Physics II (PH 102)

Electromagnetism (Lecture 3)

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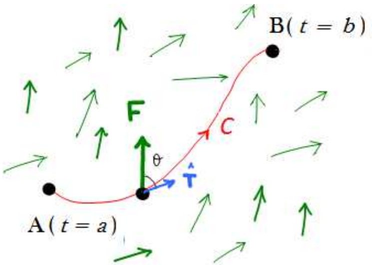
Line Integral of Vector Fields



Definition

Let $\mathbf{r}(t) = g(t)\hat{\mathbf{i}} + h(t)\hat{\mathbf{j}} + k(t)\hat{\mathbf{k}};\ t \in [a, b]$ is a parametrized curve C in $\mathbb{R}^3$ and $\mathbf{F}$ is continuous vector field over $\mathbb{R}^3$. Then the LINE INTEGRAL of the vector $\mathbf{F} = (F_x, F_y, F_z)$ over C between the end-pints A and B is given as

$$\begin{aligned}
\int_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} &= \int_{AB} \mathbf{F}(\mathbf{r}) \cdot (\hat{\mathbf{T}}\, ds) = \int_{AB} F(\mathbf{r}) \cos\theta \, ds \\
&= \int_a^b F[\mathbf{r}(t)] \cos[\theta(t)] \left(\frac{ds(t)}{dt}\right) dt = \int_a^b F[\mathbf{r}(t)] \cos[\theta(t)]\ |\mathbf{r}'(t)|\, dt \\
&= \int_a^b F[g(t), h(t), k(t)] \cos[\theta(t)] \sqrt{\left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2 + \left(\frac{dk}{dt}\right)^2}\, dt
\end{aligned}$$

Line Integral of Vector Fields

Corollary

If the Line Integral of $\mathbf{F}$ *is defined along any simple closed curve/loop* L *(that does not intesect with itself) in* $\mathbb{R}^3$*, it is termed as the CONTOUR INTEGRAL or CIRCULATION of* $\mathbf{F}$ *about* L*, and expressed as*

$$\oint_L \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \oint_L F_x dx + F_y dy + F_z dz$$

Examples

1. WORK DONE, $\Delta W_{\mathrm{AB}} = \int_{\mathrm{AB}} \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r}$ is the most familar example in Physics of a line integral of a force field $\mathbf{F}(\mathbf{r})$.
2. For a CONSERVATIVE FIELD $\mathbf{F}_{\mathrm{consv.}}$ the net work done about *EVERY* closed path vanishes:

$$\Delta W_{\mathrm{Loop}} = \oint \mathbf{F}_{\mathrm{consv.}}(\mathbf{r}) \cdot d\mathbf{r} = 0$$

Examples of Line Integral of Vector Fields

Example

Consider the inverse square force field, $\mathbf{F}(\mathbf{r}) = \alpha\mathbf{r}/r^3$, where $\alpha > 0$ is a constant and $\mathbf{r}$ is the position vector. Find the work done in moving a particle along the unit circle C: $\mathbf{r}(\theta) = (\cos\theta, \sin\theta)$; $\theta \in [0, 2\pi]$.

The given path is *circular and closed* (end-point coincides with starting point), with unit radius, $|\mathbf{r}(\theta)| = r(\theta) = 1$. Thus, the work done is

$$\begin{aligned}
\Delta W &= \oint_C \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\left[\mathbf{r}(\theta)\right] \cdot \left(\frac{d\mathbf{r}(\theta)}{d\theta}\right) d\theta \\
&= \alpha \int_0^{2\pi} \left(\frac{\cos\theta\hat{\mathbf{i}} + \sin\theta\hat{\mathbf{j}}}{r(\theta)^3}\right) \cdot (-\sin\theta\hat{\mathbf{i}} + \cos\theta\hat{\mathbf{j}}) d\theta = 0
\end{aligned}$$

- This is a NECESSARY but not a sufficient condition for "consevativeness" of $\mathbf{F}(\mathbf{r})$, since work done must be zero about EVERY closed path.
- NECESSARY & SUFFICIENT condition: What is $\mathrm{curl}\,\mathbf{F}$?
- The inverse square field with $\nabla \times \mathbf{F} = \mathbf{0}$ is a conservative field.

