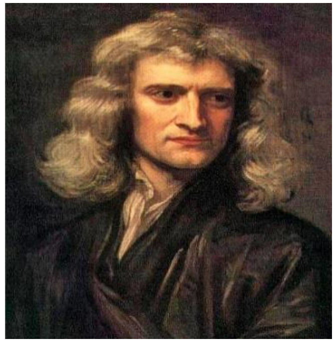
PH101

Lecture 10

Variational principle, Brachistochrone problem

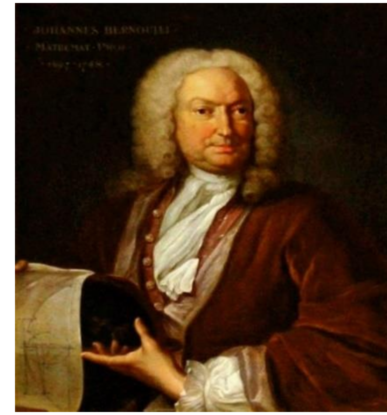
History of Variational Calculus (Wikipedia/Rana&Joag)

Pierre de Fermat
(1607 –1665)



Isaac Newton
(1642 – 1727**)**

Jacob Bernoulli
(1655 – 1705)
Algebra



Johann (**Jean** or **John)**
Bernoulli (1667 –1748)
Variational calculus

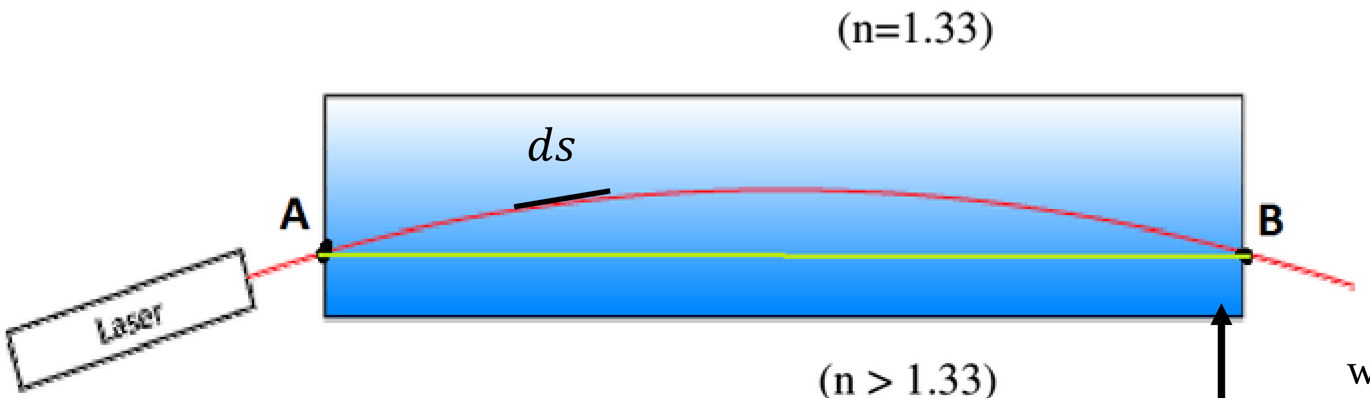
Leonhard Euler
(1707-1783)

Joseph-Louis
Lagrange

Daniel Bernoulli (1700 –1782)
Bernoulli's principle on fluids

Fermat's principle of least (extremum) time ~1662

Which path light follows when it passes through a medium?



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Salt solution in a container with vertically varying refractive index $[n(z)]$

Fermat's principle: Light travels between two points along the path that requires the least time, as compared to other nearby paths

$$\textit{Travels in a path for which}$$
$$\delta\int_A^B dt = \delta\int_A^B \frac{ds}{v} = 0$$

Where, $ds \rightarrow Elementary\ length\ along\ any\ possible\ path$

$$n = \frac{\text{speed of light in vacuum } (c)}{\text{speed of light in medium } (v)}$$

