ME101: Engineering Mechanics (3 1 0 8)

2019-20 (II Semester)

LECTURE: 7-8

Distributed forces: Center of Gravity

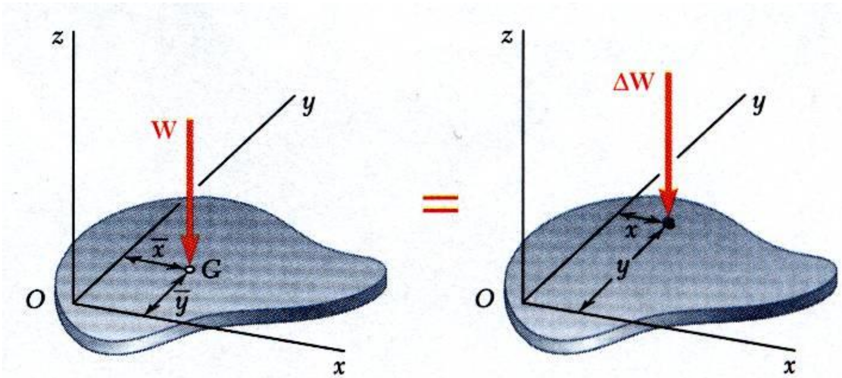
- The earth exerts a gravitational force on each of the particles forming a body. These forces can be replace by a single equivalent force equal to the weight of the body and applied at the *center of gravity* for the body.

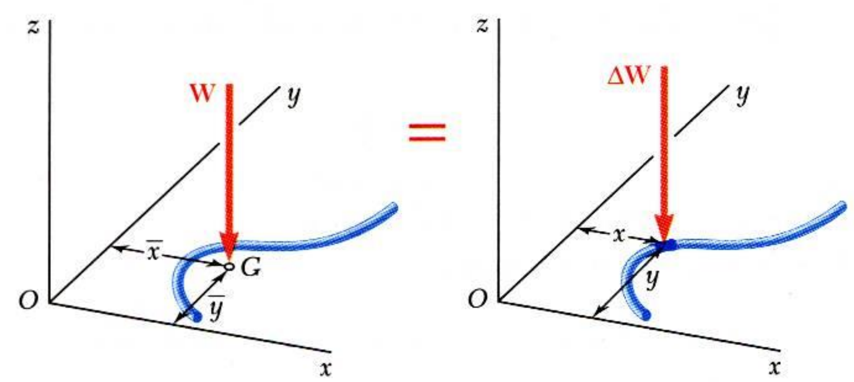
- The *centroid of an area* is analogous to the center of gravity of a body. The concept of the *first moment of an area* is used to locate the centroid.

- Determination of the area of a *surface of revolution* and the volume of a *body of revolution* are accomplished with the *Theorems of Pappus-Guldinus*.

Center of Gravity of a 2D Body

- Center of gravity of a plate
- Center of gravity of a wire

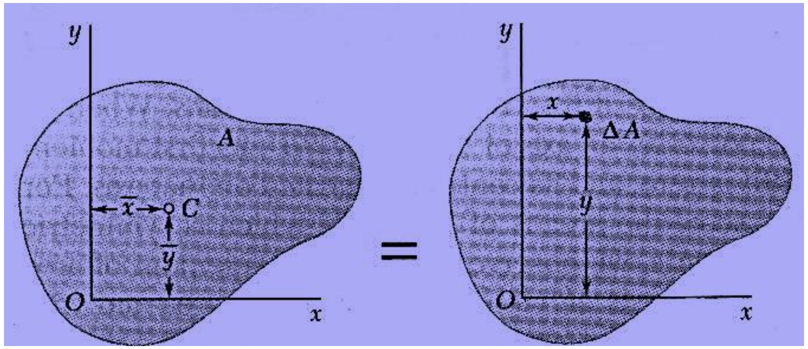




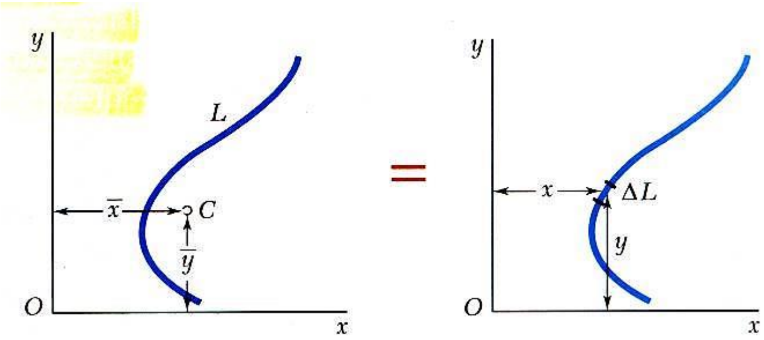
$$\sum M_y \quad \bar{x}W = \sum x\Delta W$$
$$= \int x\,dW$$
$$\sum M_x \quad \bar{y}W = \sum y\Delta W$$
$$= \int y\,dW$$

Centroids & First Moments of Areas/Lines

- Centroid of an area



- Centroid of a line



$$\bar{x}W = \int x\,dW$$

$$\bar{x}(\gamma A t) = \int x(\gamma t)dA$$

$$\bar{x}A = \int x\,dA = Q_y$$

$$= \text{first moment with respect to } y$$

$$\bar{y}A = \int y\,dA = Q_x$$

$$= \text{first moment with respect to } x$$

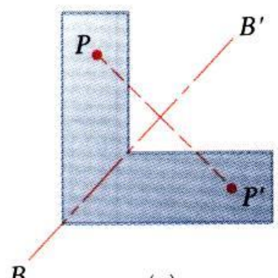
$$\bar{x}W = \int x\,dW$$

$$\bar{x}(\gamma L a) = \int x(\gamma a)dL$$

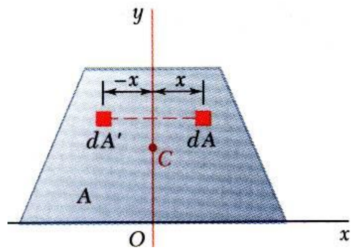
$$\bar{x}L = \int x\,dL$$

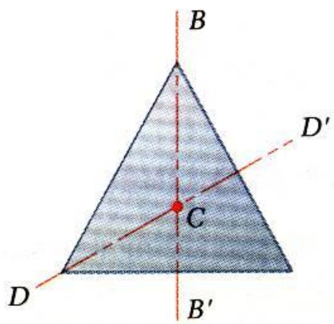
$$\bar{y}L = \int y\,dL$$

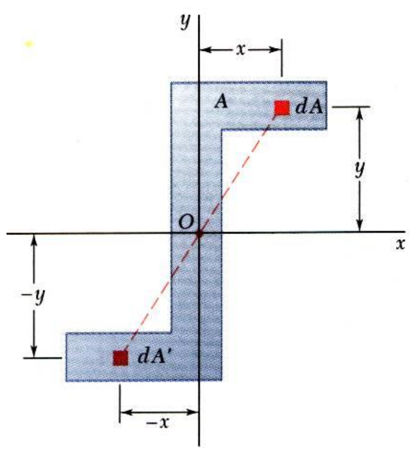
First Moments of Areas and Lines



(a)

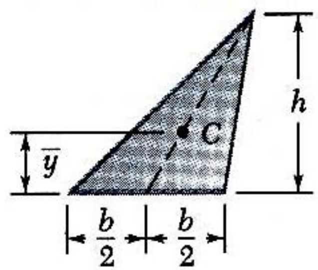
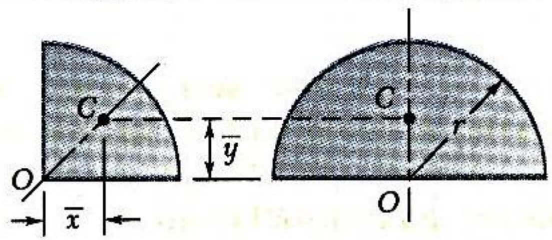
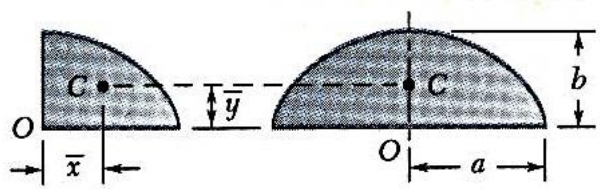




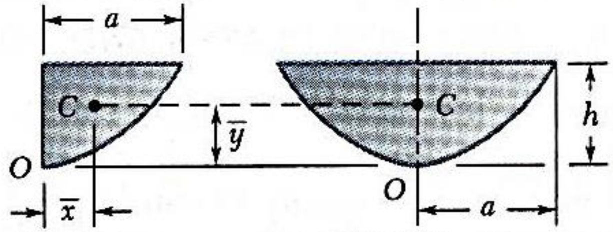
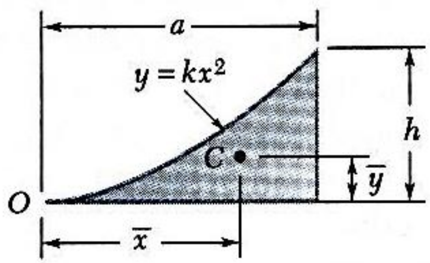
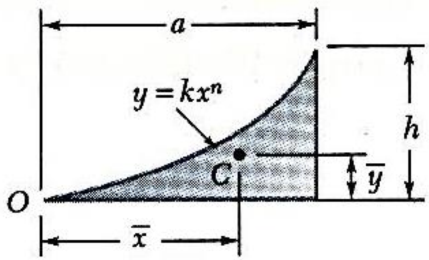


- An area is symmetric with respect to an axis *BB'* if for every point *P* there exists a point *P'* such that *PP'* is perpendicular to *BB'* and is divided into two equal parts by *BB'*.
- The first moment of an area with respect to a line of symmetry is zero.
- If an area possesses a line of symmetry, its centroid lies on that axis
- If an area possesses two lines of symmetry, its centroid lies at their intersection.
- An area is symmetric with respect to a center *O* if for every element *dA* at (x,y) there exists an area *dA'* of equal area at ($-x$,$-y$).
- The centroid of the area coincides with the center of symmetry.

