Physics II (PH 102)

Electromagnetism (Lecture 4)

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Jan 2020

1st Fundamental Theorem for Gradients

Theorem

If ϕ is a differentiable scalar field with continuous gradient $\mathbf{F} = \nabla\phi$ in $\mathbb{R}^3$ and A and B are any two points in this 3D space, then the total change in ϕ in going from A and B is

$$\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_A^B \nabla\phi \cdot d\mathbf{r} = \int_A^B d\phi = \phi(B) - \phi(A)$$

over any smooth path C joining A and B.

Note: *Here we used the* ***CHAIN RULE:***

$$d\phi(x, y, z) = \left(\frac{\partial\phi}{\partial x}\right) dx + \left(\frac{\partial\phi}{\partial y}\right) dy + \left(\frac{\partial\phi}{\partial z}\right) dz = \nabla\phi \cdot d\mathbf{r}$$

In other words, the integral of the gradiant of a function over some interval is given by the value of the function at the bounderies.

Corollary

(1) $\int_A^B \nabla\phi \cdot d\mathbf{r}$ is independent of path C .

Corollary

(2) $\oint \nabla\phi \cdot d\mathbf{r} = 0$, for EVERY closed path ($\because$ end points are identical.)

2nd Fundamental Theorem for Gradients

Theorem

Let $\mathbf{F}$ *be a continuous vector field over* $\mathbb{R}^3$ *such that its line integral between any two points in space is independent of the path. Also, let* ϕ *be a scalar field over* $\mathbb{R}^3$ *such that*

$$\phi(\mathbf{r}) = \int_{\mathbf{a}}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

where $\mathbf{a} = (a_x, a_y, a_z)$ *is some fixed reference point in the 3D space. Then it follows that* $\nabla\phi = \mathbf{F}$.

Corollary

(1) If $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ *for EVERY closed path, then* $\nabla\phi = \mathbf{F}$. *The field* $\mathbf{F}$ *is then said to be CONSERVATIVE, while the field* ϕ *is the SCALAR POTENTIAL.*

Corollary

(2) Since $\nabla\phi = \mathbf{F}$, *it must be that* $\nabla \times \mathbf{F} = \nabla \times (\nabla\phi) = 0$.

Fundamental Theorem for Divergence

Theorem

Gauss' Theorem: Let V be a closed bounded region in $\mathbb{R}^3$ whose boundary is the smooth or piecewise smooth closed surface S with $\hat{\mathbf{N}}_{\text{out}}$ being the unit outward normal. If $\mathbf{F}$ is a vector function with continuous partial derivatives in V, then the volume integral of its divergence over V is equal to the surface integral of the outer normal component of $\mathbf{F}$ over the bounding surface S, i.e.,

$$\iiint_V (\nabla \cdot \mathbf{F})\, dv = \oiint_S \mathbf{F} \cdot \hat{\mathbf{N}}_{\text{out}}\, dS.$$

Corollary

(1) If $\oiint \mathbf{F} \cdot d\mathbf{S} = 0$ for EVERY closed surface, then $\nabla \cdot \mathbf{F} = 0$ IDENTICALLY, in which case $\mathbf{F}$ is SOLENOIDAL.

Corollary

(2) If there exists a vector field $\mathbf{A}$, such that $\mathbf{F} = \nabla \times \mathbf{A}$, then the identity $\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$ implies $\oiint \mathbf{F} \cdot d\mathbf{S} = 0$ for EVERY closed surface in which case $\mathbf{A}$ is termed as the VECTOR POTENTIAL of the field $\mathbf{F}$.

Corollaries (1) & (2): For every SOLENOIDAL vector field there exists a VECTOR POTENTIAL and vice versa.

Verification of Gauss' Theorem (Simple Example)

Example

Let $\mathbf{V} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}} \equiv \mathbf{r}$ and S be the surface of the sphere, $x^2 + y^2 + z^2 = a^2$, enclosing the region V. Verify Gauss' Theorem.

