PH 102: Physics II

Lecture 14 (Post midsem, Spring 2020)

Debasish Borah
IIT Guwahati

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Division
Lec 1	05-03-2020	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 2	11-03-2020	Application of Ampere's Law, Magnetic Vector Potential	5.3, 5.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 1	12-03-2020	Lec 1		
Tut 2	17-03-2020	Lec 2		
Lec 3	18-03-2020	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic materials, magnetization	5.4, 6.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 4	19-03-2020	Field of a magnetized object, Boundary conditions	6.2, 6.3, 6.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 3	24-03-2020	Lec 3, 4		
Lec 5	25-03-2020	Ohm's law, motional emf, electromotive force	7.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 6	26-03-2020	Faraday's law, Lenz's law, Self & Mutual inductance, Energy stored in magnetic field	7.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 4	31-03-2020	Lec 5, 6		
Lec 7	01-04-2020	Maxwell's equations	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 8	02-04-2020	Discussions, Problem solving	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
	07-04-2020	Quiz II		
Lec 9	08-04-2020	Continuity equation, Poynting theorem	8.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 10	16-04-2020	Wave solution of Maxwell's equations, polarisation	9.1, 9.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 5	21-04-2020	Lec 9, 10		
Lec 11	22-04-2020	Electromagnetic waves in matter, reflection & transmission: normal incidence	9.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 12	23-04-2020	Reflection & transmission: oblique incidence	9.3, 9.4	I, II (4-4:55 pm) III, IV (11-11:55

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				am)
Tut 6	28-4-2020	Lec 11, 12		
Lec 13	29-04-2020	Relativity and electromagnetism: Galilean & special relativity	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 14	30-04-2020	Discussions, problem solving	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)

Transformation of electromagnetic fields

$$\bar{E}_x = E_x,\ \bar{E}_y = \gamma(E_y - vB_z),\ \bar{E}_z = \gamma(E_z + vB_y)$$
$$\bar{B}_x = B_x,\ \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z),\ \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

Will these transformations keep the form of the Maxwell's equations same in two inertial frames?

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\epsilon_0}, \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$
$$\vec{\nabla}\cdot\vec{B} = 0, \vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}$$

$$\xleftrightarrow{\ ?\ }$$

$$\vec{\nabla}'\cdot\vec{\bar{E}} = \frac{\rho'}{\epsilon_0}, \vec{\nabla}'\times\vec{\bar{E}} = -\frac{\partial\vec{\bar{B}}}{\partial t'}$$
$$\vec{\nabla}'\cdot\vec{\bar{B}} = 0, \vec{\nabla}'\times\vec{\bar{B}} = \mu_0\vec{J}' + \mu_0\epsilon_0\frac{\partial\vec{\bar{E}}}{\partial t'}$$

To check this, one has to know how $(\rho, \vec{J})$ **are related to** $(\rho', \vec{J}')$

Transformation of electromagnetic fields

$$\bar{E}_x = E_x,\ \bar{E}_y = \gamma(E_y - vB_z),\ \bar{E}_z = \gamma(E_z + vB_y)$$

$$\bar{B}_x = B_x,\ \bar{B}_y = \gamma(B_y + \frac{v}{c^2}E_z),\ \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y)$$

$E \leftrightarrow Bc$

$$\bar{B}_x = B_x,\ \bar{B}_y c = \gamma(B_y c - vE_z/c),\ \bar{B}_z c = \gamma(B_z c + vE_y/c)$$

$$\bar{E}_x = E_x,\ \bar{E}_y/c = \gamma(E_y/c + \frac{v}{c^2}B_z c),\ \bar{E}_z/c = \gamma(E_z/c - \frac{v}{c^2}B_y c)$$

$y \leftrightarrow z$

$$\bar{B}_x = B_x,\ \bar{B}_z c = \gamma(B_z c - vE_y/c),\ \bar{B}_y c = \gamma(B_y c + vE_z/c)$$

$$\bar{E}_x = E_x,\ \bar{E}_z/c = \gamma(E_z/c + \frac{v}{c^2}B_y c),\ \bar{E}_y/c = \gamma(E_y/c - \frac{v}{c^2}B_z c)$$

- The transformations are symmetric with respect to interchange of E’s with B's and y’s with z’s.

