DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
Even Semester of the Academic Year 2019-2020
MA 101 Mathematics I
Problem Sheet 4: Multiple integrals and applications, Green's Theorem, Stokes Theorem and Divergence Theorem.
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1. Evaluate the integrals:

(a) $\displaystyle\int_0^3\int_0^1 \sqrt{x+y}dx\ dy,$

(b) $\displaystyle\iint_R \frac{xy^2}{x^2+1}dA,\ R=\{(x,y)|\ 0\le x\le 1,\ -3\le y\le 3\}$

(c) $\displaystyle\iint_R x\sin(x+y)dA,\ R=\left[0,\frac{\pi}{6}\right]\times\left[0,\frac{\pi}{3}\right].$

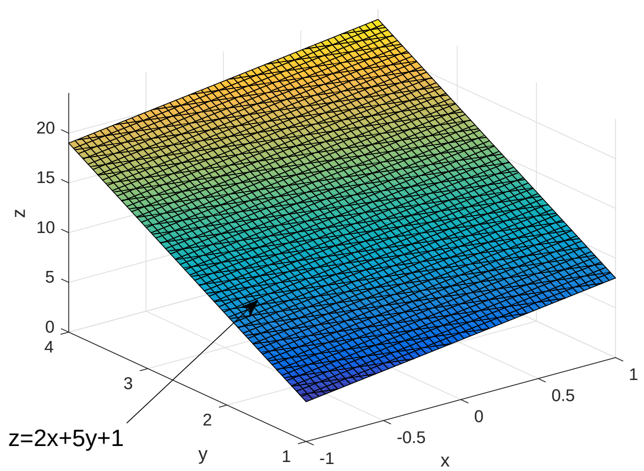
Solution:
(a) $\displaystyle\int_0^3\int_0^1 \sqrt{x+y}dx\ dy = \int_0^3\left[\frac{2}{3}(x+y)^{\frac{3}{2}}\right]_0^1 dy$
$\displaystyle= \frac{2}{3}\int_0^3\left[(1+y)^{\frac{3}{2}}-y^{\frac{3}{2}}\right]dy$
$\displaystyle= \frac{2}{3}\cdot\frac{2}{5}\left[(1+y)^{\frac{5}{2}}-y^{\frac{5}{2}}\right]_0^3$
$= \frac{4}{15}(31-9\sqrt{3}).$

(b) Ans $= 9\ln(2)$.

(c) Ans $= \frac{1}{2}(\sqrt{3}-1)-\frac{\pi}{12}$.

2. Find the volume of the solid lying under the plane $z = 2x+5y+1$ and above the rectangle $\{(x,y)|\ -1\le x\le 1,\ 1\le y\le 4\}$.

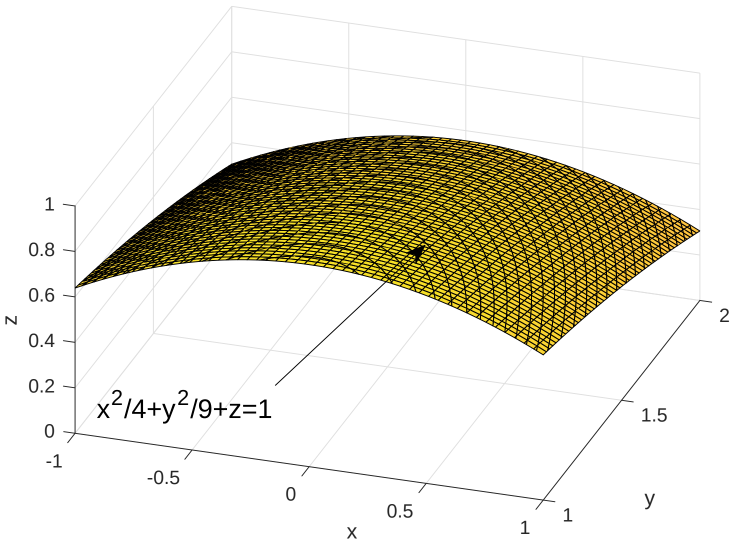
Solution: The required volume is given by:
$\displaystyle V=\iint_R z\ dA = \int_1^4\int_{-1}^1 (2x+5y+1)dx\ dy = 81.$



3. Find the volume of the solid lying under the elliptic paraboloid $\dfrac{x^2}{4}+\dfrac{y^2}{9}+z=1$ and above the square $R=[-1,1]\times[-2,2]$.

Solution: The required volume is given by:
$\displaystyle V=\iint_R z\ dA = \int_{-2}^2\int_{-1}^1 (1-\frac{x^2}{4}-\frac{y^2}{9})dx\ dy = \frac{166}{27}.$

4. Evaluate the iterated integrals:



(a) $\displaystyle\int_0^1 \int_y^{e^y} \sqrt{x}dx\ dy$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} e^{\sin\theta} dr\ d\theta.$

Solution:

(a) $\displaystyle\int_0^1 \int_y^{e^y} \sqrt{x}dx\ dy$

$\displaystyle= \frac{2}{3}\int_0^1 \left((e^y)^{\frac{3}{2}} - y^{\frac{3}{2}}\right) dy$

$\displaystyle= \frac{2}{3}\left(\frac{2}{3}e^{\frac{3y}{2}} - \frac{2}{5}y^{\frac{5}{2}}\right)_0^1$

$\displaystyle= \frac{4}{9}e^{\frac{3}{2}} - \frac{32}{45}.$

(b) $\displaystyle\int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} e^{\sin\theta} dr\ d\theta$

$\displaystyle= \int_0^{\frac{\pi}{2}} \left[re^{\sin\theta}\right]_0^{\cos\theta} d\theta$

$\displaystyle= \int_0^{\frac{\pi}{2}} e^{\sin\theta}\cos\theta d\theta$

$\displaystyle= \left[e^{\sin\theta}\right]_0^1 = e - 1.$

5. Evaluate the double integrals:

(a) $\displaystyle\iint_D \frac{2y}{x^2+1} dA,\ R = \{(x,y) |\ 0 \le x \le 1,\ 0 \le y \le \sqrt{x}\}$

(b) $\displaystyle\iint_D x\cos y dA$, D is bounded by $y = 0, y = x^2, x = 1$

(c) $\displaystyle\iint_D y^3 dA$, D is the triangular region with vertices (0,2), (1,1) and (3,2).

Solution:

(a) The required integral is equal to:

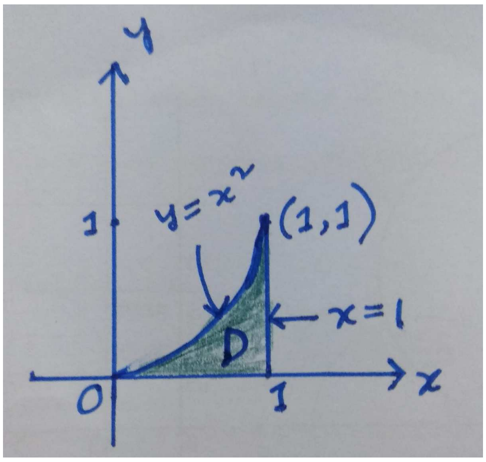
$\displaystyle\int_0^1 \int_0^{\sqrt{x}} \frac{2y}{x^2+1} dy\ dx$

$\displaystyle= \int_0^1 \frac{1}{x^2+1}\left[y^2\right]_0^{\sqrt{x}} dx$

$\displaystyle= \frac{1}{2}\left[ln(x^2+1)\right]_0^1 = \frac{1}{2}ln(2).$

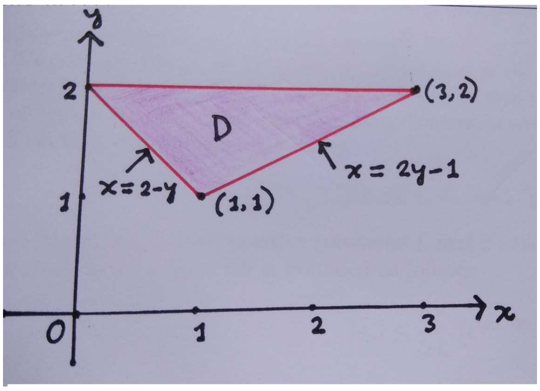
(b) The required integral is equal to:

$$\int_0^1 \int_0^{x^2} x \cos y \, dy \, dx$$
$$= \int_0^1 x \sin(x^2) dx$$
$$= \frac{1}{2} \left[-\cos(x^2) \right]_0^1$$
$$= \tfrac{1}{2}(1 - \cos(1)).$$



(c) The required integral is equal to:

$$\int_1^2 \int_{2-y}^{2y-1} y^3 dx \, dy \quad = \left(\int_0^1 \int_{2-x}^{2} y^3 dy \, dx + \int_1^3 \int_{\frac{1+x}{2}}^{2} y^3 dy \, dx \right)$$
$$= \int_1^2 y^3(3y - 3) dy$$
$$= \tfrac{147}{20}.$$



6. Find the volume of the given solids.

(a) Under the paraboloid $z = x^2 + y^2$ and above the region bounded by $y = x^2$ and $x = y^2$ in the first octant.

