

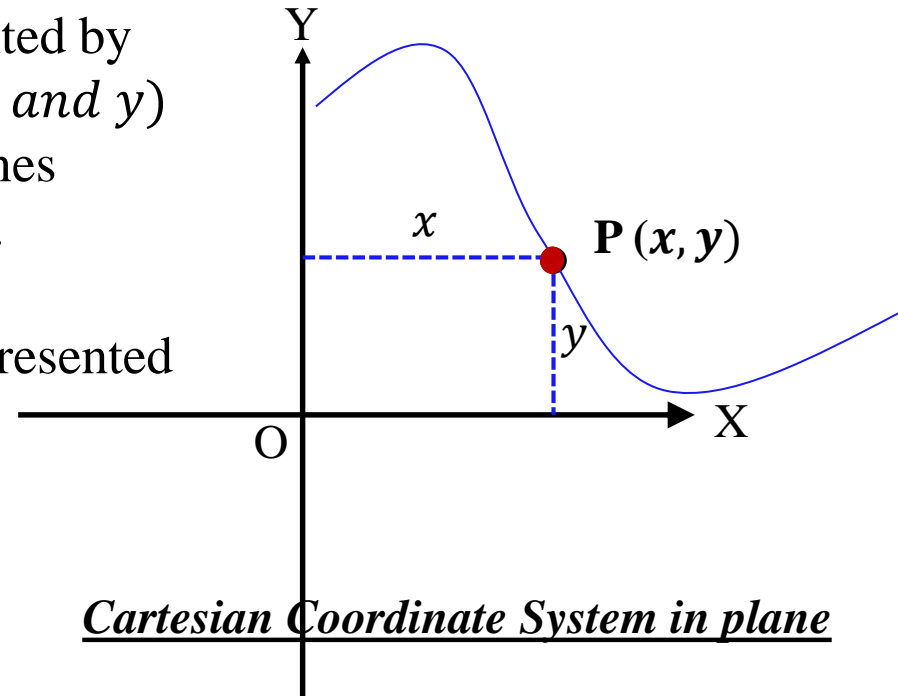
PH101

Lecture 2

Coordinate systems

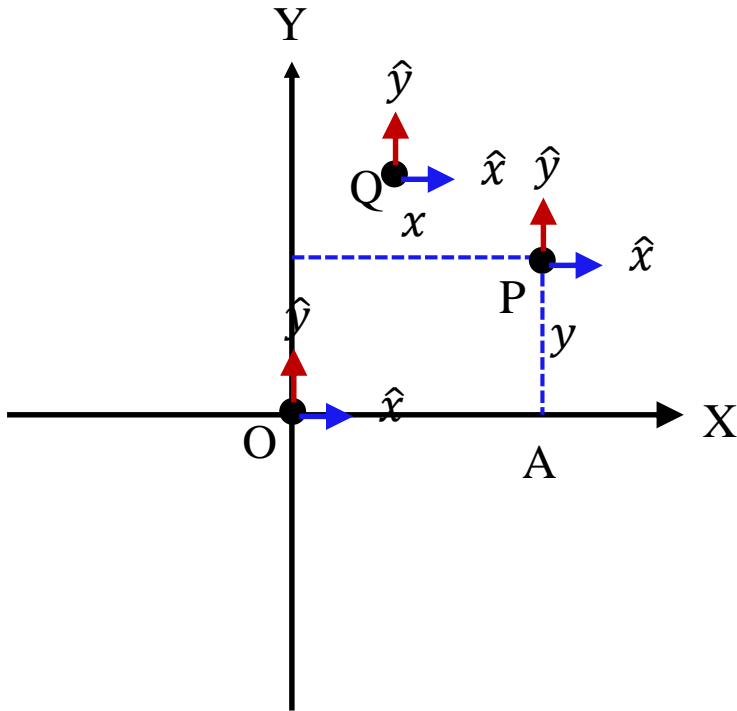
Cartesian coordinate System in plane

- Instantaneous position P can be represented by measuring its perpendicular distances (x and y) from two perpendicularly intersecting lines (called Axes) passing through the origin.
- In Cartesian coordinate position P is represented by (x, y) .



All points in the plane are referred with respect to same set of axes X and Y .

