DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
Even Semester of the Academic Year 2019-2020
MA 101 Mathematics I
Problem Sheet 5: Line Integrals and applications, Green's Theorem, Stokes Theorem and Divergence Theorem.
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1. Calculate the line integral of the vector field along the path described:

   (a) $f(x,y) = (x^2+y^2)\mathbf{i} + (x^2-y^2)\mathbf{j}$ from $(0,0)$ to $(2,0)$ along the curve $y = 1-|1-x|$

   (b) $f(x,y,z) = 2xy\mathbf{i} + (x^2+z)\mathbf{j} + (y+z)\mathbf{k}$ from $(1,0,2)$ to $(3,4,1)$ along a line segment

   (c) $f(x,y,z) = x\mathbf{i} + y\mathbf{j} + (xz-y)\mathbf{k}$, along the path described by $\alpha(t) = t^2\mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k}$, $0 \le t \le 1$.

   Solution:
   (a) Note that the parametric representation of the curve $\mathbf{r}(x)$ is given by:
   $\mathbf{r}(x) = x\mathbf{i} + x\mathbf{j}$ for $0 \le x \le 1$,
   $\mathbf{r}(x) = x\mathbf{i} + (2-x)\mathbf{j}$ for $1 \le x \le 2$.
   $$\int_C f\cdot d\mathbf{r} = \int_C f(\mathbf{r}(x))\cdot\mathbf{r}'(x)dx = \int_0^1 \big((x^2+x^2)1 + (x^2-x^2)1\big)\,dx$$
   $$+\int_1^2 \big((x^2+(2-x)^2)1 + (x^2-(2-x)^2)(-1)\big)\,dx$$
   $= \frac{4}{3}$.

   (b) Note that the parametrization of the line is given by:
   $\mathbf{r}(t) = <1,0,2> + t(<3,4,1> - <1,0,2>) = <2t+1, 4t, 2-t>$ for $0 \le t \le 1$.
   Hence the line integral is given by:
   $$\int_C f\cdot d\mathbf{r} = \int_C f(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_0^1 \big(2(2t+1)(4t)2 + ((2t+1)^2+2-t)4 + (4t+2-t)(-1)\big)\,dt$$
   $= 40 - \frac{3}{2} = \frac{77}{2}$.

   (c) The line integral is given by:
   $$\int_C f\cdot d\mathbf{r} = \int_0^1 f(\mathbf{r}(t))\cdot\mathbf{r}'(t)dt = \int_0^1 \big((t^2)2t + (2t)2 + (4t^5-2t)12t^2\big)\,dt = \frac{5}{2}.$$

2. Find the line integral of $f(x,y,z) = z$ with respect to arc length of the curve given by $\mathbf{r}(t) = (t\cos t)\mathbf{i} + (t\sin t)\mathbf{j} + t\mathbf{k}$, $0 \le t \le 1$.

   Solution: $\displaystyle\int_C f ds = \int_0^1 t\,|\mathbf{r}'(t)|\,dt = \int_0^1 t\sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + 1}\;dt$
   $$= \int_0^1 t\sqrt{2+t^2}\;dt = \frac{1}{3}(3^{\frac{3}{2}} - 2^{\frac{3}{2}}).$$

3. For each of the following vector fields show that $\mathbf{f}$ is not a gradient vector in $\mathbf{R}^2$. Then for each of the following find a closed path C such that $\oint_C \mathbf{f} \neq 0$ and if possible find a closed path C such that $\oint_C \mathbf{f} = 0$. ($\oint_C \mathbf{f}$ is also used to represent $\oint_C \mathbf{f}\cdot d\mathbf{r}$.)

   (a) $\mathbf{f}(x,y) = y\mathbf{i} - x\mathbf{j}$

   (b) $\mathbf{f}(x,y) = \frac{y}{(x^2+y^2)}\mathbf{i} - \frac{x}{(x^2+y^2)}\mathbf{j}$, for $(x,y) \neq (0,0)$.

