

[2<sup>pnts.</sup>] 1.

Given  $\mathbf{a} = \langle 4, -5, 3 \rangle$  and  $\mathbf{b} = \langle 2, 1, -2 \rangle$ , express  $\mathbf{a}$  as the sum of a vector  $\mathbf{a}_{\parallel}$  parallel to  $\mathbf{b}$  and a vector  $\mathbf{a}_{\perp}$  perpendicular to  $\mathbf{b}$ . Verify that  $\mathbf{a}_{\perp} \perp \mathbf{b}$ . (HINT: See the adjacent figure.)

Soln.:

From the figure, we see that

$$\mathbf{a}_{\parallel} = \text{comp}_{\mathbf{b}} \mathbf{a} \frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} \mathbf{b} = \frac{8 - 5 - 6}{9} \mathbf{b} = \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle$$

and

$$\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\parallel} = \left\langle \frac{14}{3}, -\frac{14}{3}, \frac{7}{3} \right\rangle$$

Clearly  $\mathbf{a}_{\perp} \cdot \mathbf{b} = \frac{28}{3} - \frac{14}{3} - \frac{14}{3} = 0$ . Hence the verification.

**Aliter:** We have  $\mathbf{a}_{\parallel} = \lambda \mathbf{b}$  for some  $\lambda \neq 0$  and  $\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\parallel} = \mathbf{a} - \lambda \mathbf{b}$ .

Now  $\mathbf{a}_{\perp} \cdot \mathbf{b} = 0$  or  $(\mathbf{a} - \lambda \mathbf{b}) \cdot \mathbf{b} = 0$  yields  $\lambda = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|^2} = \frac{8 - 5 - 6}{9} = -\frac{1}{3}$ . The remaining part is similar to the above answer.

[2<sup>pnts.</sup>] 2. Find the point on the curve  $\mathbf{r}(t) = 5 \sin t \hat{\mathbf{i}} + 5 \cos t \hat{\mathbf{j}} + 12t \hat{\mathbf{k}}$  at a distance  $26\pi$  units along the curve from its origin ( $t = 0$ ) in the direction of increasing arc-length.

Soln.:

The arc length along the curve from its origin to the point corresponding to  $t = T$  is given by  $\int_0^T |\mathbf{r}'(t)| dt = \int_0^T \left( \sqrt{25 \cos^2 t + 25 \sin^2 t + 144} \right) dt = 13T$

Thus  $13T = 26\pi$  or  $T = 2\pi$ .

Hence the required point is  $(5 \sin(2\pi), 5 \cos(2\pi), 24\pi)$ , i.e.,  $(0, 5, 24\pi)$ .

[1<sup>pnts.</sup>] 3. Find the unit tangent vector of the curve  $\mathbf{r}(t)$  at  $(0, 1, 0)$ , where  $\mathbf{r}(t) = \langle \cos(t), \sin(t), t \cos(t) \rangle$ ,  $0 \leq t \leq 2\pi$ .

Soln.:

$(0, 1, 0)$  of the curve  $\mathbf{r}(t)$  corresponds to  $t = \frac{\pi}{2}$ .

Since  $\mathbf{r}'\left(\frac{\pi}{2}\right) = \left\langle -1, 0, -\frac{\pi}{2} \right\rangle$ , the unit tangent vector of  $\mathbf{r}(t)$  at  $(0, 1, 0)$  is  
 $= \left\langle \frac{-1}{\sqrt{1 + (\frac{\pi}{2})^2}}, 0, \frac{-\frac{\pi}{2}}{\sqrt{1 + (\frac{\pi}{2})^2}} \right\rangle$ .

[3<sup>pnts.</sup>] 4. Let

$$f(x, y) = \begin{cases} \frac{x^2y^3}{(x^2+y^2)^2} & \text{if } x^2 + y^2 \neq 0 \\ 0 & \text{if } x^2 + y^2 = 0. \end{cases}$$

Check whether  $f$  is differentiable at  $(0, 0)$ .

Soln.:

$$f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = 0.$$

Similarly  $f_y(0, 0) = 0$ .

$f$  is differentiable at  $(0, 0)$  if

$$\lim_{(h_1, h_2) \rightarrow (0, 0)} \frac{f(h_1, h_2) - f(0, 0) - f_x(0, 0)h_1 - f_y(0, 0)h_2}{\sqrt{h_1^2 + h_2^2}} = 0.$$

$$\frac{f(h_1, h_2) - f(0, 0) - f_x(0, 0)h_1 - f_y(0, 0)h_2}{\sqrt{h_1^2 + h_2^2}} = \frac{h_1^2 h_2^3}{(h_1^2 + h_2^2)^{\frac{5}{2}}}.$$

Consider  $(h_1, h_2) \rightarrow (0, 0)$  along a line  $L$  with slope  $m$ ,  $m \neq 0$ ,

$$\text{then } \lim_{(h_1, h_2) \rightarrow (0, 0)} \frac{h_1^2 h_2^3}{(h_1^2 + h_2^2)^{\frac{5}{2}}} \text{ along } L$$

$$= \lim_{h_1 \rightarrow 0} \frac{m^3 h_1}{(1 + m^2)^{\frac{5}{2}} |h_1|}, \text{ which does not exist ( or } \neq 0\text{),}$$

hence  $f$  is not differentiable at  $(0, 0)$ .