Unit vectors in plane Cartesian



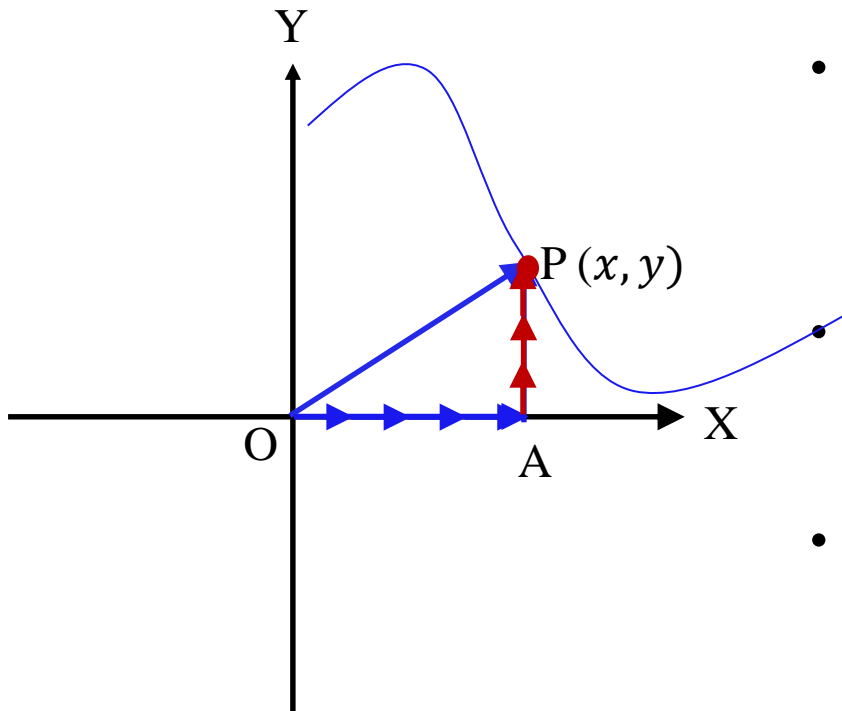
- In Cartesian coordinate system for plane, one can define two unit vectors \hat{x} and \hat{y} at every point in the plane.
- \hat{x} and \hat{y} are unit vectors in the increasing direction of x and y , thus they are parallel to coordinate axes X and Y respectively.
- \hat{x} and \hat{y} are orthogonal, and directed in the same direction at every points.

Another way of looking unit vector Cartesian coordinate in plane

\hat{x} is the unit vector perpendicular to $x = \text{constant}$ line (surface)

\hat{y} is the unit vector perpendicular to $y = \text{constant}$ line (surface)

Position vector in terms of Cartesian components



- Position P can be represented either by vector \vec{r} (\overrightarrow{OP}) or through Cartesian coordinate (x, y) .

- From vector addition rule
$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

- $\overrightarrow{OA} = (\text{magnitude of OA}) (\text{unit vector along/parallel to X-direction}) = x \hat{x}$
- $\overrightarrow{AP} = (\text{magnitude of AP}) (\text{unit vector along/parallel to Y-direction}) = y \hat{y}$

Thus $\overrightarrow{OP} = \vec{r} = x \hat{x} + y \hat{y}$

We may also use the notation:

$$\hat{x} = \hat{i}$$

$$\hat{y} = \hat{j}$$

Velocity and acceleration in Cartesian

Velocity $\vec{v} = \frac{d\vec{r}}{dt}$

Velocity in Cartesian coordinate

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt} (x \hat{x} + y \hat{y})$$

$$= \dot{x} \hat{x} + x \frac{d\hat{x}}{dt} + \dot{y} \hat{y} + y \frac{d\hat{y}}{dt}$$
$$\vec{v} = \dot{x} \hat{x} + \dot{y} \hat{y}$$

Since,

$$\frac{d\hat{x}}{dt} = \frac{d\hat{y}}{dt} = 0$$

Acceleration $\vec{a} = \frac{d\vec{v}}{dt} = \ddot{x} \hat{x} + \ddot{y} \hat{y}$

Newton's second law in vector form

$$\vec{F} = F_x \hat{x} + F_y \hat{y} = m \frac{d\vec{v}}{dt} = m(\ddot{x} \hat{x} + \ddot{y} \hat{y})$$

