PH 102, Electromagnetism,

Post Mid Semester

Lecture 6

Electrodynamics

Electromagnetic induction:

Faraday’s Law, inductance
and
Energy in magnetic field.

D. J. Griffiths: 7.2

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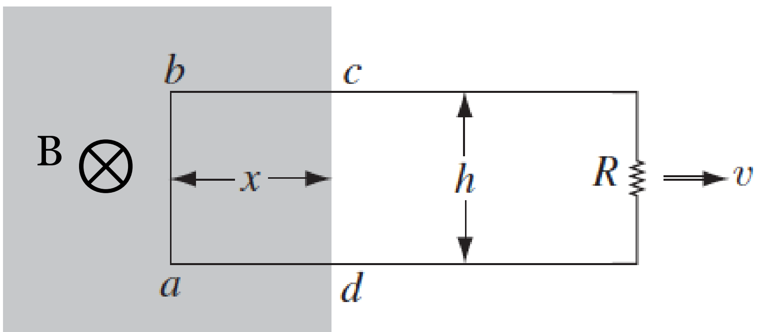
Motional emf

Generators exploit motinal emf's: Move a wire through a magnetic field.



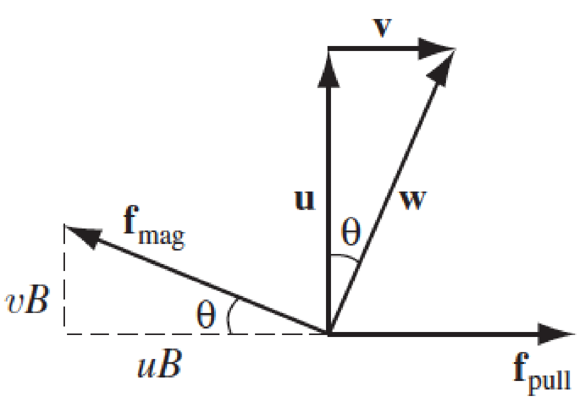
Charges in ab experience a magnetic force qvB
Driving current in loop in the clockwise direction.

The emf is, $\mathcal{E} = \oint \mathbf{f}_{\text{mag}} \cdot d\mathbf{l} = vBh$

In bc , ad force is ⊥ to the wire.

This emf is established by magnetic force, but they are not doing any work.

Who is doing the work?



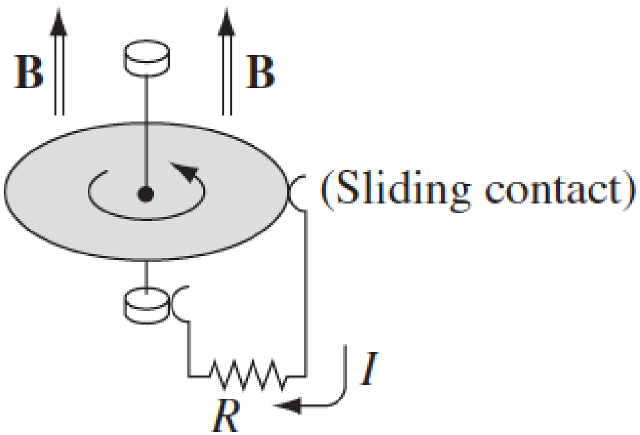
emf generated in a moving loop

$$\varepsilon = \mathrm{v}\bar{B}h = Bh(-\frac{dx}{dt}) = -\frac{d}{dt}(\underbrace{Bhx}_{\Phi}) = -\frac{d}{dt}\Phi$$

$$\mathcal{E} = -\frac{d\Phi}{dt}.$$

Flux rule for motional emf.

Example 7.4. A metal disk of radius a rotates with angular velocity ω about a vertical axis, through a uniform field $\mathbf{B}$, pointing up. A circuit is made by connecting one end of a resistor to the axle and the other end to a sliding contact, which touches the outer edge of the disk (Fig. 7.15). Find the current in the resistor.



