Department of Mathematics
Indian Institute of Technology Guwahati
MA 101: Mathematics I
Tutorial Sheet-1
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1. Find the supremum and the infimum of the following sets:

(a) $\left\{1 - \dfrac{(-1)^n}{n} : n \in \mathbb{N}\right\}$

Solution. The elements of the given set A are $2, 1/2, 4/3, 3/4, 6/5, 5/6, \ldots$. Clearly, $\inf(A) = \min(A) = 1/2$ and $\sup(A) = \max(A) = 2$. $\square$

(b) $\left\{1 - \dfrac{1}{n} : n \in \mathbb{N}\right\}$

Solution. Let A be the given set. Clearly, $0 \leq x$ for all $x \in A$ and $0 \in A$. Hence, $\inf(A) = \min(A) = 0$. We next observe that $x < 1$ for all $x \in A$. Hence, 1 is an upper bound of A. Suppose that $u < 1$ and u is an upper bound of A. Then $1 - u > 0$ and by Archimedean property, there exists $n_0 \in \mathbb{N}$ such that $n_0(1 - u) > 1$. This gives $u < 1 - \frac{1}{n_0}$. This is a contradiction to the fact that u is an upper bound of A. Thus, any number smaller than 1 is not an upper bound. This proves that 1 is the least upper bound of A. $\square$

(c) $\left\{\dfrac{1}{n} - \dfrac{1}{m} : n, m \in \mathbb{N}\right\}$

Solution. Taking $n = 1$ and $m \in \mathbb{N}$, we see that the numbers $0, 1/2, 2/3, 3/4, \ldots$ are in the set A. Similarly, taking $m = 1$ and $n \in \mathbb{N}$, we see that 0, $-1/2$, $-2/3$, $-3/4, \ldots \in A$. Thus, $-1 < x < 1$ for every $x \in A$. Suppose that $u < 1$ and u is an upper bound of A. Then $1 - u > 0$ and by Archimedean property, there exists $m_0 \in \mathbb{N}$ such that $m_0(1 - u) > 1$. This gives $u < 1 - \frac{1}{m_0}$. Since $1 - \frac{1}{m_0} \in A$, we get a contradiction to the fact that u is an upper bound of A. Thus, $\sup(A) = 1$.

Similarly, suppose that $v > -1$ and v is a lower bound of A. By Archimedean property we have $n_0(1 + v) > 1$ for some $n_0 \in \mathbb{N}$. This gives $v > \frac{1}{n_0} - 1$. Since $\frac{1}{n_0} - 1 \in A$, we get a contradiction to the fact that v is a lower bound of A. Thus, $\inf(A) = -1$. $\square$

(d) $\left\{\dfrac{m}{m+n} : m, n \in \mathbb{N}\right\}$

Solution. First note that $0 < \frac{m}{m+n} < 1$. Thus, 0 is a lower bound. We guess that $\inf(A) = 0$ because $\frac{1}{1+n}$ is in the set and it approaches 0 when n is very large. Suppose that $v > 0$ and v is a lower bound. Using the Archimedean property we can find an n such that $\frac{1}{n+1} < v$, which gives a contradiction. This proves that 0 is the greatest lower bound. Similarly, we can show that $\sup(A) = 1$. $\square$

2. Let A and B be two bounded subsets of $\mathbb{R}$. The sum of A and B is defined by

$$A + B = \{a + b : a \in A, b \in B\}.$$

Show that

(a) $\inf(A + B) = \inf(A) + \inf(B)$

Solution. Since A and B are bounded, so they are bounded below. Let $\alpha = \inf(A)$ and $\beta = \inf(B)$. Then $\alpha \leq x$ for all $x \in A$ and $\beta \leq y$ for all $y \in B$. Let $z \in A + B$. Then $z = a + b$ for some $a \in A$ and $b \in B$. Thus, $\alpha + \beta \leq z$ for all $z \in A+B$. This proves that $\alpha+\beta$ is a lower bound of $A+B$. Let $\gamma = \inf(A+B)$. Our claim is that $\gamma = \alpha + \beta$. Since $\gamma \geq \alpha + \beta$, we only need to prove that $\gamma \leq \alpha+\beta$. Let $\varepsilon > 0$ be an arbitrary real number. Since $\alpha = \inf(A)$, so $\alpha+\varepsilon/2$ is not a lower bound of A and hence $a < \alpha + \varepsilon/2$ for some $a \in A$. Similarly, $b < \beta+\varepsilon/2$ for some $b \in B$. Thus, $\gamma \leq a+b < \alpha+\beta+\varepsilon$ for every $\varepsilon > 0$. This proves that $\gamma \leq \alpha + \beta$. Hence, $\inf(A + B) = \gamma = \alpha + \beta = \inf(A) + \inf(B)$. □