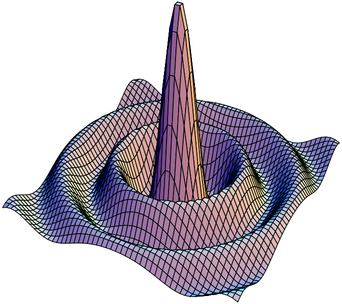
How to describe Surfaces in 3D?

- $F(x, y, z) = c = const.$ is used to represent the general quation of a surface in 3D, where F is a real smooth function of x, y and z.
- Surface can be OPEN or CLOSED types.
- $z = f(x, y)$ is a typical form of an *open surface* in 3D space, where f is a real smooth function of x and y.

Examples

(a) $z = const.$ is an open plane surface parallel to XY plane

(b) Another open surface: $z = \sin\left(\sqrt{x^2 + y^2}\right) / \sqrt{x^2 + y^2}$



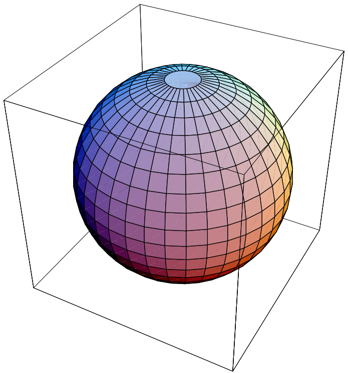
Surfaces with 2D Parametric Representations

Example

UNIT SPHERE: $F(x, y, z) = x^2 + y^2 + z^2 = 1$

- The two *open half surfaces* described by $z = \pm\sqrt{1 - x^2 - y^2}$.
- **PARAMETRIC REPRESENTATION:** Alternatively, it can be described in terms of two real parameters θ and ϕ as:

$$\mathbf{r}(\theta, \phi) = \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}},\ \ \theta \in [0, \pi],\ \ \phi \in [0, 2\pi].$$



Surface Parameterizations

Examples

1. CYLINDER: $x^2 + y^2 = a^2$, $-1 \leq z \leq 1$ has radius a and height 2 units is described as

$$\mathbf{r}(\phi, z) = a\cos\phi\,\hat{\mathbf{i}} + a\sin\phi\,\hat{\mathbf{j}} + z\hat{\mathbf{k}},\ \phi \in [0, 2\pi],\ z \in [-1, 1].$$

2. REGULAR CONE: $z = \sqrt{x^2 + y^2}$ of height H is described as

$$\mathbf{r}(\phi, z) = z\cos\phi\,\hat{\mathbf{i}} + z\sin\phi\,\hat{\mathbf{j}} + z\hat{\mathbf{k}},\ \phi \in [0, 2\pi],\ z \in [0, H].$$

3. PARABOLOID: $r = x^2 + y^2$ of height H is described as

$$\mathbf{r}(r, \phi) = r\cos\phi\,\hat{\mathbf{i}} + r\sin\phi\,\hat{\mathbf{j}} + r^2\hat{\mathbf{k}},\ r \in [0, H],\ \phi \in [0, 2\pi].$$

4. HYPERBOLOID: $z = x^2 - y^2$ is described as

$$\mathbf{r}(u, v) = u\sec v\,\hat{\mathbf{i}} + u\tan v\,\hat{\mathbf{j}} + u^2\hat{\mathbf{k}},\ u \in [0, \infty],\ v \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

How to represent Elemental Area on an Open Surface?

- Let S be a patch of area on a smooth two-sided open surface, $z = f(x, y)$.
- Let R be the projection on the xy-plane with unit normal vector $\hat{\mathbf{k}}$.
- $\hat{\mathbf{N}}$ be the unit normal vector at any point on the surface.
- The projection of dS is the rectangular patch of area $dx\,dy$, i.e.,

$$
\begin{aligned}
dx\,dy &= |\hat{\mathbf{k}} \cdot \hat{\mathbf{N}}|\, dS = \hat{\mathbf{k}} \cdot d\mathbf{S} \\
d\mathbf{S} &= \hat{\mathbf{N}}\, dS = \hat{\mathbf{N}} \left(\frac{dx\,dy}{|\hat{\mathbf{k}} \cdot \hat{\mathbf{N}}|} \right)
\end{aligned}
$$

