PH 102, Electromagnetism,

Post Mid Semester

Lecture 8.

Electrodynamics

Maxwell's Equations:
Problem Solving & Discussions

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Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0},$$

$$\nabla \times \mathbf{B} - \mu_0\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}.$$

Maxwell’s Equations

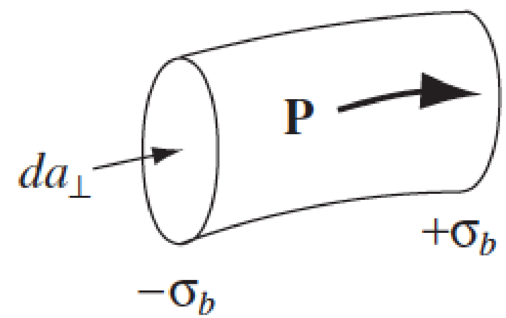
Maxwell’s Equations in Matter : Electric and Magnetic Polarization

$$\rho_b = -\nabla \cdot \mathbf{P} \qquad \mathbf{J}_b = \nabla \times \mathbf{M}$$

electric/magnetic polarization produces bound charge density/current

Polarization current: P changes with time

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp$$



Polarization current, $\mathbf{J}_p = \dfrac{\partial \mathbf{P}}{\partial t}$

$\mathbf{J_b}$ connected to magnetization but $\mathbf{J_p}$ connected to change of electric polarization

$$\nabla \cdot \mathbf{J}_p = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P}) = -\frac{\partial \rho_b}{\partial t}$$

Continuity Equation for $\mathbf{J}_p$

Maxwell's Equations

Maxwell's Equations in Matter :

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.$$

$$\boxed{\mathbf{J}_d \equiv \frac{\partial \mathbf{D}}{\partial t}}$$

Maxwell's Equations

Boundary Conditions: General Conditions

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

For linear media: Boundary conditions in terms of **E** and **B** alone,

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0},$$

$$B_1^{\perp} - B_2^{\perp} = 0, \qquad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.$$

Theory of reflection and refraction!

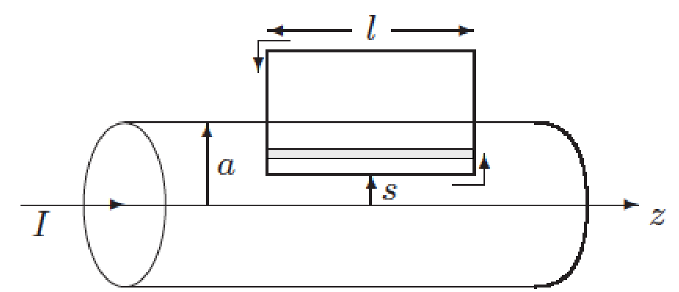
(1) An alternating current $I = I_0 \cos(\omega t)$ flows down a long straight wire, and returns along a coaxial conducting tube of radius a. Assuming that the field goes to zero as $s \to \infty$,

a) (a) Find E(s, t).
b) Find the displacement current density J_d .
c) Integrate it to get the total displacement current,
d) Compare I_d and I. If the outer cylinder were, say, 2 mm in diameter, how high would the frequency have to be, for I_d to be 1% of I ?

Solution:

a)

Using the "Amperian loop"



$$\oint \mathbf{E}\cdot d\mathbf{l} = El = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \mathbf{B}\cdot d\mathbf{a} = -\frac{d}{dt}\int_s^a \frac{\mu_0 I}{2\pi s'} l\, ds'$$

Outside, B =0, so here E = 0
(like B outside a solenoid).

$$\therefore E = -\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln\left(\frac{a}{s}\right).$$

$$\boxed{\frac{dI}{dt} = -I_0\omega\sin\omega t,}$$

$$\boxed{\mathbf{E} = \frac{\mu_0 I_0 \omega}{2\pi}\sin(\omega t)\ln\left(\frac{a}{s}\right)\hat{\mathbf{z}}.}$$

a) (a) Find E(s, t).
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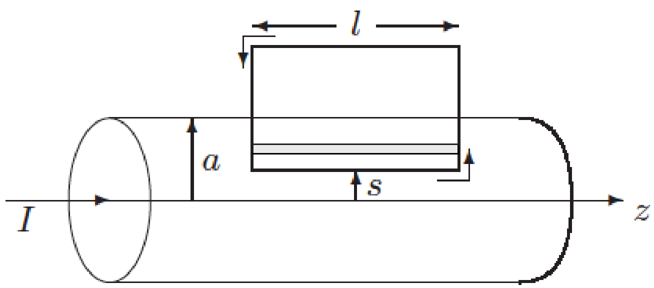


Solution:

$$\mathbf{E} = \frac{\mu_0 I_0 \omega}{2\pi} \sin(\omega t) \ln\left(\frac{a}{s}\right) \hat{\mathbf{z}}.$$