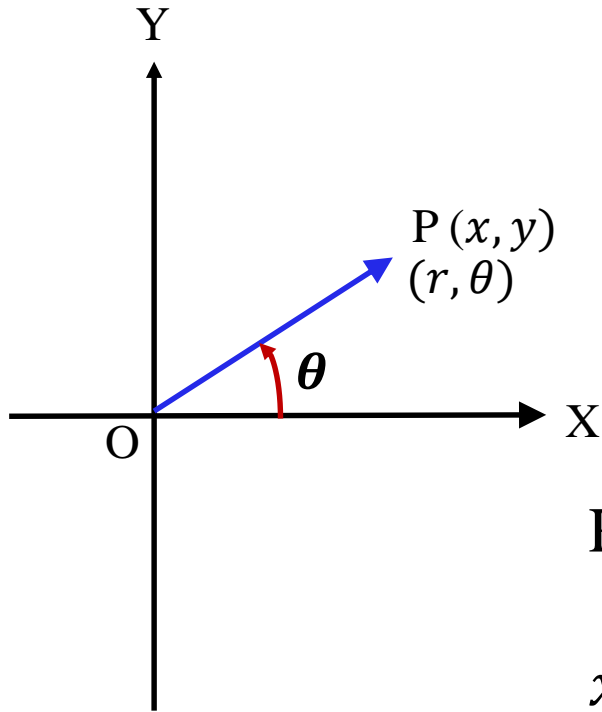
Standard Notations:

$$\frac{dx}{dt} = \dot{x}$$

$$\frac{dr}{dt} = \dot{r}$$

$$\frac{d^2\theta}{dt^2} = \ddot{\theta}$$

Plane polar coordinate



Each point P (x, y) on the plane can also be represented by its distance (r) from the origin O and the angle (θ) OP makes with X-axis.

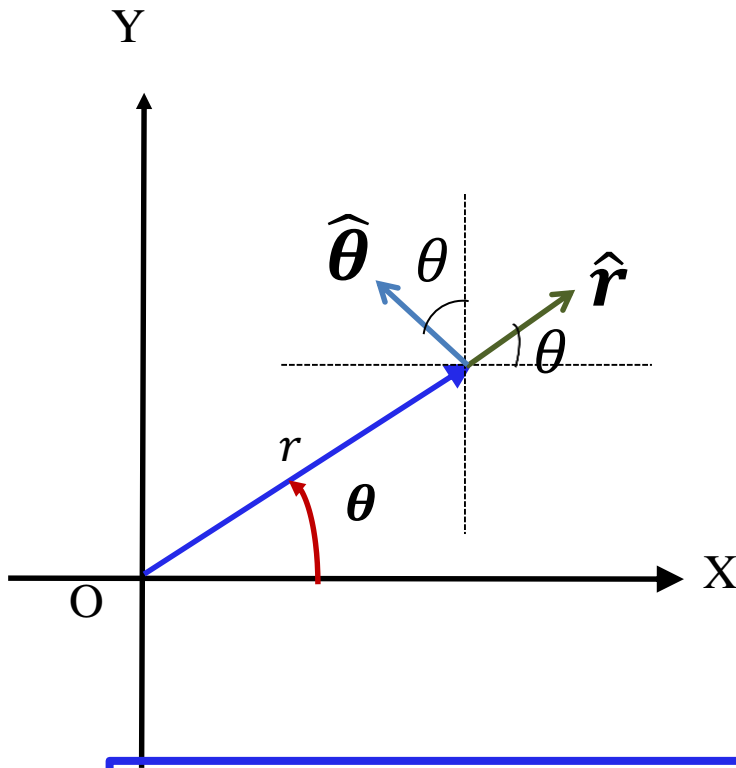
Relationship with Cartesian coordinates

$$x = r \cos \theta \quad \& \quad y = r \sin \theta$$

Thus ,

$$r = (x^2 + y^2)^{1/2}$$
$$\theta = \tan^{-1} \frac{y}{x}$$

Unit vector in plane polar coordinate



□ For plane polar unit vectors:

$$\hat{r} \text{ and } \hat{\theta}$$

associated to **each point** in the plane.

□ \hat{r} and $\hat{\theta}$ are unit vector along increasing direction of coordinate r and θ .