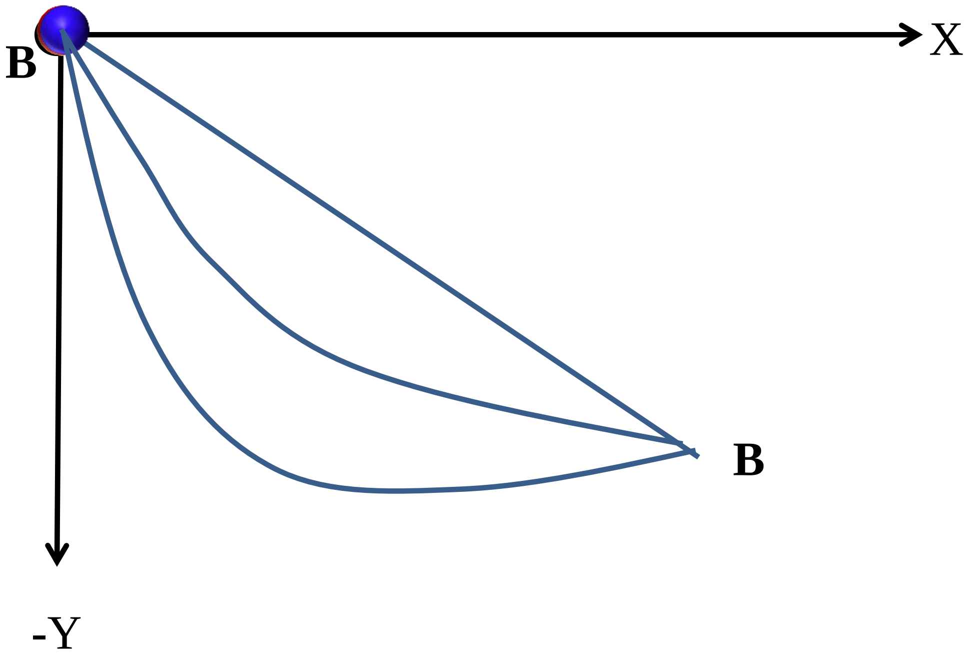
Needs a condition for minima of an integral!

Pierre de Fermat
(1607 –1665)

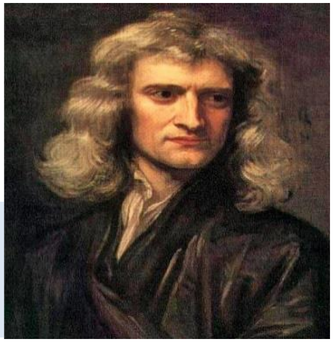
Jean Bernoulli’s challenge!
“Brachistochrone”

❑ What should be the shape of a stone’s trajectory (or, of a roller coaster track) so that released from point A it reaches point B in the shortest possible time? **Brachistochrone problem!** ~1696

Brachisto~ shortest **Chrone** ~ time



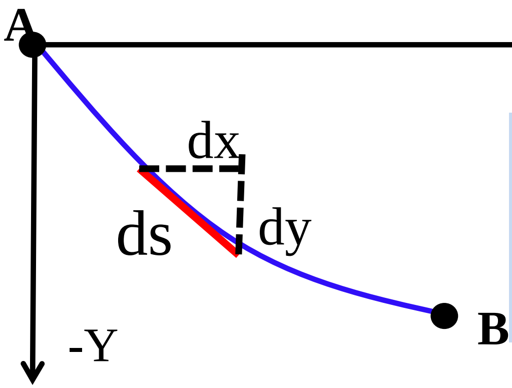
Jean Bernoulli
(1667 –1748)
Variational calculus



Isaac Newton
(1642 – 1727)

Jean Bernoulli's challenge!
"Brachistochrone"

- What should be the shape of a stone's trajectory (or, of a roller coaster track) so that released from point 1 it reaches point 2 in the shortest possible time? **Brachistochrone problem!** ~1696



$$ds = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{(1 + y'^2)}\, dx$$

$$v = \sqrt{2gy}$$

- Time (from 1 to 2) $I = \int_A^B \frac{ds}{v} = \int_A^B \frac{\sqrt{(1+y'^2)}}{\sqrt{2gy}}\, dx = \int_A^B \boldsymbol{F(y, y', x)dx}$

Cycloid

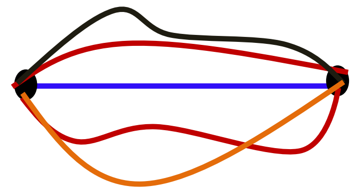
Wiki

Needs a condition for minima of an integral!

