PH101

Lecture 8

Conservation laws, D'Alembert's principle of virtual work, Derivation of Lagrange's equation from Newton's law using D'Alembert's principle

Symmetry and conservation laws

Cyclic coordinates, symmetries and conservation laws

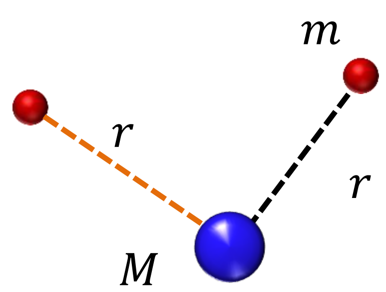
Example: For planetary motion

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}$$

Here, θ is cyclic coordinate $\left(i, e. \frac{\partial L}{\partial \theta} = 0\right)$ corresponding generalized (canonical) momentum $\boldsymbol{p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}}$=**Constant**; [Using Lagrange's eqn. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$]

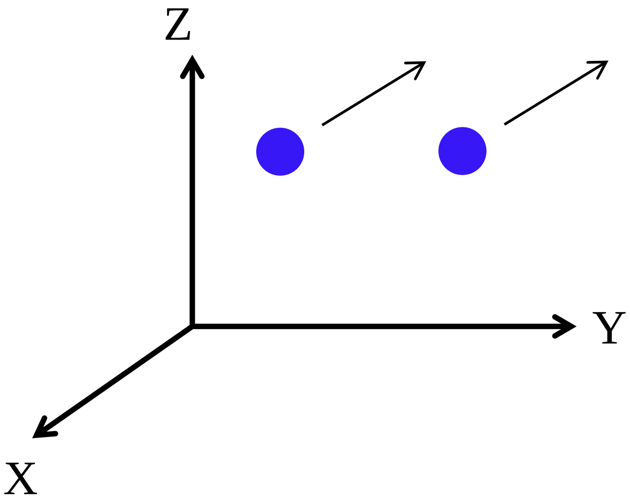
This generalized momentum is nothing but angular momentum.

Another way of saying the same statement '$\boldsymbol{\theta}$ is cyclic':
L is independent of rotation angle θ, the system has rotational symmetry,
the system remains the same after any change in θ.



Conclusion: Conservation of angular momentum is related to rotational symmetry of the system.

Cyclic coordinates, symmetries and conservation laws



Lagrangian of a point mass moving in general direction under gravitational field.

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

x and y are cyclic coordinates, thus $p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$ and $p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$ both are constant of motion .

In this case, generalized momentum is the linear momentum.

In other words the system has translation symmetry in x and y coordinates.
Any change in either x or y will not change the Lagrangian.

Conclusion: Conservation of linear momentum is associated with translational symmetry of the system.

Symmetry and conservation laws

Translational symmetry → **Conservation of linear momentum**

Rotational symmetry → **Conservation of angular momentum**

? → **Conservation of energy**

If L does not explicitly depend on time, then energy of the system is conserved, provided potential energy is velocity independent.

If L does not have explicit time dependence, change in time does not cause any change in Lagrangian i,e, $\frac{\partial L}{\partial t} = 0$,

$E = T + U = constnat$ **(looking for proof !)**

Translational symmetry in time → **Conservation of energy**

Time-translation symmetry leads to energy conservation: Proof

Lagrangian $L = L(q_1, \dots q_n, \dot{q}_1, \dots . \dot{q}_n, t)$

Using the chain rule of partial differentiation

$$dL = \sum_j \frac{\partial L}{\partial \dot{q}_j} d\dot{q}_j + \sum_j \frac{\partial L}{\partial q_j} dq_j + \frac{\partial L}{\partial t} dt$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \frac{d\dot{q}_j}{dt} + \sum_j \frac{\partial L}{\partial q_j} \frac{dq_j}{dt} + \frac{\partial L}{\partial t} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \frac{\partial L}{\partial t}$$

