Double integrals over general regions

$$\boxed{1} \qquad F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{if } (x, y) \text{ is in } R \text{ but not in } D \end{cases}$$

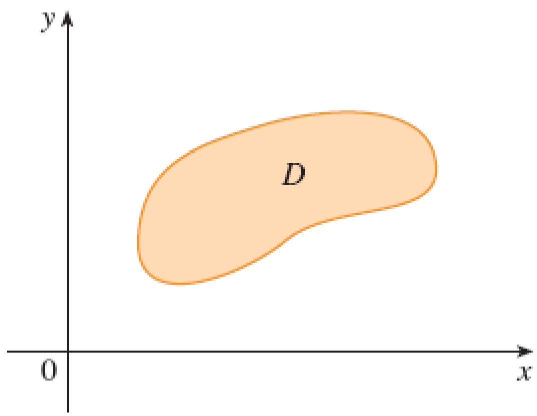


FIGURE 1

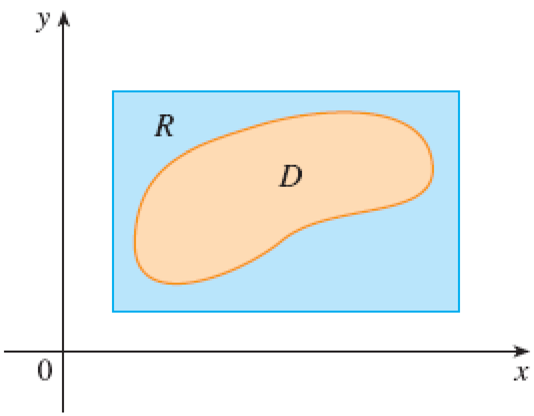


FIGURE 2

If F is integrable over R, then we define the **double integral of f over D** by

$$\boxed{2} \qquad \iint_D f(x, y)\, dA = \iint_R F(x, y)\, dA \qquad \text{where } F \text{ is given by Equation 1}$$

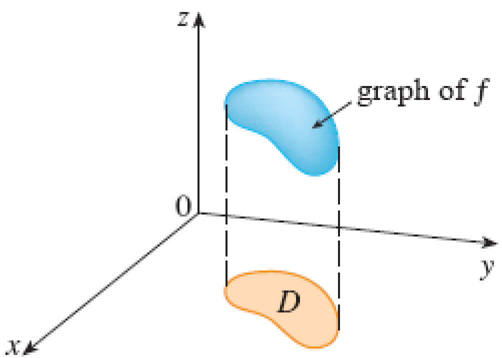


FIGURE 3

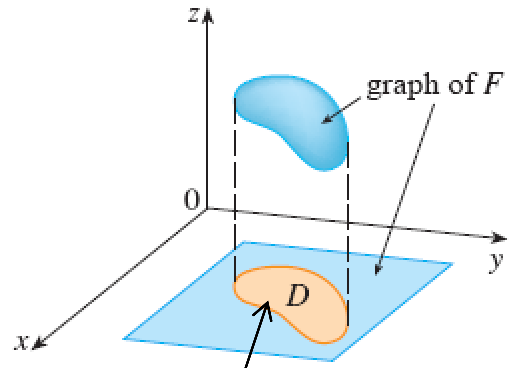


FIGURE 4

Any comment on the nature of F(x,y) on the boundary of D?

30 October 2019

A plane region D is said to be of **type I** if it lies between the graphs of two continuous functions of x, that is,

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

where g_1 and g_2 are continuous on $[a, b]$. Some examples of type I regions are shown in Figure 5.

