# Physics II

## Electromagnetism (Lecture 14)

Udit Raha

Indian Institute of Technology Guwahati

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# Macroscopic Electric Fields In Dielectrics

- In vacuum, the TRUE Electric field $\mathbf{E}_{\mathrm{True}} \equiv \mathbf{E}_{\mathrm{vac}}$ is unambiguously calculated which in general has contributions both from distant free as well as bound charge distributions $\rho_{\mathrm{tot}} = \rho_f + \rho_b$.
- Within matter, the MICROSCOPIC Background Electric field $\mathbf{E}_{\mathrm{Micr}}$, due to ALL "elementary" charges (e.g., electrons, ions, nuclei, ...), is utterly complicated if not impossible to calculate. The net in-medium field is

$$\mathbf{E}_{\mathrm{True}} = \mathbf{E}_{\mathrm{vac}} + \mathbf{E}_{\mathrm{Micr}}$$

- Then it becomes crucial to define a realistic MACROSCOPIC Field:

### Definition

MACROSCOPIC Electric field: *It is defined as the space average field over an arbitrary macroscopic volume $\mathcal{V}$ of matter which is large enough to contain a statistically large number ($\gtrsim 10^4 - 10^5$) of atoms or molecules of that material, yet small enough compared to the dimensions of the material sample, in order to preserve all significant large-scale spatial variations in the field, i.e.,*

$$\boldsymbol{\mathcal{E}}(\mathbf{r}) \equiv \langle \mathbf{E}_{\mathrm{True}}(\mathbf{r}) \rangle_{\mathcal{V}} = \frac{1}{\mathcal{V}} \iiint_{\mathcal{V}} \mathbf{E}_{\mathrm{True}}(\mathbf{r}' - \mathbf{r}) dv',$$

*where, for convenience, the integral is defined over a spherical region $\mathcal{V}$.*

## Macroscopic Electric Fields In Dielectrics (contd.)

The entire dielectric medium can be thought of being composed of sufficiently finely grained spherical Averaging volumes (like, close-packing of marbles), such that each spherical volume contains a statistically large number of atoms or molecules.

## Macroscopic Fields and Potential In Dielectrics

- We henceforth work with Macroscopic Electric field $\boldsymbol{\mathcal{E}}(\mathbf{r}) \equiv \langle \mathbf{E}_{\mathrm{True}}(\mathbf{r}) \rangle_{\mathcal{V}}$, in dielectrics, which is a conservative field derivable from a coresponding Macroscopic Potential $\mathscr{V}(\mathbf{r}) = \langle V_{\mathrm{True}}(\mathbf{r}) \rangle_{\mathcal{V}}$, such that

$$\nabla \times \boldsymbol{\mathcal{E}}(\mathbf{r}) = \nabla \times [-\nabla \mathscr{V}(\mathbf{r})] = 0 \qquad \& \qquad \oint_{\mathrm{Loop}} \boldsymbol{\mathcal{E}}(\mathbf{r}) \cdot d\mathbf{r} = 0.$$

- **Modified Gauss's Differential Law:**

$$\begin{aligned} \epsilon_0 \nabla \cdot \boldsymbol{\mathcal{E}}(\mathbf{r}) &= \rho_{\mathrm{tot}}(\mathbf{r}) = \rho_b(\mathbf{r}) + \rho_f(\mathbf{r}) \\ &= -\nabla \cdot \mathbf{P}(\mathbf{r}) + \rho_f(\mathbf{r}) \\ \nabla \cdot (\epsilon_0 \boldsymbol{\mathcal{E}} + \mathbf{P}) &= \rho_f \\ \Longrightarrow \quad \nabla \cdot \mathbf{D} &= \rho_f \end{aligned}$$

- The field $\mathbf{D} \equiv \epsilon_0 \boldsymbol{\mathcal{E}} + \mathbf{P}$ is termed as the ELECTRIC DISPLACEMENT.
- **Modified Gauss's Integral Law:**

$$\oiint_S \mathbf{D} \cdot d\mathbf{S} = \iiint_{\mathcal{V}} \nabla \cdot \mathbf{D}\, dv = \iiint_{\mathcal{V}} \rho_f \, dv = Q_{f,\,\mathrm{encl}}$$

where $S$ is an arbitrary closed surface bounding a region of dielectric $\mathcal{V}$ with total enclosed free charge $Q_{f,\,\mathrm{encl}}$.

