PH 102 : Electromagnetism **Department of Physics**
Tutorial 5 **Indian Institute of Technology Guwahati**
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1. In complex notation, we use the Complex wave function which, as discussed in the class, is given by $\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$ with $\tilde{A} = Ae^{i\delta}$ being the complex amplitude. Use the method of separation of variables to solve the wave equation and to show that any wave can be expressed as a linear combination of sinusoidal waves:

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}dk.$$

Solution:

Let us assume the solution of the wave equation ($\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$) is of the form $f(z,t) = Z(z)T(t)$. Using this separation of variable and dividing by ZT, $\frac{1}{Z}\frac{d^2Z}{dz^2} = \frac{1}{v^2 T}\frac{d^2T}{dt^2}$. The left side depends only on z, and the right side only on t, so both sides must be constant (say, $-k^2$). Then the equations are,

$$\frac{d^2Z}{dz^2} = -k^2 Z; \qquad \frac{d^2T}{dt^2} = -(kv)^2 T.$$

The solutions being,

$$Z(z) = Ae^{ikz} + Be^{-ikz}; \qquad T(t) = Ce^{ikvt} + De^{-ikvt}.$$

$$\begin{aligned} f(z,t) &= (Ae^{ikz} + Be^{-ikz})(Ce^{ikvt} + De^{-ikvt}) \\ &= A_1 e^{i(kz+kvt)} + A_2 e^{i(kz-kvt)} + A_3 e^{i(-kz+kvt)} + A_4 e^{i(-kz-kvt)}. \end{aligned}$$

k is real to not to blow up Z and T and with no loss of generality one can assume $k > 0$. Therefore the general linear combination of separable solution is,

$$f(z,t) = \int_0^{\infty}[A_1(k)e^{i(kz+\omega t)} + A_2(k)e^{i(kz-\omega t)} + A_3(k)e^{i(-kz+\omega t)} + A_4(k)e^{i(-kz-\omega t)}]dk.$$

where $\omega \equiv kv$ and by allowing k to run negative one can combine the third term with the first and the second with the fourth, but $\omega = |k|v$ remains positive;

$f(z,t) = \int_{-\infty}^{\infty}[A_1(k)e^{i(kz+\omega t)} + A_2(k)e^{i(kz-\omega t)}]dk$, so f has both real and imaginary part and we are interested in the real one.

$$\begin{aligned} Re(f) = \int_{-\infty}^{\infty}[&Re(A_1)\cos(kz+\omega t) - Im(A_1)\sin(kz+\omega t) \\ &+ Re(A_2)\cos(kz-\omega t) - Im(A_2)\sin(kz-\omega t)]dk. \end{aligned}$$

Now, k goes negative hence both terms include waves traveling in both directions and it is enough to keep only one term. The term, $\cos(kz+\omega t) = \cos(-kz-\omega t)$, combines with the $\cos(kz-\omega t)$, as the negative k is picked up from the other half of the integration. Similarly, the second, $\sin(kz+\omega t) = -\sin(-kz-\omega t)$, combines with the $sin(kz-\omega t)$. Thus the general solution, can be written in the form

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz-\omega t)} dk,$$

here $\tilde{f}$ is the real part.

2. The linearly polarised wave is denoted by $\vec{\tilde{f}}(z,t) = \tilde{A}e^{i(kz-\omega t)}\hat{n}$. Linear polarisation results from the combination of horizontally and vertically polarised waves of the same phase. If the two components are of equal amplitude, but out of phase by $\pi/2$ (say, $\delta_v = 0, \delta_h = \pi/2$), the result is a circularly polarised wave. In that case:
(a) At a fixed point z, show that the string moves in a circle about the z axis. Does it go clockwise (right circular polarised) or counterclockwise (left circular polarised), as you look down the axis toward the origin? How would you construct a wave circling the other way?
(b) Sketch the string at time $t = 0$.
(c) How would you shake the string in order to produce a circularly polarised wave?
Solution: (a) The vertical polarisation is $\vec{f_v}(z,t) = A\cos(kz-\omega t)\,\hat{x}$ and the hori-

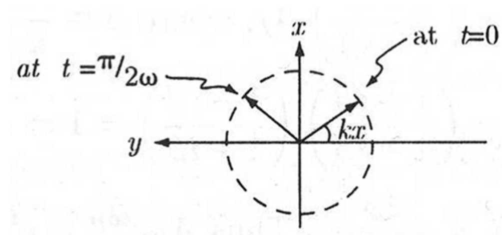

Figure 1: Figure for solution to problem 2.

zontal one is $\vec{f_h}(z,t) = A\cos(kz-\omega t+90^\circ)\,\hat{y} = -A\sin(kz-\omega t)\,\hat{y}$.

