PH 101: Physics I

Module 2: Special Theory of Relativity-Basics

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RECAP **Lorentz Transformations**

$$\gamma_v = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

$$x' = \gamma(x - vt);\ \ y' = y;\ \ z' = z;\ \ t' = \gamma_v(t - \frac{v}{c^2}x)$$

Law of addition of velocities

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad u_y' = \frac{u_y}{\gamma_v(1 - \frac{u_x v}{c^2})} \qquad u_z' = \frac{u_z}{\gamma_v(1 - \frac{u_x v}{c^2})}$$

Distance between events (a Lorentz invariant)

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$ds'^2 = ds^2$$

Definition of relativistic momentum

Should we continue to define the momentum as $\mathbf{p} = m\,\mathbf{u}$, where m is a velocity independent constant called the rest mass?

It can be shown that this choice is not suitable as total momentum conserved in one reference frame can be shown to be **not** conserved in a moving reference frame related to the earlier one by a Lorentz transformation.

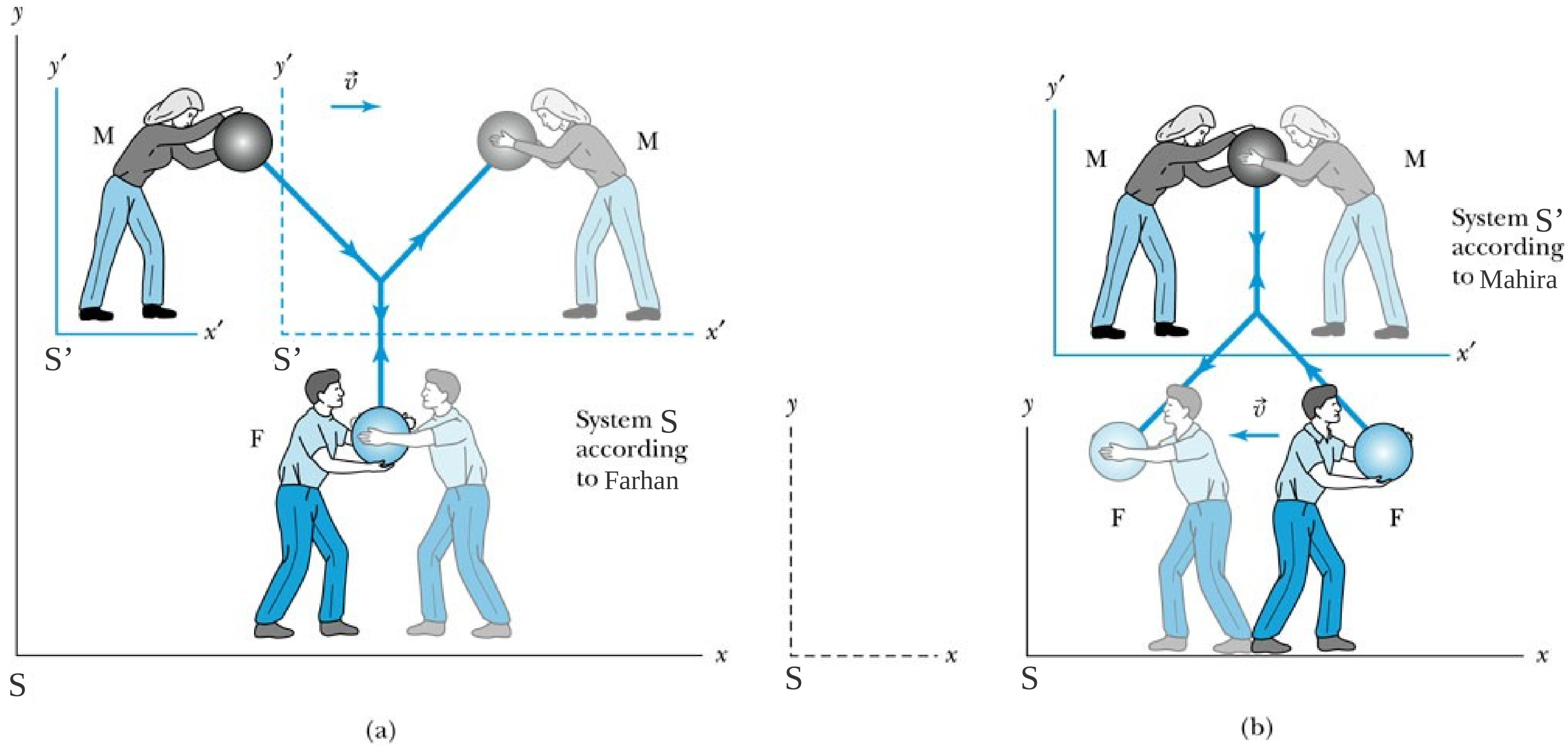
Given that the momentum has to be parallel to the velocity of the particle we write

