PART A ANSWERS:

Question	1	2	3	4	5	6	7
Answer	2	π	$\mathbb{R}-\{0\}$	$\{1\}$ or $\{(1,13)\}$	2	$\frac{\pi}{4}$	$p<3$

PART B ANSWERS:

1. Consider a function f defined as:

$$f(x,y)=\begin{cases}\frac{5x^3y}{(x^2+y^2)^{\frac{3}{2}}} & \text{if} \quad (x,y)\neq(0,0)\\ 0 & \text{if} \quad (x,y)=(0,0)\end{cases}$$

[2pnts.] (a) Check whether f is continuous at $(0,0)$ by using the ϵ-δ definition of continuity of a function at a point.

Soln.:
$|f(x,y)-0| = 5\dfrac{x^2\,|xy|}{(x^2+y^2)^{\frac{3}{2}}} \leq 5\dfrac{(x^2+y^2)\,|xy|}{(x^2+y^2)^{\frac{3}{2}}} \leq 5\dfrac{(x^2+y^2)}{(x^2+y^2)^{\frac{1}{2}}} = 5\sqrt{x^2+y^2}$ for $(x,y)\neq(0,0)$.
Given $\epsilon>0$ take any $\delta<\frac{\epsilon}{5}$, then
$0<\sqrt{x^2+y^2}<\delta \Rightarrow |f(x,y)-0|<\epsilon$,
hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)=0$.
Since $f(0,0)=0$, f is continuous at $(0,0)$.

[3pnts.] (b) If possible find two unit vectors $\mathbf{u}$ and $\mathbf{v}$ such that $D_{\mathbf{u}}f(0,0)$ (the directional derivative of f at $(0,0)$ along $\mathbf{u}$) exists but $D_{\mathbf{v}}f(0,0)$ does not exist.

Soln.:
The directional derivative of f at $(0,0)$ along $\mathbf{u}=\langle 1,0\rangle$ is given by:
$D_{\mathbf{u}}f(0,0)=f_x(0,0)=\lim\limits_{t\to0}\dfrac{f(t,0)-f(0,0)}{t}=0.$
Or choose $\mathbf{u}=<0,1>$ then $D_{\mathbf{u}}f(0,0)=f_y(0,0)=0$.
If $\mathbf{v}=\langle v_1,v_2\rangle$ is such that $v_1\neq0, v_2\neq0$ then
$D_{\mathbf{v}}f(0,0)=\lim\limits_{t\to0}\dfrac{f(t(v_1,v_2))-f(0,0)}{t}$, provided this limit exists.

$$\frac{f(t(v_1,v_2))-f(0,0)}{t}=\frac{\frac{5t^4v_1^3v_2}{(t^2v_1^2+t^2v_2^2)^{\frac{3}{2}}}}{t}=\frac{t5v_1^3v_2}{|t|(v_1^2+v_2^2)^{\frac{3}{2}}}.$$

Since $\lim\limits_{t\to0,t>0}\dfrac{f(t(v_1,v_2))-f(0,0)}{t}=\dfrac{5v_1^3v_2}{(v_1^2+v_2^2)^{\frac{3}{2}}}$ and $\lim\limits_{t\to0,t<0}\dfrac{f(t(v_1,v_2))-f(0,0)}{t}=-\dfrac{5v_1^3v_2}{(v_1^2+v_2^2)^{\frac{3}{2}}}$,
$D_{\mathbf{v}}f(0,0)$ does not exist.

REMARKS:
Question 1(a): Marks are deducted for not writing the $\epsilon-\delta$ definition of continuity or for giving wrong definition.
If modulus signs are not given in appropriate places, then marks are deducted.

Question 1(b): No mark is given for first writing that $D_u f(0,0) = (\nabla f) \cdot \mathbf{u}$ and then wrongly concluding that $D_u f(0,0)$ exists for all unit vectors u at $(0,0)$.
However if it wrongly concluded while trying to derive from the definition that $D_u f(0,0)$ exists for all unit vectors u at $(0,0)$, then one mark is awarded.
Similarly no mark is awarded if f_x, f_y is calculated for $(x,y) \neq (0,0)$ and then substituting in those expressions $(x,y) = (0,0)$ to get the partial derivatives at $(0,0)$.

[2pnts.] 2. (a) Find the equations of the tangent plane and normal line to the level surface S given by the equation $f(x,y,z) = 8$, at the point $(1,2,1)$, where $f(x,y,z) = x^2 + y^2 - z^2 + 3xz$.

Soln.:
Since the level surface is of the form $f(x,y,z) = 8$,
where $f(x,y,z) = x^2 + y^2 - z^2 + 3xz$,
the tangent plane of the level surface at $(x_0, y_0, z_0) = (1,2,1)$ is of the form:
$(z - z_0)f_z(x_0, y_0, z_0) + (x - x_0)f_x(x_0, y_0, z_0) + (y - y_0)f_y(x_0, y_0, z_0) = 0.$ **(1)**
where $f_x(x_0, y_0, z_0) = 2x_0 + 3z_0,\ f_y(x_0, y_0, z_0) = 2y_0$
and $f_z(x_0, y_0, z_0) = -2z_0 + 3x_0$.
(1) implies $\Rightarrow (z-1)1 + (x-1)5 + (y-2)4 = 0$ or $z + 5x + 4y = 14$.