Transformation of electromagnetic fields: A general proof

- The symmetry of the Lorentz transformations of E and B indicate that they may be part of one mathematical entity. Since E, B have a total of six components, all of them can't be fit inside a vector in four spacetime dimensions.

- E, B can be part of bigger entity like tensors (In simple terms, they can be thought of as components of a 4x4 antisymmetric matrix, for example).

- Since current and charge densities produce E, B, they can also be part of the same entity. Since **J** and ρ have a total of four components, they can be thought of as parts of a four dimensional vector in four spacetime dimensions.

- Let us assume the Lorentz transformation of $(\rho, \vec{J})$ to be

$$\rho'(\vec{r'}, t') = A_{00}\rho(\vec{r}, t) + A_{01}J_x(\vec{r}, t), \; J'_x(\vec{r'}, t') = A_{11}J_x(\vec{r}, t) + A_{10}\rho(\vec{r}, t)$$

- Demanding the equation of continuity $\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$ to be same in all inertial frames

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = \frac{\partial \rho'}{\partial t'} + \vec{\nabla}' \cdot \vec{J}' \quad \Longrightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} = \frac{\partial \rho'}{\partial t'} + \frac{\partial J'_x}{\partial x'}$$

- Using the Lorentz transformations: $x' = \gamma(x - vt), \; t' = \gamma(t - \frac{vx}{c^2})$ and their inverse $x = \gamma(x' + vt'), \; t = \gamma(t' + \frac{vx'}{c^2})$ we can write the new derivatives as

$$\frac{\partial}{\partial t'} = \frac{\partial t}{\partial t'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial t'}\frac{\partial}{\partial x} = \gamma(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x})$$

$$\frac{\partial}{\partial x'} = \frac{\partial t}{\partial x'}\frac{\partial}{\partial t} + \frac{\partial x}{\partial x'}\frac{\partial}{\partial x} = \gamma(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t}), \; \frac{\partial}{\partial y'} = \frac{\partial}{\partial y}, \; \frac{\partial}{\partial z'} = \frac{\partial}{\partial z}$$

- Using these we get

$$\gamma(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x})(A_{00}\rho(\vec{r}, t) + A_{01}J_x(\vec{r}, t)) + \gamma(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t})(A_{11}J_x(\vec{r}, t) + A_{01}\rho(\vec{r}, t))$$

$$= \frac{\partial \rho(\vec{r}, t)}{\partial t} + \frac{\partial J_x(\vec{r}, t)}{\partial x}$$

This gives rise to four equations in terms of the unknown coefficients:

$$\gamma A_{00} + \gamma \frac{v}{c^2} A_{10} = 1,\ v A_{00} + A_{10} = 0,\ A_{01} + \frac{v}{c^2} A_{11} = 0,\ \gamma v A_{01} + \gamma A_{11} = 1$$

which can be solved simultaneously to get

$$A_{00} = A_{11} = \gamma,\ A_{10} = -\gamma v,\ A_{01} = -\gamma \frac{v}{c^2}$$

and hence $\rho'(\vec{r'}, t') = \gamma(\rho(\vec{r}, t) - \frac{v}{c^2} J_x(\vec{r}, t)),\ J'_x(\vec{r'}, t') = \gamma(J_x(\vec{r}, t) - v\rho(\vec{r}, t))$

which is similar to the Lorentz transformations of (t, x).

Now, the Maxwell's equations (with source terms) in the new frame are:

$$\vec{\nabla}' \cdot \vec{E}' = \frac{\rho'}{\epsilon_0},\ \vec{\nabla}' \times \vec{B}' = \mu_0 \vec{J}' + \mu_0 \epsilon_0 \frac{\partial \vec{E}'}{\partial t'}$$

$$\implies \frac{\partial E'_x}{\partial x'} + \frac{\partial E'_y}{\partial y'} + \frac{\partial E'_z}{\partial z'} = \frac{\rho'}{\epsilon_0},\ \frac{\partial B'_z}{\partial y'} - \frac{\partial B'_y}{\partial z'} = \mu_0 J'_x + \mu_0 \epsilon_0 \frac{\partial E'_x}{\partial t'}$$

$$\frac{\partial B'_x}{\partial z'} - \frac{\partial B'_z}{\partial x'} = \mu_0 J'_y + \mu_0 \epsilon_0 \frac{\partial E'_y}{\partial t'},\ \frac{\partial B'_y}{\partial x'} - \frac{\partial B'_x}{\partial y'} = \mu_0 J'_z + \mu_0 \epsilon_0 \frac{\partial E'_z}{\partial t'}$$