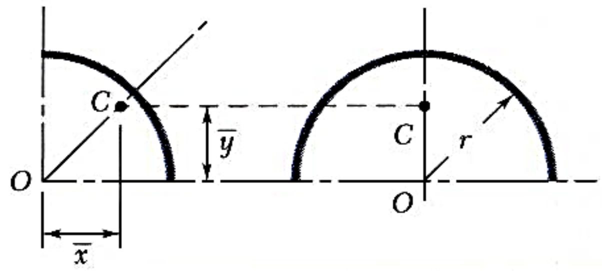
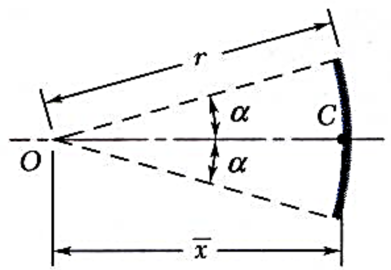
Centroids of Common Shapes of Areas

Shape		$\bar{x}$	$\bar{y}$	Area
Triangular area			$\frac{h}{3}$	$\frac{bh}{2}$
Quarter-circular area		$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{4}$
Semicircular area		0	$\frac{4r}{3\pi}$	$\frac{\pi r^2}{2}$
Quarter-elliptical area		$\frac{4a}{3\pi}$	$\frac{4b}{3\pi}$	$\frac{\pi ab}{4}$
Semielliptical area		0	$\frac{4b}{3\pi}$	$\frac{\pi ab}{2}$

Centroids of Common Shapes of Areas

Semiparabolic area		$\frac{3a}{8}$	$\frac{3h}{5}$	$\frac{2ah}{3}$
Parabolic area		0	$\frac{3h}{5}$	$\frac{4ah}{3}$
Parabolic spandrel		$\frac{3a}{4}$	$\frac{3h}{10}$	$\frac{ah}{3}$
General spandrel		$\frac{n+1}{n+2}a$	$\frac{n+1}{4n+2}h$	$\frac{ah}{n+1}$

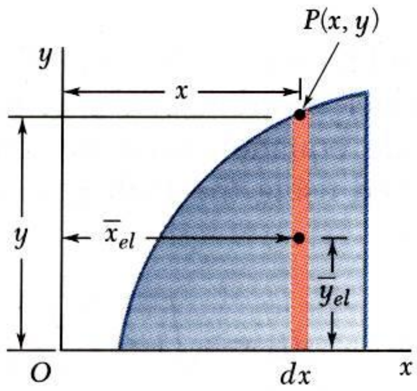
Centroids of Common Shapes of Lines

Shape		$\bar{x}$	$\bar{y}$	Length
Quarter-circular arc		$\frac{2r}{\pi}$	$\frac{2r}{\pi}$	$\frac{\pi r}{2}$
Semicircular arc		0	$\frac{2r}{\pi}$	πr
Arc of circle		$\frac{r \sin \alpha}{\alpha}$	0	$2\alpha r$

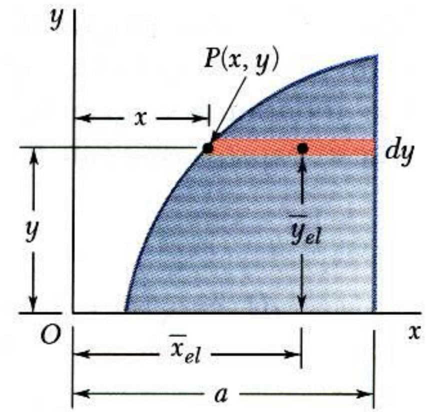
Determination of Centroids by Integration

$$\bar{x}A = \int x\,dA = \iint x\,dx\,dy = \int \bar{x}_{el}\,dA$$
$$\bar{y}A = \int y\,dA = \iint y\,dx\,dy = \int \bar{y}_{el}\,dA$$

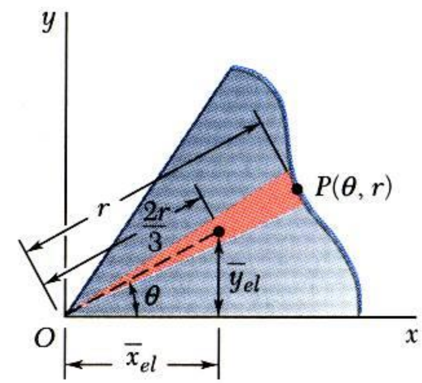
- Double integration to find the first moment may be avoided by defining *dA* as a thin rectangle or strip.



$$\bar{x}A = \int \bar{x}_{el}\,dA = \int x\left(y\,dx\right)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int \frac{y}{2}\left(y\,dx\right)$$

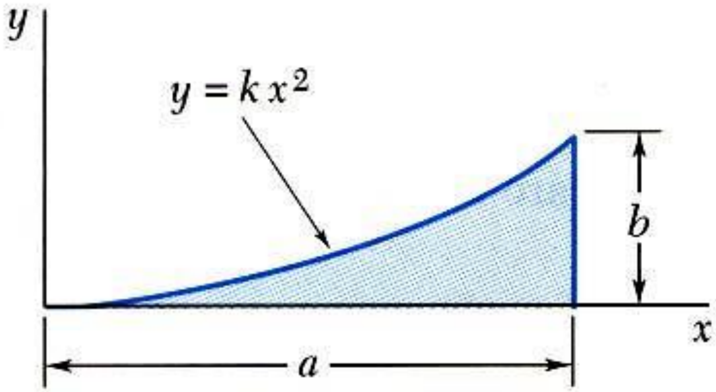


$$\bar{x}A = \int \bar{x}_{el}\,dA = \int \frac{a+x}{2}\left[\left(a-x\right)dy\right]$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int y\left[\left(a-x\right)dy\right]$$



$$\bar{x}A = \int \bar{x}_{el}\,dA = \int \frac{2r}{3}\cos\theta\left(\frac{1}{2}r^2 d\theta\right)$$
$$\bar{y}A = \int \bar{y}_{el}\,dA = \int \frac{2r}{3}\sin\theta\left(\frac{1}{2}r^2 d\theta\right)$$

Sample Problem

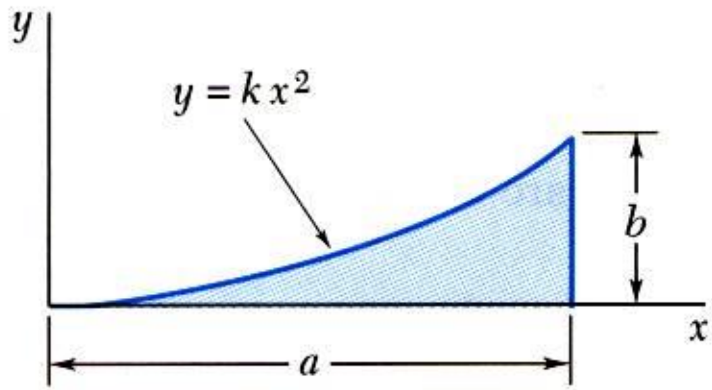


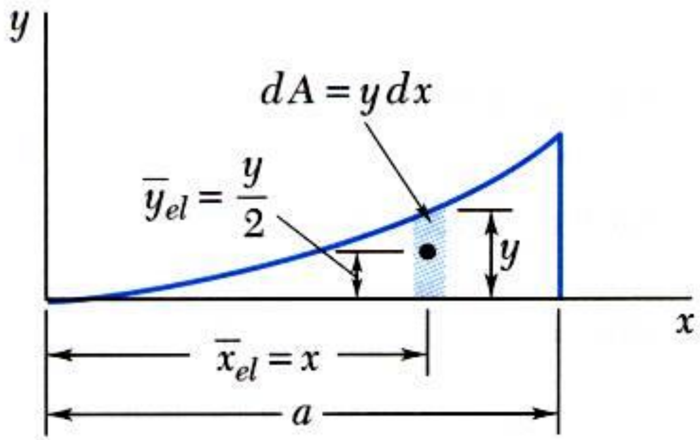
Determine by direct integration the location of the centroid of a parabolic spandrel.

SOLUTION:

- Determine the constant k.
- Evaluate the total area.
- Using either vertical or horizontal strips, perform a single integration to find the first moments.
- Evaluate the centroid coordinates.

Sample Problem





SOLUTION:

- Determine the constant k.

$$y = k\,x^2$$

$$b = k\,a^2 \;\Rightarrow k = \frac{b}{a^2}$$

$$y = \frac{b}{a^2}x^2 \quad or \quad x = \frac{a}{b^{1/2}}y^{1/2}$$

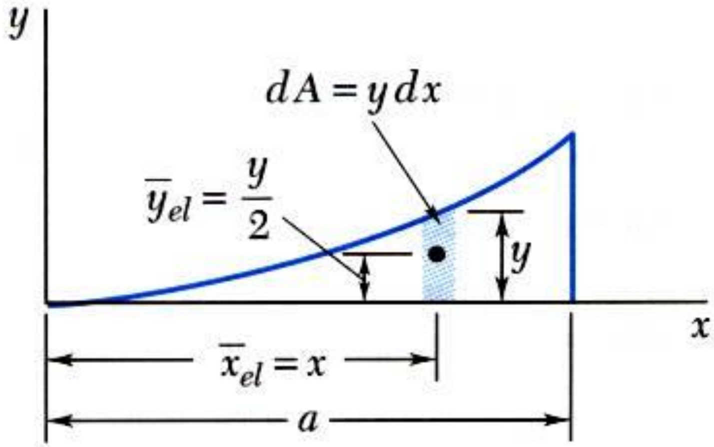
- Evaluate the total area.

$$A = \int dA$$

$$= \int y\,dx = \int_0^a \frac{b}{a^2}x^2 dx = \left[\frac{b}{a^2}\frac{x^3}{3}\right]_0^a$$

$$= \frac{ab}{3}$$

Sample Problem



- Using vertical strips, perform a single integration to find the first moments.

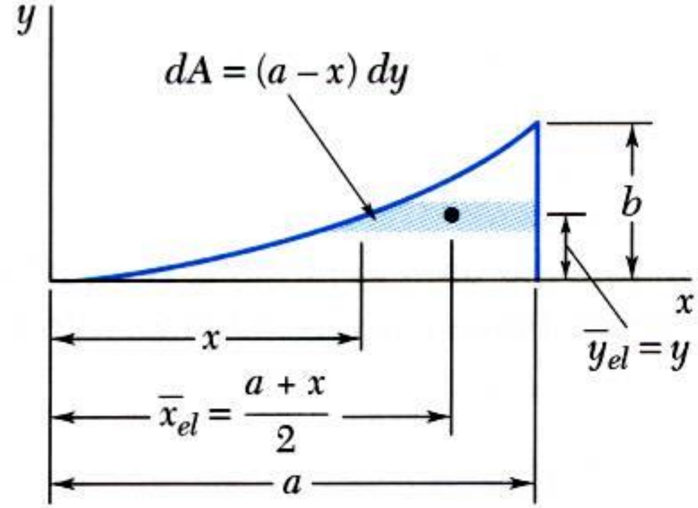
$$Q_y = \int \bar{x}_{el}\, dA = \int xy\, dx = \int_0^a x\left(\frac{b}{a^2}x^2\right)dx$$

$$= \left[\frac{b}{a^2}\frac{x^4}{4}\right]_0^a = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el}\, dA = \int \frac{y}{2} y\, dx = \int_0^a \frac{1}{2}\left(\frac{b}{a^2}x^2\right)^2 dx$$

$$= \left[\frac{b^2}{2a^4}\frac{x^5}{5}\right]_0^a = \frac{ab^2}{10}$$

Sample Problem



- Or, using horizontal strips, perform a single integration to find the first moments.

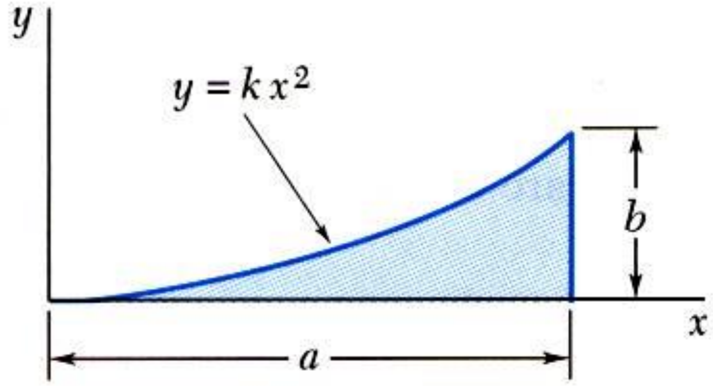
$$Q_y = \int \bar{x}_{el}\, dA = \int \frac{a+x}{2}(a-x)dy = \int_0^b \frac{a^2 - x^2}{2} dy$$

$$= \frac{1}{2}\int_0^b \left(a^2 - \frac{a^2}{b} y \right) dy = \frac{a^2 b}{4}$$

$$Q_x = \int \bar{y}_{el}\, dA = \int y(a-x)dy = \int y\left(a - \frac{a}{b^{1/2}} y^{1/2} \right) dy$$

$$= \int_0^b \left(ay - \frac{a}{b^{1/2}} y^{3/2} \right) dy = \frac{ab^2}{10}$$

Sample Problem



- Evaluate the centroid coordinates.