Now, $f_v^2 + f_h^2 = A^2$, so $\vec{f} = \vec{f_v} + \vec{f_h}$ lies on a circle of radius A. At $t = 0$, $\vec{f} = A\cos(kz)\,\hat{x} - A\sin(kz)\,\hat{y}$ and at $t = \frac{\pi}{2\omega}$, $\vec{f} = A\cos(kz-90^\circ)\,\hat{x} - A\sin(kz-90^\circ)\,\hat{y} = A\sin(kz)\,\hat{x} + \cos(kz)\,\hat{y}$. It is circling *counterclockwise*, to make it circling the other way, use $\delta_h = -\pi/2$.
(b) The sketch of the string at $t = 0$ is shown in figure 2.

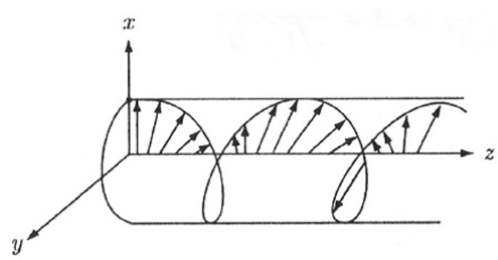

Figure 2: Figure for solution to problem 2.

(c) To produce a circularly polarised wave in a string, one needs to shake it around in a circle instead of up and down.

3. A paradoxical case of Poynting's theorem occurs when a static electric field is applied perpendicularly to a static magnetic field, as in the case of a pair of electrodes placed within a magnetic circuit with N turns, (see figure 3).
(a) What are $\vec{E}, \vec{H}$ and $\vec{S}$?
(b) What is the energy density stored in the system?
(c) Verify Poynting's theorem.

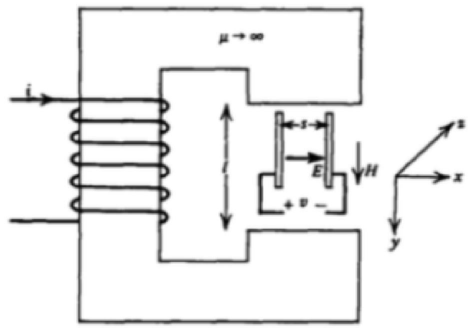

Figure 3: Figure for problem 3.

Solution:
(a) The electric field would be in the x-direction, $E_x = \frac{v}{s}$.
The magnetic field $B_y = \frac{\mu N i}{l}$ and $H_y = \frac{Ni}{l}$. Therefore, $S_z = E_x H_y = \frac{Nvi}{ls}$.
(b) Energy density $(w) = \frac{1}{2}(\mu H_y^2 + \epsilon E_x^2) = \frac{1}{2}\mu(\frac{Ni}{l})^2 + \frac{1}{2}\epsilon(\frac{v}{s})^2$.
(c) Poynting's theorem: $\vec{\nabla}.\vec{s} = \frac{\partial w}{\partial t} = 0$ as both the fields are static. Thus, $\vec{\nabla}.\vec{s} + \frac{\partial w}{\partial t} = 0$

4. A uniformly distributed volume current of thickness $2d$, $J_o \cos(\omega t)\hat{x}$ is a source of plane waves (see figure 4).
(a) From Maxwell's equations obtain a single differential equation relating E_x to J_x.

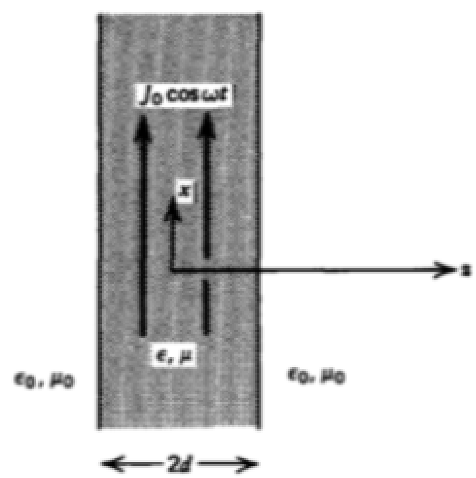

Figure 4: Figure for problem 4.

