

**Question 1:** This question contains two parts; answer them in the corresponding space provided.

a) Consider a step function potential, defined as

$$V(x) = \begin{cases} -1, & -\pi \leq x \leq 0 \\ +1, & 0 < x \leq \pi, \end{cases}$$

whose representation in the Fourier space is  $V(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$ . Using Fourier trick determine the coefficients  $a_0$ ,  $a_m$  and  $b_m$ . [4]

Fourier series expansion for  $V(x)$  is given by

$$V(x) = \frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$$

Multiplying both sides by  $\sin nx$  and integrating over  $\mathbb{D}[-\pi, \pi]$ :

$$\int_{-\pi}^{+\pi} V(x) \sin nx dx = \frac{a_0}{2} \int_{-\pi}^{+\pi} \sin nx dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{+\pi} \cos mx \sin nx dx + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{+\pi} \sin mx \sin nx dx$$

$$2 \times \frac{\pi}{2} \delta_{m,n} = \pi \delta_{m,n}$$

$$\Rightarrow \int_{-\pi}^0 (-1) \sin nx dx + \int_0^\pi \sin nx dx = \pi b_n$$

$$\Rightarrow \pi b_n = 2 \int_0^\pi \sin nx dx$$

$$\Rightarrow b_n = \frac{2}{\pi} \left[ -\frac{\cos nx}{n} \right]_0^\pi = \frac{2}{\pi n} (1 - \cos n\pi) \quad \boxed{\text{Mark } 2+}$$

$$\therefore b_n = \frac{2}{\pi n} (1 - \cos n\pi) = \begin{cases} \frac{4}{n\pi} & \text{for } n = \text{odd} \\ 0 & \text{for } n = \text{even} \end{cases}$$

Next multiplying both sides by  $\cos nx$  and integrating over  $\mathcal{D}[-\pi, \pi]$ :

$$\int_{-\pi}^{\pi} dx V(x) \cos nx = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos nx dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx \cos nx dx + \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx \cos nx dx$$

$2 \times \frac{\pi}{2} \delta_{m,n}$

$$\Rightarrow - \int_{-\pi}^0 dx \cos nx + \int_0^\pi \cos nx dx = \pi a_n$$

$$\text{or, } \pi a_n = 0 \Rightarrow a_n = 0$$

$$\therefore a_n = 0 \quad \longrightarrow$$

Mark  
1+

Finally, integrating both sides over the domain  $\mathcal{D} [-\pi, \pi]$ :

$$\int_{-\pi}^{\pi} v(x) dx = \frac{a_0}{2} \int_{-\pi}^{\pi} dx + \sum_{m=1}^{\infty} a_m \int_{-\pi}^{\pi} \cos mx dx$$

$$+ \sum_{m=1}^{\infty} b_m \int_{-\pi}^{\pi} \sin mx dx$$

$\overbrace{\hspace{10em}}$   
 $\overbrace{\hspace{10em}}$

$$\Rightarrow - \int_{-\pi}^0 dx + \int_0^{\pi} dx = \frac{a_0}{2} \cdot 2\pi$$

$$\text{or, } \pi a_0 = -\pi + \pi = 0$$

$$\Rightarrow \boxed{a_0 = 0}$$

$\rightarrow$  [Mark 1+]

- b) For a given vector field  $\vec{V} = r\hat{r} + r \cos^2 \theta \hat{\phi}$ , find the volume charge density if it represents an electric field for a collection of static charges. [2]

The given vector field is  $\vec{V} = r\hat{r} + r \cos^2 \theta \hat{\phi}$ .

First check if it represents the electric field for a collection of static charges.