Using the primed derivatives in terms of the unprimed ones and
$\rho'(\vec{r'},t') = \gamma(\rho(\vec{r},t) - \frac{v}{c^2}J_x(\vec{r},t)),\ J'_x(\vec{r'},t') = \gamma(J_x(\vec{r},t) - v\rho(\vec{r},t)), J'_y = J_y,\ J'_z = J_z$
in the second equation on previous page, we get

$$\frac{\partial B'_z}{\partial y} - \frac{\partial B'_y}{\partial z} = \mu_0\gamma(J_x - v\rho) + \mu_0\epsilon_0\gamma(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x})E'_x \qquad (1)$$

Now, using Maxwell's equations in unprimed frame:

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}, \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x + \mu_0\epsilon_0\frac{\partial E_x}{\partial t} \qquad (2)$$

in equations (1), we get

$$\frac{\partial B'_z}{\partial y} - \frac{\partial B'_y}{\partial z} = \gamma(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \mu_0\epsilon_0\frac{\partial E_x}{\partial t})$$
$$-\gamma\mu_0 v\epsilon_0(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}) + \mu_0\epsilon_0\gamma(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x})E'_x$$

$$\implies \frac{\partial}{\partial y}(B'_z - \gamma B_z + \gamma\frac{v}{c^2}E_y) - \frac{\partial}{\partial z}(B'_y - \gamma B_y - \gamma\frac{v}{c^2}E_z) = \gamma\frac{v}{c^2}\frac{\partial}{\partial x}(E'_x - E_x) + \gamma\mu_0\epsilon_0\frac{\partial}{\partial t}(E'_x - E_x)$$

$$\implies B'_z = \gamma(B_z - \frac{v}{c^2}E_y),\ B'_y = \gamma(B_y + \frac{v}{c^2}E_z),\ E'_x = E_x$$

Using the transformations of $(\rho, \vec{J})$ in the first Maxwell's equation in primed frame, we get

$$\gamma(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t})E'_x + \frac{\partial E'_y}{\partial y} + \frac{\partial E'_z}{\partial z} = \frac{1}{\epsilon_0}\gamma(\rho - \frac{v}{c^2}J_x) \qquad (3)$$

Using (2) in (3) we get

$$\gamma(\frac{\partial}{\partial x} + \frac{v}{c^2}\frac{\partial}{\partial t})E'_x + \frac{\partial E'_y}{\partial y} + \frac{\partial E'_z}{\partial z} = \gamma(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}) - \gamma v(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} - \mu_0\epsilon_0\frac{\partial E_x}{\partial t})$$

$$\Longrightarrow \gamma\frac{\partial}{\partial x}(E'_x - E_x) + \gamma\frac{v}{c^2}\frac{\partial}{\partial t}(E'_x - E_x) + \frac{\partial}{\partial y}(E'_y - \gamma E_y + \gamma v B_z) + \frac{\partial}{\partial z}(E'_z - \gamma E_z - \gamma v B_y) = 0$$

$$\Longrightarrow E'_x = E_x,\ E'_y = \gamma(E_y - vB_z),\ E'_z = \gamma(E_z + vB_y)$$

Similarly, using one of the equations involving parallel component of magnetic field, we can show that

$$B'_x = B_x$$

For another simple derivation of the Lorentz transformations of the electromagnetic fields please see:

Lorentz transformations of the electromagnetic field for beginners, Rafael Ferraro, American Journal of Physics **65**, 412 (1997).

