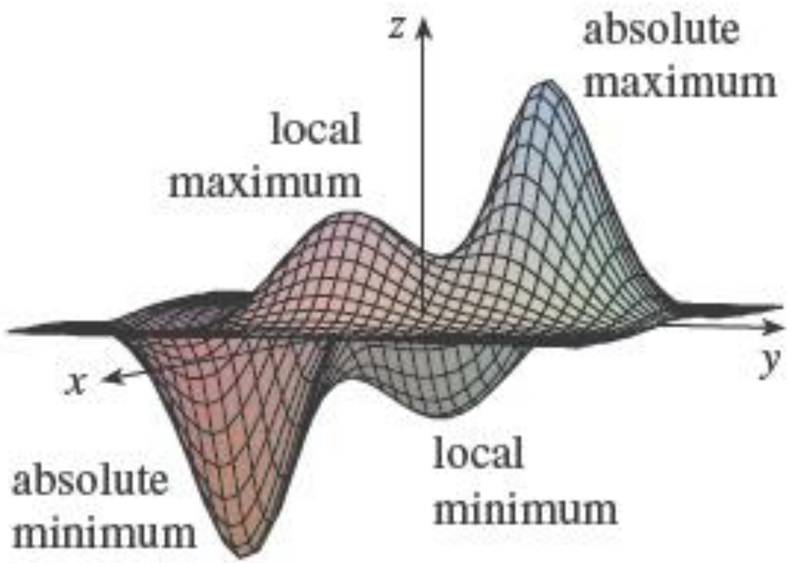
MAXIMUM AND MINIMUM VALUES

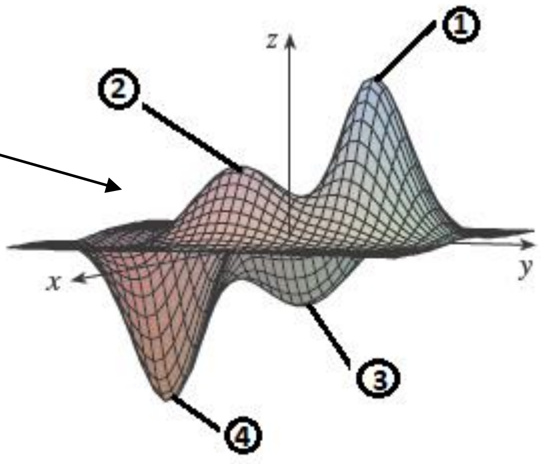
1 Definition A function of two variables has a **local maximum** at (a, b) if $f(x, y) \leqslant f(a, b)$ when (x, y) is near (a, b). [This means that $f(x, y) \leqslant f(a, b)$ for all points (x, y) in some disk with center (a, b).] The number $f(a, b)$ is called a **local maximum value**. If $f(x, y) \geqslant f(a, b)$ when (x, y) is near (a, b), then f has a **local minimum** at (a, b) and $f(a, b)$ is a **local minimum value**.



2 **Theorem** If f has a local maximum or minimum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

What is the geometric significance of it?

How would you visualize tangent planes at those marked locations?



Critical Points

A point (a, b) is called a **critical point** (or *stationary point*) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if one of these partial derivatives does not exist. Theorem 2 says that if f has a local maximum or minimum at (a, b), then (a, b) is a critical point of f.

However, as in single-variable calculus, not all critical points give rise to maxima or minima. At a critical point, a function could have a local maximum or a local minimum or neither.