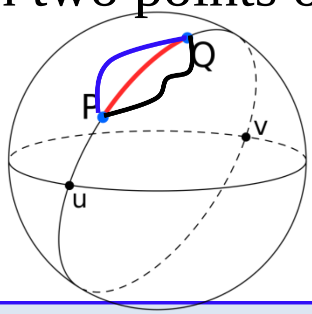
Extremums!

Can we prove mathematically

- Shortest distance between two points is a straight line?



- Shortest path between two points on the surface of a sphere is along the **great-circle**?



- To answer these questions, one need to know necessary condition that the integral $\mathbf{I} = \int_{x_1}^{x_2} F(y, y', x)dx$, where $y = y(x), y' = \frac{dy}{dx}$ is **stationary**

(ie, an **extremum!** – either a maximum or a minimum**!)**.

- **Interestingly we are already familiar with the condition!**

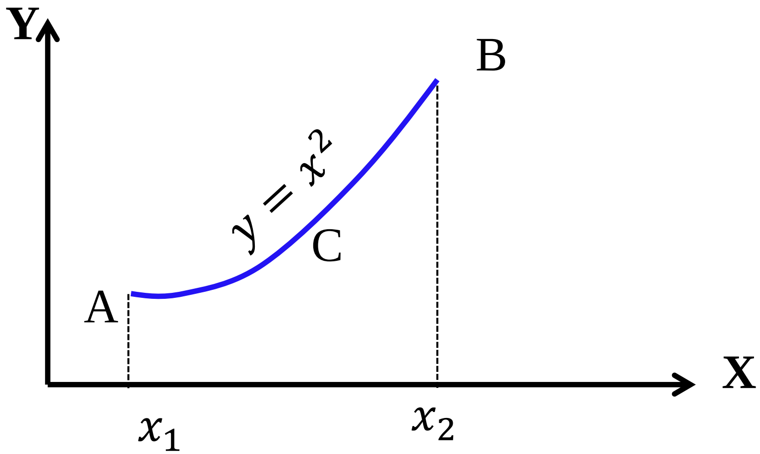
Integration of a functional

- Let's consider a function $F(y, y', x)$, **where $y = y(x)$ is a function of x**; $y' = \frac{dy}{dx}$. I,e. $F(y, y', x)$ is a function of a function known as **functional**.

Can you find out the value of this integration I = $\int_{x_1}^{x_2} F(y, y', x)\boldsymbol{dx}$ **?**

- It is surely possible if you know the integration path y $= y(x)$.

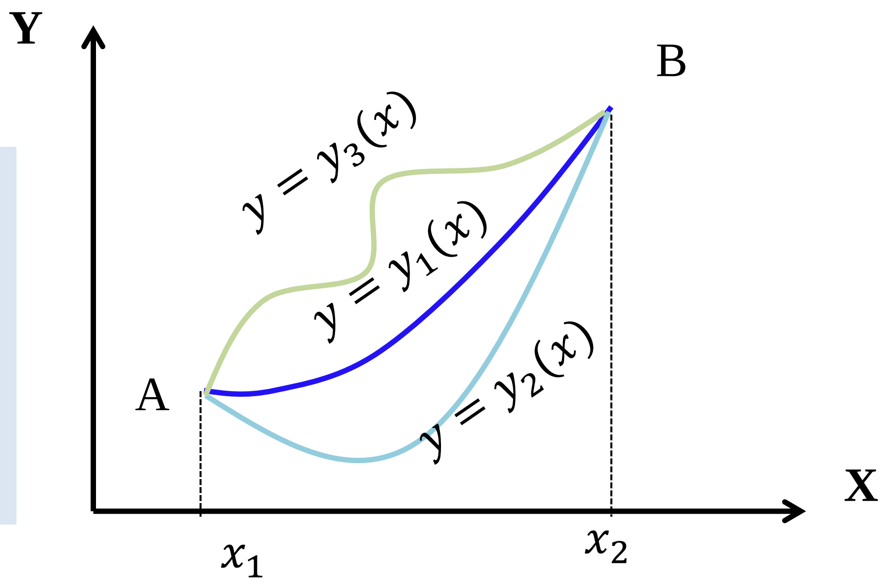
Example: If $F = xy + x^2 y'$ and **you know the integration path $y = y(x) = x^2$** (for example), then $F = xx^2 + x^2(2x) = 3x^3$, hence $I = \int_{x_1}^{x_2} 3x^2 dx = [{x_2}^3 - {x_1}^3]$