   Solution: (a) Since $\dfrac{\partial P}{\partial y}$ and $\dfrac{\partial Q}{\partial x}$ (where $P(x,y) = y$ and $Q(x,y) = -x$) are continuous functions but $\dfrac{\partial P}{\partial y} = 1 \neq \dfrac{\partial Q}{\partial x} = -1$, $\mathbf{f}$ is not a gradient vector.
   A necessary condition for $\oint_C \mathbf{f}$ to be equal to 0 for every closed path C in $\mathbf{R}^2$ is:
   $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ for all $(x,y) \in \mathbf{R}^2$ (or $\mathbf{f}$ is the gradient vector of some scalar function).
   In this case since $\dfrac{\partial P}{\partial y} = 1 \neq \dfrac{\partial Q}{\partial x} = -1$ for all $(x,y) \in \mathbf{R}^2$, so for all smooth simple closed curves C in $\mathbf{R}^2$, $\oint_C \mathbf{f} \neq 0$.
   Take for example a positively oriented circle C of radius $r > 0$ centered at the origin,
   $C = r\cos t\mathbf{i} + r\sin t\mathbf{j}$, $0 \le t \le 2\pi$
   then $\displaystyle\oint_C \mathbf{f} = \int_0^{2\pi} (r\sin t(-r\sin t) - r\cos t(r\cos t))\,dt = -2\pi r^2 \neq 0.$
   (You can also use Green's theorem to check this).

(b) $\dfrac{\partial P}{\partial y}, \dfrac{\partial Q}{\partial x} \quad \left(= \dfrac{x^2-y^2}{(x^2+y^2)^2}\right)$ are continuous for all $(x,y) \in \mathbf{R}^2 \setminus \{(0,0)\}$,
where $P(x,y) = \dfrac{y}{(x^2+y^2)}$ and $Q(x,y) = -\dfrac{x}{(x^2+y^2)}$.
Since $\mathbf{f}$ is not defined at $(0,0)$, so $\mathbf{f}$ is not the gradient vector of any scalar function defined throughout $\mathbf{R}^2$, but since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ for all $(x,y) \neq (0,0)$,
f is the gradient vector of some scalar function in any open simply connected subset of $\mathbf{R}^2$ not containing the origin.

If we take C as the positively oriented unit circle centered at $(2,0)$ such that the closed disc enclosed by C does not contain the origin then check that $\oint_C \mathbf{f} = 0$.
(You can also use Green's theorem to check this).
Whereas if we choose C as the positively oriented unit circle centered at the origin, that is $C = \cos t\mathbf{i} + \sin t\mathbf{j}$, $0 \leq t \leq 2\pi$ then
(**) $\displaystyle\oint_C \mathbf{f} = \int_0^{2\pi} (\sin t(-\sin t) - \cos t(\cos t))\, dt = -2\pi \neq 0 \Rightarrow$ $\mathbf{f}$ is not a gradient vector in $\mathbf{R}^2 - \{(0,0)\}$.

4. Show that each of the following functions F is a gradient vector and find an f for each F such that $F = \nabla f$.

   (a) $F(x,y) = 3x^2y\mathbf{i} + x^3\mathbf{j}$

   (b) $F(x,y) = (\sin y - y\sin x + x)\mathbf{i} + (\cos x + x\cos y + y)\mathbf{j}$.

Solution:
(a) Since P, Q are continuously differentiable functions, (where $P(x,y) = 3x^2y$ and $Q = x^3$) and $\dfrac{\partial P}{\partial y} = 3x^2 = \dfrac{\partial Q}{\partial x}$, F is the gradient of some scalar function f.
Note that if f is such that F is the gradient vector of f, then
$\dfrac{\partial f}{\partial x} = P(x,y) = 3x^2y$ and $\dfrac{\partial f}{\partial y} = Q(x,y) = x^3$, hence
$f(x,y) = \int P(x,y)dx + g(y)$, where g is independent of x.
Also $f(x,y) = \displaystyle\int Q(x,y)dy + h(x)$, where h is independent of y.
$f(x,y) = \displaystyle\int P(x,y)dx + g(y) = x^3y + g(y) = \int Q(x,y)dy + h(x) = x^3y + h(x)$,
suggests that h, g should be such that, $h = g = c$.
Hence F is the gradient vector of any f of the form $f(x,y) = x^3y + c$.

