Physics II (PH 102)

Electromagnetism (Lecture 7)

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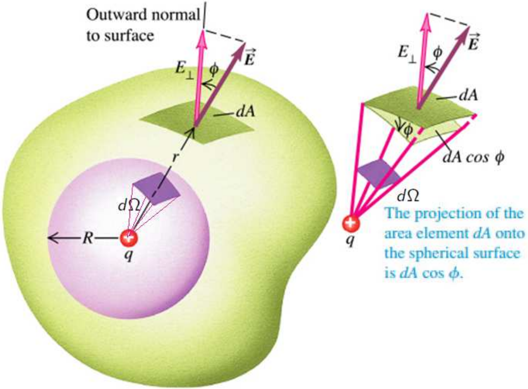
Concept of Electric Flux
Definition

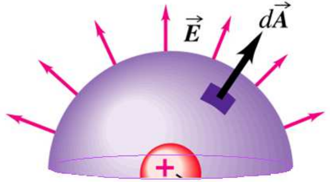
Let S be any arbitrary simple surface (open or closed) and $\mathbf{E}$ is the Electric field in the region containing S. Then, the *total flux of* $\mathbf{E}$ *through the surface* is defined as the surface integral of the outward normal component of $\mathbf{E}$ on S:

$$\phi_S = \iint_S \mathbf{E}\cdot d\mathbf{A} = \iint_S E\, dA \cos\phi = \frac{q}{4\pi\epsilon_0}\iint_S \left(\frac{dA\cos\phi}{r^2}\right) = \frac{q}{4\pi\epsilon_0}\underbrace{\int d\Omega}_{\text{Solid angle}} = \frac{q}{\epsilon_0},$$

Thus, the result is independent of the specific geometry of the surface S.



Electric Flux due to Point Charge



Example

Consider a charge q placed at the origin. Find the Electric flux through the upper hemispherical surface of radius R centered at the origin.

Let $d\mathbf{A} = R^2 \sin\theta d\theta d\phi\,\hat{\mathbf{r}}$ be an elementary area on the hemesphere at $\mathbf{r} = R\,\hat{\mathbf{r}}$, where $|\mathbf{r}| = R$, and unit normal to dA is $\hat{\mathbf{r}}$. Hence, the Electric flux is

$$\begin{aligned}
\phi_S &= \iint_S \mathbf{E}(\mathbf{r})\cdot d\mathbf{A} \ ; \quad \mathbf{E}(\mathbf{r} = R\,\hat{\mathbf{r}}) = \frac{q}{4\pi\epsilon_0}\left(\frac{\hat{\mathbf{r}}}{R^2}\right) \\
&= \frac{q}{4\pi\epsilon_0}\int_0^{\pi/2}\int_0^{2\pi}\left(\frac{\hat{\mathbf{r}}}{R^2}\right)\cdot\hat{\mathbf{r}}\,R^2 \sin\theta d\theta d\phi \\
&= \frac{q}{2\epsilon_0}
\end{aligned}$$

Gauss's Integral Law $\leftarrow$ Differential Law

Let $\mathbf{E}$ be the Electric field defined over a volume V with volume charge density ρ, then using the **differential form of Gauss's law** we have

$$\begin{aligned} \nabla \cdot \mathbf{E}(\mathbf{r}) &= \frac{\rho(\mathbf{r})}{\epsilon_0} \\ \iiint_V \nabla \cdot \mathbf{E}(\mathbf{r})\, dv &= \iiint_V \frac{\rho(\mathbf{r})}{\epsilon_0}\, dv = \frac{Q_{\text{enclosed}}}{\epsilon_0} \end{aligned}$$

where Q_{enclosed} is the total charge enclosed within V. If V be bounded by a closed surface S, then according to the Gauss's Divergence Theorem

$$\iiint_V \nabla \cdot \mathbf{E}(\mathbf{r})\, dv = \frac{Q_{\text{enclosed}}}{\epsilon_0} = \oiint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \phi_S$$

Gauss's Integral Law:

The surface integral of the outward normal component of the Electric field, i.e., the total normal Electric flux, ϕ_S over a closed surface S enclosing a total charge Q_{enclosed} is equal the ratio $Q_{\mathrm{enclosed}}/\epsilon_0$, and this result is independent of the specific geometry (shape) of the surface. Mathematically,

$$\phi_S = \oiint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{A} = \frac{Q_{enclosed}}{\epsilon_0}$$

What is the Recipe for Application of Gauss's Integral Law?

