Department of Mathematics
Indian Institute of Technology Guwahati

MA 101: Mathematics I

The Real Number System

July-December 2019

We denote the set of natural numbers by $\mathbb{N}$, the set of integers by $\mathbb{Z}$, and the set of rational numbers by $\mathbb{Q}$, and we assume familiarity with each of these sets:

$$\mathbb{N} = \{1, 2, 3, \ldots\}, \quad \mathbb{Z} = \{0, \pm 1, \pm 2, \pm 3, \ldots\}, \quad \mathbb{Q} = \{m/n : m \in \mathbb{Z}, n \in \mathbb{N}\}.$$

We now mention a fundamental property of the set of natural numbers. The **Well-Ordering Property of** $\mathbb{N}$ states that every nonempty subset of $\mathbb{N}$ has a least element. To be precise, given a nonempty subset S of $\mathbb{N}$, there exists $m \in S$ such that $m \leq k$ for all $k \in S$. The element m is the least element of S.

The set of real numbers, denoted by $\mathbb{R}$, is best described more geometrically by setting up a one-to-one correspondence with points of a line that stretches infinitely in both directions. A more formal and technically sound definition requires more background; we postpone it for later. Instead, assuming this geometrical correspondence, we list three sets of axioms that the set of real numbers follow:

- FIELD AXIOMS
  1. (Associative laws) $x + (y + z) = (x + y) + z$ and $x(yz) = (xy)z$ for all $x, y, z \in \mathbb{R}$
  2. (Commutative laws) $x + y = y + x$ and $xy = yx$ for all $x, y \in \mathbb{R}$
  3. (Identities) $x + 0 = x = 0 + x$ and $x \cdot 1 = x = 1 \cdot x$ for all $x \in \mathbb{R}$
  4. (Inverses) $x+(-x) = 0 = (-x)+x$ for all $x \in \mathbb{R}$ and $x \cdot \frac{1}{x} = 1 = \frac{1}{x} \cdot x$ for all $x \in \mathbb{R} \setminus \{0\}$
  5. (Distributive laws) $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$ for all $x, y, z \in \mathbb{R}$

- ORDER AXIOMS
  1. For each $x, y \in \mathbb{R}$, exactly one of $x > y,\ x = y,\ x < y$ holds
  2. If $x \geq y$, then $x + z \geq y + z$ for all $z \in \mathbb{R}$
  3. If $x \geq y$ and $z \geq 0$, then $xz \geq yz$

From the *Order Axioms* one can derive the usual inequalities satisfies by the set of real numbers. One of the most important properties dealing with inequalities is the following:

Property 1. *If $x + \varepsilon \geq y$ holds for all $\varepsilon > 0$, then $x \geq y$ also holds.*

Proof. To see why this must be true, assume otherwise. Then $y - x > 0$. So with $\varepsilon = \frac{1}{2}(y-x) > 0$ the hypothesis implies $\frac{1}{2}(x+y) = x + \frac{1}{2}(y-x) \geq y$. Thus $y - x \leq 0$, which is a contradiction. $\square$

- ABSOLUTE VALUE: The absolute value $|a|$ of a real number a is defined as $\max\{a, -a\}$. In particular, $|a| = |-a| \geq 0$ for all $a \in \mathbb{R}$, with equality if and only if $a = 0$. Two of the most significant properties satisfied by the absolute value function $|\cdot|$ are: (i) $|ab| = |a||b|$ for each pair $a, b \in \mathbb{R}$, and (ii) (Triangle Inequality) $|a + b| \leq |a| + |b|$ for each pair $a, b \in \mathbb{R}$. Geometrically, $|a-b|$ denotes the distance between the real numbers a and b. In particular, $|a|$ denotes the distance of the real number a from the origin. Viewed thus, the inequality $|x-a| < \varepsilon$ easily translates to $a - \varepsilon < x < a + \varepsilon$ and the inequality $|x-a| > \varepsilon$ translates to $x > a + \varepsilon$ or $x < a - \varepsilon$.

