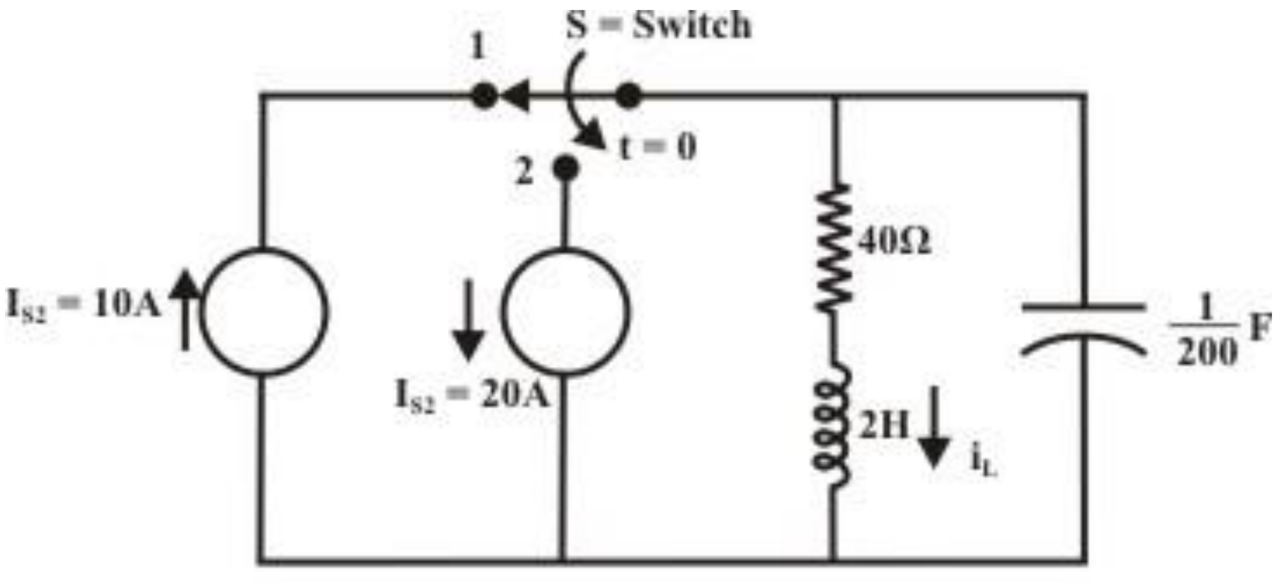
Lecture 7

Initial Conditions

RLC Circuits

Initial Conditions

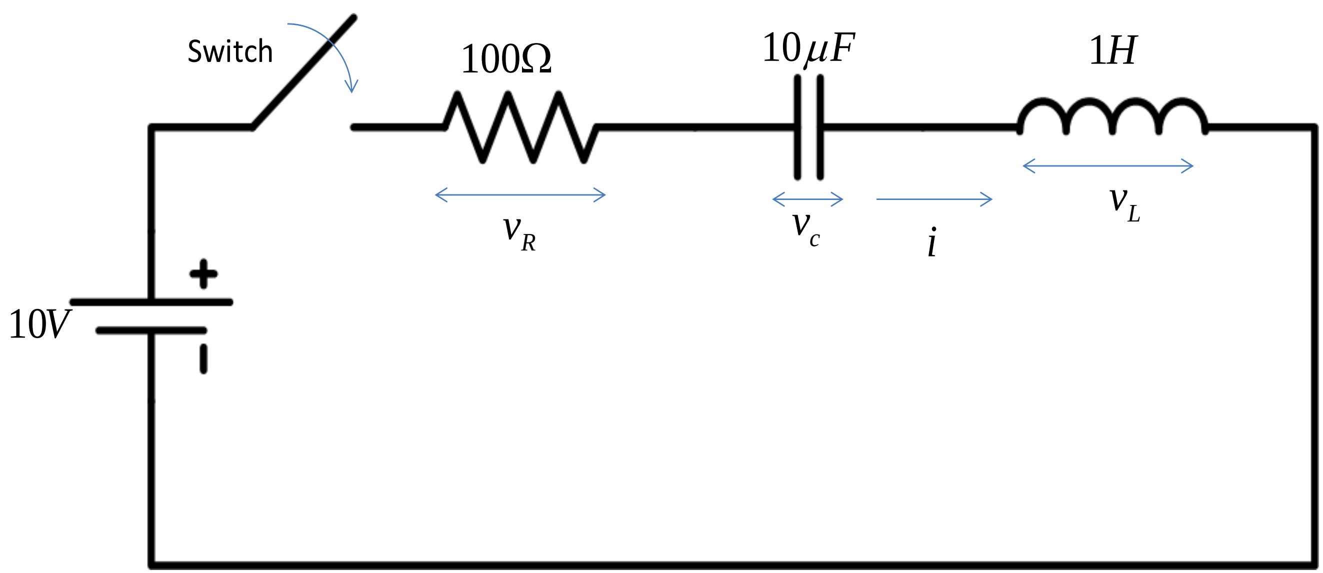


Values for: $(a)\ i_L(0^-);\ (b)\ v_c(0^+);\ (c)\ v_R(0^+);\ (d)\ i_L(\infty)$

Values are 10 A, 400 V, 400 V, -20 A

Before closing the switch, the energy storage elements