Gauss's Integral Law can ONLY be applied to problems with HIGH DEGREE OF SYMMETRY where one can construct GAUSSIAN SURFACES.

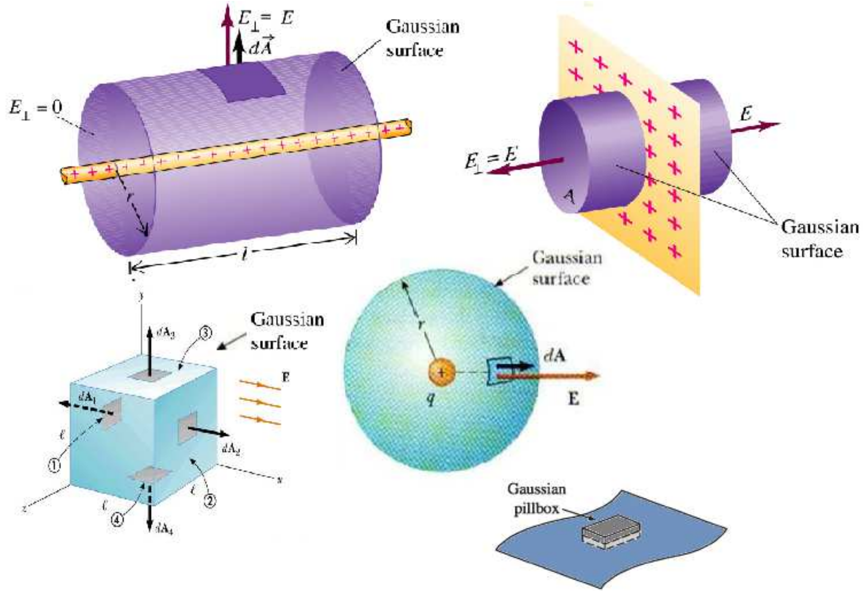
Definition

GAUSSIAN SURFACE: *This is a hypothetical closed surface enclosing the given source charges or charge distributions, which is so chosen to exploit the symmetry of a given problem. Without any apparent symmetry such a surface can not be constructed.*

- First contruct a fully closed GAUSSIAN SURFACE. Open surfaces, like discs, can not enclose charge in a 3D volume.
- The surface must include the point where the Electric field is calculated.
- The surface is chosen in such a way that for every point on that surface the Electric field $\mathbf{E}$ is constant.
- 4 types of symmetries can be exploited: (1) Spherical, (2) Cylindrical (3) Cubical and (4) Planar. Accordingly, the Gaussian surfaces are constructed **spherical**, **cylindrical**, **cubical** and **pillbox** shaped.
- Surface integration drastically simplifies, since $\mathbf{E}$ being constant can be taken outside the integral.

$$\phi_s = \oiint_S \mathbf{E} \cdot d\mathbf{A} = \mathbf{E} \cdot \hat{n} \oiint_S dA = E_n(\text{Surface Area})$$

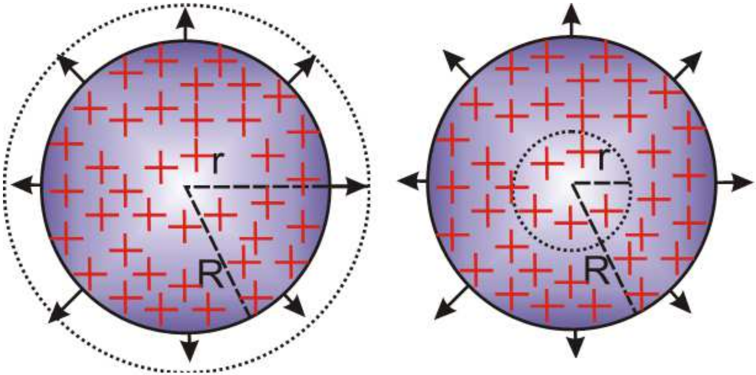
Typical Gaussian Surfaces



Applications of Gauss's Law

Example

Consider a uniformly charged *insulating* sphere of radius R and charge Q. Calculate Electric fields both outside and inside the sphere.



