Department of Mathematics
Indian Institute of Technology Guwahati
MA 101: Mathematics I
Solutions of Tutorial Sheet-2
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1. Let (x_n) be a convergent sequence in $\mathbb{R}$ with limit $\ell \in \mathbb{R}$ and let $\alpha \in \mathbb{R}$.

   (a) If $x_n \geq \alpha$ for all $n \in \mathbb{N}$, then show that $\ell \geq \alpha$.

   (b) If $\ell > \alpha$, then show that there exists $n_0 \in \mathbb{N}$ such that $x_n > \alpha$ for all $n \geq n_0$.

   (c) If (x_n) and (y_n) are convergent sequences and $x_n \geq y_n$ for all $n \in \mathbb{N}$, then $\lim\limits_{n\to\infty} x_n \geq \lim\limits_{n\to\infty} y_n$.

   (Note that ℓ can be equal to α in (a) even if $x_n > \alpha$ for all n.)

   Solution. (a) If possible, let $\ell < \alpha$. Then $\alpha - \ell > 0$ and since $x_n \to \ell$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - \ell| < \alpha - \ell$ for all $n \geq n_0$. This implies that $x_n < \ell + \alpha - \ell = \alpha$ for all $n \geq n_0$, which is a contradiction. Hence $\ell \geq \alpha$.
   (b) Since $\ell - \alpha > 0$ and since $x_n \to \ell$, there exists $n_0 \in \mathbb{N}$ such that $|x_n - \ell| < \ell - \alpha$ for all $n \geq n_0$. This implies that $x_n > \ell - (\ell - \alpha) = \alpha$ for all $n \geq n_0$.
   (c) We have $x_n - y_n \geq 0$ for all n. The result follows readily from part (a).

   (Note that although $\frac{1}{n} > 0$ for all $n \in \mathbb{N}$, $\lim\limits_{n\to\infty} \frac{1}{n} = 0$ and thus ℓ can be equal to α in (a) even if $x_n > \alpha$ for all n.) $\square$

2. Let (x_n) be a convergent sequence of positive real numbers such that $\lim\limits_{n\to\infty} x_n < 1$. Show that $\lim\limits_{n\to\infty} x_n^n = 0$.

   Solution. If $\ell = \lim\limits_{n\to\infty} x_n$, then $\varepsilon = \frac{1}{2}(1-\ell) > 0$ and so there exists $n_0 \in \mathbb{N}$ such that $|x_n - \ell| < \frac{1}{2}(1-\ell)$ for all $n \geq n_0$. Hence $0 < x_n < \frac{1}{2}(1+\ell)$ for all $n \geq n_0$ $\Rightarrow 0 < x_n^n < (\frac{1+\ell}{2})^n$ for all $n \geq n_0$. Since $\frac{1}{2}(1+\ell) < 1$, $\lim\limits_{n\to\infty} (\frac{1+\ell}{2})^n = 0$. Therefore by Sandwich theorem, $\lim\limits_{n\to\infty} x_n^n = 0$. $\square$

3. If $|\alpha| < 1$, then the sequence (α^n) converges to 0.

   Solution. If $\alpha = 0$, then $\alpha^n = 0$ for all $n \in \mathbb{N}$ and so (α^n) converges to 0. Now we assume that $\alpha \neq 0$. Since $|\alpha| < 1$, $\frac{1}{|\alpha|} > 1$ and so $\frac{1}{|\alpha|} = 1 + h$ for some $h > 0$. For all $n \in \mathbb{N}$, we have

   $$(1+h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \cdots + h^n > nh.$$

   This implies $|\alpha|^n = \frac{1}{(1+h)^n} < \frac{1}{nh}$ for all $n \in \mathbb{N}$. Given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ satisfying $n_0 > \frac{1}{h\varepsilon}$. Then $|\alpha^n - 0| = |\alpha|^n < \frac{1}{n_0 h} < \varepsilon$ for all $n \geq n_0$ and hence (α^n) converges to 0.