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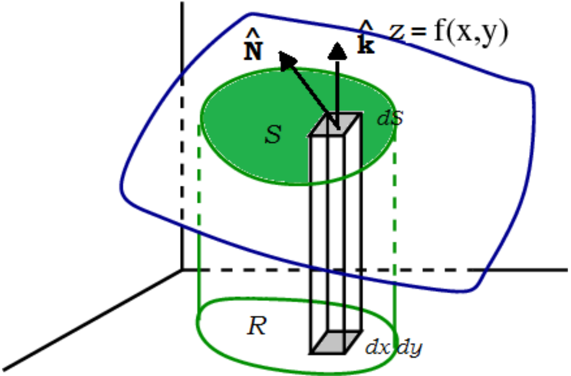
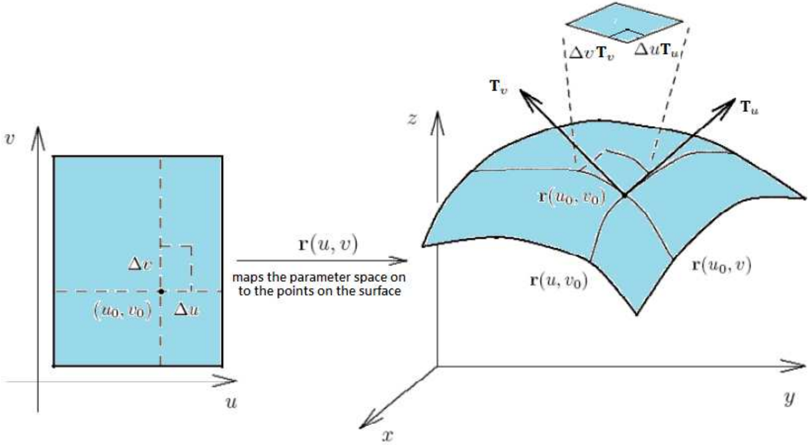


Elemental Area on a Parametrized Surface $\mathbf{r}(u, v) : D \to \mathbb{R}^3$

- We need only Orthogonal parametrizations such that if the *parameter lines* meet orthogonally in the 2-dim *abstract* parameter domain $D \in \mathbb{R}^2$, then the *co-ordinate lines* on the surface S also meet orthogonally .
- Non-orthogonal parametrizations are cumbersome and not useful.

$$\text{Tangent Vectors}: \quad \mathbf{T}_u = \left(\frac{\partial \mathbf{r}}{\partial u}\right)_{(u_0, v_0)} \quad ; \quad \mathbf{T}_v = \left(\frac{\partial \mathbf{r}}{\partial v}\right)_{(u_0, v_0)}$$



Finding Elemental Area on a Parametrized Surface

Example

- Paraboloid of Revolution:

  $$\mathbf{r}(r,\phi) = r\cos\phi\,\hat{\mathbf{i}} + r\sin\phi\,\hat{\mathbf{j}} + r^2\,\hat{\mathbf{k}}$$

- Parameter domain:

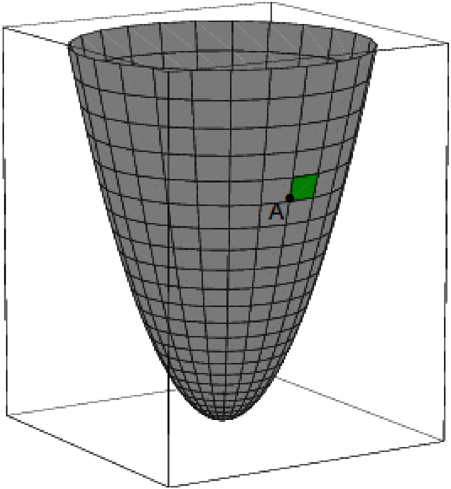
  $$D = \{r \times \phi \,|\, r \in [0,3],\ \phi \in [0,2\pi]\}$$

- Elemental area at **A** shown in green:

  $$\mathbf{r}(A) = \mathbf{r}(r=2,\phi=0) = 2\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$$

- Tangent vectors at A on the co-ordinate lines:

  $$\mathbf{T}_r = \left(\frac{\partial \mathbf{r}}{\partial r}\right)_A = \hat{\mathbf{i}} + 4\hat{\mathbf{k}} \quad ; \quad \mathbf{T}_\phi = \left(\frac{\partial \mathbf{r}}{\partial \phi}\right)_A = 2\hat{\mathbf{j}}$$



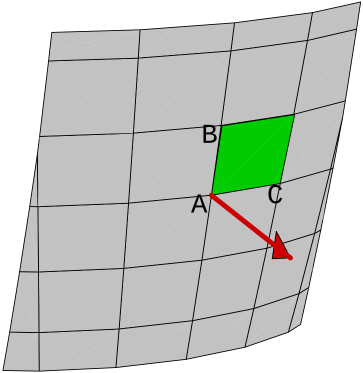
Finding Elemental Area on a Parametrized Surface (contd.)

