DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
Odd Semester of the Academic Year 2019-2020
MA 101 Mathematics II
Problem Sheet 2: Partial derivatives, tangent and normals, differentials, gradient, directional derivatives and chain rules etc.
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1. Let $f(x,y) = \begin{cases} \frac{x^2-xy}{x+y} & \text{if} \quad x+y \neq 0 \\ 0 & \text{if} \quad x+y=0. \end{cases}$

   Find

   (a) $f_x(0,0)$, $f_y(0,0)$

   (b) $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$, and check whether it is equal to $f_x(0,0)$.

   Solution: Note that $f_x(0,0)$ if it exists, is given by:
   $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{h-0}{h} = 1.$

   Similarly $f_y(0,0)$ if it exists, is given by:
   $f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k)-f(0,0)}{k} = \lim\limits_{k\to 0} \dfrac{0}{k} = 0.$

   (b) Also $f_x(x,y)$ (for $x+y \neq 0$) if it exists, is given by:
   $f_x(x,y) = \lim\limits_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h}.$
   Since (x,y) is such that $x+y \neq 0$, if h sufficiently small such that $x+y+h \neq 0$, then
   $$\frac{f(x+h,y)-f(x,y)}{h} = \frac{\frac{(x+h)^2-(x+h)y}{(x+h)+y} - \frac{x^2-xy}{x+y}}{h} = \frac{hx^2+2hxy-hy^2+h^2x+h^2y}{h(x+y)(x+h+y)}$$
   $$= \frac{x^2+2xy-y^2+hx+hy}{(x+y)(x+h+y)}.$$

   Since $\lim\limits_{h\to 0} x^2+2xy-y^2+hx+hy = x^2+2xy-y^2$ and
   $\lim\limits_{h\to 0}(x+y)(x+h+y) = (x+y)^2 \neq 0$ for $x+y\neq 0$,
   $f_x(x,y) = \lim\limits_{h\to 0} \dfrac{f(x+h,y)-f(x,y)}{h} = \dfrac{x^2+2xy-y^2}{(x+y)^2} = 1 - \dfrac{2y^2}{(x+y)^2},$
   for $x+y\neq 0$.
   Clearly $\lim\limits_{(x,y)\to(0,0)} f_x(x,y) = \lim\limits_{(x,y)\to(0,0)} 1 - \dfrac{2y^2}{(x+y)^2},$
   does not exist (consider $(x,y)\to(0,0)$ such that $y=mx$).

2. Let $f(x,y) = \sqrt{x^2+y^2}$.

   (a) Find $f_x(x,y)$ and $f_y(x,y)$ for $(x,y)\neq(0,0)$

(b) Show that $f_x(0,0)$ and $f_y(0,0)$ do not exist.

Solution: (a) Note that $f_x(x,y)$ if it exists, is given by

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{f(x+h,y)-f(x,y)}{h} = \frac{\sqrt{(x+h)^2+y^2}-\sqrt{x^2+y^2}}{h}$$

$$= \frac{(\sqrt{(x+h)^2+y^2}-\sqrt{x^2+y^2})(\sqrt{(x+h)^2+y^2}+\sqrt{x^2+y^2})}{h(\sqrt{(x+h)^2+y^2}+\sqrt{x^2+y^2})} = \frac{h+2x}{\sqrt{(x+h)^2+y^2}+\sqrt{x^2+y^2}},$$

for $(x,y)\neq(0,0)$.

Hence $f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h} = \frac{x}{\sqrt{x^2+y^2}}$, for $(x,y)\neq(0,0)$.

Similarly $f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k} = \frac{y}{\sqrt{x^2+y^2}}$, for $(x,y)\neq(0,0)$.

Aliter: $f_x(x,y)$ for $(x,y)\neq(0,0)$ may be obtained directly (without going through first principles) by taking the single variable derivative of $f(x,y)$ with respect to x by treating y as constant.
Similarly $f_y(x,y)$ for $(x,y)\neq(0,0)$ may be obtained directly (without going through first principles) by taking the single variable derivative of $f(x,y)$ with respect to y by treating x as constant.

(b) Also $f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$, does not exist.
Similarly $f_y(0,0)$ does not exist.

3. Let

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0). \end{cases}$$

(a) Calculate $f_x(x,y)$ and $f_y(x,y)$ at all points where $(x,y)\neq(0,0)$.

(b) Compute all first and second order partial derivatives at $(0,0)$ if they exist.

(c) Show that f is discontinuous at $(0,0)$.

Solution: (a)

$$f_x(x,y) = \frac{y^3(y^6-x^2)}{(x^2+y^6)^2} \text{ and } f_y(x,y) = \frac{3xy^2(x^2-y^6)}{(x^2+y^6)^2}, \text{ for } (x,y)\neq(0,0). \qquad \mathbf{(1)}$$

$$f_x(x,0) = \lim_{h\to 0} \frac{f(x+h,0)-f(x,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0 \text{ for } x\neq 0.$$

$$f_y(0,y) = \lim_{k\to 0} \frac{f(0,y+k)-f(0,y)}{k} = \lim_{k\to 0} \frac{0}{k} = 0 \text{ for } y\neq 0.$$

$$f_y(x,0) = \lim_{k\to 0} \frac{f(x,k)-f(x,0)}{k} = \lim_{k\to 0} \frac{\frac{xk^3}{x^2+k^6}-0}{k} = \lim_{k\to 0} \frac{xk^2}{x^2+k^6} = 0 \text{ for } x\neq 0.$$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y)-f(0,y)}{h} = \lim_{h\to 0} \frac{\frac{hy^3}{h^2+y^6}-0}{h} = \frac{y^3}{y^6} = \frac{1}{y^3} \text{ for } y\neq 0.$$

Aliter: The above expressions for
$f_x(x,0) = 0$ for $x\neq 0$,

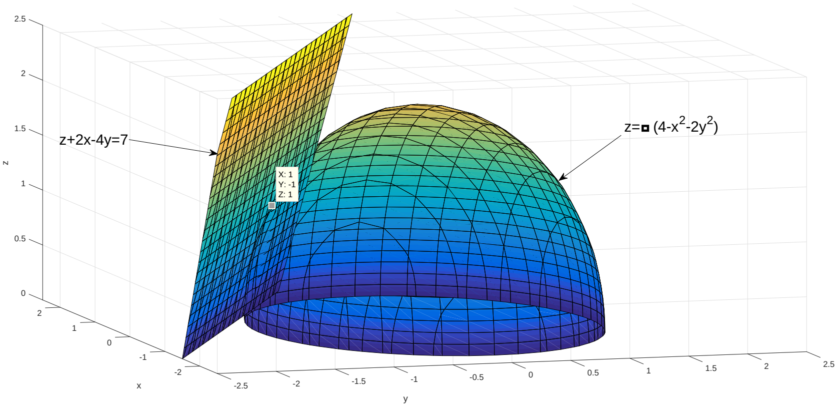
$f_y(0, y) = 0$ for $y \neq 0$,
$f_y(x, 0) = 0$ for $x \neq 0$,
$f_x(0, y) = \dfrac{1}{y^3}$ for $y \neq 0$,
could be obtained directly by suitable substitutions in **(1)**.