(b) Under the surface $z = xy$ and above the triangle with vertices (1,1), (4,1) and (1,2).

(c) Bounded by the cylinder $x^2 + z^2 = 9$ and the planes $x = 0, y = 0, z = 0, x + 2y = 2$ in the first octant.

(d) Bounded by the planes $x = 0, y = 0, z = 0$, and $x + y + z = 1$.

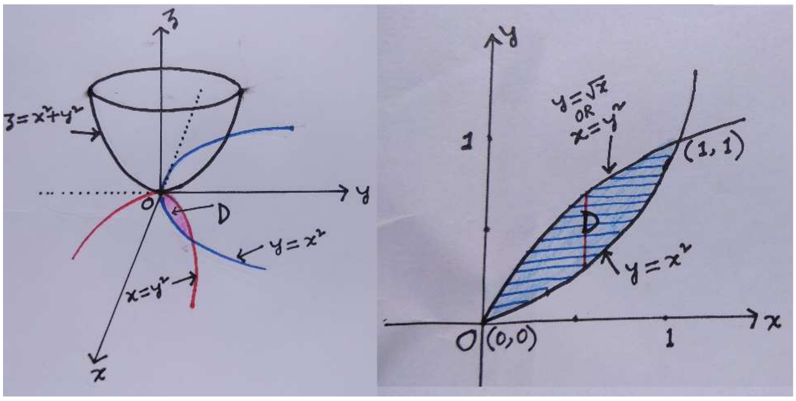
Solution: (a) The required volume is:

$= \iint_R z\ dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2+y^2)dy\ dx \quad \left(= \int_0^1 \int_{y^2}^{\sqrt{y}} (x^2+y^2)dx\ dy \right)$

$= \int_0^1 \left[x^2y + \frac{y^3}{3} \right]_{x^2}^{\sqrt{x}} dx$

$= \int_0^1 (x^{\frac{5}{2}} + \frac{x^{\frac{3}{2}}}{3} - x^4 - \frac{x^6}{3})dx$

$= \frac{18}{105}$



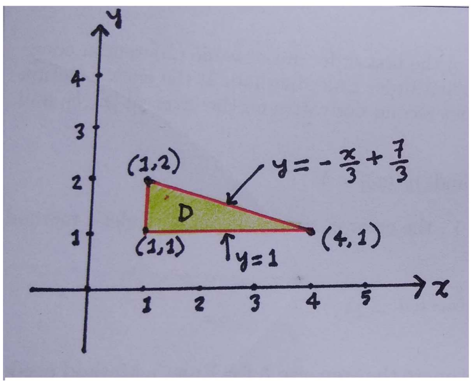
(b) The required volume is:

$= \iint_R z\ dA = \int_1^2 \int_1^{7-3y} xydx\ dy \qquad \left(= \int_1^4 \int_1^{\frac{7-x}{3}} xydy\ dx \right)$

$= \int_1^2 \frac{1}{2} y \left[x^2 \right]_1^{7-3y} dy$

$= \int_1^2 \frac{1}{2} y \left((7-3y)^2 - 1 \right) dy$

$= \frac{31}{8}$.



(c) The required volume is:

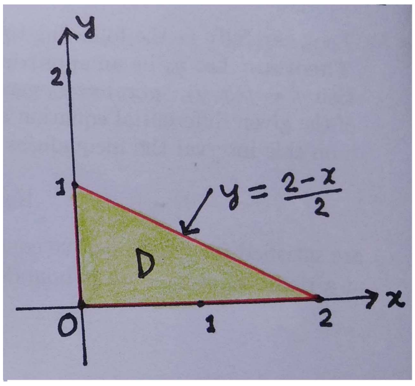
$= \iint_R z\ dA = \int_0^2 \int_0^{\frac{2-x}{2}} \sqrt{9-x^2}dy\ dx \quad \left(= \int_0^1 \int_0^{2-2y} \sqrt{9-x^2}dx\ dy \right)$

$= \int_0^2 \sqrt{9-x^2} \left(\frac{2-x}{2} \right) dx$

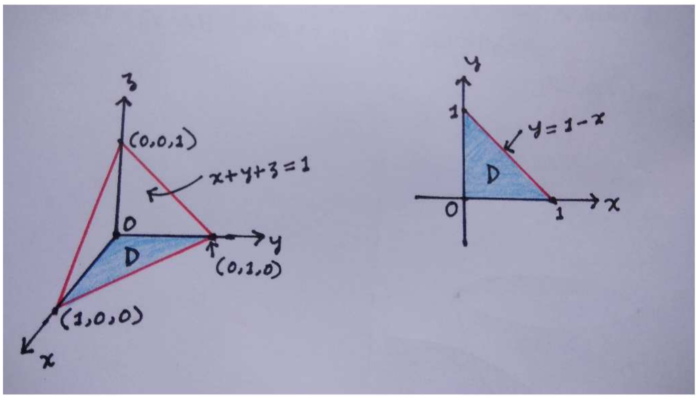
$= \int_0^2 \sqrt{9-x^2} \left(\frac{2-x}{2} \right) dx$

$= \int_0^2 \sqrt{9-x^2}dx - \frac{1}{2} \int_0^2 x\sqrt{9-x^2}dx$

$= (\sqrt{5} + \frac{9}{2}sin^{-1}(\frac{2}{3}) + \frac{1}{6}(5^{\frac{3}{2}} - 27)$.



(d) The required volume is:
$= \iint_R z\,dA = \int_0^1 \int_0^{1-y} (1-x-y)dx\,dy \quad \left(= \int_0^1 \int_0^{1-x} (1-x-y)dy\,dx\right)$
$= \frac{1}{6}$.



7. Get an upper bound and lower bound of each of the integrals given below by using the result that if m, M are such that $m \leq f(x,y) \leq M$ for all $(x,y) \in D$ then $m \times A(D) \leq \iint_D f(x,y)dA \leq M \times A(D)$, where $A(D)$ is the area of D.

   (a) $\iint_D \sqrt{x^3+y^3}dA,\ D = [0,1] \times [0,1]$

   (b) $\iint_D e^{x^2+y^2}dA$, D being the disk with center at origin and radius 0.5.

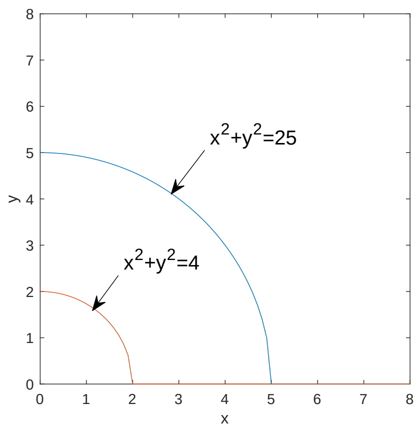
Solution: (a) Note that if there exists m and M such that $m \leq f(x,y) \leq M$ for all $(x,y) \in D$ then
$m \times A(D) \leq \iint_D f(x,y)dA \leq M \times A(D)$, where $A(D)$ is the area of D.
Since $0 \leq f(x,y) = \sqrt{x^3+y^3} \leq \sqrt{2}$ for all $(x,y) \in D$,
$0 \leq \iint_D \sqrt{x^3+y^3}dA \leq \sqrt{2} \times 1$.

(b) As in part (a),
since $0 \leq x^2+y^2 \leq 0.5$ for all $(x,y) \in D$,
$1 \leq f(x,y) = e^{x^2+y^2} \leq e^{0.5}$ for all $(x,y) \in D$
$\Rightarrow 1 \times A(D) \leq \iint_D e^{x^2+y^2}dA \leq e^{0.5} \times A(D)$, where $A(D)$ is the area of the disc
$\Rightarrow 1 \times \frac{\pi}{4} \leq \iint_D e^{x^2+y^2}dA \leq e^{0.5} \times \frac{\pi}{4}$.