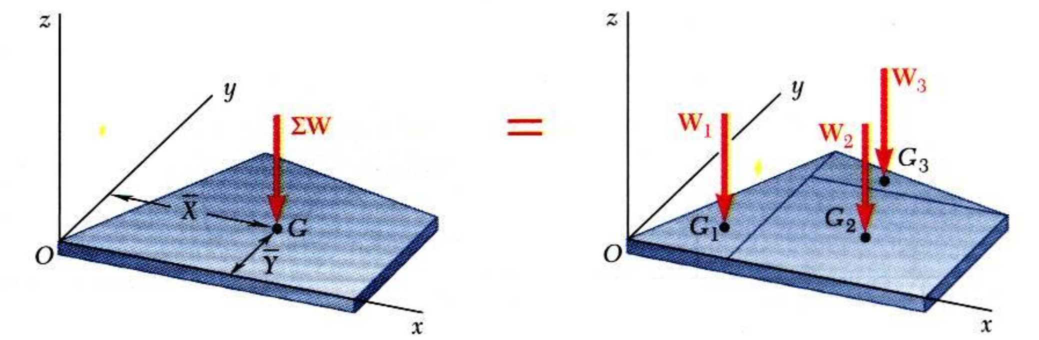
$$\bar{x}A = Q_y$$

$$\bar{x}\frac{ab}{3} = \frac{a^2 b}{4} \qquad \boxed{\bar{x} = \frac{3}{4}a}$$

$$\bar{y}A = Q_x$$

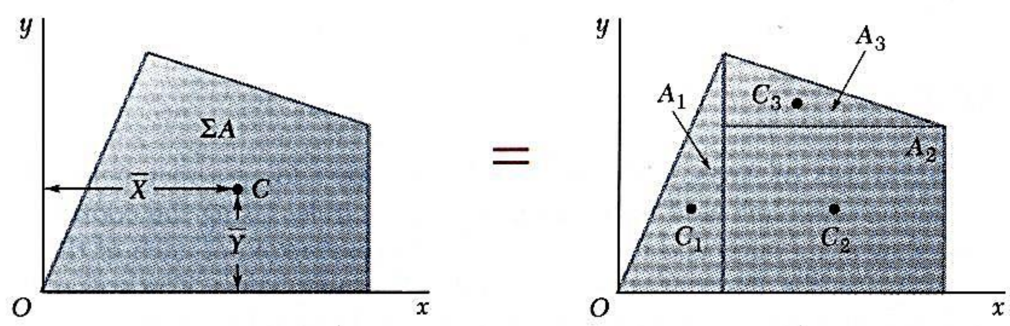
$$\bar{y}\frac{ab}{3} = \frac{ab^2}{10} \qquad \boxed{\bar{y} = \frac{3}{10}b}$$

Composite Plates and Areas



- Composite plates

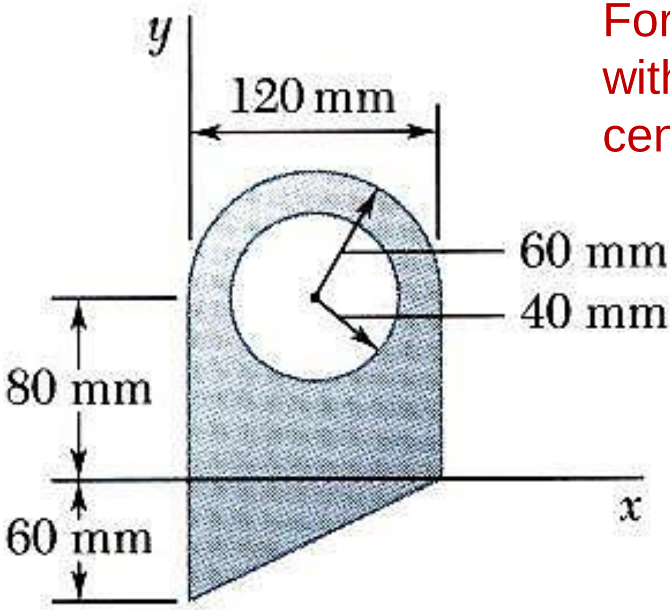
$$\overline{X}\sum W = \sum \bar{x} W$$
$$\overline{Y}\sum W = \sum \bar{y} W$$



- Composite area

$$\overline{X}\sum A = \sum \bar{x} A$$
$$\overline{Y}\sum A = \sum \bar{y} A$$

Sample Problem

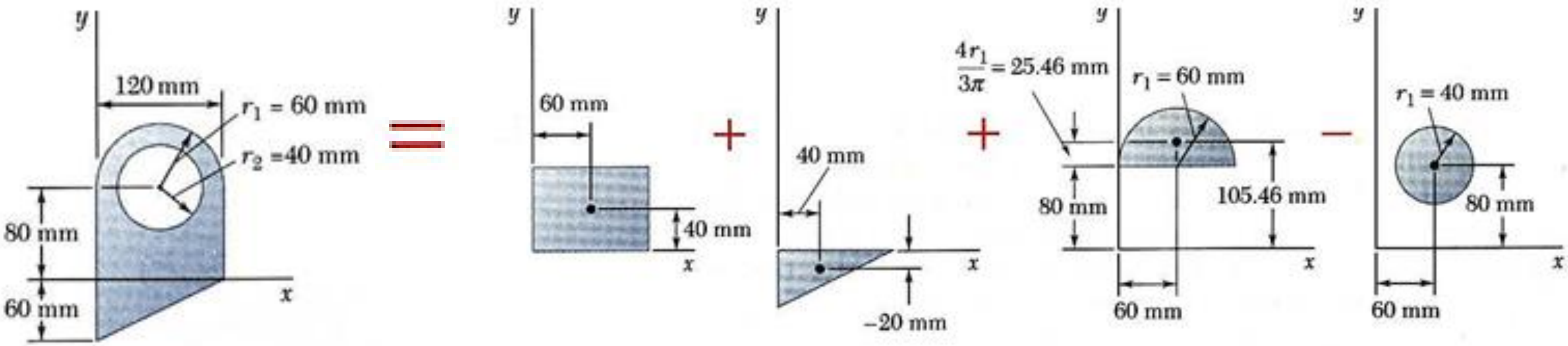


For the plane area shown, determine the first moments with respect to the *x* and *y* axes and the location of the centroid.

SOLUTION:

- Divide the area into a triangle, rectangle, and semicircle with a circular cutout.
- Calculate the first moments of each area with respect to the axes.
- Find the total area and first moments of the triangle, rectangle, and semicircle. Subtract the area and first moment of the circular cutout.
- Compute the coordinates of the area centroid by dividing the first moments by the total area.

Sample Problem



Component	A, mm²	$\bar{x}$, mm	$\bar{y}$, mm	$\bar{x}A$, mm³	$\bar{y}A$, mm³
Rectangle	$(120)(80) = 9.6 \times 10^3$	60	40	$+576 \times 10^3$	$+384 \times 10^3$
Triangle	$\frac{1}{2}(120)(60) = 3.6 \times 10^3$	40	−20	$+144 \times 10^3$	-72×10^3
Semicircle	$\frac{1}{2}\pi(60)^2 = 5.655 \times 10^3$	60	105.46	$+339.3 \times 10^3$	$+596.4 \times 10^3$
Circle	$-\pi(40)^2 = -5.027 \times 10^3$	60	80	-301.6×10^3	-402.2×10^3
	$\Sigma A = 13.828 \times 10^3$			$\Sigma\bar{x}A = +757.7 \times 10^3$	$\Sigma\bar{y}A = +506.2 \times 10^3$

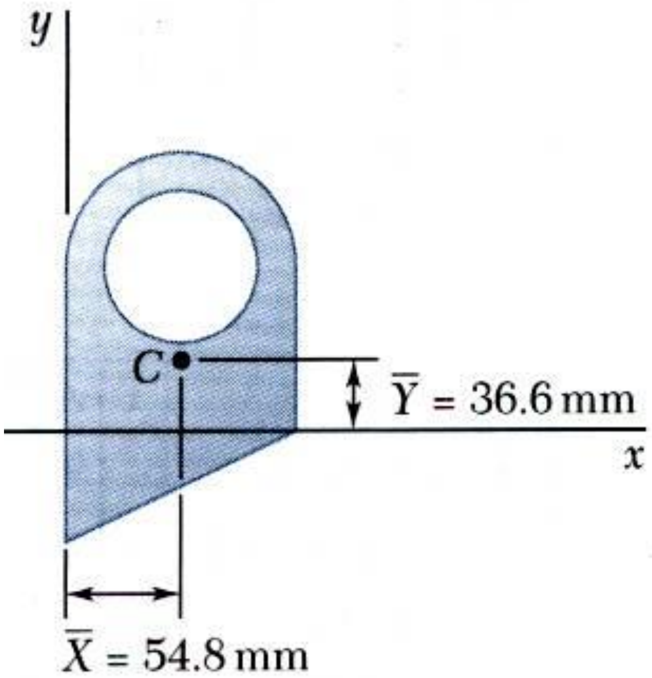
Find the **total area** and **first moments** of the triangle, rectangle, and semicircle. **Subtract the area and first moment of the circular cutout.**

$$Q_x = +506.2 \times 10^3\,\text{mm}^3$$

$$Q_y = +757.7 \times 10^3\,\text{mm}^3$$

Sample Problem

Compute the coordinates of the area centroid by dividing the first moments by the total area.



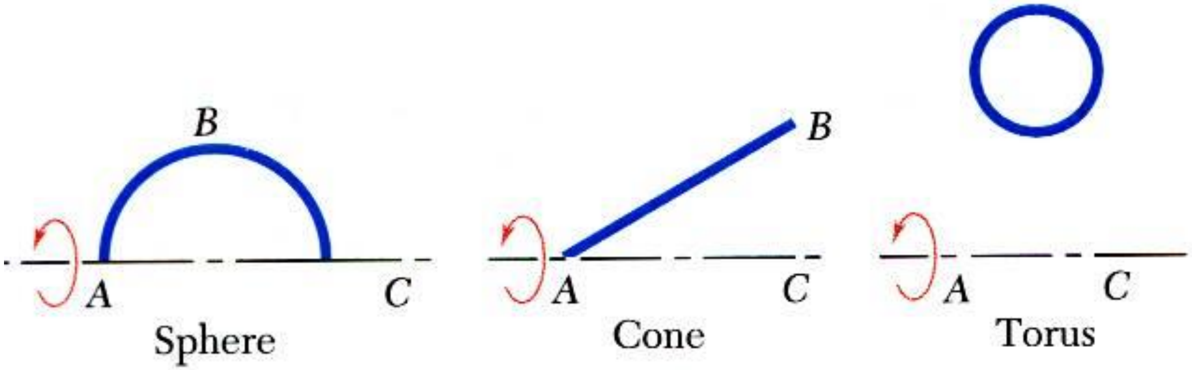
$$\bar{X} = \frac{\sum \bar{x} A}{\sum A} = \frac{+757.7 \times 10^3\,\text{mm}^3}{13.828 \times 10^3\,\text{mm}^2}$$

