PH101

Lecture 13

Hamiltonian
Hamilton's Equation of motion
Guide Lines for Mid Sem exam

Hamiltonian

Hamiltonian: A new function corresponding to a dynamical system, which is function of generalized coordinate, generalized momentum and time

$$\boldsymbol{H(p_j, q_j, t)}$$

Newtonian → Lagrangian → Hamiltonian

- ❑ Physics is not different, it is another way of looking the system
- ❑ Main difference is the view point
  - Symmetries and invariance more apparent
- ❑ Hamiltonian formalism is inevitable for
  - Hamilton-Jacobi theory
  - Quantum mechanics
  - Quantum Statistical mechanics

A quick review of previous class

- $L = L(q_1, \dots q_n, \dot{q}_1, \dots . \dot{q}_n, t)$

- Using the chain rule of partial differentiation

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \dot{q}_j + \frac{\partial L}{\partial t}$$

- Using Lagrange's eqn.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j\right) + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L\right) + \frac{\partial L}{\partial t} = 0$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

$$\frac{d}{dt}\left(\sum_j p_j \dot{q}_j - L\right) + \frac{\partial L}{\partial t} = 0$$

Hamiltonian $H(q_j, \dot{q}_j, t)$

$$\frac{d}{dt}\left(\sum_j p_j \dot{q}_j - L\right) + \frac{\partial L}{\partial t} = 0$$

- Can introduce new function

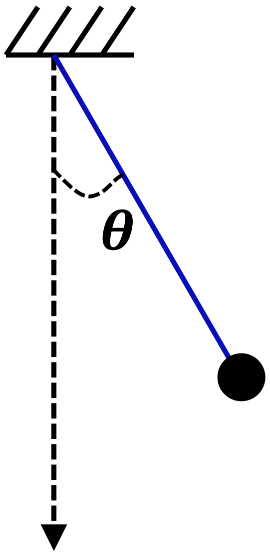
$$h(q_j, p_j, \dot{q}_j, t) = \sum_j p_j \dot{q}_j - L(q_j, \dot{q}_j, t)$$

- If $\dot{q}_j$ is substituted with p_j using their relation obtained from $\boldsymbol{p_j} = \frac{\partial \boldsymbol{L}}{\partial \dot{\boldsymbol{q}}_j}$, then the function is known as **Hamiltonian**

$$H(q_j, p_j, t) = \sum_j p_j \dot{q}_j - L$$

$\boldsymbol{h(q_j, p_j, \dot{q}_j, t)}$ → **Substitute $\dot{q}_j$ with p_j** → $\boldsymbol{H(q_j, p_j, t)}$

Hamiltonian Example 1: Simple Pendulum



Step 1: Find Lagrangian of the system

$$L = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl\ cos\theta$$

Step 2: Find generalized momentum (p_j) using $p_j = \frac{\partial L}{\partial \dot{q}_j}$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} == \frac{\partial}{\partial \dot{\theta}} \left(\frac{1}{2} m l^2 \dot{\theta}^2 + mgl\ cos\theta \right) = m l^2 \dot{\theta}$$

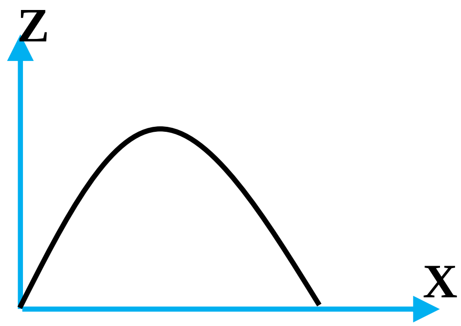
Step 3: Find the function $h(q_j, p_j, \dot{q}_j, t\ ) = \sum p_j \dot{q}_j - L$

$$h = p_\theta \dot{\theta} - \frac{1}{2} m l^2 \dot{\theta}^2 - mgl\ cos\theta$$

Step 4: Find Hamiltonian $H(q_j, p_j, t)$ from h by replacing $\dot{q}_j$ with p_j using step-2

$$H(q_j, p_j, t) = p_\theta \frac{p_\theta}{ml^2} - \frac{1}{2} m l^2 \left(\frac{p_\theta}{ml^2} \right)^2 - mgl\ cos\theta = \frac{{p_\theta}^2}{2ml^2} - mgl\ cos\theta$$