- Outward unit normal on S: Define $F(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0$. Then,

$$\hat{\mathbf{N}} = \left(\frac{\nabla F}{|\nabla F|}\right)_S = \left[\frac{2(x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})}{\sqrt{4(x^2 + y^2 + z^2)}}\right]_S = \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}}{a} = \frac{\mathbf{r}}{a} = \hat{\mathbf{r}}$$

- $\left(\mathbf{V} \cdot \hat{\mathbf{N}}\right)_S \, dS = (\mathbf{r} \cdot \hat{\mathbf{r}})_S \, dS = a \, dS$
- Closed Surface Integral:

$$\oiint_S \mathbf{V} \cdot \hat{\mathbf{N}} \, dS = a \oiint_S dS = a(4\pi a^2) = 4\pi a^3$$

- $\nabla \cdot \mathbf{V} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 3$
- Volume Integral:

$$\iiint_V \nabla \cdot \mathbf{V} \, dv = 3 \iiint_V dv = 3\left(\frac{4}{3}\pi a^3\right) = 4\pi a^3$$

- Hence, Gauss' Theorem is verified.

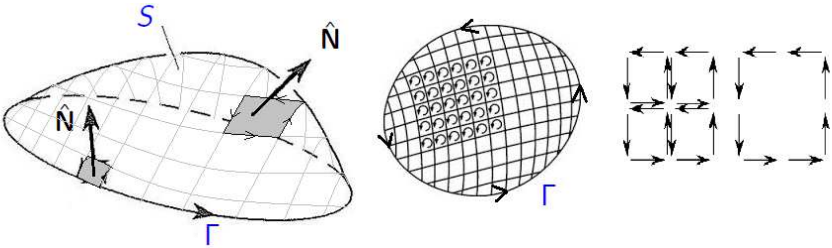
Fundamental Theorem For Curl

Theorem

Stokes' Theorem: Let S be a smooth orientable (i.e., two sided) open surface in $\mathbb{R}^3$ bounded by simple (i.e., nonintersecting), smooth or piecewise smooth closed curve Γ. If $\mathbf{F}$ is a continuously differentiable vector field, then the surface integral of the normal component of its curl over the surface S is equal to the circulation of $\mathbf{F}$ about Γ, i.e.,

$$\iint_S (\nabla \times \mathbf{F}) \cdot \hat{\mathbf{N}}\, dS = \oint_\Gamma \mathbf{F} \cdot d\mathbf{r},$$

where for the surface S the direction of unit normal vector $\hat{\mathbf{N}}$ is determined by the right hand rule (traversing Γ in the positive direction.)



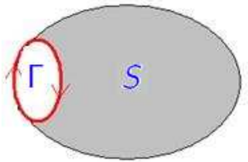
Fundamental Theorem for Curl (contd.)

Corollary

(1) The integral $\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ is independent of the geometry of the bounded open surface S, and depends ONLY on the nature of boundary curve Γ.

Corollary

(2) For EVERY closed surface S, $\oiint (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0$ IDENTICALLY, since for ALL closed surfaces there are no boundary curves.



Corollary

(3) If $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for EVERY closed loop, then $\nabla \times \mathbf{F} = 0$ IDENTICALLY, in which case $\mathbf{F}$ is IRROTATIONAL.

Verification of Stokes' Theorem (Simple Example)

Example

Let $\mathbf{F} = (2xz + 3y^2)\hat{\mathbf{j}} + 4yz^2\hat{\mathbf{k}}$ and S be the square in yz-plane, i.e., $x = 0$, with $0 \le (y, z) \le 1$. Verify Stokes' Theorem.