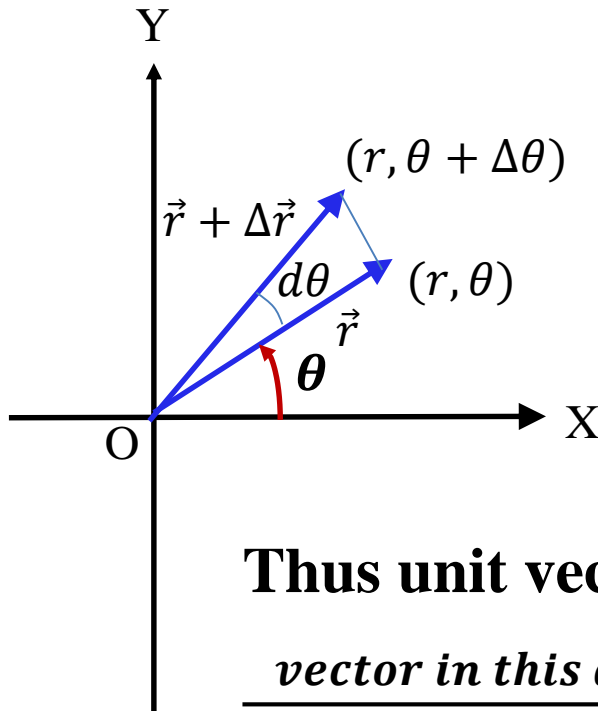
$$\hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y}$$

$$\hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y}$$

\hat{r} and $\hat{\theta}$ are **orthogonal** $\hat{r} \cdot \hat{\theta} = 0$

Their directions depend on location.

Unit vector in plane polar coordinate



In the limit $\Delta\theta \rightarrow 0$, keeping r constant, $\frac{\Delta\vec{r}}{\Delta\theta} (\text{I.e. } \frac{\partial\vec{r}}{\partial\theta})$ gives a vector in the increasing direction of ' θ '

Thus unit vector in the θ direction is $\hat{\theta} =$

$$\frac{\text{vector in this direction}}{\text{magnitude of that vector}} = \frac{\frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\theta}\right|}$$

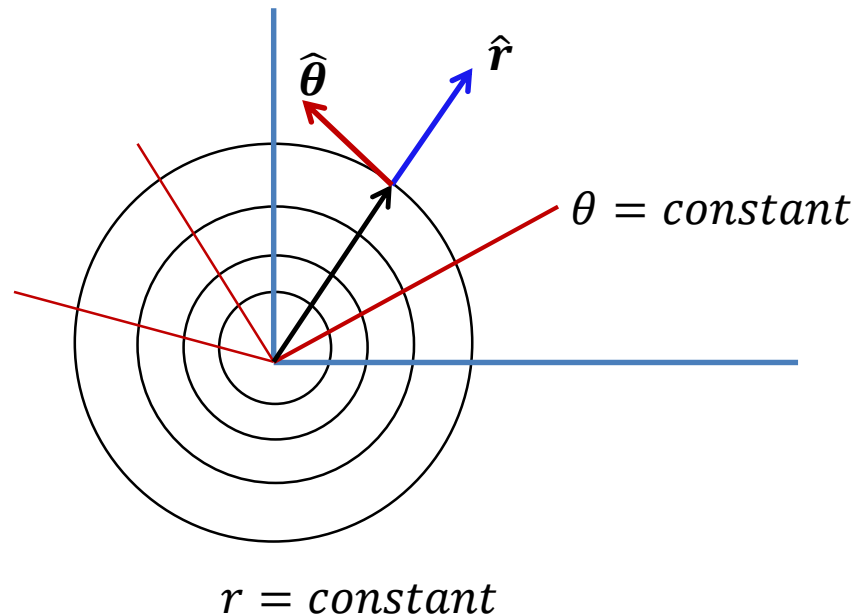
Similarly $\hat{r} = \frac{\text{vector in this direction}}{\text{magnitude of that vector}} = \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|}$