$$\boxed{\bar{X} = 54.8\,\text{mm}}$$

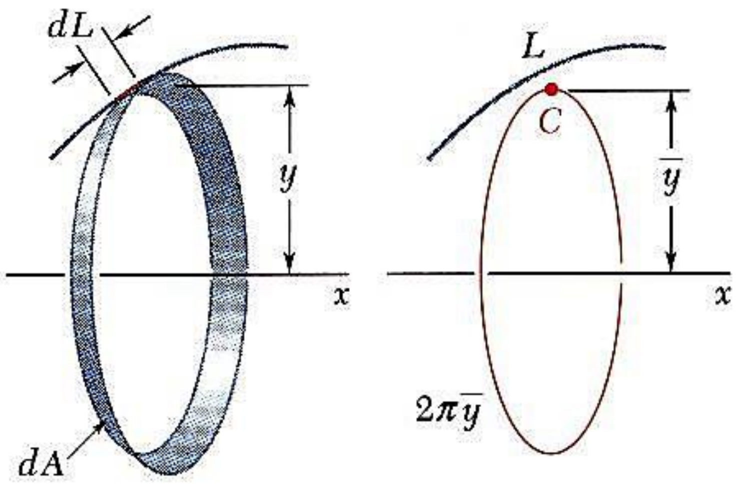
$$\bar{Y} = \frac{\sum \bar{y} A}{\sum A} = \frac{+506.2 \times 10^3\,\text{mm}^3}{13.828 \times 10^3\,\text{mm}^2}$$

$$\boxed{\bar{Y} = 36.6\,\text{mm}}$$

Theorems of Pappus-Guldinus



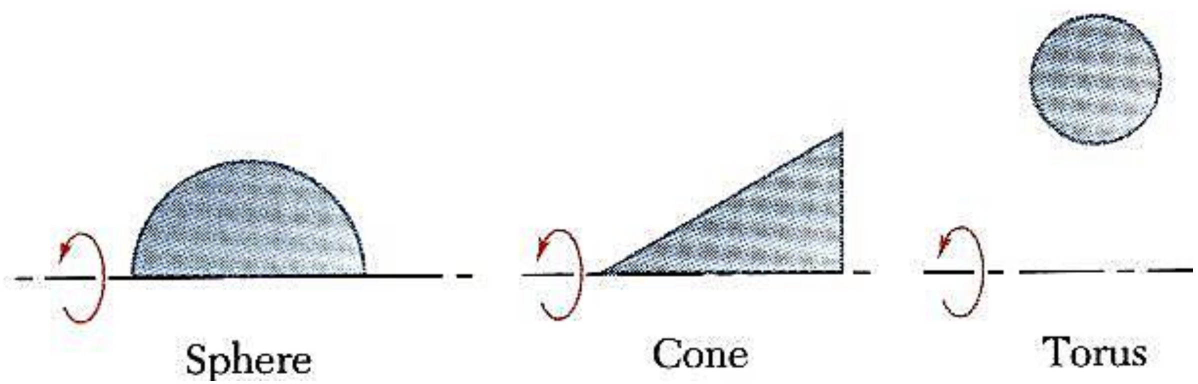
- Surface of revolution is generated by rotating a plane curve about a fixed axis.



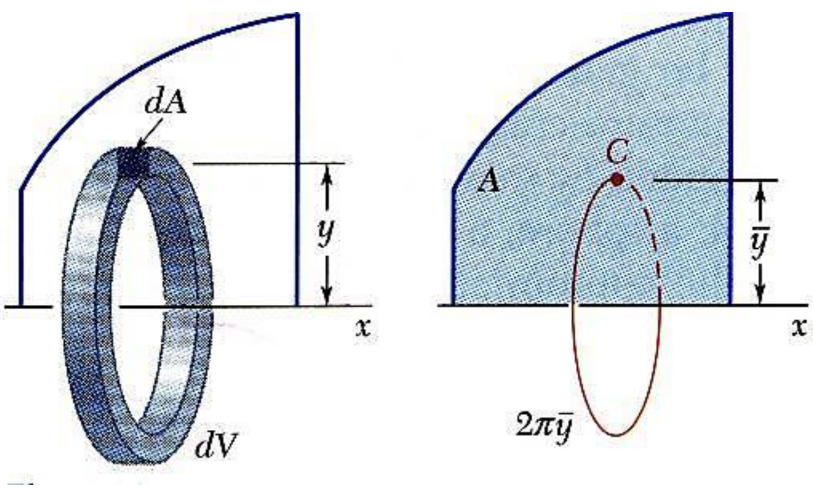
- Area of a surface of revolution is equal to the length of the generating curve times the distance traveled by the centroid through the rotation.

$$A = 2\pi \bar{y} L$$

Theorems of Pappus-Guldinus



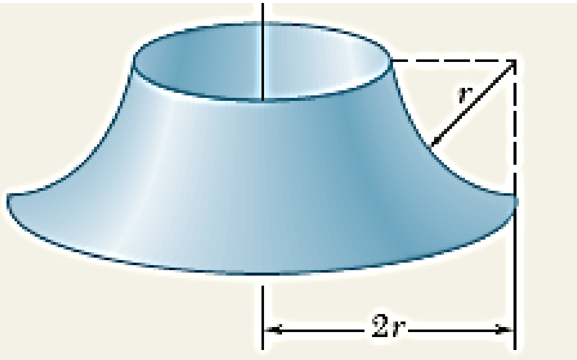
- Body of revolution is generated by rotating a plane area about a fixed axis.



- Volume of a body of revolution is equal to the generating area times the distance traveled by the centroid through the rotation.

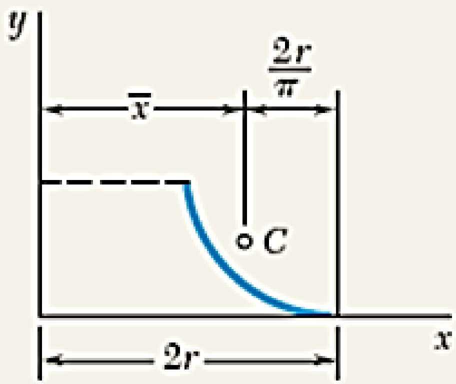
$$V = 2\pi \bar{y} A$$

Sample Problem



Determine the area of the surface of revolution shown, which is obtained by rotating a quarter-circular arc about a vertical axis.

SOLUTION



According to Theorem I of Pappus-Guldinus, the area generated is equal to the product of the length of the arc and the distance traveled by its centroid.

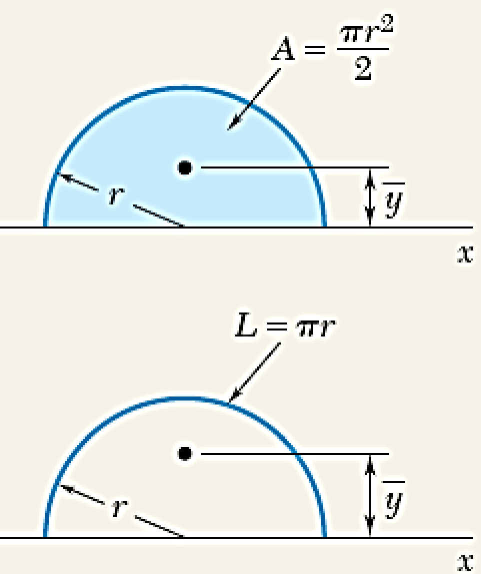
$$\bar{x} = 2r - \frac{2r}{\pi} = 2r\left(1 - \frac{1}{\pi}\right)$$

$$A = 2\pi\bar{x}L = 2\pi\left[2r\left(1 - \frac{1}{\pi}\right)\right]\left(\frac{\pi r}{2}\right)$$

$$A = 2\pi r^2(\pi - 1)$$

Sample Problem

Using the theorems of Pappus-Guldinus, determine (*a*) the centroid of a semicircular area, (*b*) the centroid of a semicircular arc. We recall that the volume and the surface area of a sphere are $\frac{4}{3}\pi r^3$ and $4\pi r^2$, respectively.



SOLUTION

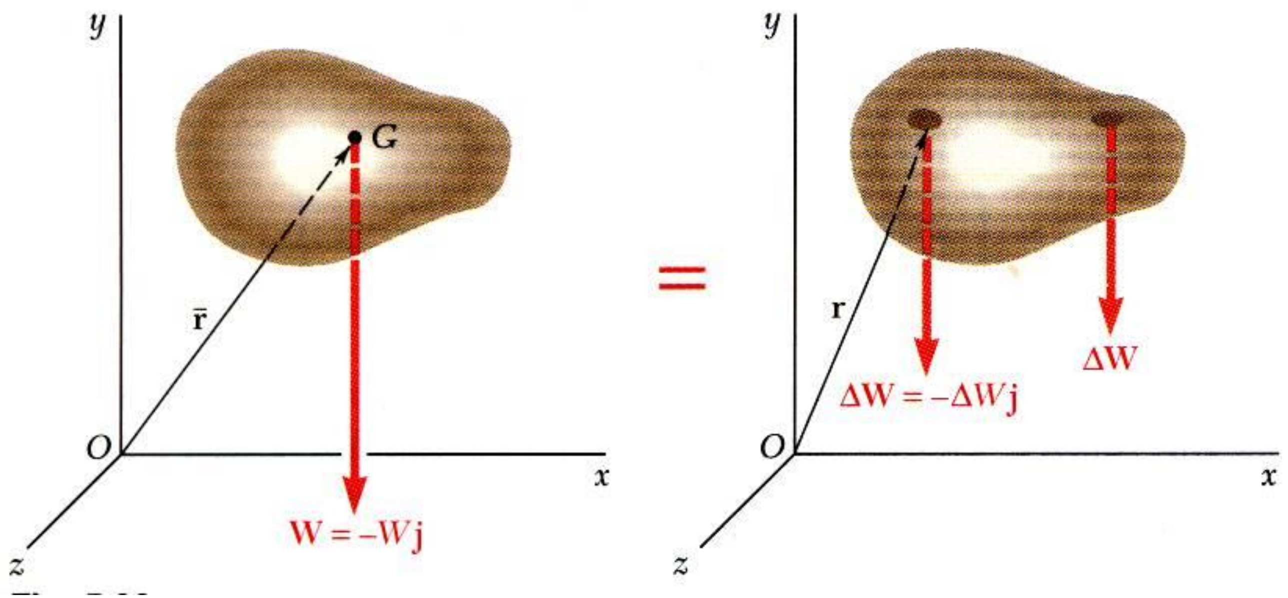
The volume of a sphere is equal to the product of the area of a semicircle and the distance traveled by the centroid of the semicircle in one revolution about the x axis.

$$V = 2\pi \bar{y} A \qquad \tfrac{4}{3}\pi r^3 = 2\pi \bar{y}\left(\tfrac{1}{2}\pi r^2\right) \qquad \bar{y} = \frac{4r}{3\pi} \quad \blacktriangleleft$$

Likewise, the area of a sphere is equal to the product of the length of the generating semicircle and the distance traveled by its centroid in one revolution.

$$A = 2\pi \bar{y} L \qquad 4\pi r^2 = 2\pi \bar{y}(\pi r) \qquad \bar{y} = \frac{2r}{\pi} \quad \blacktriangleleft$$

