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4. Consider the one-dimensional normalised wave functions $\psi_0(x)$ and $\psi_1(x)$ with the properties $\psi_0(-x) = \psi_0(x) = \psi_0^*(x)$ and $\psi_1(x) = N\dfrac{d\psi_0(x)}{dx}$. Consider another wave function $\psi(x) = c_1\psi_0(x) + c_2\psi_1(x)$, with $|c_1|^2 + |c_2|^2 = 1$. N, c_1 and c_2 are known constants. [2+6+4]

(a) Check if $\psi_0(x)$ and $\psi_1(x)$ are orthogonal to each other and $\psi(x)$ is normalised.

(b) Compute the expectation values of $\hat{x}$ and $\hat{p}$ in the states ψ_0 and ψ_1 separately.

(c) Compute the expectation value of $\hat{T}$(the kinetic energy operator) in the state ψ_0.

a) as $\psi_0(x)$ is even function, $\therefore \dfrac{d\psi_0}{dx}$ is odd.

$\therefore \psi_1(-x) = -\psi_1(x)$ — (1) $\Rightarrow \psi_1^2(x)$ is even.

$\psi_1^*(x) = N\dfrac{d\psi_0^*(x)}{dx} = N\dfrac{d\psi_0}{dx} = \psi_1(x)$ — (2)

Now, $\psi_0(x)$ and $\psi_1(x)$ are normalized

$\therefore \psi_0(\infty) = \psi_0(-\infty) = \psi_1(\infty) = \psi_1(-\infty) = 0$ — (3)

where $\psi_0(\infty)$ denotes the value of $\psi_0(x)$ as $x \to \infty$

$$\therefore \langle\psi_0|\psi_1\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\psi_1(x)dx = \int_{-\infty}^{\infty}\psi_0(x)\psi_1(x)dx$$

$$= \int_{-\infty}^{\infty}\psi_0(x) . N\frac{d\psi_0(x)}{dx} . dx = N\int_{-\infty}^{\infty}\psi_0(x)d\psi_0(x)$$

$$= N\left[\frac{\psi_0^2(x)}{2}\right]_{-\infty}^{\infty} = \frac{N}{2}\left[\psi_0^2(\infty) - \psi_0^2(-\infty)\right]$$

$= 0$

$\therefore$ as $\langle\psi_0|\psi_1\rangle = 0 \Rightarrow$ $\psi_0(x)$ and $\psi_1(x)$ are orthogonal

$\psi^*(x) = c_1^*\psi_0^*(x) + c_2^*\psi_1^*(x) = c_1^*\psi_0(x) + c_2^*\psi_1(x)$

$$\therefore \int_{-\infty}^{\infty}\psi^*(x)\psi(x)dx = \int_{-\infty}^{\infty}|c_1|^2|\psi_0(x)|^2dx + \int_{-\infty}^{\infty}|c_2|^2|\psi_1(x)|^2dx + c_1^*c_2\langle\psi_0|\psi_1\rangle + c_1c_2^*\langle\psi_1|\psi_0\rangle$$

$= |c_1|^2 + |c_2|^2 + 0 + 0 = 1$

$\left(\int_{-\infty}^{\infty}|\psi_0(x)|^2dx = \int_{-\infty}^{\infty}|\psi_1(x)|^2dx = 1\right)$ $\therefore$ (as ψ_0 and ψ_1 are normalized)

$\therefore$ $\psi(x)$ is also normalized.

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b) i) For $\psi_0(x)$

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\, x\, \psi_0(x)\,dx = \int_{-\infty}^{\infty} x\,\psi_0^2(x)\,dx = 0 \text{ (odd integrand)}$$

$$\langle \hat{p}\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\,\hat{p}\,\psi_0(x)\,dx = \int_{-\infty}^{\infty} \psi_0(x)\left(-i\hbar \frac{d\psi_0(x)}{dx}\right)dx$$

$$= -i\hbar \int_{-\infty}^{\infty} \psi_0(x)\,d\psi_0(x) = -i\hbar\left[\frac{\psi_0^2(x)}{2}\right]_{-\infty}^{\infty} = \underline{\underline{0}}$$

$\therefore$ (from ③)

ii) For $\psi_1(x)$

$$\langle x\rangle = \int_{-\infty}^{\infty} \psi_1^*(x)\, x\, \psi_1(x)\,dx = \int_{-\infty}^{\infty} x\,\psi_1^2(x)\,dx = 0 \text{ (odd integrand)}$$

$\therefore$ (from ①)

$$\langle p\rangle = \int_{-\infty}^{\infty} \psi_1^*(x)\,\hat{p}\,\psi_1(x)\,dx = -i\hbar \int_{-\infty}^{\infty} \psi_1(x)\,d\psi_1(x)$$

$$= -i\hbar\left[\frac{\psi_1^2(x)}{2}\right]_{-\infty}^{\infty}$$

$$= \underline{\underline{0}} \quad \therefore \text{(from ③)}$$

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c) $$\langle \hat{T}\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\,\hat{T}\,\psi_0(x)\,dx$$

$$= \cancel{\int_{-\infty}^{\infty} \psi_0(x)} \int_{-\infty}^{\infty} \psi_0(x)\left(-\frac{\hbar^2}{2m}\frac{d^2\psi_0(x)}{dx^2}\right)dx$$

$$= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} \psi_0(x)\frac{d^2\psi_0(x)}{dx^2}\,dx$$

Using by parts, we get

$$\langle \hat{T}\rangle = -\frac{\hbar^2}{2m}\left(\left(\psi_0(x)\frac{d\psi_0}{dx}\right)_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{d\psi_0}{dx}\cdot\frac{d\psi_0}{dx}\cdot dx\right)$$

$$= -\frac{\hbar^2}{2m}\left(0 - \int_{-\infty}^{\infty} \frac{\psi_1(x)\cdot\psi_1(x)}{N^2}\,dx\right)$$

$$= \frac{\hbar^2}{2mN^2}\int_{-\infty}^{\infty} \psi_1^*(x)\,\psi_1(x)\,dx \quad \therefore \text{(from 3)} \quad \therefore \text{(from 2)}$$

$$\boxed{\langle \hat{T}\rangle = \frac{\hbar^2}{2mN^2}}$$ (since ψ_1 is normalised)

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