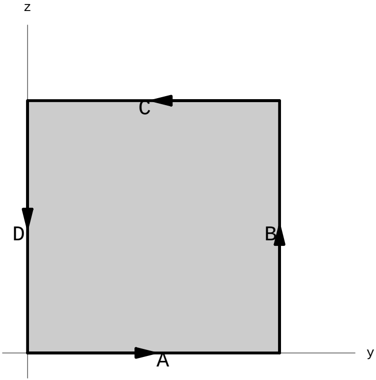
- $d\mathbf{S} = dydz\hat{\mathbf{i}} \to +$ sign chosen by right-hand rule
- $\nabla \times \mathbf{F} = (4z^2 - 2x)\hat{\mathbf{i}} + 2z\hat{\mathbf{k}}$
- Surface Integral: Evaluate with $x = 0$ on S

$$\begin{aligned}\iint\limits_S (\nabla \times \mathbf{F})_{x=0} \cdot d\mathbf{S} &= 4\int_{z=0}^{z=1} z^2 dz \int_{y=0}^{y=1} dy \\ &= 4/3\end{aligned}$$

- Contour Integral: Break it into 4 Line Integrals $\oint_{ABCD} (\mathbf{F} \cdot d\mathbf{r})_{x=0} = \oint_{ABCD} \left(3y^2 dy + 4yz^2 dz\right)$

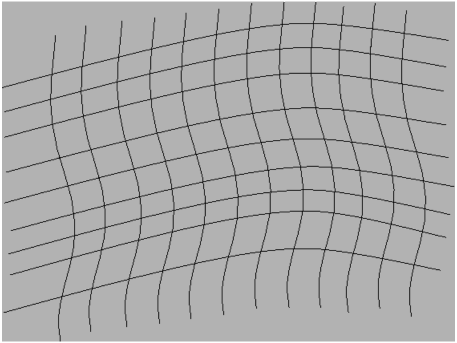
$$\int_A \mathbf{F} \cdot d\mathbf{r} = 1 \quad , \quad \int_B \mathbf{F} \cdot d\mathbf{r} = 4/3,$$
$$\int_C \mathbf{F} \cdot d\mathbf{r} = -1 \quad , \quad \int_D \mathbf{F} \cdot d\mathbf{r} = 0.$$

- $\oint_{ABCD} \mathbf{F} \cdot d\mathbf{r} = 4/3$. Hence, Stokes' Theorem is verified



General Curvilinear Co-ordinate System

- In 3D geometry, *Curvilinear Co-ordinate Systems* refer to a systems where the co-ordinate lines are curved, unlike the familiar *Rectangular Cartesian Co-ordinate System* (x, y, z).
- The curvilinear system could be *orthogonal* in which co-ordinate lines always intersect at right angles (Spherical Polar, Cylindrical, Parabolic Cylindical, Paraboloidal, Elliptic Cylindrical, Ellipsoidal, ...).
- Skew or *non-orthogonal* co-ordinate sytems are much complicated and seldom useful in physical applications.



General Curvilinear Co-ordinate System (contd.)

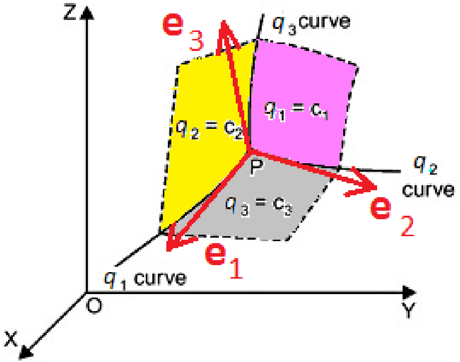
- Consider the co-ordinates of a point P in 3D space. The curvilinear coordinates, say $P(q_1, q_2, q_3)$ may be derived from the Cartesian coordinates $P(x, y, z)$ though certain unique & invertible relations in terms of smooth functions $f_{1,2,3}$ and $g_{1,2,3}$ called *Co-ordinate Transformations*:

$$
\begin{aligned}
q_1 &= f_1(x, y, z) \quad ; \quad x = g_1(q_1, q_2, q_3) \equiv f_1^{-1}, \\
q_2 &= f_2(x, y, z) \quad ; \quad y = g_2(q_1, q_2, q_3) \equiv f_2^{-1}, \\
q_3 &= f_3(x, y, z) \quad ; \quad z = g_3(q_1, q_2, q_3) \equiv f_3^{-1}.
\end{aligned}
$$

- The choice of the co-ordinate systems are fixed *only for convenience purpose*, often utilizing the *constraints/symmetries* of applications.
- *Cuboidial*, *Spherical* and *Cylindrical* symmetries are very common in Physical (electrodynamical) application, hence we shall deal with Spherical Polar and Cylindical curvilinear co-ordinate systems and study their transformations to and from Cartesian system.