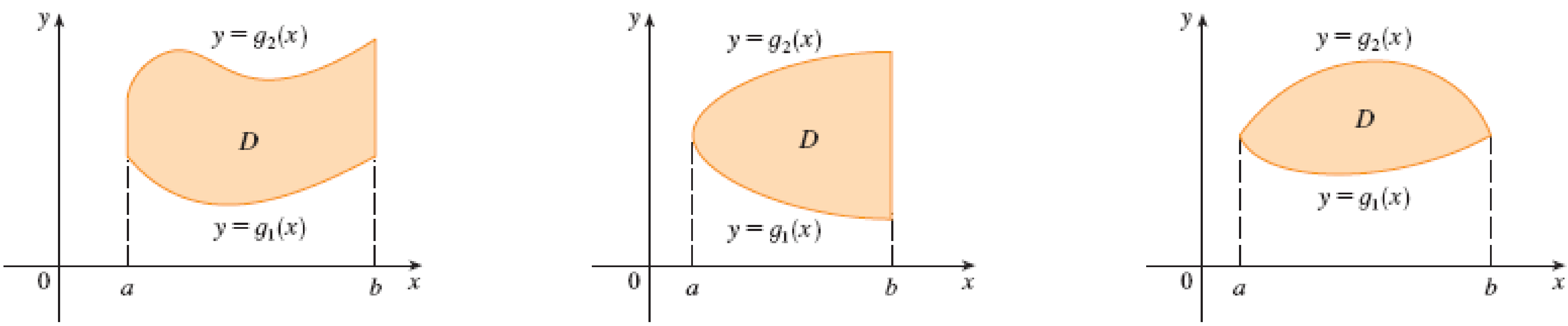


FIGURE 5 Some type I regions

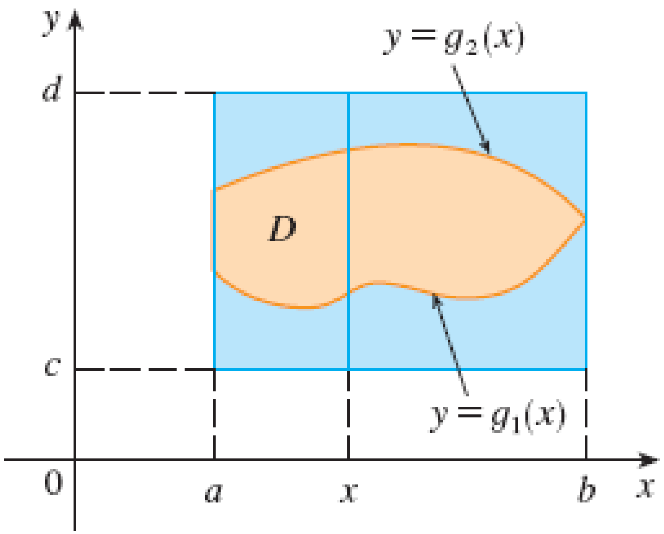


FIGURE 6

$\boxed{3}$ If f is continuous on a type I region D such that

$$D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$$

then
$$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$$

We also consider plane regions of **type II**, which can be expressed as

$$\boxed{4} \qquad D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$$

where h_1 and h_2 are continuous. Two such regions are illustrated in Figure 7.

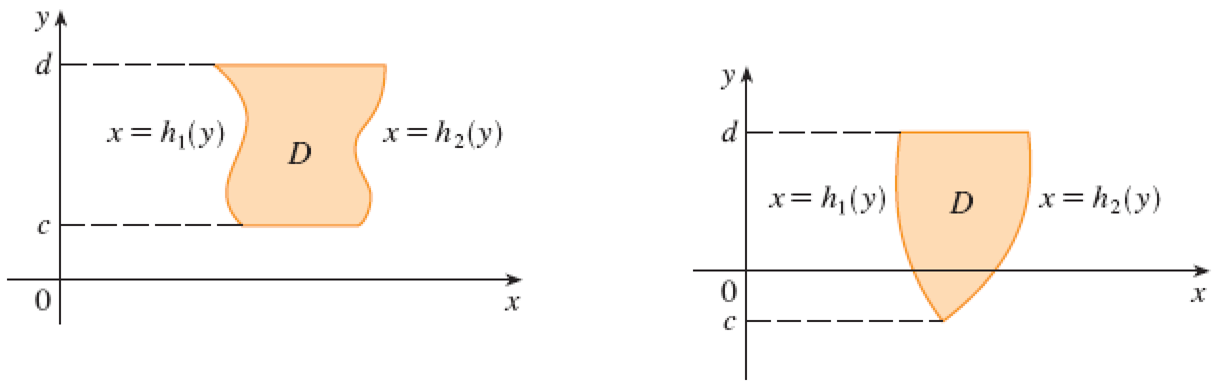


FIGURE 7
Some type II regions

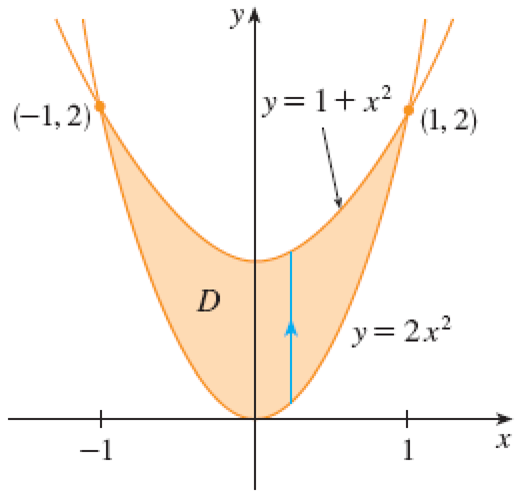
Using the same methods that were used in establishing $\boxed{3}$, we can show that

$$\boxed{5} \qquad \iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$$

where D is a type II region given by Equation 4.