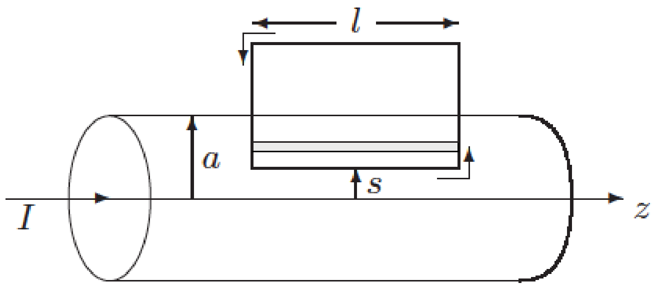
(b)

$$J_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$= \epsilon_0 \frac{\mu_0 I_0 \omega^2}{2\pi} \cos(\omega t) \ln(a/s)\, \hat{\mathbf{z}}.$$

$$= \frac{\mu_0 \epsilon_0}{2\pi} \omega^2 I \ln(a/s)\, \hat{\mathbf{z}}.$$

(c)



$$J_d = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{\mu_0 \epsilon_0}{2\pi} \omega^2 I \ln(a/s)\,\hat{\mathbf{z}}.$$

$$I_d = \int \mathbf{J}_d \cdot d\mathbf{a} = \frac{\mu_0 \epsilon_0 \omega^2 I}{2\pi} \int_0^a \ln(a/s)(2\pi s\, ds) = \mu_0 \epsilon_0 \omega^2 I \int_0^a (s \ln a - s \ln s) ds$$

$$= \mu_0 \epsilon_0 \omega^2 I \left[(\ln a) \tfrac{s^2}{2} - \tfrac{s^2}{2} \ln s + \tfrac{s^2}{4} \right] \Big|_0^a = \mu_0 \epsilon_0 \omega^2 I \left[\cancel{\tfrac{a^2}{2} \ln a} - \cancel{\tfrac{a^2}{2} \ln a} + \tfrac{a^2}{4} \right]$$

$$= \frac{\mu_0 \epsilon_0 \omega^2 I a^2}{4}.$$

(d) Compare I_d and I. If the outer cylinder were, say, 2 mm in diameter, how high would the frequency have to be, for I_d to be 1% of I ?

(d)

$$I_d = \frac{\mu_0\epsilon_0\omega^2 I a^2}{4}. \quad \Rightarrow \quad \frac{I_d}{I} = \frac{\mu_0\epsilon_0\omega^2 a^2}{4}$$

where, $\mu_0\epsilon_0 = 1/c^2,\ I_d/I = (\omega a/2c)^2.$

Given, $a = 10^{-3}\,\mathrm{m},$ and $\frac{I_d}{I} = \frac{1}{100},$ $\Rightarrow$ $\boxed{\frac{\omega a}{2c} = \frac{1}{10},}$

$$\omega = \frac{2c}{10a} = \frac{3\times10^8\ \mathrm{m/s}}{5\times10^{-3}\ \mathrm{m}},\ \text{or}\ \omega = 0.6\times10^{11}/\mathrm{s}$$

$$\nu = \frac{\omega}{2\pi} \approx 10^{10}\,\mathrm{Hz},$$

Microwave, way above the radio frequencies

The frequency required so that the displacement current is comparable (~ 1%) is very high @ mircowave. Hence, in the early days of Faraday no experimental signal could appear giving clue for the displacement current!

(2) Consider the following fields, $\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt - r)\hat{\mathbf{r}}; \;\; \mathbf{B}(\mathbf{r},t) = \mathbf{0}$

Show that these fields satisfy all of Maxwell's equations, and determine ρ and **J**. Describe the physical situation that gives rise to these fields.

Solution:

Physically, this is the field of a point charge +q at the origin, out to an expanding spherical shell of radius *vt*. Outside the shell the field is zero. Evidently the shell carries the opposite charge, -q.

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt - r)\hat{\mathbf{r}}; \;\; \mathbf{B}(\mathbf{r},t) = \mathbf{0}$$

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$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt-r)\hat{\mathbf{r}};$$

$$\boxed{\nabla\cdot(f\mathbf{A}) = f(\nabla\cdot\mathbf{A}) + \mathbf{A}\cdot(\nabla f)}$$

$$\nabla\cdot\mathbf{E} = \theta(vt-r)\nabla\cdot\left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}\right) + \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}\cdot\nabla[\theta(vt-r)]$$

$$\boxed{\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r}).}$$

$$= \frac{q}{\epsilon_0}\delta^3(\mathbf{r})\theta(vt-r) + \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})\frac{\partial}{\partial r}\theta(vt-r).$$

$$\boxed{\delta^3(\mathbf{r})\theta(vt-r) = \delta^3(\mathbf{r})\theta(t), \text{ and } \frac{\partial}{\partial r}\theta(vt-r) = -\delta(vt-r)}$$

$$\rho = \epsilon_0\nabla\cdot\mathbf{E} = q\delta^3(\mathbf{r})\theta(t) - \frac{q}{4\pi r^2}\delta(vt-r).$$

$$\boxed{\rho = \epsilon_0 \nabla \cdot \mathbf{E} = q\delta^3(\mathbf{r})\theta(t) - \frac{q}{4\pi r^2}\delta(vt - r).}$$

For t < 0 the field and the charge density are zero everywhere.

$$\nabla \cdot \mathbf{B} = 0, \text{ and } \nabla \times \mathbf{E} = \mathbf{0}$$

(since E has only an r component and it is independent of θ and φ)

$$\nabla \times \mathbf{B} = \mathbf{0} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \partial \mathbf{E}/\partial t.$$

$$\mathbf{J} = -\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = -\epsilon_0 \left\{ \frac{q}{4\pi\epsilon_0 r^2} \frac{\partial}{\partial t} \left[\theta(vt - r)\right] \right\} \hat{\mathbf{r}} = \quad -\frac{q}{4\pi r^2} v\delta(vt - r)\, \hat{\mathbf{r}}.$$

The stationary charge at the origin doesn't contribute to **J**, of course; for the expanding shell we have $\mathbf{J} = \rho \mathbf{v}$.