(b) $\sup(A + B) = \sup(A) + \sup(B)$

Solution. Since A and B are bounded, so they are bounded above. Let $\alpha = \sup(A)$ and $\beta = \sup(B)$. Then $\alpha \geq x$ for all $x \in A$ and $\beta \geq y$ for all $y \in B$. Let $z \in A + B$. Then $z = a + b$ for some $a \in A$ and $b \in B$. Thus, $\alpha + \beta \geq z$ for all $z \in A + B$. This proves that $\alpha + \beta$ is an upper bound of $A + B$. Let $\gamma = \sup(A + B)$. Our claim is that $\gamma = \alpha + \beta$. Since $\gamma \leq \alpha + \beta$, we only need to prove that $\gamma \geq \alpha + \beta$. Let $\varepsilon > 0$ be an arbitrary real number. Since $\alpha = \sup(A)$, so $\alpha - \varepsilon/2$ is not an upper bound of A and hence $a > \alpha - \varepsilon/2$ for some $a \in A$. Similarly, $b > \beta-\varepsilon/2$ for some $b \in B$. Thus, $\gamma \geq a+b > \alpha+\beta-\varepsilon$, that is, $\gamma + \varepsilon > \alpha + \beta$ for every $\varepsilon > 0$. This proves that $\gamma \geq \alpha + \beta$. Hence, $\sup(A + B) = \gamma = \alpha + \beta = \sup(A) + \sup(B)$. □

3. Let A be a nonempty bounded subset of $\mathbb{R}$. For a real number x, we define $xA = \{xa : a \in A\}$. Prove that:

(a) If $x > 0$, then $\inf(xA) = x \cdot \inf(A)$ and $\sup(xA) = x \cdot \sup(A)$.

Solution. Let $\inf(A) = \alpha$. Since $x > 0$, so $x\alpha \leq xa$ for all $a \in A$. Hence, $x\alpha$ is a lower bound of xA. Let $B = xA$ and $\gamma = \inf(xA) = \inf(B)$. Then we have $x\alpha \leq \gamma$. That is, for any bounded set A we have $x \cdot \inf(A) \leq \inf(xA)$ for all $x > 0$. This implies that $\frac{1}{x} \cdot \inf(B) \leq \inf(\frac{1}{x}B)$ for all $x > 0$. But $\frac{1}{x}B = A$, and hence $\inf(xA) \leq x \cdot \inf(A)$ for all $x > 0$. This proves that $\inf(xA) = x \cdot \inf(A)$ if $x > 0$. The supremum case follows similarly. □

(b) If $x < 0$, then $\inf(xA) = x \cdot \sup(A)$ and $\sup(xA) = x \cdot \inf(A)$.

Solution. Let $\sup(A) = \alpha$. Since $x < 0$, so $x\alpha \leq xa$ for all $a \in A$. Hence, $x\alpha$ is a lower bound of xA. Let $\gamma = \inf(xA)$. Then we have $x\alpha \leq \gamma$. Again, $\gamma \leq xa$ for all $a \in A$. Since $x < 0$, so $\frac{1}{x}\gamma \geq a$ for all $a \in A$. Thus, $\frac{1}{x}\gamma$ is an upper bound of A, and hence $\frac{1}{x}\gamma \geq \alpha$. This gives $\gamma \leq x\alpha$ and completes the proof of $\gamma = x\alpha$. The other case follows similarly. □

4. Let q_1 and q_2 be two distinct real numbers. Then show that there exists an irrational number between them.

Solution. Let $q_1 < q_2$. Consider the real numbers $q_1\sqrt{2}$ and $q_2\sqrt{2}$. Then, we know that there is a rational number r such that $q_1\sqrt{2} < r < q_2\sqrt{2}$. This gives $q_1 < r/\sqrt{2} < q_2$. Thus, $r/\sqrt{2}$ is an irrational number lying between q_1 and q_2. $\square$

5. Use the Archimedean property to show that

$$\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}.$$

Solution. Let $I = \displaystyle\bigcap_{n\in\mathbb{N}} \left(-\frac{1}{n}, \frac{1}{n}\right)$. Clearly, $0 \in I$. Suppose that $x < 0$. Using the Archimedean property we can find an n_0 such that $n_0(-x) > 1$. This gives $x < -\frac{1}{n_0}$ and hence $x \notin (-\frac{1}{n_0}, \frac{1}{n_0})$. This proves that $x \notin I$ if $x < 0$. Similarly, if $x > 0$ then using the Archimedean property we can find an m_0 such that $m_0 x > 1$. This gives $x > \frac{1}{m_0}$ and hence $x \notin (-\frac{1}{m_0}, \frac{1}{m_0})$. This proves that $x \notin I$ if $x > 0$. Therefore, 0 is the only element in I. $\square$