Outside the charged Sphere ($r > R$)

Construct a Gaussian surface $S_{\rm out}$ of raduis r outside the charged sphere with the same center, then the total enclosed charge is Q. Thus,

$$\begin{aligned} \oiint_{S_{\rm out}} \mathbf{E} \cdot d\mathbf{A} &= |\mathbf{E}| \oiint_{S_{\rm out}} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dA = |\mathbf{E}|\, 4\pi r^2 = \frac{Q}{\epsilon_0} \\ \mathbf{E}(r) &= \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}}. \end{aligned}$$

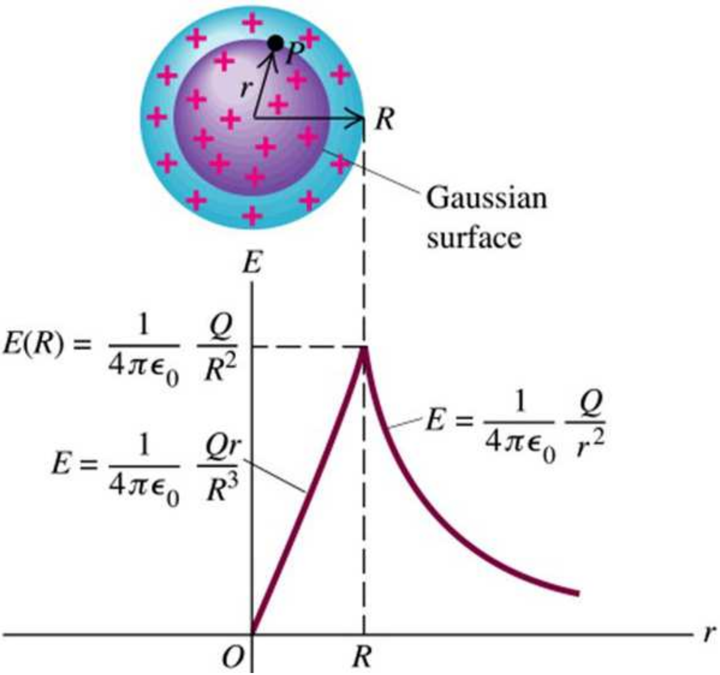
Inside the charged Sphere ($r < R$)

Construct a Gaussian surface $S_{\rm in}$ of raduis r inside the charged sphere with the same center. The amount of charge enclosed is

$$q_{\rm enclosed} = \frac{Q}{\frac{4}{3}\pi R^3} \left(\frac{4}{3} \pi r^3 \right) = \frac{Q\, r^3}{R^3}.$$

Thus,

$$\begin{aligned} \oiint_{S_{\rm in}} \mathbf{E} \cdot d\mathbf{A} &= |\mathbf{E}| \oiint_{S_{\rm in}} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dA = |\mathbf{E}|\, 4\pi r^2 = \frac{1}{\epsilon_0} \left(\frac{Q\, r^3}{R^3} \right) \\ \mathbf{E}(r) &= \frac{1}{4\pi\epsilon_0} \frac{Q\, r}{R^3} \hat{\mathbf{r}}. \end{aligned}$$


P
r
R
Gaussian surface
E
$E(R) = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2}$
$E = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$
$E = \frac{1}{4\pi\epsilon_0}\frac{Qr}{R^3}$
O
R
r

Curl of Electric Field due to a Point Charge

Suppose a Point Source Charge of magnitude q is placed at the origin. Electric field at a Field point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}.$$