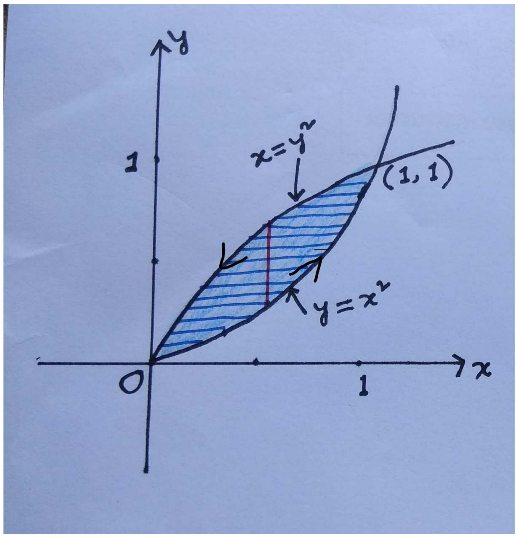
(b) Since P, Q are continuously differentiable functions, (where $P(x,y) = \sin y - y\sin x + x$ and $Q(x,y) = \cos x + x\cos y + y$)
and $\dfrac{\partial P}{\partial y} = \cos y - \sin x = \dfrac{\partial Q}{\partial x}$, F is the gradient of some scalar function f.

Note that if f is such that F is the gradient vector of f, then
$\dfrac{\partial f}{\partial x} = P(x,y)$ and $\dfrac{\partial f}{\partial y} = Q(x,y)$ hence
$f(x,y) = \int P(x,y)dx + g(y)$, where g is independent of x.
Also $f(x,y) = \int Q(x,y)dy + h(x)$, where h is independent of y. Since
$\displaystyle\int P(x,y)dx + g(y) = x\sin y + y\cos x + \frac{x^2}{2} + g(y) = \int Q(x,y)dy + h(x) = y\cos x + x\sin y + \frac{y^2}{2} + h(x)$,
if we choose $g(y) = \frac{y^2}{2}$ and $h(x) = \frac{x^2}{2}$,
then $f(x,y) = x\sin y + y\cos x + \dfrac{x^2}{2} + \dfrac{y^2}{2}$ is such that $\nabla f = F$.

5. Use Green's theorem to evaluate the line integral along the given positively oriented curve:

   (a) $\displaystyle\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy$, C is the boundary of the region enclosed by the parabolas $y = x^2$ and $x = y^2$.

   Solution: By Green's theorem:
   $$\int_C (y + e^{\sqrt{x}})dx + (2x + \cos y^2)dy = \int_0^1 \int_{x^2}^{\sqrt{x}} \left(\frac{\partial}{\partial x}(2x + \cos y^2) - \frac{\partial}{\partial y}(y + e^{\sqrt{x}})\right) dydx$$
   $$= \int_0^1 \int_{x^2}^{\sqrt{x}} (2-1)\, dydx$$
   $= \frac{1}{3}$.



(b) $\int_C xydx + 2x^2dy$, C consists of the line segment from $(-2,0)$ to $(2,0)$ and top half of the circle $x^2 + y^2 = 4$.

Solution: By Green's theorem:
$\int_C xydx + 2x^2dy = \int_{-2}^{2}\int_0^{\sqrt{4-x^2}} \left(\frac{\partial}{\partial x}(2x^2) - \frac{\partial}{\partial y}(xy)\right) dydx$
$= \int_{-2}^{2}\int_0^{\sqrt{4-x^2}} (4x - x)\, dydx$ (*)
$= \int_{-2}^{2} \sqrt{4-x^2}3xdx = \frac{3}{2}\left[-\frac{2}{3}(4-x^2)^{\frac{3}{2}}\right]_{-2}^{2} = 0.$
Aliter: By converting (*) to polar coordinates we get:
$= \int_0^{\pi}\int_0^2 3r\cos\theta rdrd\theta$
$= 0.$

6. Use Green's theorem to find out the work done by the force $\mathbf{F}(x,y) = x(x+y)\mathbf{i} + xy^2\mathbf{j}$ in moving a particle from the origin along x-axis to $(1,0)$ and then along a line segment to $(0,1)$, and then back to the origin along y-axis.