So calculate  $\vec{\nabla} \times \vec{V}$ :

$$\begin{aligned}
 \text{Here } \vec{\nabla} \times \vec{V} &= \frac{1}{r \sin \theta} \left[ \frac{\partial}{\partial \theta} ( \sin \theta \cdot r \cos^2 \theta ) \right] \hat{r} \\
 (\text{In spherical polar coordinates}) &+ \frac{1}{r} \left[ \frac{1}{\sin \theta} \frac{\partial}{\partial \phi} (r) - \frac{\partial}{\partial r} (r^2 \cos^2 \theta) \right] \hat{\theta} \\
 &+ \frac{1}{r} \left[ - \frac{\partial}{\partial \theta} (r) \right] \hat{\phi} \\
 &= \frac{1}{r \sin \theta} r \left[ \cos^3 \theta - 2 \sin^2 \theta \cos \theta \right] \hat{r} \\
 &- \frac{1}{r} \cos^2 \theta \cdot (2r) \hat{\theta}
 \end{aligned}$$

$$= \left( \frac{\cos^3\theta - 2\sin^2\theta \cos\theta}{\sin\theta} \right) \hat{r} - 2\cos^2\theta \hat{\theta} \rightarrow \boxed{\begin{array}{l} \text{Mark} \\ \frac{1}{2} + \end{array}}$$

Since  $\vec{\nabla} \times \vec{v} \neq 0$ ,  $\vec{v}$  does not represent an electrostatic electric field.

$\Rightarrow$  The question of volume charge density does not arise here.

$\rightarrow \boxed{\begin{array}{l} \text{Mark} \\ \frac{1}{2} \end{array}}$

★

**Question 2:** Consider a solid of uniform mass density  $\rho$  that is bounded below by the  $xy$  plane, above by the sphere  $x^2 + y^2 + z^2 = 4a^2$ , and laterally by the half-cylinder  $s = 2a \cos \phi$  with  $\phi \in [0, \pi/2]$ . Here  $(x, y, z)$  are in Cartesian coordinates, while  $(s, \phi, z)$  are in cylindrical coordinates.

- (a) Write down the expression for the differential moment of inertia  $dI_z(x, y, z)$  about the  $z$ -axis in Cartesian coordinates.  
(b) Express the above in cylindrical coordinates, i.e.,  $dI_z(s, \phi, z)$ .  
(c) Find the limits of the volume integration in cylindrical coordinates that you will need to obtain the total moment of inertia  $I_z$  about  $z$ -axis.  
(d) Determine the above moment of inertia of the system. [ $\frac{1}{2} + \frac{1}{2} + 1 + 4 = 6$ ]

(a) Consider a small volume  $dV$ .

It contains  $dm = \rho dV$  amount of mass.

∴ The differential form of moment of inertia of this  $dm$  mass about  $z$ -axis is (in Cartesian coordinates)

$$\begin{aligned} dI_z &= dm(x^2 + y^2) \\ &= \rho(x^2 + y^2) dV. \quad \rightarrow \left[ \begin{array}{c} \text{Mark} \\ \frac{1}{2} + \end{array} \right] \end{aligned}$$

(b) In Cylindrical coordinates we have  $dV = s ds d\phi dz$ .

$$\begin{aligned} \therefore dI_z &= \rho(x^2 + y^2) s ds d\phi dz = \rho s^2 \cdot s ds d\phi dz \\ &= \rho s^3 ds d\phi dz \end{aligned}$$

$$\left[ \begin{array}{c} \text{Mark} \\ \frac{1}{2} + \end{array} \right]$$

(c) To find the total moment of inertia, we have to integrate the above within the allowed region of mass.

For the present case, the limits of integrations are:

$$\left. \begin{aligned} s = 0 &\rightarrow 2a \cos \phi \\ z = 0 &\rightarrow \sqrt{4a^2 - r^2 - y^2} = \sqrt{4a^2 - s^2} \end{aligned} \right\} \quad \begin{array}{l} \text{Mark} \\ \boxed{1+} \end{array}$$

and  $\phi = 0 \rightarrow +\pi/2$ .

