On Fixed Points

MATH 4LT3 Project

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December 7, 2023

Introduction

Fixed points, as the name implies, are essentially points that remain fixed under a certain set of transforms that attempt to map an object into itself. The idea is that these fixed points show , sometimes deep, underlying behaviors of these morphisms. We strictly define a use introduce the notion of fixed point as thus:

Fixed Point (Walker 2017) Let X be a set and let $f : X \to X$ be a function that maps X into itself. (Such a function is often called an operator, a transformation, or a transform on X, and the notation $T(x)$ or even Tx is often used.) A fixed point of f is an element $x \in X$ for which $f(x) = x$.

Fix points arise in many areas of mathematics, and computer science. This essay explores some key ideas related to their corresponding fixed point theorems. We shall first start the exploration of what a fixed point is by considering a rather accessible and more well known example of it.

Brouwer's Fixed Point Theorem - An Example

We introduce the Brouwer fixed point theorem here for elaboration. Brouwer's Fixed Point Theorem is a theorem of algebraic topology that was stated and proved in 1912 by the Dutch mathematician L.E.J. Brouwer (Carlson, n.d.). Brouwer investigated the behaviour of endomorphisms of the ball of unit radius in n-dimensional Euclidean space. Brouwer's fixed point theorem asserts that for any such function f there is at least one point x such that $f(x) = x$; in other words, such that the function f maps x to itself. For an analogy, consider,the case that one is holding a map of Canada and is situated in Canada (in the real world), there will be a point on the map that sits directly above the place in the real world it represents, regardles of twisting, bending or crumpling of the map. Furthermore, Brouwer's Fixed point theorem can be seen a generalization of one-dimensional case of the intermediate value theorem (Carlson, n.d.). From this we see that fixed point theorems can be used to see into the behavior of the morphism (how points change in manipulating the map) and the object we are operating on (mainly geo-location).

Continuous Least Fixed Point Theorem

Now that we have established what a fixed point is, and what a fixed point theorem is. We expand our discussion to how fixed points can be helpful in set theory, and how it can be applied. Here we shall discuss the Continuous Least Fixed Point Theorem (CLFPT). In a discussion about set theory fixed points are useful notions when writing proofs involving the use of posets and ordering. As set theory forms a strong connection with the computer science. The CLFPT in this regard is fundamental to the study of computation. We define the continuous least fixed point thus (check appendix for a reference for what it means to be a monotone, countably continuous, inductive poset [1,2,3]):

Continuous Least Fixed Point Theorem (CLFPT) (Moschovakis 2006, 6.21) Every countably continuous, monotone [1] mapping $\pi : P \to P$ on an inductive poset [2] into itself has exactly one strongly least fixed point x_*, which is characterized by the two properties

$$\pi(x^*) = x^*$$
$$(\forall y \in P)[\pi(y) \leq y \implies x^* \leq y]$$

It is important to note that to apply the CLFPT, we must pose the problem which we are trying to solve as a question of existence and,perhaps, uniqueness of solutions for an equation of the form $\pi(x) = x$, where $\pi : P \to P$ is monotone and countably continuous on some inductive poset P. Which will require the bulk of efforts when utilizing it (Moschovakis 2006).

It is the case that the CLFPT is a corollary to the Recursion Theorem on the natural numbers. In essence CLFPT implies the Recursion Theorem by a direct argument (We consider an application of the CLFPT to prove the Recursion Theorem in the appendix [7]).

GCD

The CLFPT can thus be used for working in computer science for proving the correctness of algorithms and other results, due to the reliance on recursion in that field. For instance the *Euclidean algorithm* given thus:

Proposition (Moschovakis 2006, 6.33). (1) There exists exactly one partial function $f : \mathbb{N} \times \mathbb{N} \rightharpoonup \mathbb{N}$ with domain of definition $(n, m)|n, m \neq 0$ which satisfies the following identities for all $0 < n < m$:

$$\begin{aligned} f(m, n) &= f(n, m) \\ f(n, n) &= n \\ f(n, m) &= f(n, m - n) \end{aligned}$$

(2) The unique f^* which satisfies the system above computes the greatest common divisor of any two natural numbers different from 0, $f^*(n, m) = \gcd(n, m) =$ the largest k which divides evenly both natural numbers n, m.

The above proposition is important because it yields a characterization of *gcd* which suggests a specific method for computing it. For example:

$$\begin{aligned} \gcd(231, 165) &= \gcd(165, 231) = \gcd(165, 66) \\ &= \gcd(66, 165) = \gcd(66, 99) = \gcd(66, 33) = \gcd(33, 66) \\ &= \gcd(33, 33) = 33 \end{aligned}$$

The given example is merely an instance of the following generalization: the characterization of a partial function f as the least solution of a system of simple identities usually yields an algorithm (Moschovakis 2006). This aspect makes CLFPT incredibly useful for computer science.

Fixed point theorem by Zermelo

By extending the CLFPT we can define another fixed point theorem by zermelo. We would in this case require the application of the Hartog's theorem to construct a well-ordered set to provide a proof for this (though it is ommitted here)

Fixed Point Theorem (Zermelo) (Moschovakis 2006, 7.35) Every expansive [8] mapping $\pi : P \to P$ on an inductive poset has at least one fixed point, i.e., some $x^* \in P$ satisfies the equation $x^* = \pi(x^*)$.

Knaster-Tarski Priniciple and Axiom Of Choice

Fixed points are more than just useful in defining recursion. They can be used for producing equivalent statements to the axiom of choice. We present a fixed point that eventually produces an equivalent statement to the Axiom of Choice (AC).

A set-theoretical fixed point theorem for maps in the power set of a set proved in 1927 by Knaster and improved by Tarski led to the following theorem referred to as the Knaster–Tarski principle in the literature (We should note that isotone and monotone will be used interchangeably to mean the same).

