PH 102: Physics II

Lecture 13 (Post midsem, Spring 2020)

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LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Division
Lec 1	05-03-2020	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 2	11-03-2020	Application of Ampere's Law, Magnetic Vector Potential	5.3, 5.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 1	12-03-2020	Lec 1		
Tut 2	17-03-2020	Lec 2		
Lec 3	18-03-2020	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic materials, magnetization	5.4, 6.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 4	19-03-2020	Field of a magnetized object, Boundary conditions	6.2, 6.3, 6.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 3	24-03-2020	Lec 3, 4		
Lec 5	25-03-2020	Ohm's law, motional emf, electromotive force	7.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 6	26-03-2020	Faraday's law, Lenz's law, Self & Mutual inductance, Energy stored in magnetic field	7.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 4	31-03-2020	Lec 5, 6		
Lec 7	01-04-2020	Maxwell's equations	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 8	02-04-2020	Discussions, Problem solving	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
	07-04-2020	Quiz II		
Lec 9	08-04-2020	Continuity equation, Poynting theorem	8.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 10	16-04-2020	Wave solution of Maxwell's equations, polarisation	9.1, 9.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 5	21-04-2020	Lec 9, 10		
Lec 11	22-04-2020	Electromagnetic waves in matter, reflection & transmission: normal incidence	9.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 12	23-04-2020	Reflection & transmission: oblique incidence	9.3, 9.4	I, II (4-4:55 pm) III, IV (11-11:55

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

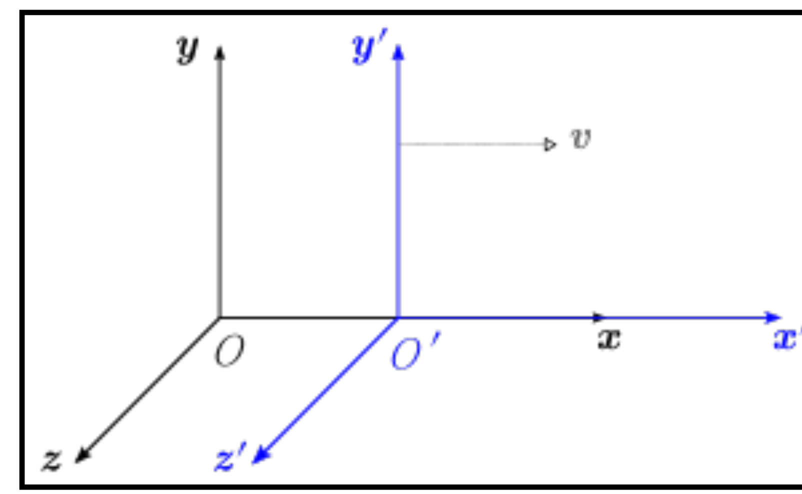
				am)
Tut 6	28-4-2020	Lec 11, 12		
Lec 13	29-04-2020	Relativity and electromagnetism: Galilean & special relativity	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 14	30-04-2020	Discussions, problem solving	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)

Coordinate Transformations

Consider a frame (blue) moving with respect to another frame (black) along the x direction with uniform speed v.

Let the coordinates be related by Galilean Transformations:

$$x' = x - vt$$
$$y' = y$$
$$z' = z$$
$$t' = t$$



How do the fields and the Maxwell's equations look like in the new frame?

According to Newtonian principle, the force is same in the two frames moving with constant velocity with respect to each other.

The Lorentz force on a charged particle moving with speed u in the black frame is

$$F = q(E - uB) = q(E' - u'B')$$

For simplicity, we assume E, B to be in y and z directions respectively and u to be in x direction. This gives rise to the direction of $q(\vec{u} \times \vec{B})$ in the negative y direction.

Here $u' = u - v$ and hence

$$E' - u'B' = (E' + vB') - uB' = E - uB$$

(E', B') are fields as measured in the moving frame

In the equality: $(E' + vB') - uB' = E - uB$ one term on each side is independent of the velocity of the charged particle (u) and other is dependent on it. For this to be valid in the most general case we must have

$$E = E' + vB', \; B = B'$$

Use this transformation and check if the Maxwell's equations are invariant under it.

Consider the Ampere's law in the absence of any source:

$$\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

Since E, B are assumed to be in y and z directions respectively $\vec{E} = E(x,t)\hat{y}, \;\; \vec{B} = B(x,t)\hat{z}$

we can write $\vec{\nabla} \times \vec{B} = -\frac{\partial B}{\partial x}\hat{y}, \frac{\partial \vec{E}}{\partial t} = \frac{\partial E}{\partial t}\hat{y}$

Thus, the Ampere's law in un-primed frame is $\frac{\partial B}{\partial x} = -\mu_0 \epsilon_0 \frac{\partial E}{\partial t}$

Using chain rule:

$$\frac{\partial B}{\partial x} = \frac{\partial B}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial B}{\partial t'}\frac{\partial t'}{\partial x}$$