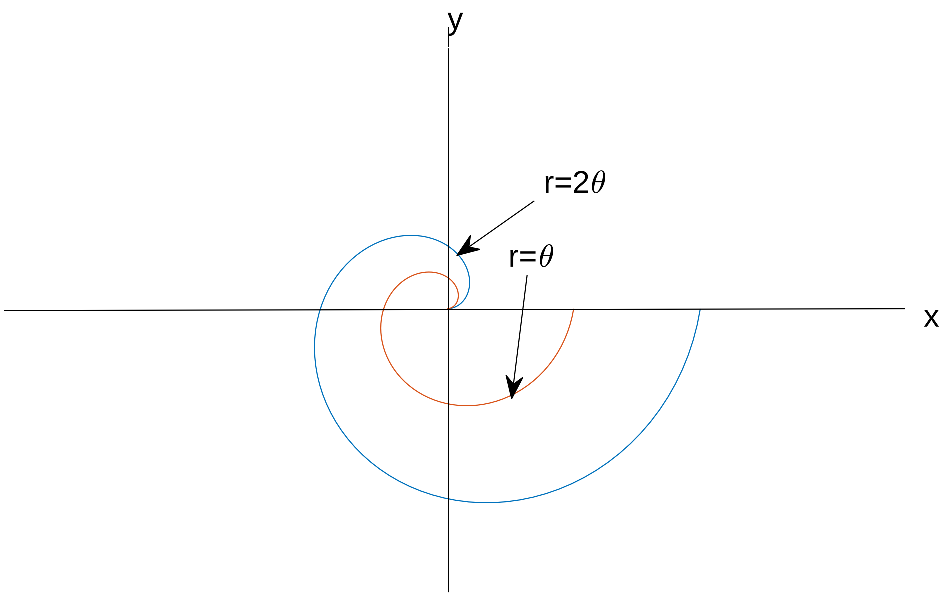
8. Using polar coordinates, find:

(a) $\iint_R xydA$, where R is the region in the first quadrant that lies between the circles $x^2+y^2=4$ and $x^2+y^2=25$.

Solution: The required integral when converted to polar coordinates is given by:
$\iint_R xydA = \int_2^5 \int_0^{\frac{\pi}{2}} r^2 \sin\theta\cos\theta \left| J\left(\frac{x,y}{r,\theta}\right)\right| d\theta dr$
$\int_2^5 \int_0^{\frac{\pi}{2}} r^2 \sin\theta\cos\theta r d\theta dr = \frac{609}{8}$.



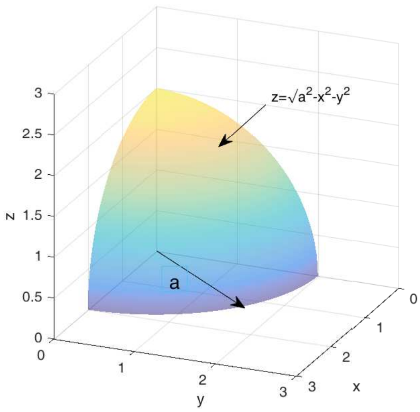
(b) $\iint_D (x^2+y^2)dA$, where D is the region bounded by the spirals $r=\theta$ and $r=2\theta$ for $0\le\theta\le 2\pi$.



Solution: As in part (a) the integral after changing to polar coordinates is given by:
$\iint_D (x^2+y^2)dA = \int_0^{2\pi}\int_\theta^{2\theta} r^2 \left| J\left(\frac{x,y}{r,\theta}\right)\right| drd\theta = \int_0^{2\pi}\int_\theta^{2\theta} r^3 drd\theta = 24\pi^5$.

(c) The volume of a sphere of radius a.

Solution: The volume of a sphere of radius a is eight times its volume in the first octant (the shaded portion in the figure).
The equation for a sphere of radius a centered at the origin is given by:
$x^2+y^2+z^2=a^2$, so for the upper hemisphere $z=\sqrt{a^2-x^2-y^2}$ and for the lower hemisphere $z=-\sqrt{a^2-x^2-y^2}$.
Hence the required volume is
$= 8\int_0^a \int_0^{\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}dxdy$ (which is not easy to calculate).
After changing to polar coordinates the integral is given by:



$$8\int_0^a\int_0^{\frac{\pi}{2}}\sqrt{a^2-r^2}\,rd\theta dr$$
$$=\tfrac{4}{3}\pi a^3.$$

(d) The volume of the closed region enclosed by the cylinder $x^2+y^2=4$ and the ellipsoid $4x^2+4y^2+z^2=64$.

Aliter of this solution on page 18

Solution: The required volume is given by:

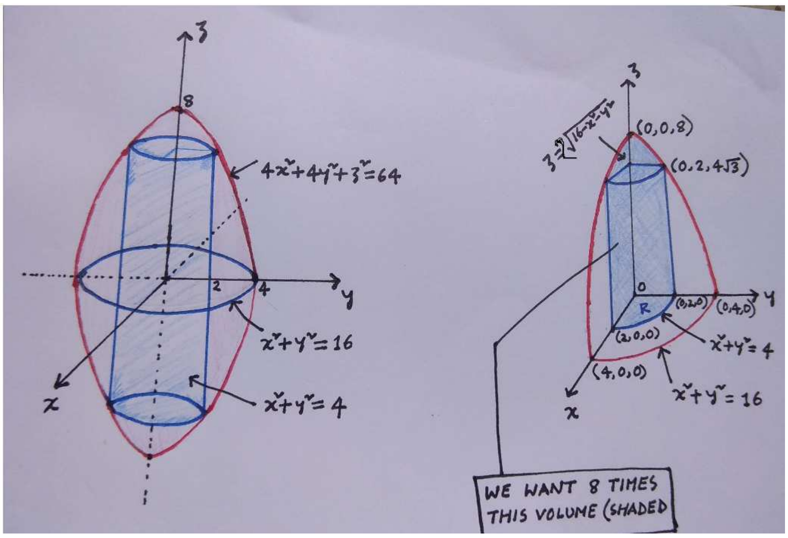
$$8\iint_R\sqrt{64-4x^2-4y^2}\,dA,$$

where $R=\{(x,y):x^2+y^2\le 4\}$.

$$=8\int_0^2\int_0^{\sqrt{4-x^2}}\sqrt{64-4x^2-4y^2}\,dydx \quad \text{(which is not easy to calculate).}$$

After changing to polar coordinates the integral is given by:

$$=8\int_0^{\frac{\pi}{2}}\int_0^2\sqrt{64-4r^2}\,rdrd\theta=-4\pi\left[\frac{2}{3}(16-r^2)^{\frac{3}{2}}\right]_0^2$$
$$=\frac{8\pi}{3}(64-24\sqrt{3}).$$



(e) $$\int_0^1\int_0^{\sqrt{1-x^2}}e^{x^2+y^2}\,dydx.$$

Solution: The required integral after changing to polar coordinates is given by:

$$\int_0^{\frac{\pi}{2}}\int_0^1 e^{r^2}\,rdrd\theta$$
$$=\tfrac{\pi}{4}(e-1).$$

(f) $$\int_0^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}x^2y^2\,dxdy.$$

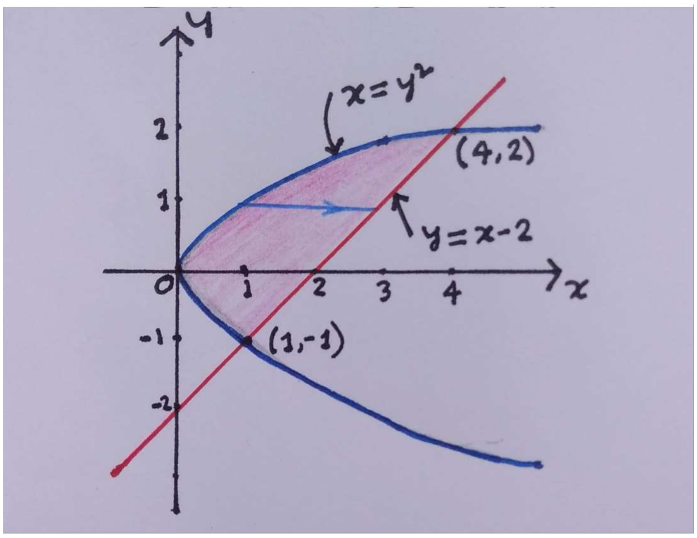
Solution: The required integral after changing to polar coordinates is given by:

$$2\int_0^{\frac{\pi}{2}}\int_0^2 r^2\cos^2\theta\sin^2\theta\, rdrd\theta$$

$= \frac{4\pi}{3}$.

9. Find the mass, center of mass and the moments of inertia I_x, I_y and I_o of the lamina D bounded by the parabola $x = y^2$ and the line $y = x - 2$; the density is $\rho(x, y) = 3$.