If L does not contain explicit time dependence, i,e. $L = L(q_1, \dots q_n, \dot{q}_1, \dots . \dot{q}_n)$, Hence $\frac{\partial L}{\partial t} = 0$

Thus

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \sum_j \frac{\partial L}{\partial q_j} \dot{q}_j = \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \sum_j \frac{d}{dt} (\frac{\partial L}{\partial \dot{q}_j}) \dot{q}_j$$

Using Lagrange's eqn $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = \frac{\partial L}{\partial q_j}$

$$\frac{dL}{dt} = \sum_j \frac{d}{dt} (\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j)$$

$$\frac{d}{dt}\left(L - \sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j\right) = 0$$

Equation 1

Proof continue…

Now, $L = T - U$

If U does not depend on generalized velocity ($\frac{\partial U}{\partial \dot{q}_j} = 0$) and only function of generalized coordinate , then $\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$

$$Thus\ from\ eqn\ 1.\ \frac{d}{dt}\left(L - \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j\right) = 0;\ \frac{d}{dt}\left(L - \sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j\right) = 0$$

Now, check that for a single free particle $T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$\sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = m(\dot{x}\dot{x} + \dot{y}\dot{y} + \dot{z}\dot{z}) = 2T$$

The relationship is true for general case as well, which can be obtained immediately from Euler's theorem. If $f(x_i)$ is a homogeneous function of the n_{th} degree of set of variables x_i, then $\sum_j \frac{\partial f}{\partial x_j} x_j = nf$.

Here, T is a function of 2nd degree of generalized velocities $\dot{q}_j$, *thus* $\sum_j \frac{\partial T}{\partial \dot{q}_j}\dot{q}_j = 2T$

$\frac{d}{dt}(L - 2T) = 0; \frac{d}{dt}(T - U - 2T) = 0\ hence\ \frac{d}{dt}(T + U) = 0;\quad T + U = constant$

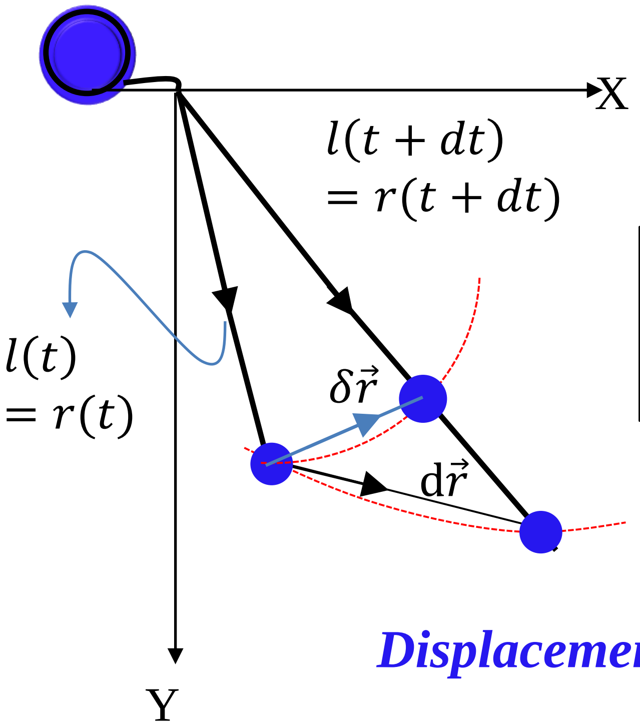
Conclusion, $T + U = E = constant$,

$if\ L\ does\ not\ explicitely\ depend\ on\ time\ and\ potential\ is\ velocity\ indepdent$

D'Alembert's principle of virtual work and proof of newton's law

Real vs Virtual displacement

Simple pendulum with a variable string length $\boldsymbol{l(t)}$ [Time dependent constraint]



Real displacement of the bob in time d$\boldsymbol{t}$ is given by
$\mathrm{d}\vec{r} = \vec{r}(t+dt) - \vec{r}(t)$

Let's **imagine** any **instantaneous displacement** at time t (that it, *without allowing the real time to change* $dt = 0$) which is **consistent with the constrain relation at time $\boldsymbol{t}$**?