$$\mathbf{p} = \Gamma(u)\; m\; \mathbf{u}$$

where $\Gamma(u)$ is a scalar function of $u = |\mathbf{u}|$.

The goal now is to find $\Gamma(u)$ in such a way that if momentum conserved in one reference frame it is also seen to be conserved in a Lorentz transformed reference frame.

Relativistic Momentum



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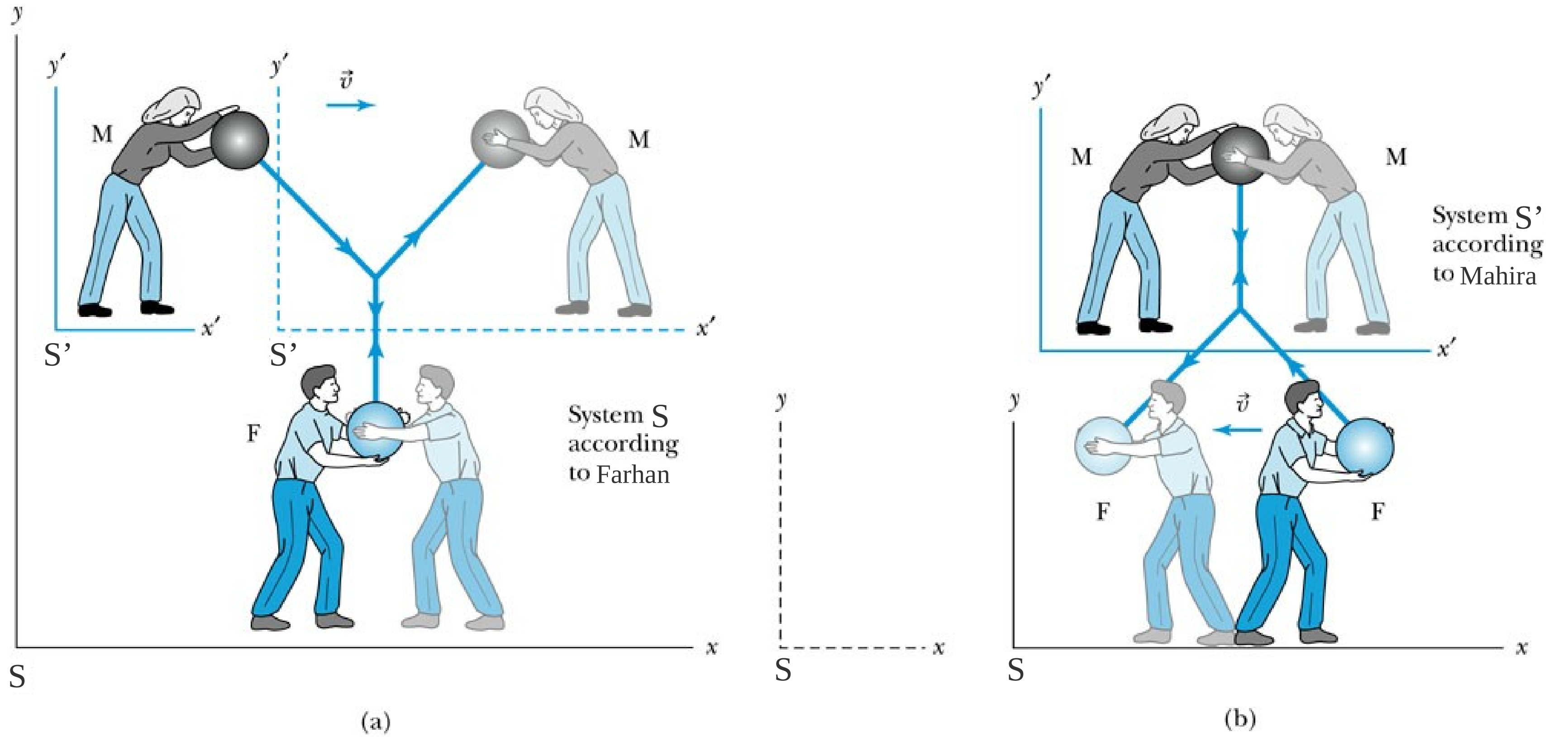
Consider two observers ‘Farhan (F)’ and ‘Mahira (M)’ each have a ball of mass m. ‘F’ is in the static frame S, while, ‘M’ is in the frame S’ which is moving with a velocity v in the right direction.