(d) Total moment of inertia:

$$I_z = \rho \int_{\phi=0}^{+\pi/2} \int_{s=0}^{2a \cos \phi} \int_{z=0}^{\sqrt{4a^2 - s^2}} s^3 ds d\phi dz$$

$$= \rho \int_{\phi=0}^{+\pi/2} d\phi \int_{s=0}^{2a \cos \phi} ds s^3 \sqrt{4a^2 - s^2}$$

### Method I

$$2a \cos \phi \int_{s=0}^{2a \cos \phi} ds s^3 \sqrt{4a^2 - s^2}$$

$$\text{Put } 4a^2 - s^2 = m^2$$

$$\therefore sds = -mdm$$

$$\text{and } m = 2a \rightarrow 2a \sin\phi$$

$$2a \sin\phi$$

$$= \int_{m=2a}^{2a \sin\phi} (-mdm) (4a^2 - m^2) \cdot m$$

$$2a \sin\phi$$

$$= \int_{2a}^{2a \sin\phi} (m^4 - 4a^2 m^2) dm$$

$$= \left[ \frac{m^5}{5} - \frac{4a^2 m^3}{3} \right]_{2a}^{2a \sin\phi}$$

$$= \left( \frac{32a^5 \sin^5 \phi}{5} - \frac{32a^5 \sin^3 \phi}{3} - \frac{32a^5}{5} + \frac{32a^5}{3} \right)$$

$$= \frac{64}{15} a^5 + 32a^5 \left( \frac{\sin^5 \phi}{5} - \frac{\sin^3 \phi}{3} \right)$$

$$\therefore I_z = \rho \int_{\phi=0}^{\pi/2} d\phi \left[ \frac{64a^5}{15} + 32a^5 \left( \frac{\sin^5 \phi}{5} - \frac{\sin^3 \phi}{3} \right) \right]$$

$$= \frac{32\pi a^5}{15} p + 32a^5 p \int_0^{\pi/2} d\phi \left( \underbrace{\frac{\sin^5 \phi}{5}}_{\frac{1}{5} \times \frac{8}{15}} - \underbrace{\frac{\sin^3 \phi}{3}}_{\frac{1}{3} \times \frac{2}{3}} \right)$$

$$= \frac{32\pi a^5}{15} p + 32a^5 p \left( \frac{8}{75} - \frac{2}{9} \right)$$

$$\frac{72 - 150}{75 \times 9} = -\frac{78}{75 \times 9}$$

$$= \frac{32\pi a^5}{15} p - \frac{32 \times 78}{75 \times 9} a^5 p$$

$$= \frac{32 a^5 p}{15} \left( \pi - \frac{78}{5 \times 9} \right)$$

$$= \frac{32 a^5 p}{15} \left( \pi - \frac{26}{15} \right)$$

[ Marks  
4 + ]

METHOD-2

$$I_z = \rho \int_0^{\pi/2} d\phi \int_0^{2a \cos \phi} ds \sqrt[3]{(4a^2 - s^2)}$$

Substitution Type  $s = 2a \cos x$   $\left\{ \begin{array}{l} s : 0 \rightarrow 2a \cos \phi \\ x : \frac{\pi}{2} \rightarrow \phi \end{array} \right.$

$$I_z = -\rho (2a)^5 \int_0^{\pi/2} d\phi \int_{\pi/2}^{\phi} \cos^3 x \sin^2 x dx$$

$$= -\rho (2a)^5 \int_0^{\pi/2} d\phi \int_{\pi/2}^{\phi} (1 - \sin^2 x) \sin^2 x \cos x dx$$

$$= -(2a)^5 \rho \int_0^{\pi/2} d\phi \int_{\pi/2}^{\phi} (\sin^2 x - \sin^4 x) \cos x dx$$

$$= -(2a)^5 \rho \int_0^{\pi/2} d\phi \left( \frac{1}{3} \sin^3 x - \frac{1}{5} \sin^5 x \right) \Big|_{\pi/2}^{\phi}$$

$$= -(2a)^5 \rho \int_0^{\pi/2} \left[ \frac{1}{3} \sin^3 \phi - \frac{1}{5} \sin^5 \phi - \frac{2}{15} \right] d\phi$$