did not have any stored energy in them. If switch is closed at t=0, find

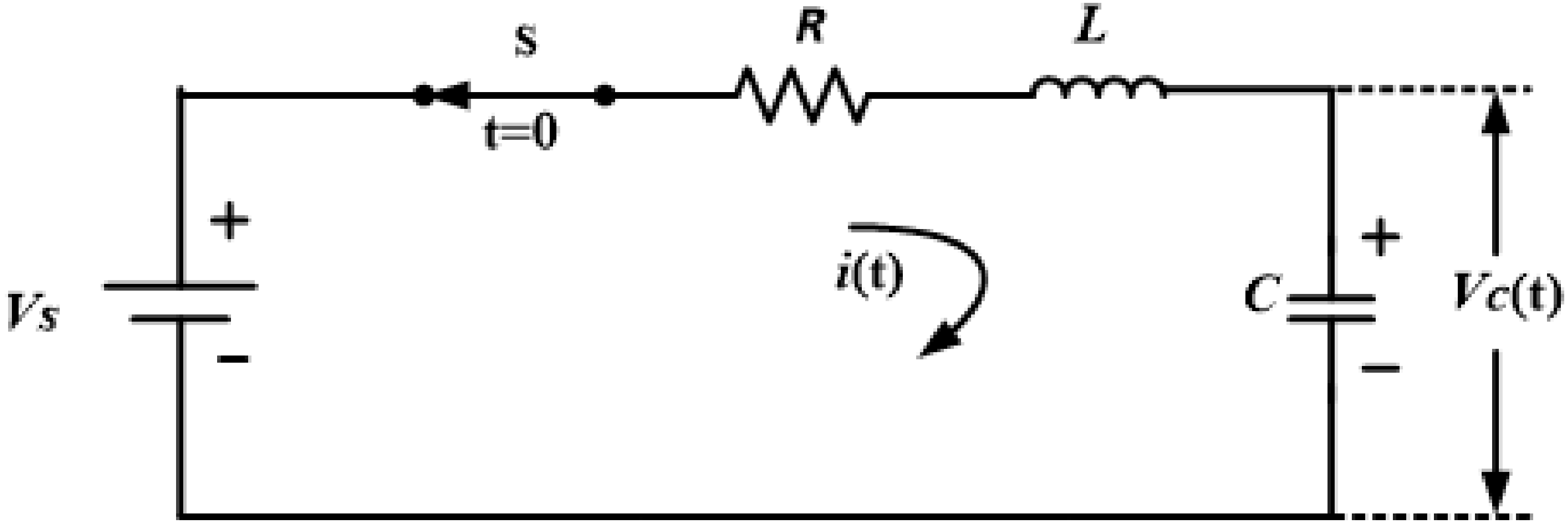
a) $i(0^+)$ b) $\dfrac{di}{dt}(0^+)$ c) $\dfrac{d^2i}{dt^2}(0^+)$ d) $V_L(0^+)$ and e) $V_c(0^+)$.



a) 0 A b) 10 A/s c) -1000 A/s^2 d) 10 V e) 0 V

RLC Circuits

Derive an expression for *Vc*(t) for t > 0.