Solution: The work done is given by $\int_C \mathbf{F}\cdot d\mathbf{r}$, where C is the given curve.
By Green's theorem $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1\int_0^{-x+1} \left(\frac{\partial}{\partial x}(xy^2) - \frac{\partial}{\partial y}(x(x+y))\right) dydx$
$= \int_0^1\int_0^{-x+1} \left(y^2 - x\right) dydx$
$= -\frac{1}{12}.$

7. Let D be a region bounded by a simple closed path C in the xy-plane. Use Green's theorem to prove that the coordinates of the centroid $(\bar{x},\bar{y})$ of D are

$$\bar{x} = \frac{1}{2A}\oint x^2dy,\quad \bar{y} = -\frac{1}{2A}\oint y^2dx \quad \text{where } A \text{ is the area of } D.$$

Solution: By definition $\bar{x} = \frac{\iint_D xdA}{A}$, $\bar{y} = \frac{\iint_D ydA}{A}$, where $A = \text{Area}(D)$.
But $\iint_D xdA = \iint_D \left(\frac{\partial}{\partial x}(\frac{1}{2}x^2) - \frac{\partial}{\partial y}(0)\right) dA \quad (P = 0,\ Q = \frac{1}{2}x^2).$
By Green's theorem $\iint_D xdA = \frac{1}{2}\oint_C x^2dy$, hence $\bar{x} = \frac{1}{2A}\oint x^2dy.$
Similarly $\iint_D ydA = \iint_D \left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(\frac{1}{2}y^2)\right) dA \quad (Q = 0,\ P = \frac{1}{2}y^2).$
By Green's theorem $\iint_D ydA = \frac{1}{2}\oint_C y^2dx$, hence $\bar{y} = \frac{1}{2A}\oint y^2dx.$

8. If $\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j}$ and $r = |\mathbf{r}|$, let

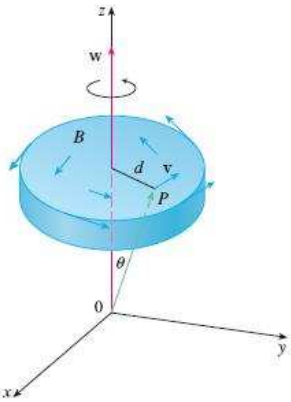
$$f(x,y) = \frac{\partial(logr)}{\partial y}\mathbf{i} - \frac{\partial(logr)}{\partial x}\mathbf{j}$$

for $r > 0$.
Let C be a smooth simple closed curve in the annulus $1 < x^2 + y^2 < 25$, then find all possible values of the line integral of f along C.

Solution: Let C_1 denote the positively oriented unit circle centered at the origin.
Note that $f(x, y) = \dfrac{y}{(x^2 + y^2)}\mathbf{i} - \dfrac{x}{(x^2 + y^2)}\mathbf{j}$, for $(x, y) \neq (0, 0)$.
$\dfrac{\partial P}{\partial y}, \dfrac{\partial Q}{\partial x} \quad \left(= \dfrac{x^2 - y^2}{(x^2 + y^2)^2} \right)$ are continuous for all $(x, y) \in \mathbf{R}^2 \setminus \{(0, 0)\}$,
where $P(x, y) = \dfrac{y}{(x^2 + y^2)}$ and $Q(x, y) = -\dfrac{x}{(x^2 + y^2)}$.

By Green's theorem $\displaystyle\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0 = \oint_C (P(x, y)dx + Q(x, y)dy)$,
if C is such that the origin is not inside C.
By Green's theorem again

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0 = \oint_C (P(x, y)dx + Q(x, y)dy) - \oint_{C_1} (P(x, y)dx + Q(x, y)dy),$$
if C is such that the origin is inside C, and C is positively oriented.
Hence $\oint_C (P(x, y)dx + Q(x, y)dy) = -2\pi$. (refer to (**) of problem 3)
$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = 0 = -\oint_C (P(x, y)dx + Q(x, y)dy) - \oint_{C_1} (P(x, y)dx + Q(x, y)dy),$$
if C is such that the origin is inside C, and C is negatively oriented.
Hence $\oint_C (P(x, y)dx + Q(x, y)dy) = 2\pi$.
Hence the only possible values of the line integral are $0, \pm 2\pi$.



9. The exercise demonstrates a connection between curl vector and rotations. Let $\mathbf{B}$ be a rigid body rotating about z-axis. The rotation can be described by the vector $\mathbf{w} = \omega\mathbf{k}$, where ω is the angular speed of $\mathbf{B}$, that is, the tangential speed at any point P in B divided by the distance d from the axis of rotation. Let $\mathbf{r} =< x, y, z >$ be the position vector of P.

   (a) By considering the angle θ in the figure, show that the velocity field of B is given by $\mathbf{v} = \mathbf{w} \times \mathbf{r}$.