$$= \frac{(2a)^5}{15} \rho \pi - \frac{(2a)^5}{1} \rho \int_0^{\pi/2} \left[ \frac{1}{3} (1 - \cos^2 \phi) \sin \phi \right] d\phi + \frac{(2a)^5}{5} \rho \int_0^{\pi/2} [(1 - \cos^2 \phi)^2 \sin \phi] d\phi$$

$$= \frac{(2a)^5}{15} \rho \pi - \frac{(2a)^5}{3} \rho \int_0^{\pi/2} [\sin \phi - \cos^2 \phi \sin \phi] d\phi$$

$$+ \frac{(2a)^5}{5} \rho \int_0^{\pi/2} [\sin \phi + \cos^4 \phi \sin \phi - 2 \cos^2 \phi \sin \phi] d\phi$$

$$= \frac{(2a)^5}{15} \rho \pi - \frac{(2a)^5}{3} \rho \left[ -\cos \phi + \frac{\cos^3 \phi}{3} \right] \Big|_0^{\pi/2} + \frac{(2a)^5}{5} \rho \left[ -\cos \phi - \frac{1}{5} \cos^5 \phi + \frac{2}{3} \cos^3 \phi \right] \Big|_0^{\pi/2}$$

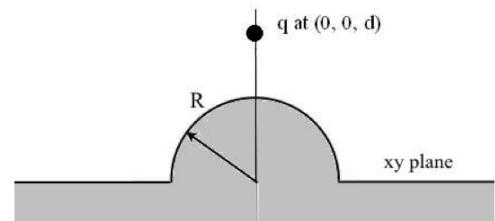
$$= \rho \frac{(2a)^5}{15} \pi - \rho (2a)^5 \frac{2}{9} + \rho (2a)^5 \frac{8}{75} = \rho \frac{(2a)^5}{15} \left[ \pi - \frac{26}{15} \right]$$

**Question 3:** A conducting grounded sheet lies in the  $xy$  plane with a hemispherical dome (radius  $R$ ) centered at the origin. A point charge  $q$  is placed at the distance  $d (> R)$  on the  $z$  axis.

a) Write down the image charges and their locations. (Don't derive, just guess on the basis of the classroom discussions and the tutorial problems.)

b) Find the potential at a point  $\vec{r} = (x, y, z)$  for  $z > 0$  and  $r > R$ .

c) Find the induced surface charge density and the corresponding total charge induced on the flat surface of the conductor, i.e., excluding the induced charges on the dome.



[2+1+3=6]

(a) Image charges:

$$q_1 = -q \text{ at } (0, 0, -d)$$

$$q_2 = -\frac{qR}{d} \text{ at } (0, 0, +\frac{R^2}{d})$$

$$q_3 = +\frac{qR}{d} \text{ at } (0, 0, -\frac{R^2}{d})$$

} → [Marks  
2+]

(b) The potential at pt.  $\vec{r} = (x, y, z)$  for  $z > 0$  and  $r > R$  is

$$V(r) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} + \frac{(-q)}{\sqrt{x^2 + y^2 + (z+d)^2}} + \frac{(-\frac{qR}{d})}{\sqrt{x^2 + y^2 + (z - \frac{R^2}{d})^2}} + \frac{qR/d}{\sqrt{x^2 + y^2 + (z + \frac{R^2}{d})^2}} \right] \rightarrow [Marks  
1+]$$

(c) Take a point on the flat section of the conductor.

The coordinates of this pt. are  $(x, y, 0)$ ;  $|x|, |y| \geq R$

Now, induced charge density is given by:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial n} \Big|_{z=0}$$

$$= -\epsilon_0 \hat{n} \cdot \vec{\nabla} V \Big|_{z=0}$$

Here we have  $\hat{n} = \hat{z}$ .

$$\therefore \sigma = -\epsilon_0 \frac{\partial V}{\partial z} \Big|_{z=0}$$

$$= \frac{q}{4\pi} \left[ -\frac{2d}{(x^2 + y^2 + d^2)^{3/2}} + \frac{2R^3}{d^2(x^2 + y^2 + R^2/d^2)^{3/2}} \right] \rightarrow \boxed{\begin{matrix} \text{Marks} \\ 1+ \end{matrix}}$$