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial t}$$

For Galilean transformations:

$$\frac{\partial x'}{\partial x} = 1, \;\; \frac{\partial x'}{\partial t} = -v$$

$$\frac{\partial t'}{\partial t} = 1, \;\; \frac{\partial t'}{\partial x} = 0$$

Using these, the field derivatives in Ampere's law are:

$$\frac{\partial B}{\partial x} = \frac{\partial B}{\partial x'}, \quad \frac{\partial E}{\partial t} = \frac{\partial E}{\partial t'} - v\frac{\partial E}{\partial x'}$$

Using the transformations of fields we get:

$$\frac{\partial B}{\partial x} = \frac{\partial B}{\partial x'}, \quad \frac{\partial E}{\partial t} = \frac{\partial E'}{\partial t'} + v\frac{\partial B'}{\partial t'} - v\frac{\partial E'}{\partial x'} - v^2\frac{\partial B'}{\partial x'}$$

Thus the Maxwell's equation in the un-primed frame given by

$$\frac{\partial B}{\partial x} = -\mu_0\epsilon_0\frac{\partial E}{\partial t}$$

becomes

$$\frac{\partial B}{\partial x'} = -\mu_0\epsilon_0\frac{\partial E'}{\partial t'} + \mu_0\epsilon_0\left(v\frac{\partial E'}{\partial x'} - v\frac{\partial B'}{\partial t'} + v^2\frac{\partial B'}{\partial x'}\right)$$

in the primed frame.

Maxwell's equations are not invariant under Galilean Transformations!

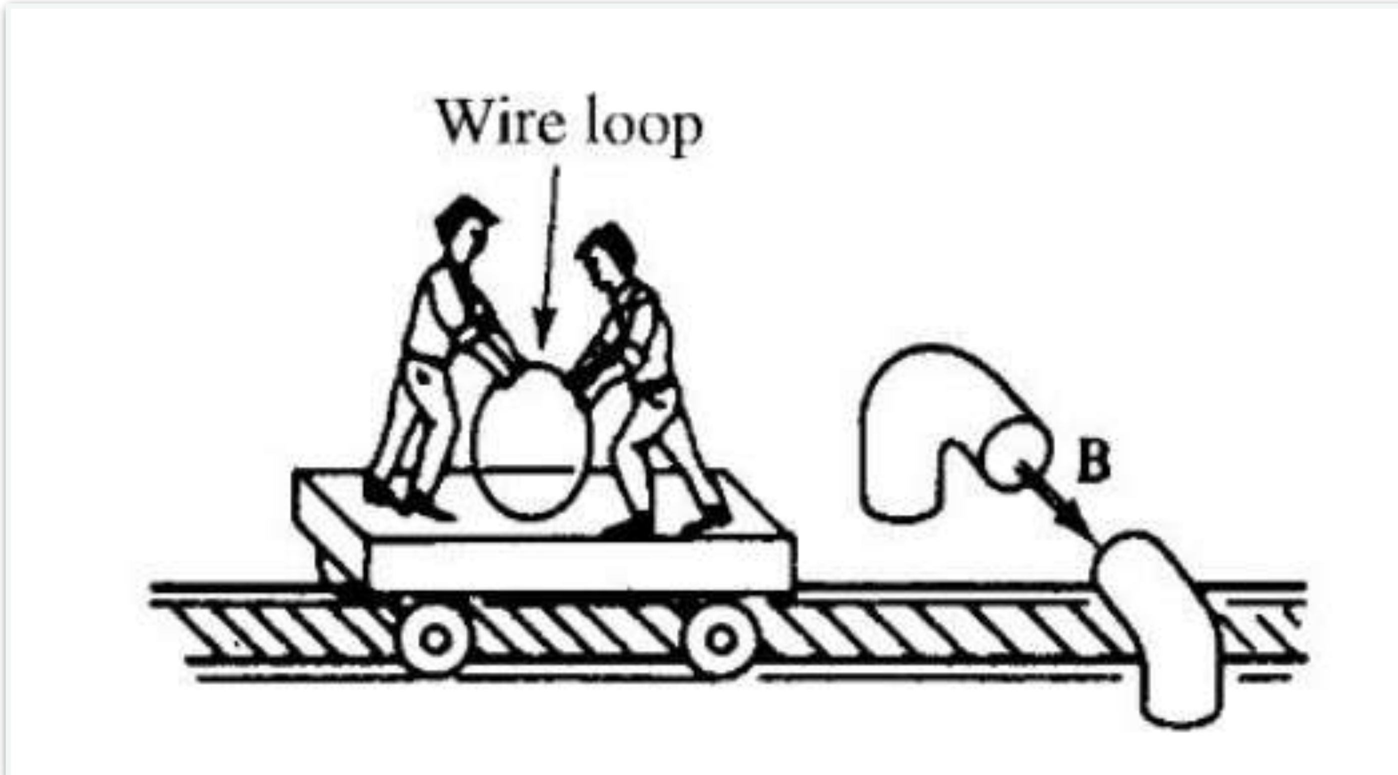
- Although Newtonian mechanics is invariant under Galilean transformations and a particle experiences same force in all inertial frames, the same invariance is not applicable to electromagnetic force.