   (b) Show that $\mathbf{v} = -\omega y\mathbf{i} + \omega x\mathbf{j}$.

   (c) Show that $\nabla \times \mathbf{v} = 2\mathbf{w}$.

Solution:
(a) Given that $\omega = \frac{|\mathbf{v}|}{d}$, and $\sin\theta = \frac{d}{|\mathbf{r}|}$.
Therefore $|\mathbf{v}| = d\omega = \omega|\mathbf{r}|\sin\theta = |\mathbf{w} \times \mathbf{r}|$. **(1)**
Since $\mathbf{v}$ is orthogonal to both $\mathbf{w}$ and $\mathbf{r}$ and its direction is same as that of $\mathbf{w} \times \mathbf{r}$ therefore from (1) it follows:
$\mathbf{v} = \mathbf{w} \times \mathbf{r}$.
(b) Since $\mathbf{w} = \omega\mathbf{k}$ and $\mathbf{r}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,
$$\mathbf{v} = \mathbf{w} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & \omega \\ x & y & z \end{vmatrix} = -\omega y\mathbf{i} + \omega x\mathbf{j}.$$
(c) $$\nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix} = 2\mathbf{w}.$$

10. Use Stokes' Theorem to evaluate

(a) $\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S}$ where $\mathbf{F} = xyz\mathbf{i} + xy\mathbf{j} + x^2yz\mathbf{k}$ and S consists of the top and the four sides (but not the bottom) of the cube with vertices $(\pm1, \pm1, \pm1)$, oriented outward.

Solution: $\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \iint_S (\nabla\times\mathbf{F})\cdot \mathbf{n}dS$.

By Stokes' theorem, $\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \int_{C_1} Pdx + Qdy + Rdz + \int_{C_2} Pdx + Qdy + Rdz$

$+ \int_{C_3} Pdx + Qdy + Rdz + \int_{C_4} Pdx + Qdy + Rdz$

where $C_i, i = 1, 2, \ldots 4$ are the four sides of the bottom surface of the cube, and $P(x, y, z) = xyz$, $Q(x, y, z) = xy$, $R(x, y, z) = x^2yz$.

Since for $C_1, y = -1, z = -1$: For C_2, $x = 1, z = -1$: For C_3, $y = 1, z = -1$: For C_4 $x = -1, z = -1$,

$$\int_{C_1} Pdx + Qdy + Rdz + \int_{C_2} Pdx + Qdy + Rdz + \int_{C_3} Pdx + Qdy + Rdz$$

$$+ \int_{C_4} Pdx + Qdy + Rdz$$

$$= \int_{C_1} Pdx + \int_{C_2} Qdy + \int_{C_3} Pdx + \int_{C_4} Qdy$$

$$= \int_{-1}^{1} (-1)(-1)xdx + \int_{-1}^{1} (-1)(1)ydy + \int_{1}^{-1} (-1)(1)xdx + \int_{1}^{-1} (-1)(-1)ydy = 0.$$

(b) $\int_C \mathbf{F}\cdot d\mathbf{r}$ where $\mathbf{F} = 2z\mathbf{i} + 4x\mathbf{j} + 5y\mathbf{k}$ and C is the curve of intersection of the plane $z = x + 4$ and the cylinder $x^2 + y^2 = 4$.

Solution: $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS$.

$$\nabla\times\mathbf{F} = (\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k} = 5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k},$$

where $P(x, y, z) = 2z$, $Q(x, y, z) = 4x$, $R(x, y, z) = 5y$.

The surface $\mathbf{r}(x, y)$ is of the form: $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{j}$, where $f(x, y) = x + 4$, hence

$$\frac{\partial\mathbf{r}}{\partial x}\times\frac{\partial\mathbf{r}}{\partial y} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k} = -\mathbf{i} + \mathbf{k}$$

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS = \iint_D (5\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})\cdot(-\mathbf{i} + \mathbf{k})\,dA,$$

where D is the disc of radius 2 centered at the origin.

Hence $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS = \iint_D (-5 + 4)dA = -\int_0^2\int_0^{2\pi} rd\theta dr = -4\pi.$

11. Calculate the work done by the force field

$$\mathbf{F}(x, y, z) = (x^x + z^2)\mathbf{i} + (y^y + x^2)\mathbf{j} + (z^z + y^2)\mathbf{k}$$

when a particle moves under its influence around the edge of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies in the first octant, in a counterclockwise direction as viewed from above.