   Alternative proof: Given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ satisfying $n_0 > \frac{\log \varepsilon}{\log |\alpha|}$. Then for all $n \geq n_0$, we have $|\alpha^n - 0| = |\alpha|^n \leq |\alpha|^{n_0} < \varepsilon$ and hence (α^n) converges to 0. $\square$

4. Show that the sequence $((2^n + 3^n)^{\frac{1}{n}})$ converges to 3.

Solution. We have $3^n < 2^n + 3^n < 2 \cdot 3^n$ for all $n \in \mathbb{N}$. Hence, $3 < (2^n + 3^n)^{\frac{1}{n}} < 2^{\frac{1}{n}} \cdot 3$ for all $n \in \mathbb{N}$. Since $2^{\frac{1}{n}} \to 1$ (done in the class), hence by Sandwich theorem, the given sequence converges to 3. □

5. Let (a_n) be a sequence of real numbers such that each of the subsequences (a_{2n}), (a_{2n-1}) and (a_{3n}) converges. Show that (a_n) is convergent.

Solution. Let $a_{2n} \to a, a_{2n-1} \to b$ and $a_{3n} \to c$. Clearly, (a_{6n}) is a subsequence of (a_{2n}) and (a_{3n}). Hence, $a_{6n} \to a$ and $a_{6n} \to c$. This implies $a = c$.
Again, $(a_{3(2n-1)})$ is a subsequence of (a_{2n-1}) and (a_{3n}). Hence, $a_{3(2n-1)} \to b$ and $a_{3(2n-1)} \to c$. This implies $b = c$, and hence $a = b = c$. Since (a_{2n}) and (a_{2n-1}) converge to the same limit, it follows that (a_n) is convergent. □

6. If (a_n) is a bounded sequence and (b_n) is another sequence which converges to 0, show that the product $(a_n b_n)$ converges to 0.

Solution. Since (a_n) is bounded, so there is a positive number M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$. Since (b_n) converges to 0, so for given $\varepsilon/M > 0$, there exists $n_0 \in \mathbb{N}$ such that $|b_n| < \varepsilon/M$ for all $n \geq n_0$. Now, for $n \geq n_0$, we have

$$|a_n b_n| = |a_n||b_n| < M \cdot \frac{\varepsilon}{M} \quad \Rightarrow |a_n b_n| < \varepsilon \ \text{ for all } \ n \geq n_0.$$

This proves that $a_n b_n \to 0$. □

Remark 1. *Note that if (a_n) is not bounded then the result need not be true. For example, take $a_n = n^2$ and $b_n = \frac{1}{n}$.*

7. Let (a_n) be a sequence of real numbers. Define the sequence (s_n) by $s_n = \dfrac{1}{n}\sum_{i=1}^{n} a_i$.

   (a) If (a_n) is bounded, then show that (s_n) is also bounded.

   Solution. We have $|a_n| \leq M$ for all $n \in \mathbb{N}$. Then we obtain $|s_n| \leq M$ for all $n \in \mathbb{N}$. □

   (b) If (a_n) is monotone, then show that (s_n) is also monotone.

   Solution. We have

$$s_{n+1} - s_n = \frac{1}{n+1}\sum_{i=1}^{n+1} a_i - \frac{1}{n}\sum_{i=1}^{n} a_i = \frac{a_{n+1}}{n+1} - \frac{1}{n(n+1)}\sum_{i=1}^{n} a_i$$

   Now, if (a_n) is increasing, then $a_{n+1} \geq a_i$ for all $i = 1, 2, \ldots, n$.
   Hence, $s_{n+1} \geq s_n$ for all n. Similarly, if (a_n) is decreasing, then $a_{n+1} \leq a_i$ for all $i = 1, 2, \ldots, n$. Hence, $s_{n+1} \leq s_n$ for all n. □

   (c) If (a_n) converges to ℓ, then show that the sequence (s_n) also converges to ℓ.