Now curl of Electric field in Cartesian system will be

$$\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial_x & \partial_y & \partial_z \\ (x/r^3) & (y/r^3) & (z/r^3) \end{vmatrix} \\
[\nabla \times \mathbf{E}(\mathbf{r})]_x &= \frac{q}{4\pi\epsilon_0}\left[\partial_y\left(\frac{z}{r^3}\right) - \partial_z\left(\frac{y}{r^3}\right)\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\left(-\frac{3yz}{r^5}\right) - \left(\frac{-3zy}{r^5}\right)\right] = 0.
\end{aligned}$$

Similarly, the other components: $[\nabla \times \mathbf{E}(\mathbf{r})]_y = [\nabla \times \mathbf{E}(\mathbf{r})]_z = 0$. Thus,

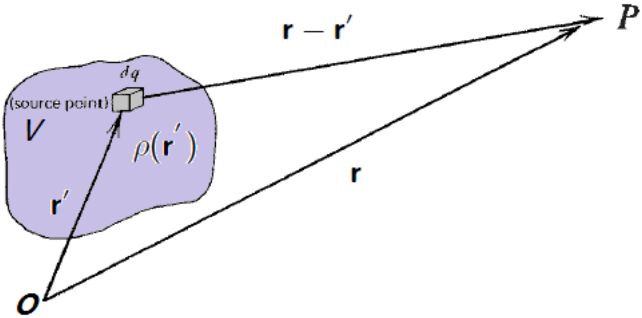
$$\nabla \times \mathbf{E}(\mathbf{r}) = 0$$

This is a general result in ELECTROSTATICS, but not necesarily in ELECTRODYNAMICS where $\nabla \times \vec{\mathcal{E}}(\mathbf{r}, t) \neq 0$

Curl of Electric Field due to a Continuous Volume Distribution

Now we extend the result to arbitrary Volume Charge Distribution V with volume density ρ. Electric field at the target point P is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\rho(\mathbf{r}')\left(\mathbf{r}-\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|^3} dv'$$



Curl of Electric Field due to a continuous distribution (contd.)

Curl with respect to which variable, $\mathbf{r}$ or $\mathbf{r}'$, i.e., is it $\nabla\times$ or $\nabla'\times$?

Here we are interested in the curl with respect to TARGET POINT variable $\mathbf{r}$:

$$\begin{aligned}\nabla \times \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\nabla \times \iiint\limits_V \frac{\rho(\mathbf{r}')\left(\mathbf{r}-\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|^3}dv' \\ &= \frac{1}{4\pi\epsilon_0}\iiint\limits_V \rho(\mathbf{r}')\left(\nabla \times \frac{\left(\mathbf{r}-\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|^3}\right)dv'\end{aligned}$$

$$\left(\nabla \times \frac{\left(\mathbf{r}-\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|^3}\right)_x = \left(-3(z-z')\frac{(y-y')}{|\mathbf{r}-\mathbf{r}'|^5} + 3(y-y')\frac{(z-z')}{|\mathbf{r}-\mathbf{r}'|^5}\right) = 0$$

The other components similarly vanish.

Curl of Electric field $\mathbf{E}$ is ALWAYS zero in electrostatics, and therefore $\mathbf{E}$ can be derived from the gradient of an arbitrary scalar field $\phi(\mathbf{r})$:

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \qquad \Longrightarrow \qquad \mathbf{E}(\mathbf{r}) = \nabla\phi(\mathbf{r})$$

Note: In electrodynamics, for *time varying* e.m. fields, $\nabla \times \vec{\mathcal{E}}(\mathbf{r}, t) \neq 0$.

Concept of Electrostatic Potential Function

Electric field **E** is by convention taken as the negative gradient of the Electrostatic Potential V .

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= -\nabla V(\mathbf{r}) \\ \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} = -\nabla V(\mathbf{r}) \cdot d\mathbf{r} &= -dV(\mathbf{r}) \end{aligned}$$

Integrating from a reference point **ref** to the point **r** along arb. path C:

$$\int_C \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}' = -\int_{\mathbf{ref}}^{\mathbf{r}} \nabla V(\mathbf{r}') \cdot d\mathbf{r}' = -\int_{\mathbf{ref}}^{\mathbf{r}} dV = -V(\mathbf{r}) + \cancelto{0}{V(\mathbf{ref})}$$

The **ref** point is so chosen that the Potential at that location is zero, which is conventionally taken at $\mathbf{ref} = \infty$ for finite charge distributions and for infinitely extended charge distributions the **ref** point may be chosen arbitrarily.