CG of a 3D Body: Centroid of a Volume



- Center of gravity *G*

$$-W\vec{j} = \sum\left(-\Delta W\vec{j}\right)$$

$$\vec{r}_G \times \left(-W\vec{j}\right) = \sum\left[\vec{r} \times \left(-\Delta W\vec{j}\right)\right]$$
$$\vec{r}_G W \times \left(-\vec{j}\right) = \left(\sum \vec{r}\Delta W\right) \times \left(-\vec{j}\right)$$

$$W = \int dW \qquad \vec{r}_G W = \int \vec{r}\, dW$$

- Results are independent of body orientation,

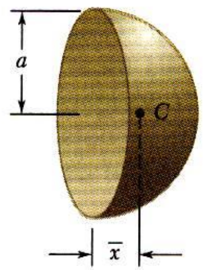
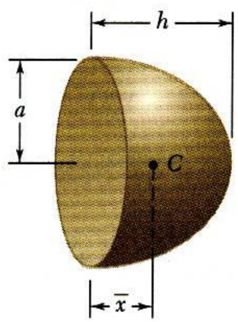
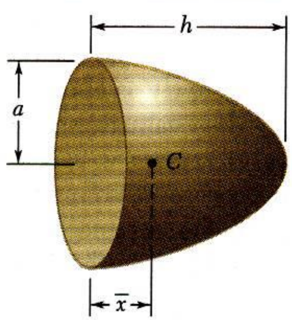
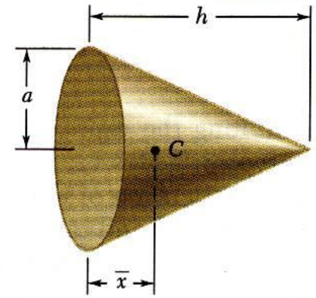
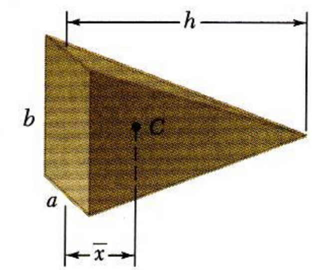
$$\bar{x}W = \int x\,dW \quad \bar{y}W = \int y\,dW \quad \bar{z}W = \int z\,dW$$

- For homogeneous bodies,

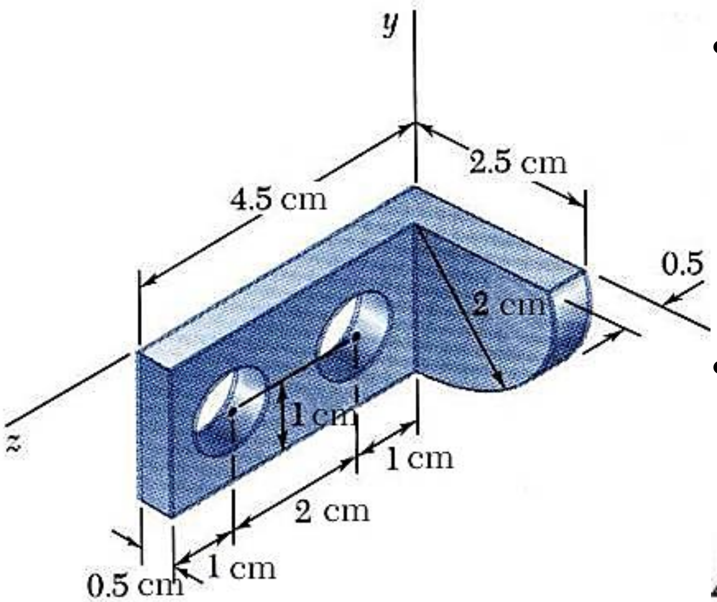
$$W = \gamma V \text{ and } dW = \gamma dV$$

$$\bar{x}V = \int x\,dV \quad \bar{y}V = \int y\,dV \quad \bar{z}V = \int z\,dV$$

Centroids of Common 3D Shapes

Shape		$\bar{x}$	Volume
Hemisphere		$\frac{3a}{8}$	$\frac{2}{3}\pi a^3$
Semiellipsoid of revolution		$\frac{3h}{8}$	$\frac{2}{3}\pi a^2 h$
Paraboloid of revolution		$\frac{h}{3}$	$\frac{1}{2}\pi a^2 h$
Cone		$\frac{h}{4}$	$\frac{1}{3}\pi a^2 h$
Pyramid		$\frac{h}{4}$	$\frac{1}{3}abh$

Composite 3D Bodies

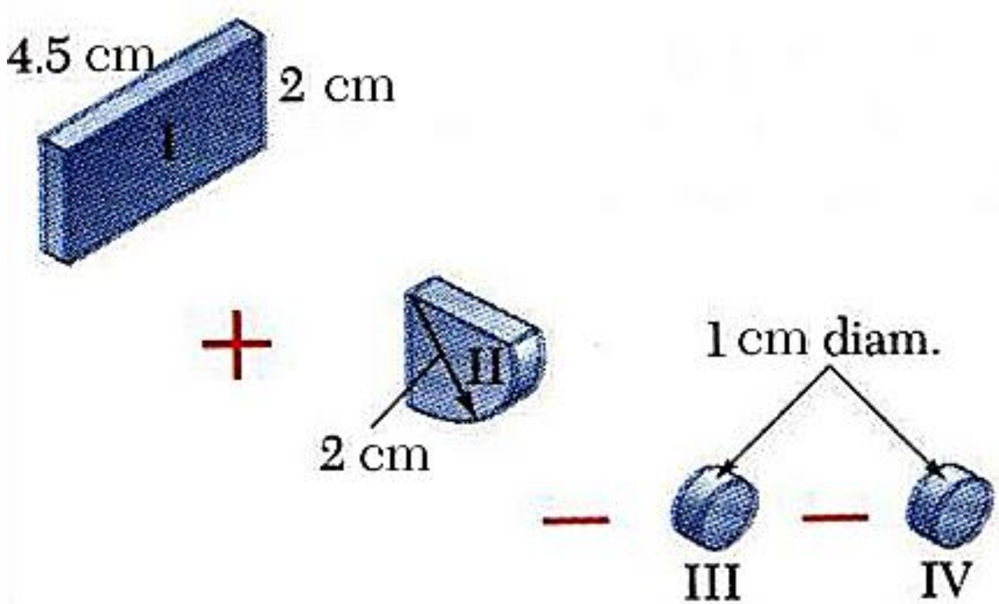


- Moment of the total weight concentrated at the center of gravity G is equal to the sum of the moments of the weights of the component parts.

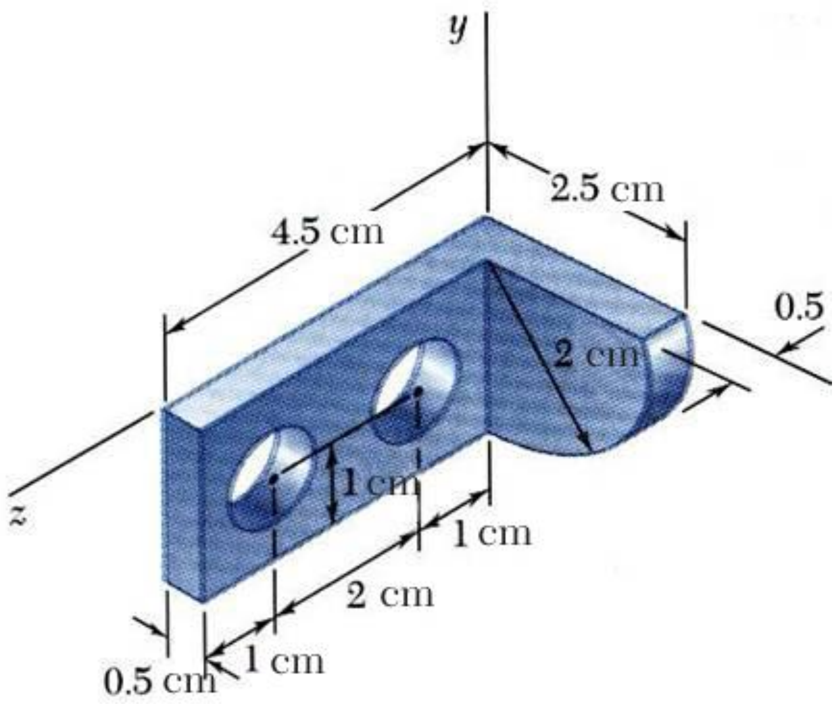
$$\bar{X}\sum W = \sum \bar{x}W \quad \bar{Y}\sum W = \sum \bar{y}W \quad \bar{Z}\sum W = \sum \bar{z}W$$

- For homogeneous bodies,

$$\bar{X}\sum V = \sum \bar{x}V \quad \bar{Y}\sum V = \sum \bar{y}V \quad \bar{Z}\sum V = \sum \bar{z}V$$



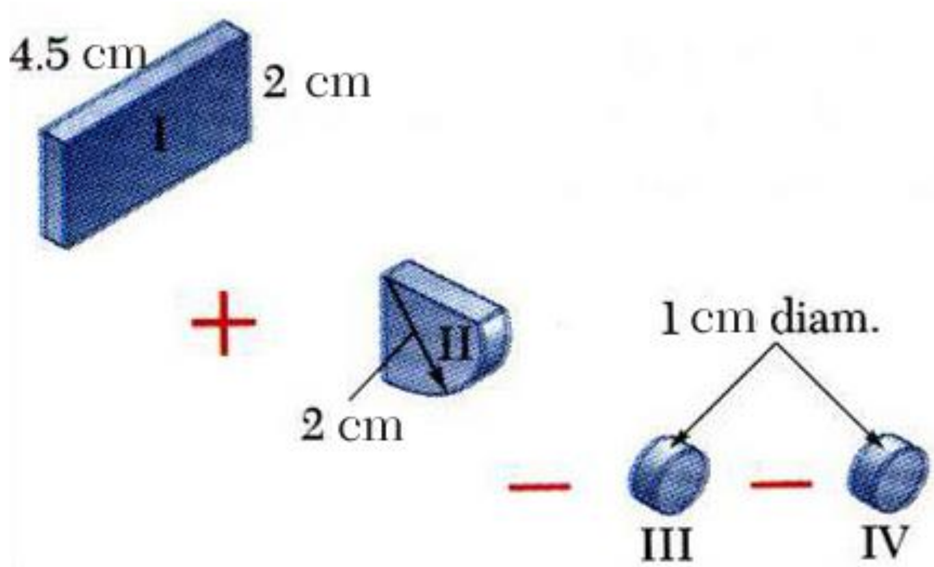
Sample Problem



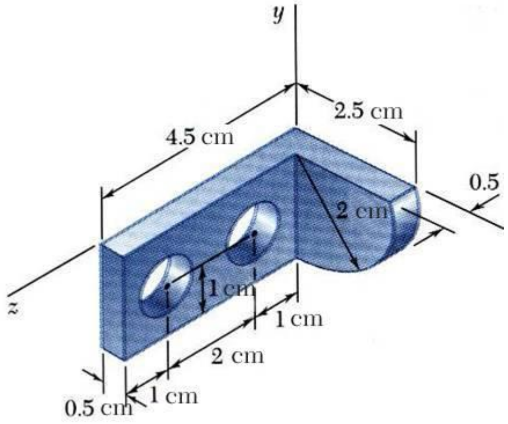
Locate the center of gravity of the steel machine element. The diameter of each hole is 1 cm.