Total charge on the flat surface:

On the  $xy$ -plane, polar coordinates  $(s, \phi)$  can be used. Then  $x^2 + y^2 = s^2$

and  $da = \text{surface element}$   
 $= s ds d\phi$

$$\therefore Q_{\text{total}} = \int \sigma da$$

$$= \frac{q}{4\pi} \int \left[ -\frac{2d}{(s^2 + d^2)^{3/2}} + \frac{2R^3}{d^2(s^2 + R^2/d^2)^{3/2}} \right] s ds d\phi$$

$$= \frac{q}{4\pi} \cdot 2\pi \int \left[ \frac{-2d}{(s^2 + d^2)^{3/2}} + \frac{2R^3}{d^2(s^2 + R^4/d^2)^{3/2}} \right] s ds$$

$$= \frac{q}{2} \int_R^\infty \left[ \frac{-2d}{(s^2 + d^2)^{3/2}} + \frac{2R^3}{d^2(s^2 + R^4/d^2)^{3/2}} \right] s ds$$

Substituting,  $s^2 + d^2 = x^2$  and  $s^2 + \frac{R^4}{d^2} = Y^2$  one

obtains:

$$Q_{\text{total}} = -qd \int_{\sqrt{R^2+d^2}}^{\infty} \frac{dx}{x^2} + \frac{qR^3}{d^2} \int_{\sqrt{R^2+\frac{R^4}{d^2}}}^{\infty} \frac{dy}{y^2}$$

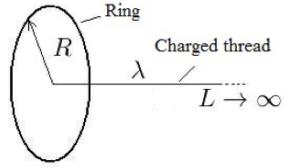
$$= -qd \left[ -\frac{1}{x} \right]_{\sqrt{R^2+d^2}}^{\infty} + \frac{qR^3}{d^2} \left[ -\frac{1}{Y} \right]_{\sqrt{R^2+\frac{R^4}{d^2}}}^{\infty}$$

$$= \frac{q}{2} \frac{2}{d} \left[ \frac{R^2}{d(1+\frac{R^2}{d^2})^{1/2}} - \frac{d^2}{(d^2+R^2)^{1/2}} \right]$$

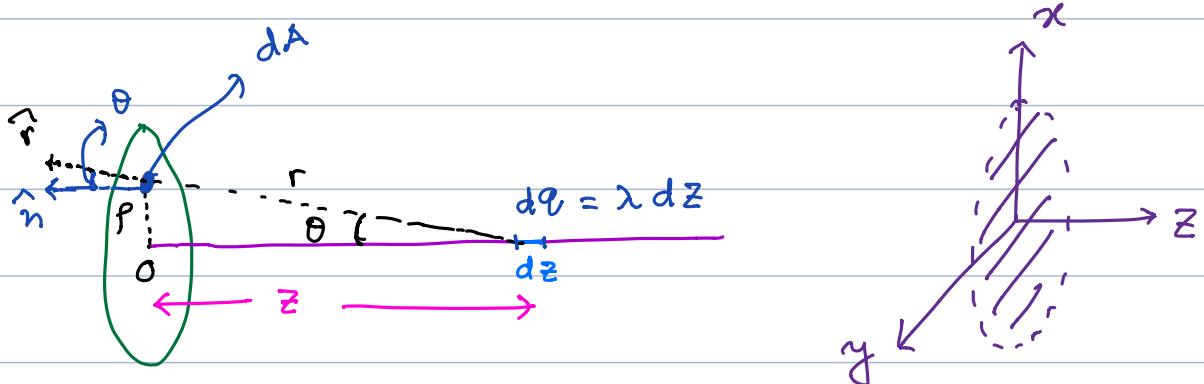
$$= \frac{q}{d} \frac{R^2 - d^2}{\sqrt{R^2 + d^2}} \rightarrow \begin{bmatrix} \text{Mark} \\ 2+ \end{bmatrix}$$

**Question 4:** A very long uniformly charged thread of length  $L$  oriented along the axis of a ring of radius  $R$  rests on its center with one of the ends. The charge of the thread per unit length is  $\lambda$ .