Definition

The Electrostatic Potential at any point in an existing electric field is equal to the work done by an external agent against the repulsive electric forces in carrying a unit positive test charge from infinity to that point, i.e.,

$$V(\mathbf{r}) = -\int_{\infty}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}'$$

Electrostatic Potential: Facts

- SI Unit: **Joules per Coulomb** (J/C) or **volt** (V).
- Sign Convension: A positive Potential implies *work done by the external agent on the Electrostatic field*, and a negative Potential implies *work done by the Electrostatic field*.
- Corollary from **Stokes' Theorem**: Since $\nabla \times \mathbf{E}(\mathbf{r}) = 0$, the circulation of the Electric field about any closed path is zero, and so is the net work done $\Longrightarrow$ conservative nature of Electric field.

$$\iint_S [\nabla \times \mathbf{E}(\mathbf{r})] \cdot d\mathbf{S} = \oint_L \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} = 0.$$

- Corollary from **Fundamental Gradient Theorem**: The line integral of the Electric field is path-independent and depends only on the end points $\Longrightarrow$ *potential difference* between the given end points is uniquely given by

$$\int_{\mathbf{r}=\mathbf{a}}^{\mathbf{r}=\mathbf{b}} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{r} = -\int_{\mathbf{a}}^{\mathbf{b}} \nabla V(\mathbf{r}) \cdot d\mathbf{r} = -\int_{\mathbf{a}}^{\mathbf{b}} dV = V(\mathbf{a}) - V(\mathbf{b}) \equiv \Delta V_{ab}.$$

Electrostatic Potential: Facts (contd.)

- **Linear Superposition Principle** holds for Potentials: The total Electrostatic Potential at any point is the sum of the Electrostatic Potentials due to all the source charges/charge distributions separately, i.e.,

$$V_{\mathrm{Total}} = V_1 + V_2 + \cdots = \sum_i V_i$$

- General Charge Distribution: For a localized charge distribution with volume, surface and linear densities, ρ, σ, λ, respectively, as well as discrete point charges q_i, the resulting Electrostatic Potential is the **Superposition of Potentials** due to the independent distributions:

$$\begin{aligned} V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i}{|\mathbf{r}-\mathbf{r}_i^{'}|} + \frac{1}{4\pi\epsilon_0}\int_C \frac{\lambda(\mathbf{r}^{'})}{|\mathbf{r}-\mathbf{r}^{'}|}dl^{'} \\ &\quad + \frac{1}{4\pi\epsilon_0}\iint_S \frac{\sigma(\mathbf{r}^{'})}{|\mathbf{r}-\mathbf{r}^{'}|}dS^{'} + \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\mathbf{r}^{'})}{|\mathbf{r}-\mathbf{r}^{'}|}dv'. \end{aligned}$$

Poisson's and Laplace's Equations

- We have seen that $\mathbf{E}(\mathbf{r}) = -\nabla V(\mathbf{r})$ and $\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}$.
- Combining the two yields the **POISSON'S EQUATION**:

$$\begin{aligned} \nabla \cdot \mathbf{E}(\mathbf{r}) &= \nabla \cdot [-\nabla V(\mathbf{r})] = -\nabla^2 V(\mathbf{r}) \\ \nabla^2 V(\mathbf{r}) &= -\frac{\rho(\mathbf{r})}{\epsilon_0}. \end{aligned}$$