The symmetric equations of the normal line is given by:
$$\frac{(z - z_0)}{f_z(x_0, y_0, z_0)} = \frac{(x - x_0)}{f_x(x_0, y_0, z_0)} = \frac{(y - y_0)}{f_y(x_0, y_0, z_0)}, \quad \text{or}$$
$$\frac{(z-1)}{1} = \frac{(x-1)}{5} = \frac{(y-2)}{4}.$$

[1pnts.] (b) If $\mathbf{r}(t) = \langle t-1, 2t^2, t^3 \rangle$, $0 \leq t \leq 2$ is a space curve C, then find $\frac{d}{dt}(f(\mathbf{r}(t)))$ at $t = 1$, where f is as in part (a) above.

Soln.:
By chain rule, $\dfrac{d}{dt}(f(\mathbf{r}(t))) = \dfrac{\partial f}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial f}{\partial y}\dfrac{dy}{dt} + \dfrac{\partial f}{\partial z}\dfrac{dz}{dt}$
$= (2(t-1) + 3t^3) \times 1 + (4t^2) \times 4t + (-2t^3 + 3(t-1)) \times 3t^2.$
Hence $\dfrac{d}{dt}(f(\mathbf{r}(t)))|_{t=1} = 13.$

3. Let $f(x,y) = x^2 + y^2 + x^2y - 4$ be a scalar function and $R = \{(x,y) \mid x^2 + y^2 \leq 12\}$.

[2pnts.] (a) Find all points of local minimum and local maximum of f in the interior of R (interior of R is defined by the constraint, $x^2 + y^2 < 12$).

Soln.:
Since $f(x,y) = x^2 + y^2 + x^2y - 4$,
$f_x(x,y) = 2x + 2xy$ and $f_y(x,y) = 2y + x^2$.
The critical points are given by the following equations:
$f_x(x,y) = f_y(x,y) = 0$, which gives $x(y+1) = 0$ and $2y = -x^2$,
$\Rightarrow (0,0), (\sqrt{2}, -1), (-\sqrt{2}, -1)$ are the only critical points of f, all in the interior of R.
$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (2 + 2y)(2) - (2x)^2 = 4(1 + y - x^2)$.
At $(0,0)$, $D(0,0) = 4 > 0$, and $f_{xx}(0,0) = 2 > 0$ hence $(0,0)$ is a point of local minimum.
Since $D(\sqrt{2}, -1) = D(-\sqrt{2}, -1) < 0$, $(\sqrt{2}, -1), (-\sqrt{2}, -1)$ are saddle points of f hence $(0,0)$ is the only point of local minimum in the interior of R.

[3pnts.] (b) Find the absolute minimum value (m) and the absolute maximum value (M) of f in R.

Soln.:
To find critical points on the boundary of R we need to use Lagrange's multiplier technique.
Let λ be such that
$\nabla f(x,y) = \lambda \nabla g(x,y)$, where $g(x,y) = x^2 + y^2 - 12$.
$\Rightarrow (2x + 2xy, 2y + x^2) = \lambda(2x, 2y)$ which gives:
$2x(y+1) = \lambda(2x)$ (1)
$2y + x^2 = \lambda(2y)$ (2).
Since $(0,0)$ does not satisfy $x^2 + y^2 = 12$, so at least one of x, y is nonzero.

If $x \neq 0$ then from (1) it follows, $\lambda = y + 1$.
From (2) $y \neq 0$ and $2y + (12 - y^2) = 2y(y+1)$ or $y = \pm 2$ which implies $x = \pm 2\sqrt{2}$, and gives the points $(\pm 2\sqrt{2}, \pm 2)$.

If $x = 0$ then (2) implies $\lambda = 1$ and $y = \pm 2\sqrt{3}$ and gives the points $(0, \pm 2\sqrt{3})$
Since $f(0,0) = -4$, $f((\pm 2\sqrt{2}, 2)) = 24$, $f((\pm 2\sqrt{2}, -2)) = -8$, $f(0, 2\sqrt{3}) = 8 = f(0, -2\sqrt{3})$, hence the absolute maximum and minimum values are 24 and -8.

[1pnts.] (c) Is it possible to find a rectangular region D inside the circle $x^2 + y^2 = 12$, such that the absolute minimum value of f in D is $m + \frac{1}{10}$ and the absolute maximum value of f in D is $M + \frac{1}{13}$ (where m and M are as defined in part (b)) ? Justify.

Soln.:
Since $D \subseteq R$, if M is the absolute maximum of f in R, $f(x,y) \leq M < M + \frac{1}{13}$ for all $(x,y) \in D$, hence $M + \frac{1}{13}$ cannot be the maximum value of f in D.

REMARKS:

Question 3(b): If any set of points is missed out as solutions of lagrange's equations (even if they do not give the absolute minimum or maximum) one mark is deducted.

Question 3(c): No mark is awarded for writing just Yes or No. Marks given only if the justification is correct.

[2pnts.] 4. (a) Evaluate $\iiint_E x^2 dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$, and below the cone $z^2 = 9x^2 + 9y^2$.