Transformation of electromagnetic fields

Denoting
$$\vec{E} = \vec{E}_{\|} + \vec{E}_{\perp},\ \vec{E'} = \vec{E'}_{\|} + \vec{E'}_{\perp}$$
$$\vec{B} = \vec{B}_{\|} + \vec{B}_{\perp},\ \vec{B'} = \vec{B'}_{\|} + \vec{B'}_{\perp}$$

the transformations of the fields, in general, can be denoted as

$$\vec{E'}_{\|} = \vec{E}_{\|},\ \vec{E'}_{\perp} = \gamma(\vec{E}_{\perp} + \vec{v} \times \vec{B}_{\perp})$$
$$\vec{B'}_{\|} = \vec{B}_{\|},\ \vec{B'}_{\perp} = \gamma(\vec{B}_{\perp} - \frac{1}{c^2}\vec{v} \times \vec{E}_{\perp})$$

If **B**=0 in the unprimed frame (say, a point charge at rest) then

$$\vec{E'}_{\|} = \vec{E}_{\|},\ \vec{E'}_{\perp} = \gamma\vec{E}_{\perp},\ \vec{B'}_{\|} = 0,\ \vec{B'}_{\perp} = -\frac{\gamma}{c^2}\vec{v} \times \vec{E}_{\perp}$$

$$\Longrightarrow\ \vec{B'} = -\frac{1}{c^2}\vec{v} \times \vec{E'}$$ Since $\vec{v} \times \vec{E}_{\|} = 0$

Similarly, if there exists a frame in which E=0, then in the moving frame

$$\vec{E}' = \vec{v} \times \vec{B}'$$

Field of a point charge moving with constant speed v:

1. In the unprimed frame, where the charge is at rest, B=0.

2. In the lab frame, where the charge is moving, there exists a magnetic field perpendicular to electric field and to the direction of motion.

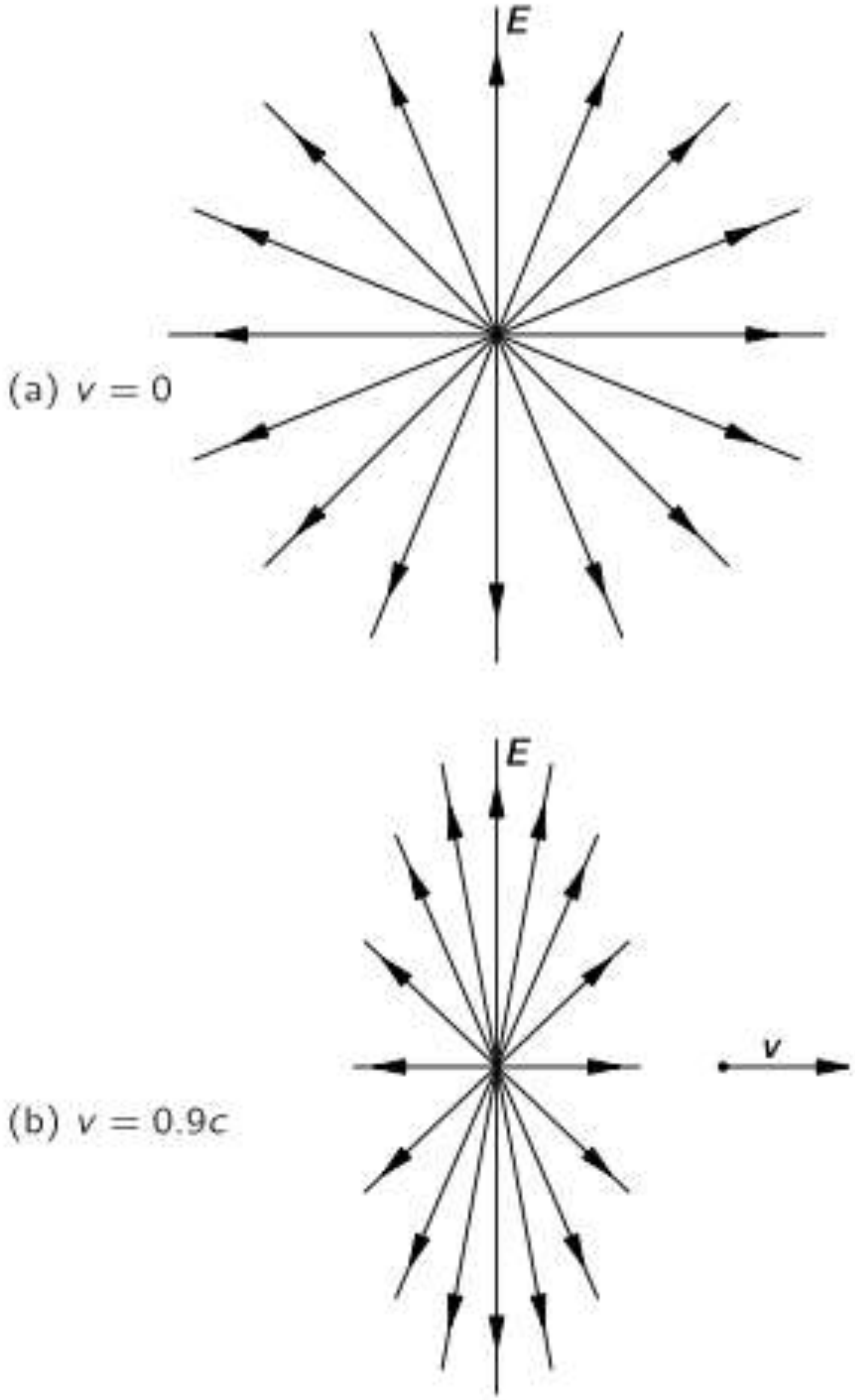
3. The electric field in the lab frame is radial from the instantaneous position of the charge.