**Aliter:**  $f_x(0, 0) = \lim_{h \rightarrow 0} \frac{f(h, 0) - f(0, 0)}{h} = 0.$

Similarly  $f_y(0, 0) = 0$ .

$f$  is differentiable at  $(0, 0)$ , if (and only if)

$$(\Delta z =) f(\Delta x, \Delta y) - f(0, 0) = f_x(0, 0)\Delta x + f_y(0, 0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

where  $\epsilon_1, \epsilon_2 \rightarrow 0$  whenever  $(\Delta x, \Delta y) \rightarrow (0, 0)$ .

Since  $f(0, 0) = f_x(0, 0) = f_y(0, 0) = 0$ ,  $f$  is differentiable at  $(0, 0)$  if

$$\Delta z = \frac{(\Delta x)^2 (\Delta y)^3}{((\Delta x)^2 + (\Delta y)^2)^2} = \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

for some  $\epsilon_1, \epsilon_2$  such that  $\epsilon_1, \epsilon_2 \rightarrow 0$  whenever  $(\Delta x, \Delta y) \rightarrow (0, 0)$ .

Consider  $(\Delta x, \Delta y) \rightarrow (0, 0)$  along a line  $L$  with slope  $m$ ,  $(\Delta y = m\Delta x)$ ,  $m \neq 0$ .

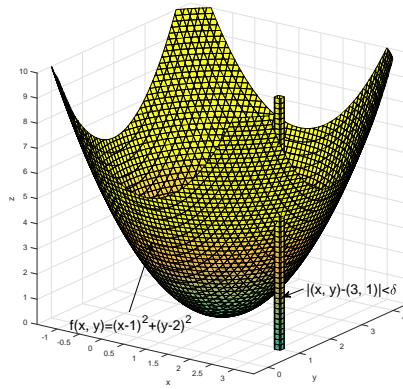
If  $f$  is differentiable at  $(0, 0)$  then  $\Delta z = \epsilon \Delta x$  for some  $\epsilon (= \epsilon_1 + m\epsilon_2)$  such that  $\epsilon \rightarrow 0$  whenever  $\Delta x \rightarrow 0$ .

$$\text{Since } \Delta z = \frac{(\Delta x)^2 (\Delta y)^3}{((\Delta x)^2 + (\Delta y)^2)^2} = \Delta x \frac{m^3 (\Delta x)^4}{(1 + m^2)^2 (\Delta x)^4}, \text{ along } L$$

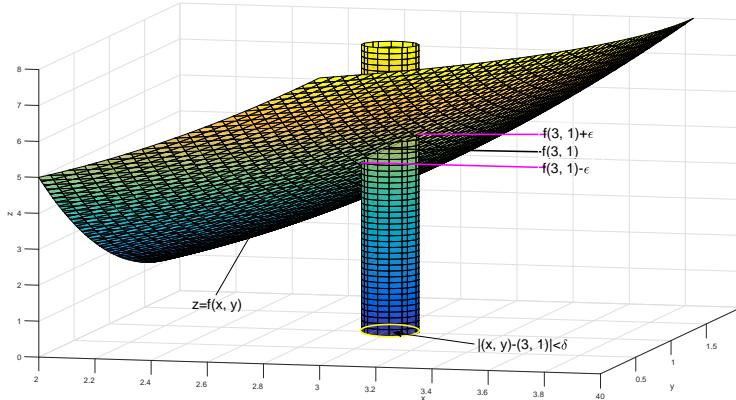
$$\epsilon = \frac{m^3 (\Delta x)^4}{(1 + m^2)^2 (\Delta x)^4} = \frac{m^3}{(1 + m^2)^2}.$$

Since  $\epsilon$  does not tend to 0 as  $\Delta x \rightarrow 0$ ,  $f$  is not differentiable at  $(0, 0)$ .

- [2pnts.] 5. Let  $f(x, y) = (x - 1)^2 + (y - 2)^2$ . Given  $\epsilon > 0$ , find a  $\delta > 0$  (explicitly in terms of  $\epsilon$ ) such that,  $\sqrt{(x - 3)^2 + (y - 1)^2} < \delta$  implies  $|f(x, y) - f(3, 1)| < \epsilon$ .



(a)



(b)

Figure 1: Geometrical interpretation of problem 5: (a) Global view and (b) Close up view.

Soln.:

$$\begin{aligned}
 |f(x, y) - f(3, 1)| &= |(x - 1)^2 + (y - 2)^2 - 2^2 - 1^2| \\
 &= |(x - 3)^2 + (y - 1)^2 + 4(x - 3) - 2(y - 1)| \\
 &\leq ((x - 3)^2 + (y - 1)^2) + 4|(x - 3)| + 2|(y - 1)| \\
 &\leq ((x - 3)^2 + (y - 1)^2) + 4\sqrt{(x - 3)^2 + (y - 1)^2} + 2\sqrt{(x - 3)^2 + (y - 1)^2} \quad (***) \\
 &\leq 7\sqrt{(x - 3)^2 + (y - 1)^2} \text{ if } (x, y) \text{ is such that } \sqrt{(x - 3)^2 + (y - 1)^2} \leq 1. \\
 \text{Hence for any } \delta \text{ such that, } 0 < \delta \leq \min\left\{1, \frac{\epsilon}{7}\right\}, \\
 \sqrt{(x - 3)^2 + (y - 1)^2} < \delta \text{ implies } |f(x, y) - f(3, 1)| < \epsilon.
 \end{aligned}$$