Soln.:
By using cylindrical coordinates the required integral is given by:

$$\int_0^2 \int_0^{2\pi} \int_0^{3r} r^2 \cos^2\theta \left| J\left(\frac{x,y,z}{r,\theta,z}\right) \right| dz d\theta dr$$

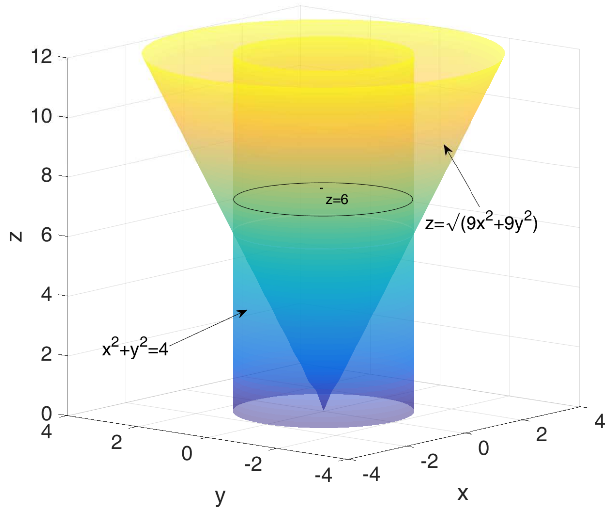
$$= \int_0^2 \int_0^{2\pi} \int_0^{3r} r^3 \cos^2\theta dz d\theta dr$$

$$= \int_0^2 \int_0^{2\pi} r^3 \cos^2\theta \times 3r d\theta dr$$

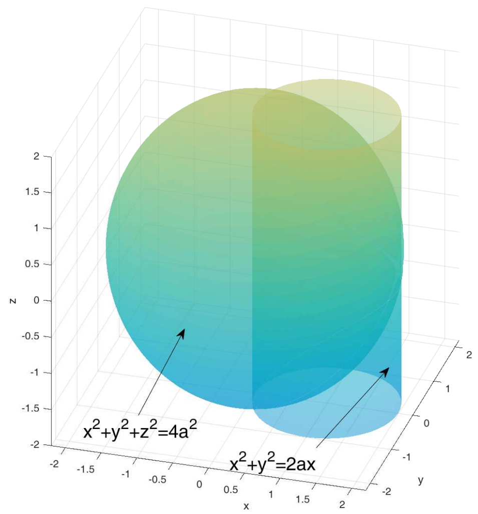
$$= \int_0^2 \int_0^{2\pi} 3r^4 \frac{(\cos 2\theta + 1)}{2} d\theta dr$$

$$= 3\left[\frac{r^5}{5}\right]_0^2 \left[\frac{\sin 2\theta}{4} + \frac{\theta}{2}\right]_0^{2\pi}$$

$= \frac{96\pi}{5}$.



[3pnts.] (b) Find the area of the surface of the portion of the sphere $x^2 + y^2 + z^2 = 4a^2$, that lies inside the cylinder $x^2 + y^2 = 2ax$.



Soln.:
Let $A(S)$ be the surface area of the portion of the surface that lies above the plane $z = 0$. Then, the required surface area is nothing but twice the area $A(S)$ of the graph of the function $z = \sqrt{4a^2 - x^2 - y^2}$ inside the cylinder $x^2 + y^2 = 2ax$.
Thus the surface can be parametrized by $x = x$, $y = y$ and $z = \sqrt{4a^2 - x^2 - y^2}$.
$\therefore \dfrac{\partial z}{\partial x} = \dfrac{1}{2}(4a^2 - x^2 - y^2)^{-\frac{1}{2}}(-2x) = -\dfrac{x}{\sqrt{4a^2 - x^2 - y^2}}$,
$\dfrac{\partial z}{\partial y} = \dfrac{1}{2}(4a^2 - x^2 - y^2)^{-\frac{1}{2}}(-2y) = -\dfrac{y}{\sqrt{4a^2 - x^2 - y^2}}$.
Thus $A(S) = \displaystyle\iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$, where D is the region in the xy plane bounded by the circle $x^2 + y^2 = 2ax$.

Changing to polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, the surface area reduces to

$$A(S) = \iint_D \frac{2|a|}{\sqrt{4a^2 - x^2 - y^2}} dA = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{r=0}^{2a\cos\theta} \frac{2|a|}{\sqrt{4a^2 - r^2}} r\,dr\,d\theta$$

$$= -2|a| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left[\left(4a^2 - r^2\right)^{\frac{1}{2}}\right]_0^{2a\cos\theta} d\theta = -2|a| \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2|a\sin\theta| - 2|a|)d\theta$$

$$= 4a^2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (1 - |\sin\theta|)d\theta = 8a^2 \int_0^{\frac{\pi}{2}} (1 - \sin\theta)d\theta = 8a^2 \left[\theta + \cos\theta\right]_0^{\frac{\pi}{2}} = 8a^2 \left(\frac{\pi}{2} - 1\right).$$

Here we have assumed that that $a > 0$ so that $-\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$.

Hence the required surface area is $8a^2(\pi - 2)$.

ALITER: We can use the usual parametrization of the surface of the sphere by the vector $\mathbf{r}(\phi, \theta) = 2|a|\sin\phi\cos\theta\hat{\mathbf{i}} + 2|a|\sin\phi\sin\theta\hat{\mathbf{j}} + 2|a|\cos\phi\hat{\mathbf{k}}$ so that

$\mathbf{r}_\phi = 2|a|\cos\phi\cos\theta\hat{\mathbf{i}} + 2|a|\cos\phi\sin\theta\hat{\mathbf{j}} - 2|a|\sin\phi\hat{\mathbf{k}}$ and $\mathbf{r}_\theta = -2|a|\sin\phi\sin\theta\hat{\mathbf{i}} + 2|a|\sin\phi\cos\theta\hat{\mathbf{j}}$.

Hence $|\mathbf{r}_\phi \times \mathbf{r}_\theta| = 4a^2\sin\phi$.

Now, for the region D in the ϕ-θ plane, $0 \le \phi \le \dfrac{\pi}{2}$ and for each fixed ϕ, $x^2 + y^2 \le 2ax$, $4a^2\sin^2\phi \le 4a|a|\sin\phi\cos\theta \Rightarrow \sin\phi(\sin\phi - \cos\theta) \le 0$. But $0 \le \phi \le \dfrac{\pi}{2}$, so $\cos\theta \ge \sin\phi$ or $\phi - \dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2} - \phi$.