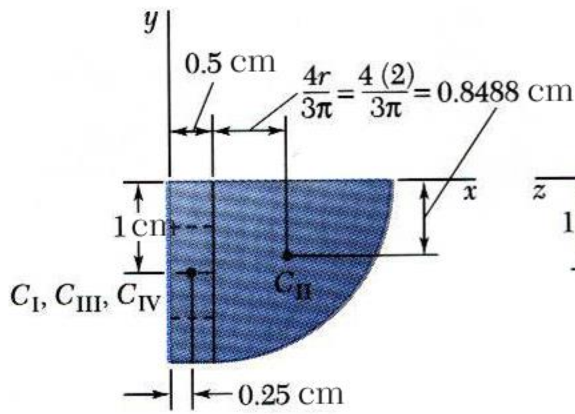
SOLUTION:

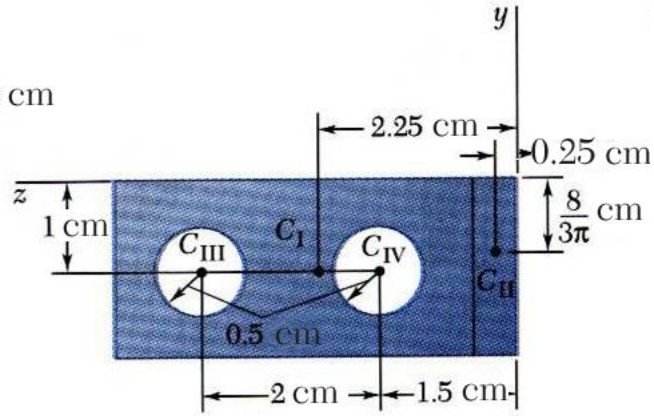
- Form the machine element from a rectangular parallelepiped and a quarter cylinder and then subtracting two 1-in. diameter cylinders.

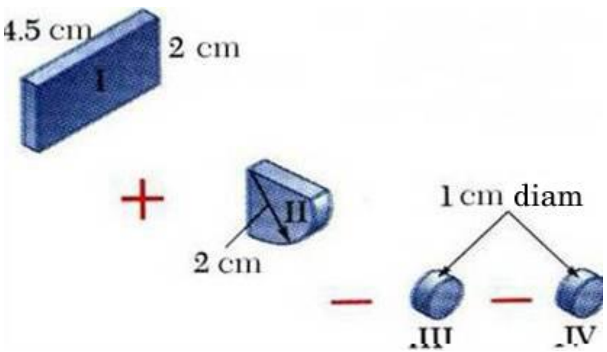


Sample Problem





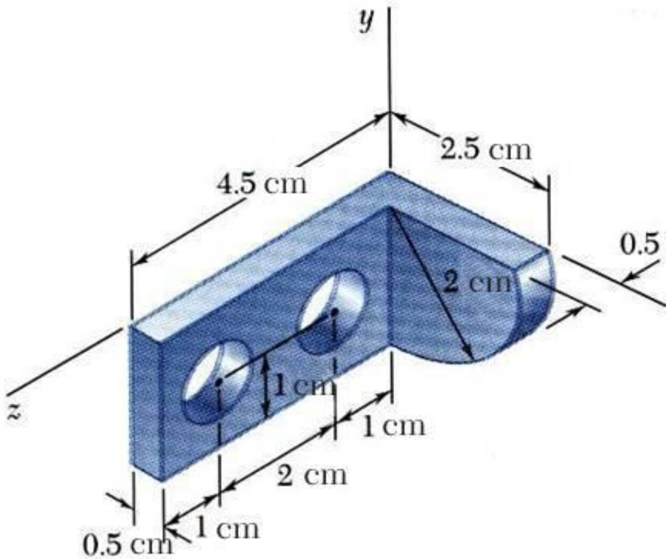




	V, cm³	$\bar{x}$, cm	$\bar{y}$, cm	$\bar{z}$, cm	$\bar{x}V$, cm⁴	$\bar{y}V$, cm⁴	$\bar{z}V$, cm⁴
I	$(4.5)(2)(0.5) = 4.5$	0.25	−1	2.25	1.125	−4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	−0.8488	0.25	2.119	−1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	3.5	−0.098	0.393	−1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	1.5	−0.098	0.393	−0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$

Sample Problem

	V, cm^3	$\bar{x}$, cm	$\bar{y}$, cm	$\bar{z}$, cm	$\bar{x}V$, cm^4	$\bar{y}V$, cm^4	$\bar{z}V$, cm^4
I	$(4.5)(2)(0.5) = 4.5$	0.25	−1	2.25	1.125	−4.5	10.125
II	$\frac{1}{4}\pi(2)^2(0.5) = 1.571$	1.3488	−0.8488	0.25	2.119	−1.333	0.393
III	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	3.5	−0.098	0.393	−1.374
IV	$-\pi(0.5)^2(0.5) = -0.3927$	0.25	−1	1.5	−0.098	0.393	−0.589
	$\Sigma V = 5.286$				$\Sigma\bar{x}V = 3.048$	$\Sigma\bar{y}V = -5.047$	$\Sigma\bar{z}V = 8.555$



$$\bar{X} = \sum \bar{x}V / \sum V = \left(3.08\ \text{cm}^4\right) / \left(5.286\ \text{cm}^3\right)$$

$$\boxed{\bar{X} = 0.577\ \text{cm}}$$

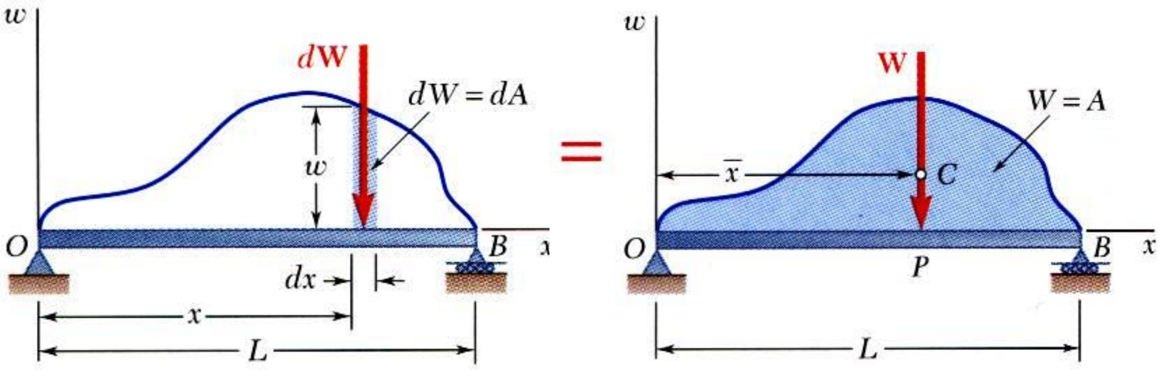
$$\bar{Y} = \sum \bar{y}V / \sum V = \left(-5.047\ \text{cm}^4\right) / \left(5.286\ \text{cm}^3\right)$$

$$\boxed{\bar{Y} = 0.577\ \text{cm}}$$

$$\bar{Z} = \sum \bar{z}V / \sum V = \left(1.618\ \text{cm}^4\right) / \left(5.286\ \text{cm}^3\right)$$

$$\boxed{\bar{Z} = 0.577\ \text{cm}}$$

Distributed Loads on Beams



$$W = \int_0^L w\,dx = \int dA = A$$

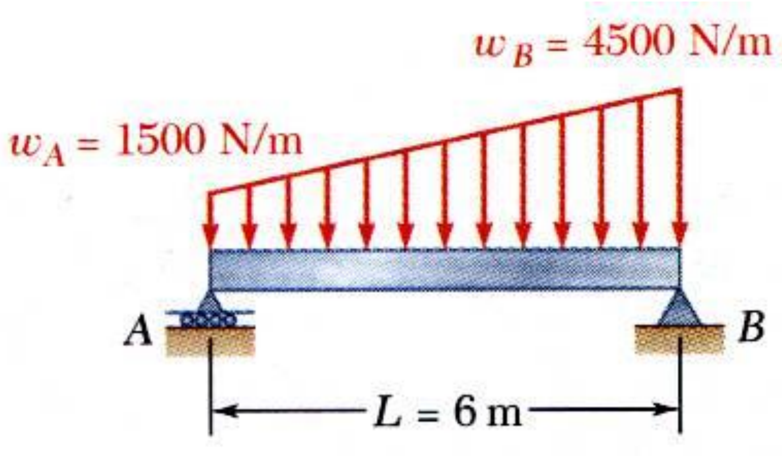
- A distributed load is represented by plotting the load per unit length, *w* (N/m) . The total load is equal to the area under the load curve.

$$(OP)W = \int x\,dW$$

$$(OP)A = \int_0^L x\,dA = \bar{x}A$$

- A distributed load can be replace by a concentrated load with a magnitude equal to the area under the load curve and a line of action passing through the area centroid.

Sample Problem

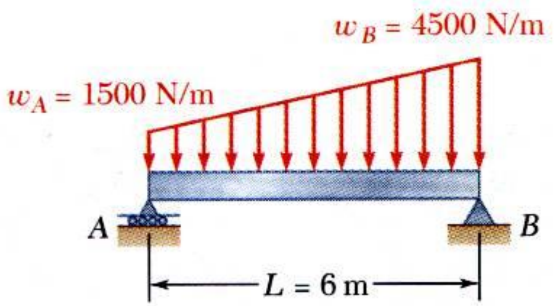


A beam supports a distributed load as shown. Determine the equivalent concentrated load and the reactions at the supports.

<u>SOLUTION</u>:

- The magnitude of the concentrated load is equal to the total load or the area under the curve.
- The line of action of the concentrated load passes through the centroid of the area under the curve.
- Determine the support reactions by summing moments about the beam ends.

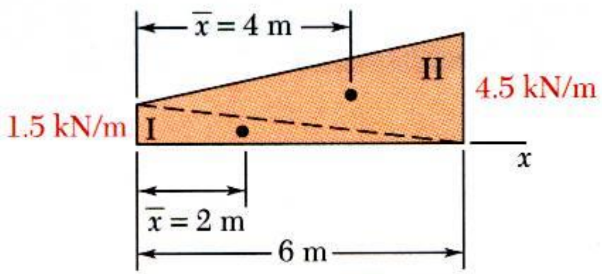
Sample Problem



SOLUTION:

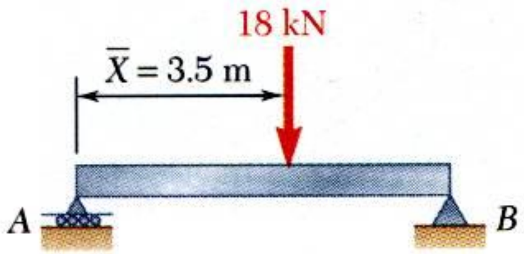
- The magnitude of the concentrated load is equal to the total load or the area under the curve.

$$\boxed{F = 18.0\ \text{kN}}$$



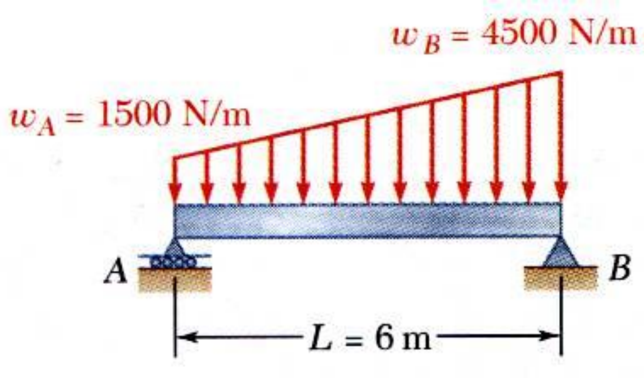
- The line of action of the concentrated load passes through the centroid of the area under the curve.