3. (a)

Show that Maxwell's equations with magnetic charge are invariant under the duality transformation,

$$\begin{aligned}
\mathbf{E}' &= \mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha, \\
c\mathbf{B}' &= c\mathbf{B}\cos\alpha - \mathbf{E}\sin\alpha, \\
cq_e' &= cq_e\cos\alpha + q_m\sin\alpha, \\
q_m' &= q_m\cos\alpha - cq_e\sin\alpha,
\end{aligned}$$

where $c = 1/\sqrt{\epsilon_0\mu_0}$ and α is an arbitrary rotation angle in "**E**/**B**-space." Charge and current densities transform in the same way as q_e and q_m.

3. (b)

Show that the force law, $\mathbf{F} = q_e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right)$, is also invariant under the duality transformation.

$$\begin{aligned}
&\text{(i)}\quad \nabla\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho_e, && \text{(iii)}\quad \nabla\times\mathbf{E} = -\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}, \\
&\text{(ii)}\quad \nabla\cdot\mathbf{B} = \mu_0\rho_m, && \text{(iv)}\quad \nabla\times\mathbf{B} = \mu_0\mathbf{J}_e + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}.
\end{aligned}$$

$$\boxed{\begin{aligned} \mathbf{E}' &= \mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha, \\ c\mathbf{B}' &= c\mathbf{B}\cos\alpha - \mathbf{E}\sin\alpha, \\ cq_e' &= cq_e\cos\alpha + q_m\sin\alpha, \\ q_m' &= q_m\cos\alpha - cq_e\sin\alpha, \end{aligned}}$$

3. (a)

$$\begin{aligned} \boldsymbol{\nabla}\cdot\mathbf{E}' &= (\boldsymbol{\nabla}\cdot\mathbf{E})\cos\alpha + c(\boldsymbol{\nabla}\cdot\mathbf{B})\sin\alpha = \frac{1}{\epsilon_0}\rho_e\cos\alpha + c\mu_0\rho_m\sin\alpha \\ &= \frac{1}{\epsilon_0}(\rho_e\cos\alpha + c\mu_0\epsilon_0\rho_m\sin\alpha) = \frac{1}{\epsilon_0}\left(\rho_e\cos\alpha + \frac{1}{c}\rho_m\sin\alpha\right) = \frac{1}{\epsilon_0}\rho_e'. \\ \boldsymbol{\nabla}\cdot\mathbf{B}' &= (\boldsymbol{\nabla}\cdot\mathbf{B})\cos\alpha - \frac{1}{c}(\boldsymbol{\nabla}\cdot\mathbf{E})\sin\alpha = \mu_0\rho_m\cos\alpha - \frac{1}{c\epsilon_0}\rho_e\sin\alpha \\ &= \mu_0\left(\rho_m\cos\alpha - \frac{1}{c\mu_0\epsilon_0}\rho_e\sin\alpha\right) = \mu_0(\rho_m\cos\alpha - c\rho_e\sin\alpha) = \mu_0\rho_m'. \end{aligned}$$

$$\begin{aligned} \boldsymbol{\nabla}\times\mathbf{E}' &= (\boldsymbol{\nabla}\times\mathbf{E})\cos\alpha + c(\boldsymbol{\nabla}\times\mathbf{B})\sin\alpha = \left(-\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}\right)\cos\alpha + c\left(\mu_0\mathbf{J}_e + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\sin\alpha \\ &= -\mu_0(\mathbf{J}_m\cos\alpha - c\mathbf{J}_e\sin\alpha) - \frac{\partial}{\partial t}\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right) = -\mu_0\mathbf{J}_m' - \frac{\partial\mathbf{B}'}{\partial t}.\ \checkmark \\ \boldsymbol{\nabla}\times\mathbf{B}' &= (\boldsymbol{\nabla}\times\mathbf{B})\cos\alpha - \frac{1}{c}(\boldsymbol{\nabla}\times\mathbf{E})\sin\alpha = \left(\mu_0\mathbf{J}_e + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right)\cos\alpha - \frac{1}{c}\left(-\mu_0\mathbf{J}_m - \frac{\partial\mathbf{B}}{\partial t}\right)\sin\alpha \\ &= \mu_0\left(\mathbf{J}_e\cos\alpha + \frac{1}{c}\mathbf{J}_m\sin\alpha\right) + \mu_0\epsilon_0\frac{\partial}{\partial t}(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha) = \mu_0\mathbf{J}_e' + \mu_0\epsilon_0\frac{\partial\mathbf{E}'}{\partial t}.\ \checkmark \end{aligned}$$

$$\boxed{\begin{aligned}\mathbf{E}' &= \mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha,\\ c\mathbf{B}' &= c\mathbf{B}\cos\alpha - \mathbf{E}\sin\alpha,\\ cq_e' &= cq_e\cos\alpha + q_m\sin\alpha,\\ q_m' &= q_m\cos\alpha - cq_e\sin\alpha,\end{aligned}}$$