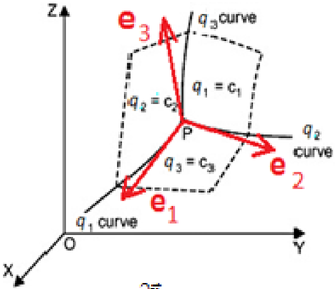
WARNING: All physics should be independent of the co-ordinate system used.

Typical Orthogonal Curvilinear System in 3D (q_1, q_2, q_3); $q_i \in \mathbb{R}$



- The curved surfaces $q_1 = c_1 = const.$, $q_2 = c_2 = const.$, and $q_3 = c_3 = const.$ are called co-ordinate surfaces. Any point $\mathrm{P}(q_1, q_2, q_3)$ is the intersection of the three such co-ordinate surfaces.
- The orthogonal set of curves formed by the intersection of pairs of co-ordinate surfaces are called co-ordinate lines/axes.
- The unit vector $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3)$, unlike the Cartesian ones $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$, do not point in specific directions in space. Their directions are instead specified by the tangents to the co-ordinate lines at each point $\mathrm{P}(q_1, q_2, q_3)$.

Unit vectors $(\hat{e}_1, \hat{e}_2, \hat{e}_3)$ as the Orthonormal Basis



Unit vector

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \equiv \vec{r}(x, y, z)$$

$$\vec{r} = \vec{r}(q_1, q_2, q_3)$$

$$\vec{A}(q_1, q_2, q_3) = A_1\hat{e}_1 + A_2\hat{e}_2 + A_3\hat{e}_3$$

A tangent vector to the q_i *curve at P is* $\dfrac{\partial \vec{r}}{\partial q_i}$

$$\frac{\partial \vec{r}}{\partial q_i} = h_i\,\hat{e}_i \quad ; \quad \hat{e}_i \rightarrow \text{unit vector tangent to the coordinate curves}$$

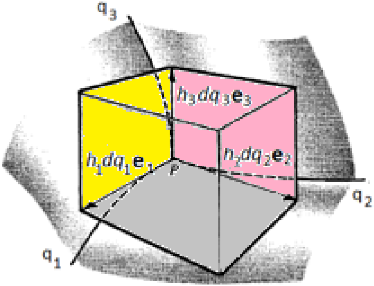
$h_i \rightarrow$ *scale factor*

A unit tangent vector in the direction of q_1-axis : $\hat{e}_1 = \dfrac{\partial \vec{r}}{\partial q_1} \Big/ \left|\dfrac{\partial \vec{r}}{\partial q_1}\right|$

$$\hat{i} = \frac{\partial \vec{r}}{\partial x} \Rightarrow h_x = \left|\frac{\partial \vec{r}}{\partial x}\right| = 1$$

$$\hat{e}_1 \cdot \hat{e}_1 = 1, \quad \hat{e}_2 \cdot \hat{e}_2 = 1, \quad \hat{e}_3 \cdot \hat{e}_3 = 1,$$
$$\hat{e}_1 \cdot \hat{e}_2 = 0, \quad \hat{e}_1 \cdot \hat{e}_3 = 0, \quad \hat{e}_2 \cdot \hat{e}_3 = 0,$$
$$\hat{e}_1 \times \hat{e}_2 = \hat{e}_3, \quad \hat{e}_2 \times \hat{e}_3 = \hat{e}_1, \quad \hat{e}_3 \times \hat{e}_1 = \hat{e}_2,$$
$$\hat{e}_1 \times \hat{e}_1 = 0, \quad \hat{e}_2 \times \hat{e}_2 = 0, \quad \hat{e}_3 \times \hat{e}_3 = 0$$