- **Exercise**: Calculate the force between two line charges of uniform charge density placed parallel to each other with distance of separation d. Show that the force in the two frames connected by Galilean transformations are not same.

- The same is reflected by the fact that the **Maxwell's equations do not remain invariant under Galilean transformations.**

- Recall other shortcomings/fallacies of Galilean relativity from PH 101.

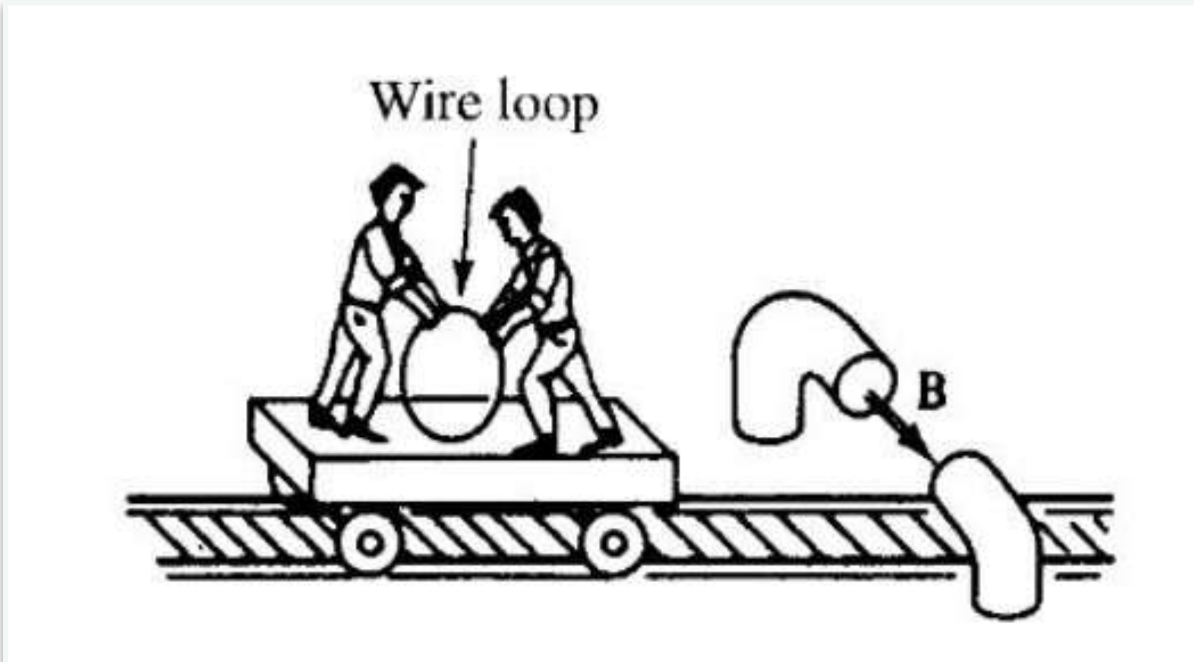
Relativity & Electromagnetism



As the train and hence the loop passes through the region of magnetic field B, a motional emf is induced in the loop.

1. How will the observers moving on the train interpret this?

2. How will an observer at rest outside the rail will interpret this?



- The loop is at rest w.r.t the observers moving on the rail. The change in magnetic field through the loop induces an electric field: Faraday's law.

- For an observer outside, emf is due to the magnetic force on the charged particles inside the loop which are moving.

Coincidence?

Recall Faraday's three experiments: Lecture 6

To Einstein's predecessors:

- The agreement between two different observers/ interpretations was a coincidence, though they thought that one of the interpretations was incorrect.

- The electric and magnetic fields were strains in an invisible medium called ether, which fills all space. The speed of the charged particles inside the loop should be measured with respect to the ether.

- The observer moving with the train and his/her interpretation is wrong as the frame is moving relative to ether.

- No such ether was found experimentally (Michelson-Morley experiment)!

Einstein, who did not believe the agreement between two different interpretations to be a coincidence, wrote in his paper on special theory of relativity (1905)

It is known that Maxwell's electrodynamics—as usually understood at the present time—when applied to moving bodies, leads to asymmetries which do not appear to be inherent in the phenomena. Take, for example, the reciprocal electrodynamic action of a magnet and a conductor. The observable phenomenon here depends only on the relative motion of the conductor and the magnet, whereas the customary view draws a sharp distinction between the two cases in which either one or the other of these bodies is in motion. For if the magnet is in motion and the conductor at rest, there arises in the neighborhood of the magnet an electric field ... producing a current at the places where parts of the conductor are situated. But if the magnet is stationary and the conductor in motion, no electric field arises in the neighborhood of the magnet. In the conductor, however, we find an electromotive force ... which gives rise—assuming equality of relative motion in the two cases discussed—to electric currents of the same path and intensity as those produced by the electric forces in the former case.