KVL => $$L\frac{di(t)}{dt}+R\,i(t)+v_c(t)=V_s$$ and $$i(t)=C\frac{dv_c(t)}{dt}$$

Substitution yields

$$LC\frac{d^2v_c(t)}{dt^2}+RC\frac{dv_c(t)}{dt}+v_c(t)=V_s$$

Complete solution made of transient and steady state

$$v_c(t) = v_{cn}(t) + v_{cf}(t) = \left(A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \right) + A$$

To find the natural response, consider the homogeneous diff eqn

$$LC \frac{d^2 v_c(t)}{dt^2} + RC \frac{dv_c(t)}{dt} + v_c(t) = 0$$

$$\Rightarrow \frac{d^2 v_c(t)}{dt^2} + \frac{R}{L} \frac{dv_c(t)}{dt} + \frac{1}{LC} v_c(t) = 0$$

Using $\alpha = \frac{d}{dt}$, $\alpha^2 = \frac{d^2}{dt^2}$

$$\alpha^2 + \frac{R}{L}\alpha + \frac{1}{LC} = 0$$

< = Quadratic Equation

Soln of quadratic equation

$$\alpha_1 = \left(-\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \right)$$

$$\alpha_2 = \left(-\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \right)$$

Discriminant positive

When $\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} > 0$

Roots are distinct with NEGATIVE real parts and giving the natural or transient response of the form

$$v_{cn}(t) = A_1\, e^{\alpha_1 t} + A_2\, e^{\alpha_2 t}$$

Overdamped Response

Critically damped Response

$$\text{When } \left(\frac{R}{2L}\right)^2 - \frac{1}{LC} = 0$$

Roots are EQUAL with NEGATIVE real parts and giving the natural or transient response of the form

$$v_{cn}(t) = \left(A_1 t + A_2\right) e^{\alpha t}$$

Underdamped Response

$$\text{When}\left(\frac{R}{2L}\right)^2 - \frac{1}{LC} < 0$$

Roots are complex conjugates:

$$\alpha_1 = \left(-\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\right) = \beta + j\gamma$$

$$\alpha_2 = \left(-\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2}\right) = \beta - j\gamma$$

$$v_{cn}(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} = A_1 e^{(\beta + j\gamma)} + A_2 e^{(\beta - j\gamma)}$$

$$= e^{\beta t}\left[\left(A_1 + A_2\right)\cos\left(\gamma t\right) + j\left(A_1 - A_2\right)\sin\left(\gamma t\right)\right]$$

$$= e^{\beta t}\left[B_1 \cos\left(\gamma t\right) + B_2 \sin\left(\gamma t\right)\right] \text{ where } B_1 = A_1 + A_2 \;;\; B_2 = j\left(A_1 - A_2\right)$$

Further simplification gives expression for transient response

$$e^{\beta t}\; K \sin\left(\gamma t + \theta\right)$$

with $K = \sqrt{B_1{}^2 + B_2{}^2}$ *and* $\theta = \tan^{-1}\left(\dfrac{B_1}{B_2}\right)$

Complete solution

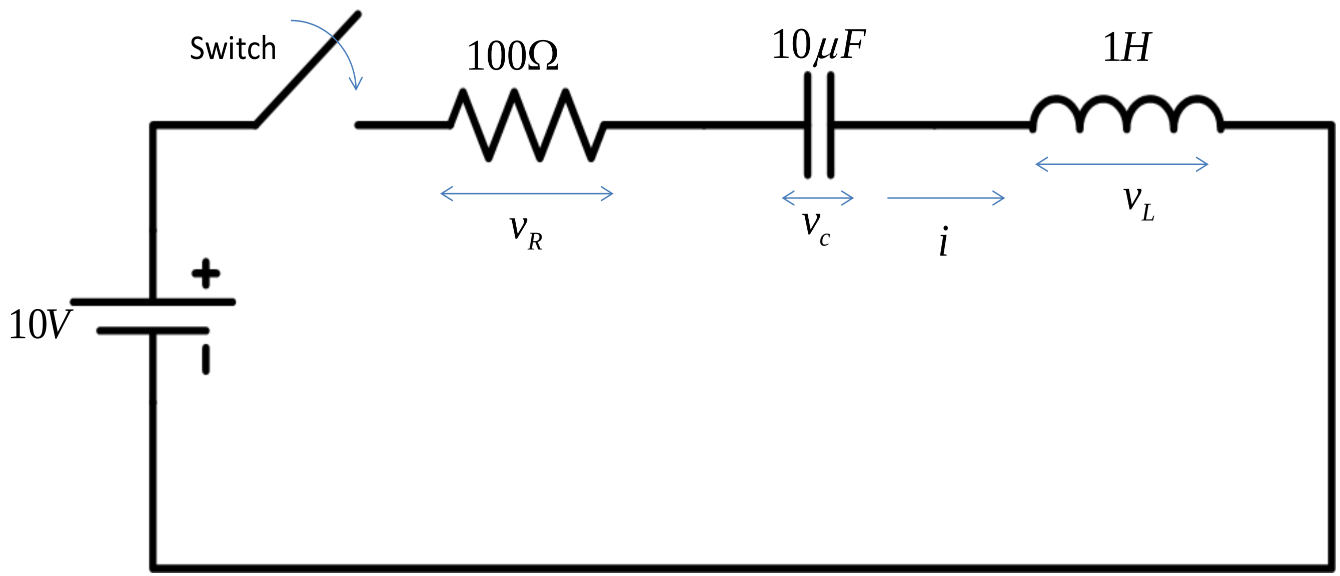
$$v_c(t) = v_{cn}(t) + v_{cf}(t) = \left(A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \right) + A$$