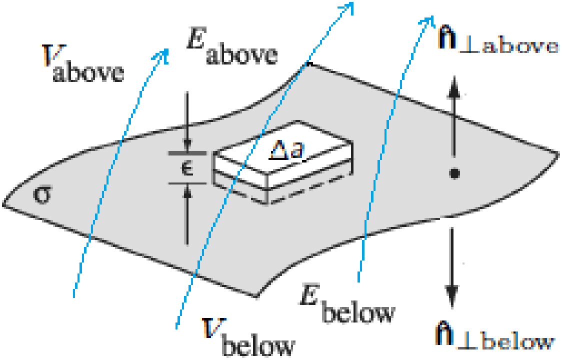
- In regions where there are no charge distributions, $\rho = 0$, we obtain the **LAPLACE'S EQUATION**, $\nabla^2 V(\mathbf{r}) = 0$, e.g., in Cartesian System

$$\boxed{\nabla^2 V(\mathbf{r}) = \frac{\partial^2 V(x,y,z)}{\partial^2 x} + \frac{\partial^2 V(x,y,z)}{\partial^2 y} + \frac{\partial^2 V(x,y,z)}{\partial^2 z} = 0.}$$

- **Boundary Valued Problems:**
  To obtain UNIQUE solutions to such 2^{nd} order *Partial Differential Equations* (PDEs), require specification of Potentials, Electric fields or charge configurations across *boundaries* or *interface* between different media, e.g., conductors or dielectrics with different physical properties. These specifications are termed as BOUNDARY CONDITIONS.

Boundary Conditions on **E** and V

Consider a arbitrarily shaped smooth *interface* with surface charge density σ. Construct a thin **wafer-like Gaussian pillbox** across the interface of vanishing thickness $\epsilon \to 0$ and infinitesimally small upper and lower "lid" surface areas Δa. The pillbox is contructed arbitrarily close, stradling to the interface so that the surface looks "locally flat" such that **E** is 'almost' constant on all its surfaces.



Here we shall find the relations between $\mathbf{E}_{above}$ & $\mathbf{E}_{below}$ and V_{above} & V_{below}.

Boundary Conditions on $\mathbf{E}$

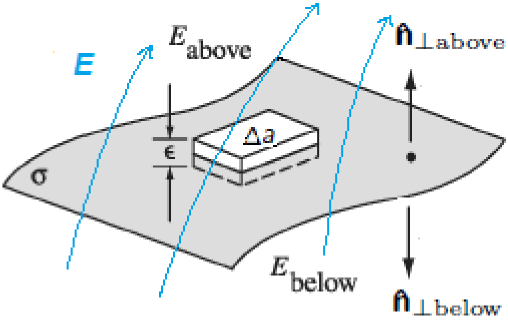
- Applying Gauss's law,

$$\lim_{\epsilon\to 0}\left[\ \oiint_{\text{pillbox}} \mathbf{E}\cdot d\mathbf{S}\right] = \left(\mathbf{E}_{\text{above}}\cdot\hat{\mathbf{n}}_{\perp\text{above}} + \mathbf{E}_{\text{below}}\cdot\hat{\mathbf{n}}_{\perp\text{below}}\right)\Delta a = \frac{1}{\epsilon_0}Q_{\text{encl}} = \frac{1}{\epsilon_0}\sigma\Delta a$$

$$\left(\mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}}\right)\cdot\hat{\mathbf{n}}_{\perp\text{above}} = \frac{\sigma}{\epsilon_0}$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{\sigma}{\epsilon_0}$$

Note: There is no contribution from the sides of the pillbox as $\epsilon \to 0$.

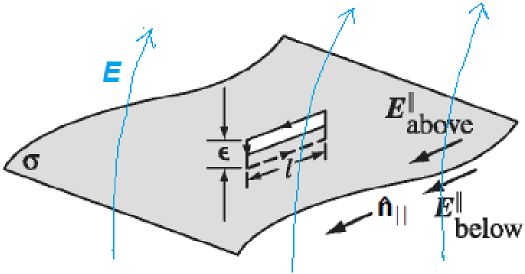


Boundary Conditions on E (contd.)