Hamiltonian Example 2: Projectile



Step 1: Find Lagrangian of the system

$$L = \frac{1}{2} m(\dot{x}^2 + \dot{z}^2) - mgz$$

Step 2: Find generalized momentum (p_j) using $p_j = \frac{\partial L}{\partial \dot{q}_j}$

$$p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} \quad ; \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z}$$

Step 3: Find the function $h(q_j, p_j, \dot{q}_j, t) = \sum p_j \dot{q}_j - L$

$$h(x, z, \dot{x}, \dot{z}, t) = p_x \dot{x} + p_z \dot{z} - \frac{1}{2} m(\dot{x}^2 + \dot{z}^2) + mgz$$

Step 4: Find Hamiltonian $H(q_j, p_j, t)$ from h by replacing $\dot{q}_j$ with p_j using step-2

$$H(x, z, p_x, p_z, t) = p_x \frac{p_x}{m} + p_z \frac{p_z}{m} - \frac{1}{2} m \left(\frac{{p_x}^2}{m^2} + \frac{{p_z}^2}{m^2} \right) + mgz$$

$$= \frac{{p_x}^2}{2m} + \frac{{p_z}^2}{2m} + mgz$$

Hamilton's equations

- ❑ At ant instant t, state of a dynamical system is can be described by a function Lagrangian $L\left(q_j, \dot{q}_j, t\right)$

The system evolve in time following Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

- ❑ At ant instant t, state of a dynamical system is can be described by a function Hamiltonian $H\left(p_j, q_j, t\right)$

The system evolve in time following Hamilton's equation

Hamilton's equation?

Hamilton’s equations

$$\boldsymbol{H(q_j,p_j,t)} = \sum_j p_j\dot{q}_j - L$$

$$\boldsymbol{dH(q_j,p_j,t)} = d\left[\sum_j p_j\dot{q}_j - L\right]$$

$$\boldsymbol{L.H.S = \; dH(q_j,p_j,t)}$$

$$\boldsymbol{= \sum_j \frac{\partial H}{\partial q_j} dq_j + \sum_j \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt}$$

$$R.H.S = d\left[\sum_j p_j\dot{q}_j - L\right] = d\sum_j p_j\dot{q}_j - dL(q_j,\dot{q}_j,t)$$

$$= \sum_j (\dot{q}_j dp_j + p_j d\dot{q}_j) - \left[\sum_j \left(\frac{\partial L}{\partial q_j} dq_j + \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j\right) + \frac{\partial L}{\partial t} dt\right]$$

Hamilton's equations

$$dH(q_j, p_j, t) = d\left[\sum_j p_j \dot{q}_j - L\right]$$

$$\sum_j \frac{\partial H}{\partial q_j} dq_j + \sum_j \frac{\partial H}{\partial p_j} dp_j + \frac{\partial H}{\partial t} dt$$

$$= \sum_j (\dot{q}_j dp_j + p_j d\dot{q}_j) - \sum_j \frac{\partial L}{\partial q_j} dq_j - \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j - \frac{\partial L}{\partial t} dt$$

From EL eqn.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$$

$$\frac{d}{dt}(p_j) = \frac{\partial L}{\partial q_j}$$

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

$$= \sum_j (\dot{q}_j dp_j + \cancel{p_j d\dot{q}_j}) - \sum_j \dot{p}_j dq_j - \sum_j \cancel{p_j d\dot{q}_j} - \frac{\partial L}{\partial t} dt$$

$$= \sum_j \dot{q}_j dp_j - \sum_j \dot{p}_j dq_j - \frac{\partial L}{\partial t} dt$$

This relationship true for any arbitrary values of dp_j, dq_j and dt

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}; \quad \dot{q}_j = \frac{\partial H}{\partial p_j} \quad and \; \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Hamilton's equations

Few facts about Hamilton's equations

Hamilton's equations $\Rightarrow$ $\dot{p}_j = -\frac{\partial H}{\partial q_j}$; $\dot{q}_j = \frac{\partial H}{\partial p_j}$

- ❑ Hamilton equations are first order differential equations

❑ $j \to 1 \ldots . n$ for a system of n −degree of freedom.
Thus there are **2n number of first order Hamilton's equations**
(n number for $\dot{p}_j$ and another n number for $\dot{q}_j$)

- ❑ **A comparison with Lagrangian**: Lagrange's equations are second order differential equations and the number of equations is n (no. of degrees of freedom)

- ❑ There is nothing new. Just have rearranged the equations to give momentum much importance than generalized velocity.

- ❑ **Hamiltonian concept**: Extremely important for quantum mechanics, quantum statistical mechanics.