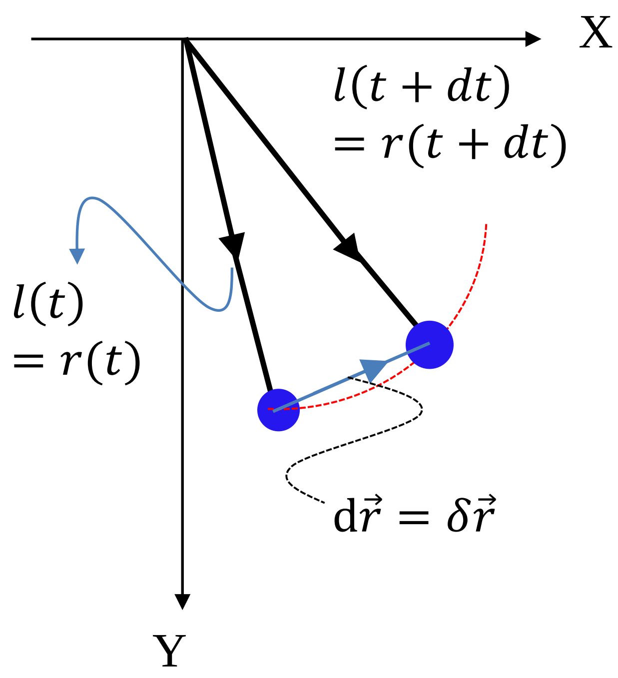
Displacement without allowing time to change **-Seems absurd?**

As you are making an imagination of displacement at time t $(dt = 0)$, this displacement must not change the constrain condition at time t [i,e, $l(t)$ remains same] as shown.

Imaginary, instantaneous displacement which is consistent with the constrain relation at a given instant (i,e. without allowing real time to change) is called *Virtual displacement* and denoted by $\delta\vec{r}$ (for infinitesimal case)

Real vs Virtual displacement

- If the constrain is not time dependent, the real and virtual displacements matches each other, only the difference you need to remember that **virtual displacement is imagined without allowing time to change**.



Virtual displacement in generalized coordinates

- Consider a system of N particles with k constrains, DOF, $\boldsymbol{n = 3N - k}$
- Cartesian coordinates, $\vec{r}_i = \vec{r}_i\,(x_i, y_i, z_i)$ $\qquad (i = 1, \dots \dots, N)$
- Generalized coordinates q_j $\qquad (j = 1, \dots\dots, n)$
- Virtual displacements of the particles $\delta\vec{r}_1, \delta\vec{r}_2. \dots\dots, \delta\vec{r}_N$
- Virtual displacements of the particles in the generalized coordinates $\delta q_1, \delta q_2. \dots\dots, \delta q_n$ can be found from given transformation relations

$$\boldsymbol{\vec{r}_1 = \vec{r}_1(q_1, q_2, \dots\dots, q_n, t)}$$
$$\boldsymbol{\vec{r}_2 = \vec{r}_2(q_1, q_2, \dots\dots, q_n, t)}$$
$$\boldsymbol{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots}$$
$$\boldsymbol{\vec{r}_N = \vec{r}_N(q_1, q_2, \dots\dots, q_n, t)}$$

$$\delta\vec{r}_i = \sum_{j=1}^{n} \frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$$

$3N$ coordinates, not independent

$n = 3N - k$ generalized coordinates, independent

Note: There is no $\frac{\partial\vec{r}_i}{\partial t}\delta t$, as virtual displacement is instantaneous without allowing time to change, δt=0

Virtual work done

- **Real work done**: Work done due to real displacement ($d\vec{r}$) of a particle acted on by total force $\vec{F}$ is given by

$$dW = \vec{F}.d\vec{r}$$

- As you can always **imagine** an instantaneous displacement (without allowing time to change), known as virtual displacement ($\delta\vec{r}$), and hence you can always define a scalar function

$$\delta W = \vec{F}.\delta\vec{r}$$

This scalar function is called as **Virtual work.**

Note: '**Virtual work**' is different from '**Real work**', as virtual displacement is imagined without allowing time to change.