(b) $f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(h,0) - f(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0}{h} = 0.$
$f_y(0,0) = \lim\limits_{k \to 0} \dfrac{f(0,k) - f(0,0)}{k} = \lim\limits_{k \to 0} \dfrac{0}{k} = 0.$
Hence $f_{xx}(0,0) = \lim\limits_{h \to 0} \dfrac{f_x(h,0) - f_x(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0}{h} = 0.$
$f_{yy}(0,0) = \lim\limits_{k \to 0} \dfrac{f_y(0,k) - f_y(0,0)}{k} = \lim\limits_{k \to 0} \dfrac{0}{k} = 0.$
$f_{xy}(0,0) = \lim\limits_{k \to 0} \dfrac{f_x(0,k) - f_x(0,0)}{k} = \lim\limits_{k \to 0} \dfrac{\frac{1}{k^3} - 0}{k} = \lim\limits_{k \to 0} \dfrac{1}{k^4}$, does not exist.
$f_{yx}(0,0) = \lim\limits_{h \to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h} = \lim\limits_{h \to 0} \dfrac{0}{h} = 0.$

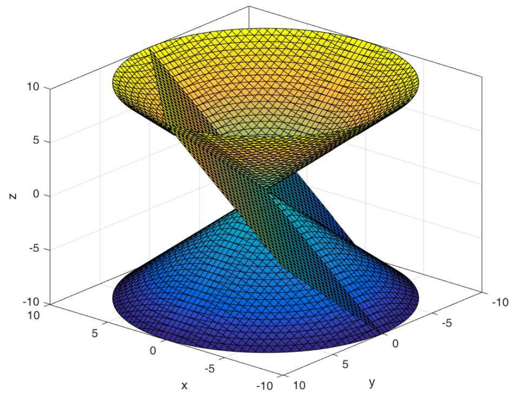
(c) We had already seen (tut-1, 11(d)) that $\lim\limits_{(x,y) \to (0,0)} f(x,y)$ does not exist, hence f is not continuous at $(0,0)$.

4. Find the equation of the tangent plane to the surface $z = \sqrt{4 - x^2 - 2y^2}$ at the point $(1, -1, 1)$.

Solution: $(z - z_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.
$\Rightarrow (z - z_0) = \left(\dfrac{-x_0}{\sqrt{4 - x_0^2 - 2y_0^2}} \right)(x - x_0) + \left(\dfrac{-2y_0}{\sqrt{4 - x_0^2 - 2y_0^2}} \right)(y - y_0)$
$\Rightarrow (z - 1) = (-1)(x - 1) + (2)(y + 1).$



5. It is geometrically evident that every plane tangent to the cone $z^2 = x^2 + y^2$ pass through the origin. Show this by the method of calculus.



Solution: Since the level surface is of the form $f(x, y, z) = 0$,
where $f(x, y, z) = z^2 - (x^2 + y^2)$,
the tangent plane of the level surface at (x_0, y_0, z_0) is of the form:
$(z - z_0)f_z(x_0, y_0, z_0) + (x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) = 0$.
$\Rightarrow (z - z_0)(2z_0) + (x - x_0)(-2x_0) + (y - y_0)(-2y_0) = 0$,
$\Rightarrow zz_0 - xx_0 - yy_0 = 0$ since $z_0^2 - x_0^2 - y_0^2 = 0$.
Clearly $(x, y, z) = (0, 0, 0)$ satisfies the above equation for all points (x_0, y_0, z_0) of the cone.

6. Find the equations of the tangent plane and normal line to the given surface at the specified point

   (a) $x^2 + y^2 - z^2 - 2xy + 4xz = 4$, $(1, 0, 1)$.

   (b) $z + 1 = xe^y \cos z$, $(1, 0, 0)$.

Solution:
(a) Since the level surface is of the form $f(x, y, z) = 4$,
where $f(x, y, z) = x^2 + y^2 - z^2 - 2xy + 4xz$,
the tangent plane of the level surface at $(x_0, y_0, z_0) = (1, 0, 1)$ is of the form:
$(z - z_0)f_z(x_0, y_0, z_0) + (x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) = 0$. **(1)**
where $f_x(x_0, y_0, z_0) = 2x_0 - 2y_0 + 4z_0$, $f_y(x_0, y_0, z_0) = 2y_0 - 2x_0$
and $f_z(x_0, y_0, z_0) = -2z_0 + 4x_0$.
(1) implies $\Rightarrow (z - 1)2 + (x - 1)6 + y(-2) = 0$ or $2z + 6x - 2y = 8$.
The symmetric equations of the normal line is given by:
$$\frac{(z - z_0)}{f_z(x_0, y_0, z_0)} = \frac{(x - x_0)}{f_x(x_0, y_0, z_0)} = \frac{(y - y_0)}{f_y(x_0, y_0, z_0)}, \quad \text{or}$$
$$\frac{(z - 1)}{2} = \frac{(x - 1)}{6} = \frac{(y - 0)}{-2}.$$

(b) Since the level surface is of the form $f(x, y, z) = -1$,
where $f(x, y, z) = z - xe^y \cos z$,
the tangent plane of the level surface at $(x_0, y_0, z_0) = (1, 0, 0)$ is of the form:
$(z - z_0)f_z(x_0, y_0, z_0) + (x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) = 0$, **(1)**
where $f_x(x_0, y_0, z_0) = -e^{y_0} \cos z_0$, $f_y(x_0, y_0, z_0) = -x_0e^{y_0} \cos z_0$

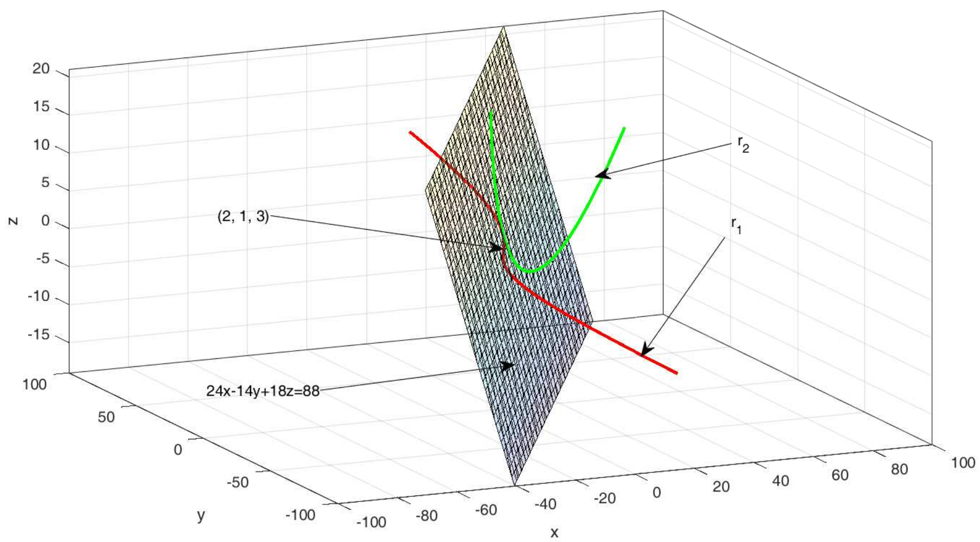
and $f_z(x_0, y_0, z_0) = 1 + x_0 e^{y_0} \sin z_0$.
(1) $\Rightarrow (z-0)1 + (x-1)(-1) + (y-0)(-1) = 0$ or $z - x - y = -1$.