- Find the flux  $\Phi_L$  of the electric field vector  $\vec{E}$  across the ring.
- Determine the flux  $\Phi_\infty$  through the ring in the infinite length limit (i.e.,  $L \rightarrow \infty$ ).



[5+1=6]



The electric field on the elementary area  $d\vec{A} = dA \hat{n}$  due to elementary charge  $dq = \lambda dz$  is

$$d\vec{E} = \frac{dq}{4\pi\epsilon_0 r^2} \hat{r}.$$

∴ The flux through  $d\vec{A}$  area for the electric field  $d\vec{E}$  is :

$$d\Phi = d\vec{E} \cdot d\vec{A}$$

$$= \frac{dq}{4\pi\epsilon_0 r^2} (\hat{r} \cdot \hat{n}) dA$$

$$= \frac{dq}{4\pi\epsilon_0} \frac{dA \cos\theta}{r^2} . \quad \boxed{\frac{\text{Mark}}{1+}}$$

$\therefore$  The flux through the ring due to charge  $dq = \lambda dz$  is

$$\Delta\Phi = \int \frac{dq}{4\pi\epsilon_0} \frac{dA \cos\theta}{r^2}$$

(area of  
the ring)

Here  $dA = \rho d\rho d\phi$  with  $\rho = 0 \rightarrow R$   
 $\phi = 0 \rightarrow 2\pi$ ;

and  $\cos\theta = \frac{z}{r}$  with  $r = \sqrt{\rho^2 + z^2}$ .

$$\therefore \Delta\Phi = \frac{dq}{4\pi\epsilon_0} \int \rho d\rho d\phi \cdot \frac{z}{r^3}$$

$$= \frac{z dq}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_0^R \frac{\rho d\rho}{(\rho^2 + z^2)^{3/2}}$$

$$= \frac{2\pi z dq}{4\pi\epsilon_0} \int_0^z \frac{m dm}{m^3}$$

Put  $\rho^2 + z^2 = m^2$

$\therefore \rho d\rho = mdm$

$$= \frac{z dq}{2\epsilon_0} \left[ -\frac{1}{m} \right]_z^{\sqrt{z^2 + R^2}}$$

$$= \frac{z dq}{2\epsilon_0} \left[ \frac{1}{z} - \frac{1}{\sqrt{z^2 + R^2}} \right]$$

$$= \frac{dq}{2\epsilon_0} \left[ 1 - \frac{z}{\sqrt{z^2 + R^2}} \right] = \frac{\lambda dz}{2\epsilon_0} \left[ 1 - \frac{z}{\sqrt{z^2 + R^2}} \right]$$

Thus the total flux through the ring due to length L of the charged thread is L

$$\Phi_L = \frac{\lambda}{2\epsilon_0} \int_0^L \left( 1 - \frac{z}{\sqrt{z^2 + R^2}} \right) dz \xrightarrow{\text{Mark } \frac{2+}{2+}}$$

$$\text{Substituting, } z = R \tan \theta \Rightarrow dz = R \sec^2 \theta d\theta$$

$$\Phi_L = \frac{\lambda}{2\epsilon_0} \int_0^{\tan^{-1}(L/R)} \left( 1 - \frac{R \tan \theta}{R \sec \theta} \right) R \sec^2 \theta d\theta$$

$$= \frac{\lambda R}{2\epsilon_0} \int_0^{\tan^{-1}(L/R)} (\sec^2 \theta - \tan \theta \sec \theta) d\theta$$

$$= \frac{\lambda R}{2\epsilon_0} \left[ \tan \theta - \sec \theta \right]_0^{\tan^{-1}(L/R)}$$

$$= \frac{\lambda R}{2\epsilon_0} \left[ \frac{L}{R} - \sec(\tan^{-1}(L/R)) + 1 \right]$$

