Department of Mathematics
Indian Institute of Technology Guwahati
MA 101: Mathematics I
Solutions of Tutorial Sheet-4
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1. Examine whether the following series are convergent.

   (a) $\sum\limits_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$

   Solution. Taking $x_n = \frac{n^n}{2^{n^2}}$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} |x_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{n}{2^n} = 0 < 1$ (since $\lim\limits_{n\to\infty} \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{1}{2} < 1$). Hence by the root test, the given series is convergent. $\square$

   (b) $\sum\limits_{n=1}^{\infty} (\frac{n}{n+1})^{n^2}$

   Solution. Taking $x_n = (\frac{n}{n+1})^{n^2}$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} |x_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$. Hence by the root test, the given series is convergent. $\square$

   (c) $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{\sqrt{n}+1}{n+1}$

   Solution. For $n \in \mathbb{N}$, the inequality $\frac{\sqrt{n+1}+1}{n+2} < \frac{\sqrt{n}+1}{n+1}$ is equivalent to the inequality $(n+1)^{\frac{3}{2}} < (n+2)\sqrt{n} + 1$. Since $n(n+2)^2 - (n+1)^3 = n^2 + n - 1 > 0$ for all $n \in \mathbb{N}$, we get $(n+1)^{\frac{3}{2}} < (n+2)\sqrt{n} + 1$ for all $n \in \mathbb{N}$ and hence $\frac{\sqrt{n+1}+1}{n+2} < \frac{\sqrt{n}+1}{n+1}$ for all $n \in \mathbb{N}$. Consequently the sequence $\left(\frac{\sqrt{n}+1}{n+1}\right)$ is decreasing. Also, $\frac{\sqrt{n}+1}{n+1} = \frac{\frac{1}{\sqrt{n}}+\frac{1}{n}}{1+\frac{1}{n}} \to 0$. Hence by Leibniz's test, the given series converges. $\square$

2. Examine whether the series $\sum\limits_{n=2}^{\infty} \frac{1}{(\log n)^{\log n}}$ is convergent.

   Solution. We have $(\log n)^{\log n} = (e^{\log(\log n)})^{\log n} = (e^{\log n})^{\log(\log n)} = n^{\log(\log n)}$ for all $n \geq 2$. Also, $\log(\log n) > 2$ for all $n > e^{e^2}$. We choose $n_0 \in \mathbb{N}$ such that $n_0 > e^{e^2}$. Then $\frac{1}{(\log n)^{\log n}} = \frac{1}{n^{\log(\log n)}} \leq \frac{1}{n^2}$ for all $n \geq n_0$. Since $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent, by comparison test, the given series is convergent. $\square$

3. Examine whether the following series are conditionally convergent.

   (a) $\sum\limits_{n=1}^{\infty} (-1)^n(\sqrt{n^2+1} - n)$

   Solution. Let $x_n = \sqrt{n^2+1} - n$ for all $n \in \mathbb{N}$. Then $x_n > 0$ for all $n \in \mathbb{N}$ and $x_n = \frac{1}{\sqrt{n^2+1}+n} = \frac{\frac{1}{n}}{\sqrt{1+\frac{1}{n^2}}+1} \to 0$. Also, $x_{n+1} = \frac{1}{\sqrt{(n+1)^2+1}+(n+1)} < \frac{1}{\sqrt{n^2+1}+n} = x_n$ for all $n \in \mathbb{N}$. That is, the sequence (x_n) is decreasing. Therefore by Leibniz's test, $\sum\limits_{n=1}^{\infty} (-1)^{n+1} x_n$ is convergent and hence the given series is convergent.

Again, if $y_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n^2}}+1} = \frac{1}{2} \neq 0$. Since $\sum_{n=1}^{\infty} y_n$ is not convergent, by limit comparison test, $\sum_{n=1}^{\infty} x_n$ is not convergent, that is, $\sum_{n=1}^{\infty} |(-1)^n(\sqrt{n^2+1}-n)|$ is not convergent. Thus the given series is conditionally convergent. □

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2+(-1)^n}$

Solution. By comparison test, the series $\sum_{n=2}^{\infty} \left|\frac{(-1)^n}{n^2+(-1)^n}\right| = \sum_{n=2}^{\infty} \frac{1}{n^2+(-1)^n}$ is convergent, since $0 < \frac{1}{n^2+(-1)^n} < \frac{2}{n^2}$ for all $n \geq 2$ and $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is convergent. Thus the given series is absolutely convergent, and hence the series is not conditionally convergent. □

(c) $\sum_{n=1}^{\infty} (-1)^n \frac{a^2+n}{n^2}$, where $a \in \mathbb{R}$

Solution. Let $a \in \mathbb{R}$ and let $x_n = \frac{a^2+n}{n^2}$ for all $n \in \mathbb{N}$. Then $x_n > 0$ for all $n \in \mathbb{N}$ and $x_n = \frac{a^2}{n^2} + \frac{1}{n} \to 0$. Also, $x_{n+1} = \frac{a^2}{(n+1)^2} + \frac{1}{n+1} < \frac{a^2}{n^2} + \frac{1}{n} = x_n$ for all $n \in \mathbb{N}$, that is, the sequence (x_n) is decreasing. Therefore by Leibniz's test, it follows that the given series is convergent.
Again, if $y_n = \frac{1}{n}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} (\frac{a^2}{n} + 1) = 1 \neq 0$. Since $\sum_{n=1}^{\infty} y_n$ is not convergent, by limit comparison test, $\sum_{n=1}^{\infty} x_n$ is not convergent, that is, $\sum_{n=1}^{\infty} |(-1)^n \frac{a^2+n}{n^2}|$ is not convergent. Thus the given series is conditionally convergent. □

4. Find all $x \in \mathbb{R}$ for which the series $\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{2^n n^2}$ converges.