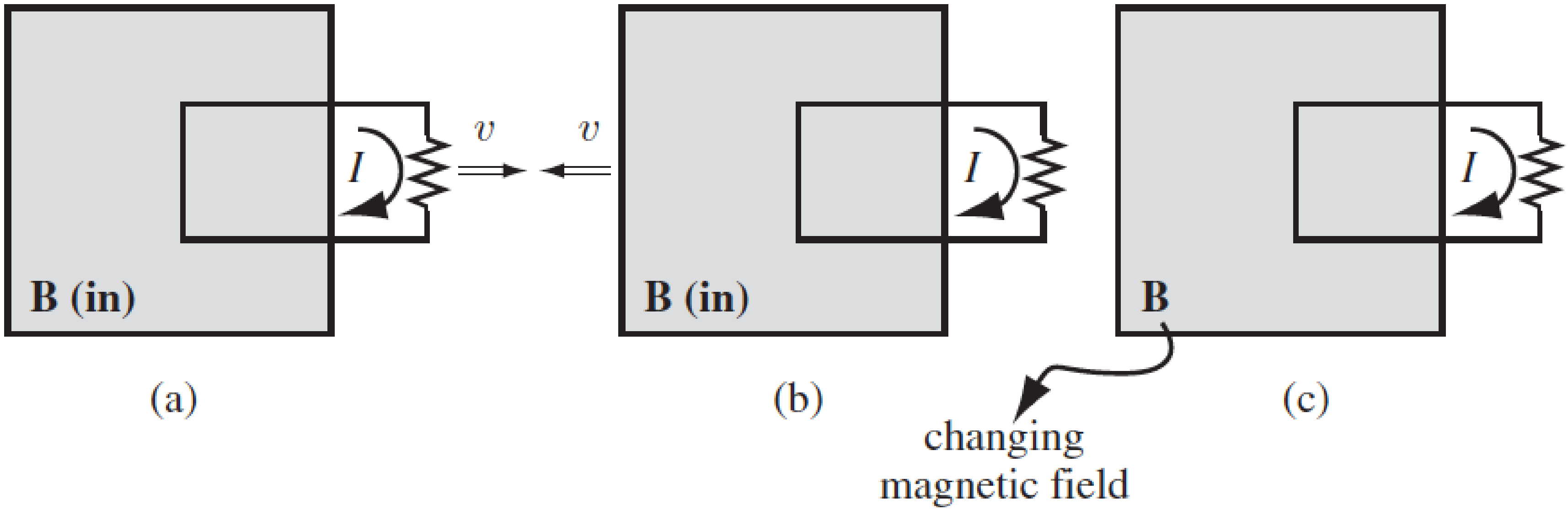
The speed of a point on the disk at a distance s from the axis is $v = \omega s$

force per unit charge is $\mathbf{f}_{\text{mag}} = \mathbf{v} \times \mathbf{B} = \omega s B \hat{\mathbf{s}}$.

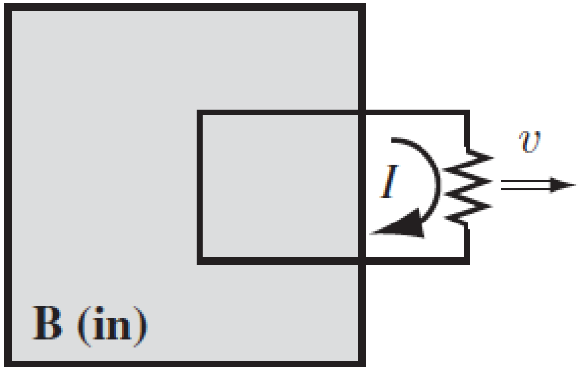
Emf, $$\mathcal{E} = \int_0^a f_{\text{mag}}\, ds = \omega B \int_0^a s\, ds = \frac{\omega B a^2}{2}$$

Current in the resistor $$I = \frac{\mathcal{E}}{R} = \frac{\omega B a^2}{2R}.$$

Faraday's Law



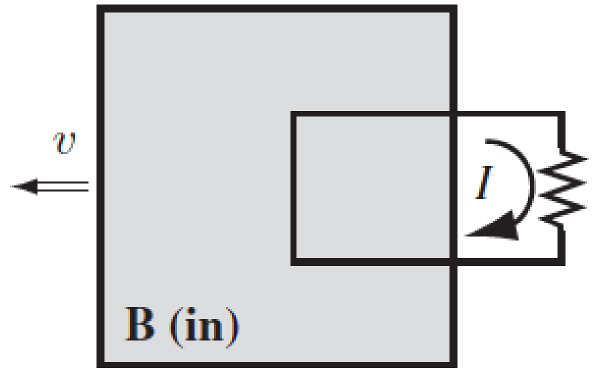
Faraday's Law



(a)

(a) Case of motional emf; $\mathcal{E} = -\dfrac{d\Phi}{dt}$

Faraday's Law



(b)

(b) 2^{nd} Experiment has the same emf, relative motion of loop and the magnet.