Examples of this sort, together with unsuccessful attempts to discover any motion of the earth relative to the "light medium," suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest.[2]

Conclusion: Speed of light in vacuum is a universal constant, the same in all directions, regardless of the motion of the observer or the source

Recap of STR

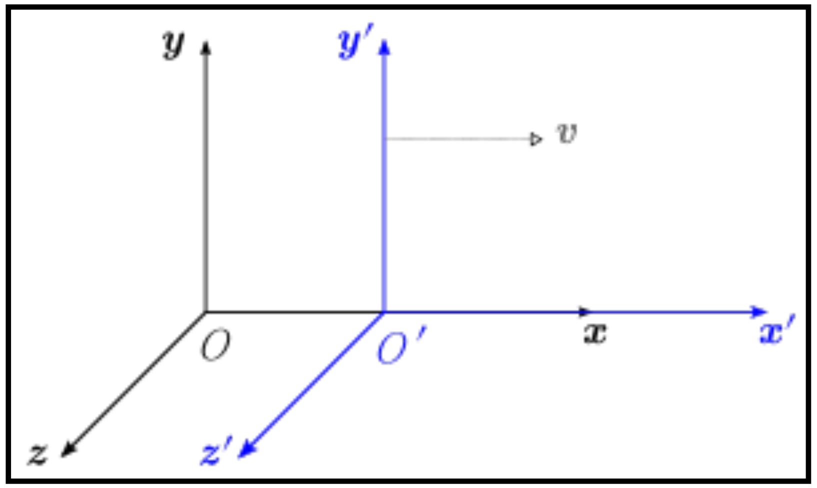
PH 101

Postulates:

- **The principle of relativity**: The laws of physics apply in all inertial reference systems.
- **The universal speed of light**: The speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

While Galilean transformations between two inertial frames can not accommodate these postulates, the Lorentz transformations can:

$$
\begin{aligned}
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z \\
t' = \gamma\left(t - \frac{vx}{c^2}\right), \gamma &= 1/\sqrt{1 - \frac{v^2}{c^2}}
\end{aligned}
$$



Consequences of STR

PH 101

- Distance between spacetime events (line element) is a Lorentz invariant $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$

- Notion of simultaneity is relative. Two simultaneous events in one frame may not be so for an observer moving with uniform speed.

- Clocks run slower as observed from a moving frame: $$\triangle t = \triangle\tau / \sqrt{1 - \frac{v^2}{c^2}}$$ Time Dilation

- Objects appear to be shorted when viewed from a moving frame: $$L = L_0\sqrt{1 - \frac{v^2}{c^2}}$$ Length Contraction

Consequences of STR

- Velocity transformations: For a particle moving with speed u in frame S, an observer in the frame S' will measure its velocity as

$$u'_x = \frac{u_x - v}{1 - \frac{v u_x}{c^2}}, u'_y = \frac{u_y}{\gamma\left(1 - \frac{v u_x}{c^2}\right)}, u'_z = \frac{u_z}{\gamma\left(1 - \frac{v u_x}{c^2}\right)}$$

- Relativistic momentum (ensures that momentum conserved in one frame is also conserved in another inertial frame, related by Lorentz transformations):

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Consequences of STR

- Relativistic Energy:

$$E = \frac{mc^2}{\sqrt{1 - \frac{u^2}{c^2}}}$$

$$E^2 = p^2c^2 + m^2c^4$$

- Rest Energy:

$$E_{\text{rest}} = mc^2$$

- Kinetic Energy:

$$E_{\text{kinetic}} = mc^2 \left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right)$$

Transformation of momentum-energy

$$p'_x = \gamma \left(p_x - v \frac{E}{c^2} \right)$$

$$p'_y = p_y$$

$$p'_z = p_z$$

$$\frac{E'}{c^2} = \gamma \left(\frac{E}{c^2} - \frac{v p_x}{c^2} \right)$$

Which is similar to Lorentz transformations of spacetime coordinates if we replace x, y, z, t by

$$p_x, p_y, p_z, \frac{E}{c^2}$$

Transformation of force

$$F'_x = \frac{dp'_x}{dt'} = \frac{\gamma(dp_x - vdE/c^2)}{\gamma(dt - vdx/c^2)} = \frac{\frac{dp_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \frac{F_x - \frac{v}{c^2}\frac{dE}{dt}}{1 - \frac{v}{c^2}u_x}$$

$$F'_{y,z} = \frac{dp'_y}{dt'} = \frac{\frac{dp_{y,z}}{dt}}{\gamma\left(1 - \frac{v}{c^2}\frac{dx}{dt}\right)} = \frac{F_{y,z}}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}$$

Using

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1 - u^2/c^2}}\right) = \frac{m\vec{u}}{(1 - u^2/c^2)^{3/2}} \cdot \frac{d\vec{u}}{dt}$$

$$= \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1 - u^2/c^2}}\right) \cdot \vec{u} = \frac{d\vec{p}}{dt} \cdot \vec{u}$$

We can write

$$F'_x = \frac{F_x - \frac{v}{c^2}(\vec{u} \cdot \vec{F})}{1 - \frac{v}{c^2}u_x}$$

Therefore, for u=0:

$$F'_\perp = F_\perp/\gamma, F'_\parallel = F_\parallel$$

How do electromagnetic fields transform?