$\mathbf{r}(r,\phi) = r\cos\phi\hat{\mathbf{i}} + r\sin\phi\hat{\mathbf{j}} + r^2\hat{\mathbf{k}}$; $r \in [0,3], \phi \in [0,2\pi]$, and $\mathbf{A} \equiv \mathbf{r}(2,0) = 2\hat{\mathbf{i}} + 4\hat{\mathbf{k}}$

Line elements at A:

$$\overrightarrow{AB} = \mathbf{T}_r\, dr = \left(\frac{\partial \mathbf{r}}{\partial r}\right)_A dr = (\hat{\mathbf{i}} + 4\hat{\mathbf{k}})\, dr$$

$$\overrightarrow{AC} = \mathbf{T}_\phi\, d\phi = \left(\frac{\partial \mathbf{r}}{\partial \phi}\right)_A d\phi = 2\hat{\mathbf{j}}\, d\phi$$



Outward Normal vector at A:

$$\mathbf{N} = \mathbf{T}_\phi \times \mathbf{T}_r = \left(\frac{\partial \mathbf{r}}{\partial \phi}\right)_A \times \left(\frac{\partial \mathbf{r}}{\partial r}\right)_A = 8\hat{\mathbf{i}} - 2\hat{\mathbf{k}}$$

Scalar area element:

$$dS = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = |\mathbf{N}|\, dr\, d\phi = 2\sqrt{17}\, dr\, d\phi$$

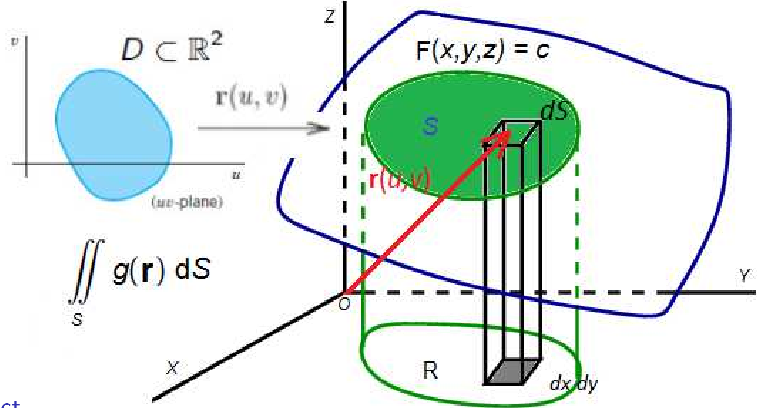
Vector area element:

$$d\mathbf{S} \equiv \hat{\mathbf{N}}\, dS = \overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{N}\, dr\, d\phi = \left(8\hat{\mathbf{i}} - 2\hat{\mathbf{k}}\right) dr\, d\phi$$

Surface Integrals of Scalar Fields

Definition

A SURFACE INTEGRAL of a continuous scalar field, $g = g(x, y, z)$ is the *generalization* of a 2D definite integral where the *doman of integration* is a *smooth or piecewise smooth* surface $S : F(x, y, z) = c$, or parametrized as $\mathbf{r} = \mathbf{r}(u, v)$, with $(u, v) \in D \subset \mathbb{R}^2$.



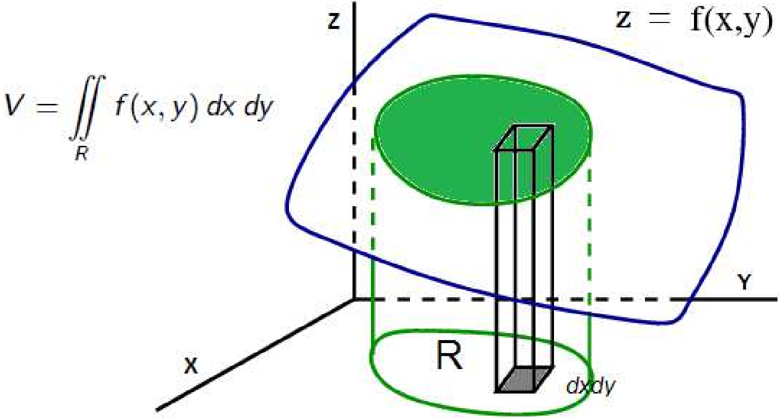
Fact

Surface integral CAN NOT be evaluated without reducing to double integral!