The symmetric equations of the normal line is given by:
$$\frac{(z-z_0)}{f_z(x_0,y_0,z_0)} = \frac{(x-x_0)}{f_x(x_0,y_0,z_0)} = \frac{(y-y_0)}{f_y(x_0,y_0,z_0)}$$
$$\frac{(z-0)}{1} = \frac{(x-1)}{-1} = \frac{(y-0)}{-1}.$$

7. Suppose we are seeking the equation of the tangent plane to a surface S at the point $P = (2, 1, 3)$. You don't have an equation for S, but you know that the curves

$$\mathbf{r}_1(t) = \left\langle 2 + 3t, 1 - t^2, 3 - 4t + t^2 \right\rangle$$
$$\mathbf{r}_2(t) = \left\langle 1 + t^2, 2t^3 - 1, 2t + 1 \right\rangle$$

both lie on S. Find an equation of the tangent plane at P.
Solution: Clearly $\mathbf{r}_1(0)$ corresponds to $(2, 1, 3)$, $\mathbf{r}_2(1)$ corresponds to $(2, 1, 3)$.



Also the tangent plane to the surface S at $(2, 1, 3)$ contains the tangent lines to each of the curves $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$ at $(2, 1, 3)$.
The tangent vectors at $(2, 1, 3)$ for $\mathbf{r}_1, \mathbf{r}_2$ at $(2, 1, 3)$ are given by:

$$\mathbf{r}_1'(0) = \langle 3, 0, -4 \rangle,$$
$$\mathbf{r}_2'(1) = \langle 2, 6, 2 \rangle,$$

respectively.
Hence a normal to the surface is one which is orthogonal to both these tangent vectors and is given by:
$$\mathbf{r}_1'(0) \times \mathbf{r}_2'(1) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 0 & -4 \\ 2 & 6 & 2 \end{vmatrix} = \langle 24, -14, 18 \rangle.$$

Hence an equation of the tangent plane is given by:
$24(x-2)+(-14)(y-1)+18(z-3)=0.$

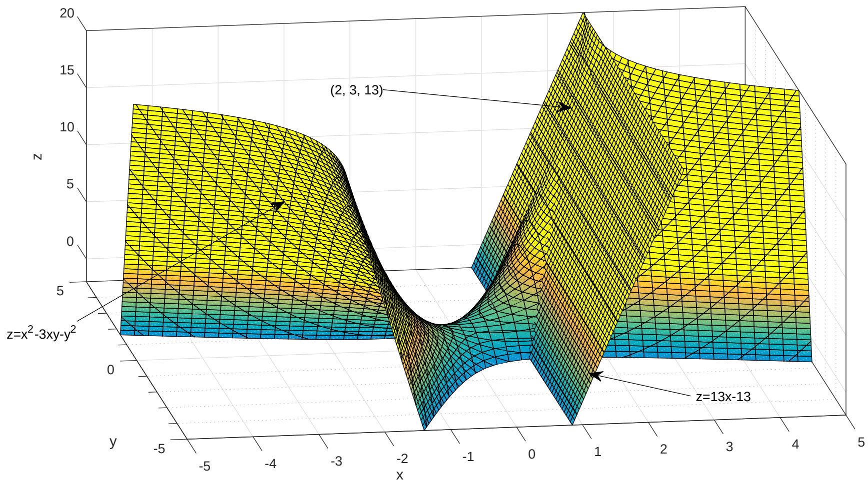
8. Show that the sum of the x-, y-, and z-intercepts of any tangent plane (at any point of the surface wherever it is defined) to the surface $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{c}$ is a constant.

Solution: The equation of the tangent plane at (x_0, y_0, z_0) is given by
$(z-z_0)f_z(x_0,y_0,z_0)+(x-x_0)f_x(x_0,y_0,z_0)+(y-y_0)f_y(x_0,y_0,z_0)=0.$
$\Rightarrow \dfrac{(z-z_0)}{2\sqrt{z_0}}+\dfrac{(x-x_0)}{2\sqrt{x_0}}+\dfrac{(y-y_0)}{2\sqrt{y_0}}=0,$
provided none of x_0, y_0, z_0 is equal to 0.
$\Rightarrow \dfrac{z}{\sqrt{z_0}}+\dfrac{x}{\sqrt{x_0}}+\dfrac{y}{\sqrt{y_0}}=(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})=\sqrt{c}.$
$\Rightarrow \dfrac{x}{\sqrt{cx_0}}+\dfrac{y}{\sqrt{cy_0}}+\dfrac{z}{\sqrt{cz_0}}=1.$
Hence the sum of the intercepts is
$(\sqrt{x_0}+\sqrt{y_0}+\sqrt{z_0})\sqrt{c}=c.$

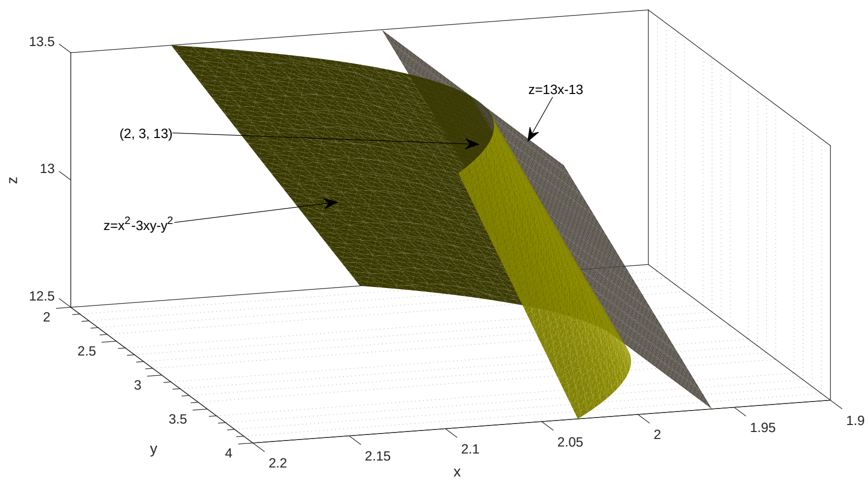
9. If $z=f(x,y)=x^2+3xy-y^2$,
   (a) write the expression for the differential dz at (x,y,z);
   (b) and if x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

Solution:
(a) The differential dz at (x,y,z) is given by:
$dz=\dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy$
$=(2x+3y)dx+(3y-2x)dy.$

(b) Now $dx=\triangle x=2.05-2=0.05$, $dy=\triangle y=-3+2.96=-0.04$.
Hence at $(2,3,f(2,3))$, $dz=(2\times 2+3\times 3)(0.05)+(3\times 2-2\times 3)(-0.04)=0.65$
$\triangle z=f(2.05,2.96)-f(2,3)=0.6449.$



(a)



(b)

10. Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

(a) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2},\ (1, 2, -2),\ \mathbf{v} = \langle -6, 6, -3 \rangle$

(b) $g(x, y, z) = x \tan^{-1}\left(\dfrac{y}{z}\right),\ (1, 2, -2),\ \mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}.$

