PH 102: Physics II

Lecture 6 (Post midsem, Spring 2020)

IIT Guwahati
Debasish Borah

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Division
Lec 1	05-03-2020	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 2	11-03-2020	Application of Ampere's Law, Magnetic Vector Potential	5.3, 5.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 1	12-03-2020	Lec 1		
Tut 2	17-03-2020	Lec 2		
Lec 3	18-03-2020	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic materials, magnetization	5.4, 6.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 4	19-03-2020	Field of a magnetized object, Boundary conditions	6.2, 6.3, 6.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 3	24-03-2020	Lec 3, 4		
Lec 5	25-03-2020	Ohm's law, motional emf, electromotive force	7.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 6	26-03-2020	Faraday's law, Lenz's law, Self & Mutual inductance, Energy stored in magnetic field	7.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 4	31-03-2020	Lec 5, 6		
Lec 7	01-04-2020	Maxwell's equations	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 8	02-04-2020	Discussions, Problem solving	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
	07-04-2020	Quiz II		
Lec 9	08-04-2020	Continuity equation, Poynting theorem	8.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 10	16-04-2020	Wave solution of Maxwell's equations, polarisation	9.1, 9.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 5	21-04-2020	Lec 9, 10		
Lec 11	22-04-2020	Electromagnetic waves in matter, reflection & transmission: normal incidence	9.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 12	23-04-2020	Reflection & transmission: oblique incidence	9.3, 9.4	I, II (4-4:55 pm) III, IV (11-11:55

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

				am)
Tut 6	28-4-2020	Lec 11, 12		
Lec 13	29-04-2020	Relativity and electromagnetism: Galilean & special relativity	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 14	30-04-2020	Discussions, problem solving	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)

Electromagnetic Induction

- Faraday's three experiments:

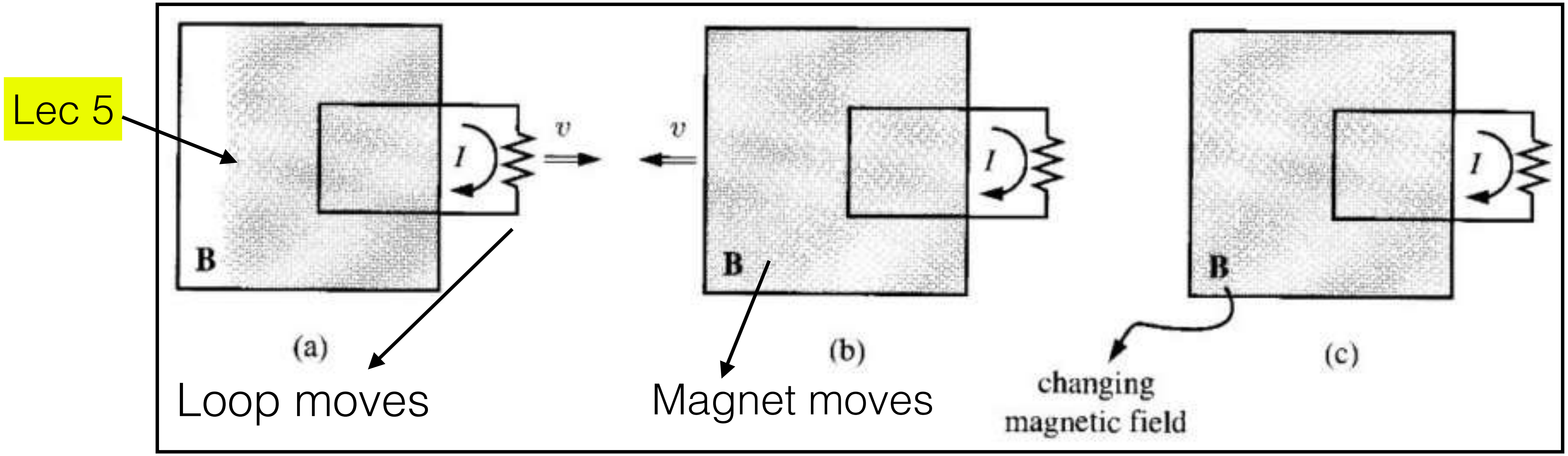


Figure 7.20, Introduction to Electrodynamics, D. J. Griffiths

- In the first experiment (a), the loop moves and hence the magnetic force on the charged particles induces the observed emf and the resulting current. What happens in experiment (b) and (c) where the loop was stationary but still emf gets induced!

Faraday's three experiments

- The first experiment is clearly an example of motional emf, that can be expressed by the flux rule discussed before:

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Lecture 5

- The way emf arises in the second experiment can be understood well in terms of relativity. A relative motion between the magnet and the loop is all that matters. (Though the theory of relativity was not yet there during Faraday's time (1831)!)

- For stationary loop (and hence stationary charged particles), the force that drives the current can not be magnetic!

$$\vec{F} = q(\vec{E} + \underset{\searrow 0}{\vec{v}} \times \vec{B})$$

- Similarly the result of the third experiment also can not be explained in terms of magnetic force!

Electromagnetic Induction: Faraday's Law

Faraday's law: ***A changing magnetic field induces an electric field*****.**

Since, the induced emf, empirically is equal to the rate of change of flux

$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

$$\Longrightarrow \oint \vec{E} \cdot d\vec{l} = -\int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a}$$

Faraday's law in integral form

Using Stokes' theorem: $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ Faraday's law in differential form

This is consistent with the results from electrostatics/magnetostatics: $\vec{\nabla} \times \vec{E} = 0$ for constant $\vec{B}$

Electromagnetic Induction

All three experiments of Faraday can be simultaneously explained under the **universal flux rule**:

"*Whenever (and for whatever reason) the magnetic flux through a loop changes, an emf will appear in the loop, given by the flux rule*"

The underlying mechanism can however, be different: in experiment (a), the emf is induced by the magnetic force acting on charged particles. In (b) and (c), it is the electric force (induced by the changing magnetic field) that is responsible for the emf.

Emf in a circuit

True in free space too. Circuit is not required. It is more general.

However,

$$\mathcal{E} = -\frac{d\Phi}{dt} \not\equiv \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Flux Rule

Faraday's Law

In Faraday's 1st experiment: the loop (circuit) moves and the resulting emf is due to magnetic force (*magnetic emf*).

In Faraday's 2nd, 3rd experiment: the induced electric field does the job.

$$f = \frac{F}{q} = \vec{E} + \vec{v} \times \vec{B}$$

Induced due to changing B

Wire moves

Coincidence?