Three types of Responses

Initial Conditions

Before closing the switch, the energy storage elements did not have any stored energy in them. If switch is closed at t=0, find

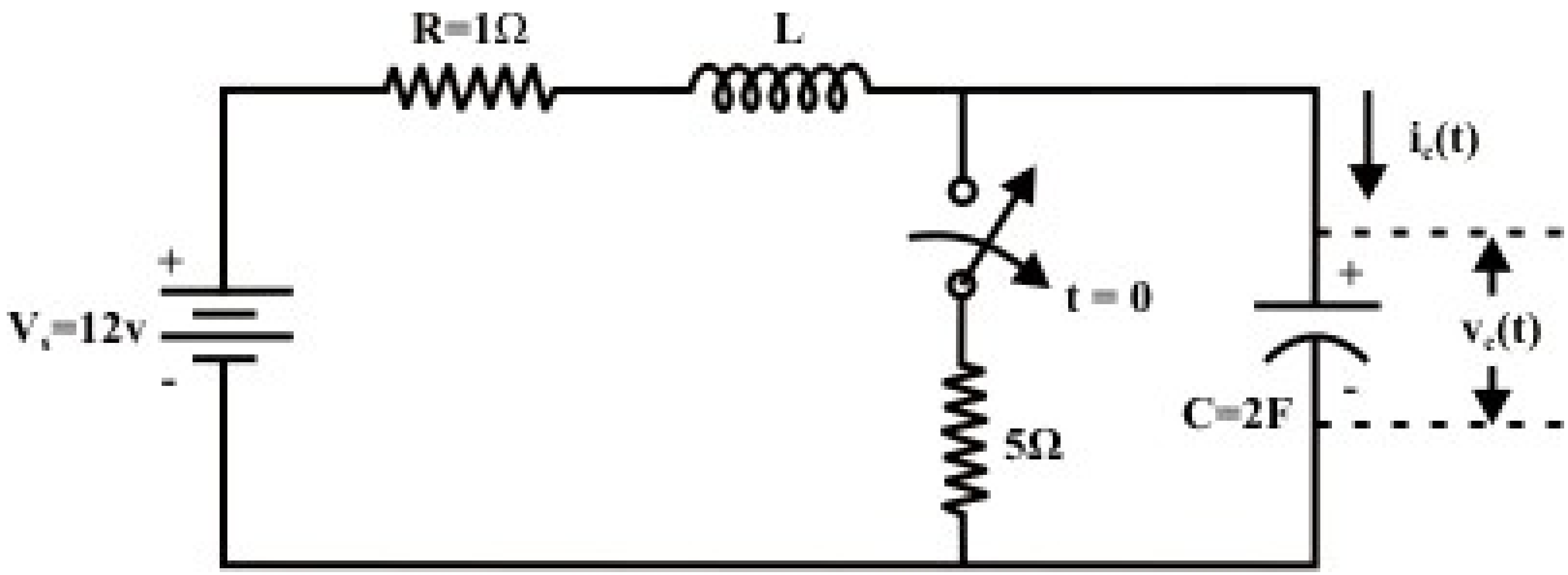
a) $i(0^+)$ b) $\frac{di}{dt}(0^+)$ c) $\frac{d^2 i}{dt^2}(0^+)$ d) $V_L(0^+)$ and e) $V_c(0^+)$.



a) 0 A b) 10 A/s c) -1000 A/s^2 d) 10 V e) 0 V

Example

Switch is closed for a sufficiently long time before opening at $t=0$. Find the expression for $v_c(t)$ and $i_c(t)$. Take $L = 0.2$ H.