Solution: (a) The directional derivative of f at $(1, 2, -2)$ along the unit vector $\mathbf{u} = \left\langle -\dfrac{2}{3}, \dfrac{2}{3}, -\dfrac{1}{3} \right\rangle$ if it exists, is equal to the following limit:

$$D_{\mathbf{u}} f(1, 2, -2) = \lim_{t \to 0} \frac{f((1, 2, -2) + t(-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})) - f(1, 2, -2)}{t}.$$

$$\frac{f((1, 2, -2) + t(\frac{-2}{3}, \frac{2}{3}, -\frac{1}{3})) - f(1, 2, -2)}{t} = \frac{\sqrt{(1 - \frac{2}{3}t)^2 + (2 + \frac{2}{3}t)^2 + (-2 - \frac{1}{3}t)^2} - 3}{t}.$$

$$= \frac{\sqrt{9 + t^2 + \frac{8}{3}t} - 3}{t}.$$

By applying L'Hospital's rule we get the limit as
$= \lim_{t\to 0} \dfrac{2t+\frac{8}{3}}{2\sqrt{9+t^2+\frac{8}{3}t}} = \dfrac{4}{9}.$

(b) The directional derivative of f at $(1,2,-2)$ along the unit vector $\mathbf{u} = \left\langle \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}, \frac{\mathbf{1}}{\sqrt{\mathbf{3}}}, -\frac{\mathbf{1}}{\sqrt{\mathbf{3}}} \right\rangle$ is equal to the following limit, if it exists.

$$D_{\mathbf{u}}f(1,2,-2) = \lim_{t\to 0} \frac{f((1,2,-2)+t(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})) - f(1,2,-2)}{t}.$$

$\dfrac{f((1,2,-2)+t(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}})) - f(1,2,-2)}{t}$

$= \dfrac{(1+\frac{t}{\sqrt{3}})tan^{-1}(-1) - tan^{-1}(-1)}{t}$, for $t \neq 0$ and t sufficiently small.

$= \dfrac{\frac{t}{\sqrt{3}}tan^{-1}(-1)}{t} = \dfrac{1}{\sqrt{3}}tan^{-1}(-1) = -\dfrac{\pi}{4\sqrt{3}},$

$\Rightarrow \lim_{t\to 0} \dfrac{f((1,2,-2)+t(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{-\sqrt{3}})) - f(1,2,-2)}{t} = -\dfrac{\pi}{4\sqrt{3}}.$

Aliter: Note that if the existence of all directional derivatives of f at (x_0,y_0,z_0) is already guaranteed (say when f is differentiable at (x_0,y_0,z_0)) then one can directly calculate the value of the directional derivative by the formula given below:
$D_{\mathbf{u}}f(x_0,y_0,z_0) = [\nabla f]_{(x_0,y_0,z_0)} \cdot \mathbf{u}.$

11. Find the directional derivatives of the scalar field $f(x,y) = x^3 - 3xy$ along the parabola $y = x^2 - x + 2$ at the point (1,2).

Solution: A unit vector along the tangent to the parabola at $(1,2)$ can be obtained as follows:
The slope of the tangent vector of the parabola at $(1,2)$ is given by
$\dfrac{dy}{dx} |_{x=1} = (2x-1)_{x=1} = 1.$
Hence the tangent line of the parabola at $(1,2)$ is given by $y-2 = 1(x-1)$ or $y - x = 1$.
Any two points on this tangent line can be taken as $P_0(x_0, 1+x_0)$ and $P_1(2x_0, 1+2x_0)$ (you can choose any arbitrary points in such a way that $y = 1+x$).
Therefore the vector $\overrightarrow{P_0P_1}$ joining the points P_0 and P_1 is given by
$< 2x_0 - x_0, (1+2x_0) - (1+x_0) > = < x_0, x_0 >$ so that a unit vector along this tangent is $\left\langle \dfrac{x_0}{\sqrt{x_0^2+x_0^2}}, \dfrac{x_0}{\sqrt{x_0^2+x_0^2}} \right\rangle = \left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle$. (You can also obtain the unit tangent by parametrizing the parabola by the vector $\mathbf{r}(t) = < t, t^2 - t + 1 >$, so that a unit vector at the point $(1,2)$ which corresponds to $t = 1$ is given by $\hat{u} = \left.\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\right|_{t=1} =$
$\left.\left\langle \dfrac{1}{\sqrt{1+(2t-1)^2}}, \dfrac{2t-1}{\sqrt{1+(2t-1)^2}} \right\rangle\right|_{t=1} = \left\langle \dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}} \right\rangle.$
Note that depending upon the orientation of the parabola, you may take this unit

vector as $\left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$ or $\left\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right\rangle$.)

Since f is clearly differentiable throughout $\mathbf{R}^2$ (* the partial derivatives $f_x = 3x^2 - 3y$ and $f_y = -3x$ are continuous throughout $\mathbf{R}^2$), therefore the directional derivatives exist along any direction $\mathbf{u}$ and is given by

$D_{\mathbf{u}}f(1,2) = [\nabla f]_{(1,2)} \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \langle -3, -3 \rangle \cdot \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = -3\sqrt{2}.$

(* For any $(x_0, y_0) \in \mathbf{R}^2$
$|f_x(x,y) - f_x(x_0,y_0)| = 3\left|(x^2 - x_0^2) - (y - y_0)\right| \le 3(\left|x^2 - x_0^2\right| + |y - y_0|)$
$= 3(|x - x_0|\,|x + x_0| + |y - y_0|)$.
Since (x, y) is sufficiently close to (x_0, y_0), we can assume that $|x - x_0| < 1$, then $|x + x_0| < 2\,|x_0| + 1 = M$, say.
Then $|f_x(x,y) - f_x(x_0,y_0)| < 3(M\,|x - x_0| + |y - y_0|) \le (3M + 1)\sqrt{(x - x_0)^2 + (y - y_0)^2}$
Hence given $\epsilon > 0$ take $\delta = \dfrac{\epsilon}{3M + 1}$,
then $|f_x(x,y) - f_x(x_0,y_0)| < \epsilon$ if $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$,
which implies f_x is continuous at (x_0, y_0).
Similarly one can show that f_y is continuous at (x_0, y_0).)

12. Let $f(x,y) = \dfrac{x}{|x|}\sqrt{x^2 + y^2}$ if $x \neq 0$ and $f(x,y) = 0$ if $x = 0$. Show that f is continuous at $(0,0)$ and the directional derivatives exist thereat, but it is not differentiable at $(0,0)$.

Solution: $|f(x,y) - f(0,0)| = |f(x,y)| = \dfrac{|x|\sqrt{x^2 + y^2}}{|x|} = \sqrt{x^2 + y^2}$ if $x \neq 0$
$= 0$ if $x = 0$.
Given any $\epsilon > 0$ take $\delta = \epsilon$, then
$|f(x,y) - f(0,0)| \le \sqrt{x^2 + y^2} < \epsilon$ if $\sqrt{x^2 + y^2} < \delta$.
Hence f is continuous at $(0,0)$.
The directional derivative of f at $(0,0)$ along the unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$, if it exists, is given by:

$D_{\mathbf{u}}f(0,0) = \lim\limits_{t \to 0} \dfrac{f(tu_1, tu_2) - f(0,0)}{t}$.