Double Integral is different from Surface Integral !

Definition

A DOUBLE INTEGRAL is essentially a 2D definite integral where the *doman of integration* is the region $R \subset \mathbb{R}^2$ on the co-ordinate xy-plane for the given surface $S : z = f(x, y)$. Here the integral yields the volume of the cylindrical region under the surface.



Surface Integral of Scalar Fields (with Surface Parameterization)

Definition

The SURFACE INTEGRAL of a continuous scalar function $g(\mathbf{r})$ over a smooth or piecewise smooth surface S, and parametrized as $\mathbf{r} = \mathbf{r}(u, v)$, with $(u, v) \in D \subset \mathbb{R}^2$, is given as

$$\boxed{\iint_S g(\mathbf{r})\, dS = \iint_D g\left[\mathbf{r}(u, v)\right] |\mathbf{N}|\, du\, dv = \iint_D g\left[\mathbf{r}(u, v)\right]\, |\mathbf{T}_u \times \mathbf{T}_v|\, du\, dv}$$

where, $dS = |\mathbf{N}|\, du\, dv$ and $|\mathbf{N}| = |\mathbf{T}_u \times \mathbf{T}_v|$ is the *magnification/scale factor* termed as the JACOBIAN of transformation.

Corollary

- In particular the surface area is of S is obtained with $g(\mathbf{r}) = 1$, i.e.,

$$\text{Area} = \iint_S 1\, dS = \iint_D |\mathbf{T}_u \times \mathbf{T}_v|\, du\, dv = \iint_D \left| \left(\frac{\partial \mathbf{r}}{\partial u} \right) \times \left(\frac{\partial \mathbf{r}}{\partial v} \right) \right| du\, dv$$

- CLOSED SURFACE INTEGRAL over surface S enclosing some volume:

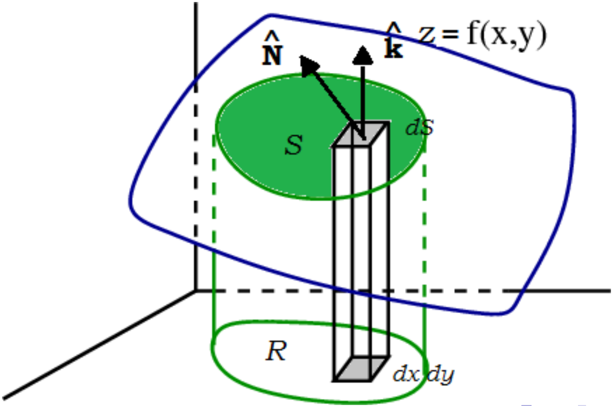
$$\oiint_S g(\mathbf{r})\, dS$$

Surfaces Without Parameterization: Surface Integral → Double Integral

Association between dS and elemental projected area on any co-ordinate plane:

- $\hat{\mathbf{N}}$ be the unit normal vector at any point on the surface.
- Projective Correspondence of dS with the elemental area $dx\,dy$ on R

$$\begin{aligned} dx\,dy &= |\hat{\mathbf{N}} \cdot \hat{\mathbf{k}}|\, dS \\ dS &= \frac{dx\,dy}{|\hat{\mathbf{N}} \cdot \hat{\mathbf{k}}|} \end{aligned}$$



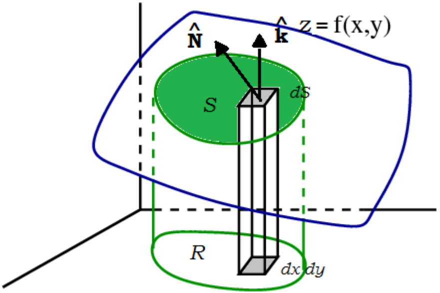
Surface Integrals of Scalar Fields in Cartesian System