3. (b)

$$\mathbf{F}' = q_e'(\mathbf{E}' + \mathbf{v}\times\mathbf{B}') + q_m'(\mathbf{B}' - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}')$$

$$= \left(q_e\cos\alpha + \frac{1}{c}q_m\sin\alpha\right)\left[(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha) + \mathbf{v}\times\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right)\right]$$

$$+ (q_m\cos\alpha - cq_e\sin\alpha)\left[\left(\mathbf{B}\cos\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\right) - \frac{1}{c^2}\mathbf{v}\times(\mathbf{E}\cos\alpha + c\mathbf{B}\sin\alpha)\right]$$

$$= q_e\Big[(\mathbf{E}\cos^2\alpha + c\mathbf{B}\sin\alpha\cos\alpha - c\mathbf{B}\sin\alpha\cos\alpha + \mathbf{E}\sin^2\alpha)$$

$$+\mathbf{v}\times\left(\mathbf{B}\cos^2\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha + \frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha + \mathbf{B}\sin^2\alpha\right)\Big]$$

$$+q_m\Big[\left(\frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha + \mathbf{B}\sin^2\alpha + \mathbf{B}\cos^2\alpha - \frac{1}{c}\mathbf{E}\sin\alpha\cos\alpha\right)$$

$$+\mathbf{v}\times\left(\frac{1}{c}\mathbf{B}\sin\alpha\cos\alpha - \frac{1}{c^2}\mathbf{E}\sin^2\alpha - \frac{1}{c^2}\mathbf{E}\cos^2\alpha - \frac{1}{c}\mathbf{B}\sin\alpha\cos\alpha\right)\Big]$$

$$= q_e(\mathbf{E} + \mathbf{v}\times\mathbf{B}) + q_m\left(\mathbf{B} - \frac{1}{c^2}\mathbf{v}\times\mathbf{E}\right) = \mathbf{F}. \quad \text{qed}$$

LT can transform E ⬅➡ B

RECAP

Magnetostatics:

Lorentz force law

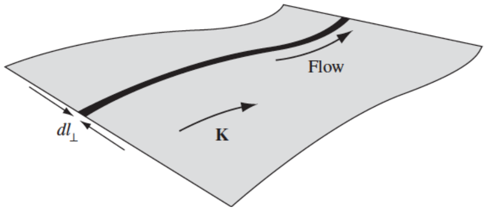
$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})].$$

Magnetic forces do no work.

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt = 0.$$

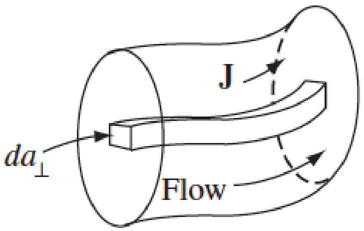
Current

$$\mathbf{I} = \lambda \mathbf{v}. \qquad \mathbf{K} = \sigma \mathbf{v}. \qquad \mathbf{J} = \rho \mathbf{v}.$$



Continuity Equation

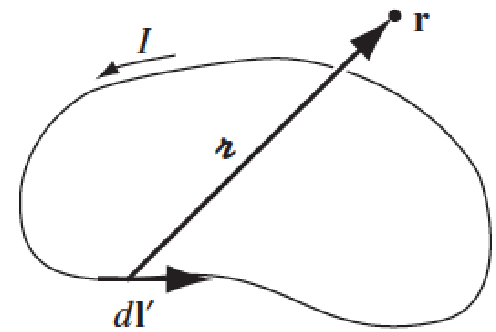
$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = 0.$$



Magnetostatics:

Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I}\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}'\times\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.$$



Ampere's Law

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enc}}, \qquad \nabla\times\mathbf{B} = \mu_0\mathbf{J}.$$

Ampere's Law is useful, only when symmetry enables pulling ***B*** outside integral

Magnetic vector potential :

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

The Vector Potential (**A**)

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Freedom in choosing ***A***

As, $\nabla \times (\nabla \lambda) = 0$, we can transform $\mathbf{A}_0$ to, $\mathbf{A} = \mathbf{A}_0 + \nabla \lambda$

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

Freedom in choosing ***A*** to eliminate Div. A = 0

$$\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2 \lambda.$$

Choose, λ is such that, $\nabla^2 \lambda = -\nabla \cdot \mathbf{A}_0.$

Magnetic vector potential :

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

The Vector Potential (**A**)

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

Freedom in choosing ***A*** to eliminate Div. A = 0

3 Poisson's eqs.

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}. \qquad \text{and} \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \, d\tau'.$$

Magnetic vector potential :

Magnetostatic Boundary Conditions

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0(\mathbf{K} \times \hat{\mathbf{n}}),$$