If time is not allowed to change, no real displacement ($d\vec{r}$) is possible and hence no real work (dW) is possible without allowing time to change

Virtual work done for a system of particles

- Consider a system of particles and $\vec{F}_1, \vec{F}_2, \ldots, \vec{F}_N$ are the forces on $1,2\ldots.N_{th}$ particles, then

Total virtual work done

$$\delta W = \sum_{i=1}^{N} \vec{F}_i . \delta \vec{r}_i$$

- Here, force on each particle, $\vec{F}_i$ is the sum of external force and also forces of constrains.

$$\vec{F}_i = \vec{F}_{ie} + \vec{f}_{ic}$$

Where,

- $\vec{F}_{ie}$ is the external applied force on i_{th} particle.
- $\vec{f}_{ic}$ is the constrain force

Virtual work for a dynamical system

- Newton's second law for a single particle reads as

$$m\ddot{\vec{r}} = \vec{F}$$

$$m\ddot{\vec{r}} = \vec{F}_e + \vec{f}_c$$

- Taking dot product with an infinitesimal virtual displacement $\delta\vec{r}$

$$m\ddot{\vec{r}} \cdot \delta\vec{r} = (\vec{F}_e + \vec{f}_c) \cdot \delta\vec{r}$$

$$\left(\vec{\boldsymbol{F}}_{\boldsymbol{e}} - \boldsymbol{m}\ddot{\vec{\boldsymbol{r}}}\right) \cdot \boldsymbol{\delta}\vec{\boldsymbol{r}} + \vec{\boldsymbol{f}}_{\boldsymbol{c}} \cdot \boldsymbol{\delta}\vec{\boldsymbol{r}} = \mathbf{0} \qquad \boxed{1}$$

Total force$(\vec{F})$ =
$Applied\ force(\vec{F}_e)$
$+constrain\ force\ (\vec{f}_c)$

Now, consider a general case of system of N particles having virtual displacements, $\delta\vec{r}_1$, $\delta\vec{r}_2$,….., $\delta\vec{r}_N$,

Eqn.1 for i_{th} particle $\Rightarrow$ $\left(\vec{\boldsymbol{F}}_{\boldsymbol{ie}} - \boldsymbol{m_i}\ddot{\vec{\boldsymbol{r}}}_{\boldsymbol{i}}\right) \cdot \boldsymbol{\delta}\vec{\boldsymbol{r}}_{\boldsymbol{i}} + \vec{\boldsymbol{f}}_{\boldsymbol{ic}} \cdot \boldsymbol{\delta}\vec{\boldsymbol{r}}_{\boldsymbol{i}} = \mathbf{0}$

Summing for all the particles of the system

$$\sum_{i=1}^{N}\left(\vec{\boldsymbol{F}}_{\boldsymbol{ie}} - \boldsymbol{m_i}\ddot{\vec{\boldsymbol{r}}}_{\boldsymbol{i}}\right) \cdot \boldsymbol{\delta}\vec{\boldsymbol{r}}_{\boldsymbol{i}} + \boxed{\sum_{i=1}^{N}\vec{\boldsymbol{f}}_{\boldsymbol{ic}} \cdot \boldsymbol{\delta}\vec{\boldsymbol{r}}_{\boldsymbol{i}}} = \mathbf{0}$$