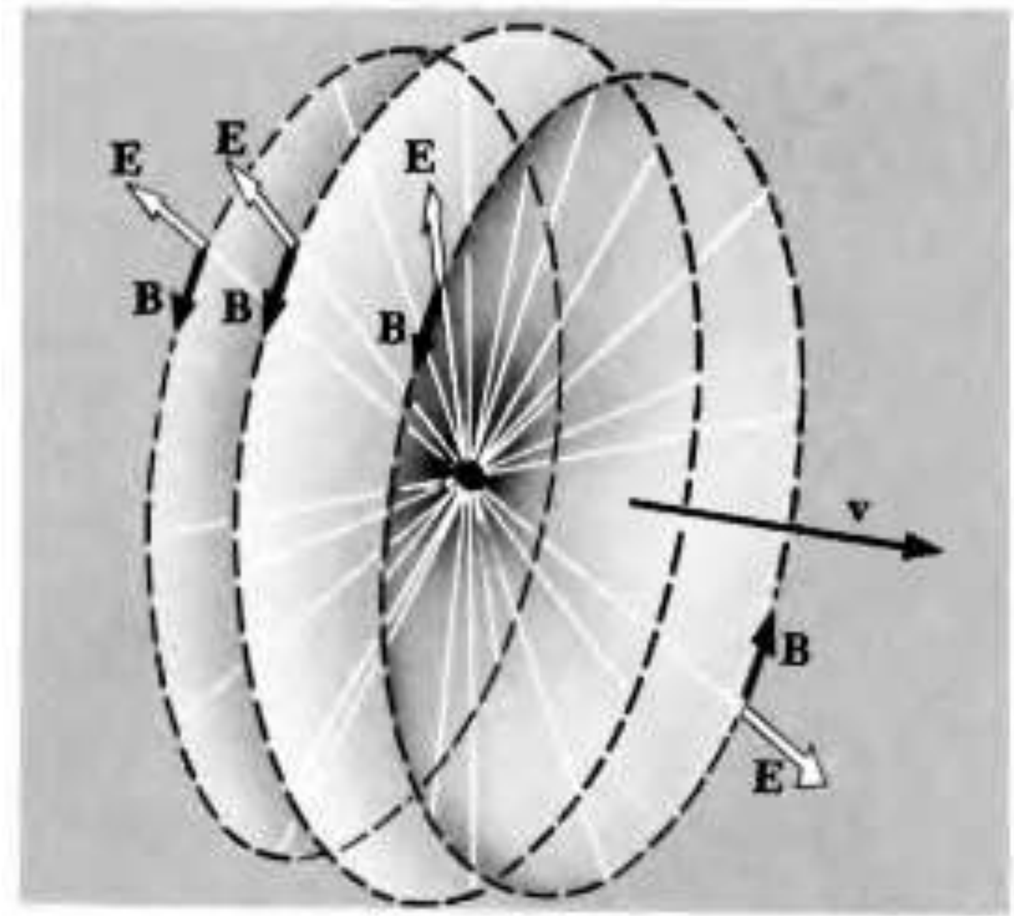
4. The magnetic field lines are circles around the direction of motion.