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

See 17-2, Feynman Lectures in Physics: Exception to Flux Rule!

The fact that Faraday's law is valid in free space too, have serious implications in electro-magnetic waves: upcoming lectures!

Lenz's Law

The induced current will flow in such a direction that the flux it produces tends to cancel the change in flux: **Nature abhors a change in flux.**

A conducting loop likes to maintain a constant flux through it, it opposes any attempt to change the flux by sending a current around in such a direction as to oppose the change: a kind of inertial phenomenon.

Example 7.6 (Introduction to Electrodynamics, D. J. Griffiths):The metal ring on top jumps in the air after the current is turned on in the solenoidal coil.

Flux through the ring is zero initially.

After current is turned on, it is non-zero.

Change in flux induces an emf.

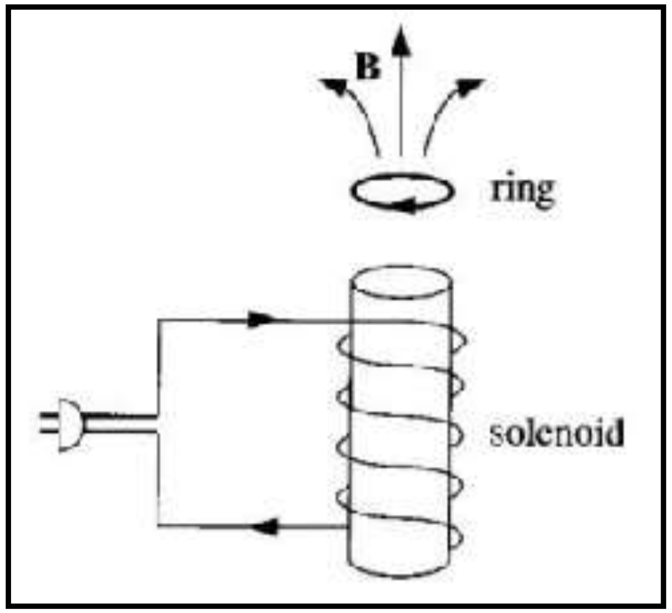


Figure 7.23, Introduction to Electrodynamics, D. J. Griffiths

Will the magnetic property of the ring play a role here?

Direction of current is opposite to that in the solenoid in accordance with Lenz's law. Since opposite currents repel, the loops jumps in the air.

Lenz’s Law

As the bar magnet approaches the loop, it experiences a repulsive force due to the induced emf (in accordance with Lenz’s law).

Since like poles repel, the loop must behave as if it were a bar magnet with its north pole pointing up.

Using the right-hand thumb rule, the direction of induced current can be found to be counterclockwise as viewed from above.

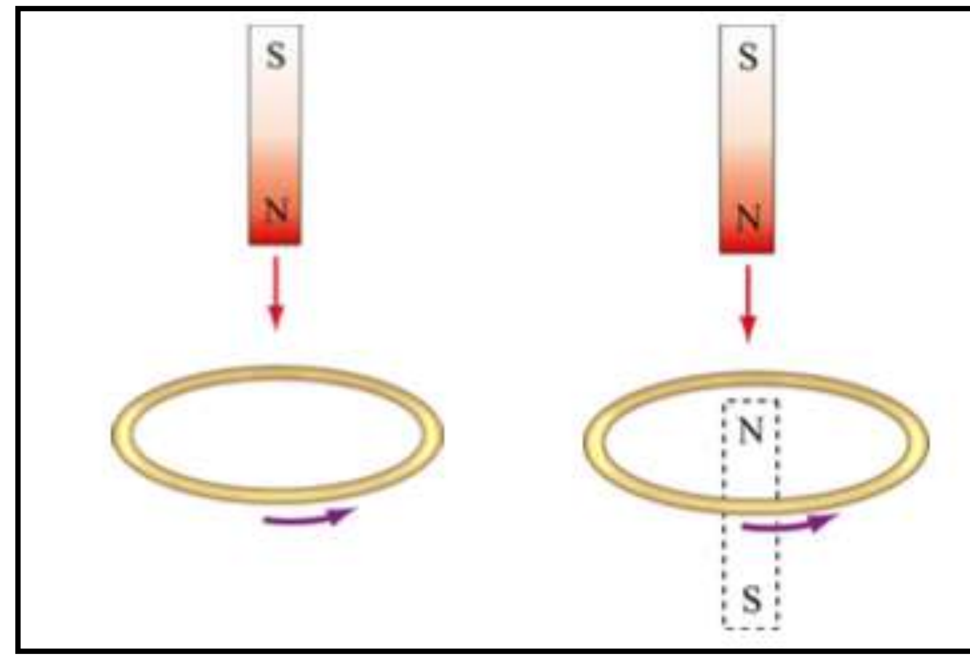
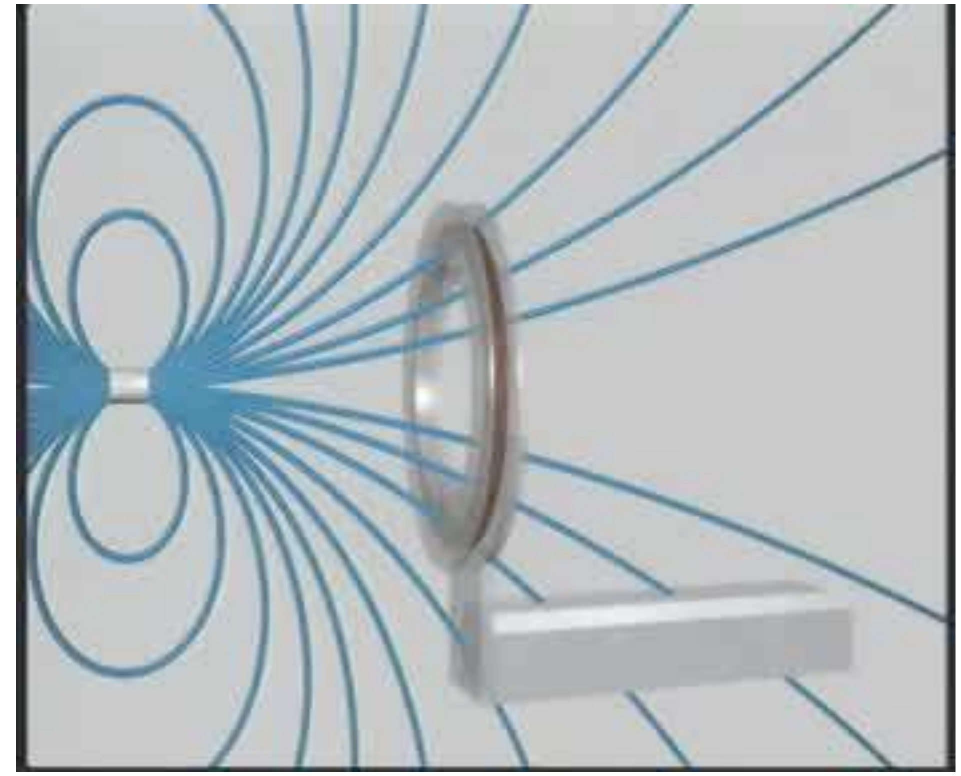
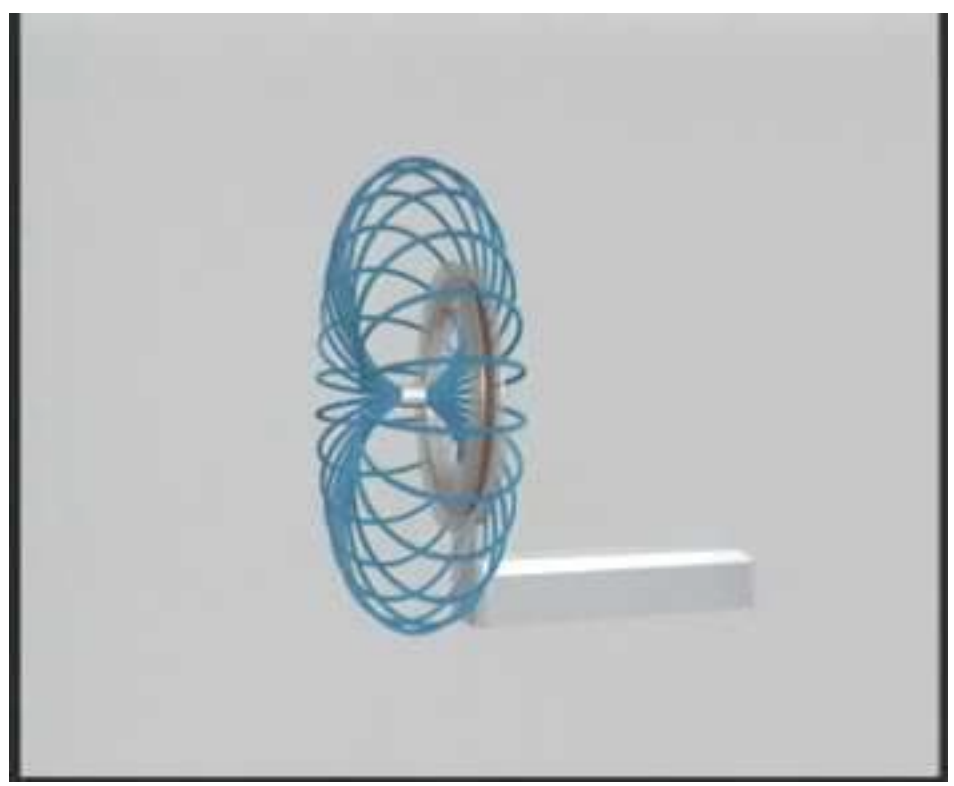