V EXAMPLE 1 Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

$$\frac{32}{15}$$



EXAMPLE 2 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

$$\frac{216}{35}$$

Figure 11 shows the solid whose volume is calculated in Example 2. It lies above the xy-plane, below the paraboloid $z = x^2 + y^2$, and between the plane $y = 2x$ and the parabolic cylinder $y = x^2$.

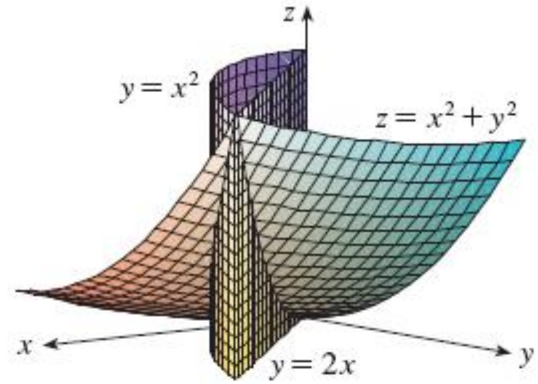


FIGURE 11

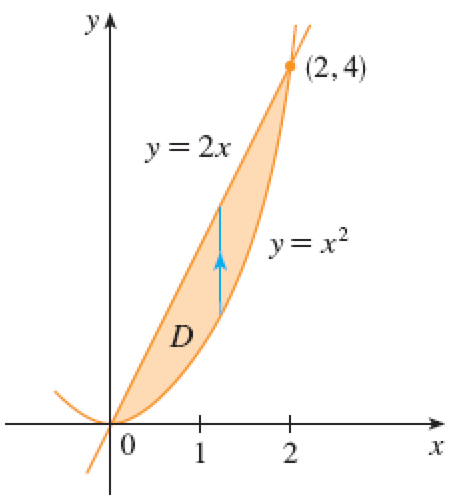


FIGURE 9
D as a type I region

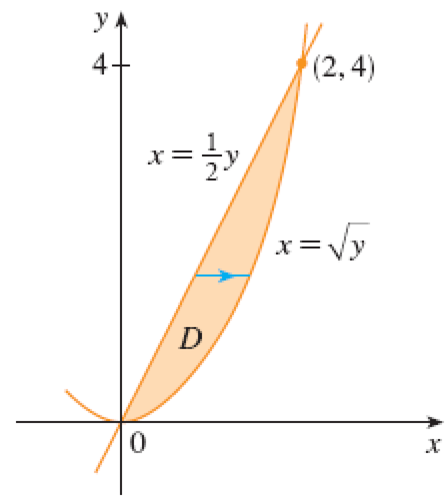


FIGURE 10
D as a type II region

EXAMPLE 4 Find the volume of the tetrahedron bounded by the planes $x + 2y + z = 2$, $x = 2y$, $x = 0$, and $z = 0$.

$$\frac{1}{3}$$

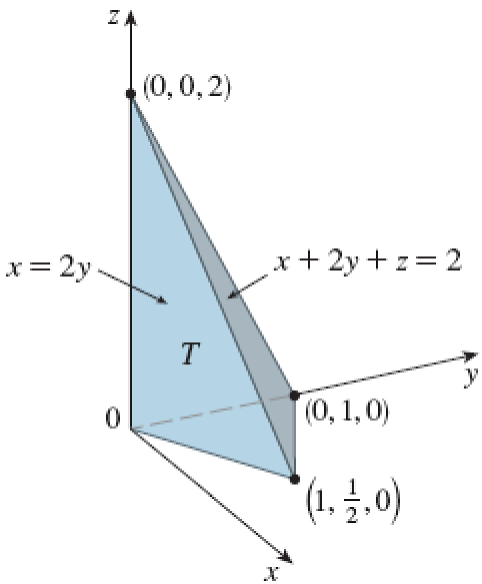


FIGURE 13

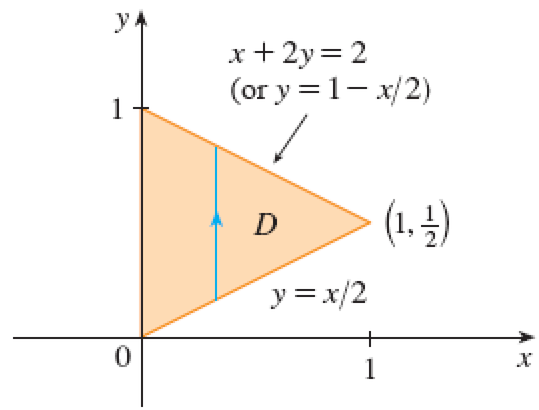


FIGURE 14

Properties of Double Integrals

$\boxed{6}$ $$\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$$

$\boxed{7}$ $$\iint_D c\,f(x,y)\,dA = c\iint_D f(x,y)\,dA$$

If $f(x,y) \geqslant g(x,y)$ for all (x,y) in D, then

$\boxed{8}$ $$\iint_D f(x,y)\,dA \geqslant \iint_D g(x,y)\,dA$$

The next property of double integrals is similar to the property of single integrals given by the equation $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