Unit vector in plane polar coordinate

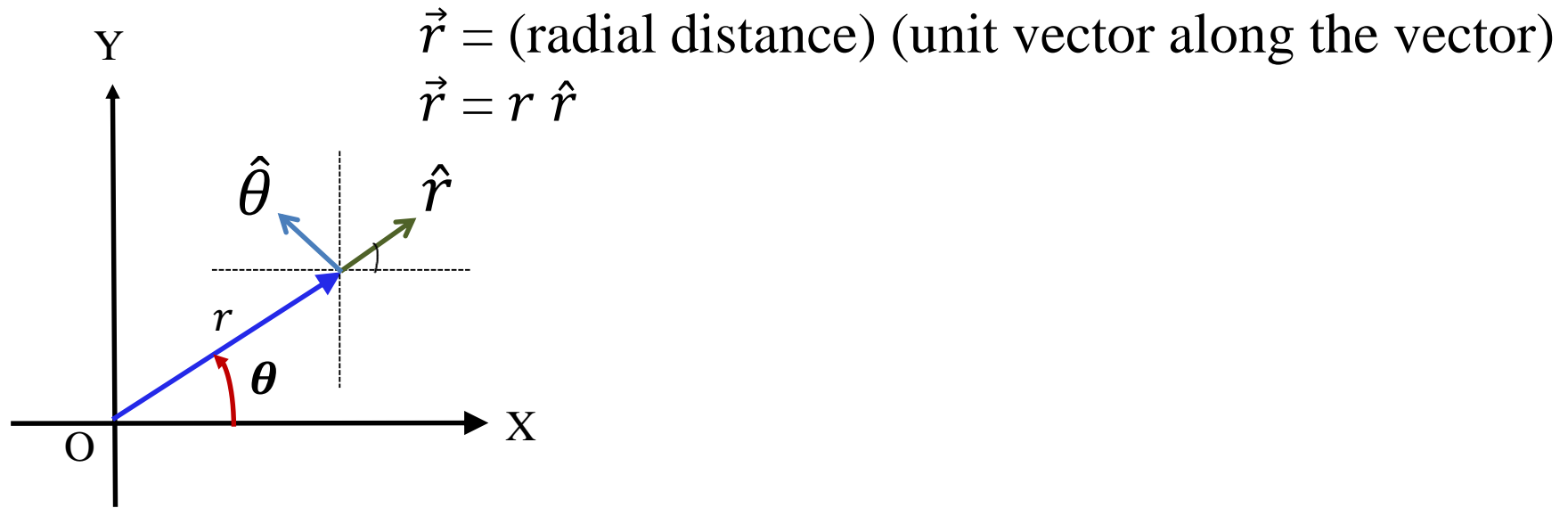
The unit vectors in the polar coordinate can also be viewed in another way.

\hat{r} is the unit vector perpendicular to $r = \text{constant}$ surface and points in the increasing direction of r .

Similarly, $\hat{\theta}$ is the unit vector perpendicular to $\theta = \text{constant}$ surface (i.e. tangential to $r = \text{constant}$) and points in the increasing direction of θ .



Unit vector in plane polar coordinate



Unit vectors in polar coordinate are function of θ only.

$$\frac{\partial \hat{r}}{\partial \theta} = \frac{\partial}{\partial \theta} (\hat{x} \cos \theta + \hat{y} \sin \theta) = -\hat{x} \sin \theta + \hat{y} \cos \theta = \hat{\theta}$$

$$\frac{\partial \hat{\theta}}{\partial \theta} = \frac{\partial}{\partial \theta} (-\hat{x} \sin \theta + \hat{y} \cos \theta) = -(\hat{x} \cos \theta + \hat{y} \sin \theta) = -\hat{r}$$

Velocity in plane polar coordinate

$$\text{Velocity } \vec{v} = \frac{d\vec{r}}{dt}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt} (r \hat{r}) = \frac{dr}{dt} \hat{r} + r \frac{d\hat{r}}{dt}$$

$$= \dot{r} \hat{r} + r \frac{\partial \hat{r}}{\partial \theta} \frac{d\theta}{dt}$$

Since,

$$\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$$

$$\boxed{\vec{v} = \dot{r} \hat{r} + r \dot{\theta} \hat{\theta}}$$

Radial component \dot{r} and
tangential/transverse component $r\dot{\theta}$

Acceleration in plane polar coordinate

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Note: $\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$ & $\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$

$$= \frac{d}{dt} (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \frac{dr}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\frac{\partial \hat{r}}{\partial \theta}\frac{d\theta}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{\partial \hat{\theta}}{\partial \theta}\frac{d\theta}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

Radial component of acceleration: $\ddot{r} - r\dot{\theta}^2$

(Note: $-r\dot{\theta}^2$ is the familiar *Centripetal* contribution!)

Tangential component: $2\dot{r}\dot{\theta} + r\ddot{\theta}$

(Note: $2\dot{r}\dot{\theta}$ is called the *Coriolis* contribution!)