# Modified Gauss's Law in Dielectrics: Summary

$$\begin{aligned}
\nabla \cdot \mathbf{D} &= \rho_f \\
\nabla \cdot \boldsymbol{\mathcal{E}} &= \frac{\rho_{tot}}{\epsilon_0} = \frac{\rho_b + \rho_f}{\epsilon_0} \\
\oiint_{\text{surface}} \mathbf{D} \cdot d\mathbf{S} &= Q_{f,\,\text{encl}} \\
\oiint_{\text{surface}} \boldsymbol{\mathcal{E}} \cdot d\mathbf{S} &= \frac{1}{\epsilon_0} Q_{tot,\,encl} = \frac{1}{\epsilon_0} \left( Q_b + Q_f \right)_{\text{encl}} \\
\text{The Constitutive Relation:} \quad \mathbf{D} &= \epsilon_0 \boldsymbol{\mathcal{E}} + \mathbf{P}
\end{aligned}$$

**Warning!**

- Henceforth, we revert back to using the old symbol $\mathbf{E} \longleftrightarrow \boldsymbol{\mathcal{E}}$ for the Macroscopic Electric Field keeping in mind that it is NOT the same as the True Electric Field $\mathbf{E}_{\text{True}}$ within a dielectric which in general includes the Microscopic Background Field $\mathbf{E}_{\text{Micr}}$.
- For free space (vacuum), they are equivalent, i.e., $\boldsymbol{\mathcal{E}} \equiv \mathbf{E}_{\text{True}} \Rightarrow \mathbf{E}$.

# Polarized Sphere

## Example

Consider an uncharged dielectric sphere with a "frozen-in" Polarization $\mathbf{P} = \frac{k}{r}\hat{\mathbf{r}}$, where $k$ is a constant. Find the Electric field as a function of $r$.

- **Method I:** The bound volume & surface charge densities:

$$\rho_b(r) = -\nabla \cdot \left(\frac{k}{r}\hat{\mathbf{r}}\right) = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{k}{r}\right) = -\frac{k}{r^2} \quad ; \quad \sigma_b = (\mathbf{P}\cdot\hat{\mathbf{r}})_{r=R} = \frac{k}{R}$$

- Total bound charge:

$$\begin{aligned} Q_b &= Q_b^{(\text{volume})} + Q_b^{(\text{surface})} = \iiint\limits_{\mathcal{V}} \rho_b(r) dv' + \oint\!\!\oint_S \sigma_b da' \\ &= \int_0^R \left(-\frac{k}{r'^2}\right) 4\pi r'^2 dr' + \oint\!\!\oint \frac{k}{R} da' = -4\pi kR + 4\pi kR = 0 \end{aligned}$$

- **Field outside sphere** $(r \geq R)$: Since $(Q_f + Q_b)_{encl} = 0$, then applying Gauss's Integral Law for $\mathbf{E}$:

$$\begin{aligned} \oint\!\!\oint_{\mathbb{S}(r\geq R)} \mathbf{E}\cdot d\mathbf{a}' &= \frac{1}{\epsilon_0} Q_{tot,\, encl}(r) = 0 \\ \mathbf{E}_{r\geq R}(\mathbf{r}) &= 0 \end{aligned}$$

# Polarized Sphere (contd.)

- **Field inside sphere** ($r < R$): $\sigma_b$ does not contribute in the bulk, then using Gauss's Integral Law for **E**:

$$\oint\!\!\oint_{s(r<R)} \mathbf{E}\cdot d\mathbf{a}' = \frac{1}{\epsilon_0} Q_{tot,\ encl}(r) = \frac{1}{\epsilon_0}\iiint_V \rho_b(r)dv' = \frac{1}{\epsilon_0}\iiint_V \left(-\frac{k}{r'^2}\right) dv'$$

$$E(4\pi r^2) = \frac{1}{\epsilon_0}\int_0^r \left(-\frac{k}{r'^2}\right) 4\pi r'^2 dr' \implies \mathbf{E}(\mathbf{r}) = -\frac{k}{\epsilon_0 r}\hat{\mathbf{r}}$$