If $D = D_1 \cup D_2$, where D_1 and D_2 don't overlap except perhaps on their boundaries (see Figure 17), then

9
$$\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA$$

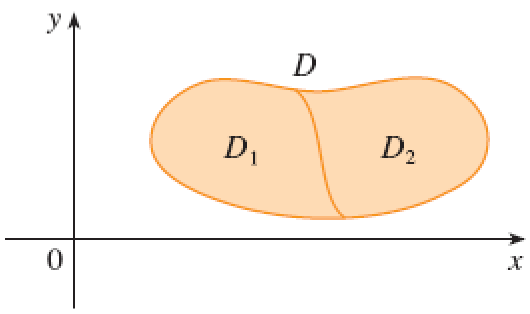


FIGURE 17

Property 9 can be used to evaluate double integrals over regions D that are neither type I nor type II but can be expressed as a union of regions of type I or type II. Figure 18 illustrates this procedure. (See Exercises 55 and 56.)

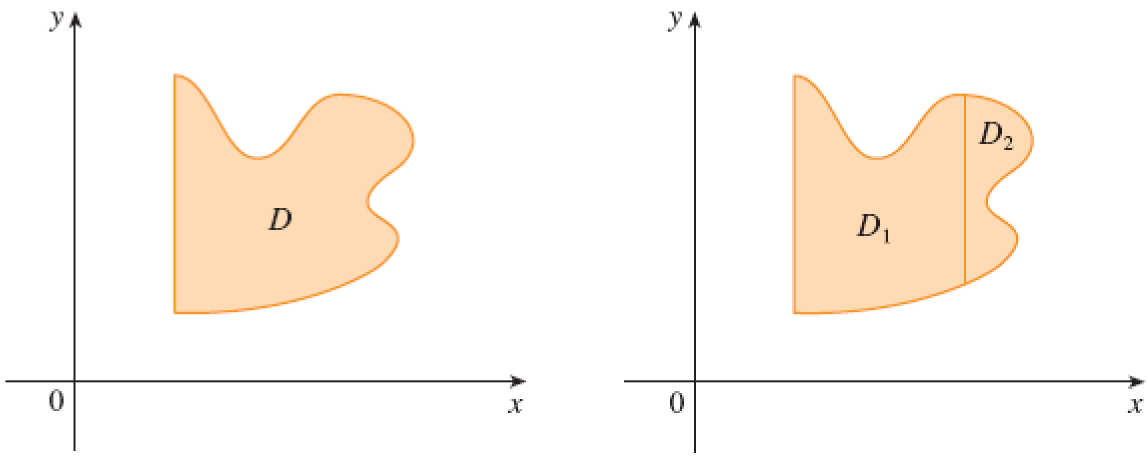


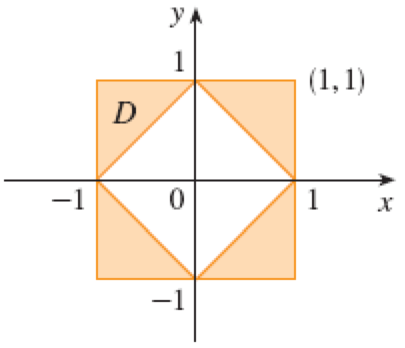
FIGURE 18

(a) D is neither type I nor type II.

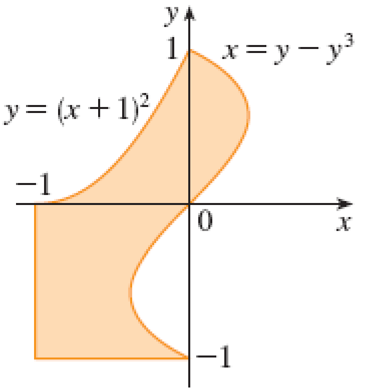
(b) $D = D_1 \cup D_2$, D_1 is type I, D_2 is type II.

55–56 Express D as a union of regions of type I or type II and evaluate the integral.

55. $\iint_D x^2\, dA$



56. $\iint_D y\, dA$



55. 1, 56. 2/15

The next property of integrals says that if we integrate the constant function $f(x, y) = 1$ over a region D, we get the area of D:

$$\textbf{(10)} \qquad \boxed{\iint_D 1\, dA = A(D)}$$

For instance, if D is a type I region and we put $f(x, y) = 1$ in Formula 3, we get

$$\iint_D 1\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} 1\, dy\, dx = \int_a^b [g_2(x) - g_1(x)]\, dx = A(D)$$

by Equation 5.1.2.

Finally, we obtain an analogue of Property 8 of single integrals by combining Properties 7, 8, and 10.