Solution. If $x = 1$, then the given series becomes $0 + 0 + \cdots$, which is clearly convergent. Let $x(\neq 1) \in \mathbb{R}$ and let $a_n = \frac{(-1)^n(x-1)^n}{2^n n^2}$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{2}|x-1|$. Hence by ratio test, $\sum_{n=1}^{\infty} a_n$ converges (absolutely) if $\frac{1}{2}|x-1| < 1$, that is, if $x \in (-1, 3)$ and does not converge if $\frac{1}{2}|x-1| > 1$, that is, if $x \in (-\infty, -1) \cup (3, \infty)$. If $\frac{1}{2}|x-1| = 1$, that is, if $x \in \{-1, 3\}$, then $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges and hence $\sum_{n=1}^{\infty} a_n$ converges. Therefore the set of $x \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} a_n$ converges is $[-1, 3]$. □

5. Show that the series $\sum_{n=1}^{\infty} \frac{a^n}{a^n+n}$ is convergent if $0 < a < 1$ and is not convergent if $a > 1$.

Solution. If $0 < a < 1$, then $0 < \frac{a^n}{a^n+n} < a^n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} a^n$ is convergent. Hence by comparison test, $\sum_{n=1}^{\infty} \frac{a^n}{a^n+n}$ is convergent if $0 < a < 1$. Again, if $a > 1$, then

$\frac{a^n}{a^n+n} = \frac{1}{1+\frac{n}{a^n}} \to 1 \neq 0$ and hence $\sum\limits_{n=1}^{\infty} \frac{a^n}{a^n+n}$ is not convergent if $a > 1$. (We have used that $\lim\limits_{n\to\infty} \frac{n}{a^n} = 0$, which follows from the fact that $\lim\limits_{n\to\infty} \frac{n+1}{a^{n+1}} \cdot \frac{a^n}{n} = \frac{1}{a} < 1$.) $\square$

6. If $\alpha\ (\neq 0) \in \mathbb{R}$, then show that the series $\sum\limits_{n=1}^{\infty} (-1)^n \sin(\frac{\alpha}{n})$ is conditionally convergent.

Solution. We choose $n_0 \in \mathbb{N}$ such that $\frac{|\alpha|}{n_0} < \frac{\pi}{2}$. Then for all $n \geq n_0$, $\sin(\frac{\alpha}{n})$ has the same sign as that of α. Since the sine function is increasing in $(0, \frac{\pi}{2})$, it follows that the sequence $\left(\sin(\frac{|\alpha|}{n})\right)_{n=n_0}^{\infty}$ is decreasing. Also, $\lim\limits_{n\to\infty} \sin(\frac{|\alpha|}{n}) = 0$. Hence by Leibniz's test, $\sum\limits_{n=n_0}^{\infty} (-1)^n \sin(\frac{\alpha}{n})$ is convergent. Consequently $\sum\limits_{n=1}^{\infty} (-1)^n \sin(\frac{\alpha}{n})$ is convergent. Again, $\sum\limits_{n=1}^{\infty} |(-1)^n \sin(\frac{\alpha}{n})| = \sum\limits_{n=1}^{\infty} |\sin(\frac{\alpha}{n})|$ is not convergent by limit comparison test, since (using $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$) $\lim\limits_{n\to\infty} \frac{|\sin(\alpha/n)|}{1/n} = |\alpha| \lim\limits_{n\to\infty} \left|\frac{\sin(\alpha/n)}{\alpha/n}\right| = |\alpha| \neq 0$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is not convergent. Therefore the given series is conditionally convergent. $\square$

7. For $p \in \mathbb{R}$, the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^p}$ is convergent iff $p > 0$.

Solution. For $p \leq 0$, $|(-1)^{n+1} \frac{1}{n^p}| = \frac{1}{n^p} \not\to 0$ and so $(-1)^{n+1} \frac{1}{n^p} \not\to 0$. Hence the given series is not convergent if $p \leq 0$. If $p > 0$, then $(\frac{1}{n^p})$ is a decreasing sequence of positive real numbers with $\frac{1}{n^p} \to 0$ and hence the given series converges by Leibniz's test. $\square$

8. (Rearrangement of series). If $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = s$, then prove that $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots = \frac{3}{2}s$.

Solution. We first note that by Leibniz's test, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges. Let

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = s. \tag{1}$$

Then the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots = \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \cdots)$ converges to $\frac{1}{2}s$. It follows that the series

$$0 + \frac{1}{2} - 0 - \frac{1}{4} + 0 + \frac{1}{6} - 0 - \frac{1}{8} + \cdots = \frac{1}{2}s \tag{2}$$

Hence the series $(1+0) + (-\frac{1}{2} + \frac{1}{2}) + (\frac{1}{3} - 0) + (-\frac{1}{4} - \frac{1}{4}) + (\frac{1}{5} + 0) + \cdots$, which is the sum of the series (1) and (2), converges to $s + \frac{1}{2}s = \frac{3}{2}s$. Therefore it follows that $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots = \frac{3}{2}s$. $\square$