5. When the velocity of the charge is very high, the electric field lines are folded together into a thin disk, the circular magnetic field lines are folded together in this disk.

6. The magnitude of B is nearly equal to the magnitude of E.

Field of a moving charge





Credit: Feynman Lectures in Physics, Berkeley Physics Course, E Purcell

Example 12.13 (Introduction to Electrodynamics, D J Griffiths): A point charge q is at rest at the origin of a coordinate system S. What is the electric field of this same charge in a frame S' which moves to the right with speed v relative to S along x direction?

Solution: Electric field in the rest frame:

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{(x_0^2 + y_0^2 + z_0^2)^{3/2}}(x_0\hat{x} + y_0\hat{y} + z_0\hat{z})$$

According to the transformations, the field components in new frame is

$$E'_x = E_x, E'_y = \gamma E_y, E'_z = \gamma E_x, \gamma = 1/\sqrt{1 - v^2/c^2}$$

The old coordinates (x_0, y_0, z_0) are related to the new coordinates by usual Lorentz transformation

$$x_0 = \gamma(x + vt) = \gamma R_x, y_0 = y = R_y, z_0 = z = R_z$$

where R is a vector from charge to the point P where field is measured.

Net electric field in S' is

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{\gamma q \vec{R}}{(\gamma^2 R^2 \cos^2\theta + R^2 \sin^2\theta)^{3/2}}$$
$$= \frac{1}{4\pi\epsilon_0} \frac{q(1 - v^2/c^2)}{(1 - (v^2/c^2)\sin^2\theta)^{3/2}} \frac{\hat{R}}{R^2}$$