Now both the observers throw the ball in vertical direction with speed u_0 in such a way that an elastic collision takes place between the balls and after collision they collect their respective balls. The process of collsion that will be seen by F and M are depicted in Figs. (a) and (b). Collision is analyzed from Farhan frame (shown in (a)) and from Mahira frame (shown in (b)).

Relativistic Momentum



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Collision process analyzed by Farhan:

The momentum of the ball thrown by Farhan is entirely in the *y*-direction. Therefore,

$$p_{Fy} = mu_0$$

Farhan will notice the change in momentum of the his ball which is given by:

$$\Delta p_F = \Delta p_{Fy} = -2mu_0$$

Relativistic Momentum

Before the collision, the momentum of Mahira's ball as measured by Farhan is

$$p_{Mx} = mv$$

$$u_{Mx} = v$$

$$u'_{Mx} = 0 \quad u'_{My} = -u_0$$

$$u_{Mx} = \frac{u'_{Mx} + v}{1 + u'_{Mx} v/c^2}$$

$$u_{My} = \frac{u'_{My}\sqrt{1 - v^2/c^2}}{1 + u'_{Mx} v/c^2}$$

$$u_{Mz} = \frac{u'_{Mz}\sqrt{1 - v^2/c^2}}{1 + u'_{Mx} v/c^2}$$

$$p_{My} = mu_{My} = -mu_0\sqrt{1 - v^2/c^2}$$

After the collision, the momentum of Mahira's ball as measured by Farhan is

$$p_{Mx} = mv$$

$$p_{My} = mu_0\sqrt{1 - v^2/c^2}$$

The change in momentum of Mahira's ball according to Farhan is

$$\Delta p_M = \Delta p_{My} = 2mu_0\sqrt{1 - v^2/c^2}$$

Relativistic Momentum

The conservation of linear momentum requires the total change in momentum of the collision,

$$\Delta p_F + \Delta p_M = 0 \qquad \Delta p_F = \Delta p_{Fy} = -2mu_0$$

However, this is clearly not true here. $\Delta p_M = \Delta p_{My} = 2mu_0\sqrt{1 - v^2/c^2}$

Hence, linear momentum is not conserved if we use the standard definition of momentum as used in Newtonian mechanics even if we use the velocity transformation equations.

There is no problem with the *x* direction, but there is some problem lying in dealing with the velocity in *y*-direction direction along in which the ball is thrown in each system.

Rather than discarding the conservation of linear momentum, let us look for a modification of the definition of linear momentum that preserves Newton's second law.