Finding Critical points and extreme values

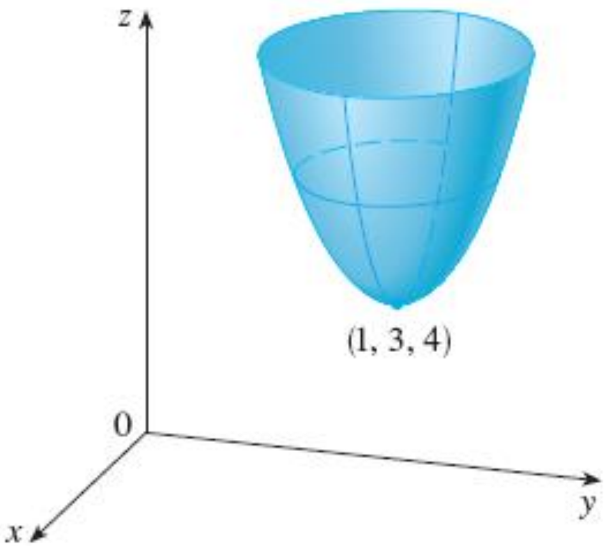


FIGURE 2
$z = x^2 + y^2 - 2x - 6y + 14$

Find the extreme values of $f(x, y) = y^2 - x^2$.

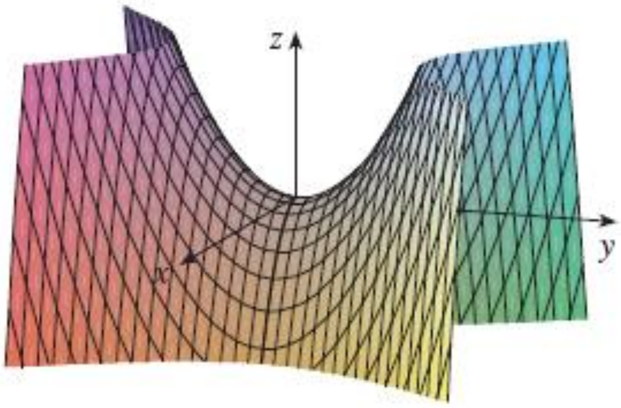
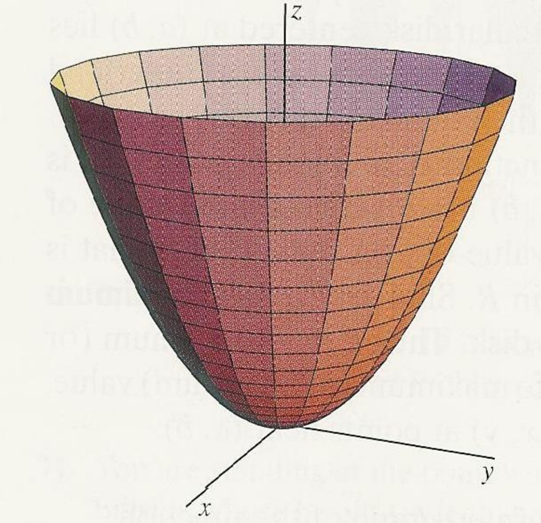
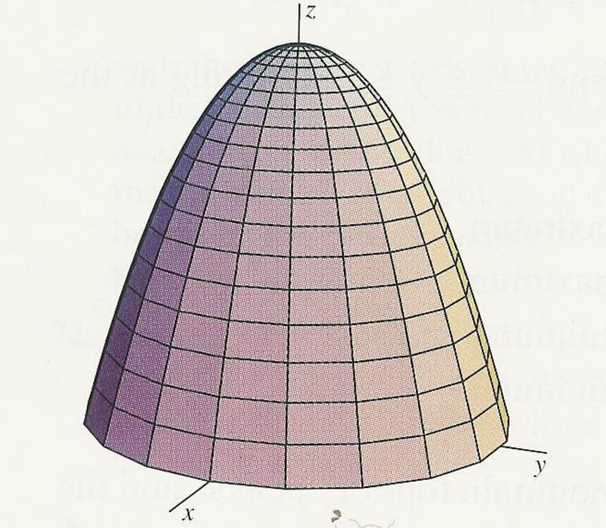