- SUPREMA AND INFIMA: We now introduce the notion of upper bound and lower bound for a set of real numbers. These ideas will be of utmost importance in later sections.

Definition 1. *Let S be a nonempty subset of $\mathbb{R}$.*

(a) *A real number u is called an* ***upper bound*** *of S if $a \leq u$ for each $a \in S$. The set S is said to be* ***bounded above*** *if it has an upper bound.*

(b) *A real number v is called a* ***lower bound*** *of S if $a \geq v$ for each $a \in S$. The set S is said to be* ***bounded below*** *if it has a lower bound.*

(c) *A set is said to be* ***bounded*** *if it is both bounded above and bounded below. A set is said to be* ***unbounded*** *if it is not bounded.*

Remark 1. *Let S be a nonempty subset of $\mathbb{R}$.*

(a) *If S is bounded, there must be real numbers u and v such that $v \leq a \leq u$ for each $a \in S$. It is often more useful to use the equivalent fact that S is bounded precisely when there is a real number M for which $|a| \leq M$ holds for every $a \in S$.*

(b) *If S has an upper bound, then it has infinitely many upper bounds, because if u is an upper bound of S, then the numbers $u+1, u+2, \ldots$ are also upper bounds of S. A similar observation is valid for lower bounds.*

Example 1. The set $S = \{x \in \mathbb{R} : x < 2\}$ is bounded above; the number 2 and any number larger than 2 is an upper bound. The set has no lower bound, so that the set is not bounded below. Thus the set is unbounded (even though it is bounded above). The set of positive integers is bounded below, but is not bounded above. The set of integers is neither bounded below nor bounded above. The set $\{1/n : n \in \mathbb{N}\}$ is bounded below by 0 and bounded above by 1, so it is a bounded set.

Definition 2 (supremum or least upper bound)**.** *If S is bounded above, then a number u is said to be a* ***supremum*** *(or a* ***least upper bound****) of S if it satisfies the following conditions:*

(a) *u is an upper bound of S*

(b) *if v is any upper bound of S, then $u \leq v$.*

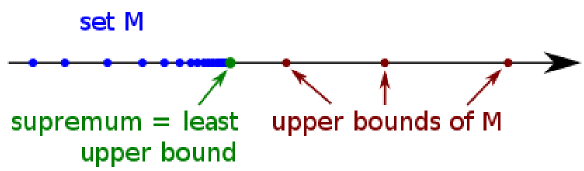
Definition 3 (infimum or greatest lower bound)**.** *If S is bounded below, then a number w is said to be an* ***infimum*** *(or a* ***greatest lower bound****) of S if it satisfies the following conditions:*

(a) *w is a lower bound of S*

(b) *if t is any lower bound of S, then $t \leq w$.*

Observe that whereas a set S that has an upper bound has infinitely many, it can have at most one least upper bound. For if u_1 and u_2 are both least upper bounds for a set S, then both $u_1 \leq u_2$ and $u_2 \leq u_1$ hold, from which it follows that $u_1 = u_2$. A similar argument can be given to show that the infimum of a set is uniquely determined. If the supremum or the infimum of a set S exists, we will denote them by

$$\sup(S) \quad \text{and} \quad \inf(S).$$



A set S has a maximum when there exists $M \in S$ such that $a \leq M$ for all $a \in S$. Observe that every nonempty set S can have at most one maximum, and that the maximum (if it exists) is also the supremum of S. On the other hand, if the supremum of S exists and $\sup(S) \in S$, then $\sup(S) = \max(S)$. Analogous observation to greatest lower bound and to minimum of a set.

