PH 101: Physics I

Module 2: Special Theory of Relativity-Basics

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RECAP

Relativistic Momentum $\vec{p} = \Gamma_u m_0 \vec{u}$

Total energy of an object of mass m moving with speed u: $E = \Gamma_u m_0 c^2$

Kinetic energy of the object: $K.E. = E - m_0c^2 = (\Gamma_u - 1)m_0c^2$

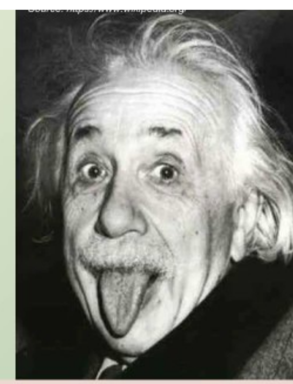
Regarding mass, Albert Einstein specifically said:

"the mass of an object increases as its speed increases…"

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

where $\sqrt{1 - \left(\frac{v}{c}\right)^2} < 1$ and m_0 = rest mass

v = speed of object



"… and because performing work on an object increases its energy (speed), then mass must also be a form of energy"

This implies that:

mass and energy are equivalent, in the sense that a *gain or loss of mass* can be regarded equally well as a *gain or loss of energy*

Mass-energy equivalence:

Mass defect- binding energy

Energy released in nuclear decay.

Nuclear Fission and Fusion

RECAP

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Now that we have everything in order let's have a go at solving the equation. We will use a mass of 1 kg to keep things simple and I will show all of the workings of the equation. So, with 1 kg of mass (around 2.2 pounds) we get:

$$
\begin{aligned}
E &= mc^2 \\
&= 1\text{kg} \times \left(3 \times 10^8 \text{ms}^{-1}\right)^2 \\
&= 1\text{kg} \times \left(3 \times 10^8 \text{ms}^{-1}\right) \times \left(3 \times 10^8 \text{ms}^{-1}\right) \\
&= 1\text{kg} \times \left(9 \times 10^{16} \text{m}^2\text{s}^{-2}\right) \\
&= 1 \times \left(9 \times 10^{16}\right) \text{kg m}^2\text{s}^{-2} \\
&= 9 \times 10^{16} \text{J}
\end{aligned}
$$

