1. Consider the propagation of en electromagnetic wave in a conducting medium where $\vec{J} = \sigma\vec{E}$ with σ representing the conductivity of the medium. Show that $\vec{E}$ would now satisfy the equation

$$\nabla^2\vec{E} - \mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

Solution:
In a conducting medium, Maxwell's equations are

$$\vec{\nabla}\cdot\vec{E} = 0, \vec{\nabla}\cdot\vec{B} = 0, \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}, \vec{\nabla}\times\vec{B} = \mu\vec{J} + \mu\epsilon\frac{\partial\vec{E}}{\partial t}$$

Taking curl of the third equation and then using the fourth one, we get

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\frac{\partial}{\partial t}(\mu\sigma\vec{E} + \mu\epsilon\frac{\partial\vec{E}}{\partial t})$$

$$\implies \vec{\nabla}(\vec{\nabla}\cdot\vec{E}) - \nabla^2\vec{E} = -\mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\implies \nabla^2\vec{E} - \mu\sigma\frac{\partial\vec{E}}{\partial t} - \mu\epsilon\frac{\partial^2\vec{E}}{\partial t^2} = 0$$

where, in the last step we have used the first Maxwell's equation.

2. Consider a plane electromagnetic wave of frequency ω, travelling in the z direction, polarised in the x direction, incident at the interface ($z = 0$) separating medium 1 and medium 2. In the derivation for reflected and transmitted amplitudes discussed in the class, the reflected and transmitted waves were assumed to have the same polarisation as the incident wave (along x direction). Prove that this has to be true always. Hint: Let the polarisation vectors of the transmitted and reflected waves be $\hat{n}_T = \cos\theta_T\hat{x} + \sin\theta_T\hat{y}, \hat{n}_R = \cos\theta_R\hat{x} + \sin\theta_R\hat{y}$, and then prove from the electromagnetic boundary conditions that $\theta_T = \theta_R = 0$.

Solution:
The incident wave is given by

$$\vec{\tilde{E}}_I(z,t) = \tilde{E}_{0I}e^{i(k_1z-\omega t)}\hat{x},\ \vec{\tilde{B}}_I(z,t) = \frac{1}{v_1}\tilde{E}_{0I}e^{i(k_1z-\omega t)}\hat{y}.$$

The reflected and transmitted waves are given by

$$\vec{\tilde{E}}_R(z,t) = \tilde{E}_{0R}e^{i(-k_1z-\omega t)}\hat{n}_R,\ \vec{\tilde{B}}_R(z,t) = \frac{1}{v_1}\tilde{E}_{0R}e^{i(-k_1z-\omega t)}(-\hat{z}\times\hat{n}_R)$$

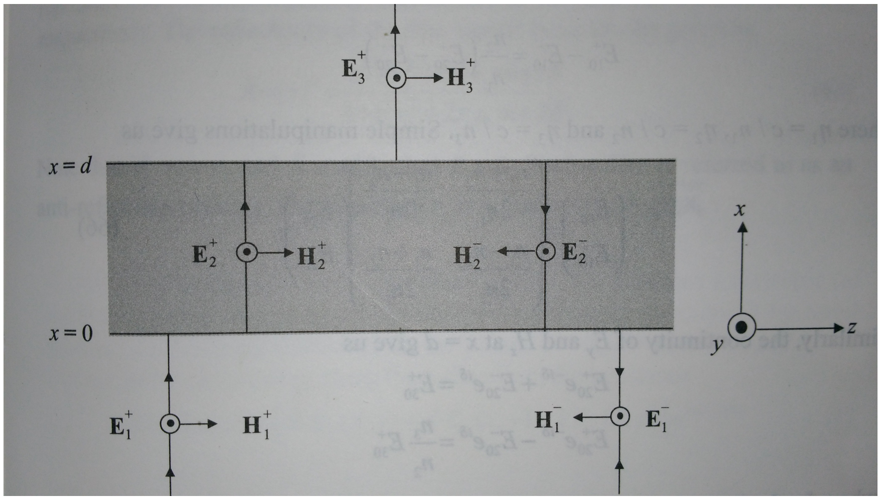

Figure 1: Figure for solution to problem 3.