FIGURE 3
$z = y^2 - x^2$

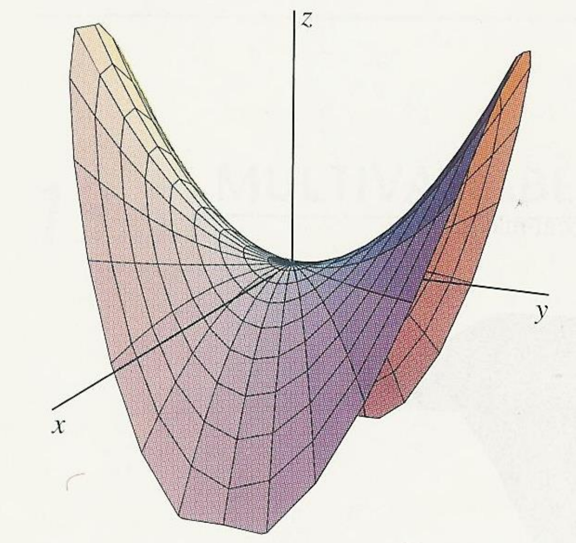
Guess the point(s) where the following surfaces may have maxima or minima.
(OR NONE)



(a) $f(x, y) = x^2 + y^2$, local minimum at (0, 0)

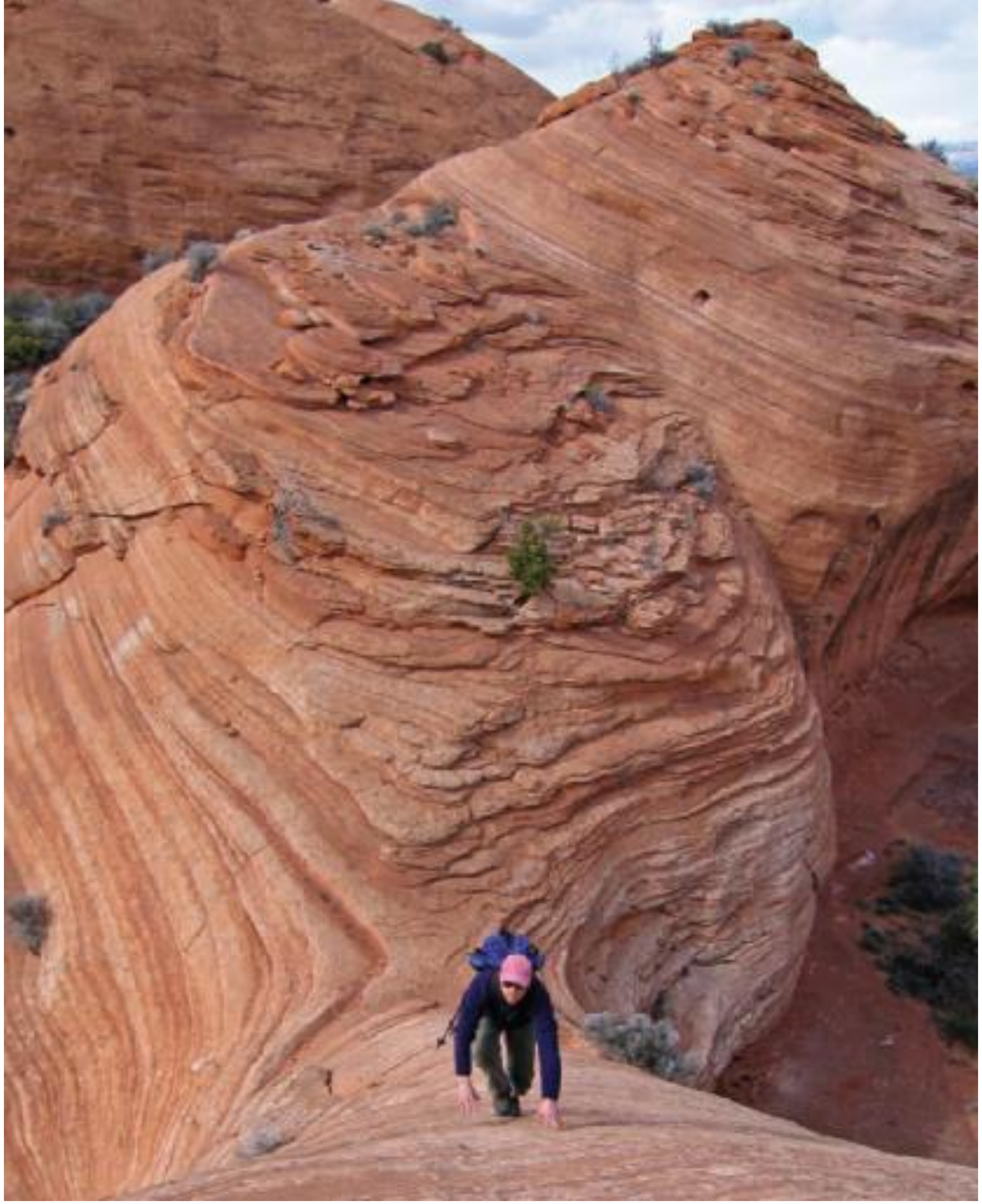


(b) $g(x, y) = 1 - x^2 - y^2$, local maximum at (0, 0)



(c) $h(x, y) = y^2 - x^2$, saddle point at (0, 0)

Do you feel like sitting on the last surface exactly at (0,0)?



A mountain pass also has the shape of a saddle. As the photograph of the geological formation illustrates, for people hiking in one direction the saddle point is the lowest point on their route, while for those traveling in a different direction the saddle point is the highest point.

Test for the existence of Maxima and Minima

3 Second Derivatives Test Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ [that is, (a, b) is a critical point of f]. Let