90,000,000,000,000,000 joules

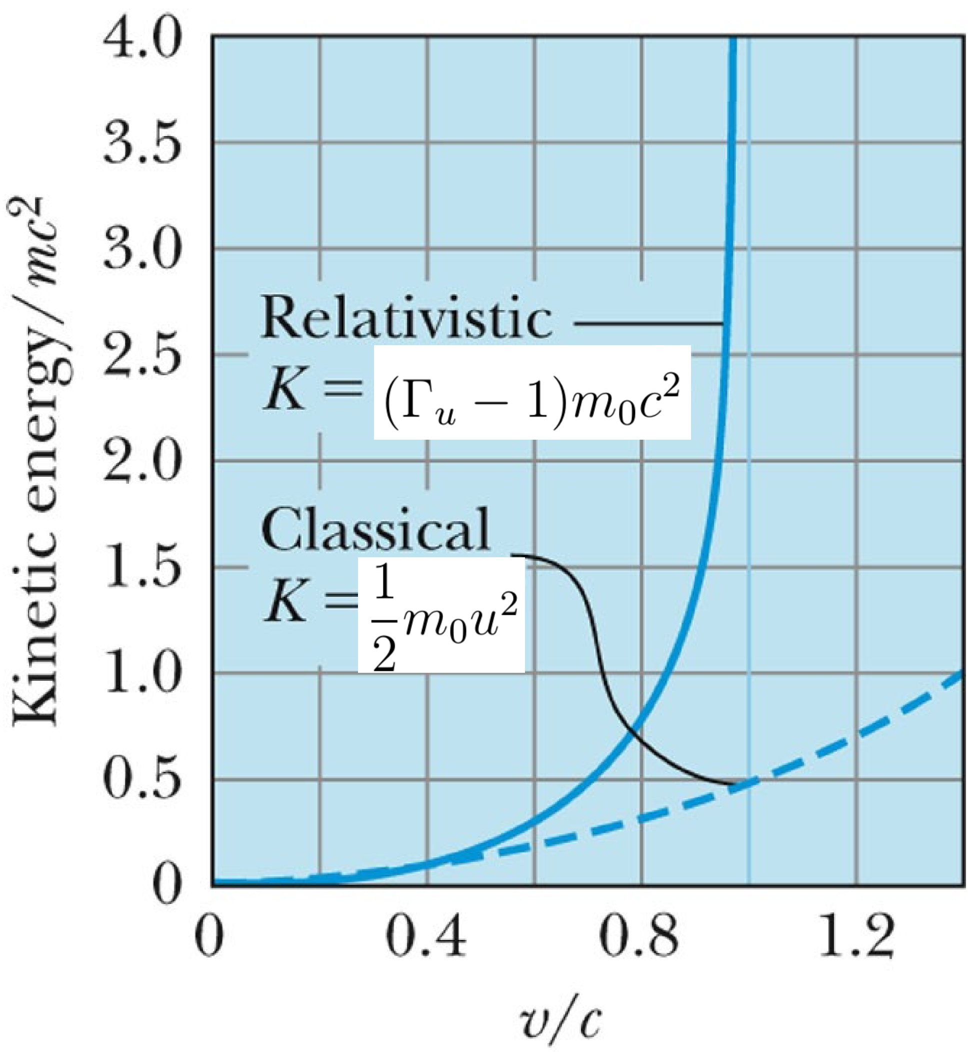
9×10^{16} J / 100 W = 9×10^{14} seconds

9×10^{14} seconds / 31,557,600 seconds = 28,519,279 years

That is a very long time!

Of course, converting mass into energy is not quite that simple

Relativistic vs Classical kinetic energy



Momentum and Energy Relation

$$K = mc^2 - m_0c^2$$

$$E_0 = m_0c^2 \text{ is the rest mass energy.}$$

$$E = K + E_0$$

$$\vec{p} = m\vec{u} = \frac{m_0\vec{u}}{\sqrt{1 - u^2/c^2}}$$

We square this result, multiply by c^2 and rearrange the result to obtain

$$p^2c^2 = m^2u^2c^2 = m^2c^4\left(\frac{u^2}{c^2}\right) = m^2c^4\beta^2 = \Gamma_u^2 m_0^2 c^4 \beta^2$$

Using $\beta^2 = 1 - \frac{1}{\Gamma_u^2}$ we have;

$$p^2c^2 = \Gamma_u^2 m_0^2 c^4\left(1 - \frac{1}{\Gamma_u^2}\right) = \Gamma_u^2 m_0^2 c^4 - m_0^2 c^4$$
$$= E^2 - m_0^2 c^4$$

For Photon m_0=0, therefore, E=pc.

Transformation of Energy and Momentum

Momentum and energy from frame S:

$$p_x = \frac{m_0 u_x}{\sqrt{1-u^2/c^2}} \qquad p_y = \frac{m_0 u_y}{\sqrt{1-u^2/c^2}} \qquad p_z = \frac{m_0 u_z}{\sqrt{1-u^2/c^2}} \qquad E = \frac{m_0 c^2}{\sqrt{1-u^2/c^2}}$$

Momentum and energy from frame S':

$$p'_x = \frac{m_0 u'_x}{\sqrt{1-u'^2/c^2}} \qquad p'_y = \frac{m_0 u'_y}{\sqrt{1-u'^2/c^2}} \qquad p'_z = \frac{m_0 u'_z}{\sqrt{1-u'^2/c^2}} \qquad E' = \frac{m_0 c^2}{\sqrt{1-u'^2/c^2}}$$

Using velocity transformation relations:

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad u'_y = \frac{u_y}{\gamma_v\left(1 - \frac{u_x v}{c^2}\right)} \qquad u'_z = \frac{u_z}{\gamma_v\left(1 - \frac{u_x v}{c^2}\right)}$$

$$\boxed{\gamma_v = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}}$$

One can express:

$$\frac{1}{\sqrt{1-u^2/c^2}} = \frac{1 + u'_x v/c^2}{\sqrt{1-u'^2/c^2}\sqrt{1-v^2/c^2}}$$

Using the above quantities we will obtain the relation between the momentum and energy measured in the frames S and S'.