$\dfrac{f(tu_1, tu_2) - f(0,0)}{t} = \dfrac{tu_1\sqrt{t^2u_1^2 + t^2u_2^2}}{|tu_1|\,t} = \dfrac{u_1}{|u_1|}$ if $u_1 \neq 0$.

$\dfrac{f(tu_1, tu_2) - f(0,0)}{t} = 0$ if $u_1 = 0$.

Hence the directional derivatives exist along every $\mathbf{u} = \langle u_1, u_2 \rangle$ at $(0,0)$.
To check the differentiability of f at $(0,0)$ we need to check that

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2) - f_x(0,0)h_1 - f_y(0,0)h_2}{\sqrt{h_1^2 + h_2^2}} = 0$$

$$\frac{f(h_1,h_2) - f_x(0,0)h_1 - f_y(0,0)h_2}{\sqrt{h_1^2 + h_2^2}} = \frac{\frac{h_1\sqrt{h_1^2+h_2^2}}{|h_1|} - h_1}{\sqrt{h_1^2 + h_2^2}}, \text{ if } h_1 \neq 0.$$

If $h_1 = h_2 \neq 0$

then $\dfrac{f(h_1,h_2) - f_x(0,0)h_1 - f_y(0,0)h_2}{\sqrt{h_1^2+h_2^2}} = \left(1 - \dfrac{1}{\sqrt{2}}\right)\left(\dfrac{h_1}{|h_1|}\right).$

Hence $\displaystyle\lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2) - f_x(0,0)h_1 - f_y(0,0)h_2}{\sqrt{h_1^2+h_2^2}}$ does not exist.

Hence f is not differentiable at $(0,0)$.

13. Show that the following functions are differentiable at the respective points mentioned below:

    (a) Let

    $$f(x,y) = \begin{cases} \frac{x}{x+y} & \text{if} \quad x+y \neq 0 \\ 0 & \text{if} \quad x+y = 0. \end{cases}$$

    Show that f is differentiable at $(2,1)$ but not differentiable at $(0,0)$.

    (b) Show that $f(x,y) = \sqrt{x+e^y}$ is differentiable at $(3,0)$, where x, y is such that $x + e^y \geq 0$.

Solution:

(a) f is differentiable at $(2,1)$ if

$$\lim_{(h,k)\to\mathbf{0}} \frac{\left|f(2+h,1+k) - f(2,1) - [\nabla f]_{(2,1)} \cdot \langle h,k\rangle\right|}{\sqrt{h^2+k^2}} = 0.$$

$$\frac{\left|f(2+h,1+k) - f(2,1) - [\nabla f]_{(2,1)} \cdot \langle h,k\rangle\right|}{\sqrt{h^2+k^2}} = \frac{\left|\frac{2+h}{3+h+k} - \frac{2}{3} - \frac{h}{9} + \frac{2k}{9}\right|}{\sqrt{h^2+k^2}}$$

$= \dfrac{|hk + 2k^2 - h^2|}{9(3+h+k)\sqrt{h^2+k^2}} \leq \dfrac{|hk+2k^2-h^2|}{\sqrt{h^2+k^2}}$ for sufficiently small h and k.

Hence the above expression is $\leq \dfrac{|hk|}{\sqrt{h^2+k^2}} + \dfrac{2k^2}{\sqrt{h^2+k^2}} + \dfrac{h^2}{\sqrt{h^2+k^2}} \leq 4\sqrt{h^2+k^2}.$

Also $\displaystyle\lim_{(h,k)\to\mathbf{0}} \sqrt{h^2+k^2} = 0.$

Since $\displaystyle 0 \leq \lim_{(h,k)\to\mathbf{0}} \frac{\left|f(2+h,1+k) - f(2,1) - [\nabla f]_{(2,1)} \cdot \langle h,k\rangle\right|}{\sqrt{h^2+k^2}} \leq 4 \lim_{(h,k)\to\mathbf{0}} \sqrt{h^2+k^2} = 0,$

$$\lim_{(h,k)\to\mathbf{0}} \frac{\left|f(2+h,1+k) - f(2,1) - [\nabla f]_{(2,1)} \cdot \langle h,k\rangle\right|}{\sqrt{h^2+k^2}} = 0.$$

Hence f is differentiable at $(2,1)$.

Aliter: Note that for all (x,y) such that $x+y \neq 0$, $f_x(x,y) = \dfrac{y}{(x+y)^2}$ and $f_y(x,y) = -\dfrac{x}{(x+y)^2}$ are continuous functions, hence f_x, f_y are continuous throughout some small neighborhood of $(2,1)$, hence f is differentiable at $(2,1)$.

If $x = y \neq 0$, then $f(x,x) = \dfrac{x}{x+x} = \dfrac{1}{2}$, hence $\displaystyle\lim_{x\to 0} f(x,x) = \frac{1}{2}$, but $f(0,0) = 0$, hence f is not continuous at $(0,0)$, which implies f is not differentiable at $(0,0)$.

(b) Note that for all (x, y) such that $x + e^y > 0$, $f_x(x, y) = \dfrac{1}{2\sqrt{x + e^y}}$ and $f_y(x, y) = \dfrac{e^y}{2\sqrt{x + e^y}}$ are continuous, hence f_x, f_y are continuous throughout some small neighborhood of $(3, 0)$, hence f is differentiable at $(3, 0)$.

14. Show that the following function is differentiable throughout $\mathbf{R}^2$ and find the maximum rate of change of $f(x, y) = 6 - 3x^2 - y^2$ at the point (1,2) and the direction in which it occurs.

Solution: If f differentiable throughout $\mathbf{R}^2$, then the directional derivatives exist along any direction $\mathbf{u}$ at $(1, 2)$.
Then the directional derivative of f along any unit vector $\mathbf{u} = \langle u_1, u_2 \rangle$ at $(1, 2)$, is given by
$D_{\mathbf{u}} f(1, 2) = [\nabla f]_{(1,2)} \cdot \langle u_1, u_2 \rangle = \left| [\nabla f]_{(1,2)} \right| \cos\theta$, where θ is the angle between $[\nabla f]_{(1,2)}$ and $\mathbf{u}$.
Hence the directional derivative is maximum when $\cos\theta = 1$, or $\theta = 0$ or $\mathbf{u}$ is parallel to and in the same direction as $[\nabla f]_{(1,2)}$.
Hence the required $\mathbf{u} = \left| [\nabla f]_{(1,2)} \right| = \left\langle \dfrac{-6}{\sqrt{52}}, \dfrac{-4}{52} \right\rangle$.

To show that f is differentiable:
Check that for $(x, y) \in \mathbf{R}^2$, $f_x(x, y) = -6x$ and $f_y(x, y) = -2y$ are continuous functions, hence f is differentiable throughout $\mathbf{R}^2$. (*)
((*) For any $(x_0, y_0) \in \mathbf{R}^2$
$|f_x(x, y) - f_x(x_0, y_0)| = 6\,|x - x_0| \leq 6\sqrt{(x - x_0)^2 + (y - y_0)^2}$.
Hence given $\epsilon > 0$ take $\delta = \frac{\epsilon}{6}$,
then $|f_x(x, y) - f_x(x_0, y_0)| < \epsilon$ if $\sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$
which implies f_x is continuous at (x_0, y_0).
Similarly one can show that f_y is continuous at (x_0, y_0).)

