PH 101: Physics I

Module 2: Special Theory of Relativity-Basics

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Recap from the last class

Lorentz transformation:

Relating space and time coordinates in frames S and S’, where S’ is moving with constant velocity along the x-axis.

$$x' = \gamma(x - \beta x_0); \qquad y' = y; \quad z' = z \qquad x_0 = ct \qquad \beta = \frac{v}{c} \qquad \gamma = \sqrt{\frac{1}{1-\beta^2}}$$

$$x'_0 = \gamma(x_0 - \beta x)$$

Interesting consequences:

Events simultaneous in one frame will not be simultaneous in another frame in relative motion with respect to the first.

Clocks appear to tick slower when looked at from a moving frame.

Objects appear to be shortened when measured from a moving frame.

The Pole-Barn Paradox

Paradoxes in relativity are just trick questions worded in such a way that makes the reader focus on only one aspect of relativity while distracting the reader into ignoring other aspects, leading to inconsistencies which are commonly referred to as ‘paradoxes’.

This implies that an alert student of relativity can easily see through these tricks. We shall now learn how to see through these ‘paradoxes’.

The pole-barn paradox is a famous variation on the twin paradox which must be addressed with the ideas of simultaneity in relativity. The fact that two events are simultaneous in one frame of reference does not imply that they are simultaneous as seen by an observer moving at a relativistic speed with respect to that frame. The wording of the problem below is such that it dwells only on the length contraction giving an impression of a paradox. The resolution rests on realising that in addition, there is time dilation, which put together resolves the paradox.

You own a barn 10 m long, with two doors at either end. You also have a pole 20 m long, which of course wont fit in the barn (in normal situations).

Now someone takes the pole and tries to run (at nearly the speed of light) through the barn with the pole horizontal. Special Relativity (SR) says that a moving object is contracted in the direction of motion: this is called the Lorentz Contraction. So, if the pole is set in motion lengthwise, then it will contract in the reference frame of a stationary observer.

You are that observer, sitting on the barn roof. You see the pole coming towards you, and it has contracted to a bit less than 10m, in your reference frame. So, as the pole passes through the barn, there is an instant when it is completely within the barn.

But consider the problem from the point of view of the runner. She will regard the pole as stationary, and the barn as approaching at high speed. In this reference frame, the pole is still 20m long, and the barn is less than 10 meters long. Surely the runner is in trouble if the doors close while she is inside. The pole is sure to get caught.

Let us now look at the details, in the light of Special Theory of Relativity.

Examples

Barn and pole paradox

Consider the case: You own a barn 10 m long, with two doors at either end. You also have a pole 20 m long, which of course wont fit in the barn (in normal situations).

Initially, D1 was open, D2 was closed.

A person now takes the pole and runs towards the door with a constant speed of $v = 0.9\,c$ (Wow!!)

Corresponding length contraction: $\ell = \dfrac{\ell_0}{\gamma} = \dfrac{\ell_0}{2.3}$

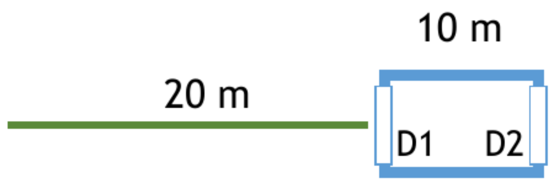
The person in the barn says he closed D1 and simultaneously opened D2, the moment the rear end of the pole crossed the first door.

The person running with the pole says D2 was opened first (while the rear end of the pole was outside the barn, and D1 was closed much later when the rear end was inside, so that she could comfortably pass through both the doors without trouble.

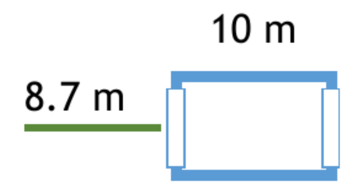
Who is right?

Galileo: Both cannot be right.

Einstein: Both can be right. Simultaneous in one frame is not simultaneous in the other frame.