$$\tilde{\vec{E}}_T(z,t) = \tilde{E}_{0T} e^{i(k_2 z - \omega t)} \hat{n}_T,\ \ \tilde{\vec{B}}_T(z,t) = \frac{1}{v_2} \tilde{E}_{0T} e^{i(k_2 z - \omega t)} (\hat{z} \times \hat{n}_T).$$

The boundary conditions

$$E_1^{\parallel} = E_2^{\parallel} \implies \tilde{E}_{0I}\hat{x} + \tilde{E}_{0R}\hat{n}_R = \tilde{E}_{0T}\hat{n}_T$$

$$\frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel} \implies \frac{1}{\mu_1 v_1}(\tilde{E}_{0I}\hat{y} - \tilde{E}_{0R}(\hat{z} \times \hat{n}_R)) = \frac{1}{\mu_2 v_2} \tilde{E}_{0T}(\hat{z} \times \hat{n}_T).$$

The y component of the first boundary condition is

$$\tilde{E}_{0R} \sin\theta_R = \tilde{E}_{0T} \sin\theta_T$$

The x component of the second boundary condition is

$$\tilde{E}_{0R} \sin\theta_R = -\frac{\mu_1 v_1}{\mu_2 v_2} \tilde{E}_{0T} \sin\theta_T$$

Both of these can be true only if

$$\sin\theta_R = \sin\theta_T = 0 \implies \theta_R = \theta_T = 0.$$

Therefore, the reflected and transmitted waves have the same polarisation as the incident wave.

3. Consider a plane wave (with its electric field to be along the y-axis) incident normally on a dielectric film of thickness d. Calculate the reflectivity of the film.

Solution:

Let us consider the electric fields in three different regions 1, 2, 3 (as shown in figure 1) to be

$$\vec{E}_1 = \hat{y} E_{10}^{+} e^{i(\omega t - k_1 x)} + \hat{y} E_{10}^{-} e^{i(\omega t + k_1 x)}$$

$$\vec{E}_2 = \hat{y}E_{20}^{+}e^{i(\omega t - k_2 x)} + \hat{y}E_{20}^{-}e^{i(\omega t + k_2 x)}, \vec{E}_3 = \hat{y}E_{30}^{+}e^{i[\omega t - k_3(x-d)]}$$

where $E_{10}^{+}, E_{10}^{-}(E_{20}^{+}, E_{20}^{-})$ represent the amplitude of the forward and backward propagating waves respectively in region 1 (2) while in region 3 (which extends upto infinity) we have only forward going waves with amplitude E_{30}^{+}. Corresponding magnetic field is given by $\vec{B} = \vec{k}\times\vec{E}/\omega$ where $\vec{k} = k\hat{x}(-k\hat{x})$ for waves propagating in $+x(-x)$ directions. Therefore,

$$\vec{B}_1 = \hat{z}\frac{k_1}{\omega}\left[E_{10}^{+}e^{i(\omega t - k_1 x)} - E_{10}^{-}e^{i(\omega t + k_1 x)}\right]$$

$$\vec{B}_2 = \hat{z}\frac{k_2}{\omega}\left[E_{20}^{+}e^{i(\omega t - k_2 x)} - \hat{y}E_{20}^{-}e^{i(\omega t + k_2 x)}\right], \vec{B}_3 = \hat{z}\frac{k_3}{\omega}E_{30}^{+}e^{i[\omega t - k_3(x-d)]}$$

From electromagnetic boundary conditions, the tangential components of $\vec{E}, \vec{B}/\mu$ are continuous at $x = 0, d$ interfaces. The continuity equation at $x = 0$ gives

$$E_{10}^{+} + E_{10}^{-} = E_{20}^{+} + E_{20}^{-}, \frac{k_1}{\omega}(E_{10}^{+} - E_{10}^{-}) = \frac{k_2}{\omega}(E_{20}^{+} - E_{20}^{-})$$