Aliter: f is differentiable at (x_0, y_0) if

$$\lim_{(h,k)\to\mathbf{0}} \frac{\left| f(x_0 + h, y_0 + k) - f(x_0, y_0) - [\nabla f]_{(x_0,y_0)} \cdot \langle h, k \rangle \right|}{\sqrt{h^2 + k^2}} = 0.$$

$$\frac{\left| f(x_0 + h, y_0 + k) - f(x_0, y_0) - [\nabla f]_{(x_0,y_0)} \cdot \langle h, k \rangle \right|}{\sqrt{h^2 + k^2}} = \frac{|-3h^2 - k^2|}{\sqrt{h^2 + k^2}} = 3\frac{h^2}{\sqrt{h^2 + k^2}} + \frac{k^2}{\sqrt{h^2 + k^2}}$$
$\leq 3\sqrt{h^2 + k^2} + \sqrt{h^2 + k^2}$ (since $h^2, k^2 \leq h^2 + k^2$).
$= 4\sqrt{h^2 + k^2}$.

Also $\displaystyle\lim_{(h,k)\to\mathbf{0}} \sqrt{h^2 + k^2} = 0$.

Since $\displaystyle 0 \leq \lim_{(h,k)\to\mathbf{0}} \frac{\left| f(x_0 + h, y_0 + k) - f(x_0, y_0) - [\nabla f]_{(x_0,y_0)} \cdot \langle h, k \rangle \right|}{\sqrt{h^2 + k^2}} \leq 4 \lim_{(h,k)\to\mathbf{0}} \sqrt{h^2 + k^2} = 0$,

$$\lim_{(h,k)\to\mathbf{0}} \frac{\left|f(x_0+h, y_0+k) - f(x_0, y_0) - [\nabla f]_{(x_0,y_0)} \cdot \langle h, k\rangle\right|}{\sqrt{h^2+k^2}} = 0.$$

15. If R is the total resistance of three resistors, connected in parallel, with resistances R_1, R_2, R_3, then
$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$
The resistances are measured in ohms as $R_1 = 100\Omega$, $R_2 = 100\Omega$ and $R_3 = 200\Omega$ at a particular instant. At any instant R_1 and R_2 are increasing at $1\Omega/s$ whereas R_3 is decreasing at $2\Omega/s$. Is R increasing or decreasing at that instant? At what rate?
Solution: At any time point t
$$\frac{-1}{R(t)^2}\frac{\partial R(t)}{\partial t} = \frac{-1}{R_1(t)^2}\frac{\partial R_1(t)}{\partial t} + \frac{-1}{R_2(t)^2}\frac{\partial R_2(t)}{\partial t} + \frac{-1}{R_3(t)^2}\frac{\partial R_3(t)}{\partial t}.$$
$$\Rightarrow \frac{\partial R(t)}{\partial t}\left(\frac{1}{100} + \frac{1}{100} + \frac{1}{200}\right)^2 = \left(\frac{1}{100^2}(1) + \frac{1}{100^2}(1) + \frac{1}{200^2}(-2)\right)$$
$$\Rightarrow \frac{\partial R(t)}{\partial t} = \frac{4}{25} \times \frac{3}{2} = .24\Omega/s.$$

16. Assume that $w = f(x, y)$, $x = r\cos\theta$ and $y = r\sin\theta$. Assuming the existence of all the required first and second order partial derivatives of w with respect to x, y, r and θ, show that
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$$
Solution: Since $w = f(x, y) = f(r\cos\theta, r\sin\theta) = g(r, \theta)$.
By the chain rule we get

$$\frac{\partial g}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$
But we will write $\frac{\partial g}{\partial r}$ as $\frac{\partial w}{\partial r}$.
$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cos\theta + \frac{\partial w}{\partial y}\sin\theta. \qquad (1)$$
Hence $\frac{\partial^2 g}{\partial r^2} = \cos\theta\left(\frac{\partial}{\partial r}(\frac{\partial w}{\partial x})\right) + \sin\theta\left(\frac{\partial}{\partial r}(\frac{\partial w}{\partial y})\right)$
But we will write $\frac{\partial^2 g}{\partial r^2}$ as $\frac{\partial^2 w}{\partial r^2}$.

By again applying the chain rule we get

$$\frac{\partial^2 w}{\partial r^2} = \cos\theta\left(\left(\frac{\partial}{\partial x}(\frac{\partial w}{\partial x})\right)\frac{\partial x}{\partial r} + \left(\frac{\partial}{\partial y}(\frac{\partial w}{\partial x})\right)\frac{\partial y}{\partial r}\right) + \sin\theta\left(\left(\frac{\partial}{\partial x}(\frac{\partial w}{\partial y})\right)\frac{\partial x}{\partial r} + \left(\frac{\partial}{\partial y}(\frac{\partial w}{\partial y})\right)\frac{\partial y}{\partial r}\right)$$
$$= \left(\cos^2\theta\frac{\partial^2 w}{\partial x^2} + \cos\theta\sin\theta\frac{\partial^2 w}{\partial y\partial x}\right) + \left(\sin\theta\cos\theta\frac{\partial^2 w}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 w}{\partial y^2}\right). \qquad (2)$$

By applying the chain rule we get

$\dfrac{\partial g}{\partial \theta} = \dfrac{\partial w}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial w}{\partial y}\dfrac{\partial y}{\partial \theta}.$
$= \dfrac{\partial w}{\partial x}(-r\sin\theta) + \dfrac{\partial w}{\partial y}(r\cos\theta).$

Again by applying the chain rule we get
$\dfrac{\partial^2 g}{\partial \theta^2} = \dfrac{\partial w}{\partial x}(-r\cos\theta) + (-r\sin\theta)\left(\dfrac{\partial^2 w}{\partial x^2}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial^2 w}{\partial y\partial x}\dfrac{\partial y}{\partial \theta}\right) + \dfrac{\partial w}{\partial y}(-r\sin\theta) +$
$(r\cos\theta)\left(\dfrac{\partial^2 w}{\partial y^2}\dfrac{\partial y}{\partial \theta} + \dfrac{\partial^2 w}{\partial x\partial y}\dfrac{\partial x}{\partial \theta}\right).$
But we will write $\dfrac{\partial^2 g}{\partial \theta^2}$ as $\dfrac{\partial^2 w}{\partial \theta^2}$.
$\dfrac{\partial^2 w}{\partial \theta^2} = (-r\cos\theta)\dfrac{\partial w}{\partial x} + (-r\sin\theta)^2\left(\dfrac{\partial^2 w}{\partial x^2}\right) + (-r\sin\theta)(r\cos\theta)\left(\dfrac{\partial^2 w}{\partial y\partial x}\right) + (-r\sin\theta)\dfrac{\partial w}{\partial y} +$
$(r\cos\theta)^2\left(\dfrac{\partial^2 w}{\partial y^2}\right) + (r\cos\theta)(-r\sin\theta)\left(\dfrac{\partial^2 w}{\partial x\partial y}\right).$ (3)

From (1) , (2) and (3) it follows that
$\dfrac{\partial^2 w}{\partial r^2} + \dfrac{1}{r}\dfrac{\partial w}{\partial r} + \dfrac{1}{r^2}\dfrac{\partial^2 w}{\partial \theta^2} = \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2}.$

17. Suppose $w = f(u)$ where $u = \dfrac{x^2 - y^2}{x^2 + y^2}$. Assuming the existence of all the required first order partial derivatives of w and u show that $xw_x + yw_y = 0$.

