxi", "glmrdftojbux", "glmrdftojbux"
  "gnparceuwlis", "gobwidkqmtfu", "gockrbjwmtfv", "gojsdvaqmwle", "goqjnbxudles", "gosurdmwelja", "govlidmwqbtf", "gotlidmwqbtf", "gotlidmwqbt
 "gpeiqdusmbwl", "gquednaljwso", "gsajvdolewqm", "gseiwcqapuml", "gtdkqjmwpafv", "gtnuidqeolxb", "guitdbqn-
  wofl", "guktidfqmbow", "gwbtmjpvqdfl", "gwecipquslma", "gwvfidmlqsao", "gxplrcnfitub", "hcofibqmusxk", "hra-
jsdfnkxov", "ibfogqvxmcsk", "ibhoqkmvxcft", "ibpgqkfdmvxs", "ibptkgnwqcfv", "icbhqwmsplfv", "icelqwmgosub",
  "icvomehwrksb"," idaukmsqheow"," idegqkusmowb"," idetqlwmbgpv"," idkhbfrnxout"," idsulrafpgwn"," idvoqextbgml",\\
 "idytqblwpgfn", "iektqbmvpwhd", "iekwbvgscorn", "ifvkqbmsxodh", "igqnblpfdsuw", "ihbksdwfvoqn", "ihclnqpuwfbs",
"ihpbnqukedxs"," iksuqemhxbdo"," ilbxqdhuofms"," ilvbqemdxohs"," imrfwbuhsdpk"," iodglbmwqtfv"," iomfbutxrgdk"," introduction introdu
 "ipelgbusqnwc", "irsbdhlymeow", "isalnqdguexo", "isepqbxukdmg", "iuehqbwmkdps", "iyatnqhfdwko", "iybpnqewkgdt",
  "iwtfqbmkpuhd", "jnakwuqdsgpe", "joqgebvwmctk", "jragvkpwmces", "jskwnqugecbp", "jsowndegqlua", "juaonetwqchk",
"kdjebnrohsuw", "kdqgwsanpfjy", "kesugdinoqaw", "kgiebdmoqvtw", "khmaqdjofsuw", "kibtgfnqdvow", "kixfqbupm-
{\tt cth","krmxcsjobgfv","ksmwcuafhqpj","lceoshiaqumw","lccunipaqshw","licmgovbfwtr","lnadqtejhwuo","lphvc-lcunipaqshw","lcmgovbfwtr","lnadqtejhwuo","lphvc-lcunipaqshw","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","lcmgovbfwtr","
 sjrbenw",\ "lwepchsmuria",\ "lwjbhfrspdnv",\ "majedvoqglws",\ "mgarcoelisuw",\ "mhqlvawtepjc",\ "mjeskbuqcgow",\ "majedvoqglws",\ "majedvoqg
  "mjpfrbukcgxs", "mleapdugsqwj", "mocwblvqjhte", "mougkiawqces", "mqoclsfjavgw", "mruagfjpdlws", "mrugfk-
 wjdpbs", "ngibqdxlsufo", "obxngvtciqkf", "ojdkxrmsgafv", "olfbndquwgti", "omriwdtbkvgf", "pdmlwfargsju", "pg-
 bicnqxfsvk", "prjncuawfkhs", "psmwcfjlbgur", "pvkacimgswfr", "qameowucjlsg", "qjecwsmgoaul", "qkenixvocsha", "prjncuawfkhs", "psmwcfjlbgur", "pvkacimgswfr", "prjncuawfkhs", "prjncuawfkhs", "psmwcfjlbgur", "pvkacimgswfr", "prjncuawfkhs", "prjncuawfkhs",
 "qwvfcbmoikhs", "rnoticvelxag", "scegaomrixuk", "sgelcbqowimu", "sgwoniuqelac", "tgeawdmoilur", "trgiwpmbe-
{\tt clv","twvoiemhqadk","txkpbfdgiurn","unkgasfqiwco","utbwiemhqocl","vjtkoqmbhcex","voekbsjwqcng","wjeovk-twvoiemhqadk","txkpbfdgiurn","unkgasfqiwco","utbwiemhqocl","vjtkoqmbhcex","voekbsjwqcng","wjeovk-twvoiemhqadk","txkpbfdgiurn","unkgasfqiwco","utbwiemhqocl","vjtkoqmbhcex","voekbsjwqcng","unkgasfqiwco","utbwiemhqocl","vjtkoqmbhcex","voekbsjwqcng","unkgasfqiwco","utbwiemhqocl","vjtkoqmbhcex","voekbsjwqcng","utbwiemhqocl","vjtkoqmbhcex","voekbsjwqcng","utbwiemhqocl","vjtkoqmbhcex","voekbsjwqcng","utbwiemhqocl","vjtkoqmbhcex,"utbwiemhqocl","voekbsjwqcng","utbwiemhqocl","vjtkoqmbhcex,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl","voekbsjwqcng,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqocl,"utbwiemhqoc
```

```
maqcsg", "wseclpmgaruj", "xoahcqesniul", "xsvnafjgkrdp" ];
```

The last data have been selected from a list that would take about 2 months to generate on an usual PC but thanks to my advisor Gert Almkvist access to fast computers have been given and made it possible to compute in just a couple of days.

References

- [1] http://www.math.rwth-aachen.de/~Martin.Schoenert/Cube-Lovers/Dan_Hoey_Symmetry_and_Local_Maxima_(long_message).html
- [2] http://www.math.ucf.edu/~reid/Rubik/optimal_solver.html
- [3] http://www-groups.dcs.st-and.ac.uk/~gap/Manuals/doc/ref/manual.pdf
- [4] http://www-gap.mcs.st-and.ac.uk/Doc/Examples/rubik.html
- [5] http://www.cubeman.org/
- [6] http://cubezzz.homelinux.org/drupal/
- [7] http://www.math.rwth-aachen.de/~Martin.Schoenert/Cube-Lovers/
- [8] http://theory.csail.mit.edu/~randall/