Solution. Given that $a_n \to \ell$. We have

$$|s_n - \ell| = |\frac{1}{n}\sum_{i=1}^{n} a_i - \ell| \leq \frac{1}{n}\sum_{i=1}^{n} |a_i - \ell|.$$

Let $\varepsilon > 0$. Since $a_n \to \ell$, so there exists a positive integer n_0 such that $|a_i - \ell| < \varepsilon/2$ for all $i \geq n_0$. Hence, for $n \geq n_0$ we have

$$|s_n - \ell| \leq \frac{1}{n}\sum_{i=1}^{n} |a_i - \ell| = \frac{1}{n}\sum_{i=1}^{n_0-1} |a_i - \ell| + \frac{1}{n}\sum_{i=n_0}^{n} |a_i - \ell| < \frac{\alpha}{n} + \varepsilon/2,$$

where $\alpha = \sum_{i=1}^{n_0-1} |a_i - \ell|$. Since $\frac{\alpha}{n} \to 0$, there exists a positive integer n_1 such that $\frac{\alpha}{n} < \varepsilon/2$ for all $n \geq n_1$. Hence, $|s_n - \ell| < \varepsilon$ for all $n \geq n_2$, where $n_2 = \max\{n_0, n_1\}$. $\square$

8. Show that the sequence (x_n) defined by $x_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ diverges to infinity.

Solution. Clearly, (x_n) is an increasing sequence. Also, for $n \in \mathbb{N}$, we have

$$\begin{aligned} x_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{n-1}+1} + \cdots + \frac{1}{2^n}\right) \\ &\geq 1 + \frac{1}{2} + \frac{2}{4} + \frac{4}{8} + \cdots + \frac{2^{n-1}}{2^n} \\ &= 1 + \frac{n}{2}. \end{aligned}$$

This proves that (x_n) is not bounded above. Hence, (x_n) diverges to infinity. $\square$

9. Let the sequence (a_n) be defined by

$$a_1 = 1, a_{n+1} = \left(\frac{3 + a_n^2}{2}\right)^{1/2}, \quad n \geq 1.$$

Show that (a_n) converges to $\sqrt{3}$.

Solution. Using the principle of mathematical induction, we find that $a_n \leq \sqrt{3}$ for all $n \geq 1$. Also, $a_n > 1$ for all n. We now find that $a_{n+1}^2 - a_n^2 = \frac{3}{2} - \frac{a_n^2}{2} \geq 0$, and hence $a_{n+1} \geq a_n$ for all n. This proves that the sequence is convergent. Let $x_n \to \ell$. Then, $\ell^2 = 3$. Since ℓ is positive, so $\ell = \sqrt{3}$. $\square$

10. Let $a_1 > 0$ and for $n \geq 1$, $a_{n+1} = \dfrac{1}{2}\left(a_n + \dfrac{2}{a_n}\right)$. Show that the sequence $\{a_n\}$ is convergent and find the limit.

Solution. Since $a_1 > 0$, we can write $a_{n+1} = \frac{1}{2}(a_n + \frac{2}{a_n}) = \frac{1}{2}(\sqrt{a_n} - \frac{\sqrt{2}}{\sqrt{a_n}})^2 + \sqrt{2}$. This implies $a_{n+1} \geq \sqrt{2}$ for all $n \in \mathbb{N}$. Thus, (a_n) is bounded below. Note that $\sqrt{2}$ need not be a lower bound. If $a_1 < \sqrt{2}$, then a_1 will be a lower bound. Now, $2a_{n+1} - a_n = \frac{2}{a_n}$. This implies $2a_{n+1} - 2a_n = \frac{2}{a_n} - a_n = \frac{2-a_n^2}{a_n} \leq 0$ for all $n \geq 2$. Thus,

$$a_2 \geq a_3 \geq \cdots \geq a_n \geq \cdots$$

Hence (a_n) is convergent. If $a_n \to \ell$, then $\ell^2 = 2$. Hence, $\ell = \sqrt{2}$. $\square$

11. For $a \in \mathbb{R}$, let $x_1 = a$ and $x_{n+1} = \dfrac{1}{4}(x_n^2 + 3)$ for all $n \geq 2$. Examine the convergence of the sequence $\{x_n\}$ for different values of a. Also, find $\lim\limits_{n\to\infty} x_n$ whenever it exists.