Newton's law in plane polar coordinate

$$\vec{F} = m \frac{d\vec{v}}{dt}$$

$$\vec{F} = F_r \hat{r} + F_\theta \hat{\theta} = m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}]$$

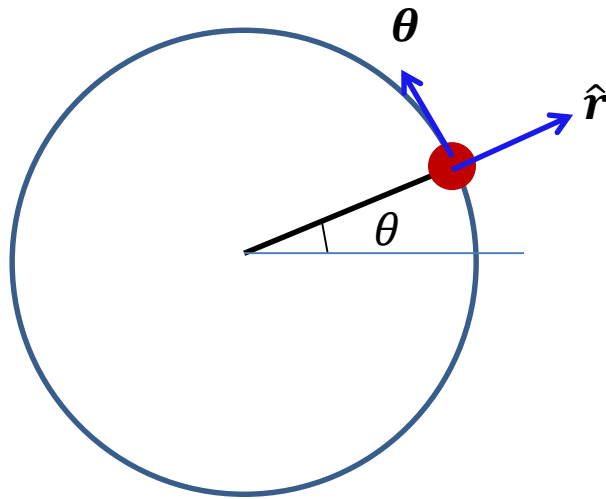
Newton's law for **radial** direction: $F_r = m(\ddot{r} - r\dot{\theta}^2)$

Newton's law for **tangential** direction: $F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$

Note: Newton's law in polar coordinates does not follow its **Cartesian form** as,

$$F_r \neq m\ddot{r} \quad \text{or} \quad F_\theta \neq m\ddot{\theta}$$

Choice of proper coordinate system makes analysis easier



Motion in circular trajectory

Equation of trajectory in polar coordinate

$$r = R = \text{constat}$$

$$\theta = \omega t + \frac{1}{2}\alpha t^2$$

The velocity components are

$$v_r = \dot{r} = 0; v_\theta = r\dot{\theta} = R(\omega + \alpha t) = v$$

Acceleration components are

$$a_r = \ddot{r} - r\dot{\theta}^2 = -R(\omega + \alpha t)^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = R\alpha = a_t$$

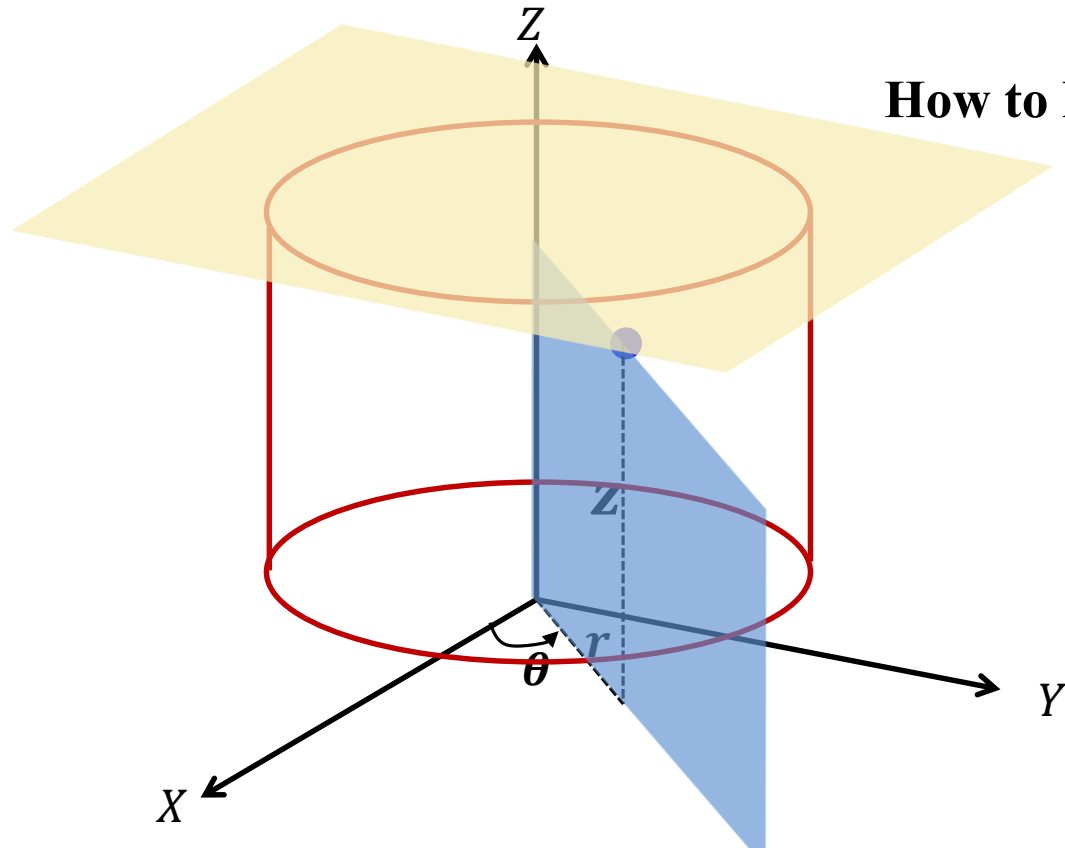
Equation of trajectory in cartesian coordinate

$$x = R \cos\left(\omega t + \frac{1}{2}\alpha t^2\right); y = R \sin\left(\omega t + \frac{1}{2}\alpha t^2\right);$$

velocity components are

$$v_x = -R(\omega + \alpha t) \sin\left(\omega t + \frac{1}{2}\alpha t^2\right); v_y = R(\omega + \alpha t) \cos\left(\omega t + \frac{1}{2}\alpha t^2\right)$$

Cylindrical coordinate system



How to locate a point 'P' in space ?