Hence $D = \left\{(\phi, \theta) |\ 0 \le \phi \le \dfrac{\pi}{2},\ \phi - \dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2} - \phi\right\}$.

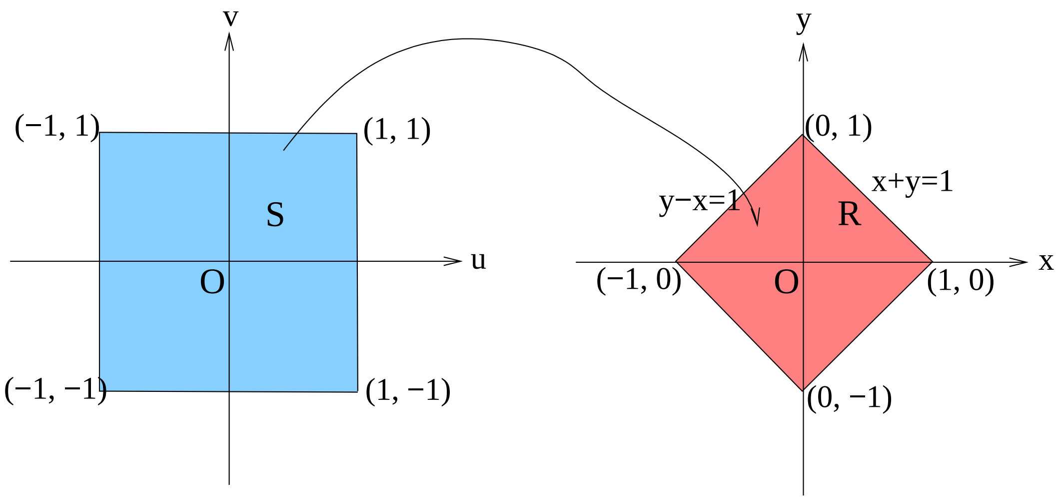
Therefore $A(S) = \displaystyle\iint_D |\mathbf{r}_\phi \times \mathbf{r}_\theta| dA$

$$= \int_{\phi=0}^{\frac{\pi}{2}} \int_{\theta=\phi-\frac{\pi}{2}}^{\frac{\pi}{2}-\phi} 4a^2\sin\phi\, d\theta\, d\phi = 4a^2 \int_0^{\frac{\pi}{2}} (\pi - 2\phi)\sin\phi\, d\phi$$

$$= 4a^2\left[(-\pi\cos\phi) - 2(\phi\cos\phi + \sin\phi)\right]_0^{\frac{\pi}{2}} = 4a^2(\pi - 2)$$

Hence the required surface area is $8a^2(\pi - 2)$.

[3pnts.] (c) Evaluate $\displaystyle\iint_R e^{x+y} dA$, by making an appropriate change of variables, where R is the region given by the inequality $|x| + |y| \le 1$.



Soln.:

Let $u = x + y$ and $v = -x + y$. Then $u + v = 2y \quad \Rightarrow y = \frac{1}{2}(u + v)$ and likewise $u - v = 2x \quad \Rightarrow x = \frac{1}{2}(u - v)$. $\therefore \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{2}$.

Now, $|u| = |x + y| \leq |x| + |y| \leq 1 \quad \Rightarrow -1 \leq u \leq 1$. Note that $x + y = 1$ yields $u = 1$ and $-x - y = 1$ yields $u = -1$. Similarly $|v| = |-x + y| \leq |x| + |y| \leq 1 \quad \Rightarrow -1 \leq v \leq 1$.

Thus R is the image of the square region S with vertices $(1, 1)$, $(-1, 1)$, $(1, -1)$ and $(-1, -1)$ as shown in the figure.

$$\therefore \iint\limits_R e^{x+y} dA = \iint\limits_S \left|\frac{\partial(x, y)}{\partial(u, v)}\right| e^u dA = \frac{1}{2} \int_{-1}^{1} \int_{-1}^{1} e^u du dv = \frac{1}{2} \left[e^u\right]_{-1}^{1} \left[v\right]_{-1}^{1} = e - e^{-1}.$$

REMARKS:

1. **Finding the value of the integral without using transformation will fetch you only partial mark.**

5. Let $\mathbf{F}(x, y) = -y\mathbf{i} + y^2 x\mathbf{j}$ be a vector field.

[1pnts.] (a) Check whether there is any nonempty open set $D \subseteq \mathbb{R}^2$ such that $\mathbf{F}$ is a conservative vector field in D.

Soln.:

Since $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$ (where $P(x, y) = -y$ and $Q(x, y) = y^2 x$) are continuous functions but $\frac{\partial P}{\partial y} = -1 \neq \frac{\partial Q}{\partial x} = y^2$, for all $(x, y) \in \mathbf{R}^2$, hence there does not exist any nonempty open set $D \subseteq \mathbf{R}^2$ in which F is a conservative vector field.

[2pnts.] (b) **If possible** find two smooth simple closed curves C_1 and C_2 each of arc length one, such that $\oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$ and $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} \neq 0$.

Soln.:

If C is a smooth positively oriented simple closed curve then by Green's theorem, $\int\int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA = \int_C P dx + Q dy$ where R is the region enclosed by C.

Since for all $(x, y) \in \mathbf{R}^2$, $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \geq 1, \quad \int\int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dA \geq Area(R) > 0$.

Hence it is not possible to find a C_1 such that $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0$.

For C_2 you can choose a circle with circumference equal to 1.