Knaster-Tarski (Subrahmanyam 2018, 3.2.2) Let $(X, \leq)$ be a poset and $f : X \to X$ be an isotone map such that (i) $b \leq f(b)$ for some $b \in X$ (ii) every chain in $X_1 = x \in X; b \leq x$ has a supremum. Then F_f , the set of fixed points of f is nonempty and contains a maximal fixed point.

In this context we can restate the Fixed point theorem of Zermelo as thus:

Theorem (Subrahmanyam 2018, 3.2.6) If every well-ordered subset A of a poset $(X, \leq)$ has an upperbound in X and $f : X \to X$ is a map such that $x \leq f(x)$ for all $x \in X$, then f has a fixed point in X.

This statement is equivelent to AC (Due to Moroianu (Subrahmanyam 2018, 54)).

Conclusion

In conclusion, a fixed point is an important tool in set theory as it is useful in describing operations and constructions on sets and orders. And it is also quite useful in computer science to derive algorithms and produce the correctness of its execution. Fixed points appear in many interesting parts of mathematics and in aspects of real life. It derives from the simple idea of self reference and how certain aspects remain unchanged under transforms that map objects to themselves. Fixed point theorems exploit the fact that these “immutable” elements exist and help construct arguments to create useful, and powerful, applications out of them.

Appendix

(Note that here I reference theorems by the same number as in the text as there are references to theorems in the book that I did not need to elaborate on)

1. **6.18.** (Moschovakis 2006) Definition. A mapping $\pi : P \to Q$ on a poset P to another is monotone if for all $x, y \in P$, $x \leq_P y \implies \pi(x) \leq_Q \pi(y)$.

2. **6.10.** (Moschovakis 2006) Definition. A chain in a poset P is any linearly ordered subset S of P, i.e., a set satisfying $(\forall x, y \in S)[x \leq y \vee y \leq x]$.
   A poset P is chain-complete or inductive if every chain in P has a least upper bound.

3. **6.19.** Definition. A monotone mapping $\pi : P \to Q$ on an inductive poset to another is countably continuous if for every non-empty, countable chain $S \subseteq P$, $\pi(\sup S) = \sup \pi[S]$.

4. **6.23.** (Moschovakis 2006) Proposition. A mapping $\pi : (A \rightharpoonup E) \to (B \rightharpoonup M)$ is continuous if and only if it satisfies both (6-13) and its converse, i.e., if for every $f : A \rightharpoonup E$ and all $y \in B$ and $v \in M$:

   $\pi(f)(y) = v \Longleftrightarrow (\exists f_0 \subseteq f)[f_0$ is finite $\& \pi(f_0)(y) = v]$. $(6 - 13^*)$

   For example, the mapping $\pi : (\mathbb{N} \to \mathbb{N}) \to (\mathbb{N} \rightharpoonup \mathbb{N})$ defined by $\pi(f) = (n \mapsto f(n) + f(n^2))$ is continuous, as for every f and n, $\pi(f)(n) = \pi(f0)(n)$, where f_0 is the restriction of f to the two-element set $\{n, n^2\}$.

5. **6.28.** (Moschovakis 2006) Lemma. If $S \subseteq (A \rightharpoonup E)$ is a non-empty chain in a partial function poset and $f_0 \subseteq \sup S$ is a finite function, then there exists some $g \in S$ such that $f_0 \subseteq g$

6. **6.29.** (Moschovakis 2006) Lemma. Every continuous mapping $\pi : (A \rightharpoonup E) \to (B \rightharpoonup M)$ is countably continuous, in fact, for every (not necessarily countable) non-empty chain $S \subseteq (A \rightharpoonup E)$,$\pi(\sup S) = \sup \pi[S]$.

7. **6.30.** (Moschovakis 2006) Proof of the Recursion Theorem from CLFPT. For each given $a \in E$ and function $h : E \to E$, we define the mapping $\pi : (N \rightharpoonup E) \to (N \rightharpoonup E)$ by the formula $\pi(f) = f'$, where

   $$f'(x) = \begin{cases} a & \text{if } x = 0 \\ h(f(x-1)) & \text{if } x > 0 \end{cases}$$

   where f is any partial function from $\mathbb{N}$ to E and we understand the definition naturally, so that $x > 0 \implies [f(x)\downarrow \iff h(f(x-1))\downarrow \iff f(x-1)\downarrow]$.

Written out in detail, the mapping π associates a set of pairs $f' \subseteq (\mathbb{N} \times E)$ with every $f \in (\mathbb{N} \to E)$ and it is defined by the equation

$$\pi(f) = \{(0,a)\} \cup \{(x, h(w)) | x > 0 \& (x-1, w) \in f\} (f : \mathbb{N} \to E). (6-14)$$

From this we get that for every f and x, $\pi(f)(x) = \pi(f_0)(x)$, where $f_0 = \{(0,a)\}$ if $x = 0$ and $f_0 = (x-1, f(x-1))$if $x > 0$,so that π is continuous by Proposition 6.23, and hence countably continuous. Thus by 6.21, it has a fixed point: that is, some partial function $f^* : N \rightharpoonup E$ exists which satisfies $f^* = \pi(f^*)$, so that, immediately,

$$f^*(0) = a, (6-15)$$
$$f^*(x+1) = h(f^*(x))(f^*(x)\downarrow). (6-16)$$

Theorem 6.21 does not guarantee that this f^* is a total function, with domain of definition the entire $\mathbb{N}$, but this can be verified by an easy induction on x using the identities (6-15) and (6-16).

8. **7.19.** (Moschovakis 2006) Definition A mapping $\pi : P \to P$ on a poset to itself is expansive, if for all $x \in P, x \leq \pi(x)$.

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