Image credit: MIT





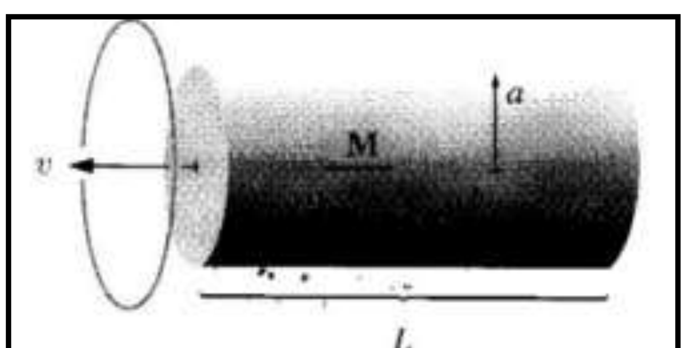
Credit: MIT

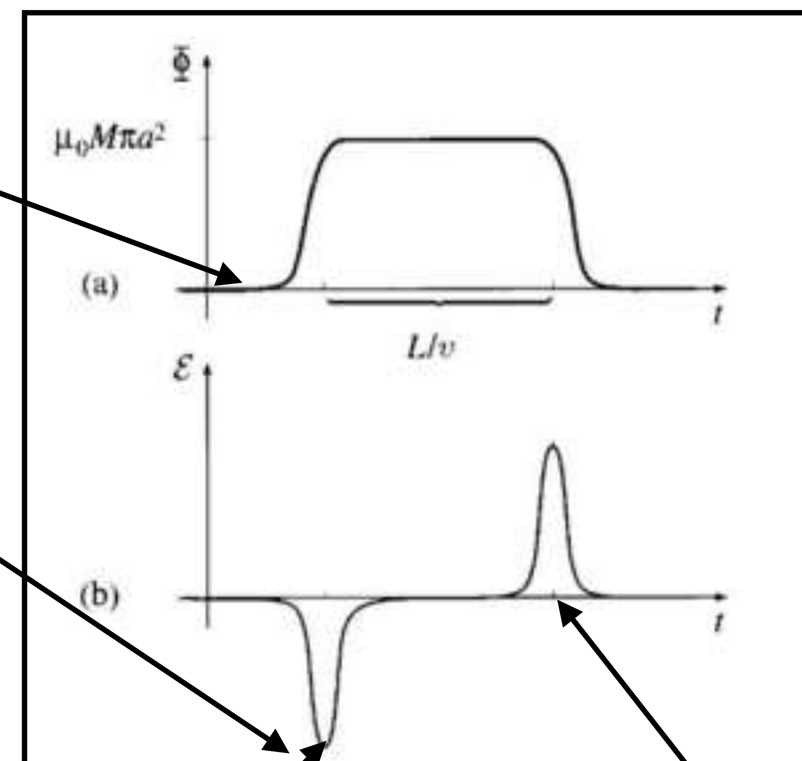
Example 7.5 (Introduction to Electrodynamics, D J Griffiths): A long cylindrical magnet of length L and radius a carries a uniform magnetisation **M** parallel to its axis. It passes at constant velocity v through a circular wire ring of slightly larger diameter. Draw the emf induced in the ring as a function of time.