Solution: By applying chain rule we get
$w_x = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial x}$ and $w_y = \dfrac{\partial f}{\partial u}\dfrac{\partial u}{\partial y}$.
$\dfrac{\partial u}{\partial x}|_{(x,y)\neq(0,0)} = \dfrac{4xy^2}{(x^2+y^2)^2}$ and $\dfrac{\partial u}{\partial y}|_{(x,y)\neq(0,0)} = \dfrac{-4yx^2}{(x^2+y^2)^2}$.
Hence $xw_x + yw_y = \dfrac{4x^2y^2}{(x^2+y^2)^2}\dfrac{\partial f}{\partial u} + \dfrac{-4x^2y^2}{(x^2+y^2)^2}\dfrac{\partial f}{\partial u} = 0.$

18. **Implicit differentiation:** If $\phi(x, y, z) = 0$ defines z as an implicit function of x and y in a region R of the xy-plane, assuming the existence of all the required partial derivatives prove that $\dfrac{\partial z}{\partial x} = -\dfrac{\phi_x}{\phi_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{\phi_y}{\phi_z}$, where $\phi_z \neq 0$. Hence find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ when $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = 1$.

Solution: Since $\phi(x, y, z) = 0$, gives z implicitly as a function of x and y we write $z = f(x, y)$. Hence $\phi(x, y, f(x, y)) = g(x, y) = 0$.
If we take $u_1(x, y) = x, u_2(x, y) = y$, and $u_3(x, y) = f(x, y) = z$, then we get,
$\phi(u_1(x, y), u_2(x, y), u_3(x, y)) = g(x, y) = 0$.
By applying chain rule we get

$$\frac{\partial \phi}{\partial u_1}\frac{\partial u_1}{\partial x} + \frac{\partial \phi}{\partial u_2}\frac{\partial u_2}{\partial x} + \frac{\partial \phi}{\partial u_3}\frac{\partial u_3}{\partial x} = \frac{\partial g}{\partial x} = 0, \text{ or}$$

$\dfrac{\partial\phi}{\partial x}\dfrac{\partial x}{\partial x}+\dfrac{\partial\phi}{\partial y}\dfrac{\partial y}{\partial x}+\dfrac{\partial\phi}{\partial z}\dfrac{\partial z}{\partial x}=0.$
$\Rightarrow \dfrac{\partial\phi}{\partial x}+\dfrac{\partial\phi}{\partial z}\dfrac{\partial z}{\partial x}=0$ (since $\dfrac{\partial y}{\partial x}=0$).
Hence $\dfrac{\partial z}{\partial x}=-\dfrac{\phi_x}{\phi_z}$ when $\phi_z\neq 0$,
and similarly $\dfrac{\partial z}{\partial y}=-\dfrac{\phi_y}{\phi_z}$, when $\phi_z\neq 0$.
If $\phi(x,y,z)=x^{\frac{2}{3}}+y^{\frac{2}{3}}+z^{\frac{2}{3}}-1$
then $\phi_x=\dfrac{2}{3}x^{-\frac{1}{3}}$, $\phi_y=\dfrac{2}{3}y^{-\frac{1}{3}}$, $\phi_z=\dfrac{2}{3}z^{-\frac{1}{3}}$.
Hence $\dfrac{\partial z}{\partial x}=-\dfrac{\phi_x}{\phi_z}=-(\dfrac{z}{x})^{\frac{1}{3}}$ and $\dfrac{\partial z}{\partial y}=-\dfrac{\phi_y}{\phi_z}=-(\dfrac{z}{y})^{\frac{1}{3}}$.

19. Suppose that $w=\dfrac{1}{r}f\left(t-\dfrac{r}{a}\right)$ and that $r=\sqrt{x^2+y^2+z^2}$. Assuming the existence of all the required second order partial derivatives, show that

$$\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}=\frac{1}{a^2}\frac{\partial^2 w}{\partial t^2}.$$

Solution: $w=\dfrac{1}{r}f\left(t-\dfrac{r}{a}\right)=g(x,y,z)$, say.
By applying the chain rule we get

$\dfrac{\partial g}{\partial x}=\dfrac{\partial w}{\partial r}\dfrac{\partial r}{\partial x}$
$\Rightarrow \dfrac{\partial^2 g}{\partial x^2}=\dfrac{\partial}{\partial x}(\dfrac{\partial w}{\partial r}\dfrac{\partial r}{\partial x})$
$=\left(\dfrac{\partial r}{\partial x}\right)\left(\dfrac{\partial}{\partial x}(\dfrac{\partial w}{\partial r})\right)+\left(\dfrac{\partial w}{\partial r}\right)\left(\dfrac{\partial}{\partial x}(\dfrac{\partial r}{\partial x})\right).$
$=\dfrac{\partial r}{\partial x}\left(\left(\dfrac{\partial}{\partial r}(\dfrac{\partial w}{\partial r})\right)\dfrac{\partial r}{\partial x}\right)+\dfrac{\partial w}{\partial r}\left(\dfrac{\partial^2 r}{\partial x^2}\right)$
$=\left(\dfrac{\partial^2 w}{\partial r^2}\right)\left(\dfrac{\partial r}{\partial x}\right)^2+\left(\dfrac{\partial w}{\partial r}\right)\left(\dfrac{\partial^2 r}{\partial x^2}\right).$
Similarly $\dfrac{\partial^2 g}{\partial y^2}=\left(\dfrac{\partial^2 w}{\partial r^2}\right)\left(\dfrac{\partial r}{\partial y}\right)^2+\left(\dfrac{\partial w}{\partial r}\right)\left(\dfrac{\partial^2 r}{\partial y^2}\right),$
and $\dfrac{\partial^2 g}{\partial z^2}=\left(\dfrac{\partial^2 w}{\partial r^2}\right)\left(\dfrac{\partial r}{\partial z}\right)^2+\left(\dfrac{\partial w}{\partial r}\right)\left(\dfrac{\partial^2 r}{\partial z^2}\right).$
$\Rightarrow \dfrac{\partial^2 w}{\partial x^2}+\dfrac{\partial^2 w}{\partial y^2}+\dfrac{\partial^2 w}{\partial z^2}=\left(\dfrac{\partial^2 w}{\partial r^2}\right)\left((\dfrac{\partial r}{\partial x})^2+(\dfrac{\partial r}{\partial y})^2+(\dfrac{\partial r}{\partial z})^2\right)+\left(\dfrac{\partial w}{\partial r}\right)\left(\dfrac{\partial^2 r}{\partial x^2}+\dfrac{\partial^2 r}{\partial y^2}+\dfrac{\partial^2 r}{\partial z^2}\right).$
Note that $\dfrac{\partial r}{\partial x}=\dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $\dfrac{\partial^2 r}{\partial x^2}=\dfrac{y^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$.
Similarly $\dfrac{\partial r}{\partial y}=\dfrac{y}{\sqrt{x^2+y^2+z^2}}$, $\dfrac{\partial^2 r}{\partial y^2}=\dfrac{x^2+z^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$
and $\dfrac{\partial r}{\partial z}=\dfrac{z}{\sqrt{x^2+y^2+z^2}}$, $\dfrac{\partial^2 r}{\partial z^2}=\dfrac{x^2+y^2}{(x^2+y^2+z^2)^{\frac{3}{2}}}$.