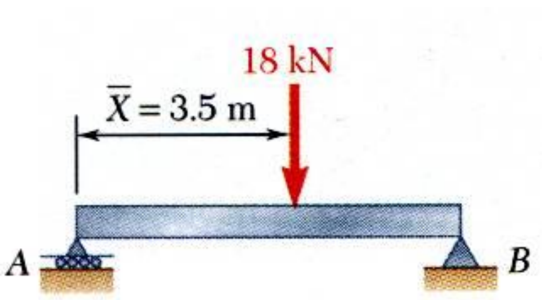
$$\bar{X} = \frac{63\ \text{kN}\cdot\text{m}}{18\ \text{kN}} \qquad \boxed{\bar{X} = 3.5\ \text{m}}$$



Component	A, kN	$\bar{x}$, m	$\bar{x}A$, kN·m
Triangle I	4.5	2	9
Triangle II	13.5	4	54
	$\Sigma A = 18.0$		$\Sigma \bar{x}A = 63$

Sample Problem





- Determine the support reactions by summing moments about the beam ends.

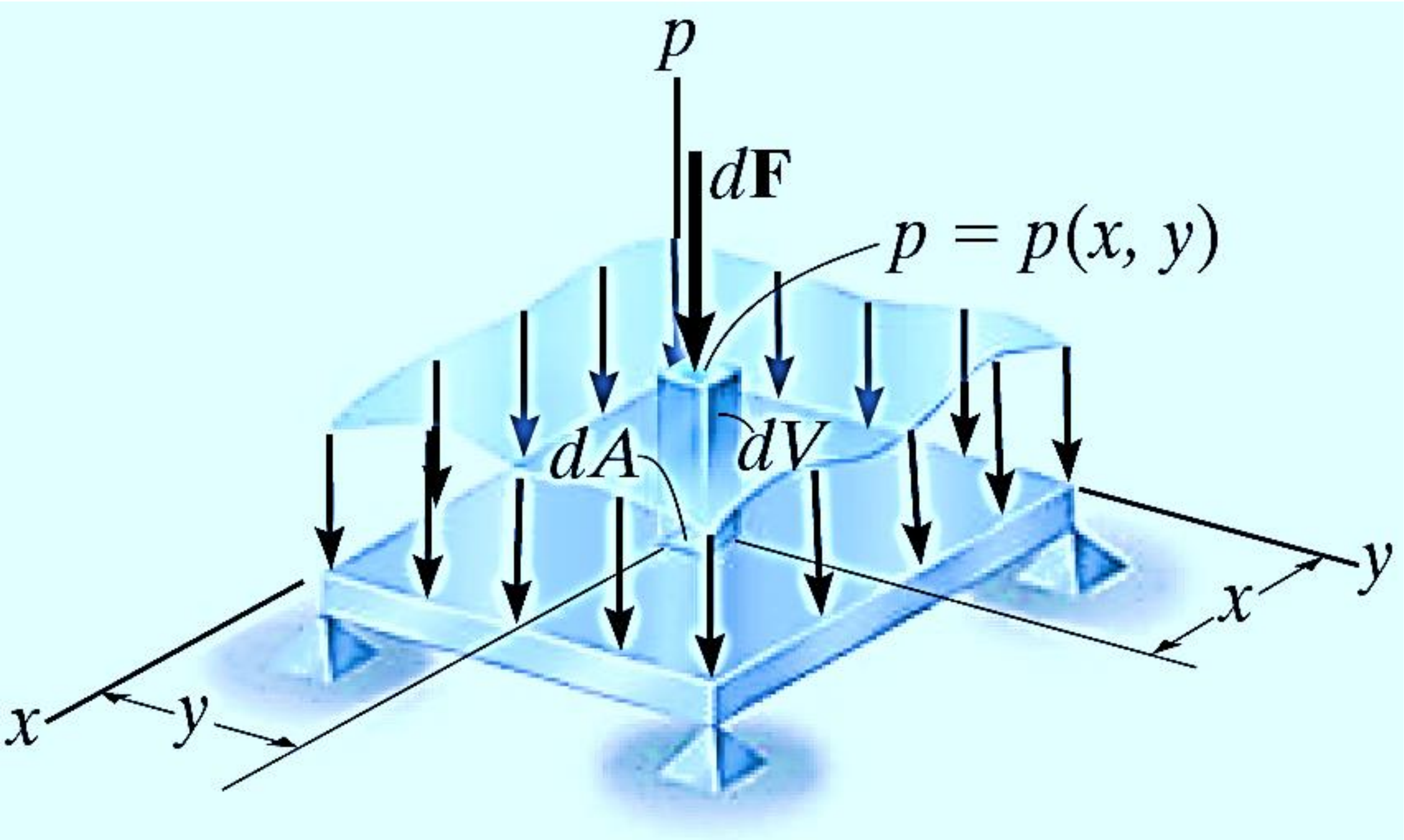
$$\sum M_A = 0: \quad B_y(6\,\mathrm{m}) - (18\,\mathrm{kN})(3.5\,\mathrm{m}) = 0$$

$$\boxed{B_y = 10.5\,\mathrm{kN}}$$

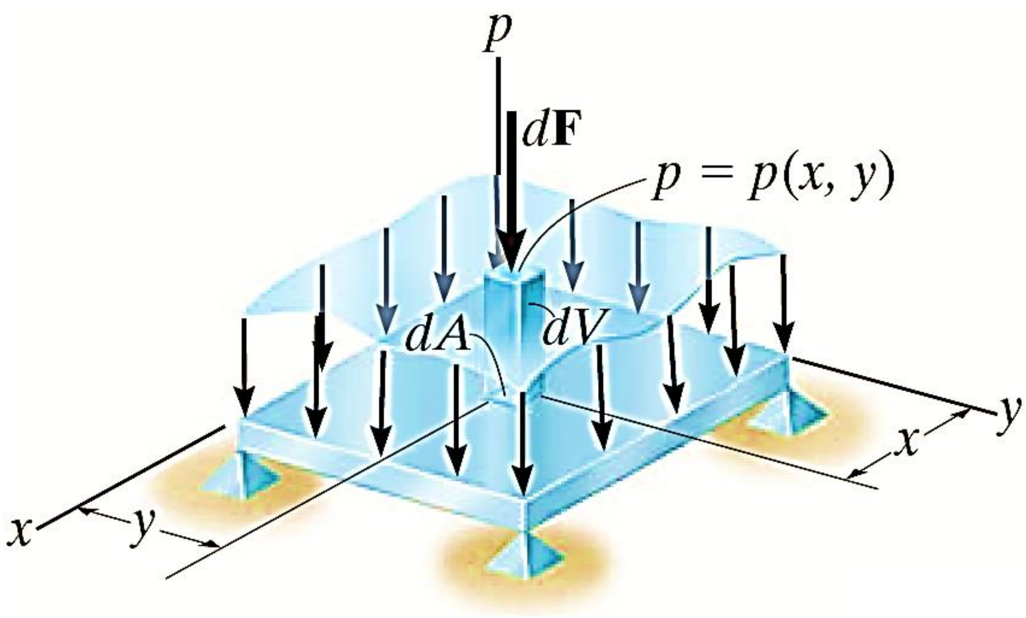
$$\sum M_B = 0: \quad -A_y(6\,\mathrm{m}) + (18\,\mathrm{kN})(6\,\mathrm{m} - 3.5\,\mathrm{m}) = 0$$

$$\boxed{A_y = 7.5\,\mathrm{kN}}$$

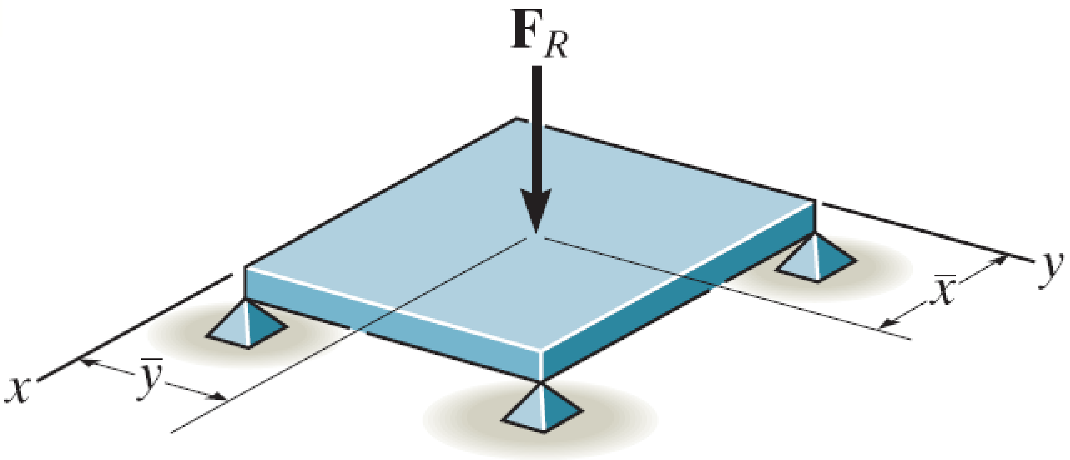
Distributed forces: Pressure load on a flat plate