Solution:
The Maxwell's equations $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ and $\nabla \times \vec{B} = \mu \vec{J}_f + \mu\epsilon \frac{\partial \vec{E}}{\partial t}$ can connect E_x to J_x. The induced electric field direction is given by the current direction $(\hat{x})$. The Magnetic field circles around and in the $y-$direction. Component wise, $\frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t}$ and $-\frac{\partial B_y}{\partial z} = \mu J_x + \mu\epsilon \frac{\partial Ex}{\partial t}$. Differentiating the later equation and using first one,

$$\mu\epsilon \frac{\partial^2 E_x}{\partial t^2} + \mu \frac{\partial J_x}{\partial t} = -\frac{\partial^2 B_y}{\partial t \partial z} = -\frac{\partial^2 B_y}{\partial z \partial t} = \frac{\partial^2 E_x}{\partial z^2}$$

$$\frac{\partial^2 E_x}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 E_x}{\partial t^2} = \mu \frac{\partial J_x}{\partial t}$$

5. A polarising filter to microwaves is essentially formed by many highly conducting parallel wires whose spacing is much smaller than a wavelength (see figure 5). That polarisation whose electric field field is transverse to the wires passes through. The incident electric field is $\vec{E} = E_x \cos(\omega t - kz)\hat{x} + E_y \sin(\omega t - kz)\hat{y}$.
(a) What is the incident magnetic field and incident power density?
(b) What are the transmitted fields and power density?
(c) Another set of polarising wires are placed parallel but a distance d and oriented at an angle ϕ to the first. What are the transmitted fields?

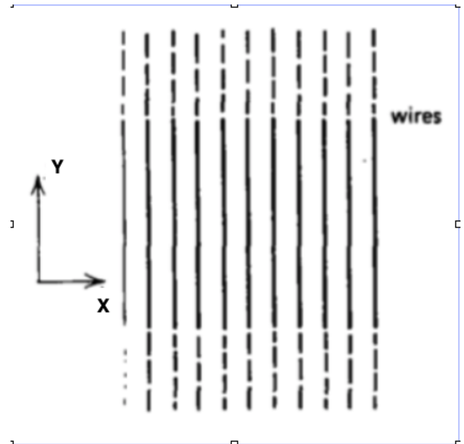

Figure 5: Figure for problem 5.

Solution:
(a) The incident electric field is $\vec{E} = E_x \cos(\omega t - kz)\hat{x} + E_y \sin(\omega t - kz)\hat{y}$. The incident magnetic field is $\vec{B} = \frac{1}{c}\hat{k} \times \vec{E} = \frac{1}{c}(E_x \cos(\omega t - kz)\hat{y} - E_y \sin(\omega t - kz)\hat{x})$.
The incident power density, $\vec{S} = \frac{1}{\mu}\vec{E} \times \vec{B} = \frac{1}{\mu c}[E_x^2 \cos^2(\omega t - kz) + E_y^2 \sin^2(\omega t - kz)]\hat{z}$.

(b) The transmitted fields are,

$$\vec{E_t} = E_x \cos(\omega t - kz)\hat{x} \ ; \quad \vec{B_t} = \frac{E_x}{c} \cos(\omega t - kz)\hat{y}$$

.

$$\vec{S_t} = \frac{E_x^2}{\mu c} \cos^2(\omega t - kz)]\hat{z}$$

(c) The other set of parallel polarising wires are kept at an angle ϕ. Thus only the $\cos\phi$ component of the electric field would get transmitted through the next filter. The fields will have the following magnitude, $|\vec{E_{t2}}| = |\vec{E_t}\cos\phi|$, $|\vec{B_{t2}}| = |\frac{\vec{E_t}\cos\phi}{c}|$ and $\vec{S_{t2}} = \frac{|\vec{E_t}|^2}{\mu c}\cos^2\phi$.

6. Consider a satellite in a stationary orbit of earth, i.e, to earth based observers the satellite would appear motionless, at a fixed position in the sky. The satellite beams a signal towards earth. The beam covers a region with area A km^2 on earth. Assume the field to be a monochoromatic plane wave with electric field amplitude E_0. Find the power delivered at the receiver on earth. What is energy density at the receiver on earth?

Solution:
The energy flux density (energy per unit time per unit area) transported by the fields is given by the Poynting vector ($\vec{S}$). Given the stationary orbit and the monochromatic plane wave nature of the signal we need to consider the time averaged quantities.

The time averaged power ($\langle P \rangle$) per unit area transported by the electromagnetic wave, i.e, intensity ($I = \frac{\langle P \rangle}{A}$) is the Poynting vector ($\vec{S}$) time averaged over a complete cycle.

$$\frac{\langle P \rangle}{A} = \langle S \rangle = \frac{1}{2} c \epsilon_0 E_0^2$$

Thus the power delivered at the receiver on earth $\frac{1}{2} c \epsilon_0 E_0^2 A$.

For a monochromatic plane wave the time averaged energy density (energy per unit volume), $\langle u \rangle = \frac{\langle S \rangle}{c} = \frac{1}{2} \epsilon_0 E_0^2$. This is the energy density at the receiver on earth.