$\Rightarrow \dfrac{\partial^2 g}{\partial x^2} + \dfrac{\partial^2 g}{\partial y^2} + \dfrac{\partial^2 g}{\partial z^2} = \left(\dfrac{\partial^2 w}{\partial r^2}\right) + \left(\dfrac{\partial w}{\partial r}\right)\dfrac{2}{r}.$ (1)

Note that $\dfrac{\partial w}{\partial r} = -\dfrac{1}{r^2} f(u) + \dfrac{1}{r}\dfrac{\partial f(u)}{\partial u}\dfrac{\partial u}{\partial r}$

$= -\dfrac{1}{r^2} f(u) + \dfrac{1}{r} f'(u)(-\dfrac{1}{a})$, where $u = t - \frac{r}{a}$, and

$\dfrac{\partial^2 w}{\partial r^2} = \left(\dfrac{2}{r^3} f(u) + (-\dfrac{1}{r^2}) f'(u)(-\dfrac{1}{a})\right) + \left((-\dfrac{1}{r^2}) f'(u)(-\dfrac{1}{a}) + (\dfrac{1}{r})(-\dfrac{1}{a}) f''(u)(-\dfrac{1}{a})\right)$,
where $u = t - \frac{r}{a}$.

Hence $\left(\dfrac{\partial^2 w}{\partial r^2}\right) + \left(\dfrac{\partial w}{\partial r}\right)\dfrac{2}{r} = \dfrac{1}{r}(-\dfrac{1}{a}) f''(u)(-\dfrac{1}{a}) = \dfrac{1}{a^2 r} f''(u).$ (2)

Also $\dfrac{\partial w}{\partial t} = \dfrac{\partial w}{\partial u}\dfrac{\partial u}{\partial t} = \dfrac{1}{r} f'(u) \times 1$, where $u = t - \frac{r}{a}$ and

$\dfrac{\partial^2 w}{\partial t^2} = \left(\dfrac{\partial}{\partial u}(\dfrac{1}{r} f'(u))\right)\dfrac{\partial u}{\partial t} = \dfrac{1}{r} f''(u).$

$\Rightarrow \dfrac{1}{a^2 r} f''(u) = \dfrac{1}{a^2}\dfrac{\partial^2 w}{\partial t^2}.$ (3)

(1) , (2) and (3) $\Rightarrow \dfrac{\partial^2 w}{\partial x^2} + \dfrac{\partial^2 w}{\partial y^2} + \dfrac{\partial^2 w}{\partial z^2} = \dfrac{1}{a^2}\dfrac{\partial^2 w}{\partial t^2}.$

(We write $\dfrac{\partial^2 g}{\partial x^2}, \dfrac{\partial^2 g}{\partial y^2}$ and $\dfrac{\partial^2 g}{\partial z^2}$ as $\dfrac{\partial^2 w}{\partial x^2}, \dfrac{\partial^2 w}{\partial y^2}$ and $\dfrac{\partial^2 w}{\partial z^2}$, respectively).

20. A function f is called **homogeneous of degree** n if it satisfies the equation $f(tx, ty) = t^n f(x, y)$ for all t, where n is a positive integer and f has continuous second order partial derivatives.

   (a) Verify that $f(x, y) = x^3 - 2xy^2 + 5y^3$ is homogeneous of degree 3.
   Solution: Can be easily worked out.

   (b) Show that if f is homogeneous of degree n, then

      i. $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x, y),$

      ii. $x^2\dfrac{\partial^2 f}{\partial x^2} + 2xy\dfrac{\partial^2 f}{\partial x \partial y} + y^2\dfrac{\partial^2 f}{\partial y^2} = n(n-1)f(x, y)$

      iii. $f_x(tx, ty) = t^{n-1} f_x(x, y).$

Solution:

(i) $f(tx, ty) = t^n f(x, y)$ for all $t \in \mathbf{R}$

$$\frac{\partial f(u,v)}{\partial t} = \frac{\partial f(u,v)}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial f(u,v)}{\partial v}\frac{\partial v}{\partial t} = nt^{n-1} f(x, y) \text{ for all } t \in \mathbf{R},$$

where $u = tx$ and $v = ty$.

$\Rightarrow \dfrac{\partial f(u,v)}{\partial t} = \dfrac{\partial f(u,v)}{\partial u} x + \dfrac{\partial f(u,v)}{\partial v} y = nt^{n-1} f(x, y).$ (1)

Hence for $t = 1$ or $u = x$ and $v = y$ we get:

$\Rightarrow \dfrac{\partial f(x,y)}{\partial x} x + \dfrac{\partial f(x,y)}{\partial x} y = nf(x, y).$

(ii)

By taking the partial derivative with respect to t on both sides of equation (1) we

get
$\dfrac{\partial^2 f(u,v)}{\partial t^2} = x\left(\left(\dfrac{\partial}{\partial u}(\dfrac{\partial f(u,v)}{\partial u})\right)\dfrac{\partial u}{\partial t} + \left(\dfrac{\partial}{\partial v}(\dfrac{\partial f(u,v)}{\partial u})\right)\dfrac{\partial v}{\partial t}\right) +$
$y\left(\left(\dfrac{\partial}{\partial v}(\dfrac{\partial f(u,v)}{\partial v})\right)\dfrac{\partial v}{\partial t} + \left(\dfrac{\partial}{\partial u}(\dfrac{\partial f(u,v)}{\partial v})\right)\dfrac{\partial u}{\partial t}\right)$
$= n(n-1)t^{n-2}f(x,y)$ for all $t \in \mathbf{R}$, where $u = tx$ and $v = ty$. (2)

(2) $\Rightarrow x^2\dfrac{\partial^2 f(u,v)}{\partial u^2} + 2xy\dfrac{\partial^2 f(u,v)}{\partial u \partial v} + y^2\dfrac{\partial^2 f(u,v)}{\partial v^2} = n(n-1)t^{n-2}f(x,y)$
(since $\dfrac{\partial^2 f(u,v)}{\partial u \partial v} = \dfrac{\partial^2 f(u,v)}{\partial v \partial u}$).

Hence for $t = 1$ or $u = x$ and $v = y$ we get:
$x^2\dfrac{\partial^2 f}{\partial x^2} + 2xy\dfrac{\partial^2 f}{\partial x \partial y} + y^2\dfrac{\partial^2 f}{\partial y^2} = n(n-1)f(x,y)$.

(iii) $f_x(tx,ty) = \lim_{h\to 0}\dfrac{f(tx+th,ty) - f(tx,ty)}{th}$
$= \lim_{h\to 0}\dfrac{t^n(f(x+h,y) - f(x,y))}{th}$
$= t^{n-1}\lim_{h\to 0}\dfrac{f(x+h,y) - f(x,y)}{h} = t^{n-1}f_x(x,y)$.