$$D = D(a, b) = f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.

(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.

(c) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum.

NOTE 1 In case (c) the point (a, b) is called a **saddle point** of f and the graph of f crosses its tangent plane at (a, b).

NOTE 2 If $D = 0$, the test gives no information: f could have a local maximum or local minimum at (a, b), or (a, b) could be a saddle point of f.

NOTE 3 To remember the formula for D, it's helpful to write it as a determinant:

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2$$

More on Saddle(s) points

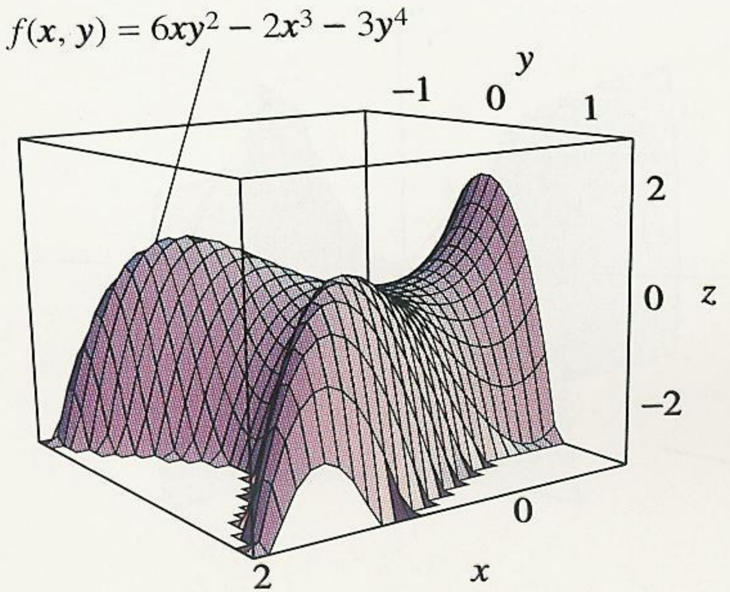


FIGURE 13.10.9 The monkey saddle of Example 3.

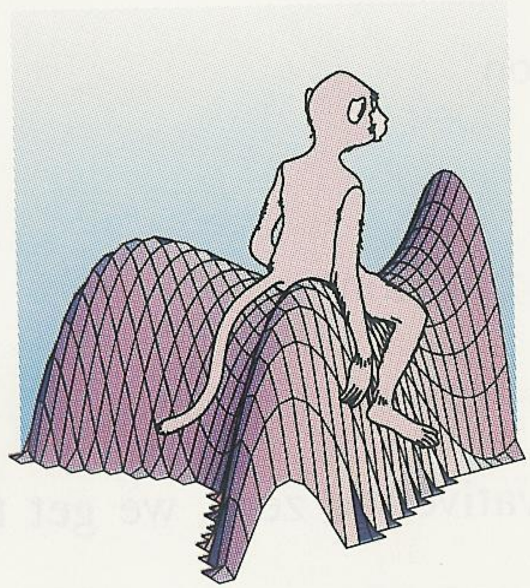


FIGURE 13.10.10 The monkey in its saddle (Example 3).