□ z -Height from the XY plane

□ (r, θ) Coordinate of the foot of the point in XY plane

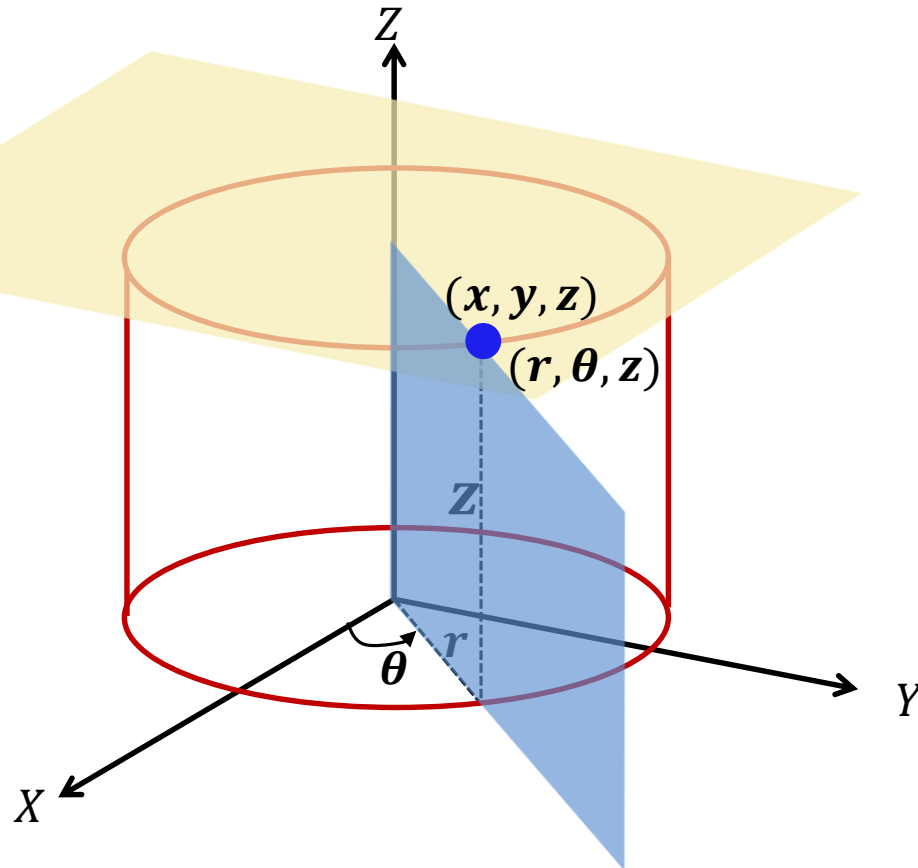
□ (r, θ, z) coordinates system is known as cylindrical coordinate system

Why the name cylindrical?

□ Point 'P' is the intersection of three surfaces: A cylindrical surface $r = \text{constant}$; A half plane containing z-axis with $\theta = \text{constant}$ and a plane $z = \text{constant}$.

Coordinate transformation: Cartesian to cylindrical

Transformation equation is very similar to polar coordinate with additional z-coordinate.