- **Method II**: Since $Q_{f,\ encl} = 0$, then applying Modified Gauss's Law for **D**:

$$\oint\!\!\oint_{\mathbb{S}} \mathbf{D}\cdot d\mathbf{a}' = Q_{f,\ encl} = 0 \implies \mathbf{D} = 0\,, \quad \forall r \text{ (everywhere)}$$

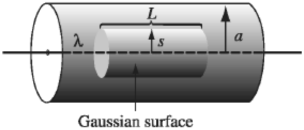
$$\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = 0 \implies \mathbf{E}(\mathbf{r}) = -\frac{\mathbf{P}}{\epsilon_0} = \begin{cases} 0 & \text{if} \quad r > R \\ -\frac{k}{\epsilon_0 r}\hat{\mathbf{r}} & \text{if} \quad r \le R \end{cases} .$$

- Notice: Method II is much quicker without reference to bound charges!

# Long Cylindrical Wire

## Example

A long straight wire, carrying uniform line charge density $\lambda$, is surrounded by rubber insulation out to radius $a$. Find the Electric Displacements and Electric fields everywhere.



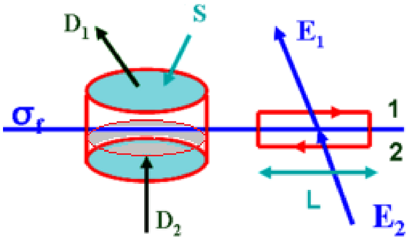

- Construct a coaxial cylindrical Gaussian surface $S$ of radius $s$ and length $L$:

$$\begin{aligned}
\oiint_S \mathbf{D} \cdot d\mathbf{S} &= Q_{f;\,\mathrm{encl}} = \lambda L \\
D(2\pi s L) &= \lambda L \\
\mathbf{D}(\mathbf{s}) &= \left(\frac{\lambda}{2\pi s}\right)\hat{\mathbf{s}}\,, \quad \forall s \text{ (everywhere)}.
\end{aligned}$$

- Note: This formula is applicable both inside and outside the cladding.
- **Electric Field inside the cladding** ($s \le a$): Polarization $\mathbf{P}$ as well as the dielectric constant being unknown, $\mathbf{E}$ can not be calculated.
- **Electric Field outside the cladding** ($s > a$): Since Polarization $\mathbf{P} = 0$, so

$$\mathbf{D} \equiv \epsilon_0 \boldsymbol{E} + \cancelto{0}{\boldsymbol{P}} \quad \Longrightarrow \quad \mathbf{E}(\mathbf{s}) = \frac{1}{\epsilon_0}\mathbf{D}(\mathbf{s}) = \left(\frac{\lambda}{2\pi\epsilon_0 s}\right)\hat{\mathbf{s}}.$$

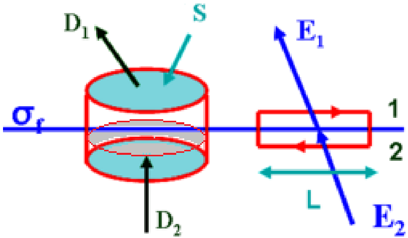
## Boundary Conditions in Dielectrics




- Consider an interface of two dielectrics media (1 & 2) with total surface charge density $\sigma_{tot} = \sigma_f + \sigma_b$ at the interface and total volume charge densities $\rho_{tot,1} = \rho_{f1} + \rho_{b1}$ and $\rho_{tot,2} = \rho_{f2} + \rho_{b2}$, in the respectively bulks.
- Consider a pillbox-shaped Gaussian surface enclosing area $S$ at the interface, with negligibly small width, $\epsilon \to 0$ in comparison with the base diameters.
- Total enclosed **free** and **bound** charges within the Gaussian surface:

$$
\begin{aligned}
Q_{f,\,encl} &= \sigma_f S + \frac{\epsilon}{2}(\rho_{f1} + \rho_{f2})S \xrightarrow{\epsilon \to 0} \sigma_f S, \\
Q_{b,\,encl} &= \sigma_b S + \frac{\epsilon}{2}(\rho_{b1} + \rho_{b2})S \xrightarrow{\epsilon \to 0} \sigma_b S.
\end{aligned}
$$