Possible integration paths

- ❑ If exact integration path $[y = y(x)]$ between A and B is not known, one can imagine infinite number of **possible paths** between these two points.

- ❑ Out of all the possible paths, in which path $\mathbf{I} = \int_{x_1}^{x_2} F(y, y', x)\,dx$ is stationary?



You need to find the condition for **an integral to be stationary,** where **variable is the integration path $[y = y(x)]$**

Review: Finding extremum of a function

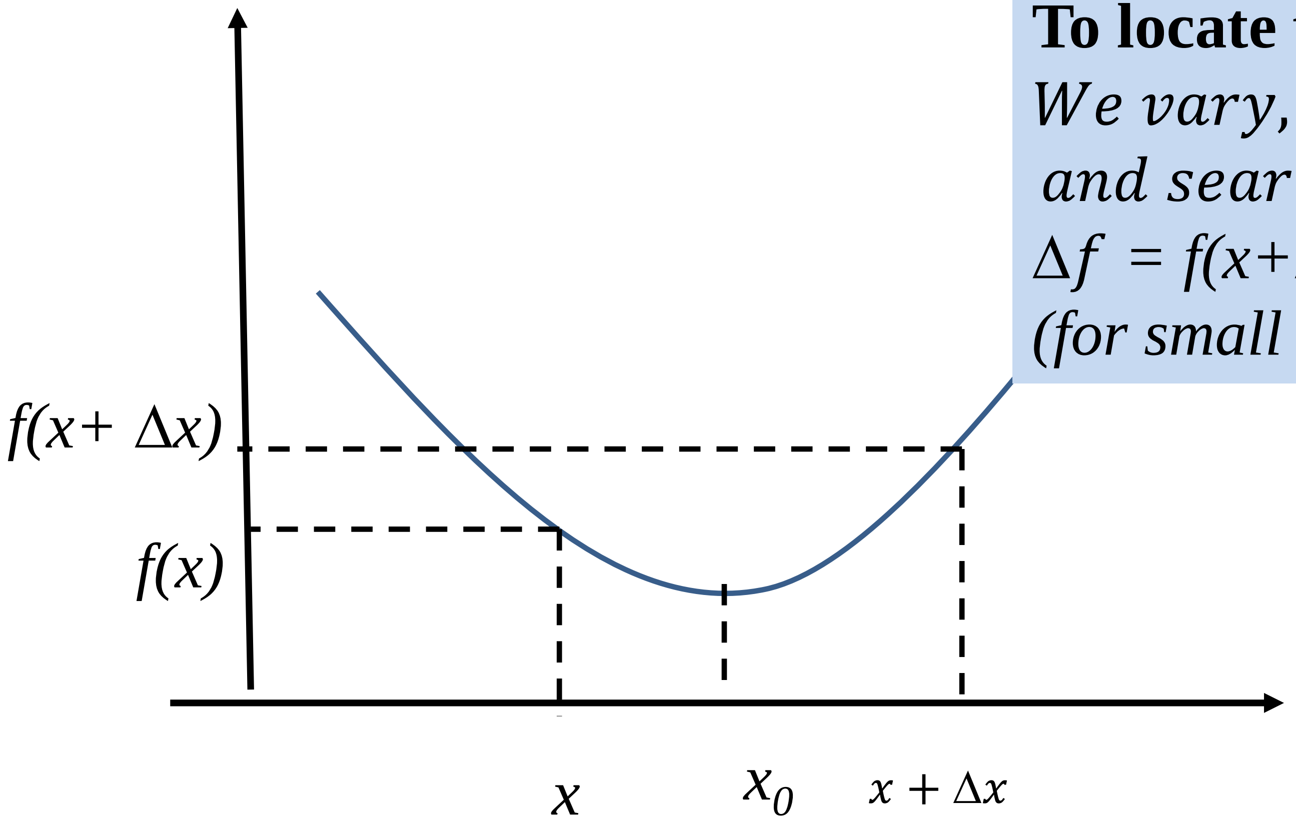
To locate the minimum, x_0

$We\ vary, \boldsymbol{x} \rightarrow$

$and\ search\ for\ the\ location$

$\Delta f = f(x+\Delta x) - f(x) \rightarrow 0$

(for small Δx)



We say the function is **stationary** at, $\boldsymbol{x_0}$

(Meaning, for small steps, Δx, *at* $\boldsymbol{x_0}$ *the value of the function does not change.* $\Delta f = \left(\frac{df}{dx}\right)_{x0} \Delta x \rightarrow 0$

Stationary condition of function vs stationary condition of integral

The stationary condition of an integral I can easily be established by reviewing the steps we follow to get **stationary condition of a function** $\boldsymbol{f(x)}$ and making an analogy.

Step 1: Suppose x [*to be determined*] is the point for which $f(x)$ is stationary.

[**Analogy**]: Let's consider a **path** $y = Y(x)$ [*to be determined*], for which the integration is stationary.