Distributed forces: Pressure load on a flat plate

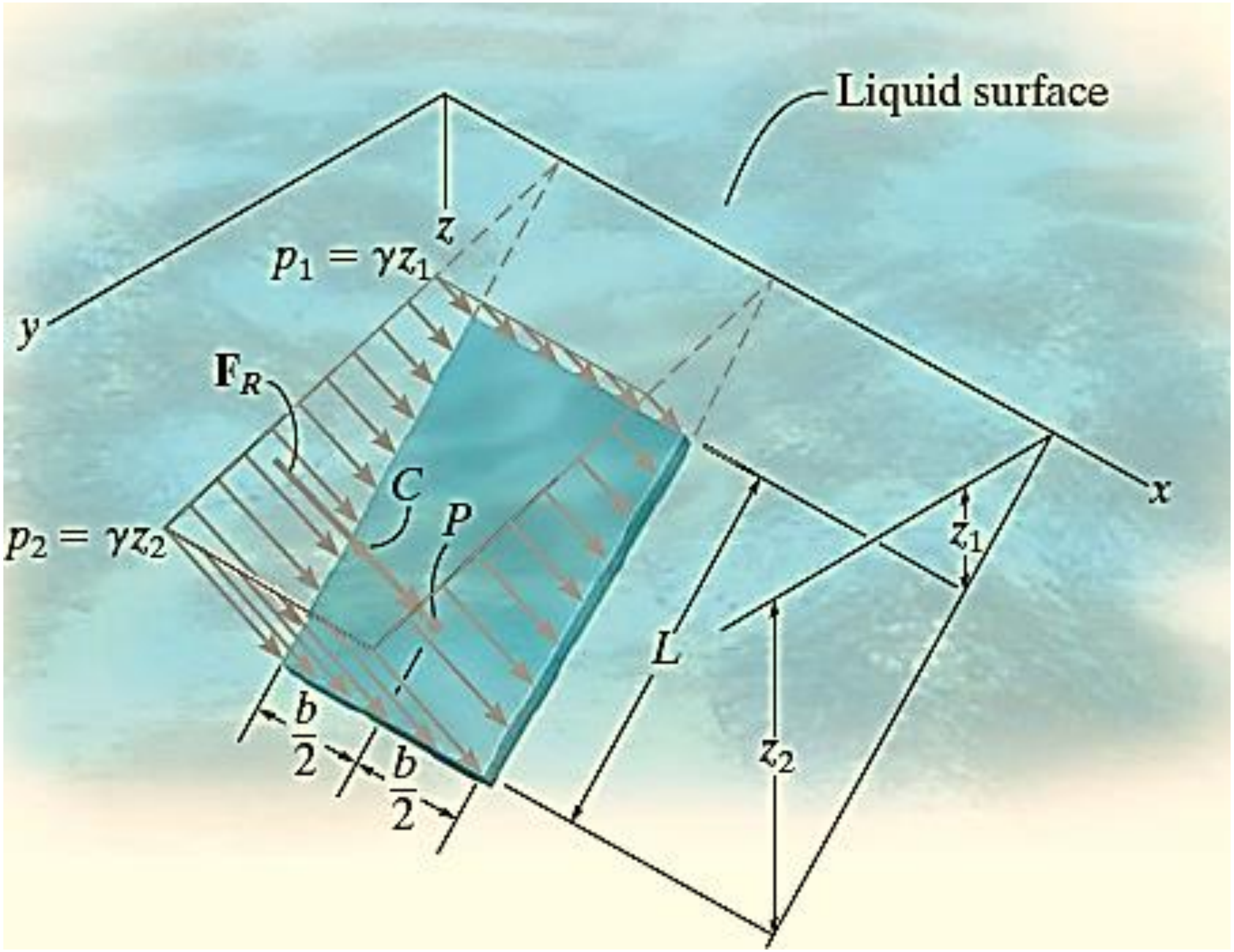


$$F_R = \int_A p(x, y)\, dA$$

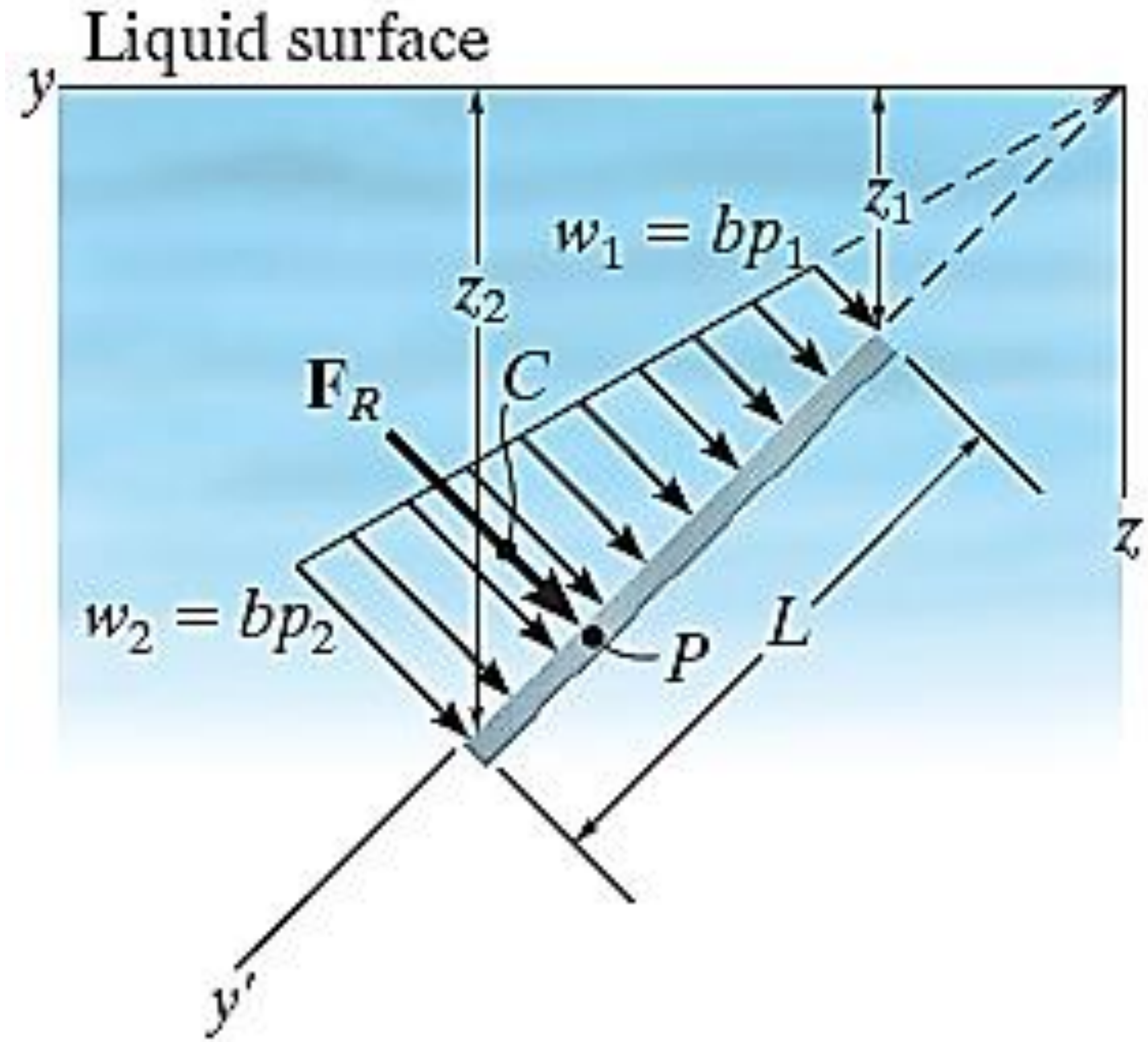


$$\bar{x} = \frac{\int_A x p(x, y)\, dA}{\int_A p(x, y)\, dA} \qquad \bar{y} = \frac{\int_A y p(x, y)\, dA}{\int_A p(x, y)\, dA}$$

Distributed forces: Hydrostatic forces

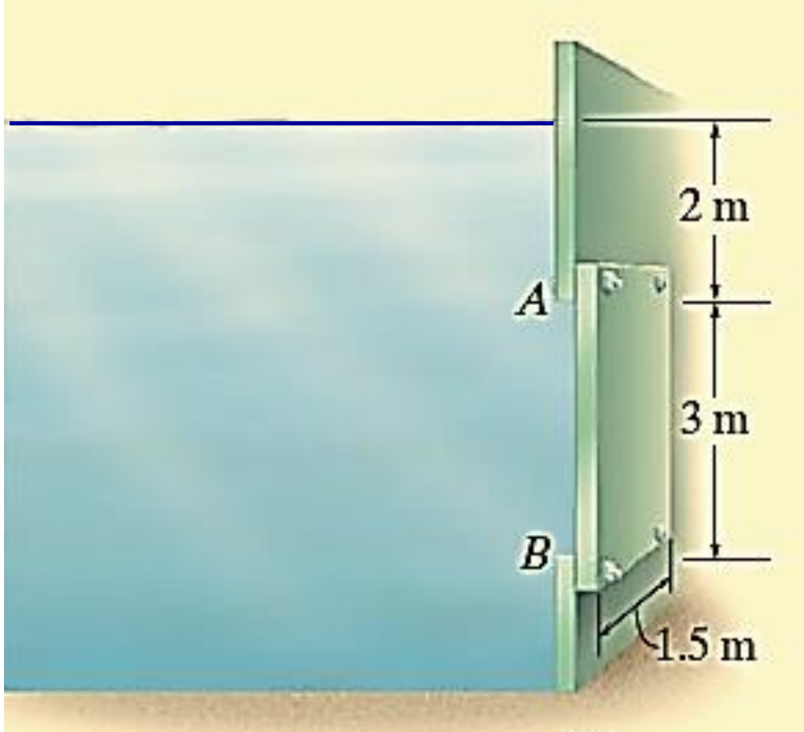


Hydrostatics of submerged bodies



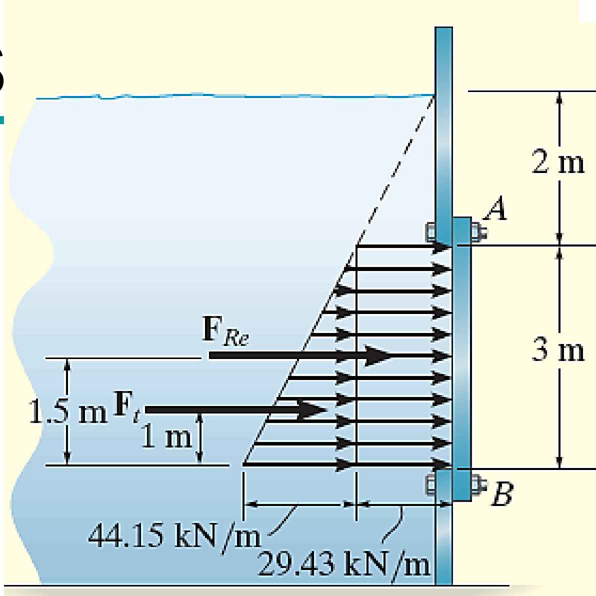
Sample problem: Hydrostatics

Determine the magnitude and location of the resultant hydrostatic force acting on the submerged rectangular plate AB shown in Fig. a. The plate has a width of 1.5 m; $\rho_w = 1000\ \mathrm{kg/m^3}$.



(a)

Sample problem: Hydrostatics



The water pressures at depths A and B are

$$p_A = \rho_w g z_A = (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(2\ \text{m}) = 19.62\ \text{kPa}$$

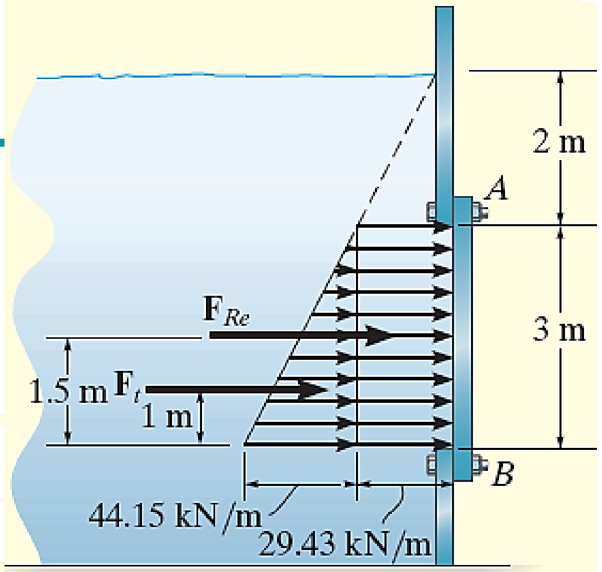
$$p_B = \rho_w g z_B = (1000\ \text{kg/m}^3)(9.81\ \text{m/s}^2)(5\ \text{m}) = 49.05\ \text{kPa}$$

Since the plate has a constant width, the pressure loading can be viewed in two dimensions as shown in Fig. The intensities of the load at A and B are

$$w_A = bp_A = (1.5\ \text{m})(19.62\ \text{kPa}) = 29.43\ \text{kN/m}$$

$$w_B = bp_B = (1.5\ \text{m})(49.05\ \text{kPa}) = 73.58\ \text{kN/m}$$

Sample problem: Hydrostatics



Consider two components of $\mathbf{F}_R$, defined by the triangle and rectangle shown in Fig. Each force acts through its associated centroid and has a magnitude of

$$F_{Re} = (29.43 \text{ kN/m})(3 \text{ m}) = 88.3 \text{ kN}$$

$$F_t = \tfrac{1}{2}(44.15 \text{ kN/m})(3 \text{ m}) = 66.2 \text{ kN}$$

Hence,

$$F_R = F_{Re} + F_t = 88.3 + 66.2 = 154.5 \text{ kN} \qquad \textit{Ans.}$$

The location of $\mathbf{F}_R$ is determined by summing moments about B,

$$\circlearrowleft +(M_R)_B = \Sigma M_B;\ (154.5)h = 88.3(1.5) + 66.2(1)$$

$$h = 1.29 \text{ m} \qquad \textit{Ans.}$$

Hydrostatic forces on curved surfaces