REMARKS:

1. **Simply giving an example without showing that $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} \neq 0$ will not be awarded any mark.**
2. **Results from part (a) does not guarantee you a non-zero value for non-conservative vector fields. Simply citing part (a) to answer part (b) this way won't fetch you any mark. For example consider the vector field $< 0, x^2 >$ around the unit circle.**
3. **Using piecewise smooth curves for answering part (b) won't be rewarded any mark.**

[4pnts.] 6. **By using** Green's theorem find the area of the enclosed region R bounded below by the parabola $y = x^2$ and above by the circle $x^2 + y^2 = 2$.

Soln.:

If we choose $P = 0$ and $Q = x$, then by Green's theorem

$Area(R) = \int_C Qdy$

where C is the piecewise smooth curve which forms the boundary of R. Note that the points where the parabola cuts the circle are given by $(x, y) = (\pm 1, 1)$.

Also a parametrization of the curve C is given by:

$\mathbf{r}(t) = \langle t, t^2 \rangle$ for $0 \le t \le 1$

$= \left\langle \sqrt{2}cos(\frac{\pi t}{4}), \sqrt{2}sin(\frac{\pi t}{4}) \right\rangle$ for $1 \le t \le 3$

$= \langle t - 4, (t-4)^2 \rangle$ for $3 \le t \le 4$.

$= \int_0^1 t \times (2t)dt + \int_1^3 \sqrt{2}cos(\frac{\pi t}{4})\sqrt{2}(\frac{\pi}{4})(cos(\frac{\pi t}{4}))dt + \int_3^4 (t-4) \times (2(t-4))dt$

$= \frac{\pi}{2} + \frac{1}{3}.$

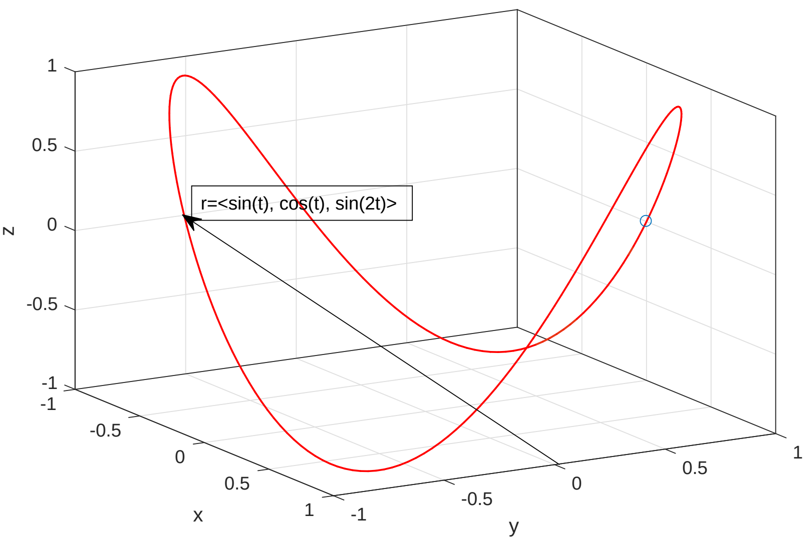
REMARKS:

1. **Finding area without using Green's theorem won't fetch you any mark.**
2. **Simply using Green's theorem without identifying P and Q will be rewarded only partial mark.**

[3pnts.] 7. (a) Evaluate

$$\oint_C (y + \sin x)dx + (z^2 + \cos y)dy + x^2 dz$$

where C is the curve $\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle$, $0 \le t \le 2\pi$.



Soln.:

Let C be the boundary of the surface S : $z = 2xy$ over the region D given by $x^2 + y^2 \le 1$ in $\mathbb{R}^2$. Then C is given by $\mathbf{r}(t) =< \sin t, \cos t, 2\sin t \cos t > \ 0 \le t \le 2\pi$.

Therefore, applying Stokes' theorem,

$$\oint_C (y+\sin x)dx + (z^2+\cos y)dy + x^2 dz$$

$$= \oint_C \mathbf{F}\cdot d\mathbf{r}, \quad \text{where} \quad \mathbf{F} =< y+\sin x, z^2+\cos y, x^2 >$$

$$= \iint_S (\nabla\times\mathbf{F})\cdot\hat{\mathbf{n}}dS, \text{ where } S \text{ is the surface } z = 2xy \text{ with } (x,y)\in D$$

$$= \iint_D (-Pz_x - Qz_y + R)\, dA \text{ with } P,\ Q,\ R \text{ being components of } \nabla\times\mathbf{F}.$$

Now, $\nabla\times\mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y+\sin x & z^2+\cos y & x^2 \end{vmatrix} = -2z\hat{\mathbf{i}} - 2x\hat{\mathbf{j}} - \hat{\mathbf{k}}.$

$\therefore$ the required integral

$$= \iint_D \{-(-2z)(2y) - (-2x)(2x) - 1\}\, dA = \iint_D (8xy^2 + 4x^2 - 1)dA \text{ [as } z = 2xy]$$