As measured when both are at rest



From the point of view of the barn

From the point of view of the pole

Solution:

Event 1: Rare end of the pole crosses D1. Both the observers synchronise the clocks, and set the origin of the space coordinates so that this event records $(0,0,0,0)$ in the two frames (S: that of the barn, S' that of the moving pole)

Event 2: Open D2. As recorded by S: $(0, 10\ m, 0, 0)$ As recorded by S': $(t', x', 0, 0)$

$$v = 0.9c \quad \Rightarrow \quad \gamma \approx 2.3 \qquad t' = -\gamma \frac{xv}{c^2} = -2.3 \times \frac{10 \times 0.9}{3 \times 10^8} s = -0.69 \times 10^{-7} s$$

In S' the Event 2 happened before the Event 1

Distance covered by the person with pole by this time $= 0.69 \times 10^{-7} \times 0.9 \times 3 \times 10^8\ m = 18.63\ m$

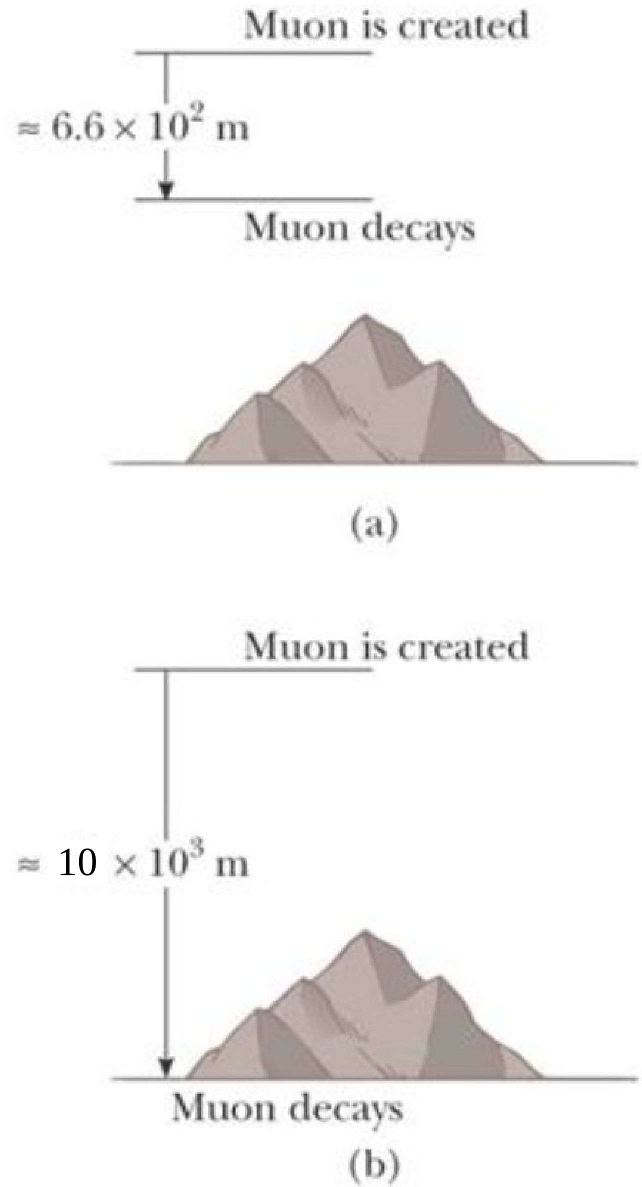
Coordinates of the D_2 (in S') when Event 2 happened $(-0.069 \mu s, 1.37\ m, 0, 0)$

Well within the barn! As also recorded by S'

Time Dilation verification: Muon Decays

- Muons are unstable particles that have the same charge as an electron, but a mass 207 times more than an electron
- Muons have a half-life of $\Delta t_p = 2.2$ μs when measured in a reference frame at rest with respect to them (a)
- Relative to an observer on the Earth, muons should have a lifetime of

  $\gamma \Delta t_p$ (b)
- A CERN experiment measured lifetimes in agreement with the predictions of relativity



Muons are produced by cosmic ray interaction with atmosphere, at about 10 - 20 km above see level.

$$\text{Mean life time } = 2.197 \times 10^{-6} s$$

With this life time muons can travel a distance of

$$vt_0 = ((2.994 \times 10^8 m/s)(2.2 \times 10^{-6} s) = 6.6 \times 10^2 m = 0.66 km$$

However, we detect them on earth in large numbers: one of them passes through each square centimeter of earth's surface on the average slightly more often than once in a minute.

Due to time dilation the life time of muon will increase by a factor γ, i.e.

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}} = \frac{2.2 \times 10^{-6}}{\sqrt{1 - (0.998c)^2/c^2}} = 34.8 \times 10^{-6} s = 34.8 \mu s$$

With this lifetime muon can travel a distance

$$vt = (2.994 \times 10^8 m/s)(34.8 \times 10^{-6} s) = 1.04 \times 10^4 m = 10.4 km$$

From muon frame of reference lifetime remains $2.2\mu s$.

The question arises howcome muon covers such a long distance as seen from earth .

Is it a paradox ?? NO !!!

