Special Theory of Relativity (PH101)
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Tutorial-7
due on Wednesday, 23rd of October, 2019 (8:00Hrs IST)

1. Two rockets of rest length L_0 are approaching the earth from opposite directions at velocities $\pm c/2$. What will be size of each rocket measured from the other rocket frame of reference?

Solution:

Let's pick one rocket (call it rocket 1) and consider how fast the other rocket (rocket 2) looks in this frame. In the Earth frame, rocket 1 has velocity $c/2$ and rocket 2 has velocity $c/2$. Applying the velocity addition law gives

$$v_2' = \frac{v_2 - v_1}{1 - v_1 v_2/c^2} = \frac{(-c/2) - (c/2)}{1 - (c/2)(-c/2)/c^2} = -\frac{4}{5}c.$$

As Rocket 2 looks like it is approaching at $\frac{4}{5}c$. Applying the Lorentz contraction relation we have

$$L' = \frac{L_0}{\gamma} = L_0\sqrt{1 - \left(\frac{4}{5}\right)^2} = \frac{3}{5}L_0.$$

2. A body of rest mass m_0 moving at speed u in positive x-direction approaches an identical body of same mass which is at rest. Find the speed V of a frame which is also moving in positive x-direction from which the total momentum of the system would be zero. Further repeat the calculation for non-relativistic range, i.e., $u \ll c$ and see whether your relativistic result reduces to that value or not.

Solution:

Two identical particles with equal and opposite velocities will have equal and opposite momenta as the relativistic momentum is given by $p = \Gamma_u m u$ where $\Gamma_u = \frac{1}{\sqrt{1 - u^2/c^2}}$. So we will look for a frame where the velocities are equal and opposite.

From a frame S' moving with a velocity V with respect to the S-frame, the body which is at rest rest will have the velocity V , while the mass that is moving with a velocity u w.r.t. S-frame will have the velocity

$$u' = \frac{u - V}{1 - \frac{uV}{c^2}}$$

If the total momentum is zero from S', then we have

$$\frac{u - V}{1 - \frac{uV}{c^2}} = V \Rightarrow uV^2 - 2c^2 V + c^2 u = 0$$

$$\Rightarrow V = \frac{2c^2 \pm \sqrt{4c^4 - 4u^2c^2}}{2u} = \frac{c^2 \pm c^2/\Gamma_u}{u}$$

If we consider + sign $V > c$ which is not accepted in STR. Considering the − sign we have,

$$V = \frac{c^2(1 - 1/\Gamma_u)}{u}$$

We can further simplify this by

$$V = u\frac{1 - 1/\Gamma_u}{u^2/c^2} = u\frac{1 - 1/\Gamma_u}{1 - 1/\Gamma_u^2} = u\frac{\Gamma_u}{1 + \Gamma_u}$$

For non-relativistic range ($\Gamma_u \simeq 1$), we have $V = u/2$, which we can easily get using the Galilean relativity also.

3. A spaceship moves away from Earth with speed v and fires a shuttle craft in the forward direction at a speed v relative to the spaceship. The pilot of the shuttle craft launches a probe in the forward direction at speed v relative to the shuttle craft.

   (i)Determine the speed of the shuttle craft relative to the Earth.
   (ii) Determine the speed of the probe relative to the Earth.

   Analyze your results for the limiting cases $v \ll c$ and $v \to c$.

   Solution:

   We consider the S frame to be attached to the Earth and the S' frame to be attached to the spaceship moving with v along the $x-$axis. The shuttle craft has speed $u'_x = v$. Therefore,

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}. \tag{1}$$

   Gives the speed $u_x = \frac{2v}{1+\beta^2}$, where $\beta = v/c$.

   Now consider S' frame is attached to the shuttle craft moving with speed,

   $v' = \frac{2v}{1+\beta^2}$ with respect to the S-frame (Earth)

   along the $x-$ axis. The probe has a speed $u'_x = v$ in S'. Its speed u_x in the S frame would be given by Eq. (1) with v replaced by v' from which we get

$$u_x = \left(\frac{3+\beta^2}{1+3\beta^2}v\right) \tag{2}$$

   It follows from Eq.(2) that $u_x \to 3v$ when $\beta \ll 1$ and $u_x \to c$ when $\beta \to 1$

4. An electron e^- ($m_0 = 0.511 MeV/c^2$) with kinetic energy 1 MeV undergoes a head-on collision with a positron e^+ at rest (A positron is an antimatter particle that has the same mass as the electron but opposite charge). In the process of collision the two particles annihilate resulting in creation of two photons of equal energy. Consider one photon makes an angle θ and another makes and $-\theta$ with the electrons direction of motion (A photon γ is a massless particle of electromagnetic radiation having energy $E = pc$). The process is denoted as

   $e^- + e^+ \rightarrow 2\gamma$

   Determine the energy E, momentum p and angle of emission θ of each photon.