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}.$$

$$(\nabla \times \vec{A})_{above} - (\nabla \times \vec{A})_{below} = \mu_0(\vec{K} \times \hat{n})$$

Multipole expansion of vector potential

Magnetic Dipole : In absence of monopoles dipole is the dominant term

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\alpha \, d\mathbf{l}' = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\mathbf{r}} \cdot \mathbf{r}')\, d\mathbf{l}'.$$

$$= \frac{\mu_0}{4\pi} \frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2},$$

$$\oint (\vec{c} \cdot \vec{r}) d\vec{l} = \vec{a} \times \vec{c}$$

$$\vec{a} = \frac{1}{2} \oint \vec{r} \times d\vec{l}$$

with $\mathbf{c} = \hat{\mathbf{r}}$

$$\oint (\hat{\mathbf{r}} \cdot \mathbf{r}')\, d\mathbf{l}' = -\hat{\mathbf{r}} \times \int d\mathbf{a}'.$$

$$\hat{r} \times \oint (\vec{r}' \times d\vec{r}') = \oint \vec{r}'(\hat{r} \cdot d\vec{r}') - \oint d\vec{r}'(\hat{r} \cdot \vec{r}')$$

$$= \oint d\left[(\vec{r}'(\hat{r} \cdot \vec{r}')\right] - \oint (\hat{r} \cdot \vec{r}') d\vec{r}' - \oint d\vec{r}'(\hat{r} \cdot \vec{r}')$$

$$= -\mathbf{2} \oint (\hat{r} \cdot \vec{r}') d\vec{r}'$$

Here ***m*** is the magnetic dipole moment

$$\mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}.$$

Magnetic dipole moment is independent of choice of origin!
No Magnetic monopole

Remember,

Electric dipole moment was independent of origin only
when the total charge vanished

Torque and forces on magnetic dipole

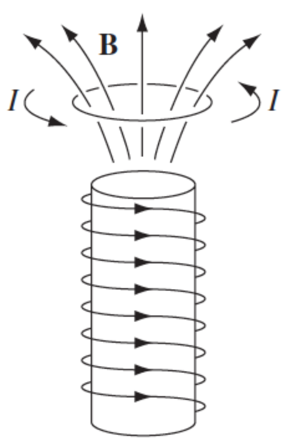
In the uniform field the net force on a current loop is zero!

$$\mathbf{F} = I \oint (d\mathbf{l} \times \mathbf{B}) = I \left(\oint d\mathbf{l}\right) \times \mathbf{B} = \mathbf{0};$$

B being constant comes outside integral and the net displacement around a closed loop also vanishes,

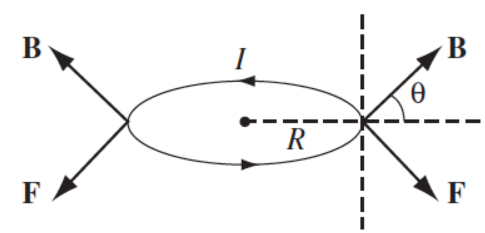
Not true for non uniform field,

Circular wire (***R***) carrying current ***I*** is suspended over a short solenoid over the fringing region.



Here B has a radial component, hence net downward force on loop.

$$F = 2\pi I R B \cos\theta.$$



$$\mathbf{F} = \nabla(\mathbf{m} \cdot \mathbf{B})$$

Multipole expansion of vector potential

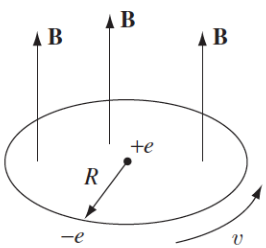
$$\boxed{\nabla \cdot \mathbf{B} = 0.}$$

No magnetic Monopole

$$\mathbf{A}_{\text{dip}}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \hat{\mathbf{r}}}{r^2}, \qquad \mathbf{m} \equiv I \int d\mathbf{a} = I\mathbf{a}.$$

Torque and forces on magnetic dipole

$$\boxed{\overline{T} = \overline{m} \times \overline{B}.}$$



$$\boxed{\Delta \mathbf{m} = -\frac{1}{2}e(\Delta v)R\,\hat{\mathbf{z}} = -\frac{e^2R^2}{4m_e}\mathbf{B}.}$$

The Field of Magnetized Object

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int_{\mathcal{V}} \frac{\mathbf{J}_b(\mathbf{r}')}{\mathscr{r}}\,d\tau' + \frac{\mu_0}{4\pi}\oint_{\mathcal{S}} \frac{\mathbf{K}_b(\mathbf{r}')}{\mathscr{r}}\,da'$$

$$\mathbf{J}_b = \nabla \times \mathbf{M}, \qquad \mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}}$$

Potential of a volume current

Potential of a surface current

Potential of a magnetized object = Potential from a volume current $\boldsymbol{J_b}$ + Potential from a surface current $\boldsymbol{K_b}$ on boundary

Ampere's law in Magnetized Materials

The total current in a medium is the summation of free and bound currents

$$\vec{J} = \vec{J}_b + \vec{J}_f$$

Ampere's law can therefore, be written as

$$\frac{1}{\mu_0}(\vec{\nabla} \times \vec{B}) = \vec{J} = \vec{J}_f + \vec{J}_b = \vec{J}_f + (\vec{\nabla} \times \vec{M})$$

$$\Longrightarrow \vec{\nabla} \times \left(\frac{1}{\mu_0}\vec{B} - \vec{M}\right) = \vec{\nabla} \times \vec{H} = \vec{J}_f$$

where $\vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$ plays a role in magnetostatics analogous to $\vec{D} = \epsilon_0\vec{E} + \vec{P}$ in electrostatics.