Step 2: Consider all possible points $(x + \Delta x)$ which differ from stationary point x by different amount Δx.

[**Analogy**] Choose a function $[Y(x) + \Delta y(x)]$ to represent all possible paths between x_1 and x_2 which differ from stationary path by different amount $(different\ value\ of\ \Delta y(x))$

Step 3: Use the fact that, variation of $f(x)$ [I,e $\Delta f = f(x + \Delta x) - f(x)$] is negligibly small near stationary point (say $say\ x = x_0$) (I,e $\Delta x \to 0$), which gives the **final condition** $\boldsymbol{\frac{dy}{dx} = 0}$

[**Analogy**]: Use the fact that variation of the integral value (I) [I,e $\delta I = I(Y + \Delta y) - I(Y)$]is negligibly small in the nearby paths [$\Delta y \to 0$], **Which will give a condition………?**

Smart choice of varied paths

- **Step 1:** Lets consider $y = Y(x)$ as the path for which integration $\mathrm{I} = \int_{x_1}^{x_2} F(y, y', x)dx$ is stationary.
- **Step2: Lets chose** $Y(x) + \Delta y(x), such\ that\ it$ can represent all possible paths between x_1 and x_2 for different Δy. How to chose this function $\boldsymbol{\Delta y(x)}$?

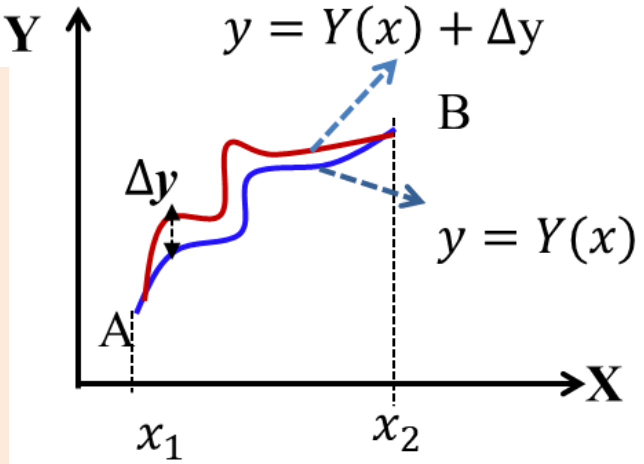
Mathematical form of Δy should be such that

(i) $Y(x) + \Delta y(x)$ can represent all possible paths but must not have variations at $\mathrm{A}(x_1)$ and $B(x_2)$ (fixed points).