- Next consider a thin rectangular closed loop C stradling across the interface of vanishing ends $\epsilon \to 0$ and side lengths l. Then, since $\mathbf{E}$ is a conservative vector field

$$
\begin{aligned}
\lim_{\epsilon \to 0} \left[\oint_C \mathbf{E} \cdot d\mathbf{l} \right] &= 0 \\
\left(\mathbf{E}_{\text{above}} \cdot \hat{\mathbf{n}}_{||} - \mathbf{E}_{\text{below}} \cdot \hat{\mathbf{n}}_{||} \right) l &= (\mathrm{E}^{||}_{\text{above}} - \mathrm{E}^{||}_{\text{below}}) l = 0 \\
\mathrm{E}^{||}_{\text{above}} &= \mathrm{E}^{||}_{\text{below}}
\end{aligned}
$$

Note: The ends give vanishing contributions since $\epsilon \to 0$.



Boundary Conditions on $\mathbf{E}$ (contd.)

- The two previous results can be combined as:

$$\begin{aligned} E^{\perp}_{\text{above}}(\mathbf{r}) - E^{\perp}_{\text{below}}(\mathbf{r}) &= \frac{\sigma(\mathbf{r})}{\epsilon_0} \\ \mathrm{E}^{||}_{\text{above}}(\mathbf{r}) &= \mathrm{E}^{||}_{\text{below}}(\mathbf{r}) \\ \implies \mathbf{E}_{\text{above}}(\mathbf{r}) - \mathbf{E}_{\text{below}}(\mathbf{r}) &= \frac{\sigma(\mathbf{r})}{\epsilon_0}\hat{\mathbf{n}} \end{aligned}$$

where $\hat{\mathbf{n}} \equiv \hat{\mathbf{n}}_{\perp\text{above}}$ is the unit normal vector above the interface.

- Alternatively taking dot products with $\hat{\mathbf{n}}$ the combined result is:

$$\nabla V_{\text{above}}(\mathbf{r}) \cdot \hat{\mathbf{n}} - \nabla V_{\text{below}}(\mathbf{r}) \cdot \hat{\mathbf{n}} = -\frac{\sigma(\mathbf{r})}{\epsilon_0}.$$

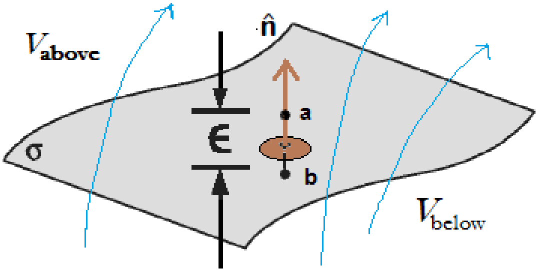
- Introducing **Normal directional derivative:** $D_{\hat{\mathbf{n}}} V \equiv \frac{\partial V}{\partial n} = \nabla V \cdot \hat{\mathbf{n}}$

$$\frac{\partial V_{\text{above}}(\mathbf{r})}{\partial n} - \frac{\partial V_{\text{below}}(\mathbf{r})}{\partial n} = -\frac{\sigma(\mathbf{r})}{\epsilon_0}.$$

Boundary Condition on V

- Consider points **a** and **b** just above and below the interface separated by an infinitesimal amount $\epsilon \to 0$. The potential difference is given by

$$\lim_{\epsilon \to 0} [V_{\text{above}}(\mathbf{a}) - V_{\text{below}}(\mathbf{b})] = \lim_{\mathbf{a} \to \mathbf{b}} \left[\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{r} \right] = 0$$



$$V_{\text{above}} = V_{\text{below}}$$

Boundary Conditions: Summary

- Boundary conditions on **E** and V apply to all types of **smooth** surfaces, **flat or curved**, or whether they happen to be **charged or not**
- Based on 2 basic principles: (1) Gauss's law and (2) Conservative nature of **E**.

Fact

1. The normal component of the Electric field, $\mathbf{E}_\perp$ is discontinuous by an amount σ/ϵ_0 across any boundary.
2. The tangential component of Electric field, $\mathbf{E}_{||}$ is continuous across any boundary.
3. The Electrostatic potential V is continuous across any boundary.
4. The **Normal derivative** of the potential, $\frac{\partial V}{\partial n}$ is discontinuous by an amount σ/ϵ_0 across any boundary.