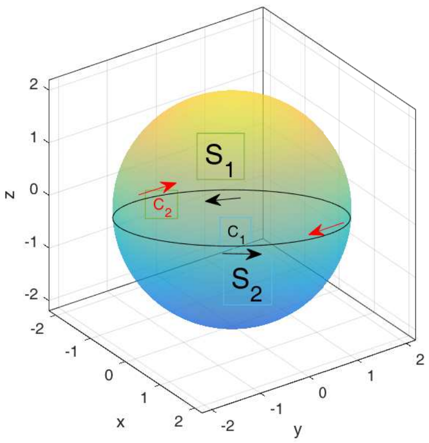
$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (8r^3\cos\theta\sin^2\theta + 4r^2\cos^2\theta - 1)rdrd\theta$$

$$= \int_{\theta=0}^{2\pi}\left[\frac{8}{5}\cos\theta\sin^2\theta + \cos^2\theta - \frac{1}{2}\right]d\theta = \int_{\theta=0}^{2\pi}\left[\frac{8}{5}\cos\theta\sin^2\theta + \frac{1}{2}\cos 2\theta\right]d\theta$$

$$= \left[\frac{8}{15}\sin^3\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = 0.$$

[2pnts.] (b) Without using Gauss' divergence theorem, show that if S is a sphere and $\mathbf{F}$ satisfies the hypothesis of Stoke's Theorem,

$$\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = 0.$$



Soln.:

The surface S of a sphere can be considered as the union of two hemispheres S_1 and S_2 lying respectively above and below the xy plane. Therefore, by applying Stokes' theorem

$$\iint_S (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \iint_{S_1\cup S_2} (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \iint_{S_1} (\nabla\times\mathbf{F})\cdot d\mathbf{S} + \iint_{S_2} (\nabla\times\mathbf{F})\cdot d\mathbf{S} = \oint_{C_1}\mathbf{F}\cdot d\mathbf{r} + \oint_{C_2}\mathbf{F}\cdot d\mathbf{r},$$

where C_1 and C_2 are the great circles having positive orientations bounding the upper and the lower hemispheres respectively.

But they are the same circles having opposite orientations. $\therefore$ $C_1 = -C_2$ so that

$$\iint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} + \oint_{-C_1} \mathbf{F} \cdot d\mathbf{r} = \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} - \oint_{C_1} \mathbf{F} \cdot d\mathbf{r} = 0.$$

8. Let S be a closed surface and let $\mathbf{r}$ denote the position vector of any point (x, y, z) measured from the origin O. Prove that

$$\iint_S \frac{\hat{\mathbf{n}} \cdot \mathbf{r}}{|\mathbf{r}|^3} dS$$

is equal to:

[3pnts.] (a) zero if O lies outside S;

[3pnts.] (b) 4π if O lies inside S (**Hint:** surround O with a very small sphere).

Soln.:

(a) When O is outside S, $|\mathbf{r}| \neq 0$ and hence the integrand $\dfrac{\hat{\mathbf{n}} \cdot \mathbf{r}}{|\mathbf{r}|^3}$ is defined inside S. Applying Gauss's divergence theorem,

$$\iint_S \frac{\hat{\mathbf{n}} \cdot \mathbf{r}}{|\mathbf{r}|^3} dS = \iint_S \left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) \cdot \hat{\mathbf{n}} dS = \iiint_V \nabla \cdot \left(\frac{\mathbf{r}}{|\mathbf{r}|^3}\right) dV$$

Now, $\nabla \cdot \left(\dfrac{\mathbf{r}}{|\mathbf{r}|^3}\right) = \nabla \cdot < x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}, y\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}, z\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} >$

$= \sum \dfrac{\partial}{\partial x}\left\{x\left(x^2+y^2+z^2\right)^{-\frac{3}{2}}\right\}$

$= \sum \left\{\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} + x\left(-\dfrac{3}{2}\right)\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}(2x)\right\}$

$= \sum \left\{\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} - 3x^2\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}\right\}$

$= 3\left(x^2+y^2+z^2\right)^{-\frac{3}{2}} - 3\left(x^2+y^2+z^2\right)\left(x^2+y^2+z^2\right)^{-\frac{5}{2}}$

$= 0.$

Hence, $\displaystyle\iint_S \frac{\hat{\mathbf{n}} \cdot \mathbf{r}}{|\mathbf{r}|^3} dS = 0.$

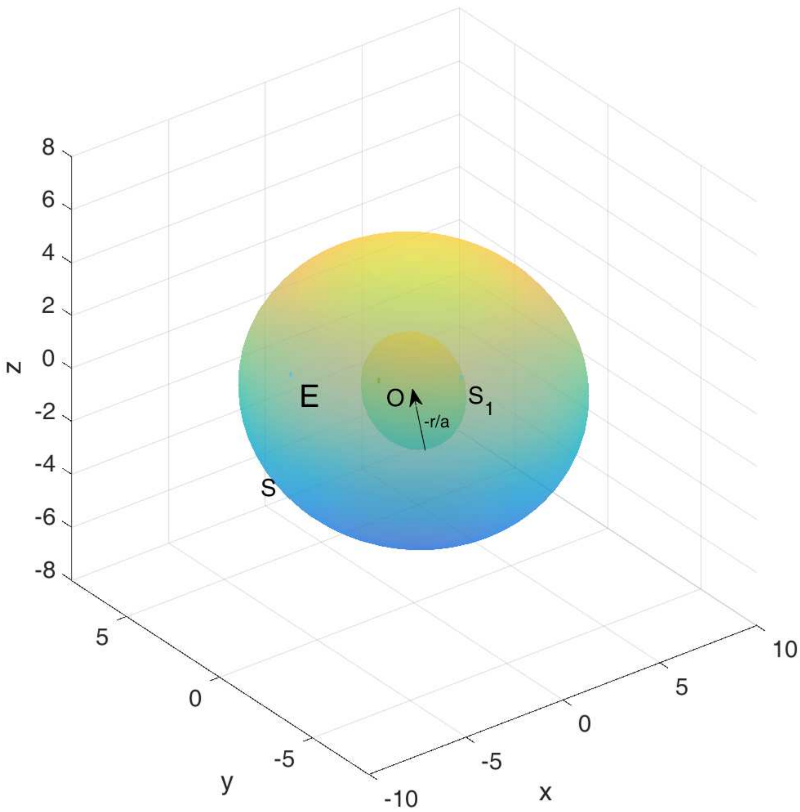
Soln.:

(b) If the origin O is inside S, then the integrand is not defined at O. Therefore, we surround O by a very small sphere S_1 of radius a.