Conservation of energy from Hamiltonian

$$\mathrm{H} = H(q_j, p_j, t)$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_j}\frac{dq_j}{dt} + \frac{\partial H}{\partial p_j}\frac{dp_j}{dt} + \frac{\partial H}{\partial t} = -\dot{p}_j\dot{q}_j + \dot{q}_j\dot{p}_j + \frac{\partial H}{\partial t}$$

$$\boldsymbol{\frac{dH}{dt} = \frac{\partial H}{\partial t}}$$

Using Hamilton's equations;

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}; \quad \dot{q}_j = \frac{\partial H}{\partial p_j}$$

- If Lagrangian does not explicitly contain time, then Hamiltonian must not have explicit time dependence, as

$$\frac{\partial L}{\partial t} = -\frac{\partial H}{\partial t} = 0 = \frac{dH}{dt}; \qquad H = constant \text{ of motion}$$

- *Remember, if potential is velocity independent*

$$\sum_j p_j\dot{q}_j = \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j = \sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = 2T$$

$$Then\ , H = \sum_j p_j\dot{q}_j - L = 2T - (T - V) = T + V = E$$

- If H *does not have explicite time dependence* $(\frac{\partial H}{\partial t} = 0)$ and $potential\ is\ velocity\ independent, then\ H = E = const$

Cyclic coordinate in Hamiltonian

- ❑ If a particular coordinate does not appear in L(but its time derivative appear), the coordinate in known as **cyclic coordinate**.

- ❑ If q_j is cyclic in L, it will also be cyclic in H

$$H(\cancel{q_j}, p_j, t) = p_j \dot{q}_j - L(\cancel{q_j}, \dot{q}_j, t)$$

- ❑ Generalized momentum corresponding to cyclic coordinate ($\frac{\partial H}{\partial q_j} = 0$) is conserved as

$$\dot{p}_j = -\frac{\partial H}{\partial q_j}$$

If $\frac{\partial H}{\partial q_j} = 0; then\ p_j = constant$

The end

Mid-Sem: Question-cum-Answer Booklet (Front Sheet)

PHYSICS-I

Department of Physics, IIT Guwahati.

Course No: PH 101 Mid-Semester Examination

Date: 19 Sept, 2019

Time 2-4 pm

Total Marks: 30

General Instructions

a) Make sure that there are **seven sheets** (including this) in this **Question-cum-Answer** Booklet.

b) Write your **Name** and **Roll Numbers** on **every sheet** in the space provided.

c) You must write the answers **ONLY IN THE SPACE PROVIDED for the given question**. Answers written **elsewhere WILL NOT be evaluated**!

d) **NO** extra **answer-sheets** will be provided!

e) Supplementary sheets provided are **ONLY** for rough work.

f) It is advised that you first solve the problems on the supplementary sheet, and then copy the key steps in the space provided for that problem in **this** Question-cum-Answer **booklet**.

g) Be **legible**! Also, make sure that your answers are systematic, logically as well as mathematically connected.

Student's Name :	
Roll No:	
Signature:	

Signature of the Invigilator:	

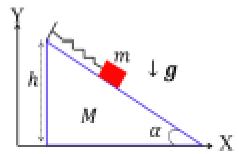
Read The Instructions Carefully

Write your name and Roll No ON EVERY sheet!

Mid-Sem: Question-cum-Answer Booklet (Q1: 2^{nd} Sheet)

Name:		Roll No:	

Q1	A wedge of mass, M, having a height, h, and angle, α, is free to move along the x-axis. Another mass, m, attached to the wedge through a light spring (of spring-constant, k, and unladen length, l) can oscillates on the slanting surface of the wedge (see figure on right). Obtain the Euler-Lagrange(E-L) equations for the system. Also, using the E-L equations obtain the frequency of oscillation of the mass, m.



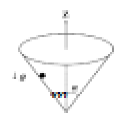
Write your name and Roll No ON EVERY sheet!

Write your answer ONLY in the Space/Page/Sheet provided.

Answers written elsewhere Will NOT be EVALUATED!

Mid-Sem: Question-cum-Answer Booklet (Q2: 3rd Sheet)

Name:		Roll No:	

Q2	A point particle of mass m is constrained to move on the inner surface of a *fixed*, open-cone of *half-angle*, α (see figure on right). Obtain the Euler-Lagrange equations for the general motion of the mass, m, under the given conditions. Mention any conserved quantity (other than the total energy).	

Answers written elsewhere Will NOT be EVALUATED!