Solution. If $\{x_n\}$ converges, then $\ell = \lim x_n$ satisfies $\ell^2 - 4\ell + 3 = 0$. Hence $\ell = 1$ or $\ell = 3$.
We have $x_{n+1} - x_n = \frac{1}{4}(x_n^2 - x_{n-1}^2)$ for all $n > 1$. Also $x_2 - x_1 = \frac{1}{4}(a-1)(a-3)$.
Case 1: If $a > 3$ then $x_2 > x_1$ and we get $x_{n+1} > x_n$ for all n. If $\{x_n\}$ converges, then $\ell = \lim x_n = \sup\{x_n : n \in \mathbb{N}\} \geq x_1 = a > 3$, which is not possible. Hence, if $a > 3$ then $\{x_n\}$ can't converge.
Case 2: If $a = 3$, then $x_n = 3$ for all $n \in \mathbb{N}$, and hence $\{x_n\}$ converges to 3.
Case 3: If $1 < a < 3$, then $x_2 < x_1$ and we get $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. Also in this case $x_n > 1$ for all $n \in \mathbb{N}$. (Because $x_{n+1} - 1 = \frac{1}{4}(x_n^2 - 1)$ for all $n \in \mathbb{N}$ and $x_1 > 1$.) Hence $\{x_n\}$ converges to 1. Note that $x_n \not\to 3$ as $\lim x_n = \inf\{x_n : n \in \mathbb{N}\} \leq x_1 = a < 3$.
Case 4: If $0 \leq a \leq 1$, then $x_2 \geq x_1$ and we get $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. Also in this case $x_n \leq 1$ for all $n \in \mathbb{N}$. Hence $\{x_n\}$ converges to 1.
Case 5: The case for $a < 0$ is treated by considering $-a$ in place of a, because x_2 is same irrespective of whether we choose $x_1 = a$ or $x_1 = -a$. Hence we can say that for $-1 \leq a \leq 0$, $x_n \to 1$, for $-3 < a < -1$, $x_n \to 1$, for $a = -3$, $x_n \to 3$ and for $a < -3$, $\{x_n\}$ does not converge. $\square$

12. Let $x_1 = 6$ and $x_{n+1} = 5 - \frac{6}{x_n}$ for all $n \in \mathbb{N}$. Examine whether the sequence (x_n) is convergent. Also, find $\lim\limits_{n\to\infty} x_n$ if (x_n) is convergent.

Solution. We have $x_1 > 3$ and if we assume that $x_k > 3$ for some $k \in \mathbb{N}$, then $x_{k+1} > 5 - 2 = 3$. Hence by the principle of mathematical induction, $x_n > 3$ for all $n \in \mathbb{N}$. Therefore (x_n) is bounded below. Again, $x_2 = 4 < x_1$ and if we assume that $x_{k+1} < x_k$ for some $k \in \mathbb{N}$, then $x_{k+2} - x_{k+1} = 6(\frac{1}{x_k} - \frac{1}{x_{k+1}}) < 0 \Rightarrow x_{k+2} < x_{k+1}$. Hence by the principle of mathematical induction, $x_{n+1} < x_n$ for all $n \in \mathbb{N}$. Therefore (x_n) is decreasing. Consequently (x_n) is convergent. Let $\ell = \lim\limits_{n\to\infty} x_n$. Then $\lim\limits_{n\to\infty} x_{n+1} = 5 - \frac{6}{\lim\limits_{n\to\infty} x_n} \Rightarrow \ell = 5 - \frac{6}{\ell}$ (since $x_n > 3$ for all $n \in \mathbb{N}$, $\ell \neq 0$) $\Rightarrow (\ell - 2)(\ell - 3) = 0 \Rightarrow \ell = 2$ or $\ell = 3$. But $x_n > 3$ for all $n \in \mathbb{N}$, so $\ell \geq 3$. Therefore $\ell = 3$. $\square$