Line (Arc), Area and Volume elements



$$\vec{r} = \vec{r}(q_1, q_2, q_3)$$

$$d\vec{r} = \frac{\partial \vec{r}}{\partial q_1} dq_1 + \frac{\partial \vec{r}}{\partial q_2} dq_2 + \frac{\partial \vec{r}}{\partial q_3} dq_3$$

$$= h_1 dq_1 \hat{e}_1 + h_2 dq_2 \hat{e}_2 + h_3 dq_3 \hat{e}_3$$

$$= ds_1 \hat{e}_1 + ds_2 \hat{e}_2 + ds_3 \hat{e}_3$$

Arc elements : $ds_1 = h_1 dq_1, \; ds_2 = h_2 dq_2, \; ds_3 = h_3 dq_3$

Volume element : $dV = h_1 h_2 h_3 \, dq_1 dq_2 dq_3$

Area elements :
$$\begin{aligned} d\vec{a}_1 &= h_2 h_3 \hat{e}_1 \, dq_2 dq_3 \\ d\vec{a}_2 &= h_1 h_3 \hat{e}_2 \, dq_1 dq_3 \\ d\vec{a}_3 &= h_1 h_2 \hat{e}_3 \, dq_1 dq_2 \end{aligned}$$

Gradient Operator (∇) for a scalar field $\Phi(\mathbf{r}) \equiv \Phi(q_1, q_2, q_3)$

$$\begin{aligned} d\Phi &= \vec{\nabla}\Phi \cdot d\vec{r} \\ &= \left(f_1\hat{e}_1 + f_2\hat{e}_2 + f_3\hat{e}_3\right)\cdot\left(h_1\hat{e}_1\,dq_1 + h_2\hat{e}_2\,dq_2 + h_3\hat{e}_3\,dq_3\right) \\ &= h_1 f_1 dq_1 + h_2 f_2 dq_2 + h_3 f_3 dq_3 \end{aligned}$$

$$d\Phi(q_1, q_2, q_3) = \frac{\partial\Phi}{\partial q_1}dq_1 + \frac{\partial\Phi}{\partial q_2}dq_2 + \frac{\partial\Phi}{\partial q_3}dq_3$$

$$f_1 = \frac{1}{h_1}\frac{\partial\Phi}{\partial q_1}, \quad f_2 = \frac{1}{h_2}\frac{\partial\Phi}{\partial q_2}, \quad and \quad f_3 = \frac{1}{h_3}\frac{\partial\Phi}{\partial q_3}$$

$$\boxed{\vec{\nabla}\Phi = \frac{\hat{e}_1}{h_1}\frac{\partial\Phi}{\partial q_1} + \frac{\hat{e}_2}{h_2}\frac{\partial\Phi}{\partial q_2} + \frac{\hat{e}_3}{h_3}\frac{\partial\Phi}{\partial q_3} \quad \Rightarrow \quad \vec{\nabla} = \frac{\hat{e}_1}{h_1}\frac{\partial}{\partial q_1} + \frac{\hat{e}_2}{h_2}\frac{\partial}{\partial q_2} + \frac{\hat{e}_3}{h_3}\frac{\partial}{\partial q_3}}$$

Divergence ($\nabla\cdot$), Curl ($\nabla\times$), and Laplacian (∇^2) Operators

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(A_1h_2h_3\right)+\frac{\partial}{\partial q_2}\left(A_2h_1h_3\right)+\frac{\partial}{\partial q_3}\left(A_3h_1h_2\right)\right]$$

$$\vec{\nabla}\times\vec{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1\hat{e}_1 & h_2\hat{e}_2 & h_3\hat{e}_3 \\ \dfrac{\partial}{\partial q_1} & \dfrac{\partial}{\partial q_2} & \dfrac{\partial}{\partial q_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix}$$

$$\vec{\nabla}^2\Phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial q_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial q_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial q_3}\right)\right]$$