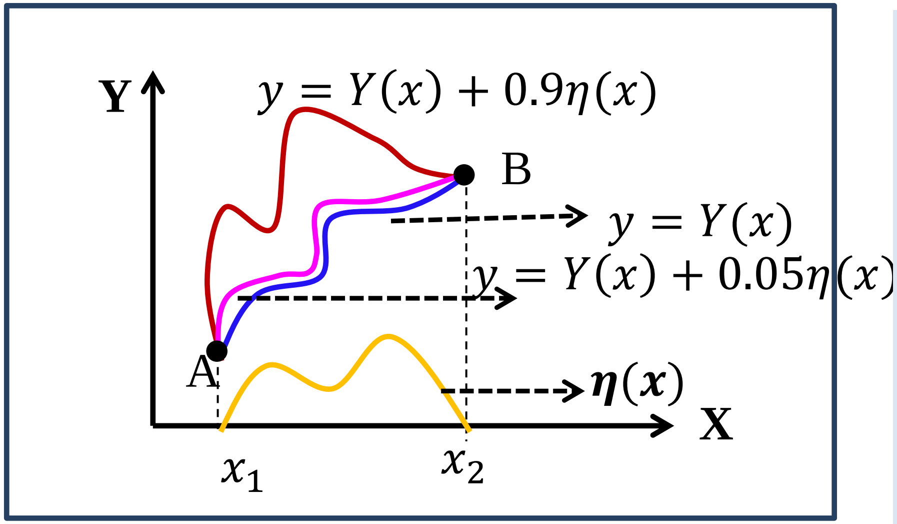
(ii) Δy goes to zero in the limiting case when the varied paths are very close to $Y(x)$.

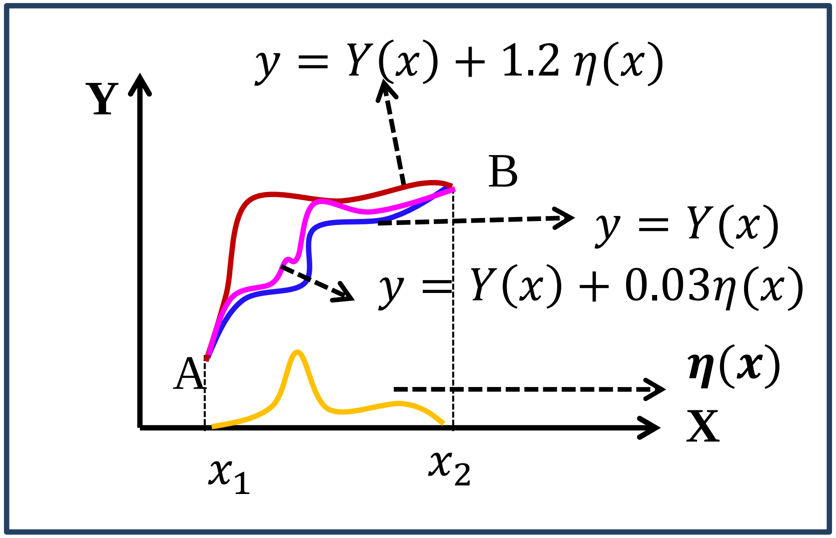
Let's check this choice $\boldsymbol{\Delta y = \epsilon\, \eta(x)}$

- where $\eta(x)$ is any **arbitrary function** of x such that $\eta(x_1) = \eta(x_2) = 0$. [condition (i) satisfied]
- ϵ is a parameter which can vary from 0 to higher value continuously. If we take **limit $\boldsymbol{\epsilon \to 0}$**, then condition (ii) satisfied.



$\boldsymbol{\eta(x)}$ and ϵ are indeed smart choice





Typical choice of arbitrary function $\boldsymbol{\eta(x)}$,

- $\eta(x_1) = \eta(x_2) = 0.$
- By varying ϵ , different strength of $\eta(x)$ can be added to $Y(x)$ to generate a range of possible paths between A and B

$[y(x,\epsilon) = Y(x) + \Delta y(x) = Y(x) + \epsilon\eta(x)]$

- $\epsilon \to 0$ can give path very close to $Y(x)$.