Solution: The magnetic field inside is same as that due to a bound surface current $\vec{K}_b = \vec{M} \times \hat{n} = M\hat{\phi}$ that is $\vec{B} = \mu_0 \vec{M}$ except near the ends.

Flux zero when magnet is far away

Emf is -ve derivative of flux and hence consists of two spikes





Flux increasing

Flux decreasing

Figure 7.21, 7.22, Introduction to Electrodynamics, D J Griffiths

The Induced Electric Field

For the induced electric field: $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}, \quad \vec{\nabla} \cdot \vec{E} = 0$

which is similar to the corresponding relations for the magnetic field:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}, \quad \vec{\nabla} \cdot \vec{B} = 0$$

Faraday-induced electric fields are therefore, determined by $-(\partial \vec{B}/\partial t)$ in exactly the same way as magnetostatic fields are determined by $\mu_0 \vec{J}$

For symmetric configurations, it is straightforward to use the integral form of Faraday's law in determining the induced electric field

$$\int \left(-\frac{\partial \vec{B}}{\partial t} \right) \cdot d\vec{a} = \int (\vec{\nabla} \times \vec{E}) \cdot d\vec{a} = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt}$$

Similar to the use of Ampere's law to find **B** in symmetric current configurations

where, the rate of change of magnetic flux through the Amperian loop plays the same role as that of $\mu_0 I_{\text{enc}}$ in Ampere's law:

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}}$$

Example 7.8 (Introduction to Electrodynamics, D. J. Griffiths): A line charge λ is glued onto the rim of a wheel of radius b, which is then suspended horizontally, so that it is free to rotate. In the central region, out to radius a, there is a uniform magnetic field $\vec{B}_0$, pointing up. What happens if the field is turned off?

Solution: According the Lenz's law, the wheel will rotate in a direction (counterclockwise) such that the resulting magnetic field tends to restore the upward magnetic flux.

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}$$

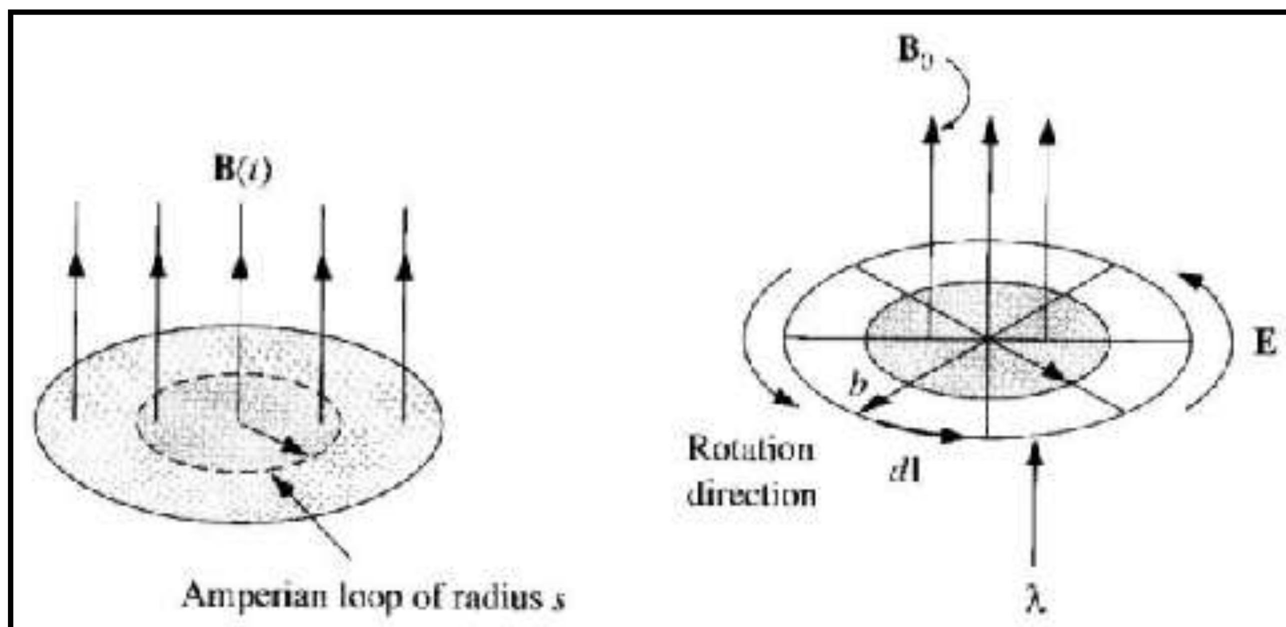


Figure 7.24, 7.25, Introduction to Electrodynamics, D. J. Griffiths

The torque on the wheel:

$$dN = b\lambda dl E$$

$$\Longrightarrow \quad N = b\lambda \oint E dl = -b\lambda\pi a^2 \frac{dB}{dt}$$

Angular momentum imparted:

$$\int N dt = -\lambda \pi a^2 b \int_{B_0}^{0} dB = \lambda \pi a^2 b B_0$$

The torque N is clearly upward (by right hand thumb rule) and positive (as dB/dt is negative)

Applications of Faraday's Law: Generators

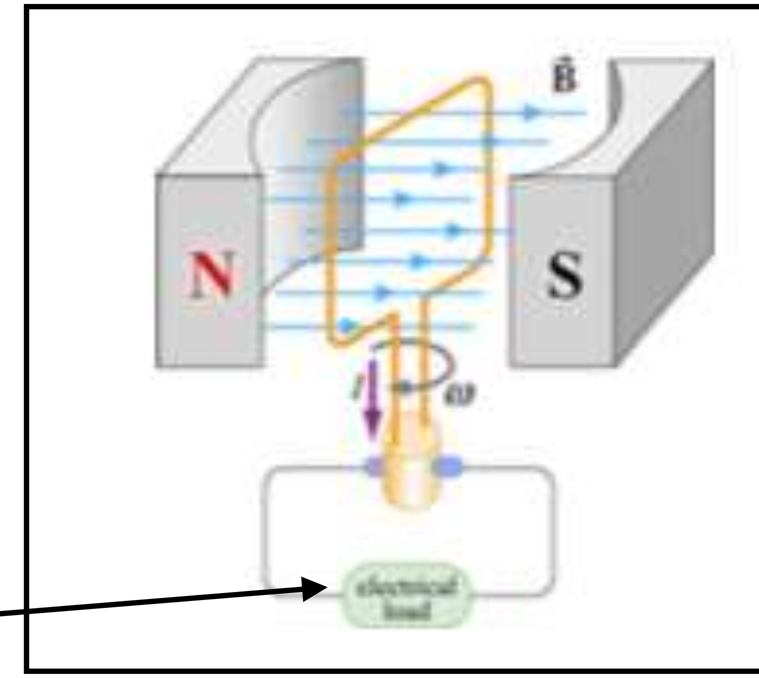
Conversion of mechanical energy to electrical energy:

$$\Phi = \int \vec{B} \cdot d\vec{a} = BA\cos\theta = BA\cos\omega t$$

$$\mathcal{E} = -N\frac{d\Phi}{dt} = NBA\omega\sin\omega t$$

$$I = \frac{\mathcal{E}}{R} = \frac{NBA\omega}{R}\sin\omega t$$

$$P = I\mathcal{E} = \frac{(NBA\omega)^2}{R}\sin^2\omega t$$



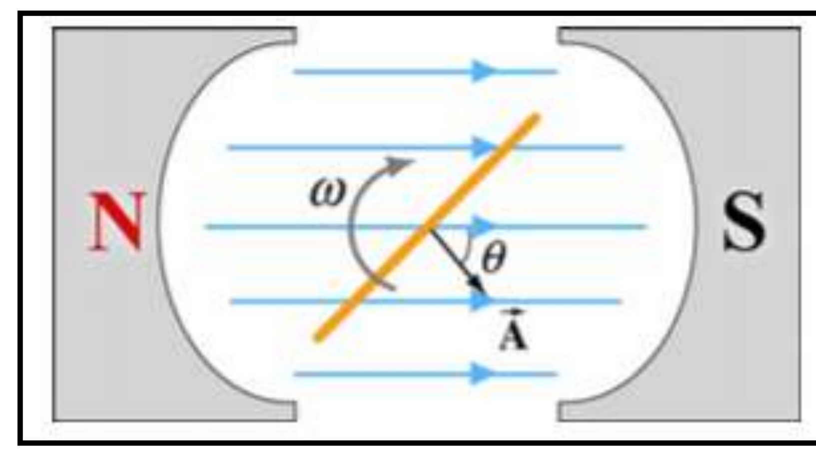
Mechanical energy required to rotate the loop:

$$\tau = mB\sin\theta = mB\sin\omega t$$

$$m = NIA = \frac{N^2A^2B\omega}{R}\sin\omega t$$

$$P_{\text{mech}} = \tau\omega = mB\omega\sin\omega t$$

$$\implies P_{\text{mech}} = \frac{(NAB\omega)^2}{R}\sin^2\omega t = P_{\text{elec}}$$



Induced current in the loop will be in a direction that opposes the rotation giving a torque $\vec{m} \times \vec{B}$ opposite to the rotation (Lenz's law). Hence the need of mechanical energy.

Image credit: MIT

Mutual Inductance

According to the Biot-Savart law, the field due to loop 1 is:

$$\vec{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\vec{l_1} \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

The magnetic flux through the loop 2:

$$\boxed{\Phi_2 = \int \vec{B}_1 \cdot d\vec{a_2} = M_{21} I_1}$$

The field and the flux are proportional to the current. The constant of proportionality in the second relation M_{21} is known as **mutual inductance** of the two loops.

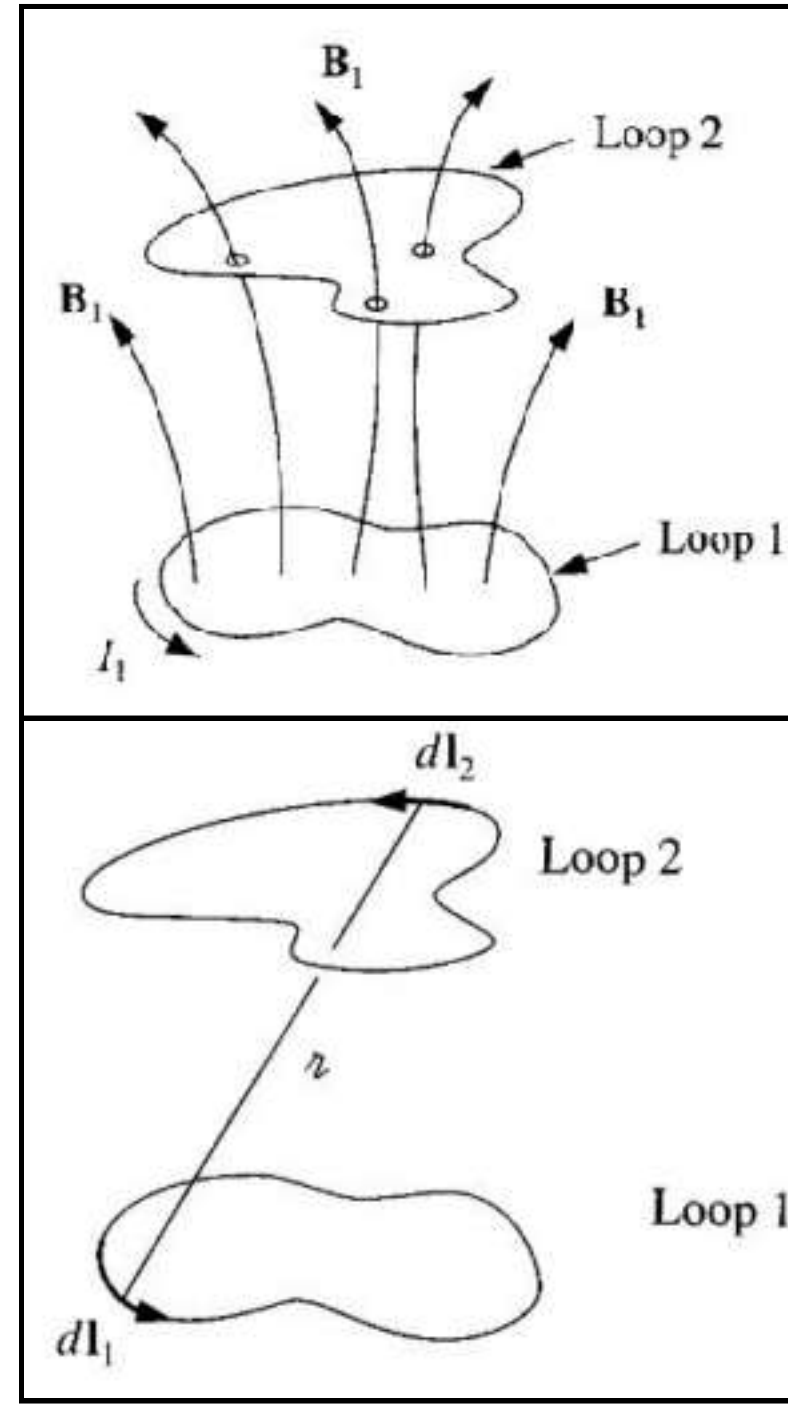


Figure 7.29, 7.30, Introduction to Electrodynamics, D. J. Griffiths

Mutual Inductance

Using Stoke's theorem in the definition of flux:

$$\Phi_2 = \int \vec{B_1} \cdot d\vec{a_2} = \int (\vec{\nabla} \times \vec{A_1}) \cdot d\vec{a_2} = \oint \vec{A_1} \cdot d\vec{l_2}$$

Now, the magnetic vector potential is given by (assuming it to be divergence-less and **J** going to zero at infinity):

$$\vec{A_1} = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\vec{l_1}}{\mathfrak{r}}$$

Lecture 2

Using this in the definition of magnetic flux above:

$$\Phi_2 = \frac{\mu_0 I_1}{4\pi} \oint \oint \frac{d\vec{l_1} \cdot d\vec{l_2}}{\mathfrak{r}} = M_{21} I_1$$

$$\Longrightarrow \boxed{M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\vec{l_1} \cdot d\vec{l_2}}{\mathfrak{r}}}$$

Neumann formula

Mutual Inductance

Neumann formula implies:

1. The mutual inductance M_{21} **depends purely on the geometry** (size, shape, relative position of the loops) of the configuration.

2. The formula for M_{21} remains unchanged if the loop indices are interchanged which implies $M_{21} = M_{12}$

$$\left(d\vec{l_1} \cdot d\vec{l_2} = d\vec{l_2} \cdot d\vec{l_1} \right)$$

Whatever the shapes and positions of the loops, the flux through the loop 2 when a current I flows through loop 1 is identical to the flux through loop 1 when the same current I flows through loop 2.

If the current in loop 1 is varied, there will be an induced emf in loop 2 given by

$$\boxed{\mathcal{E} = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}}$$

And vice versa!

Example 7.10 (Introduction to Electrodynamics, D J Griffiths): A short solenoid (length l and radius a with n_1 number of turns per unit length) lies on the axis of a very long solenoid (radius b, n_2 turns per unit length). Current I flows in the short solenoid. What is the flux through the long solenoid?

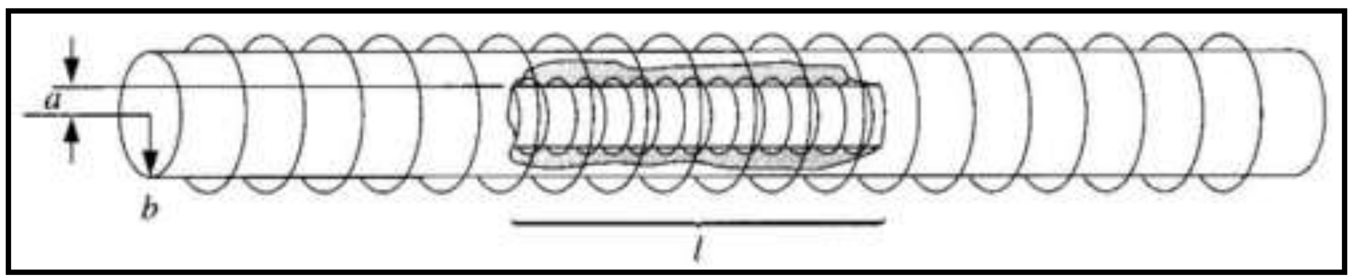


Figure 7.31, Introduction to Electrodynamics, D. J. Griffiths

Field for the short solenoid is non-uniform and complicated

Consider I to be in longer solenoid and use the equality of mutual inductance!

$$B_{\text{long}} = \mu_0 n_2 I$$

$$(\Phi_{\text{short}})_1 = B_{\text{long}} \pi a^2 = \mu_0 n_2 I \pi a^2$$

Flux through one turn

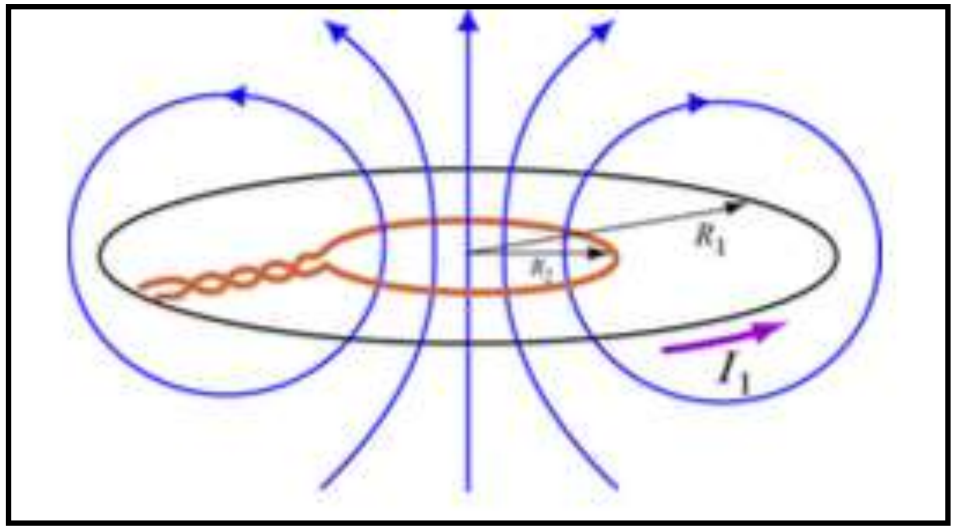
$$\Phi_{\text{short}} = \mu_0 \pi a^2 n_1 n_2 l I = MI$$

Same as the flux a current *I* in the short solenoid will put through the long one

$$\implies M = \mu_0 \pi a^2 n_1 n_2 l$$

Mutual Inductance of Two Concentric Coplanar Loops:

$$B_1 = \frac{\mu_0 I_1}{2R_1}$$



$$\Phi_{21} = B_1 A_2 = \frac{\mu_0 I_1}{2R_1}(\pi R_2^2) = \frac{\mu_0 \pi I_1 R_2^2}{2R_1}$$

$$M = \frac{\Phi_{21}}{I_1} = \frac{\mu_0 \pi R_2^2}{2R_1}$$

The area of the inner loop is taken to be very small so that the field due to outer loop remains same throughout.