Consider a parallel plate capacitor with uniform electric field in between the plates. It is at rest in frame S_0 and moving with a uniform speed in frame S.

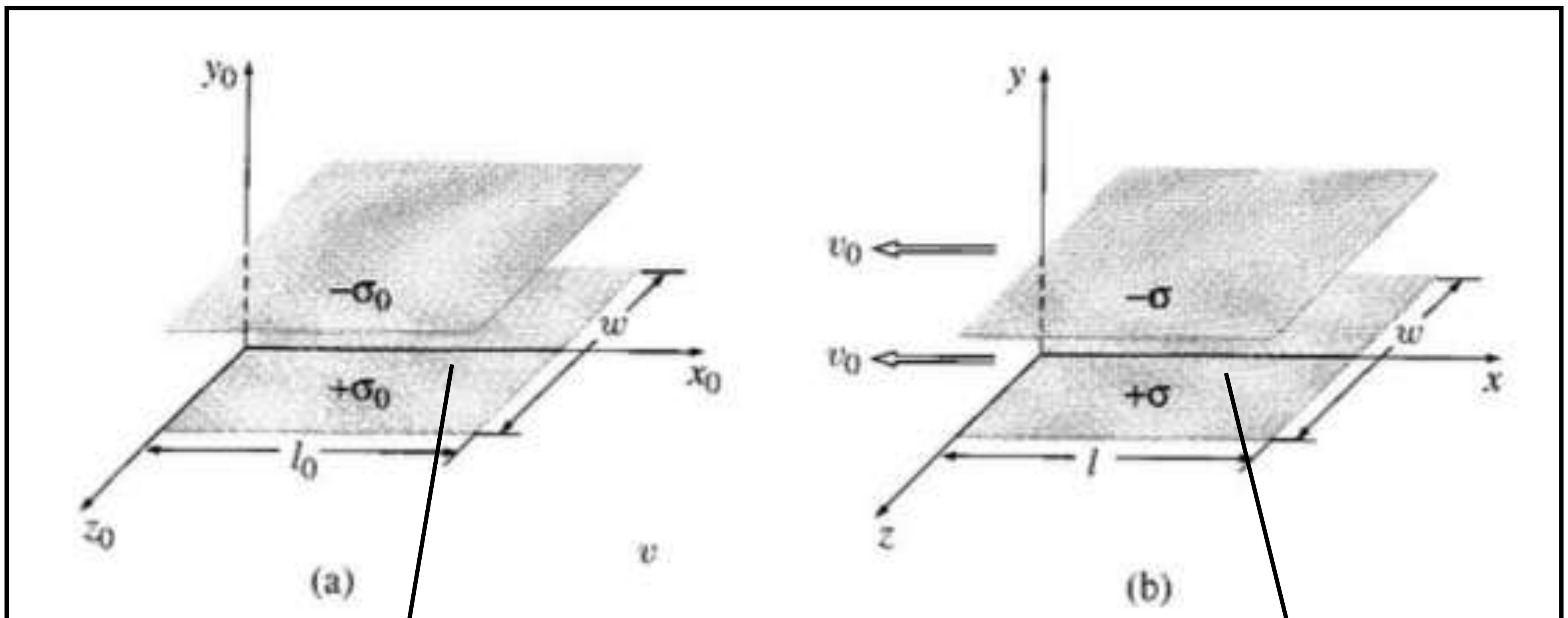


Fig 12.35, Introduction to Electrodynamics, D J Griffiths

$$\vec{E}_0 = \frac{\sigma_0}{\epsilon_0}\hat{y}$$

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{y}$$

How do electromagnetic fields transform?

- Since length along the direction of motion gets contracted, the surface charge density in the moving frame is increased by the Lorentz factor*

$$\sigma = \gamma_0 \sigma_0, \quad \gamma_0 = 1/\sqrt{1 - v_0^2/c^2}$$

- Therefore, the electric fields (perpendicular) are related by

$$E^{\perp} = \gamma_0 E_0^{\perp}$$

- To find the transformation of parallel components, consider the capacitor to be moving in a direction perpendicular to the plates. Since the field does not depend upon the separation between the plates (which gets contracted in this case), the parallel components remain same: $E^{\parallel} = E_0^{\parallel}$

*charge is Lorentz invariant just like rest mass is.

How do electromagnetic fields transform?

The previous example did not give the most general transformation of E, B as the original system at rest frame had static charges only.