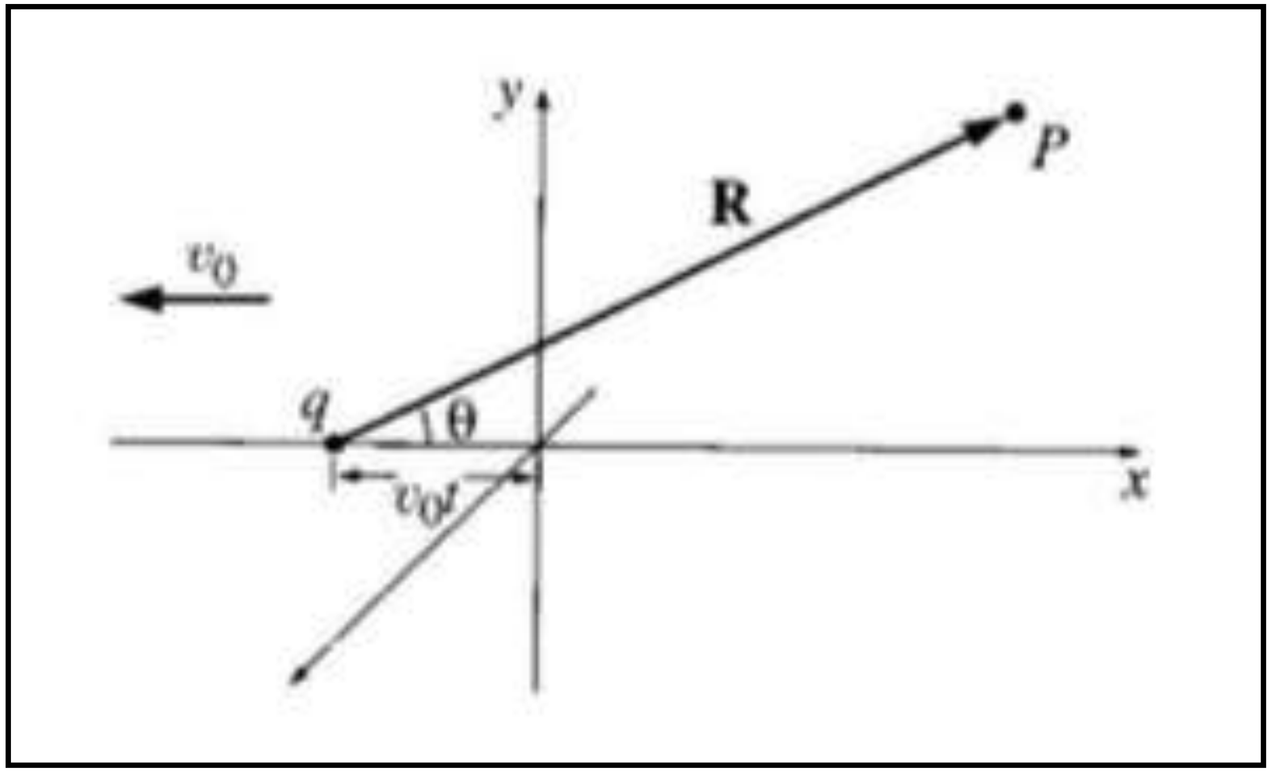


Fig 12.37, Introduction to Electrodynamics, D J Griffiths

Now, the magnetic field of the point charge in uniform motion can be found as

$$\vec{B} = -\frac{1}{c^2}(\vec{v} \times \vec{E})$$

$$\implies \vec{B} = \frac{\mu_0}{4\pi} \frac{qv(1 - v^2/c^2)\sin\theta}{(1 - (v^2/c^2)\sin^2\theta)^{3/2}} \frac{\hat{\phi}}{R^2}$$

with the corresponding field lines going counterclockwise as we face the incoming charge q (see slide no. 13). For non relativistic limit, we have

$$v \ll c$$

leading to the usual expression given by Biot-Savart law

$$\vec{B} = \frac{\mu_0}{4\pi} q \frac{\vec{v} \times \hat{R}}{R^2} \qquad I d\vec{l} = q\vec{v}$$

Plane electromagnetic wave observed from a moving frame

Let the EM wave in a frame S is given by

$$\vec{E} = E_0 \cos(kx - \omega t)\hat{y},\ \vec{B} = \frac{E_0}{c}\cos(kx - \omega t)\hat{z},\ k = \omega/c$$

The same wave is being observed from another inertial frame S' moving with respect to S with a speed v along x direction.

Using the transformations, the fields in S' can be found as

$$\bar{E}_x = \bar{E}_z = 0,\ \bar{E}_y = \gamma(E_y - vB_z) = \alpha E_0 \cos(kx - \omega t),$$

$$\bar{B}_x = \bar{B}_y = 0,\ \bar{B}_z = \gamma(B_z - \frac{v}{c^2}E_y) = \alpha\frac{E_0}{c}\cos(kx - \omega t)$$

$$\text{where}\quad \alpha = \gamma(1 - v/c) = \sqrt{\frac{1 - v/c}{1 + v/c}}$$

The new fields can be written in new coordinates using the Lorentz transformations for spacetime:

$$kx - \omega t = \gamma[k(\bar{x} + v\bar{t}) - \omega(\bar{t} + \frac{v}{c^2}\bar{x})] = \gamma[(k - \frac{\omega v}{c^2})\bar{x} - (\omega - kv)\bar{t}] = \bar{k}\bar{x} - \bar{\omega}\bar{t}$$

$$\text{where } \ \bar{k} = \gamma[(k - \frac{\omega v}{c^2}) = \gamma k(1 - v/c) = \alpha k, \ \ \bar{\omega} = \gamma\omega(1 - v/c) = \alpha\omega$$

Therefore, the EM wave observed from S' looks like

$$\vec{\bar{E}} = \bar{E}_0 \cos{(\bar{k}\bar{x} - \bar{\omega}\bar{t})}\hat{y}, \ \vec{\bar{B}} = \frac{\bar{E}_0}{c}\cos{(\bar{k}\bar{x} - \bar{\omega}\bar{t})}\hat{z},$$

$$\bar{E}_0 = \alpha E_0, \ \bar{k} = \alpha k, \ \bar{\omega} = \alpha\omega$$

Here, $\bar{\omega} = \alpha\omega = \sqrt{\frac{1 - v/c}{1 + v/c}}\omega$ is the well-known **Doppler shift.**

Speed of light (EM wave) in S' remains same $\frac{\omega}{k} = \frac{\bar{\omega}}{\bar{k}} = c$

As expected in STR