Check the results for sizeable area of inner loop.

Image credit: MIT

Self Inductance

Changing current not only induces an emf in a loop nearby, it also induces an emf in the source loop itself. The flux through the loop is again proportional to the field and hence the current: $\Phi = LI$

The constant of proportionality L is called the **self-inductance** (or simply the inductance).

Similar to mutual inductance M, self-inductance **L also depends upon the geometry of the loop**.

For a change in current, the induced emf in the loop is

$$\boxed{\mathcal{E} = -L\frac{dI}{dt}}$$

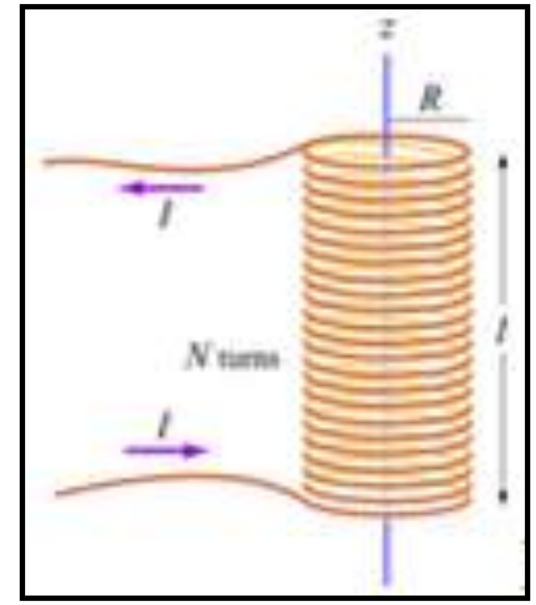
Unit of inductance is Henry (H). 1 H= 1 volt-second/ampere

Self inductance of solenoid and toroid:

$$\vec{B} = \frac{\mu_0 N I}{l}\hat{z} = \mu_0 n I \hat{z}$$

$$\Phi = BA = \mu_0 n I(\pi R^2) = \mu_0 n I \pi R^2$$

$$L_{\text{solenoid}} = \frac{N\Phi}{I} = \mu_0^2 n^2 \pi R^2 l$$



$$B = \frac{\mu_0 N I}{2\pi r}$$

$$\Phi = \int \vec{B}\cdot d\vec{a} = \int_a^b \left(\frac{\mu_0 N I}{2\pi r}\right) h dr$$

$$\implies \Phi = \frac{\mu_0 N I h}{2\pi} \ln\left(\frac{b}{a}\right)$$

$$L_{\text{toroid}} = \frac{N\Phi}{I} = \frac{\mu_0 N^2 h}{2\pi} \ln\left(\frac{b}{a}\right)$$

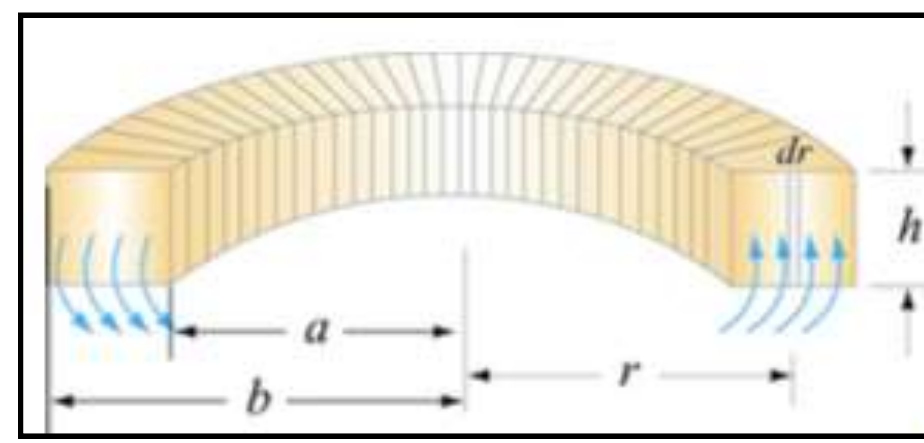


Image credit: MIT

Transformer

It is a device for raising or lowering the emf of an alternating current source.

If Φ be the flux through a single loop of either coil, so that

$$\Phi_1 = N_1\Phi, \Phi_2 = N_2\Phi$$

$$\mathcal{E}_1 = -N_1\frac{d\Phi}{dt}, \mathcal{E}_2 = -N_2\frac{d\Phi}{dt}$$

$$\implies \frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{N_2}{N_1}$$

Choosing appropriate no. of turns, any desired secondary emf can be generated.