Using $k/\omega = 1/v = n/c$, with n being the index of refraction, we can write the second equation above as

$$(E_{10}^{+} - E_{10}^{-}) = \frac{n_2}{n_1}(E_{20}^{+} - E_{20}^{-})$$

Solving these two equations, we get

$$E_{10}^{+} = \frac{n_1+n_2}{2n_1}E_{20}^{+} + \frac{n_1-n_2}{2n_1}E_{20}^{-}, E_{10}^{-} = \frac{n_1-n_2}{2n_1}E_{20}^{+} + \frac{n_1+n_2}{2n_1}E_{20}^{-}$$

Similarly, continuity equations at $x = d$ interface gives

$$E_{20}^{+}e^{-ik_2 d} + E_{20}^{-}e^{ik_2 d} = E_{30}^{+}, E_{20}^{+}e^{-ik_2 d} - E_{20}^{-}e^{ik_2 d} = \frac{n_3}{n_2}E_{30}^{+}$$

Solving them gives

$$E_{20}^{+} = \frac{n_2+n_3}{2n_2}e^{ik_2 d}E_{30}^{+}, E_{20}^{-} = \frac{n_2-n_3}{2n_2}e^{-ik_2 d}E_{30}^{+}$$

Using these in the expressions for E_{10}^{+}, E_{10}^{-}, we get

$$E_{10}^{+} = \left[\frac{n_1+n_2}{2n_1}\frac{n_2+n_3}{2n_2}e^{ik_2 d} + \frac{n_1-n_2}{2n_1}\frac{n_2-n_3}{2n_2}e^{-ik_2 d}\right]E_{30}^{+}$$

$$E_{10}^{-} = \left[\frac{n_1-n_2}{2n_1}\frac{n_2+n_3}{2n_2}e^{ik_2 d} + \frac{n_1-n_2}{2n_1}\frac{n_2-n_3}{2n_2}e^{-ik_2 d}\right]E_{30}^{+}$$

The amplitude of reflection coefficient is

$$r = \frac{E_{10}^{-}}{E_{10}^{+}} = \frac{r_1 e^{ik_2 d} + r_2 e^{-ik_2 d}}{e^{ik_2 d} + r_1 r_2 e^{-ik_2 d}}$$

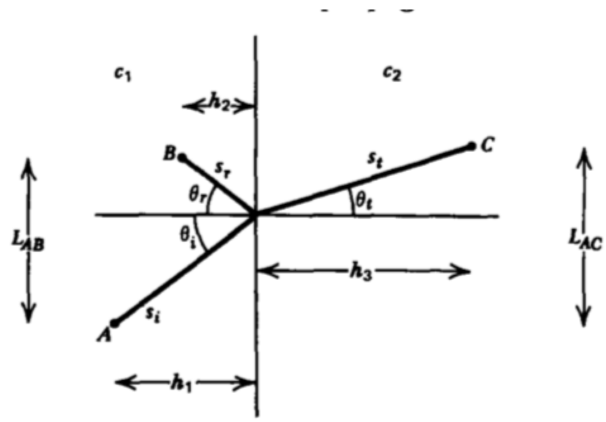

Figure 2: Figure for problem 4.

where $r_1 = (n_1 - n_2)/(n_1 + n_2), r_2 = (n_2 - n_3)/(n_2 + n_3)$. The reflectivity of the film is

$$R = |r|^2 = \frac{(r_1 \cos 2k_2 d + r_2)^2 + r_1^2 \sin^2 2k_2 d}{(\cos 2k_2 d + r_1 r_2)^2 + \sin^2 2k_2 d} = \frac{r_1^2 + r_2^2 + 2r_1 r_2 \cos 2k_2 d}{1 + r_1^2 r_2^2 + 2r_1 r_2 \cos 2k_2 d}$$