# Boundary Conditions in Dielectrics




- Applying Gauss's Law for **D**:

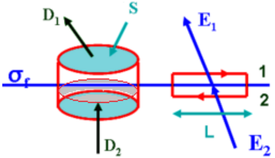
$$\lim_{\epsilon \to 0} \oiint_S \mathbf{D} \cdot d\mathbf{S} = (\mathbf{D}_1 \cdot \hat{\mathbf{n}}_1 + \mathbf{D}_2 \cdot \hat{\mathbf{n}}_2)\, S = \lim_{\epsilon \to 0} Q_{f,\, encl} = \sigma_f\, S$$

$$(\mathbf{D}_1 - \mathbf{D}_2) \cdot \hat{\mathbf{n}}_1 = D_{1\perp} - D_{2\perp} = \sigma_f .$$

- Applying Gauss's Law for **E**:

$$\lim_{\epsilon \to 0} \oiint_S \mathbf{E} \cdot d\mathbf{S} = (\mathbf{E}_1 \cdot \hat{\mathbf{n}}_1 + \mathbf{E}_2 \cdot \hat{\mathbf{n}}_2)\, S = \frac{1}{\epsilon_0} \lim_{\epsilon \to 0} Q_{tot,\, encl} = \frac{1}{\epsilon_0}(\sigma_f + \sigma_b)\, S$$

$$(\mathbf{E}_1 - \mathbf{E}_2) \cdot \hat{\mathbf{n}}_1 = E_{1\perp} - E_{2\perp} = \frac{1}{\epsilon_0}(\sigma_f + \sigma_b).$$




- Macroscopic Electric field being conservative in nature, the circulation of $\mathbf{E}$ around any closed loop must vanish. Choosing a narrow rectangular loop of length $L$ and vanishing end widths $\epsilon \to 0$ straddling across the interface,

$$\begin{aligned} \lim_{\epsilon\to 0} \oint_{\text{Loop}} \mathbf{E}\cdot d\mathbf{l} = \mathbf{E}_1\cdot\mathbf{L} + \mathbf{E}_2\cdot(-\mathbf{L}) &= 0 \\ (\mathbf{E}_1 - \mathbf{E}_2)\cdot\hat{\mathbf{n}}_{1\,||} &= 0 \\ E_{1||} &= E_{2||} \end{aligned}$$

- Since $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$, it follows that $D_{||} = \epsilon_0 E_{||} + P_{||}$ and consequently

$$D_{1||} - D_{2||} = P_{1||} - P_{2||}.$$

- Similarly, from $D_\perp = \epsilon_0 E_\perp + P_\perp$

$$\begin{aligned} D_{1\perp} - D_{2\perp} &= \epsilon_0(E_{1\perp} - E_{2\perp}) + (P_{1\perp} - P_{2\perp}) \\ \sigma_f &= (\sigma_f + \sigma_b) + (P_{1\perp} - P_{2\perp}) \\ P_{1\perp} - P_{2\perp} &= -\sigma_b. \end{aligned}$$

# Boundary Conditions in Dielectrics: SUMMARY

$$
\begin{aligned}
D_{1\perp} - D_{2\perp} &= \sigma_f, \\
D_{1||} - D_{2||} &= P_{1||} - P_{2||}, \\
P_{1\perp} - P_{2\perp} &= -\sigma_b, \\
E_{1\perp} - E_{2\perp} &= \frac{1}{\epsilon_0}(\sigma_f + \sigma_b), \\
E_{1||} &= E_{2||}, \\
V_1 &= V_2, \\
\frac{\partial V_1}{\partial n} - \frac{\partial V_2}{\partial n} &= \frac{1}{\epsilon_0}(\sigma_f + \sigma_b).
\end{aligned}
$$