Consider a more general situation where, apart from electric field due to surface charge $\pm\sigma$, there exists a magnetic field too, due to surface current $\vec{K}_{\pm} = \mp\sigma v_0 \hat{x}$

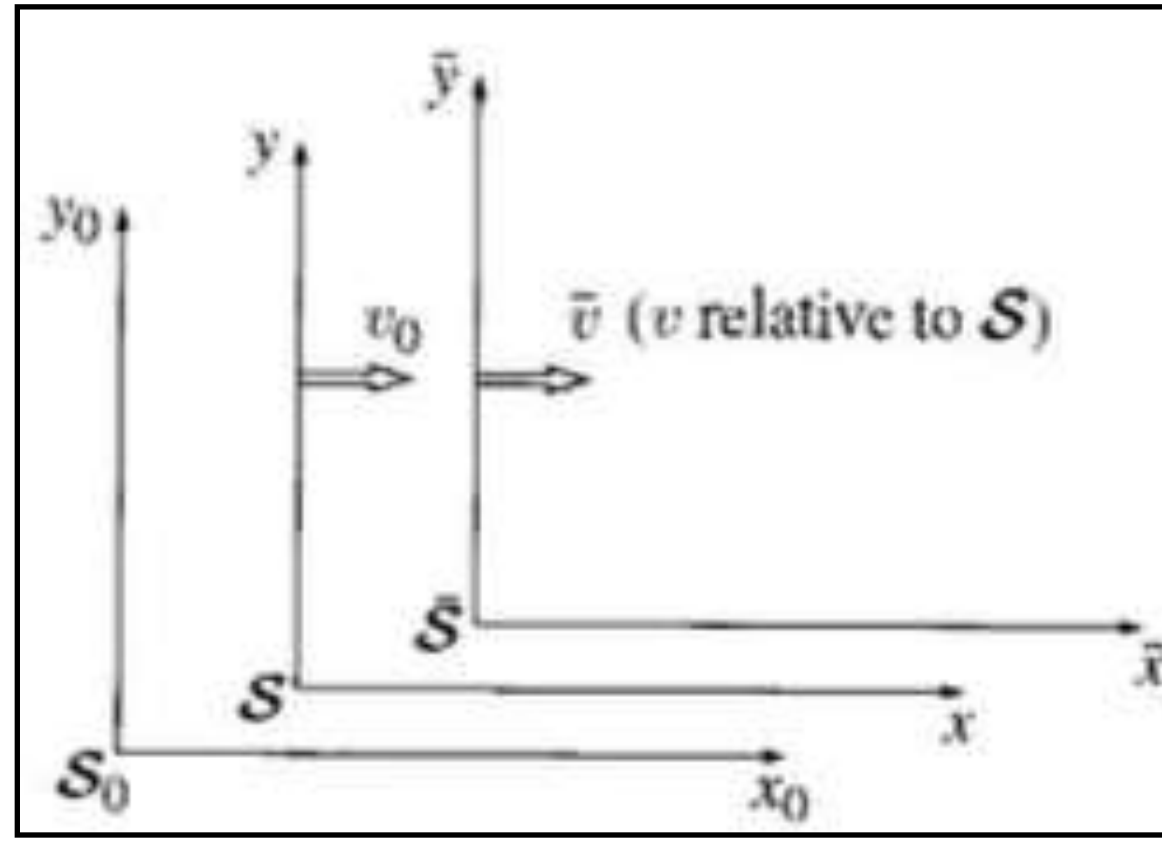


Fig 12.38, Introduction to Electrodynamics, D J Griffiths

The capacitor plates are in x-z plane as before

How do electromagnetic fields transform?

- The fields for such surface charge and surface currents:: $$E_y = \frac{\sigma}{\epsilon_0}, B_z = -\mu_0 \sigma v_0$$
- Since there are two different velocities involved: one for surface charge and one for the frame in which the fields are to be found, consider three different frames:

$$S_0(\text{at rest}), S(v_0 \text{ wrt } S_0), \bar{S}(\bar{v} \text{ wrt } S_0)$$

- Fields in the third frame: $\bar{E}_y = \frac{\bar{\sigma}}{\epsilon_0}, \bar{B}_z = -\mu_0 \bar{\sigma} \bar{v}$
- The surface charge density seen in third frame is related to that in the first frame as:

$$\bar{\sigma} = \bar{\gamma}\sigma_0, \bar{\gamma} = 1/\sqrt{1 - \bar{v}^2/c^2}$$

How do electromagnetic fields transform?

- Similarly, for second and first frame

$$\sigma = \gamma_0 \sigma_0, \gamma_0 = 1/\sqrt{1 - v_0^2/c^2}$$

- Therefore, the fields in third frame are

$$\bar{E}_y = \frac{\bar{\gamma}}{\gamma_0}\frac{\sigma}{\epsilon_0}, \bar{B}_z = -\frac{\bar{\gamma}}{\gamma_0}\mu_0 \sigma \bar{v}$$

- Using velocity transformations: $\bar{v} = \dfrac{v + v_0}{1 + v v_0/c^2}$

$$\frac{\bar{\gamma}}{\gamma_0} = \frac{\sqrt{1 - v_0^2/c^2}}{\sqrt{1 - \bar{v}^2/c^2}} = \frac{1 + v v_0/c^2}{\sqrt{1 - v^2/c^2}} = \gamma\left(1 + \frac{v v_0}{c^2}\right)$$