4. Fermat's principle of least time states that light, when reflected or refracted off an interface, will pick the path of least time to propagate between two points. A beam of light from point A is incident upon a dielectric interface at angle θ_i from the normal and is reflected through the point B at angle θ_r (see figure 2).
(a) In terms of $\theta_i, \theta_r, h_1, h_2$ and speed of light c_1, how long does it take for light to travel from A to B along this path? What other relation is there between $\theta_i, \theta_r, L_{AB}, h_1, h_2$?
(b) Find θ_i that satisfies Fermat's principle. What is θ_r?
(c) In terms of $\theta_i, \theta_r, h_1, h_2$ and the light speeds c_1, c_2 in each medium, how long does it take for light to travel from A to C?
(d) Find the relationship between θ_i and θ_t that satisfies Fermat's principle.

Solution:
(a) From figure 2, we can write $\cos\theta_i = h_1/s_i \implies s_i = h_1/\cos\theta_i$. Similarly, $s_r = h_2/\cos\theta_r$. Therefore, time taken by light to travel from A to B along the path shown is

$$t = \frac{s_i + s_r}{c_1} = \frac{1}{c_1}\left[\frac{h_1}{\cos\theta_i} + \frac{h_2}{\cos\theta_r}\right]$$

Also, $\sin\theta_i = L_1/s_i, \sin\theta_r = L_2/s_r \implies L_1 + L_2 = L_{AB} = s_i \sin\theta_i + s_r \sin\theta_r = h_1 \tan\theta_i + h_2 \tan\theta_r$.

(b) Since L_{AB} is fixed, we can write from the above expression $L_{AB} = h_1 \tan\theta_i + h_2 \tan\theta_r$ that

$$0 = \frac{h_1}{\cos^2\theta_i} d\theta_i + \frac{h_2}{\cos^2\theta_r} d\theta_r \implies \frac{d\theta_r}{d\theta_i} = -\frac{h_1}{h_2}\frac{\cos^2\theta_r}{\cos^2\theta_i}$$

To find θ_i that satisfies Fermat's principle, we use the expression for time in part (a) and differentiate it with respect to θ_i and equate it to zero:

$$0 = \frac{h_1}{\cos^2\theta_i}\sin\theta_i + \frac{h_2}{\cos^2\theta_r}\sin\theta_r\frac{d\theta_r}{d\theta_i} \implies \frac{d\theta_r}{d\theta_i} = -\frac{h_1}{h_2}\frac{\sin\theta_i}{\sin\theta_r}\frac{\cos^2\theta_r}{\cos^2\theta_i}$$

Using the above two expressions for $d\theta_r/d\theta_i$ we find $\sin\theta_i = \sin\theta_r$.

(c) Time taken by light to travel from A to C is

$$t = \frac{1}{c_1}\frac{h_1}{\cos\theta_i} + \frac{1}{c_2}\frac{h_2}{\cos\theta_t}$$

Here θ_t is also related to L_{AC} as $L_{AC} = h_1\tan\theta_i + h_2\tan\theta_t$.

(d) Similar to part (b), we can find the relationship between θ_i and θ_t that satisfies Fermat's principle by using the solution of part (c). Using $L_{AC} = h_1\tan\theta_i + h_2\tan\theta_t$, we find

$$0 = \frac{h_1}{\cos^2\theta_i}d\theta_i + \frac{h_2}{\cos^2\theta_t}d\theta_t \implies \frac{d\theta_t}{d\theta_i} = -\frac{h_1}{h_2}\frac{\cos^2\theta_t}{\cos^2\theta_i}$$

Similarly, by differentiating $t = \frac{1}{c_1}\frac{h_1}{\cos\theta_i} + \frac{1}{c_2}\frac{h_2}{\cos\theta_t}$ with respect to θ_i we get

$$0 = \frac{h_1}{c_1\cos^2\theta_i}\sin\theta_i + \frac{h_2}{c_2\cos^2\theta_t}\sin\theta_t\frac{d\theta_t}{d\theta_i} \implies \frac{d\theta_t}{d\theta_i} = -\frac{h_1c_2}{h_2c_1}\frac{\sin\theta_i}{\sin\theta_t}\frac{\cos^2\theta_t}{\cos^2\theta_i}$$

Using these two, we can find $c_2\sin\theta_i = c_1\sin\theta_t$ which is Snell's law.