From the muon frame of reference the distance from muon and earth surface will get contracted.

$$h = h_0 \sqrt{1 - v^2/c^2} = 0.66 km$$

Lorentz transformation of Velocity and acceleration

Imagine a particle moving in reference frame S with velocity

$$\vec{u} = \left(u_x, u_y, u_z\right) \quad \text{where} \quad u_x = \frac{dx}{dt}, \quad etc.$$

In the reference frame S' the velocity is $\vec{u}' = \left(u'_x, u'_y, u'_z\right)$ with $u'_x = \dfrac{dx'}{dt'}$

$$dx' = \gamma\left(dx - v\,dt\right); \qquad dy' = dy; \quad dz' = dz; \quad dt' = \gamma\left(dt - \frac{v}{c^2}dx\right)$$

$$u'_x = \frac{dx'}{dt'} = \frac{dx - v\,dt}{dt - \frac{v}{c^2}dx} = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

similarly

$$u'_y = \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

$$u'_z = \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}$$

Inverse transformations can be obtained by making v =-v and changing unprime to prime

What about acceleration ?

Inertial frames allow us to study accelerated objects, or action of force on objects.

Thus, the concept of acceleration, and the acceleration of two bodies when looked at from two different inertial frames are well within the realm of STR.

Exercise:

A body has acceleration $\vec{a} = \left(a_x, a_y, a_z\right)$ in frame S. Find the expression for acceleration in frame S' moving with speed $\vec{v}$ along x-axis.

Accelerating Frames

Accelerating frames, but are non-inertial, and require special attention. We will not go into this aspect here.

$$a'_x = \frac{a_x}{\gamma_v^3\left(1 - \frac{u_x v}{c^2}\right)^3} \qquad a_x = \frac{a'_x}{\gamma_v^3\left(1 + \frac{u'_x v}{c^2}\right)^3}$$

$$a'_y = \frac{a_y}{\gamma_v^2\left(1 - \frac{u_x v}{c^2}\right)^2} + \frac{a_x \frac{u_y v}{c^2}}{\gamma_v^2\left(1 - \frac{u_x v}{c^2}\right)^3} \qquad a_y = \frac{a'_y}{\gamma_v^2\left(1 + \frac{u'_x v}{c^2}\right)^2} - \frac{a'_x \frac{u'_y v}{c^2}}{\gamma_v^2\left(1 + \frac{u'_x v}{c^2}\right)^3}$$

$$a'_z = \frac{a_z}{\gamma_v^2\left(1 - \frac{u_x v}{c^2}\right)^2} + \frac{a_x \frac{u_z v}{c^2}}{\gamma_v^2\left(1 - \frac{u_x v}{c^2}\right)^3} \qquad a_z = \frac{a'_z}{\gamma_v^2\left(1 + \frac{u'_x v}{c^2}\right)^2} - \frac{a'_x \frac{u'_z v}{c^2}}{\gamma_v^2\left(1 + \frac{u'_x v}{c^2}\right)^3}$$

Example: Velocity addition

Spacecraft Alpha is moving at $0.90c$ with respect to the earth. If spacecraft Beta is to pass Alpha at a relative speed of $0.50c$ in the same direction, what speed must Beta have with respect to the earth?

According to the Galilean transformation, Beta would need a speed relative to the earth of $0.90c + 0.50c = 1.40c$, which we know is impossible. According to Velocity addition with $V'_x = 0.50c$ and $\boldsymbol{v} = 0.90c$, the required speed is only

$$V_x = \frac{V'_x + \boldsymbol{v}}{1 + \dfrac{\boldsymbol{v}V'_x}{c^2}} = \frac{0.50c + 0.90c}{1 + \dfrac{(0.90c)(0.50c)}{c^2}} = 0.97c$$

which is less than c. It is necessary to go less than 10 percent faster than a spacecraft traveling at $0.90c$ in order to pass it at a relative speed of $0.50c$.

Example An object moves with velocity $\vec{u}' = 0.2c\,\hat{x}$ as measured from a spaceship S_1. The spaceship itself is moving with speed 0.5c long the x-direction, as seen from the earth. What is the velocity of the object as measured from the earth?

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Exercise Two spaceships S_1 and S_2 are moving along the same direction at speeds 0.2c and 0.7c, respectively, as measured from earth. What is the speed of S_2 as measure from S_1?

Exercise Two spaceships S_1 and S_2 are moving perpendicular to each other at speeds 0.2c and 0.7c, respectively, as measured from earth. What is the speed of S_2 as measure from S_1?

We shall continue in the next class