Ampere's law in integral form for magnetised materials can therefore be written as

$$\oint \vec{H} \cdot d\vec{l} = I_{f_{enc}}$$

$$\oint \vec{D}.d\vec{a} = Q_{\mathrm{f_{enc}}}$$

Boundary Conditions

Using $\vec{\nabla} \cdot (\vec{H} + \vec{M}) = 0 \implies \oint (\vec{H} + \vec{M}) \cdot d\vec{a} = 0$

we have, $H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})$

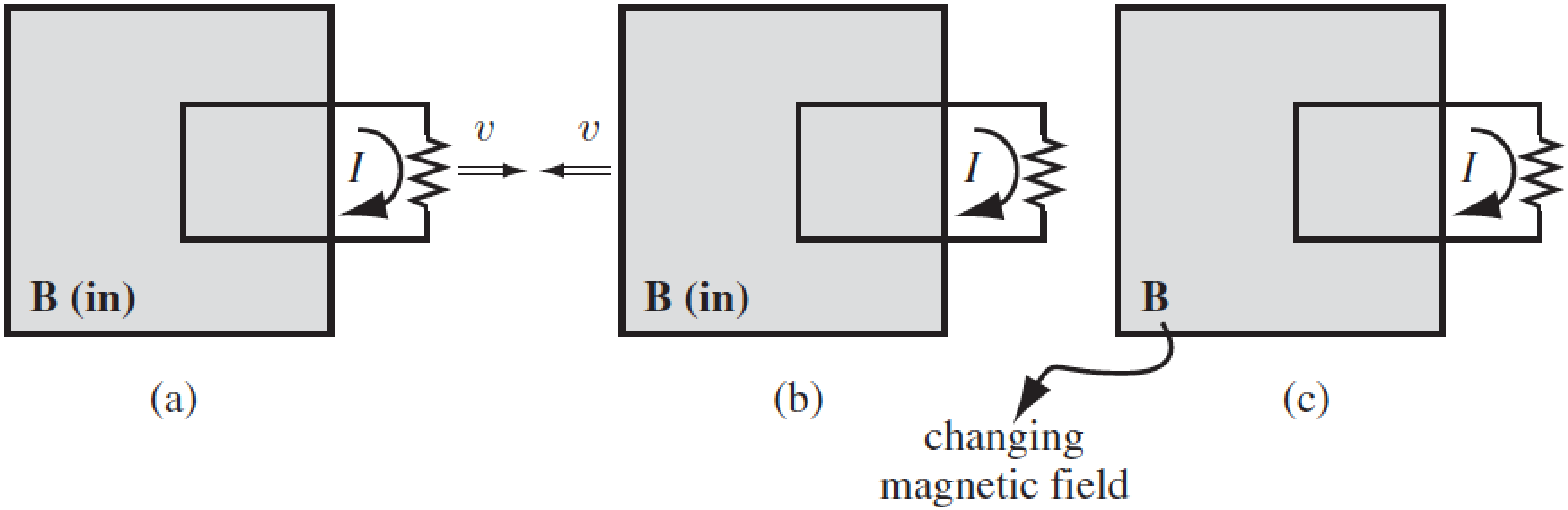
Using Ampere's law $\oint \vec{H} \cdot d\vec{l} = I_{f_{\text{enc}}}$ we have

$$\vec{H}^{||}_{\text{above}} - \vec{H}^{||}_{\text{below}} = \vec{K}_f \times \hat{n}$$

Which, in the absence of materials, become

$$B^{\perp}_{\text{above}} - B^{\perp}_{\text{below}} = 0 \qquad \vec{B}^{||}_{\text{above}} - \vec{B}^{||}_{\text{below}} = \mu_0(\vec{K}_f \times \hat{n})$$

Faraday's Law



(a) Case of motional emf;

(b) 2nd Experiment has the same emf, relative motion of loop and the magnet.