$\vec{\boldsymbol{F}}_{\boldsymbol{ie}}$ →Applied force on i_{th} particle

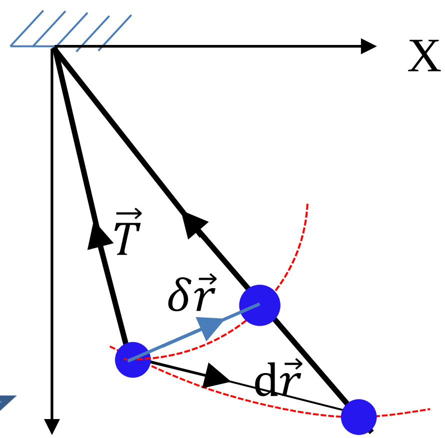
$\vec{\boldsymbol{f}}_{\boldsymbol{ic}}$ →Constrain force on i_{th} particle

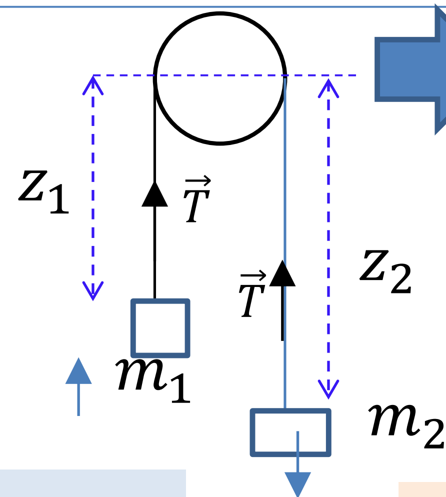
$$\sum_{i=1}^{N} \vec{f}_{ic} \cdot \delta\vec{r}_i$$

Total virtual work done by all the constrain forces

- Now, ‘virtual displacement’ is instantaneous, imaginary displacement **consistence with constrain relation**.

To keep the imaginary displacement, consistence with constrains at a given moment, either **each individual terms within this summation is zero** or **summation is zero as a whole** (even if individual terms are non-zero)





To keep the imaginary displacement, **consistence with constrains at a given moment**, the virtual displacement must be chosen along $z - direction$

and $\delta\vec{z}_2 = -\delta\vec{z}_1$

$$\sum_{i=1}^{N} \vec{f}_{ic} \cdot \delta\vec{r}_i = \vec{T} \cdot \delta\vec{z}_1 + \vec{T} \cdot \delta\vec{z}_2$$

$$= T\delta z_1 - T\delta z_1 = 0$$

Note: Individual terms are not zero

To keep the imaginary displacement, **consistence with constrains at a given moment**, the virtual displacement must be chosen perpendicular to constrain force.

$$\vec{f}_{ic} \cdot \delta\vec{r}_i = \vec{T} \cdot \delta\vec{r} = 0$$

D'Alembert's principle of virtual work

Since total virtual work done by the all the constraint forces is zero, I,e

$$\sum_{i=1}^{N} \vec{f}_{ic} \cdot \delta\vec{r}_i = 0$$

Then from equation 1

$$\sum_{i=1}^{N} (\vec{F}_{ie} - m_i\ddot{\vec{r}}_i) \cdot \delta\vec{r}_i = 0$$

D'Alembert's principle of Virtual work

$\vec{F}_{ie}$ →Applied force on i_{th} particle

Does not necessarily means that individual terms of the summation are zero as $\vec{r}_i$ **are not independent**, they are connected by constrain relation

Want to express this relation in such a way where all the terms in the summation becomes individually zero.

How to do?

Converting this relationship in terms of generalized coordinates

Quick recap of basic mathematics

$If\ u_1\,\delta x_1 + u_2\,\delta x_2 = 0$; does this always mean $u_1 = 0$ and $u_2 = 0$?

If x_1 and x_2 are independent then $u_1 = 0$ and $u_2 = 0$ for all possible variation of x_1 and x_2,
If x_1 and x_2 are independent then you can vary one without changing other. If you fix x_1 and vary x_2, and still the relation is always giving zero, then only possibility is u_1 and u_2 must be zero.
If x_1 and x_2 are not independent, changing one will change the other.

Generalization:

$$\boldsymbol{If}, \qquad \sum \boldsymbol{u_i\,\delta\,x_i = 0},$$

$then\ all\ u_i$ will be individually zero for all possible variation of the x_i only when x_i are **independent to each other.**