Let E be the region bounded by the sphere S_1 and the given surface S. Note that inner side of the surface of the sphere S_1 constitutes the surface bounding a part of the region E. The outward unit normal to this surface is given by $\hat{\mathbf{n}} = -\dfrac{\mathbf{r}}{|\mathbf{r}|} = -\dfrac{\mathbf{r}}{a}$.

As E excludes the origin, using the result of **(a)** above, we get

$\displaystyle\iint_{S \cup S_1} \frac{\hat{\mathbf{n}} \cdot \mathbf{r}}{|\mathbf{r}|^3} dS = 0$



$$\Rightarrow \iint_S \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{|\mathbf{r}|^3}dS + \iint_{S_1} \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{|\mathbf{r}|^3}dS = 0$$

$$\Rightarrow \iint_S \frac{\hat{\mathbf{n}}\cdot\mathbf{r}}{|\mathbf{r}|^3}dS = -\iint_{S_1} \frac{\left(-\frac{\mathbf{r}}{a}\right)\cdot\mathbf{r}}{|\mathbf{r}|^3}dS$$

$$= \iint_{S_1} \frac{|\mathbf{r}|^2}{a|\mathbf{r}|^3}dS = \frac{1}{a^2}\iint_{S_1} dS = \frac{1}{a^2}(4\pi a^2) = 4\pi,$$

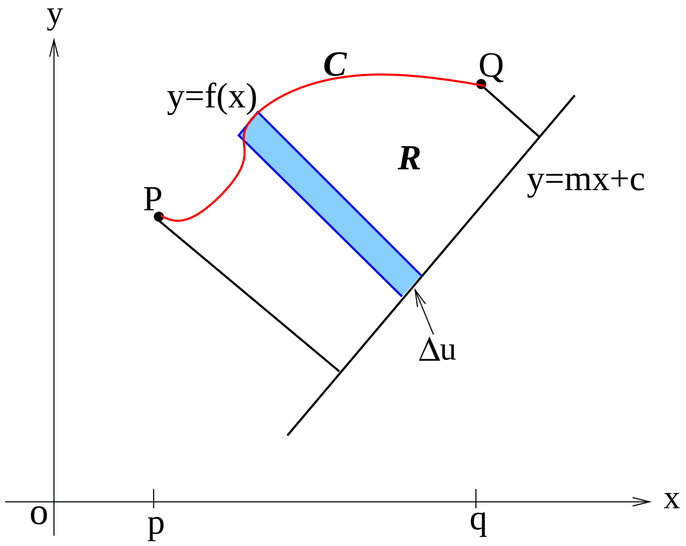
where we have made use of the fact that on the sphere S_1, $|\mathbf{r}| = a$ and the surface area of a sphere of radius "a" is given by $4\pi a^2$.

REMARKS:

1. **Many of you have done part (b) first considering the surface of a sphere independently and then proved (a) as a consequence of (b). That does not work.**
2. **Simply finding the surface area of a sphere without connecting the original surface S won't fetch you mark. Your answer makes no sense if the sphere being considered is not related to the original surface.**
3. **Considering a sphere of radius one won't fetch you full mark. It has to work for any sphere of small radius.**
4. **Roping in Gauss's theorem from Physics, for that matter, the concept of solid angle and applying it here won't fetch you any mark. If you want to use it,**

you need to prove every statement mathematically, including all the earlier concepts in use.

5. **Considering the sphere as a circle and using line integrals to forcibly get the final result won't fetch you any mark.**



[5pnts.] 9. Let C be the arc of the curve $y = f(x)$ between the points $P(p, f(p))$ and $Q(q, f(q))$. Let R be the region bounded by C, by the line $y = mx + c$ (which lies entirely below C), and by the perpendiculars to the line from P and Q (see the figure above). By using the hints in the figure below (such that the volume is $\lim_{n\to\infty} \sum_{i=1}^{n} \pi\{g(x_i)\}^2 \Delta u$) or otherwise, show that the volume of the solid obtained by rotating R about the line $y = mx + c$ is

$$\frac{\pi}{(1+m^2)^{\frac{3}{2}}} \int_p^q [f(x) - mx - c]^2 [1 + mf'(x)] dx$$

.

Soln.:

Since the line $y = mx + c$ makes an angle β with the positive x-axis, $\therefore \tan\beta = m$.

Also, the tangent to the curve $C:\ y = f(x)$ at the point $P_i(x_i, f(x_i))$ makes an angle α with the positive x-axis. $\therefore \tan\alpha = f'(x_i)$.

Let $g(x_i)$ be the length of the perpendicular drawn from the point P_i to the line $y = mx + c$ and M_i be the point where the perpendicular from the point P_i to the x-axis meets the line $y = mx + c$.

Then $\overline{P_iM_i} = f(x_i) - (mx_i + c)$ and

$\overline{P_iL_i} = g(x_i) = \overline{P_iM_i} \cos\beta = \dfrac{f(x_i) - mx_i - c}{\sqrt{1 + \tan^2\beta}} = \dfrac{f(x_i) - mx_i - c}{\sqrt{1 + m^2}}$.