Reducing to a double integral: If $\hat{\mathbf{N}}$ be the unit normal vector at any point on the smooth two-sided open surface, $S : z = f(x, y)$, then the projection of dS on R is the rectangular patch given by $dxdy = \hat{\mathbf{N}} \cdot \hat{\mathbf{k}}\, dS$. With the equation of surface written in the form

$$F(x, y, z) = f(x, y) - z = 0, \qquad \hat{\mathbf{N}} = \pm \frac{\nabla F(x, y, z)}{|\nabla F(x, y, z)|},$$

the surface integrals of a continuous scalar field $g(x, y, z)$ is given by

$$\iint_S g(x, y, z)\, dS = \iint_R g(x, y, f(x, y))\, \frac{dx\, dy}{|\hat{\mathbf{N}} \cdot \hat{\mathbf{k}}|} = \iint_R g(x, y, f)\, \frac{|\nabla F(x, y, f)|}{|\nabla F(x, y, f) \cdot \hat{\mathbf{k}}|}\, dx\, dy$$



Parametric Surface Integral of a Scalar Field

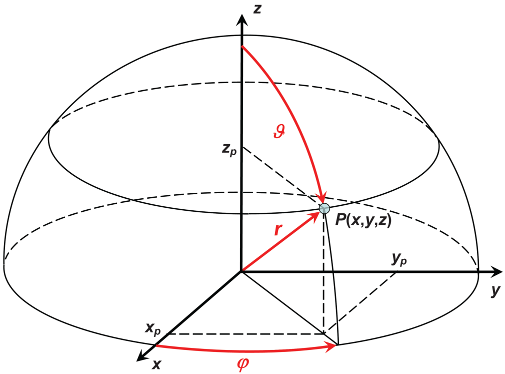
Example

Calculate the area of the upper hemispherical surface of radius a.

- Parameterization: Spherical-Polar System

$$P(x, y, z) \equiv \mathbf{r}(\theta, \phi) = a \sin\theta \cos\phi \hat{\mathbf{i}} + a \sin\theta \sin\phi \hat{\mathbf{j}} + a \cos\theta \hat{\mathbf{k}}$$

- Parameter Domain: $D = \{\theta \times \phi \,|\, \theta \in [0, \pi/2],\ \phi \in [0, 2\pi]\}$



Parametric Surface Integral of a Scalar Field (contd.)

Example

Calculate the area of the upper hemispherical surface of radius a.

- **With Parametrization:** Spherical-polar system
- $\mathbf{r}(\theta, \phi) = a\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ with $\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$
- $\mathbf{T}_\theta = \partial\mathbf{r}/\partial\theta = a\,(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$
- $\mathbf{T}_\phi = \partial\mathbf{r}/\partial\phi = a\,(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)$
- $\mathbf{N} = \mathbf{T}_\theta \times \mathbf{T}_\phi = \partial\mathbf{r}/\partial\theta \times \partial\mathbf{r}/\partial\phi = a^2\left(\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta\right)$
- JACOBIAN: $|\mathbf{N}| = |\mathbf{T}_\theta \times \mathbf{T}_\phi| = a^2\sin\theta$
- The area of hemisphere is

$$\text{Area} = \iint_S 1\, dS = \iint_D |\mathbf{T}_\theta \times \mathbf{T}_\phi|\, d\theta\, d\phi = \int_0^{\pi/2}\int_0^{2\pi} a^2\sin\theta\, d\theta\, d\phi = 2\pi a^2$$

- **Without parametrization:** Using Cartesian system
- S: $z = f(x, y) = \sqrt{a^2 - x^2 - y^2} \geq 0$ is the open upper hemisphere

Surface Integrals of a Scalar Field (without Parametrization)

Example

The equation of the upper hemispherical surface of radius a is represented as

$$F(x, y, z) = x^2 + y^2 + z^2 - a^2 = 0; \; z \geq 0$$

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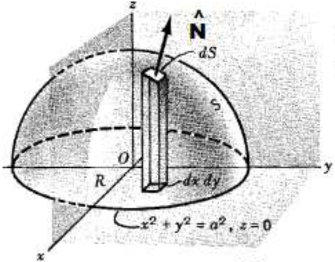


A normal to $x^2 + y^2 + z^2 = a^2$ is

$$\nabla(x^2 + y^2 + z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$$

Then the unit normal is

$$\hat{\mathbf{N}} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}$$

$$\text{Area} = \iint_R \frac{dx\,dy}{|\hat{\mathbf{N}} \cdot \mathbf{k}|} = \iint_R \frac{dx\,dy}{z/a} = a \int_{x=-a}^{a} \int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{dy\,dx}{\sqrt{a^2 - x^2 - y^2}}$$