Another possible choice of $\eta(x)$ to generate another series of possible paths between A and B by varying ϵ.

Thus arbitrary $\boldsymbol{\eta(x)}$ and ϵ can produce all possible paths.

Stationary condition of integral

Step 3: Variation of the integral value along the paths nearby to stationary path paths [, $i, e. \epsilon \to 0$], is negligibly small.

The meaning of the statement is

The difference of integral values along two nearby paths
$\delta I(\boldsymbol{\epsilon}) = [I(\epsilon) - I] \to 0, when\ \epsilon \to 0$ (paths are nearly)

Where, $\mathbf{I} = \int_{\boldsymbol{x_1}}^{\boldsymbol{x_2}} \boldsymbol{F}(\boldsymbol{Y}, \boldsymbol{Y}', \boldsymbol{x})\boldsymbol{dx}]$ along stationary path $y = Y(x)$

and

$$\mathbf{I}(\boldsymbol{\epsilon}) = \int_{\boldsymbol{x_1}}^{\boldsymbol{x_2}} \boldsymbol{F}\{(\boldsymbol{Y} + \Delta\boldsymbol{y}), (\boldsymbol{Y}' + \Delta\boldsymbol{y}'), \boldsymbol{x}\}\boldsymbol{dx}] = \int_{\boldsymbol{x_1}}^{\boldsymbol{x_2}} \boldsymbol{F}\{\boldsymbol{y}(\boldsymbol{x}, \boldsymbol{\epsilon}), \boldsymbol{y}'(\boldsymbol{x}, \boldsymbol{\epsilon}), \boldsymbol{x}\}\boldsymbol{dx}$$

Along another path $y(x, \epsilon) = Y(x) + \Delta y(x) = Y(x) + \epsilon\eta(x)$

This is equivalent to saying, $\left.\frac{\boldsymbol{dI}(\boldsymbol{\epsilon})}{\boldsymbol{d\epsilon}}\right|_{\boldsymbol{\epsilon} \to \mathbf{0}} = \mathbf{0}$

For stationary path $\left.\frac{\boldsymbol{dI(\epsilon)}}{\boldsymbol{d\epsilon}}\right|_{\boldsymbol{\epsilon\to 0}} \boldsymbol{= 0}$

$$\left.\frac{dI(\epsilon)}{d\epsilon}\right|_{\epsilon\to 0} = \frac{d}{d\epsilon}\left[\int_{x_1}^{x_2} F\{y(x,\epsilon),\boldsymbol{y}'(x,\epsilon),x\}dx\right]$$

$$= \int_{x_1}^{x_2}\left(\frac{\partial F}{\partial y}\frac{\partial y}{\partial \epsilon} + \frac{\partial F}{\partial \boldsymbol{y}'}\frac{\partial \boldsymbol{y}'}{\partial \epsilon}\right)dx = \int_{x_1}^{x_2}\frac{\partial F}{\partial y}\eta\, dx + \int_{x_1}^{x_2}\frac{\partial F}{\partial \boldsymbol{y}'}\eta'\, dx$$

Where,
$\boldsymbol{y(x,\epsilon)} = (\boldsymbol{Y} + \boldsymbol{\epsilon}\eta)$
$\boldsymbol{y'(x,\epsilon)} = \boldsymbol{Y}' + \boldsymbol{\epsilon}\eta'$

$$= \int_{x_1}^{x_2}\frac{\partial F}{\partial y}\eta\, dx + \left.\frac{\partial F}{\partial \boldsymbol{y}'}\eta\right|_{x_1}^{x_2} - \int_{x_1}^{x_2}\eta\frac{d}{dx}\left(\frac{\partial F}{\partial \boldsymbol{y}'}\right)dx$$

Integration by parts

Using $\left.\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial y'}}\eta\right|_{\boldsymbol{x_1}}^{\boldsymbol{x_2}}=0$

As $\eta(x_1) = \eta(x_2) = 0$

$$= -\int_{x_1}^{x_2}\left[\frac{d}{dx}\left(\frac{\partial F}{\partial \boldsymbol{y}'}\right) - \frac{\partial F}{\partial y}\right]\eta\, dx = 0$$