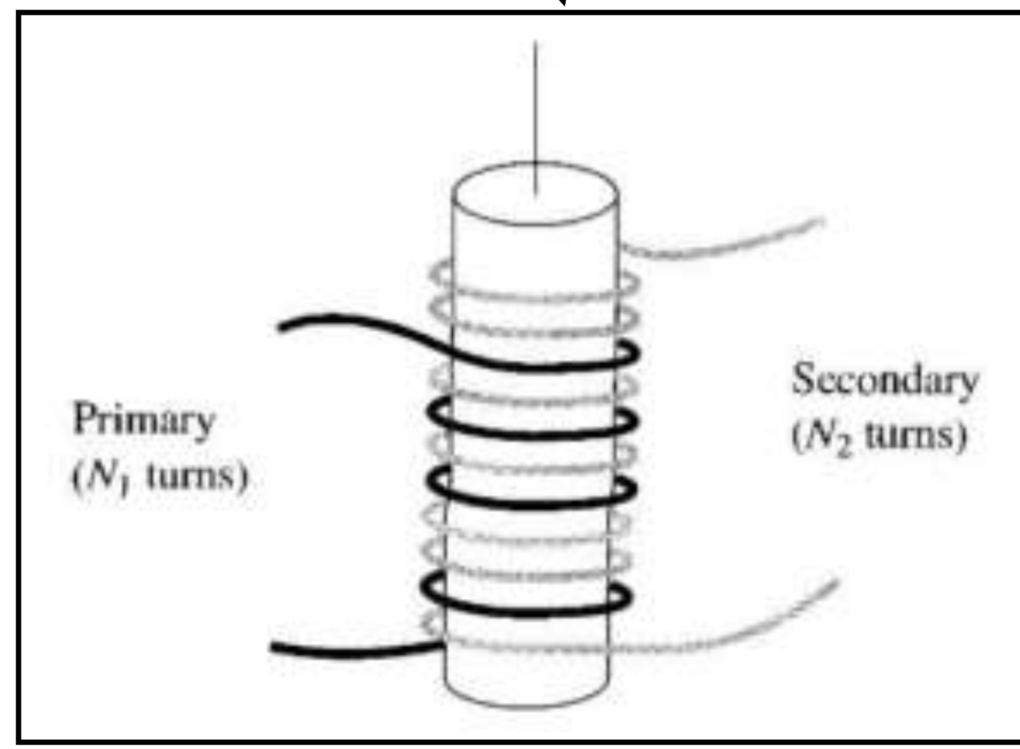


Figure 7.54, Introduction to Electrodynamics, D. J. Griffiths

Check if it is consistent with the conservation of energy principle!

Energy stored in Magnetic Field

The induced emf always opposes the change in current: one has to work against this emf (**back emf**!) to get the current going.

Evident from the Examples/Exercises discussed earlier

The work done per unit charge, against the back emf, in one trip around the circuit is $-\mathcal{E}$. Since the current *I* is equivalent to the amount of charge per unit time passing through the wire, the total work done per unit time is

$$\frac{dW}{dt} = -\mathcal{E}I = LI\frac{dI}{dt}$$

Net work done in building current from 0 to *I* is

$$W = \int_0^I LI dI = \boxed{\frac{1}{2}LI^2}$$

Energy stored in Magnetic Field

For a surface S bounded by curve C, the flux is:

$$\Phi = \int_S \vec{B} \cdot d\vec{a} = \int_S (\vec{\nabla} \times \vec{A}) \cdot d\vec{a} = \oint_C \vec{A} \cdot d\vec{l}$$

Therefore, $$W = \frac{1}{2} I(LI) = \frac{1}{2} I\Phi = \frac{1}{2} I \oint \vec{A} \cdot d\vec{l} = \frac{1}{2} \oint (\vec{A} \cdot \vec{I}) dl$$

In terms of volume current, $$W = \frac{1}{2} \oint (\vec{A} \cdot \vec{J}) d\tau$$

Using Ampere's law, $$W = \frac{1}{2\mu_0} \oint \vec{A} \cdot (\vec{\nabla} \times \vec{B}) d\tau$$

Using the product rule, $$\vec{\nabla} \cdot (\vec{A} \times \vec{B}) = \vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B})$$

$$\implies \vec{A} \cdot (\vec{\nabla} \times \vec{B}) = \vec{B} \cdot \vec{B} - \vec{\nabla} \cdot (\vec{A} \times \vec{B})$$

We can write the energy,

$$W = \frac{1}{2\mu_0} \left[\int_V B^2 d\tau - \int \vec{\nabla} \cdot (\vec{A} \times \vec{B}) d\tau \right]$$

$$\implies W = \frac{1}{2\mu_0} \left[\int_V B^2 d\tau - \oint_S (\vec{A} \times \vec{B}) \cdot d\vec{a} \right]$$

Using Gauss's law

V is the volume enclosed by surface S

Energy stored in Magnetic Field

If the surface S is chosen to be far away from the current distribution, so that J reduces to zero out there, the surface integral in the expression for W can be ignored*. Thus, for integration over all space,

$$\boxed{W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau}$$

Energy stored in magnetic field therefore, is $(B^2/2\mu_0)$ per unit volume just like the energy stored per unit volume in electric field is $(\epsilon_0 E^2/2)$. To summarise,

$$\boxed{\begin{aligned} W_{\text{electric}} &= \frac{1}{2}\int (V\rho)d\tau = \frac{\epsilon_0}{2}\int E^2 d\tau \\ W_{\text{magnetic}} &= \frac{1}{2}\int (\vec{A}\cdot\vec{J})d\tau = \frac{1}{2\mu_0}\int B^2 d\tau \end{aligned}}$$

*More details about surface term later!

Example 7.13 (Introduction to Electrodynamics, D J Griffiths): A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a, and back along the outer cylinder, radius b). Find the magnetic energy stored in a section of length l.

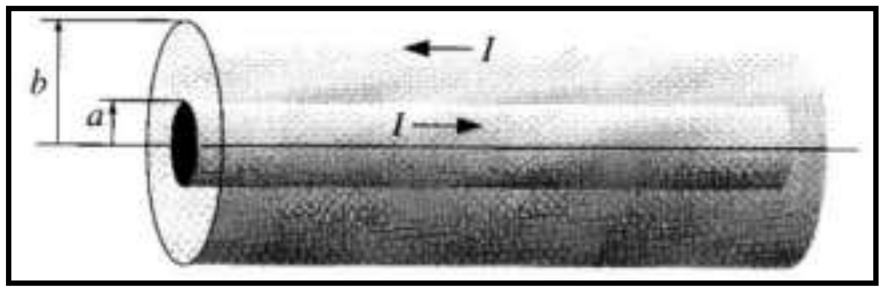


Figure 7.39, Introduction to Electrodynamics, D. J. Griffiths

Solution: Magnetic field in between the cylinders is: $\vec{B} = \frac{\mu_0 I}{2\pi r}\hat{\phi}$

Energy per unit volume: $\frac{1}{2\mu_0}B^2 = \frac{\mu_0 I^2}{8\pi^2 r^2}$

Energy in a cylindrical shell of length l, radius r, thickness dr:

$$\frac{\mu_0 I^2}{8\pi^2 r^2}(2\pi r l dr) = \frac{\mu_0 I^2 l}{4\pi}\frac{dr}{r}$$

Total energy: $W = \frac{\mu_0 I^2 l}{4\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 I^2 l}{4\pi}\ln\left(\frac{b}{a}\right)$

$$W = \frac{1}{2}LI^2 \implies L = \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right)$$

Self-inductance

Some Remarks

Discussion of energy stored in magnetic field was postponed to this lecture, as the concept of work done was not clear in magnetostatics.

Although magnetic force does not do any work on charged particles, while creating a magnetic field (from 0 to a finite value), an electric field is induced that can do the work!

Faraday's law, therefore, was necessary to calculate the energy stored in the magnetic field!

More details about work done, conservation of energy in upcoming lectures!