Solution: The work done is given by

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS.$$

$$(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z})\mathbf{i} + (\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x})\mathbf{j} + (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y})\mathbf{k} = 2(y\mathbf{i} + z\mathbf{j} + x\mathbf{k}).$$

The surface $\mathbf{r}(x, y)$ is of the form $\mathbf{r}(x, y) = x\mathbf{i} + y\mathbf{j} + f(x, y)\mathbf{j}$ where $f(x, y) = \sqrt{4 - x^2 - y^2}$, hence

$$\frac{\partial r}{\partial x}\times\frac{\partial r}{\partial y} = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k} = \frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}$$

$$\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS = \iint_D 2\,(y\mathbf{i} + z\mathbf{j} + x\mathbf{k})\cdot\left(\frac{x}{\sqrt{4 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{4 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right)dA,$$

where D is the part of the disc of radius 2 centered at the origin in the first quadrant.

Hence $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}dS = 2\iint_D \left(x + y + \frac{xy}{\sqrt{4 - x^2 - y^2}}\right)dA$

$$= 2\int_0^2\int_0^{\frac{\pi}{2}} \left((r\cos\theta + r\sin\theta) + (\frac{r^2\cos\theta\sin\theta}{\sqrt{4 - r^2}})\right) rd\theta dr = 2(I_1 + I_2) = 2(\frac{16}{3} + \frac{8}{3}) = 16.$$

12. Let S be the surface of the solid cylinder T bounded by the planes $z = 0$ and $z = 3$ and the cylinder $x^2 + y^2 = 4$. Calculate the outward flux $\iint_S \mathbf{F}\cdot\mathbf{n}dS$ given $\mathbf{F}(x, y, z) = (x^2 + y^2 + z^2)(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$.

Solution: By divergence theorem,

$\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iiint_T Div(F)dv.$

$Div(F) = \nabla\cdot F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = 5(x^2+y^2+z^2),$

where $P = (x^2+y^2+z^2)x$, $Q = (x^2+y^2+z^2)y$ and $R = (x^2+y^2+z^2)z$.

By using cylindrical coordinates we get:

$\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iiint_T Div(F)dv = \int_0^3\int_0^{2\pi}\int_0^2 5(r^2+z^2)rdrd\theta dz = 300\pi.$

13. Use Divergence Theorem to evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}dS$ where

$$\mathbf{F}(x,y,z) = z^2x\mathbf{i} + \left(\frac{1}{3}y^3 + \tan z\right)\mathbf{j} + (x^2z+y^2)\mathbf{k}$$

and S is the top half of the sphere $x^2+y^2+z^2=1$.

Solution: Let S_1 be the the unit disc centered at the origin with normal facing downward, and let $S_2 = S\cup S_1$, then S_2 has outer facing normal if S has upward facing normal.

$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS = \iint_S \mathbf{F}\cdot\mathbf{n}dS + \iint_{S_1} \mathbf{F}\cdot\mathbf{n}dS.$

By divergence theorem $\iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS = \iiint_T Div(F)dv.$

Since $Div(F) = \nabla\cdot F = \dfrac{\partial P}{\partial x} + \dfrac{\partial Q}{\partial y} + \dfrac{\partial R}{\partial z} = z^2+y^2+x^2,$

where $P = z^2x,\ Q = \frac{1}{3}y^3 + \tan z,\ R = x^2z + y^2.$

By using spherical coordinates we get:

$\iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS = \int_0^1\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \rho^2(\rho^2\sin\phi)d\phi d\theta d\rho = \dfrac{2\pi}{5}.$

Also $\iint_{S_1} \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_1} \mathbf{F}\cdot(-\mathbf{k})dS = \iint_{S_1}(x^2\times 0+y^2)(-1)dS = -\int_0^1\int_0^{2\pi}(r^2\sin^2\theta)rdrd\theta = -\dfrac{\pi}{4}$

$\iint_S \mathbf{F}\cdot\mathbf{n}dS = \iint_{S_2} \mathbf{F}\cdot\mathbf{n}dS - \iint_{S_1} \mathbf{F}\cdot\mathbf{n}dS = \dfrac{13\pi}{20}.$