   Solution:

   Using the Kinetic Energy (KE) and momentum relation we can compute the momentum using the relation

   $p = \sqrt{K(K + 2m_0c^2)}/c = 1.422 Mev/c$, when $K = 1 Mev$.
   (The above relation can be obtained from
   $E^2 = p^2c^2 + m_0^2c^4$. Substitute here $E = K + m_0c^2$.)
   The total energy E of the electron and the stationary positron before the collision is

   $E = K + 2m_0c^2 = 2.022 Mev.$

   Using energy conservation we compute the energy of the two photons emerge from the collision as

   $E_\gamma = (E/2) = 1.011 Mev.$

   The magnitude of the momentum of each photon will be $p_\gamma = E_\gamma/c = 1.011 Mev/c$.

   The momentum vectors of the photons make angles $\pm\theta$ with the x-axis. Conservation of momentum yields,

   $p = 2p_\gamma \cos\theta$ which gives $\theta = 45.3^o$

5. A bullet of rest mass $m_0 = 100 gm$ fired in the x-direction at a speed of $0.5c$ hits and get stuck on to a ball of rest mass $M_0 = 200 gm$ sitting at rest with respect to the earth. What is the speed of the combined system after the collision as per special theory of relativity (momentum defined as $\vec{p} = \Gamma_u m_0 \vec{u}$ is conserved)? (Neglect all other effects including that of air drag and gravity.)

   Solution:

   Consider the bullet is fired along the $x-$ direction.

Initial momentum of the bullet is $\vec{p}_1 = \dfrac{1}{\sqrt{1-0.5^2}} 0.1 \times 0.5c\hat{x} = 0.057735c\hat{x}$ (in SI units).

Initial momentum of the ball $= 0$.
Hence, the total initial momentum $\vec{p}_i = 0.057735c\hat{x}$.

The final momentum is given as $\vec{p}_f = \Gamma_u (M_0 + m_0)\vec{u}_f$.

According to momentum conservation; $\vec{p}_f = \vec{p}_i = 0.057735c\hat{x}$.

Therefore, $\Gamma_u(M_0 + m_0)\vec{u}_f = 0.057735c\hat{x} \Rightarrow \vec{u}_f = \dfrac{c\vec{p}_f}{\sqrt{|\vec{p}_f|^2 + (M_0+m_0)^2c^2}}$
$= \dfrac{0.057735c\vec{x}}{\sqrt{0.057735^2 + 0.3^2}} = 0.189c\vec{x}$.

6. Consider a shell of total mass $M_0 = 1$ kg at rest with respect to an observer on earth. At time $t = 0$ it explodes into three equal parts, each of which flies away with the same speed $u = 0.2c$, making an angle 120^o between each other. The velocities may be considered as $\vec{u}_1 = 0.2c\hat{x}$, $\vec{u}_2 = 0.2c\Bigg(-\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\Bigg)$, $\vec{u}_3 = 0.2c\Bigg(-\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\Bigg)$. For the observer on the earth, what is the energy and what is momentum of each of the pieces, (let us denote these by $E_1,\ E_2,\ E_3$ and $\vec{p}_1,\ \vec{p}_2,\ \vec{p}_3$ respectively)? Denoting the energy-momentum four-vectors as $p_1,\ p_2,\ p_3$ with $p_1 \equiv \left(\dfrac{E_1}{c},\ \vec{p}_1\right)$ etc., find $(p_1+p_2+p_3)\cdot(p_1+p_2+p_3)$. (In general, $p\cdot p = \dfrac{E^2}{c^2} - \vec{p}\cdot\vec{p} = m_0^2c^2$.) Consider another observer moving with velocity $\vec{v} = 0.5c\hat{x}$ with respect to the observer on the Earth. Find $E_1',\ E_2',\ E_3'$ and $\vec{p}\,'_1,\ \vec{p}\,'_2, \vec{p}\,'_3$. What are the masses of each of these parts as seen from the moving frame? Find the magnitude of the sum of the energy-momentum four-vector in this frame $((p_1'+p_2'+p_3')\cdot(p_1'+p_2'+p_3'))$.