In order to do this we need to redefine the momentum such that it takes the form,

$$\vec{p} = \Gamma(u)m\vec{u}$$

$$\Gamma(u) = \frac{1}{\sqrt{1 - u^2/c^2}}, \text{ where } u = \sqrt{u_x^2 + u_y^2 + u_z^2}$$

Relativistic Momentum

This means the correct definition of momentum in Special Relativity that ensures that momentum conserved in one reference frame is also conserved in a Lorentz transformed frame is,

$$\mathbf{p} = \frac{m\ \mathbf{u}}{\sqrt{1 - \frac{u^2}{c^2}}}$$

Note that the above definition is very different from the usual definition of momentum in Galilean Relativity viz. $\mathbf{p} = m\ \mathbf{u}$. The relativistic definition becomes the usual definition valid only in Galilean Relativity when $|\mathbf{u}| \ll c$.

$$p_x = \frac{m_0 u_x}{\sqrt{1 - u^2/c^2}} \quad p_y = \frac{m_0 u_y}{\sqrt{1 - u^2/c^2}} \quad p_z = \frac{m_0 u_z}{\sqrt{1 - u^2/c^2}}$$

Relativistic Momentum

$$\vec{p} = \Gamma(u) m \vec{u}$$

In other way this relation can be interpreted as the mass of the moving particle increases.

Following this the we can denote "m" in the above equation as m_0 which is called the rest mass or the proper mass. The first two terms can be called the modified mass or the relativistic mass which is ;

$$m = \Gamma(u) m_0 = \frac{1}{\sqrt{1 - u^2/c^2}} m_0$$

$$\vec{p} = m \vec{u}$$

Note that the proper mass is like proper length and proper time.

Here the new mass would be the function of the velcoity of ball.

Relativistic Energy

At this stage we need to redefine the work and energy in STR as the mass has become observer dependent.

Let's begin with the Newton's second law.

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(\Gamma(u)m_0\vec{u}) = \frac{d}{dt}\left(\frac{m_0\vec{u}}{\sqrt{1-u^2/c^2}}\right)$$

The workdone by a force F to move a particle from position 1 to position 2 along a path is defined as;

$$W_{12} = \int_1^2 \vec{F}\cdot d\vec{s} = K_2 - K_1$$

Where K's are the kinetic energies of the particle.

For simplicity, let's assume that the particle is at rest at position 1 (i.e. K_1=0).

Therefore, $$W = K = \int \frac{d}{dt}(\Gamma(u)m_0\vec{u})\cdot\vec{u}dt$$

Relativistic Energy

For simplicity we consider the montion in one dimension. We have

$$K = \int_{u=0}^{u=u} F dx = \int \frac{d}{dt}(mu) dx = \int d(mu) \frac{dx}{dt}$$

Where, m is the relativistic mass

$$= \int (mdu + udm)u = \int_{u=0}^{u=u} (mudu + u^2 dm) \qquad (1)$$

Using $m = m_0/\sqrt{1-u^2/c^2}$, we have

$$m^2c^2 - m^2u^2 = m_0^2c^2$$

Taking differential yields,

$$2mc^2dm - m^2 2udu - u^2 2mdm = 0$$

i.e., $mudu + u^2dm = c^2dm$

Substituting this in Eq. (1) yields,

$$K = \int_{u=0}^{u=u} c^2 dm = c^2 \int_{m=m_0}^{m=m} dm = mc^2 - m_0c^2$$

Relativistic Energy

$$K = m_0c^2\left[\frac{1}{\sqrt{1-u^2/c^2}} - 1\right]$$

Notice that as u->c the Kinetic energy tends to infinity.

Writing the total energy as $E = mc^2$ We have

$$E = m_0c^2 + K$$

Where, m_0c^2 is called the rest mass energy.

Now let's see whether K reduces to the classical result for $u/c \ll 1$

$$K = m_0c^2\left[1/\sqrt{1-u^2/c^2} - 1\right]$$

$$= m_0c^2\left[\left(1-\frac{u^2}{c^2}\right)^{-1/2} - 1\right]$$

$$K = m_0c^2\left[1+\frac{1}{2}\left(\frac{u}{c}\right)^2 + \frac{3}{8}\left(\frac{u}{c}\right)^4 + ... - 1\right]$$

$$= \frac{1}{2}m_0u^2$$

Classical Kinetic Energy

We shall continue in the next class