Solution: By definition, the mass and the coordinates of the center of mass are given by:
$M = \iint_D 3dA$, $\bar{x} = \dfrac{\iint_D 3xdA}{M}$, $\bar{y} = \dfrac{\iint_D 3ydA}{M}$.
$M = \int_{-1}^{2}\int_{y^2}^{y+2} 3dxdy = \frac{27}{2}$.
$\bar{x} = \dfrac{\iint_D 3xdA}{M} = \dfrac{\int_{-1}^{2}\int_{y^2}^{y+2} 3x\ dxdy}{M}$
$\bar{x} = \dfrac{8}{5}$.
$\bar{y} = \dfrac{\iint_D 3ydA}{M} = \dfrac{\int_{-1}^{2}\int_{y^2}^{y+2} 3y\ dxdy}{M}$
$\bar{y} = \dfrac{1}{2}$.
$I_x = \int_{-1}^{2}\int_{y^2}^{y+2} 3y^2\ dxdy = \dfrac{189}{20}$, $I_y = \int_{-1}^{2}\int_{y^2}^{y+2} 3x^2\ dxdy = \dfrac{1269}{28}$ and
$I_0 = I_x + I_y = \dfrac{1917}{35}$.



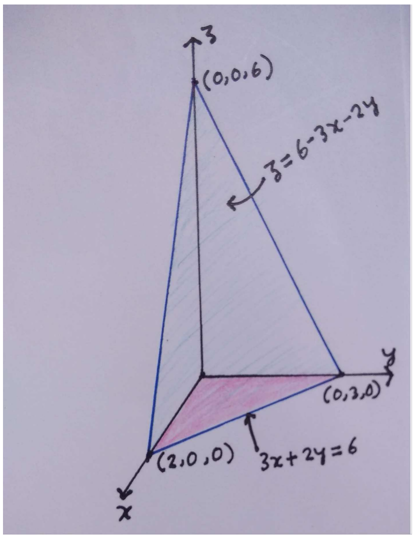
10. Find the area of the surface:

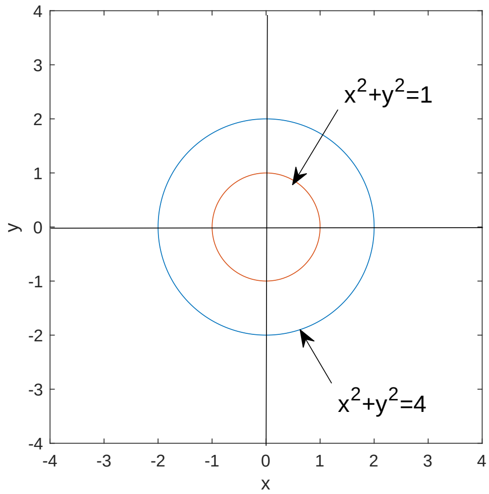
(a) The part of the plane $3x + 2y + z = 6$ that lies in the first octant.

Solution: If $z = f(x, y)$ then the required surface area is given by:
$\iint_R \sqrt{1 + f_x^2 + f_y^2}dA$
$\int_0^2\int_0^{\frac{6-3x}{2}} \sqrt{1 + (-3)^2 + (-2)^2}dydx \quad \left(\int_0^3\int_0^{\frac{6-2y}{3}} \sqrt{1 + (-3)^2 + (-2)^2}dxdy\right)$.
$= 3\sqrt{14}$.

(b) The part of the hyperbolic paraboloid $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

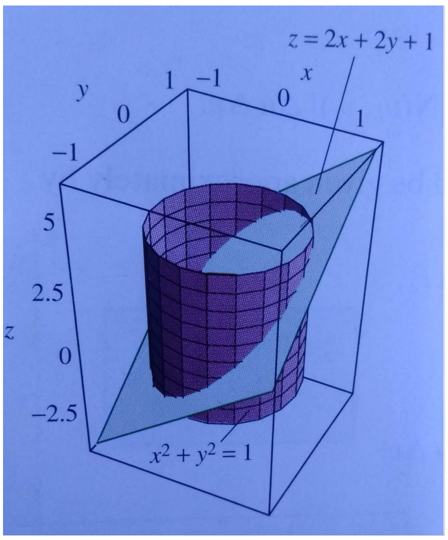
Solution: If $z = f(x, y)$ then the required area is given by:
$\iint_R \sqrt{1 + f_x^2 + f_y^2}dA. \qquad (1)$
After changing (1) to polar coordinates the above integral is equal to
$4\int_1^2\int_0^{\frac{\pi}{2}} \sqrt{1 + 4r^2}rd\theta dr$
$= \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5})$.
(The expression for (1) in rectangular coordinates is given by:
$= 4\int_1^2\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} \sqrt{1 + (-2x)^2 + (2y)^2}dydx$
which is difficult to integrate).





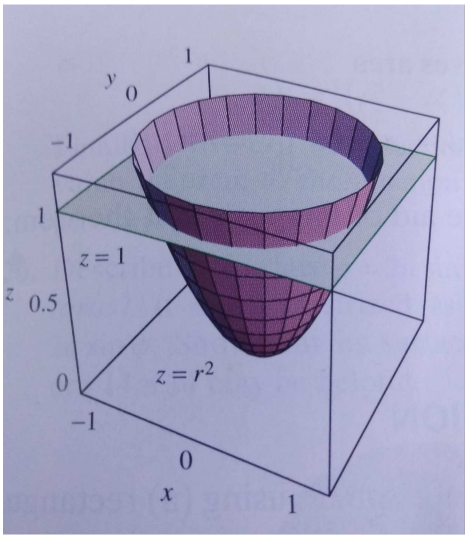
(c) Of the ellipse cut from the plane $z = 2x + 2y + 1$ by the cylinder $x^2 + y^2 = 1$.

Solution: If $z = f(x, y)$ then the required area is given by:
$\displaystyle\iint_R \sqrt{1 + f_x^2 + f_y^2}dA,$
where R is the unit disc centered at the origin.
The above integral is equal to
$\displaystyle 4\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1 + 2^2 + 2^2}dydx$
$= 3\pi.$



(d) Cut from the paraboloid $z = r^2$ by the cylinder $r = 1$.

Solution: If $z = f(x, y)$ then the required area is given by:
$$\iint_R \sqrt{1 + f_x^2 + f_y^2} dA,$$
where R is the unit disc centered at the origin.
The above integral I is equal to
$$4 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1 + f_x^2 + f_y^2} dy dx,$$
$$4 = \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1 + (2x)^2 + (2y)^2} dy dx.$$
By changing to polar coordinates we get:
$$I = 4 \int_0^1 \int_0^{\frac{\pi}{2}} \sqrt{1 + 4r^2} r d\theta dr$$
$$= 4 \int_0^1 \frac{\pi}{2} \sqrt{1 + 4r^2} r dr = \frac{\pi}{6}(5\sqrt{5} - 1).$$



11. Evaluate the triple integrals.

(a) $\iiint_E 2x dV$, where $E = \left\{(x, y, z) |\ 0 \le y \le 2,\ 0 \le x \le \sqrt{4 - y^2},\ 0 \le z \le y\right\}$

Solution: The required integral is given by:
$$\int_0^2 \int_0^y \int_0^{\sqrt{4-y^2}} 2x dx dz dy$$
$$= \int_0^2 \int_0^y \left[x^2\right]_0^{\sqrt{4-y^2}} dz dy$$
$$= \int_0^2 \int_0^y (4 - y^2) dz dy$$
$$= \int_0^2 y(4 - y^2) dy$$
$$= 4.$$

Aliter: The required integral is also given by:
$$\int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^y 2x dz dx dy$$
$$= \int_0^2 \int_0^{\sqrt{4-y^2}} 2x \left[z\right]_0^y dx dy$$
$$= \int_0^2 \int_0^{\sqrt{4-y^2}} 2xy dx dy$$
$$= \int_0^2 y \left[x^2\right]_0^{\sqrt{4-y^2}} dy$$
$$= \int_0^2 y(4 - y^2) dy$$
$$= 4.$$

(b) $\iiint_E x dV$, where E is bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.