Transformation of Energy and Momentum

$$p'_x = \frac{1}{\sqrt{1 - v^2/c^2}}\left(p_x - \frac{Ev}{c^2}\right)$$

$$p'_y = p_y$$

$$p'_z = p_z$$

$$E' = \frac{1}{\sqrt{1 - v^2/c^2}}(E - vp_x)$$

$$p_x = \frac{1}{\sqrt{1 - v^2/c^2}}\left(p'_x + \frac{E'v}{c^2}\right)$$

$$p_y = p'_y$$

Show this using the relations discussed in the previous slide

$$p_z = p'_z$$

$$E = \frac{1}{\sqrt{1 - v^2/c^2}}(E' + vp'_x)$$

Note the equilvalence of above relations with the space-time transformations.

$$x' = \frac{1}{\sqrt{1 - v^2/c^2}}(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}(t - vx/c^2)$$

$$x = \frac{1}{\sqrt{1 - v^2/c^2}}(x' + v't)$$

$$y = y'$$

$$z = z'$$

$$t = \frac{1}{\sqrt{1 - v^2/c^2}}(t' + vx'/c^2)$$

Energy-Momentum relation: Four Vector

We had noted earlier that the combination of time and spatial coordinates Collectively can be considered as components of a 4-dimentional vector.

$$X \equiv (x_0 = ct, x, y, z)$$

Under Lorentz tranformation, this vector transforms as $X' = L \cdot X$

In a similar way, we can consider energy and momentum of an object together as components of a 4-vector

$$p \equiv \left(p_0 = \frac{E}{c},\ p_x,\ p_y,\ p_z\right)$$

Generalized transformation will be

$$\begin{pmatrix} p_0' \\ p_x' \\ p_y' \\ p_z' \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v\beta & 0 & 0 \\ -\gamma_v\beta & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_0 \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

$$\begin{pmatrix} x'_0 \\ x' \\ y' \\ z' \end{pmatrix} = \begin{bmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_x\beta_y}{\beta^2} & 1+(\gamma-1)\frac{\beta_y^2}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\frac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\frac{\beta_y\beta_z}{\beta^2} & 1+(\gamma-1)\frac{\beta_z^2}{\beta^2} \end{bmatrix} \begin{pmatrix} x_0 \\ x \\ y \\ z \end{pmatrix}$$

$$\mathbf{p}' = \mathbf{L} \cdot \mathbf{p}$$

Momentum-energy relation

$$E^2 - p^2c^2 = m_0^2c^4$$

$E^2 - p^2c^2$ is an invariant quantity as can be seen from the above equation.

When a particles velocity is zero and it has no momentum. The particle still have some rest energy and that is $E_0 = m_0c^2$.

Similar to the invariance of separation of event ($dS'^2 = dS^2$), we can have the similar relation as follows:

$$p^2 = \frac{E^2}{c^2} - \vec{p} \cdot \vec{p} = \left(\vec{p}^2 + m_0^2c^2\right) - \vec{p}^2 = m_0^2c^2$$

$$p'^2 = \frac{E'^2}{c^2} - \vec{p}' \cdot \vec{p}' = \left(\vec{p}'^2 + m_0^2c^2\right) - \vec{p}'^2 = m_0^2c^2$$

$$E^2 - p^2c^2 = E'^2 - p'^2c^2$$ (Exercise)

Example

A stationary body explodes into two fragments each of mass 1.0 kg that move apart at speeds of 0.6c relative to the original body. Find the mass of the original body.

$$E_0 = m_0 c^2 = \Gamma m_1 c^2 + \Gamma m_2 c^2 = \frac{m_1 c^2}{\sqrt{1 - v_1^2/c^2}} + \frac{m_2 c^2}{\sqrt{1 - v_2^2/c^2}}$$

and

$$m_0 = \frac{E_0}{c^2} = \frac{(2)(1.0\text{ kg})}{\sqrt{1 - (0.60)^2}} = 2.5\text{ kg}$$

Example

An electron ($m_0 = 0.511$ MeV/c^2) and a photon ($m_0 = 0$) both have momenta of 2.000 MeV/c. Find the total energy of each.