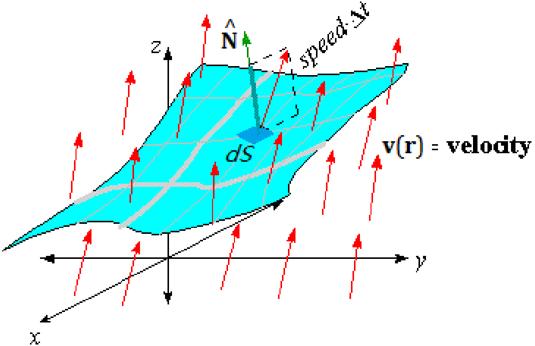
$$= \int_{\phi=0}^{2\pi} \int_{\rho=0}^{a} \frac{\rho\,d\rho\,d\phi}{\sqrt{a^2 - \rho^2}} = 2\pi a^2$$

where $x = \rho \cos\phi$, $y = \rho \sin\phi$ and $dy\,dx$ is replaced by $\rho\,d\rho\,d\phi$.

Surface Integrals of Vector Fields: Flux Integrals

Example

Consider a steady state flow of an incompressible fluid, which can be described by a velocity field $\mathbf{v}(\mathbf{r})$. What is the rate of flow of fluid across the surface?



The TOTAL FLUX yields the amount or volume of fluid flowing across the given surface in unit time, i.e.,

$$\text{Total Flux} = \iint_S \mathbf{v}(\mathbf{r}) \cdot d\mathbf{S} = \iint_S \mathbf{v}(\mathbf{r}) \cdot \hat{\mathbf{N}}\, dS$$

Parametric Flux Integrals

Definition

If S be a *smooth or piecewise smooth* surface, parametrized as $\mathbf{r} = \mathbf{r}(u, v)$ with $(u, v) \in D \subset \mathbb{R}^2$, then surface or FLUX INTEGRAL of a continuous vector field $\mathbf{F}(\mathbf{r})$ yields its *flux* through the surface S, i.e.,

$$\text{Flux of } \mathbf{F} = \iint_S \mathbf{F}(\mathbf{r}) \cdot d\mathbf{S} = \iint_S \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{N}}\, dS = \pm \iint_D \mathbf{F}\left[\mathbf{r}(u, v)\right] \cdot (\mathbf{T}_u \times \mathbf{T}_v)\, du\, dv$$

using the correspondence $dS = |\mathbf{N}|\, du\, dv$, with the "magnification/scale factor" or JACOBIAN given by the modulus of the normal vector:

$$\mathbf{N} = \pm(\mathbf{T}_u \times \mathbf{T}_v) = \pm\left(\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\right)$$

Note:

- There is always a *two-fold ambiguity* in deciding the sign of $\mathbf{N}$ for any general *two-sided* OPEN SURFACE.
- CLOSED SURFACE INTEGRAL: $\hat{\mathbf{N}} \equiv \hat{\mathbf{N}}_{\text{out}}$ is conventionally chosen as the *outward normal*, then the surface integral yields

$$\text{Net Outward Flux} = \oiint_S \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{N}}_{\text{out}}\, dS$$

Parametric Flux Integrals (contd.)

Example

Calculate the flux of $\mathbf{F}(x, y, z) = (x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}})/(x^2 + y^2 + z^2)^{3/2}$ through the same upper hemispherical surface of radius a.

- Spherical-polar Parametrization: $\mathbf{F}(\mathbf{r}) = \mathbf{r}/r^3$
- $\mathbf{r} = a\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
- Parameter domain: $D = \{\theta \times \phi \,|\, \theta \in [0, \pi/2],\ \phi \in [0, 2\pi]\}$
- $\mathbf{T}_\theta = \partial\mathbf{r}/\partial\theta = a\,(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta)$
- $\mathbf{T}_\phi = \partial\mathbf{r}/\partial\phi = a\,(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0)$
- $\mathbf{N} = \mathbf{T}_\theta \times \mathbf{T}_\phi = a^2\sin\theta\,(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$
- JACOBIAN: $|\mathbf{N}| = |\mathbf{T}_\theta \times \mathbf{T}_\phi| = a^2\sin\theta$
- FLUX: On the hemispherical surface S, $r = a$, $\hat{\mathbf{N}} = \hat{\mathbf{r}}$ and $\mathbf{N} = (a^2\sin\theta)\hat{\mathbf{r}}$

$$\begin{aligned}\text{Net Flux} = \iint_S \mathbf{F}(\mathbf{r}) \cdot \hat{\mathbf{N}}\, dS &= \iint_D [\mathbf{F}\,[\mathbf{r}(\theta, \phi)] \cdot \mathbf{N}]_S\; d\theta\, d\phi \\ &= \iint_D \left(\frac{a\hat{\mathbf{r}}}{a^3}\right) \cdot \hat{\mathbf{r}}\,(a^2\sin\theta)\, d\theta\, d\phi \\ &= \int_0^{\pi/2} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 2\pi.\end{aligned}$$