(c) 3rd scenario also an electric field gets generated and gives the same emf

$$\mathcal{E} = -\frac{d\Phi}{dt}$$

Changing magnetic field generates an electric field

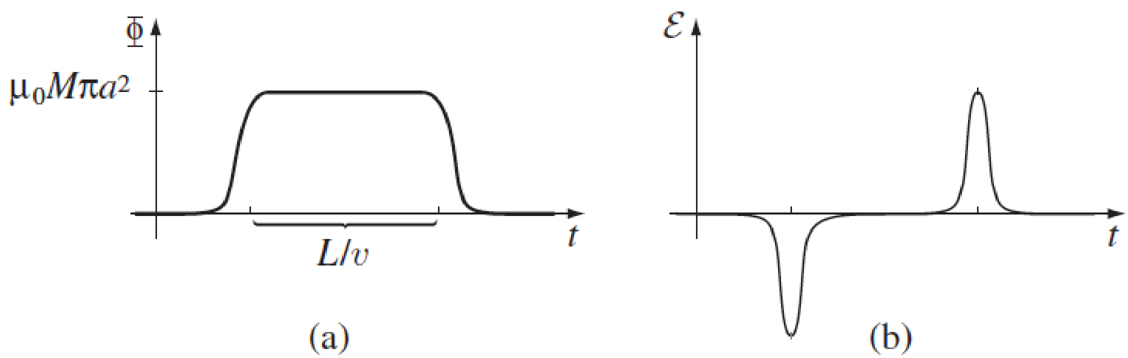
Note: In (a), $\mathbf{v} \times \mathbf{B}$ drives the current $\boldsymbol{I}$ not $\boldsymbol{E}$

In (b) & (c), induced $\boldsymbol{E}$ drives the current $\boldsymbol{I}$

Lenz's Law

The induced current flows in such a direction that the flux it produces tends to cancel the change. We can not quantify the current but can get the directions right.

Nature abhors a change in flux.

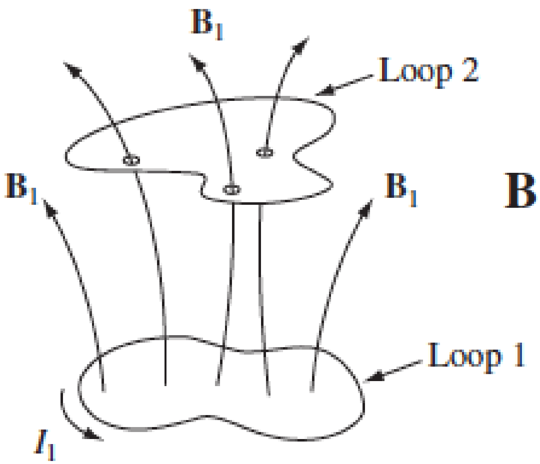


The magnet enters the ring, flux increases. The current is clockwise to generate field to the right.

Change in the flux is prevented, not flux

The magnet exits ring, flux drops, counterclockwise current to restore the field.

Inductance



$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

$\mathbf{B}_1$ is proportional to $\mathbf{I}_1$

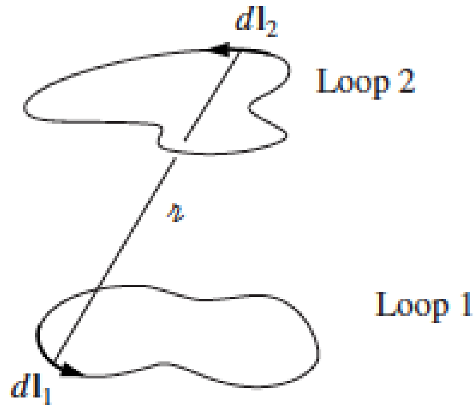
hence flux through loop2

is also proportional to $\mathbf{I}_1$

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2.$$

$$\Phi_2 = M_{21} I_1,$$

here $\mathbf{M_{21}}$ is the Mutual Inductance



$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\nabla \times \mathbf{A}_1) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2.$$

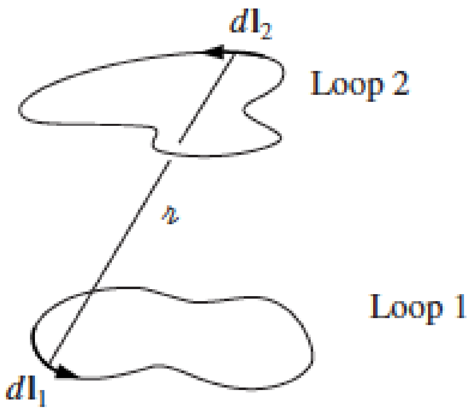
$$= \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\mathbf{l}_1}{\mathscr{r}} \right) \cdot d\mathbf{l}_2.$$

$$\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\mathscr{r}},$$

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}.$$

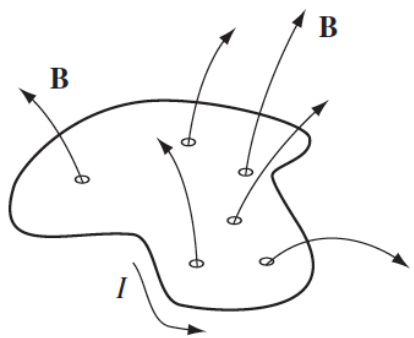
Neumann formula
for mutual inductance

Inductance



emf in loop2 due to change of current in loop1

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}.$$



Changing current will induce an emf in the source loop as well

$$\Phi = LI.$$

self inductance L also depends on the geometry of the loop!

unit is henries (H), volt-second per ampere

$$\mathcal{E} = -L\frac{dI}{dt}.$$

Back emf, Lenz's Law

Energy in Magnetic field

$$W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 \, d\tau .$$

In view of this result, we say the energy is "stored in the magnetic field," in the amount ($B_2/2\mu_0$) per unit volume.