The electron's total energy is

$$E = \sqrt{m_0{}^2c^4 + p^2c^2} = \sqrt{(0.511\ \text{MeV}/c^2)^2c^4 + (2.000\ \text{MeV}/c)^2c^2}$$
$$= \sqrt{(0.511\ \text{MeV})^2 + (2.000\ \text{MeV})^2} = 2.064\ \text{MeV}$$

The photon's total energy is

$$E = pc = (2.000\ \text{MeV}/c)c = 2.000\ \text{MeV}$$

Example

In an experiment, an electron and a positron are collided to produce a massive object, Z, of mass $M_Z = 2\times10^{-25}\ kg$

The mass of the electron and positron are the same, and can be taken to be

$$m_{e-} = m_{e+} = 10^{-30}\ kg$$

The momenta of the electron and positron are the same in magnitude, and opposite in direction, so that Z is produced at rest (zero momentum).

$$\vec{p}_{e+} = -\vec{p}_{e-}$$

What is the (magnitude of the) momentum of the electron ?

Take the value of speed of light to be $c = 3\times10^{8}\ m/s$

Solution:

Let $\vec{p}_{e+} = -\vec{p}_{e-} = p_x \hat{x}$ $\quad m_{e^+} = m_{e^-} \Rightarrow \vec{u}_{e^+} = -\vec{u}_{e^-}$ $\quad E = \dfrac{\vec{p}\cdot\vec{u}}{|\vec{u}|^2} c^2 \Rightarrow E_{e^+} = E_{e^-}$

In the rest frame of Z: $\vec{p}_Z = 0, \quad E_Z = M_Z c^2$

Energy conservation $\Rightarrow E_{e^+} + E_{e^-} = E_Z \Rightarrow E_{e^+} = E_{e^-} = \dfrac{M_Z c^2}{2} = 9\times 10^{-9}\ J$

$$|\vec{p}_{e-}| = |\vec{p}_{e+}| = \frac{\sqrt{E_{e^+}^2 - m_{e^+}^2 c^4}}{c} = \left(\sqrt{\frac{M_Z^2}{4} - m_{e^+}^2}\right)c = \left(\sqrt{10^{-50} - 10^{-60}}\right)c \approx \frac{E_{e^+}}{c}$$

The speed of electron: $E = \Gamma_u m_0 c^2 \Rightarrow \Gamma_u = E/m_0 c^2 = \dfrac{9\times 10^{-9}}{10^{-30} \times 9 \times 10^{16}} = 10^5$

$$\Rightarrow u = (\sqrt{1 - \Gamma_u^{-2}})c = \sqrt{1 - 10^{-10}})c = 0.99999999995c$$

At these speeds, the energy due to the mass of electron is negligible compared to its kinetic energy. For all practical purposes, we can neglect the mass of the electron at such speeds.

Example

An object moving with velocity $\vec{u} = 0.2c\,\hat{x}$ has momentum $|\vec{p}| = 3\times 10^{-20}\ kg\,m/s$

What is the total energy of the object?

What is the mass of the object?

Solution:

$$|\vec{p}| = p_x = \Gamma_u m_0 u_x = \Gamma_u m_0 0.2c = 3\times 10^{-20} \quad kg\,m/s,$$

$$E = \Gamma_u m_0 c^2 = \frac{\vec{p}\cdot\vec{u}}{|\vec{u}|^2}c^2 = \frac{3\times 10^{-20}}{0.2}c = 0.45\times 10^{-10} \quad J,$$

$$\Gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-0.2^2}} = 1.0206$$

$$E = \Gamma_u m_0 c^2 \Rightarrow \quad m_0 = \frac{E}{\Gamma_u c^2} = \frac{0.45\times 10^{-10}}{1.0206\times 9\times 10^{16}} = 0.4899\times 10^{-27}\ kg$$

We shall continue in the next class

Example 1

A free electron is moving with velocity $\vec{u} = \frac{1}{2\sqrt{2}} c\left(\hat{x} + \hat{y}\right)$ as seen by an observer on earth (frame: S). What is its momentum, total energy and kinetic energy? Mass of electron may be taken to be $m_e = 10^{-30}\ kg$

(a) As seen by the observer in S.
(b) As seen by an observer in S', which is moving with velocity $\vec{v} = 0.2c\ \hat{x}$