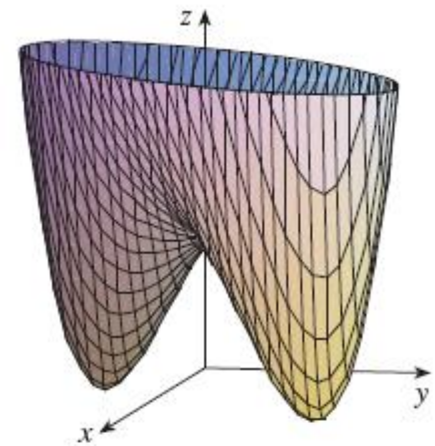
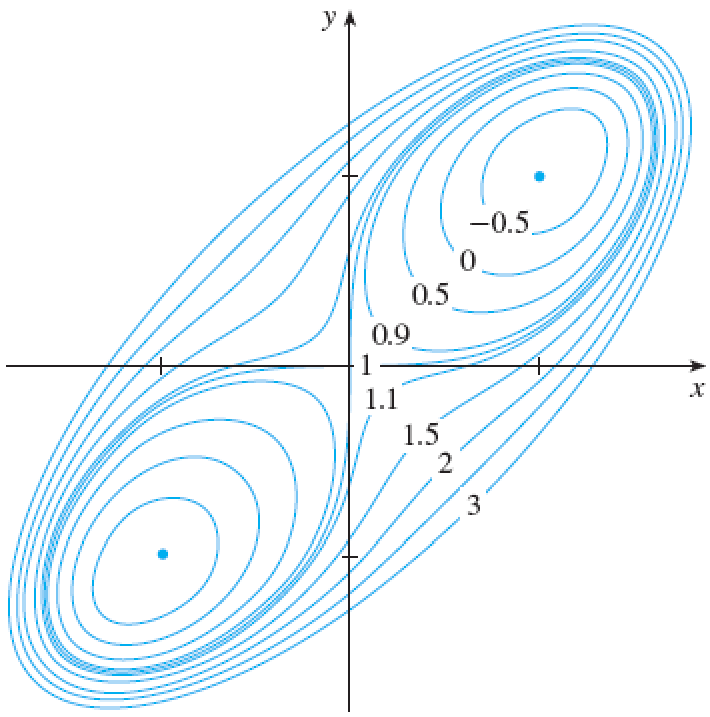


FIGURE 4
$z = x^4 + y^4 - 4xy + 1$



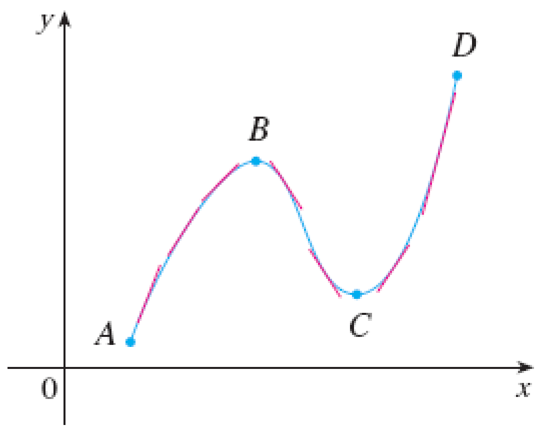
V EXAMPLE 3 Find the local maximum and minimum values and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

19. Show that $f(x, y) = x^2 + 4y^2 - 4xy + 2$ has an infinite number of critical points and that $D = 0$ at each one. Then show that f has a local (and absolute) minimum at each critical point.

20. Show that $f(x, y) = x^2 y e^{-x^2-y^2}$ has maximum values at $\left(\pm 1, 1/\sqrt{2}\right)$ and minimum values at $\left(\pm 1, -1/\sqrt{2}\right)$. Show also that f has infinitely many other critical points and $D = 0$ at each of them. Which of them give rise to maximum values? Minimum values? Saddle points?