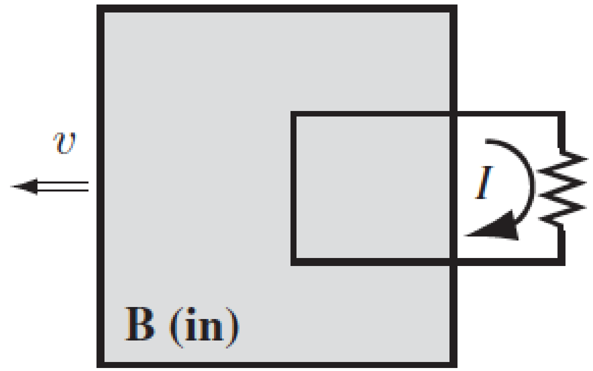
STR was still another century away! Without STR this reciprocity is great coincidence!!

Stationary charge: No Magnetic force. What field is exerting force on charges at rest?

Electric field!!

A changing magnetic field induces an electric field.

Faraday's Law



(b)

(b)

A changing magnetic field induces an electric field.

If emf is equal to the rate of change of flux $\mathcal{E} = \oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt},$

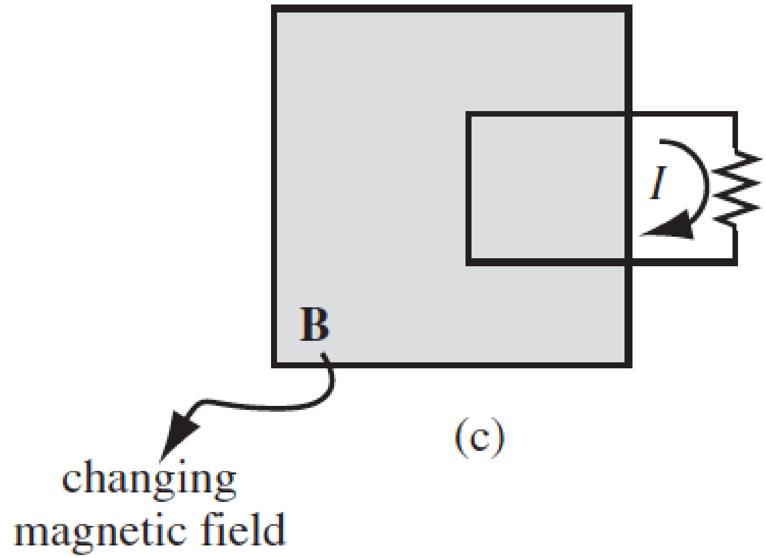
then ***E*** is connected to the change in ***B*** $\oint \mathbf{E} \cdot d\mathbf{l} = -\int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{a}.$

Faraday's law in differential form, the integral form is by Stoke's theorem

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

In the Static case, constant ***B*.**

Faraday's Law



(c)

(c) 3rd scenario also an electric field gets generated and gives the same emf $\mathcal{E} = -\frac{d\Phi}{dt}$

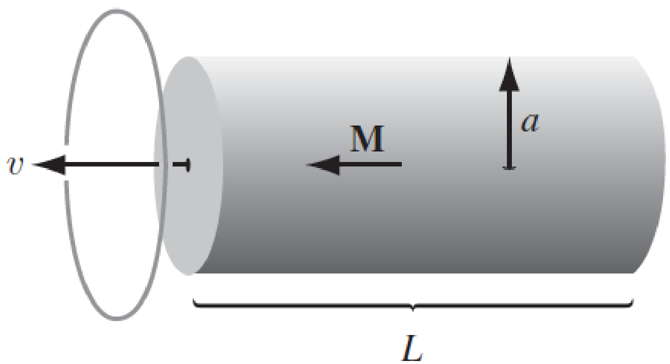
Changing magnetic field generates an electric field

Note:

Not Faraday's law! In (a), $\mathbf{v} \times \mathbf{B}$ (magnetic emf) drives the current ***I*****,** not ***E***

In (b) & (c), induced ***E*** drives the current ***I*** :(Electric Field driven)

Example 7.5. A long cylindrical magnet of length L and radius a carries a uniform magnetization $\mathbf{M}$ parallel to its axis. It passes at constant velocity v through a circular wire ring of slightly larger diameter (Fig. 7.22). Graph the emf induced in the ring, as a function of time.