Solution: The required integral is given by:

$= 4\int_0^4 \int_0^{\frac{\sqrt{x}}{2}} \int_0^{\frac{\sqrt{x-4y^2}}{2}} xdzdydx$

$= 2\int_0^4 \int_0^{\frac{\sqrt{x}}{2}} \sqrt{x-4y^2}xdydx$

$= 2\int_0^4 \int_0^{\frac{\sqrt{x}}{2}} \sqrt{x-4y^2}xdydx$

Take u such that $\sqrt{x}u = 2y$ then $\sqrt{x}du = 2dy$ then the above integral reduces to:

$2 \times \frac{1}{2}\int_0^4 \int_0^1 \sqrt{x-xu^2}\ x\sqrt{x}dudx$

$= \int_0^4 \int_0^1 \sqrt{1-u^2}\ x^2dudx$

$= \int_0^4 x^2 \left[\frac{1}{2}sin^{-1}(u)\right]_0^1 dx$

$= \frac{\pi}{4}\left[\frac{x^3}{3}\right]_0^4 = \frac{\pi 4^2}{3} = \frac{16\pi}{3}.$

Aliter: The required integral is also given by:

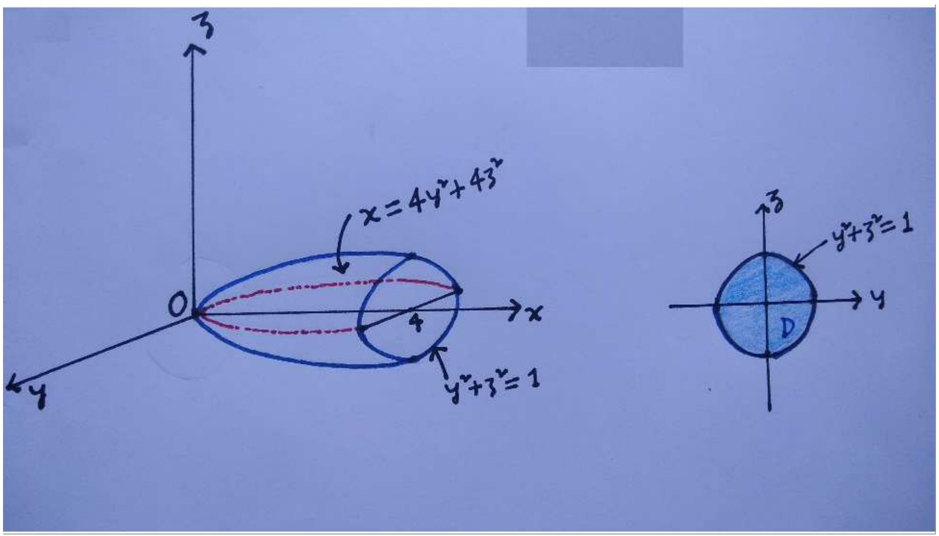
$I = \iint_D (\int_{4z^2+4y^2}^4 xdx)dA$, where D is the unit disc centered at the origin in the Y, Z plane.

Hence $I = \iint_D \frac{1}{2}(4^2 - 4z^2 - 4y^2)^2)dA$.

By changing to polar coordinates we get:

$I = 4\int_0^1 \int_0^{\frac{\pi}{2}} 8\left(1-r^4\right) rd\theta dr = \frac{16\pi}{3}$.

Aliter: $4\int_0^1 \int_{4y^2}^4 \int_0^{\frac{\sqrt{x-4y^2}}{2}} xdzdxdy$



$= 4\int_0^1 \int_{4y^2}^4 \frac{\sqrt{x-4y^2}}{2}xdxdy$

$= 2\int_0^1 \int_{4y^2}^4 \left((x-4y^2)^{\frac{3}{2}} + 4y^2\sqrt{x-4y^2}\right) dxdy = 2\int_0^1 \left[\frac{2}{5}(x-4y^2)^{\frac{5}{2}} + \frac{2}{3} \times 4y^2(x-4y^2)^{\frac{3}{2}}\right]_{4y^2}^4 dy$

$= 2\int_0^1 \left(\frac{2}{5} \times (2^5(1-y^2)^{\frac{5}{2}}) + \frac{2}{3} \times 2^3 \times (4y^2(1-y^2)^{\frac{3}{2}})\right) dy$

$= 2\int_0^1 \left(\frac{2^6}{5}(1-y^2)^{\frac{5}{2}} - \frac{2^6}{3}(1-y^2-1)(1-y^2)^{\frac{3}{2}}\right) dy$

$= 2\int_0^1 \left(-\frac{2^7}{5}(1-y^2)^{\frac{5}{2}} + \frac{2^6}{3}(1-y^2)^{\frac{3}{2}}\right) dy$

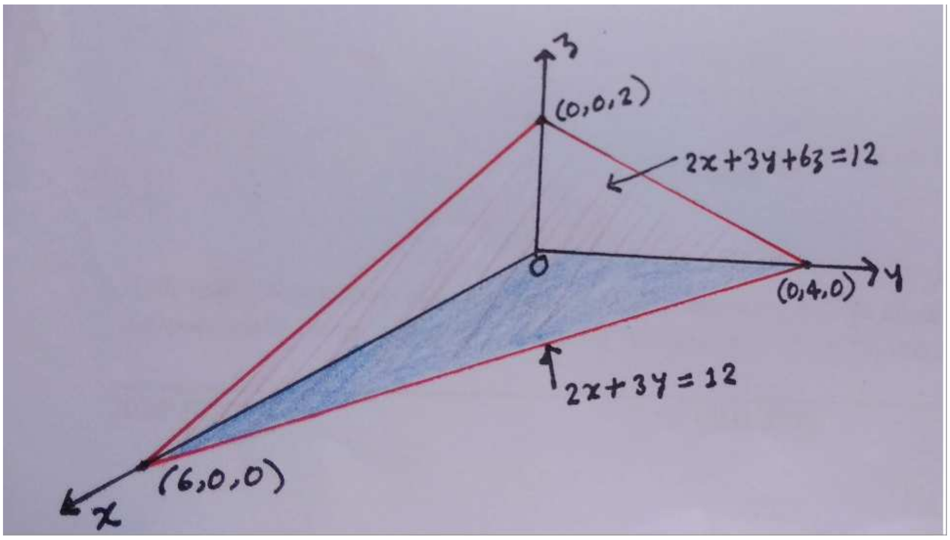
$= \frac{16\pi}{3}$.

(the above calculations are lengthy and points out the merit of using polar coordinates)

Aliter: $= 4\int_0^1 \int_0^{\sqrt{1-z^2}} \frac{1}{2}\left[x^2\right]_{4z^2+4y^2}^{4} dydz$
$= 4 \times 8 \int_0^1 \int_0^{\sqrt{1-z^2}} \left(1-(z^2+y^2)^2\right) dydz)$
(the calculations are lengthy and this representation of the integral should be avoided).

12. Use triple integral to find the volume of the tetrahedron bounded by the coordinate planes and the plane $2x+3y+6z=12$.

Solution: The required volume is given by:
$= \int_0^4 \int_0^{\frac{12-3y}{2}} \int_0^{\frac{12-3y-2x}{6}} dzdxdy$
$= 8.$



Aliter: The required volume is also given by:
$= \int_0^2 \int_0^{\frac{12-3y}{6}} \int_0^{\frac{12-3y-6z}{2}} dxdzdy$
Aliter: The required volume is also given by:
$= \int_0^6 \int_0^{\frac{12-2x}{6}} \int_0^{\frac{12-2x-6z}{3}} dydzdx.$
...etc.

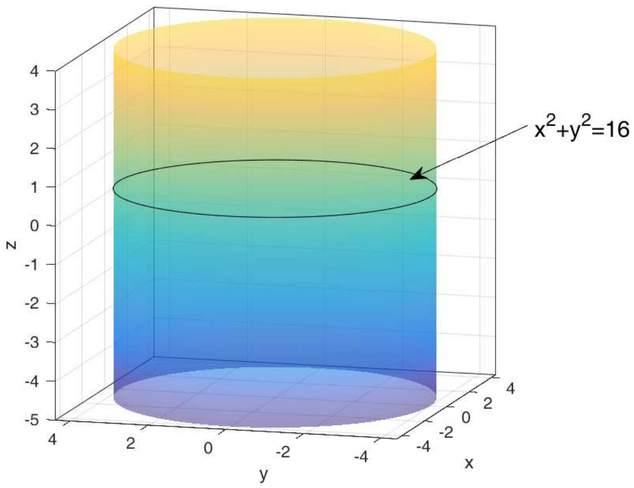
13. Use cylindrical or spherical coordinates whichever is appropriate, to:

(a) Evaluate $\iiint_E \sqrt{x^2+y^2}dV$, where E is the region that lies inside the cylinder $x^2+y^2=16$ and between the planes $z=-5$ and $z=4$.