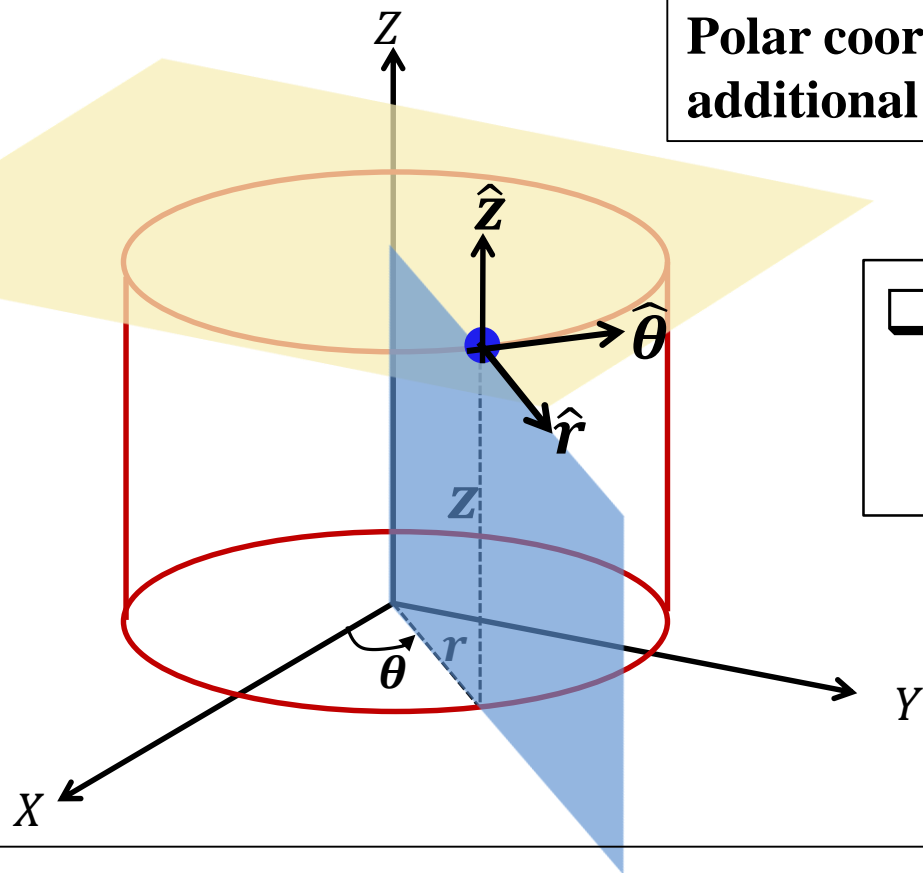
$$\begin{aligned}x &= r \cos \theta \\y &= r \sin \theta \\z &= z\end{aligned}$$

Reverse transformation

$$\begin{aligned}r &= (x^2 + y^2)^{1/2} \\ \theta &= \tan^{-1} \frac{y}{x} \\ z &= z\end{aligned}$$

Note: Instead of (r, θ) many books use notation (ρ, φ) .

Unit vectors in cylindrical coordinate system



Polar coordinate unit vectors ($\hat{r}, \hat{\theta}$) + additional unit vector in the z –direction.

□ $\hat{r}, \hat{\theta}$ and \hat{z} are unit vectors along increasing direction of coordinates r, θ and z .

$$\begin{aligned}\hat{r} &= \cos \theta \hat{x} + \sin \theta \hat{y} \\ \hat{\theta} &= -\sin \theta \hat{x} + \cos \theta \hat{y} \\ \hat{z} &= \hat{z}\end{aligned}$$

$\hat{r}, \hat{\theta}$ and \hat{z} are **orthogonal** but their directions depend on location.

$\hat{r}, \hat{\theta}$ and \hat{z} are **perpendicular** to surfaces $r = \text{constant}$; $\theta = \text{constant}$ and $z = \text{constant}$. Respectively.

Position, Velocity, Acceleration, Newton's law in cylindrical coordinate system

Vector components are very similar to polar coordinate+
z –component

Position vector

$$\vec{r} = r\hat{r} + z\hat{z}$$

Velocity

$$\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{z}\hat{z}$$

Acceleration

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{z}\hat{z}$$

Newton's law

$$\begin{aligned}\vec{F} &= F_r\hat{r} + F_\theta\hat{\theta} + F_z\hat{z} \\ &= m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{z}\hat{z}]\end{aligned}$$

Summery

- ❑ A point in plane can be represented by Cartesian coordinate $P(x, y)$ or polar coordinate $P(r, \theta)$. A point in space can be represented by (x, y, z) or (r, θ, z) or (r, θ, φ) .
- ❑ Coordinate transformation relation between *Cartesian* and *cylindrical coordinate* is given by

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

- ❑ For plane polar coordinate, transformation relation is $x = r \cos \theta; y = r \sin \theta$
- ❑ Unit vector in plane polar coordinate: $\hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y} ; \hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y}$
- ❑ Unit vectors in cylindrical coordinate: $\hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y} ; \hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y}$
, $\hat{z} = \hat{z}$
- ❑ Form of Newton's law is different in different coordinate systems.

Questions please