Recall velocity transformations: $u'_x = \dfrac{u_x - v}{1 - \frac{v u_x}{c^2}}$

Now, for an observer in second frame, the third frame is moving at speed v (along positive x axis). The first frame is moving with respect to the second frame with speed v_0 in negative x direction. Therefore using the above velocity transformation, one can find the speed of the third frame with respect to the first frame as

$$\bar{v} = \frac{v + v_0}{1 + v v_0 / c^2}$$

+ve sign as the 1st frame is moving in -ve x direction w. r. t. to the 2nd

Therefore,

$$1 - \frac{\bar{v}^2}{c^2} = \frac{\left(1 - \frac{v^2}{c^2}\right)(c^2 - v_0^2)}{c^2 \left(1 + \frac{v v_0}{c^2}\right)^2}$$

$$\implies \frac{\left(1 - \frac{v_0^2}{c^2}\right)}{\left(1 - \frac{\bar{v}^2}{c^2}\right)} = \gamma^2 \left(1 + \frac{v v_0}{c^2}\right)^2$$

- where $\gamma = 1/\sqrt{1 - v^2/c^2}$ with v being the velocity of third frame with respect to the second.

- The electric field in third frame is, therefore,

$$\bar{E}_y = \gamma \left(1 + \frac{vv_0}{c^2}\right) \frac{\sigma}{\epsilon_0} = \gamma \left(\frac{\sigma}{\epsilon_0} - \frac{v}{c^2 \epsilon_0 \mu_0} (-\mu_0 \sigma v_0) \right) = \gamma \left(E_y - \frac{v}{c^2 \epsilon_0 \mu_0} B_z \right)$$

- Similarly, the magnetic field in third frame is

$$\bar{B}_z = -\gamma \left(1 + \frac{vv_0}{c^2}\right) \mu_0 \sigma \bar{v} = -\gamma \left(1 + \frac{vv_0}{c^2}\right) \mu_0 \sigma \left(\frac{v + v_0}{1 + vv_0/c^2} \right)$$

$$\implies \bar{B}_z = \gamma \left(-\mu_0 \sigma v_0 - \mu_0 \epsilon_0 v \frac{\sigma}{\epsilon_0} \right) = \gamma (B_z - \mu_0 \epsilon_0 v E_y)$$

Transformation of electromagnetic fields

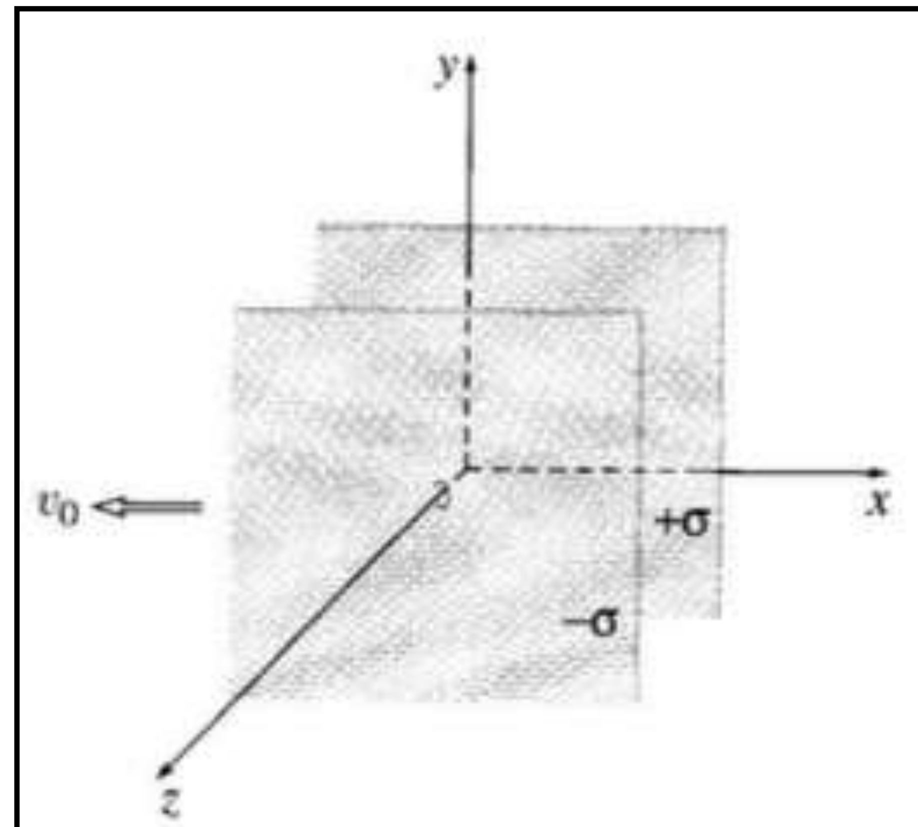
- Using $c = 1/\sqrt{\mu_0 \epsilon_0}$ the transformation of electromagnetic fields can be written as

$$\bar{E}_y = \gamma(E_y - vB_z), \quad \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

- To find the transformation for other two transverse components, consider the plates of the capacitor to be in x-y plane instead of x-z plane so that

$$E_z = \frac{\sigma}{\epsilon_0}, \quad B_y = \mu_0 \sigma v_0$$

- Verify that the transformations will be



$$\bar{E}_z = \gamma(E_z + vB_y), \quad \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z)$$

Fig 12.39, Introduction to Electrodynamics, D J Griffiths

Transformation of electromagnetic fields

- Parallel components of electric field remains unchanged, as shown already, by considering the direction of relative motion to be perpendicular to the plates of the parallel plate capacitor.