- Proof of the test for the existence of extrema.
- Analogy with one variable function.
- How derivatives affect the shape of a graph?

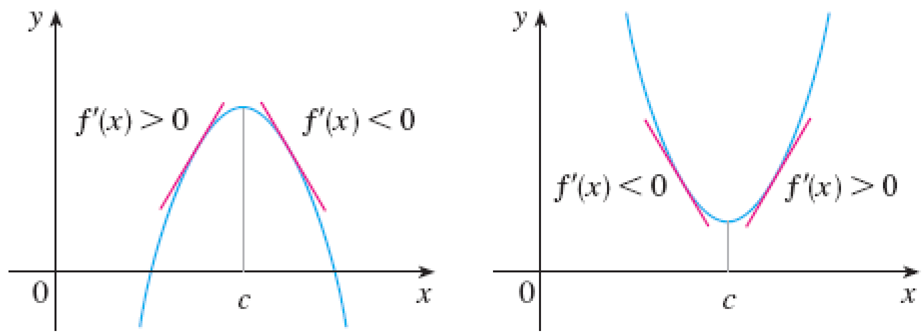
What does *f'* say about *f* ?



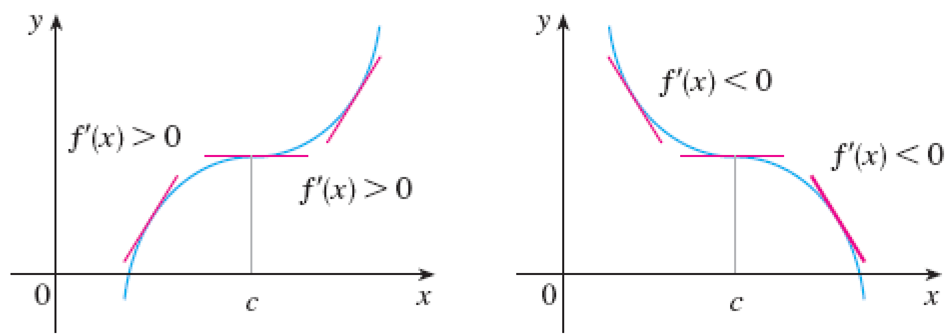
Increasing/Decreasing Test

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.



(a) Local maximum (b) Local minimum



(c) No maximum or minimum (d) No maximum or minimum

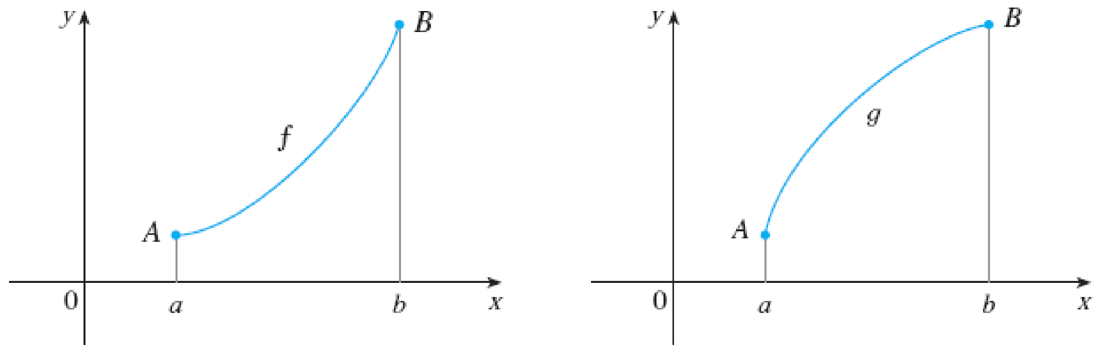
The First Derivative Test Suppose that c is a critical number of a continuous function f.

(a) If f' changes from positive to negative at c, then f has a local maximum at c.