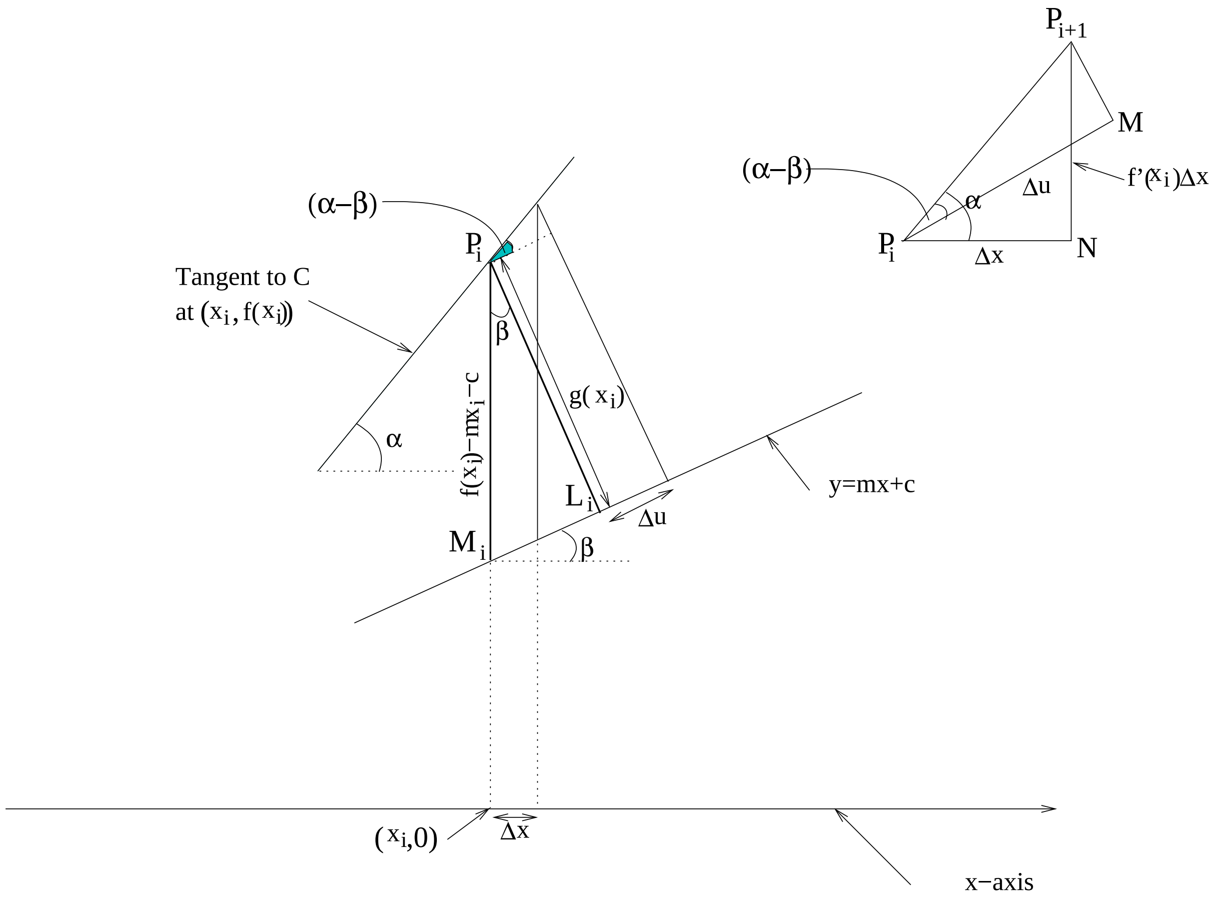
Again, from the triangle $P_iP_{i+1}M$,

$\Delta u = \overline{P_iP_{i+1}} \cos(\alpha - \beta) = \overline{P_iP_{i+1}}(\cos\alpha\cos\beta + \sin\alpha\sin\beta) \quad ....(A)$

Also from the right angled triangle $P_iP_{i+1}N$,

$\Delta x = \overline{P_iP_{i+1}} \cos\alpha$ and $\Delta y = f'(x_i)\Delta x = \overline{P_iP_{i+1}} \sin\alpha$.

Making use of the above and the fact that $\cos\beta = \dfrac{1}{\sqrt{1+m^2}}$ and $\sin\beta = \dfrac{m}{\sqrt{1+m^2}}$, equation (A) yields



$$\Delta u = \frac{1}{\sqrt{1+m^2}} \left\{1 + mf'(x_i)\right\} \Delta x.$$

$\therefore$ the volume obtained by rotating the region R about the line $y = mx + c$ is

$$\lim_{n\to\infty} \sum_{i=1}^{n} \pi\{g(x_i)\}^2 \Delta u$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \pi \frac{[f(x_i) - mx_i - c]^2}{1+m^2} \frac{[1 + mf'(x_i)]}{\sqrt{1+m^2}} \Delta x$$

$$= \frac{\pi}{(1+m^2)^{\frac{3}{2}}} \int_p^q [f(x) - mx - c]^2 [1 + mf'(x)]\, dx.$$

Hence the result.

REMARKS:

1. **Many of you have worked out this problem without referring to any figure. Only partial mark is rewarded for that.**

2. **Without giving any justifications, many of you have recognized** P_iL_i **as** $\dfrac{f(x) - mx - c}{\sqrt{1+m^2}}$ **and** Δu **as** $\dfrac{1 + mf'(x)}{\sqrt{1+m^2}}$ **and plugged that in in the given hint. You would be awarded at most** $\dfrac{1}{2}$ **mark only for that.**