- To find the same for magnetic field, consider a long solenoid, aligned parallel to the x-axis and at rest in frame S, so that the magnetic field is $B_x = \mu_0 n I$

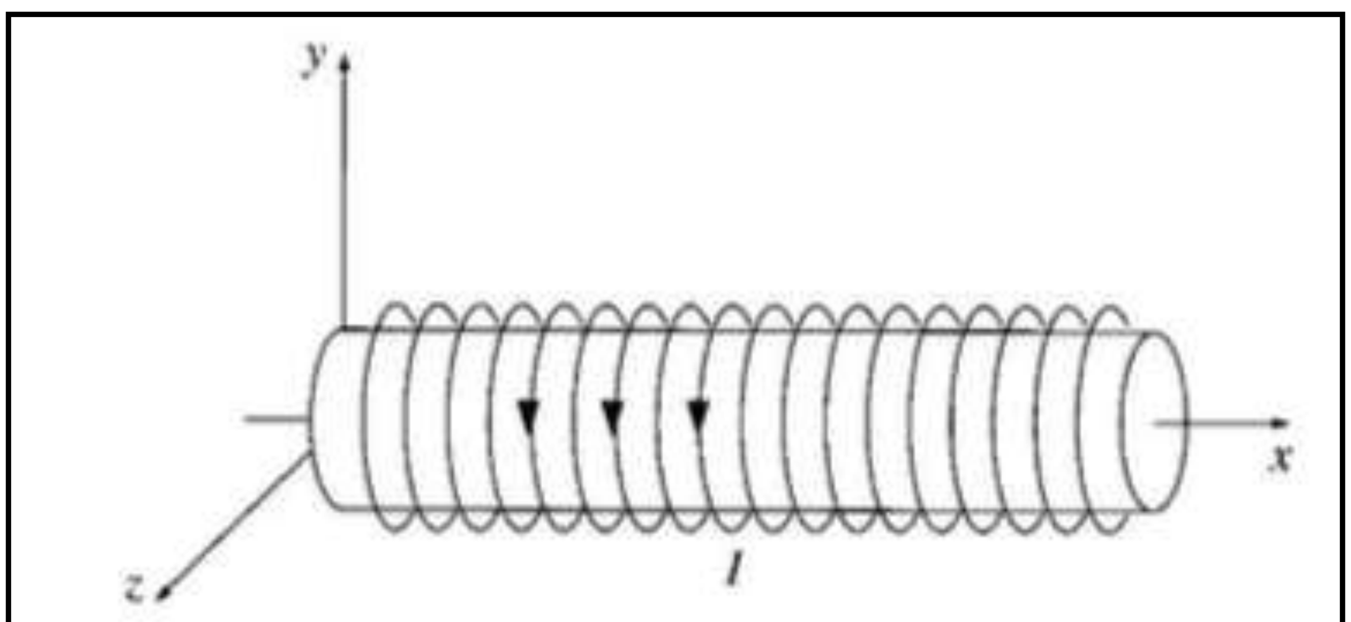


Fig 12.40, Introduction to Electrodynamics, D J Griffiths

- In a frame $\bar{S}$ moving along the x axis with speed v, the length of the solenoid contracts and hence the number of turns per unit length increases to

$$\bar{n} = \gamma n, \quad \gamma = 1/\sqrt{1 - v^2/c^2}$$

- However due to time dilation the current in the moving frame becomes

$$\bar{I} = \frac{I}{\gamma}$$

- The two Lorentz factors cancer each other, keeping the parallel components of magnetic field unchanged

$$\bar{B}_x = B_x$$

Transformation of electromagnetic fields

$$\bar{E}_x = E_x,\ \bar{E}_y = \gamma(E_y - vB_z),\ \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x,\ \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z),\ \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

- If **B**=0 in S, then

$$\vec{\bar{B}} = \gamma\frac{v}{c^2}(E_z\hat{y} - E_y\hat{z}) = \frac{v}{c^2}(\bar{E}_z\hat{y} - \bar{E}_y\hat{z})$$

$$\vec{\bar{B}} = -\frac{1}{c^2}(\vec{v}\times\vec{\bar{E}}),\quad \vec{v} = v\hat{x}$$

- If **E**=0 in S, then

$$\vec{\bar{E}} = -\gamma v(B_z\hat{y} - B_y\hat{z}) = -v(\bar{B}_z\hat{y} - \bar{B}_y\hat{z})$$

$$\vec{\bar{E}} = \vec{v}\times\vec{\bar{B}},\quad \vec{v} = v\hat{x}$$