(11) If $m \leqslant f(x, y) \leqslant M$ for all (x, y) in D, then

$$mA(D) \leqslant \iint_D f(x, y)\, dA \leqslant MA(D)$$

Volume Between Two Surfaces

Suppose now that the solid region T lies above the plane region R, as before, but *between* the surfaces $z = z_1(x, y)$ and $z = z_2(x, y)$, where $z_1(x, y) \leqq z_2(x, y)$ for all (x, y) in R (Fig. 14.3.11). Then we get the volume V of T by subtracting the volume below $z = z_1(x, y)$ from the volume below $z = z_2(x, y)$, so

$$V = \iint_R [z_2(x, y) - z_1(x, y)]\, dA. \qquad \textbf{(5)}$$

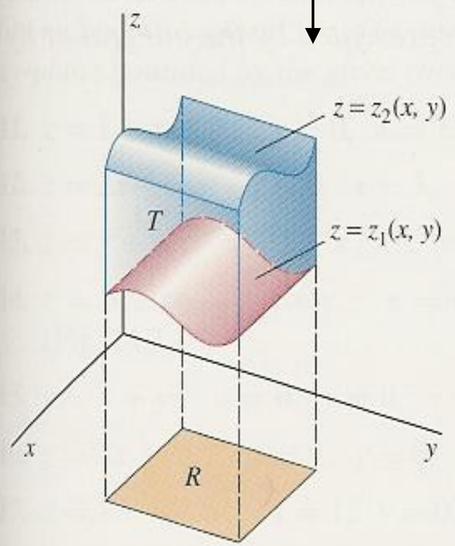
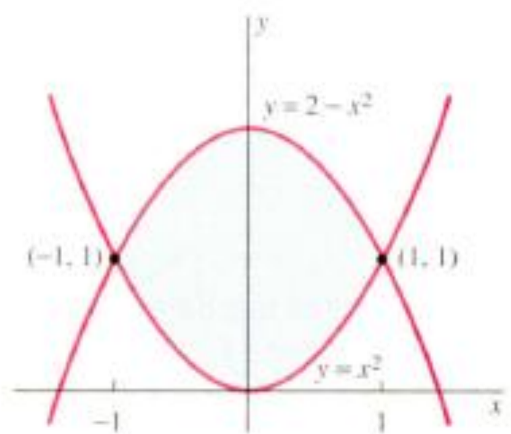


FIGURE 14.3.11 The solid T has vertical sides and is bounded above and below by surfaces.

EXAMPLE 4 Find the volume V of the solid T bounded by the planes $z = 6$ and $z = 2y$ and by the parabolic cylinders $y = x^2$ and $y = 2 - x^2$. This solid is sketched in Fig. 14.3.12.



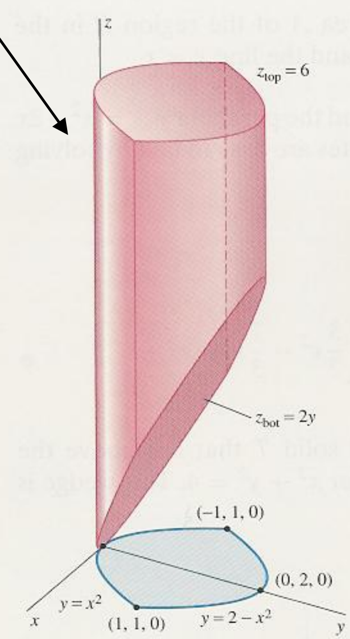
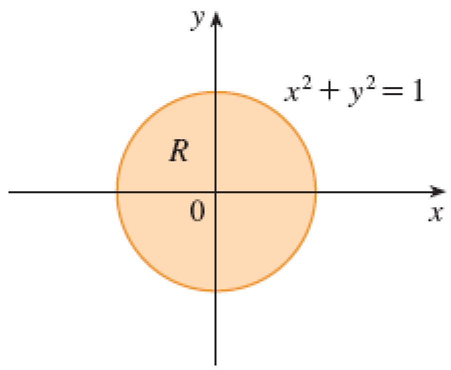
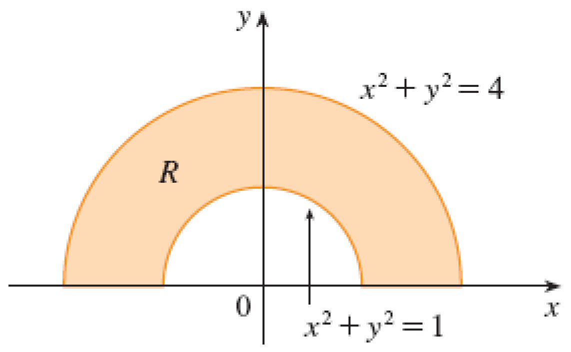


FIGURE 14.3.12 The solid T of Example 4.