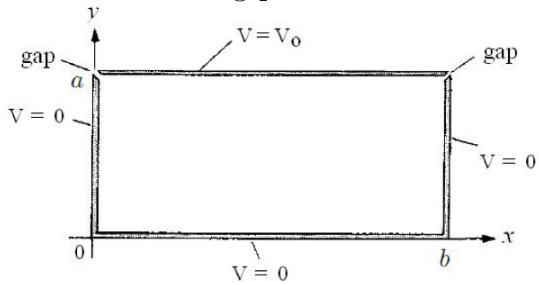
$$\therefore \boxed{\Phi_L = \frac{\lambda R}{2\epsilon_0} \left[ \frac{L}{R} - \frac{L}{R} \sqrt{1 + \frac{R^2}{L^2}} + 1 \right]} \rightarrow \boxed{\frac{\text{Mark}}{2+}}$$

(b) For  $L \rightarrow \infty$  limit: (keep upto leading order in  $(R/L)$ ):

$$\begin{aligned} \Phi_L &\simeq \frac{\lambda R}{2\epsilon_0} \left[ \frac{L}{R} - \frac{L}{R} \left( 1 + \frac{R^2}{2L^2} \right) + 1 \right] \\ &= \frac{\lambda R}{2\epsilon_0} + \mathcal{O}(R/L) \xrightarrow{L \rightarrow \infty} \frac{\lambda R}{2\epsilon_0} \end{aligned}$$

$$\therefore \boxed{\Phi_\infty = \frac{\lambda R}{2\epsilon_0}} \rightarrow \boxed{\frac{\text{Mark}}{1+}}$$

**Question 5:** Determine the electrostatic potential function for the region inside the rectangular pipe running parallel to  $z$ -axis from  $[-\infty, +\infty]$  whose cross-section is shown in the figure.



[6]

Since the given problem has a  $z$ -symmetry, the solution must be independent of  $z$ -coordinate;

$$\text{i.e. } V(x, y, z) \equiv V(x, y) .$$

$$\therefore \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0 \quad \longrightarrow \boxed{\frac{1}{2} + \text{Mark}}$$

in the 2D Laplace's equation satisfying the boundary conditions:

$$\left. \begin{array}{l} (\text{i}) \quad V(x=0, 0 \leq y \leq a) = 0 \\ (\text{ii}) \quad V(x=b, 0 \leq y \leq a) = 0 \\ (\text{iii}) \quad V(0 \leq x \leq b, y=0) = 0 \\ (\text{iv}) \quad V(0 \leq x \leq b, y=a) = V_0 \end{array} \right\} \quad \begin{array}{l} \text{Mark} \\ 1+ \end{array}$$

We now seek a variable separable solution given by the multiplicative ansatz :

$$V(x, y) = X(x) Y(y) .$$

$$\therefore \nabla^2 v = 0 \Rightarrow \frac{1}{x} x'' + \frac{1}{y} y'' = 0$$

For the given problem, by inspection we choose  $\lambda^2 > 0$  such that

$$-\frac{1}{x} x'' = \frac{1}{y} y'' = \lambda^2 > 0.$$

Hence, we obtain two 2nd order ordinary differential equations:

$$x'' + \lambda^2 x = 0$$

$$\text{and } y'' - \lambda^2 y = 0$$

The solutions are:

$$\left. \begin{aligned} x(x) &= A \sin(\lambda x) + B \cos(\lambda x) \\ y(y) &= C \sinh(\lambda y) + D \cosh(\lambda y) \end{aligned} \right\} \rightarrow \boxed{\text{Mark 1+}}$$

$$\text{From BC. (i)} : v(0, y) = 0 \Rightarrow B = 0$$

$$\therefore v(x, y) = \sin(x) (C \sinh(\lambda y) + D \cosh(\lambda y))$$

$$\text{BC. (ii)} : v(b, y) = 0 \Rightarrow \sin \lambda b = 0$$

$$\Rightarrow \lambda b = n\pi$$

$$\Rightarrow \lambda \equiv \lambda_n = \frac{n\pi}{b} \quad \text{with } n = 1, 2, 3, \dots$$

$$\text{BC. (iii)} : v(x, 0) = 0 \Rightarrow D = 0.$$

$$\therefore v_n(x, y) = C_n \sin(\lambda_n x) \sinh(\lambda_n y) \rightarrow (\times)$$