Solution:

The energy of an object moving with velocity $\vec{u}$ is given as $E = \Gamma_u m_0 c^2$ and the momentum as $\vec{p} = \Gamma_u m_0 \vec{u}$.
Situation before the explosion: Total energy available $E_i = M_0c^2 = 9\times10^{16}$J. Initial momentum, $\vec{p}_i = 0$.
As all the pieces have the same mass and move with same speed after the expolsion, they have to same energy.

i.e. $E_1 = E_2 = E_3 = \frac{E_i}{3} = 3\times10^{16}$J

Magnitude of momentum of each piece , $|\vec{p}_1| = E_1\frac{|\vec{u}_1|}{c^2}\ kg\ m/s = 3\times10^{16}\ \frac{0.2}{c} = 2\times10^7\ kg\ m/s = |\vec{p}_2| = |\vec{p}_3|$.

Therefore, $\vec{p}_1 = 2\times10^7\ \hat{x},\quad \vec{p}_2 = 2\times10^7\Bigg(-\frac{1}{2}\hat{x} + \frac{\sqrt{3}}{2}\hat{y}\Bigg),\quad \vec{p}_3 = 2\times10^7\Bigg(-\frac{1}{2}\hat{x} - \frac{\sqrt{3}}{2}\hat{y}\Bigg)$.

The rest mass of each piece is $m_{0j} = \dfrac{2\sqrt{\dfrac{2}{3}}}{5}\ kg,\ j = 1, 2, 3.$

The relativistic mass is E_j/c^2.

Here $\gamma_v = \dfrac{1}{\sqrt{1 - v^2/c^2}} = \dfrac{2}{\sqrt{3}}$.

Hence, from $p'_x = \gamma_v\left(p_x - \dfrac{v}{c^2}E\right);\ p'_y = p_y;\ p'_z = p_z;\ E' = \gamma_v\left(E - vp_x\right)$ we have

For mass m_1,

$$p'_{1,x} = \frac{2}{\sqrt{3}}\left(2 \times 10^7 - \frac{0.5}{3 \times 10^8} \times 3 \times 10^{16}\right);\ p'_{1,y} = 0;\ p'_{1,z} = 0;\ E'_1 = \frac{2}{\sqrt{3}}\left(3 \times 10^{16} - 0.5 \times 3 \times 10^8 \times 2 \times 10^7\right)$$

For mass m_2,

$$p'_{2,x} = \frac{2}{\sqrt{3}}\left(-\frac{1}{2} \times 2 \times 10^7 - \frac{0.5}{3 \times 10^8} \times 3 \times 10^{16}\right);\ p'_{2,y} = \frac{\sqrt{3}}{2} \times 2 \times 10^7;\ p'_{2,z} = 0;\ E'_2 = \frac{2}{\sqrt{3}}\left(3 \times 10^{16} + \frac{1}{2} \times 0.5 \times 3 \times 10^8 \times 2 \times 10^7\right)$$

For mass m_3,

$$p'_{3,x} = \frac{2}{\sqrt{3}}\left(-\frac{1}{2} \times 2 \times 10^7 - \frac{0.5}{3 \times 10^8} \times 3 \times 10^{16}\right);\ p'_{3,y} = -\frac{\sqrt{3}}{2} \times 2 \times 10^7;\ p'_{2,z} = 0;\ E'_3 = \frac{2}{\sqrt{3}}\left(3 \times 10^{16} + \frac{1}{2} \times 0.5 \times 3 \times 10^8 \times 2 \times 10^7\right)$$

Further simplification leads to :

For mass m_1,

$$p'_{1,x} = -2\sqrt{3} \times 10^7\ kg\ m/s;\ p'_{1,y} = 0;\ p'_{1,z} = 0;\ E'_1 = 18\sqrt{3} \times 10^{15}\ J$$

For mass m_2,

$$p'_{2,x} = -4\sqrt{3} \times 10^7\ kg\ m/s;\ p'_{2,y} = \sqrt{3} \times 10^7\ kg\ m/s;\ p'_{2,z} = 0;\ E'_2 = 21\sqrt{3} \times 10^{15}\ J$$

For mass m_3,

$$p'_{3,x} = -4\sqrt{3} \times 10^7\ kg\ m/s;\ p'_{3,y} = -\sqrt{3} \times 10^7;\ p'_{2,z} = 0;\ E'_3 = 21\sqrt{3} \times 10^{15}\ J$$

Define the 4-vector $p' = \sum_{j=1,2,3} \left(\frac{E'_j}{c},\ p'_{j,x},\ p'_{j,y},\ p'_{j,z} \right)$.

Hence, $p' = 10^8(2\sqrt{3}, -\sqrt{3}, 0, 0)$.

The length of the 4-vector is ,

$|p'|^2 = 10^{16}(12 - 3) = 9 \times 10^{16}(kg\ m/s)^2$