Volume Integrals

Definition

A volume integral is simply a 3D definitie integral or TRIPLE INTEGRAL of a continuous scalar field $f(x,y,z)$, or a vector field $\mathbf{A}(x,y,z)$, defined over a certain region of space $\mathrm{V}\subset\mathbb{R}^3$:

- Differential volume in Cartesian System : $dV \equiv dz\,dy\,dx$

$$I_{Scalar} = \iiint_{\mathrm{V}} f(\mathbf{r})\,dV = \int_{x=a}^{x=b}\left[\int_{y=g_1(x)}^{y=g_2(x)}\left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z)\,dz\right)dy\right]dx$$

$$\mathbf{I}_{Vector} = \iiint_{\mathrm{V}} \mathbf{A}(\mathbf{r})\,dV = \int_{x=a}^{x=b}\left[\int_{y=g_1(x)}^{y=g_2(x)}\left(\int_{z=f_1(x,y)}^{z=f_2(x,y)} \mathbf{A}(x,y,z)\,dz\right)dy\right]dx$$

- Differential volume with 3D Parametrization, $\mathbf{r}=\mathbf{r}(u,v,w)$ with $(u,v,w)\in D\subset\mathbb{R}^3$:

$$dV = |\mathbf{T}_u\cdot(\mathbf{T}_v\times\mathbf{T}_w)|\,du\,dv\,dw$$

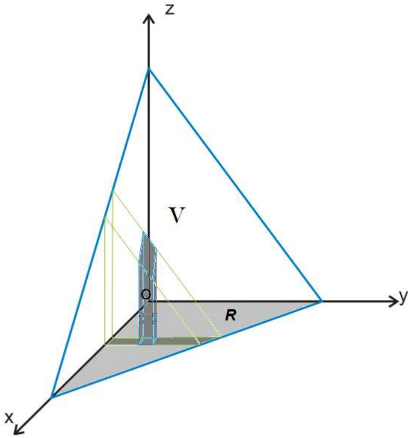
- The magnification/scale factor $J=|\mathbf{T}_u\cdot(\mathbf{T}_v\times\mathbf{T}_w)|$ is the JACOBIAN.

Volume Integrals in Cartesian System

Example

Determine the volume integral of $\phi(x, y, z) = 45x^2y$, over the closed region V bounded by the co-ordinate planes $x = 0$, $y = 0$, $z = 0$, and the plane $4x + 2y + z = 8$.

- We choose to project the region$\mathrm{V}$ onto the xy-plane, i.e., area R bounded by x-axis, y-axis and the line $4x + 2y = 8$.



- Here, it is <u>convenient</u> to first perform the z-integration, and then the double integral over the projected region R in the xy-plane.
- Limits of the integration are:

$$
\begin{aligned}
z = f_1(x,y) = 0 \quad &, \quad z = f_2(x,y) = 8 - 4x - 2y, \\
y = g_1(x) = 0 \quad &, \quad y = g_2(x) = 4 - 2x, \\
x = 0 \quad &, \quad x = 2
\end{aligned}
$$

- Volume Integral:

$$
\begin{aligned}
\iiint\limits_{\mathrm{V}} \phi(x,y,z)\, dV &= \iiint\limits_{\mathrm{V}} 45x^2 y\, dx\, dy\, dz \\
&= \iint\limits_{R} \left(\int\limits_{z=0}^{z=8-4x-2y} dz \right) 45x^2 y\, dx\, dy \\
&= 45 \int\limits_{x=0}^{x=2} x^2 \left[\int\limits_{y=0}^{y=4-2x} y(8-4x-2y)\, dy \right] dx \\
&= 45 \int\limits_{x=0}^{x=2} \frac{x^2}{3} (4-2x)^3 dx = 128
\end{aligned}
$$