Double integrals in polar coordinates



(a) $R = \{(r, \theta) \mid 0 \leq r \leq 1, 0 \leq \theta \leq 2\pi\}$



(b) $R = \{(r, \theta) \mid 1 \leq r \leq 2, 0 \leq \theta \leq \pi\}$

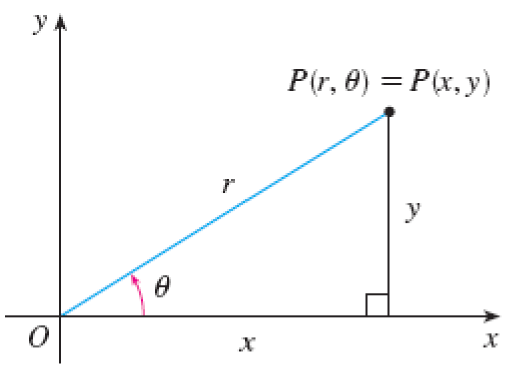


FIGURE 2

$$\boxed{r^2 = x^2 + y^2 \qquad x = r\cos\theta \qquad y = r\sin\theta}$$

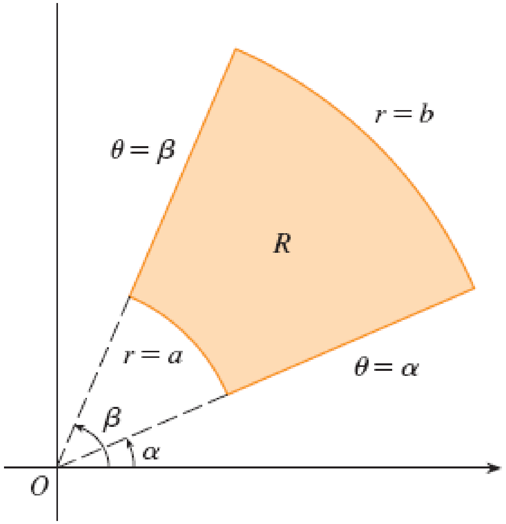


FIGURE 3 Polar rectangle

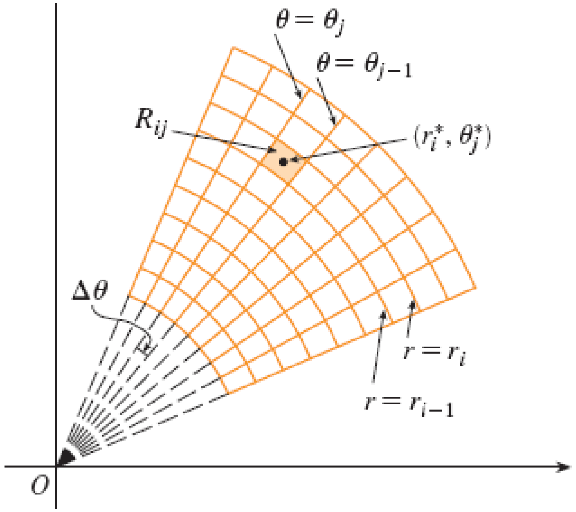


FIGURE 4 Dividing R into polar subrectangles

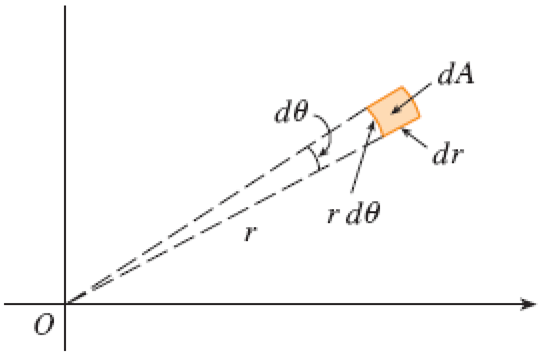


FIGURE 5

$$\boxed{1} \qquad \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\,\Delta A_i = \sum_{i=1}^{m}\sum_{j=1}^{n} f(r_i^* \cos\theta_j^*, r_i^* \sin\theta_j^*)\, r_i^*\,\Delta r\,\Delta\theta$$

$\boxed{2}$ **Change to Polar Coordinates in a Double Integral** If f is continuous on a polar rectangle R given by $0 \leqslant a \leqslant r \leqslant b$, $\alpha \leqslant \theta \leqslant \beta$, where $0 \leqslant \beta - \alpha \leqslant 2\pi$, then