See at the end which is a discussion if one takes the exponential  $\rightarrow 1^\text{st}$  for  $y(y)$

$$\therefore V(x, y) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{b}\right) \sinh\left(\frac{n\pi y}{b}\right)$$

Mark  
 $\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1\frac{1}{2}$

is the most general solution to the Laplace's equation.

Imposing the final B.c. (iv) one finds:

$$V_0 = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{b}\right) \sinh\left(\frac{n\pi a}{b}\right)$$

Multiplying both sides by  $\sin\left(\frac{m\pi x}{b}\right)$  and integrating  $0 \leq x \leq b$ :

$$\int_0^b dx V_0 \sin\left(\frac{m\pi x}{b}\right) = \sum_{n=1}^{\infty} c_n \sinh\left(\frac{n\pi a}{b}\right) \int_0^b dx \sin\left(\frac{n\pi x}{b}\right) \sin\left(\frac{m\pi x}{b}\right)$$

$\frac{b}{2} \delta_{mn}$

$$= \frac{b}{2} C_m \sinh\left(\frac{m\pi a}{b}\right)$$

$$\Rightarrow V_0 \int_0^b dx \sin\left(\frac{m\pi x}{b}\right) = \frac{b}{2} C_m \sinh\left(\frac{m\pi a}{b}\right)$$

$$\text{or, } \frac{b V_0}{m\pi} \left[ 1 - \cos(m\pi) \right] = \frac{b}{2} C_m \sinh\left(\frac{m\pi a}{b}\right)$$

$$\text{or, } C_m = \frac{2V_0}{m\pi \sinh\left(\frac{m\pi a}{b}\right)} \underbrace{\left[1 - \cos(m\pi)\right]}_{\begin{cases} 2 & \text{for } m \text{ is odd} \\ 0 & \text{for } m \text{ is even.} \end{cases}}$$

$$\Rightarrow C_n = \begin{cases} \frac{4V_0}{n\pi \sinh\left(\frac{n\pi a}{b}\right)}, & \text{if } n = \text{odd} \\ 0, & \text{if } n = \text{even} \end{cases} \rightarrow \boxed{\begin{array}{c} \text{Mark} \\ 1 \frac{1}{2} + \end{array}}$$

Thus the complete solution for the electrostatic potential is

$$V(x, y) = \frac{4V_0}{\pi} \sum_{n=1, 3, 5, \dots} \left[ \frac{\sin\left(\frac{n\pi x}{b}\right) \sinh\left(\frac{n\pi y}{b}\right)}{n \sinh\left(\frac{n\pi a}{b}\right)} \right] \boxed{\begin{array}{c} \text{Mark} \\ \frac{1}{2} + \end{array}}$$

Another form of  $\Upsilon(y)$  solution:

$$\Upsilon(y) = C'e^{xy} + D'e^{-xy}$$

Then BC (i) and (ii) implies:

$$v_n(x, y) = \sin \lambda_n x (C' e^{\lambda_n y} + D' e^{-\lambda_n y})$$

with  $\lambda_n = \frac{n\pi}{b}$ ;  $n = 1, 2, 3, \dots$

$$\text{B.C. (iii)} : v(x, 0) = 0$$

$$\Rightarrow C' = -D'$$

$$\therefore v_n(x, y) = C' \sin \lambda_n x (e^{\lambda_n y} - e^{-\lambda_n y})$$

$$= 2C' \sin \lambda_n x \sinh \lambda_n y$$

$$\equiv C_n \sin \lambda_n x \sinh \lambda_n y.$$

which is eqn (x). Rest of steps are as earlier.