Solution

Initial conditions for the capacitor voltage and inductor current:

$$v_c(0^+) = v_c(0^-) = 10 \text{ V}$$

$$i_L(0^+) = i_L(0^-) = \frac{12}{1+5} = 2 \text{ A}$$

Roots are:

$$\alpha_1 = -0.563; \ \ \alpha_2 = -4.436$$

General expression for the capacitor voltage for t > 0 is

$$v_c(t) = \ A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} + A = A_1 e^{-0.563t} + A_2 e^{-4.436t} + A$$

and

$$\frac{dv_{c(t)}}{dt} = \ \alpha_1 A_1 e^{\alpha_1 t} + \alpha_2 A_2 e^{\alpha_2 t} = \text{-}0.563\ A_1 e^{\alpha_1 t} \text{ - } 4.436 A_2 e^{\alpha_2 t}$$

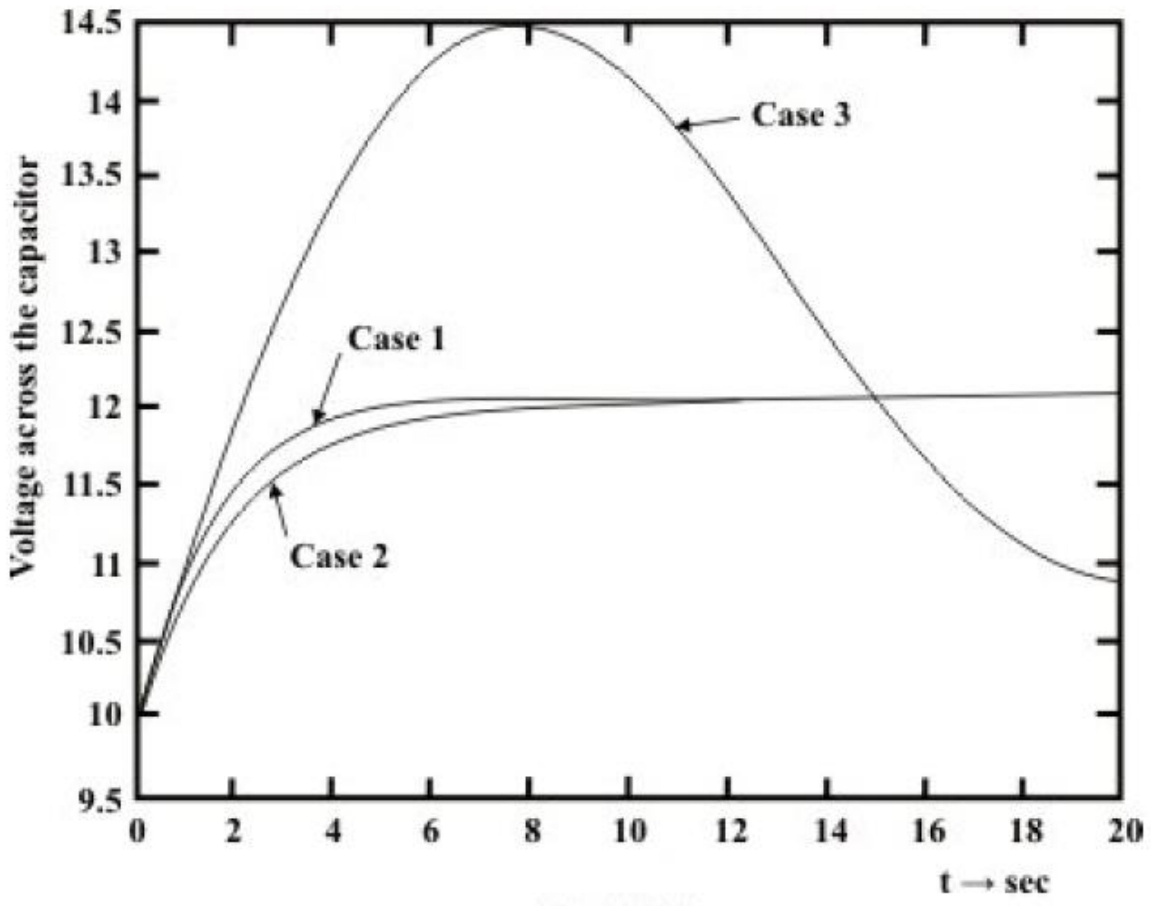
Using initial conditions

A_1 = - 2.032 , A_2 = 0.032 and A = 12

$$v_c(t) = -2.032\, e^{-0.563t} + 0.032\, e^{-4.436t} + 12$$

$$i_c(t) = C\,\frac{dv_{c(t)}}{dt} = 2(1.144e^{-0.563t} - 0.144e^{-4.436t})$$

Draw the plots for capacitor voltage for 20 sec

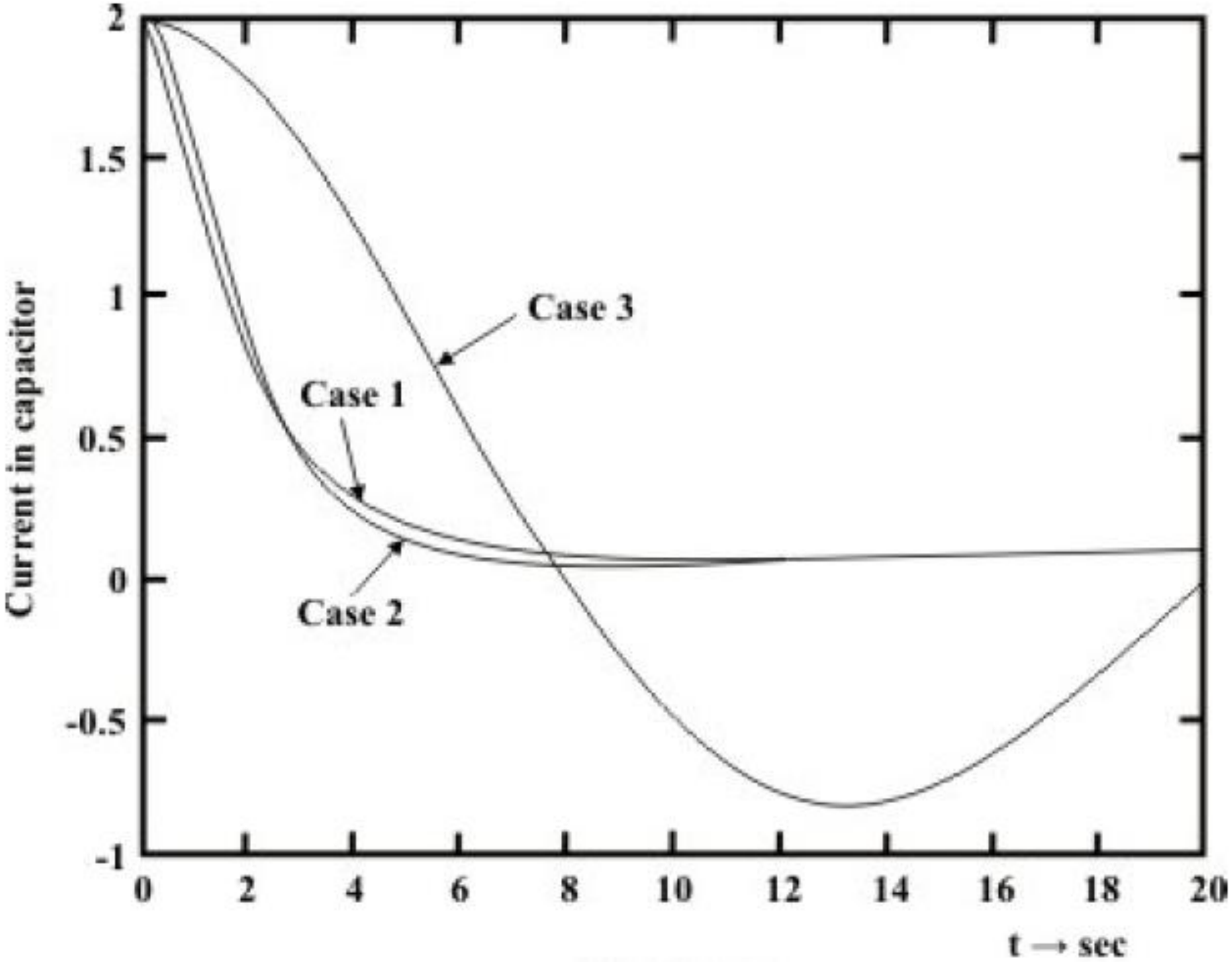


Case 1:
L = 0. 5 H

Case 2:
L = 0.2 H

Case 3:
L = 0.8 H

Draw the plots for capacitor current for 20 sec



Case 1:
L = 0. 5 H

Case 2:
L = 0.2 H

Case 3:
L = 0.8 H