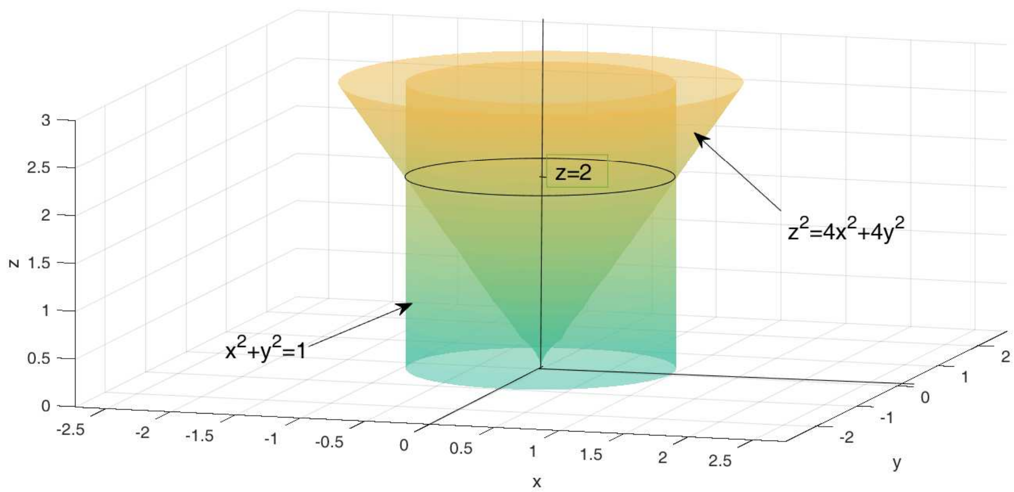
Solution: By using cylindrical coordinates the required integral is given by:
$4\int_{-5}^4 \int_0^4 \int_0^{\frac{\pi}{2}} r \left| J(\frac{x,y,z}{r,\theta,z}) \right| d\theta drdz$
$= 4\int_{-5}^4 \int_0^4 \int_0^{\frac{\pi}{2}} r^2 d\theta drdz$
$= 384\pi.$

(b) Evaluate $\iiint_E x^2 dV$, where E is the solid that lies within the cylinder $x^2+y^2=1$, above the plane $z=0$, and below the cone $z^2=4x^2+4y^2$.

Solution: By using cylindrical coordinates the required integral is given by:
$\int_0^1 \int_0^{2\pi} \int_0^{2r} r^2\cos^2\theta \left| J\left(\frac{x,y,z}{r,\theta,z}\right) \right| dzd\theta dr$
$= \int_0^1 \int_0^{2\pi} \int_0^{2r} r^3\cos^2\theta dzd\theta dr$
$= \int_0^1 \int_0^{2\pi} r^3\cos^2\theta \times 2rd\theta dr$



$$= \int_0^1 \int_0^{2\pi} 2r^4 \frac{(\cos 2\theta + 1)}{2} d\theta dr$$
$$= 2\left[\frac{r^5}{5}\right]_0^1 \left[\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right]_0^{2\pi}$$
$= \frac{2\pi}{5}$. **Aliter:** By using spherical coordinates the required integral is given by:



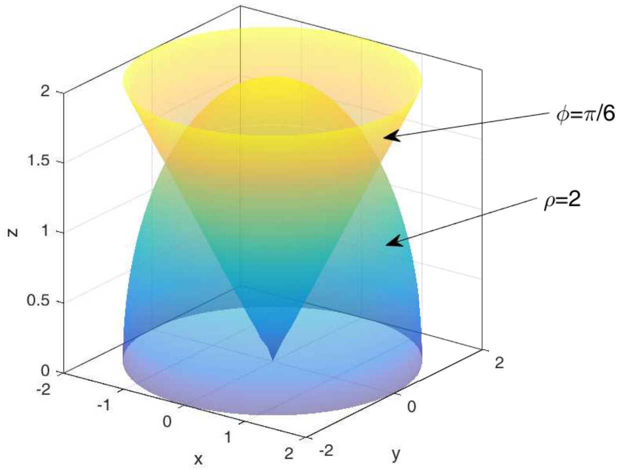
$$4\int_{\cot^{-1}2}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\phi}} \int_0^{\frac{\pi}{2}} \rho^2 \cos^2\theta \sin^2\phi \left| J\left(\frac{x,y,z}{\rho,\phi,\theta}\right)\right| d\theta d\rho d\phi$$
$$= 4\int_{\cot^{-1}2}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\phi}} \int_0^{\frac{\pi}{2}} \rho^2 \cos^2\theta \sin^2\phi \rho^2 \sin\phi \; d\theta d\rho d\phi$$
$$= 4\left[\int_{\cot^{-1}2}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\phi}} \rho^4 \sin^3\phi \; d\rho d\phi\right]\left[\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta\right]$$
$$= \pi \int_{\cot^{-1}2}^{\frac{\pi}{2}} \int_0^{\frac{1}{\sin\phi}} \rho^4 \sin^3\phi d\rho d\phi$$
$$= \pi \int_{\cot^{-1}2}^{\frac{\pi}{2}} \frac{1}{5}\frac{1}{\sin^5\phi} \sin^3\phi d\phi$$
$$= \frac{\pi}{5}\int_{\cot^{-1}2}^{\frac{\pi}{2}} cosec^2\phi d\phi = \frac{\pi}{5}\left[-\cot\phi\right]_{\cot^{-1}2}^{\frac{\pi}{2}} = \frac{2\pi}{5}.$$

Aliter: $\displaystyle\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{2\sqrt{x^2+y^2}} x^2 dz dx dy$

$= 4\displaystyle\int_0^1 \int_0^{\sqrt{1-y^2}} 2\sqrt{x^2+y^2} x^2 dx dy$ (which is difficult to compute).

(c) Evaluate $\iiint_E \sqrt{x^2+y^2+z^2} dV$, where E is bounded below by the cone $\phi = \frac{\pi}{6}$ and above by the sphere $\rho = 2$.

Solution: By using spherical coordinates the integral becomes:
$\int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{2} \rho \left| J\left(\frac{x,y,z}{\rho,\phi,\theta}\right)\right| d\rho d\phi d\theta$
$= \int_0^{2\pi}\int_0^{\frac{\pi}{6}}\int_0^{2} \rho(\rho^2 \sin\phi) d\rho d\phi d\theta$
$= 4\pi(2-\sqrt{3})$.



14. Find the volume and centroid of the solid E that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 1$.

Solution: By using spherical coordinates the volume of the enclosed solid is given by:
$\int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{1} \left| J\left(\frac{x,y,z}{\rho,\phi,\theta}\right)\right| d\rho d\phi d\theta$, (the angle $\phi = \frac{\pi}{4}$ for all points on the boundary of the cone)
$= \int_0^{2\pi}\int_0^{\frac{\pi}{4}}\int_0^{1} (\rho^2 \sin\phi) d\rho d\phi d\theta$,
$= \frac{2\pi}{3}(1-\frac{1}{\sqrt{2}})$.

15. Show that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sqrt{x^2+y^2+z^2}e^{-(x^2+y^2+z^2)} dxdydz = 2\pi$.

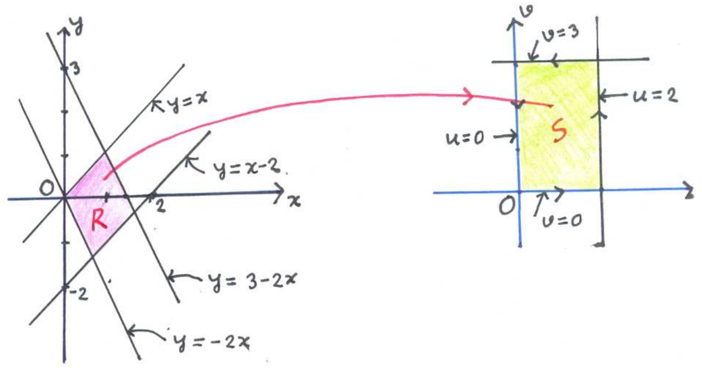
Solution: By using spherical coordinates we get:
$\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} \rho e^{-\rho^2} \left| J\left(\frac{x,y,z}{\rho,\phi,\theta}\right)\right| d\rho d\phi d\theta$
$= \int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-\rho^2}\rho(\rho^2 \sin\phi) d\rho d\phi d\theta$
$= 2\pi\left[-\cos\phi\right]_0^{\pi} \int_0^{\infty} e^{-\rho^2}\rho^3 d\rho$
$= 2\pi \times 2 \times \frac{1}{2}\left[-e^{-\rho^2}\rho^2 - e^{-\rho^2}\right]_0^{\infty}$
$= 2\pi$.

16. Use the given transformation to evaluate the integral:

(a) $\iint_R (3x+4y)dA$, where R is the region bounded by the lines $y = x$, $y = x-2$, $y = -2x$ and $y = 3-2x$; $x = \frac{1}{3}(u+v)$, $y = \frac{1}{3}(v-2u)$.