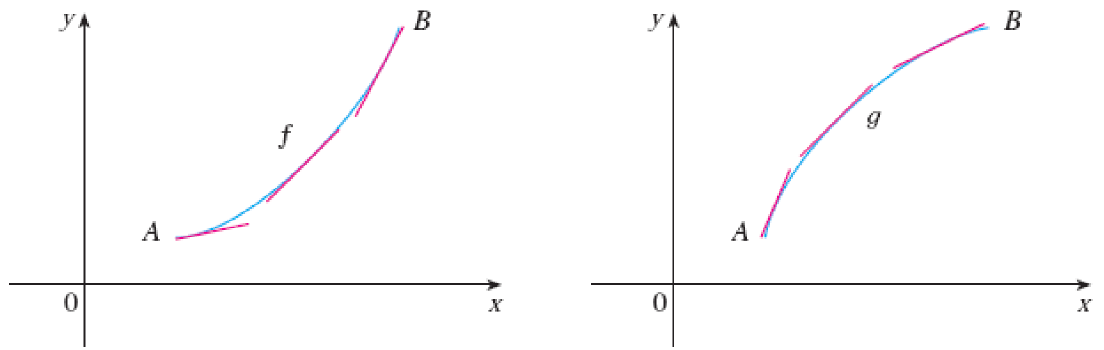
(b) If f' changes from negative to positive at c, then f has a local minimum at c.

(c) If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has no local maximum or minimum at c.

What does *f''* say about *f* ?



(a) (b)



(a) Concave upward (b) Concave downward

Definition If the graph of f lies above all of its tangents on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

Any comments on the slope of the tangents?

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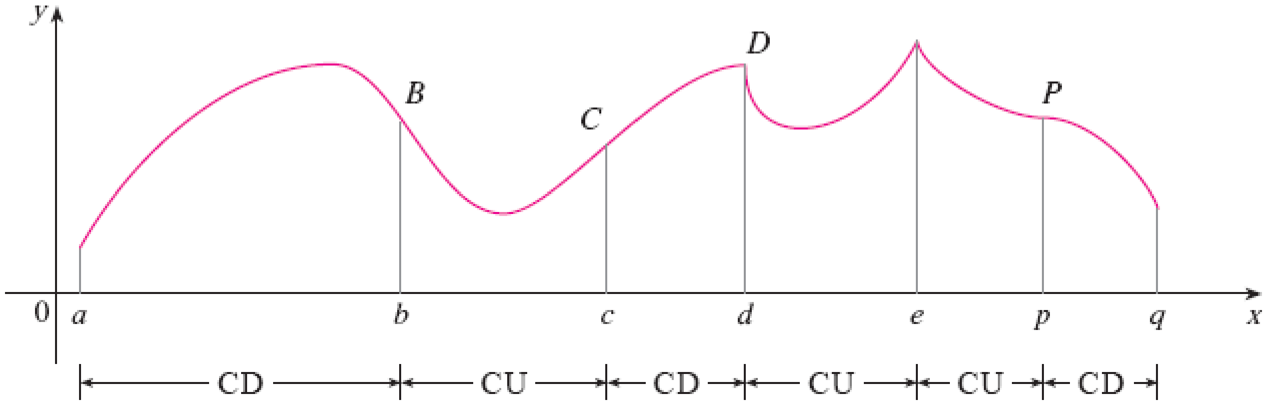

Concavity Test

(a) If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.

(b) If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

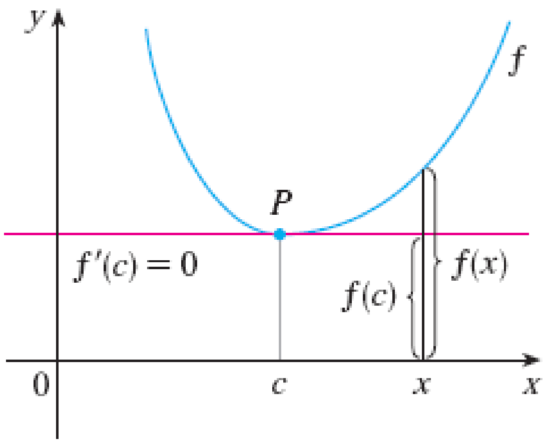

FIGURE 10
$f''(c) > 0$, f is concave upward

The Second Derivative Test Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.