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

V EXAMPLE 2 Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

$$\frac{\pi}{2}$$

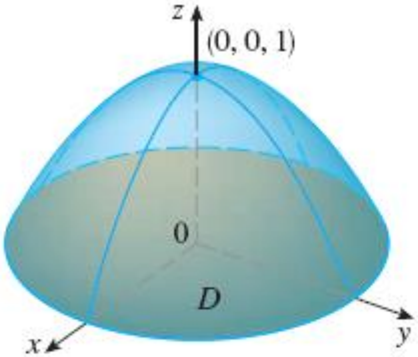


FIGURE 6

More Complicated Region

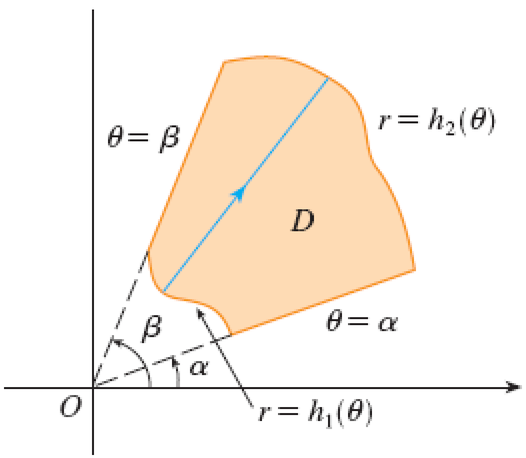


FIGURE 7

$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$

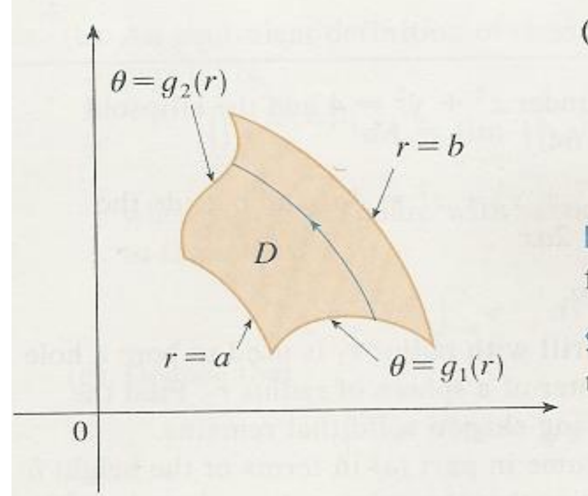


FIGURE 7

Type I polar region,

$D = \{(r, \theta) \mid a \leqslant r \leqslant b,$
$g_1(r) \leqslant \theta \leqslant g_2(r)\}$

3 If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \leqslant \theta \leqslant \beta,\ h_1(\theta) \leqslant r \leqslant h_2(\theta)\}$$

then $$\iint_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

(4) $$\iint_D f(x, y)\, dA = \int_a^b \int_{g_1(r)}^{g_2(r)} f(r\cos\theta, r\sin\theta)\, r\, d\theta\, dr$$