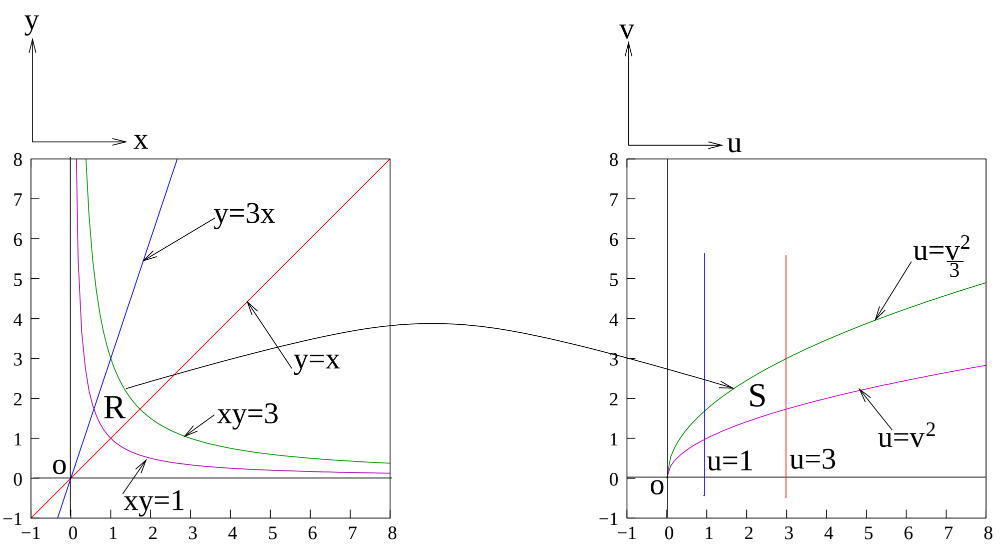
Solution: Let $I = \iint_R (3x+4y)dA$.
Then $I = \int_0^{\frac{2}{3}}\int_{-2x}^{x} (3x+4y)dydx + \int_{\frac{2}{3}}^{1}\int_{x-2}^{x} (3x+4y)dydx + \int_1^{\frac{5}{3}}\int_{x-2}^{3-2x} (3x+4y)dydx$,
calculations for which is lengthy.
By the given change of variables we get:
since $x = \frac{1}{3}(u+v), y = \frac{1}{3}(v-2u)$

$u = x - y, v = 2x + y$.
The lines $y = x$,$y = x - 2$, $y = -2x$ and $y = 3 - 2x$ gives the lines $u = 0, u = 2, v = 0, v = 3$, respectively in the changed coordinate system.
The Jacobian is given by
$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} \end{vmatrix} = \frac{1}{3}$.
Hence $I = \displaystyle\int_0^3 \int_0^2 (u + v + \frac{4}{3}(v - 2u))\frac{1}{3} dudv$
$= \frac{11}{3}$.



(b) $\displaystyle\iint_R xydA$, where R is the region in the first quadrant bounded by the lines $y = x$ and $y = 3x$ and the hyperbolas $xy = 1$, $xy = 3$; $x = \frac{u}{v}$, $y = v$.

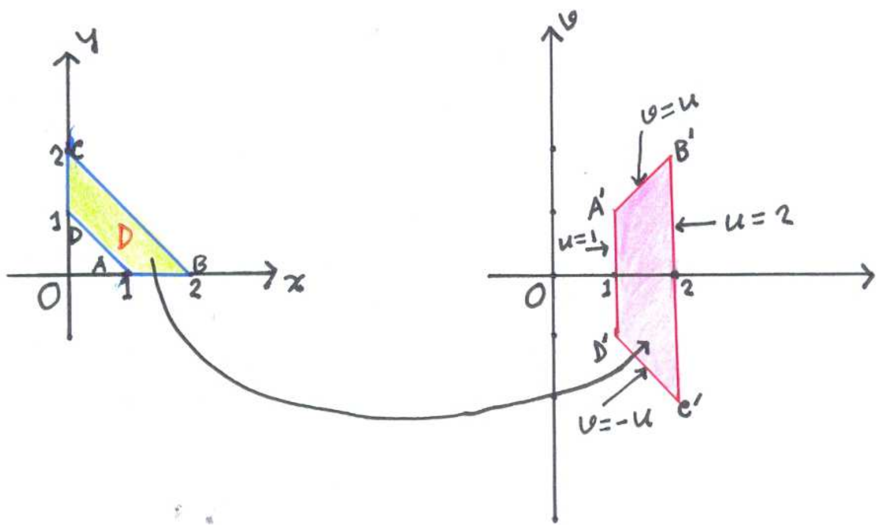
Solution: Let $I = \displaystyle\iint_R xydA$.
In rectangular coordinates $I = \displaystyle\int_{\frac{1}{\sqrt{3}}}^{1} \int_{\frac{1}{x}}^{3x} xydydx + \int_1^{\sqrt{3}} \int_x^{\frac{3}{x}} xydydx$,
which involves lengthy calculations.
If we change the variables to u, v then $u = xy$ and $v = y$.
The lines $y = x$ and $y = 3x$ gives the parabolas $u = v^2$ and $3u = v^2$ respectively and the hyperbolas $xy = 1$ and $xy = 3$ gives the lines $u = 1$ and $u = 3$, respectively in the changed coordinate system.
The Jacobian is given by:
$J\left(\frac{x,y}{u,v}\right) = \begin{vmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ 0 & 1 \end{vmatrix} = \frac{1}{v}$.
Hence $I = \displaystyle\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u \left|J(\frac{x,y}{u,v})\right| dvdu$
$= \displaystyle\int_1^3 \int_{\sqrt{u}}^{\sqrt{3u}} u \times \frac{1}{v} dvdu$
$= 2\ln 3$.



17. Evaluate the integral by making an appropriate change of variables:

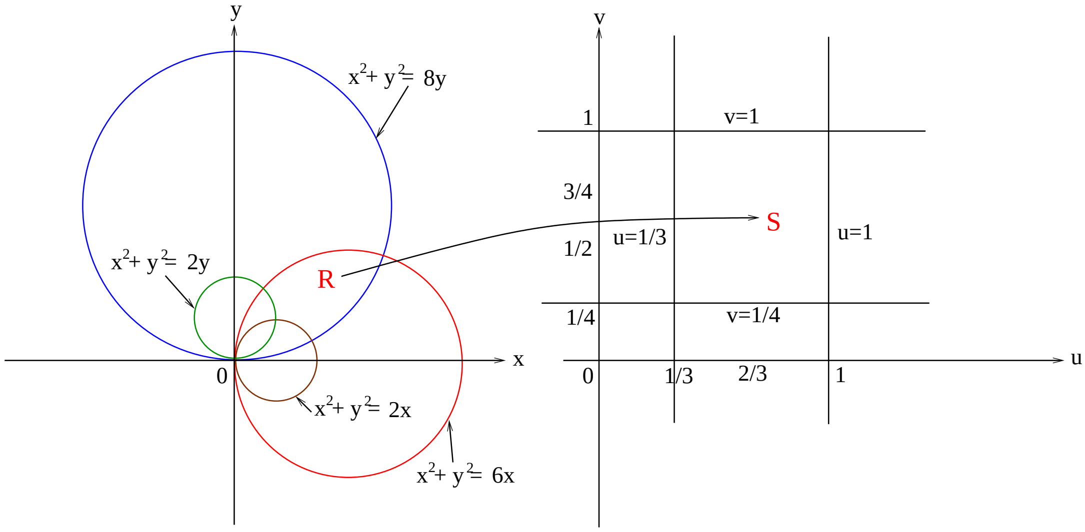
(a) $\iint_R \cos\left(\frac{y-x}{y+x}\right) dA$, where R is the trapezoidal region with vertices $(1,0)$, $(2,0)$, $(0,2)$ and $(0,1)$.

Solution: Let $I = \iint_R \cos\left(\frac{y-x}{y+x}\right) dA$.
Then $I = \int_0^1 \int_{1-y}^{2-y} \cos\left(\frac{y-x}{y+x}\right) dxdy + \int_1^2 \int_0^{2-y} \cos\left(\frac{y-x}{y+x}\right) dxdy$
(which is clearly difficult to compute).
By change of variables take $u = x+y$ and $v = y-x$, then $x = \frac{u-v}{2}$ and $y = \frac{u+v}{2}$.
The lines $x = 0, y = 0, x+y = 1, x+y = 2$ gives the lines $v = u, v = -u, u = 1, u = 2$ respectively in the changed coordinate system.
The Jacobian is given by:
$J(\frac{x,y}{u,v}) = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$.
Hence $I = \int_1^2 \int_{-u}^{u} \cos\left(\frac{v}{u}\right) \left|J(\frac{x,y}{u,v})\right| dvdu$
$= \int_1^2 \left[u \sin\left(\frac{v}{u}\right)\right]_{-u}^{u} \frac{1}{2} du$
$= \frac{3}{2}\sin(1)$.