In the light of this, it is extraordinary how similar the magnetic energy formulas are to their electrostatic counterparts:

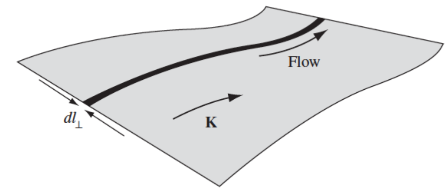
$$W_{\text{elec}} = \frac{1}{2} \int (V\rho) \, d\tau = \frac{\epsilon_0}{2} \int E^2 \, d\tau ,$$

$$W_{\text{mag}} = \frac{1}{2} \int (\mathbf{A} \cdot \mathbf{J}) \, d\tau = \frac{1}{2\mu_0} \int B^2 \, d\tau .$$

Summary: Formulae

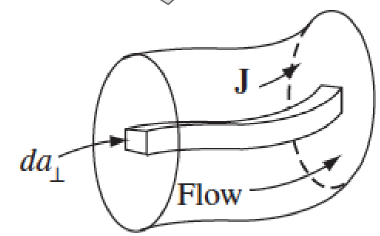
Lorentz force law:

$$\mathbf{F} = Q[\mathbf{E} + (\mathbf{v} \times \mathbf{B})].$$



Current:

$$\mathbf{I} = \lambda \mathbf{v}. \qquad \mathbf{K} = \sigma \mathbf{v}. \qquad \mathbf{J} = \rho \mathbf{v}.$$



Continuity Equation:

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = 0.$$

The Biot-Savart Law :

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.$$

surface currents: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, da'$; volume currents: $\mathbf{B}(\mathbf{r}) = \dfrac{\mu_0}{4\pi} \displaystyle\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \, d\tau'$

Ampere's law in integral and in Differential form

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$$

Absence of Magnetic Monopole:

$$\nabla \cdot \mathbf{B} = 0$$

Summary: Formulae

Magnetic Vector Potential

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

$$\vec{\nabla} \cdot \vec{A} = 0 \qquad \nabla^2 \vec{A} = -\mu_0 \vec{J} \qquad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r'})}{\mathfrak{r}} d\tau'$$

Magnetostatic Boundary Condition

$$\vec{B}_{\text{above}} - \vec{B}_{\text{below}} = \mu_0 (\vec{K} \times \hat{n})$$

Magnetic Dipole

$$\vec{m} = \frac{1}{2} \oint I(\vec{r} \times d\vec{l}) = \frac{1}{2} \int (\vec{r} \times \vec{J}) d\tau$$

$$\vec{A}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}, \qquad \vec{B}_{\text{dipole}}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} [3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}]$$

Force & Torque on Magnetic Dipole

$$\vec{F} = \vec{\nabla}(\vec{m} \cdot \vec{B}) \qquad \vec{N} = \vec{m} \times \vec{B}$$

Summary: Formulae

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{M}(\vec{r'}) \times \hat{\mathfrak{r}}}{\mathfrak{r}^2} d\tau'$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J}_b(\vec{r'})}{\mathfrak{r}} d\tau' + \frac{\mu_0}{4\pi} \oint_S \frac{\vec{K}_b(\vec{r'})}{\mathfrak{r}} da'$$

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \qquad \vec{K}_b = \vec{M} \times \hat{n}$$

$$\vec{\nabla} \times \left(\frac{1}{\mu_0} \vec{B} - \vec{M} \right) = \vec{\nabla} \times \vec{H} = \vec{J}_f$$

$$H^{\perp}_{\text{above}} - H^{\perp}_{\text{below}} = -(M^{\perp}_{\text{above}} - M^{\perp}_{\text{below}})$$

$$\vec{H}^{\parallel}_{\text{above}} - \vec{H}^{\parallel}_{\text{below}} = \vec{K}_f \times \hat{n}$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) = \mu_0(1 + \chi_m)\vec{H}$$

$$\mu = \mu_0(1 + \chi_m) \qquad \vec{J}_b = \vec{\nabla} \times \vec{M} = \vec{\nabla} \times (\chi_m \vec{H}) = \chi_m \vec{J}_f$$

Summary: Formulae

The analog to Bio-savart's law

$$\mathbf{E} = -\frac{1}{4\pi}\int \frac{(\partial \mathbf{B}/\partial t) \times \hat{\boldsymbol{\imath}}}{\imath^2}\, d\tau = -\frac{1}{4\pi}\frac{\partial}{\partial t}\int \frac{\mathbf{B} \times \hat{\boldsymbol{\imath}}}{\imath^2}\, d\tau,$$

If symmetry permits,.

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}.$$

Neumann formula for mutual inductance

$$M_{21} = \frac{\mu_0}{4\pi}\oint\oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\imath}.$$

Work against the *back emf* to get the current going. It is fixed and *recoverable!*

$$W = \frac{1}{2}LI^2.$$

(only geometry of the loop 'L' and final current I)

Energy is "stored in the magnetic field," in the amount ($B_2/2\mu_o$) per unit volume.

$$W_{\text{mag}} = \frac{1}{2}\int (\mathbf{A} \cdot \mathbf{J})\, d\tau = \frac{1}{2\mu_0}\int B^2\, d\tau.$$