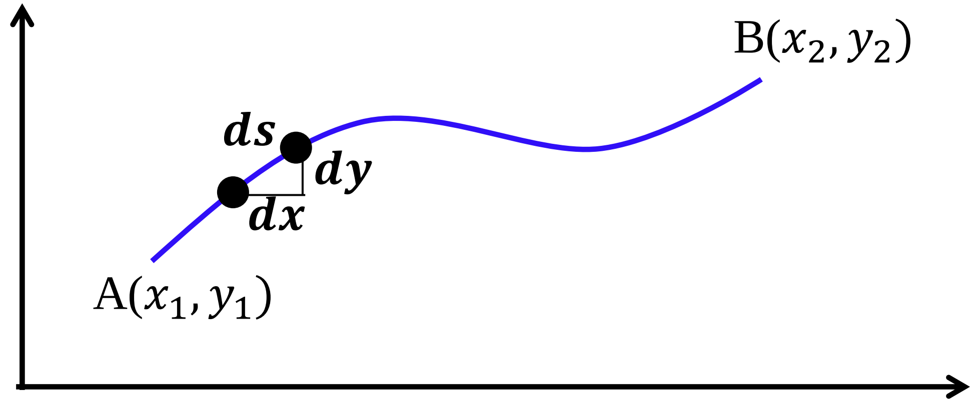
This equation is true for any possible choice of $\eta(\boldsymbol{x})$, thus

$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\left(\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial y'}}\right) - \frac{\boldsymbol{\partial F}}{\boldsymbol{\partial y}}=0$, **Euler-Lagrange equation**

This is necessary condition for $\mathrm{I} = \int_{x_1}^{x_2} F(y,\boldsymbol{y}',x)dx$ to be stationary

Application of Variational principle: Example1

- Given two points in a plane, what is the shortest path between them? You certainly know the answer: Straight line. Let's prove it using variation method



- Consider an arbitrary path $y(x)$, elementary length

$$ds = [(dx)^2 + (dy)^2]^{1/2} = \left[\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\right]^{1/2} dx = \left(1 + \boldsymbol{y'}^2\right)^{1/2} dx$$

- Total path length $\int_A^B ds = \int_{x_1}^{x_2} \left(1 + \boldsymbol{y'}^2\right)^{1/2} dx$
- Necessary condition for this integral to be stationary (maximum)

$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\left(\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial y'}}\right) - \frac{\boldsymbol{\partial F}}{\boldsymbol{\partial y}}$ =0; Here $F(y, \boldsymbol{y'}, x) = \left(1 + \boldsymbol{y'}^2\right)^{1/2}$

Application of variation principle: Example1

$$\frac{\partial F}{\partial \boldsymbol{y}'} = \frac{\partial}{\partial \boldsymbol{y}'}\left\{\left(1+\boldsymbol{y}'^2\right)^{1/2}\right\} = \boldsymbol{y}'\left(1+\boldsymbol{y}'^2\right)^{-1/2}; \frac{\partial F}{\partial y} = 0$$

Thus

$$\frac{d}{dx}\left\{\boldsymbol{y}'\left(1+\boldsymbol{y}'^2\right)^{-1/2}\right\} = 0$$

$$\boldsymbol{y}'\left(1+\boldsymbol{y}'^2\right)^{-1/2} = constant$$

$$\boldsymbol{y}'^2 = Constant\left(1+\boldsymbol{y}'^2\right),$$

$$\boldsymbol{y}'^2 = Consntant;$$

$$y(x) = mx + C, Where\ m\ and\ C\ are\ constant$$

Equation of straight line

- ❑ Shortest distance between two points in a plane is straight line.

Summery

$$\mathbf{I} = \int_{x_1}^{x_2} F(y, y', x)\,\boldsymbol{dx}$$

Necessary condition for stationary

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

- To get stationary condition of any quantity, express the quantity in terms of integral of its infinitesimal value with known integration limit, then use Euler-Lagrange equation