(b) $\iint_R \frac{1}{(x^2+y^2)^2} dxdy$, where R is the first quadrant region bounded by the circles $x^2+y^2 = 2x$, $x^2+y^2 = 6x$ and the circles $x^2+y^2 = 2y$, $x^2+y^2 = 8y$.

Solution: Let $I = \iint_R \frac{1}{(x^2+y^2)^2} dxdy$.
Note that the above integral in rectangular coordinates is not easy to compute:
Take $u = \frac{2x}{x^2+y^2}$ and $v = \frac{2y}{x^2+y^2}$, then $y = \frac{2v}{u^2+v^2}$ and $x = \frac{2u}{u^2+v^2}$.
The circles $x^2+y^2 = 2x$, $x^2+y^2 = 6x$, $x^2+y^2 = 2y$, $x^2+y^2 = 8y$, gives the lines $u = 1, u = \frac{1}{3}, v = 1, v = \frac{1}{4}$, respectively.
The Jacobian is given by:
$J(\frac{x,y}{u,v}) = \begin{vmatrix} \frac{2}{(u^2+v^2)} - \frac{4u^2}{(u^2+v^2)^2} & -\frac{4uv}{(u^2+v^2)^2} \\ -\frac{4uv}{(u^2+v^2)^2} & \frac{2}{(u^2+v^2)} - \frac{4v^2}{(u^2+v^2)^2} \end{vmatrix} = \frac{4}{(u^2+v^2)^2}$.
Hence $I = \int_{\frac{1}{4}}^{1} \int_{\frac{1}{3}}^{1} \frac{1}{16}(u^2+v^2)^2 \left|J(\frac{x,y}{u,v})\right| dudv$
$= \int_{\frac{1}{4}}^{1} \int_{\frac{1}{3}}^{1} \frac{1}{16}(u^2+v^2)^2 \frac{4}{(u^2+v^2)^2} dudv$
$= \frac{1}{8}$.



18. Let R be the solid ellipsoid with constant density δ and boundary surface

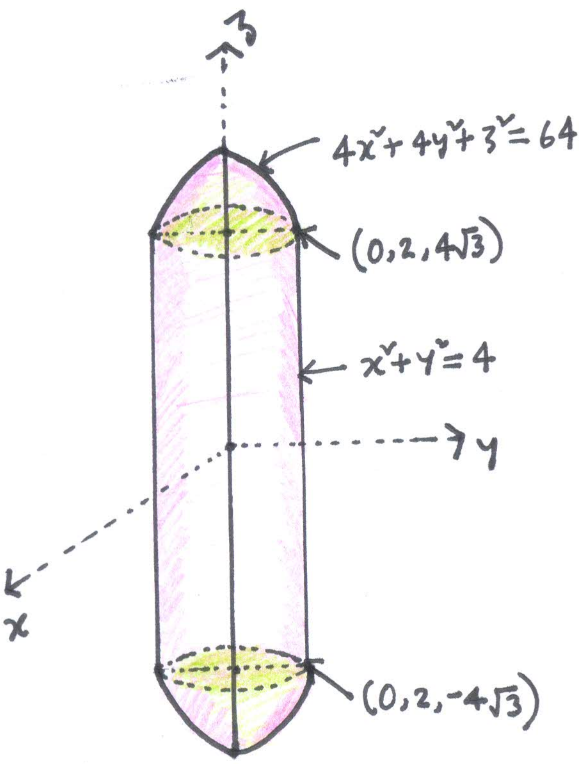
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

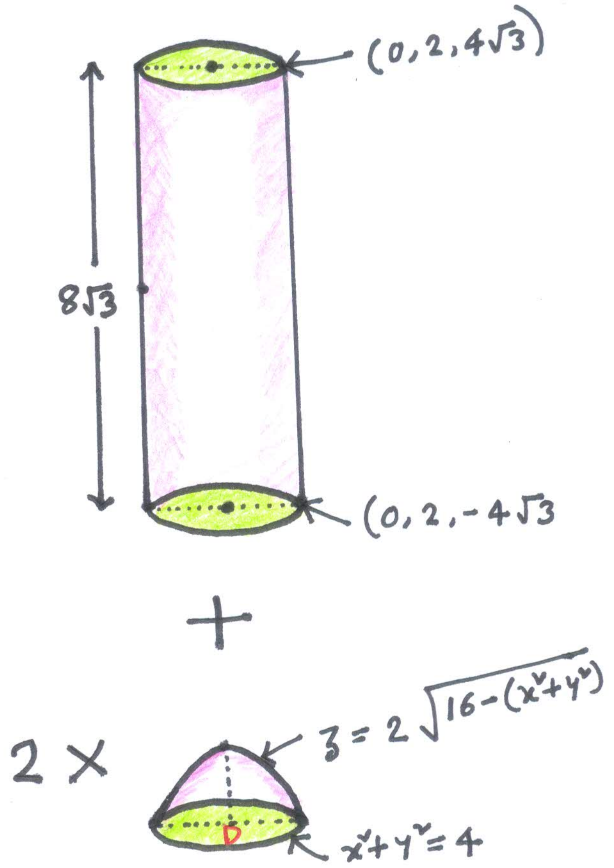
Use appropriate transformations to show that the mass M of R is $\dfrac{4}{3}\pi\delta abc$.

Solution: The mass M of the solid ellipsoid is given by:
$\displaystyle\iiint_R \delta dV = 8\delta\int_0^{a^2}\int_0^{\frac{b}{a}\sqrt{a^2-x^2}} c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}dydx$, which is not easy to calculate:

Consider $x = a\rho\sin\phi\cos\theta$, $y = b\rho\sin\phi\sin\theta$, $z = c\rho\cos\phi$.
The Jacobian is given by:
$$J(\tfrac{x,y,z}{\rho,\phi,\theta}) = \begin{vmatrix} a\sin\phi\cos\theta & a\rho\cos\phi\cos\theta & -a\rho\sin\phi\sin\theta \\ b\sin\phi\sin\theta & b\rho\cos\phi\sin\theta & b\rho\sin\phi\cos\theta \\ c\cos\phi & -c\rho\sin\phi & 0 \end{vmatrix} = abc\rho^2\sin\phi.$$
Hence the above integral is given by:
$$8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \delta\left|J\left(\frac{x,y,z}{\rho,\theta,\phi}\right)\right| d\rho d\theta d\phi,$$
$$= 8\delta\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 abc\rho^2\sin\phi d\rho d\theta d\phi,$$
$$= \frac{4}{3}\pi\delta abc.$$

ALITER: Sol^n 8(d)





The required volume is nothing but the volume of the region shown in the left which is the sum of the volume V_c of the cylinder and twice the volume V_E of the bell shaped region.

Now, the z- coordinate of the point of intersection of the ellipsoid $4x^2+4y^2+z^2=64$ and the cylinder $x^2+y^2=4$ is given by $4(4)+z^2=64 \Rightarrow z=\pm 4\sqrt{3}$.

$$\therefore \quad V_c = \{\pi(2)^2\}\, 8\sqrt{3} = 32\sqrt{3}\,\pi$$

Also, the volume V_E is bounded by the surfaces $z=4\sqrt{3}$ and $z=2\sqrt{16-(x^2+y^2)}$

$$\therefore \quad V_E = \iint_D \left(\int_{z=4\sqrt{3}}^{2\sqrt{16-x^2-y^2}} 1 \, dz \right) dA$$

$$= \iint_D \left[2\sqrt{16-x^2-y^2} - 4\sqrt{3} \right] dA$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \left[2\sqrt{16-r^2} - 4\sqrt{3} \right] r\,dr\,d\theta$$

$$= \int_0^{2\pi} \left\{ \left[-\frac{2}{3}(16-r^2)^{3/2} \right]_0^2 - 4\sqrt{3}\left[\frac{r^2}{2}\right]_0^2 \right\} d\theta$$

$$= 2\pi\left[-\frac{2}{3}(24\sqrt{3}-64) - 8\sqrt{3} \right]$$

$$= \frac{4\pi}{3}\left[64 - 36\sqrt{3}\right]$$

$\therefore$ the req.d volume $V = V_c + 2V_E$

$$= 32\sqrt{3}\,\pi + \frac{8\pi}{3}\left[64 - 36\sqrt{3}\right]$$

$$= \frac{8\pi}{3}\left[64 - 36\sqrt{3} + 12\sqrt{3}\right]$$

$$= \frac{8\pi}{3}\left[64 - 24\sqrt{3}\right]$$