- Completeness Axiom: It is not possible to prove on the basis of the field and order properties of $\mathbb{R}$ that every nonempty subset of $\mathbb{R}$ that is bounded above has a supremum in $\mathbb{R}$. However, it is a deep and fundamental property of the real number system that this is indeed the case. The **completeness property of** $\mathbb{R}$ states that every nonempty set of real numbers that has an upper bound also has a supremum in $\mathbb{R}$; and that every nonempty set of real numbers that has a lower bound also has an infimum in $\mathbb{R}$.

  Lemma 1. *Let A be a nonempty set of real numbers, and suppose sup(A) exists. Then for every $\varepsilon > 0$, there exists $a \in A$ such that sup$(A) - \varepsilon < a \leq$ sup(A).*

  Proof. The inequality $x \leq \sup(A)$ holds for every $x \in A$. Fix $\varepsilon > 0$. If there was no $a \in A$ for which $\sup(A) - \varepsilon < a$, then $x \leq \sup(A) - \varepsilon$ for every $x \in A$. But then $\sup(A) - \varepsilon$ would be an upper bound for A, and $\sup(A) - \varepsilon < \sup(A)$, which contradicts the definition of the supremum. □

  We have the following analogous lemma for infimum.

  Lemma 2. *Let A be a nonempty set of real numbers, and suppose inf(A) exists. Then for every $\varepsilon > 0$, there exists $a \in A$ such that inf$(A) \leq a <$ inf$(A) + \varepsilon$.*

- Archimedean Property: If $n \in \mathbb{N}$, then $n + 1 \in \mathbb{N}$. Hence $\mathbb{N}$ is not bounded from above in $\mathbb{N}$. It is also true that the set $\mathbb{N}$ is not bounded from above in $\mathbb{R}$. To see this, by way of contradiction assume that there exists $a \in \mathbb{R}$ such that $n \leq a$ for every $n \in \mathbb{N}$. Then by the Completeness Axiom, there exists $s = \sup(\mathbb{N})$, and by Lemma 1, there is $k \in \mathbb{N}$ with $s - 1 < k$. This implies $s < k + 1$, which is impossible since $1 + k \in \mathbb{N}$. An equivalent version of this is the following property:

  Property 2 (Archimedean property). *If x and y are positive real numbers, then there exists a positive integer n for which $nx > y$.*

Example 2. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\}$. We note that $0 < \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$. Thus, 1 is an upper bound and since $1 \in S$ we have $\sup(S) = \max(S) = 1$. Clearly, 0 is a lower bound of S. We will now use Archimedean property to prove that 0 is the greatest lower bound. To prove this, let $u > 0$. Using Archimedean property we can find an n such that $nu > 1$, that is, $u > \frac{1}{n}$. Hence, u is not a lower bound of S. Thus, any positive real number is not a lower bound of S. This proves that $\inf(S) = 0$.

- Density of rational numbers in $\mathbb{R}$: The set of rational numbers is "dense" in $\mathbb{R}$ in the sense that given any two real numbers there is a rational number between them (in fact, there are infinitely many such rational numbers).

Theorem 1 (The Density Theorem). *If x and y are any real numbers with $x < y$, then there exists a rational number $r \in \mathbb{Q}$ such that $x < r < y$.*

Proof. If $xy < 0$, then 0 is lying between them. So, we first assume that $0 < x < y$. By using the Archimedean property, we can find an n_0 in $\mathbb{N}$ such that $n_0(y - x) > 1$. Now consider the set

$$S = \{m \in \mathbb{N} : \frac{m}{n_0} > x\}.$$

Then S is nonempty (by Archimedean property) subset of $\mathbb{N}$. Using the Well-ordering principle of $\mathbb{N}$, S has a least element say m_0. Since $m_0 \in S$, so $x < \frac{m_0}{n_0}$. By minimality of m_0 we have $m_0 - 1 \notin S$, and hence $\frac{m_0-1}{n_0} \leq x$. Now

$$x < \frac{m_0}{n_0} \leq x + \frac{1}{n_0} < x + (y - x) = y.$$

Therefore,

$$x < \frac{m_0}{n_0} < y.$$

$\square$