5. In many cases the permeability of dielectric media equals that of free space. In this limit (use the general expressions derived in class) show that the reflection and transmission coefficients for waves obliquely incident upon dielectric media are

$$R = \frac{\sin^2(\theta_I - \theta_T)}{\sin^2(\theta_I + \theta_T)},\ T = \frac{\sin 2\theta_I \sin 2\theta_T}{\sin^2(\theta_I + \theta_T)}\quad (\vec{E} \text{ parallel to the interface})$$

$$R = \frac{\tan^2(\theta_T - \theta_I)}{\tan^2(\theta_T + \theta_I)},\ T = \frac{\sin 2\theta_I \sin 2\theta_T}{\sin^2(\theta_T + \theta_I)\cos^2(\theta_T - \theta_I)}\quad (\vec{B} \text{ parallel to the interface})$$

Solution:
As derived in class, the reflection and transmission coefficients for $\vec{E}$ parallel to the interface are given by

$$R = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2, T = \frac{4\alpha\beta}{(1+\alpha\beta)^2}$$

where (for $\mu_1 = \mu_2 = \mu_0$)

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I}, \beta = \frac{n_2}{n_1} = \frac{\sin\theta_I}{\sin\theta_T}$$

Using these in the expression for R, T, we get

$$R = \left(\frac{\cos\theta_I \sin\theta_T - \cos\theta_T \sin\theta_I}{\cos\theta_I \sin\theta_T + \cos\theta_T \sin\theta_I}\right)^2 = \frac{\sin^2(\theta_T - \theta_I)}{\sin^2(\theta_T + \theta_I)}$$

$$T = \frac{4\cos\theta_T \sin\theta_I \cos\theta_I \sin\theta_T}{\sin^2(\theta_T + \theta_I)} = \frac{\sin 2\theta_I \sin 2\theta_T}{\sin^2(\theta_I + \theta_T)}$$

As derived in class, the reflection and transmission coefficients for $\vec{B}$ parallel to the interface are given by

$$R = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)^2, T = \frac{4\alpha\beta}{(\alpha + \beta)^2}$$

where α, β are same as before. Using the expressions for α, β in the above expressions we get,

$$R = \left(\frac{\cos\theta_T \sin\theta_T - \sin\theta_I \cos\theta_I}{\cos\theta_T \sin\theta_T + \sin\theta_I \cos\theta_I}\right)^2 = \left(\frac{\sin 2\theta_T - \sin 2\theta_I}{\sin 2\theta_T + \sin 2\theta_I}\right)^2 = \left(\frac{\cos(\theta_T + \theta_I)\sin(\theta_T - \theta_I)}{\sin(\theta_T + \theta_I)\cos(\theta_T - \theta_I)}\right)^2$$

$$\implies R = \frac{\tan^2(\theta_T - \theta_I)}{\tan^2(\theta_T + \theta_I)}$$

$$T = \frac{4\cos\theta_T \sin\theta_I \cos\theta_I \sin\theta_T}{(\cos\theta_T \sin\theta_T + \sin\theta_I \cos\theta_I)^2} = \frac{4\sin 2\theta_I \sin 2\theta_T}{(\sin 2\theta_T + \sin 2\theta_I)^2}$$

$$\implies T = \frac{\sin 2\theta_I \sin 2\theta_T}{\sin^2(\theta_T + \theta_I)\cos^2(\theta_T - \theta_I)}$$