V EXAMPLE 4 Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

$$\frac{3\pi}{2}$$

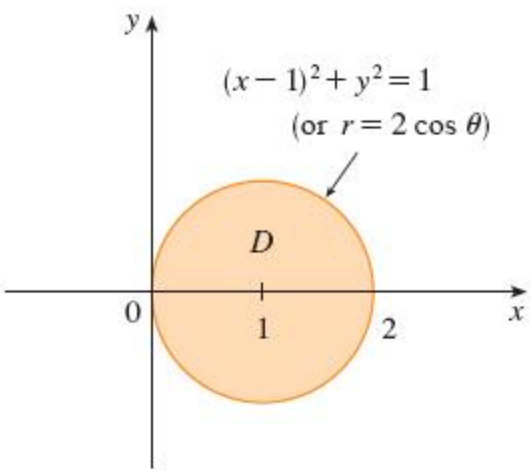


FIGURE 9

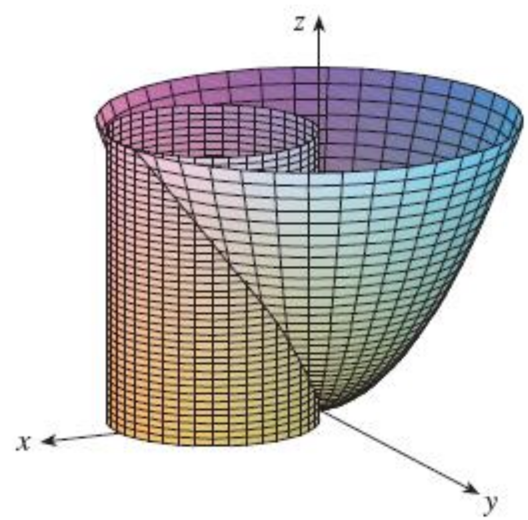


FIGURE 10

Application of Double integrals

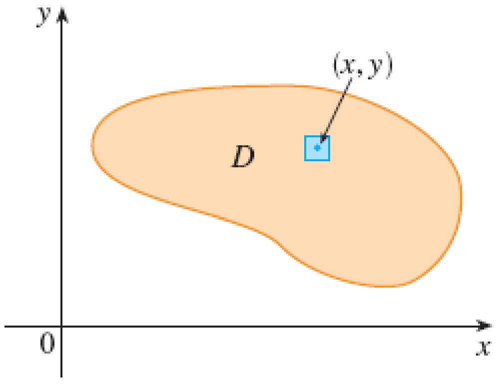


FIGURE 1

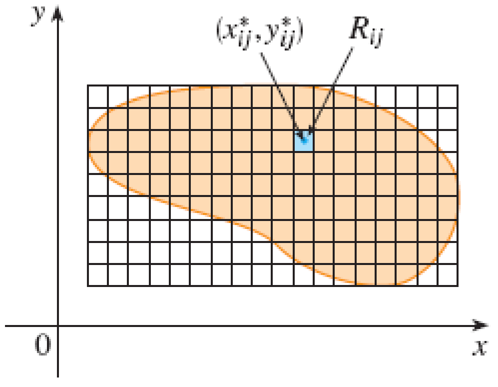


FIGURE 2

Formula for computing mass

Consider a lamina with density function ρ(x,y) that occupies a region D.

1

$$\boxed{m = \lim_{k,\, l \to \infty} \sum_{i=1}^{k} \sum_{j=1}^{l} \rho(x_{ij}^*, y_{ij}^*)\, \Delta A = \iint_D \rho(x, y)\, dA}$$

Moments and centers of mass

Consider a lamina with density function ρ(x,y) that occupies a region D.
Then the moment of the entire lamina about x-axis is

3 $$M_x = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} y_{ij}^* \, \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D y \rho(x, y) \, dA$$

Similarly, the **moment about the y-axis** is

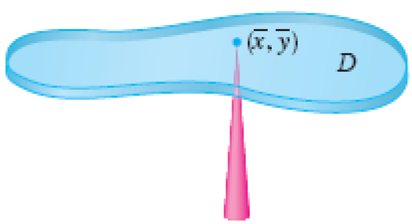
4 $$M_y = \lim_{m,\, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij}^* \, \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D x \rho(x, y) \, dA$$

5 The coordinates $(\overline{x}, \overline{y})$ of the center of mass of a lamina occupying the region D and having density function $\rho(x, y)$ are

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x \rho(x, y) \, dA \qquad \overline{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y \rho(x, y) \, dA$$

where the mass m is given by

$$m = \iint_D \rho(x, y) \, dA$$



V EXAMPLE 2 Find the mass and center of mass of a triangular lamina with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.

$$m = \frac{8}{3} \qquad \bar{x} = \frac{3}{8} \qquad \bar{y} = \frac{11}{16}$$

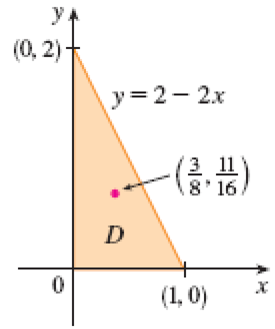
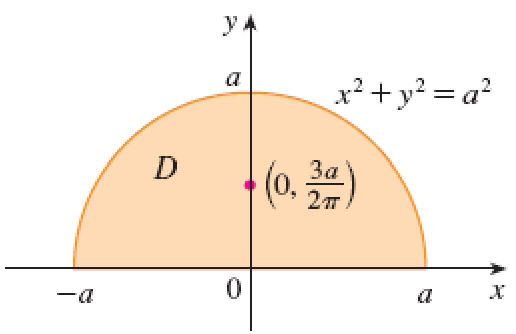


FIGURE 5

V EXAMPLE 3 The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.



Moments of inertia

The **moment of inertia** (also called the **second moment**) of a particle of mass m about an axis is defined to be mr^2, where r is the distance from the particle to the axis. We extend this concept to a lamina with density function $\rho(x, y)$ and occupying a region D by proceeding as we did for ordinary moments. We divide D into small rectangles, approximate the moment of inertia of each subrectangle about the x-axis, and take the limit of the sum as the number of subrectangles becomes large. The result is the **moment of inertia** of the lamina **about the x-axis**:

$$\boxed{6} \qquad I_x = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (y_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D y^2 \rho(x, y) \, dA$$

Similarly, the **moment of inertia about the y-axis** is

$$\boxed{7} \qquad I_y = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} (x_{ij}^*)^2 \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D x^2 \rho(x, y) \, dA$$

It is also of interest to consider the **moment of inertia about the origin**, also called the **polar moment of inertia**:

$$\boxed{8} \qquad I_0 = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \left[(x_{ij}^*)^2 + (y_{ij}^*)^2\right] \rho(x_{ij}^*, y_{ij}^*) \, \Delta A = \iint_D (x^2 + y^2) \rho(x, y) \, dA$$