Department of Mathematics
Indian Institute of Technology Guwahati
MA 101: Mathematics I
Model solutiuons of Quiz-I

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1. Let $x_n = \sin\frac{n\pi}{3}, \quad n \geq 1$.

   (a) Find $\liminf x_n$ and $\limsup x_n$. $\boxed{\mathbf{1+1}}$

   *Solution.* We have $x_1 = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, x_2 = \sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, x_3 = \sin\frac{3\pi}{3} = 0$, $x_4 = \sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2}, x_5 = \sin\frac{5\pi}{3} = -\frac{\sqrt{3}}{2}, x_6 = \sin\frac{6\pi}{3} = 0, \ldots$.
   Hence, $x_n \in \{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\}$ for all $n \geq 1$. [1]

   Since $x_n$ takes the value $-\frac{\sqrt{3}}{2}$ for infinitely many values of $n$, so $z_n = \inf\{x_k : k \geq n\} = -\frac{\sqrt{3}}{2}$ for all $n$, and hence $\liminf x_n = -\frac{\sqrt{3}}{2}$.
   Also $x_n$ takes the value $\frac{\sqrt{3}}{2}$ for infinitely many values of $n$, so $y_n = \sup\{x_k : k \geq n\} = \frac{\sqrt{3}}{2}$ for all $n$. Hence, $\limsup x_n = \frac{\sqrt{3}}{2}$. [1] □

   (b) Let $(x_{n_k})$ be a subsequence of $(x_n)$ such that $x_{n_k} \to \ell$. Find all possible values of $\ell$. To each possible values of $\ell$, find a subsequence of $(x_n)$ converging to $\ell$. $\boxed{\mathbf{1+1}}$

   *Solution.* Let $x_{n_k} \to \ell$. If $\ell \notin \{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\}$, then either $\ell < -\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2} < \ell < 0$ or $0 < \ell < \frac{\sqrt{3}}{2}$ or $\ell > \frac{\sqrt{3}}{2}$. Let $\ell < -\frac{\sqrt{3}}{2}$. Take $\varepsilon = \frac{|-\frac{\sqrt{3}}{2} - \ell|}{2}$. Then $x_n \notin (\ell - \varepsilon, \ell + \varepsilon)$ for any $n$. Hence, $(x_{n_k})$ does not converge to $\ell$. Using similar argument, it follows that $(x_{n_k})$ does not converge to $\ell$ if $\ell \notin \{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\}$. Hence, $\ell \in \{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\}$. [1]

   We have $x_{3n} = \sin\frac{3n\pi}{3} = \sin n\pi \to 0$.
   $x_{6n+1} = \sin\frac{(6n+1)\pi}{3} = \sin(2n\pi + \frac{\pi}{3}) \to \frac{\sqrt{3}}{2}$.
   $x_{6n-2} = \sin\frac{(6n-2)\pi}{3} = \sin(2n\pi - \frac{2\pi}{3}) \to -\frac{\sqrt{3}}{2}$. [1]
   □

2. Let $x_1 = 1$ and $x_{n+1} = \dfrac{3n}{3n+1}x_n^2, \ n \geq 1$.

   (a) Show that $(x_n)$ is bounded. $\boxed{\mathbf{1}}$

   *Solution.* We have $x_n^2 \geq 0$ for all $n$, and so $x_{n+1} = \dfrac{3n}{3n+1}x_n^2 \geq 0$ for all $n \geq 1$. Also, $x_1 = 1$. Hence $x_n \geq 0$ for all $n$. Suppose that $x_n \leq 1$. Then $x_{n+1} \leq 1$. Since $x_1 = 1$, by the principle of mathematical induction $x_n \leq 1$ for all $n$. Thus, $0 \leq x_n \leq 1$ for all $n$. □

   (b) Show that $(x_n)$ is decreasing. $\boxed{\mathbf{1}}$

   *Solution.* We have $x_n - x_{n+1} = x_n - \frac{3n}{3n+1}x_n^2 = x_n(1 - \frac{3n}{3n+1}x_n) \geq 0$ for all $n$. Hence $(x_n)$ is decreasing. □

   (c) Find $\lim x_n$. $\boxed{\mathbf{1}}$

   *Solution.* By monotone convergence theorem, $(x_n)$ is convergent. Suppose that $x_n \to \ell$. Then $\ell = \ell^2$. Hence, $\ell = 0$ or $\ell = 1$. We have $x_2 = \frac{3}{4}$. Since $(x_n)$ is decreasing, so $x_n \leq \frac{3}{4}$ for all $n \geq 2$. This yields $\ell \leq \frac{3}{4}$. Therefore, $\ell = 0$. □

3. Test the convergence/divergence of the infinite series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^{2019}}$. **3**

*Solution.* Here $x_n = \frac{1}{(\log n)^{2019}}$ is decreasing and positive for all $n \geq 2$. By Cauchy condensation test,

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{2019}} \text{ is convergent} \Leftrightarrow \sum_{n=2}^{\infty} 2^n \frac{1}{n^{2019}(\log 2)^{2019}} \text{ is convergent}$$

$$\Leftrightarrow \frac{1}{(\log 2)^{2019}} \sum_{n=2}^{\infty} \frac{2^n}{n^{2019}} \text{ is convergent} \qquad [1\tfrac{1}{2}]$$

Let $y_n = \dfrac{2^n}{n^{2019}}$. Then $\dfrac{y_{n+1}}{y_n} = \dfrac{2^{n+1}}{(n+1)^{2019}} \times \dfrac{n^{2019}}{2^n} = \dfrac{2}{(1+\frac{1}{n})^{2019}} \to 2.$

By Ratio test, $\displaystyle\sum_{n=1}^{\infty} y_n$ is divergent and hence $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(\log n)^{2019}}$ diverges. [1½] □