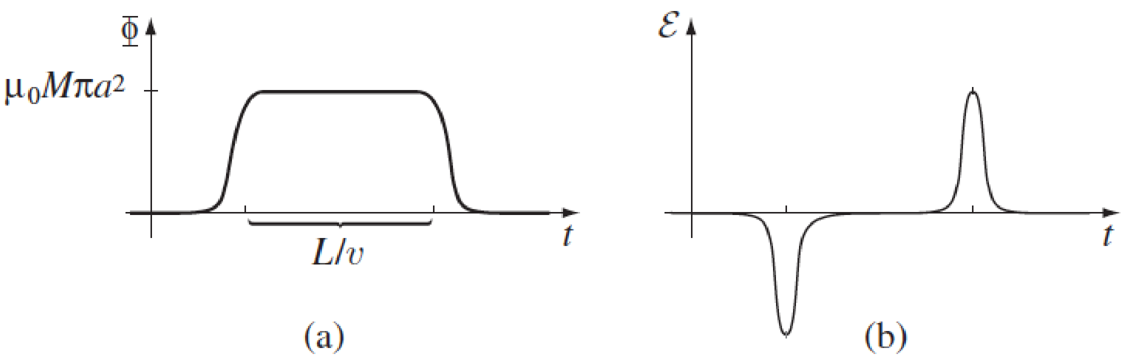


The mag field is same as a long solenoid with Surface current $\mathbf{K}_b = M\,\hat{\boldsymbol{\phi}}$.

Thus the field inside is $\mathbf{B} = \mu_0\mathbf{M}$, but spreads out near the end.

The flux through the ring is zero when the magnet is far away, the maximum flux is $\mu_0 M \pi a^2$ as the leading end passes through.

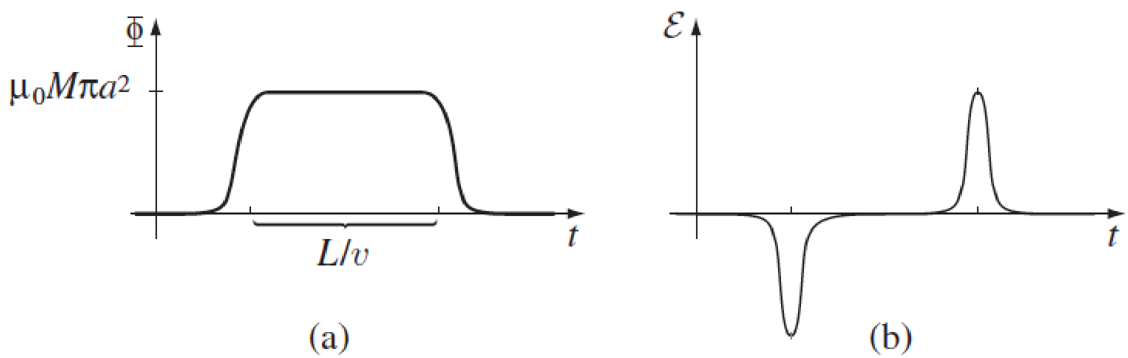
It again drops to zero as the trailing end emerges, the emf is $\mathcal{E} = -\dfrac{d\Phi}{dt}$



Lenz's Law

The induced current flows in such a direction that the flux it produces tends to cancel the change. We can not quantify the current but can get the directions right.

Nature abhors a change in flux.



The magnet enters the ring, flux increases. The current is clockwise to generate field to the right.

The magnet exits ring, flux drops, counterclockwise current to restore the field.

Change in the flux is prevented, not flux

Induced Electric Field

Two distinct kinds of electric fields:

From electric charges: static case using Coulombs Law
and
changing magnetic fields: Faraday's Law

For pure Faraday Field $\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$

[In magnetostatics, $\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$]

The analog to Bio-savart's law

$$\mathbf{E} = -\frac{1}{4\pi} \int \frac{(\partial \mathbf{B}/\partial t) \times \hat{\boldsymbol{\imath}}}{\imath^2} d\tau = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int \frac{\mathbf{B} \times \hat{\boldsymbol{\imath}}}{\imath^2} d\tau,$$

If symmetry permits, the tricks associated with Ampere's law are permissible.

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}.$$

Example 7.7. A uniform magnetic field $\mathbf{B}(t)$, pointing straight up, fills the shaded circular region of Fig. 7.25. If $\mathbf{B}$ is changing with time, what is the induced electric field?

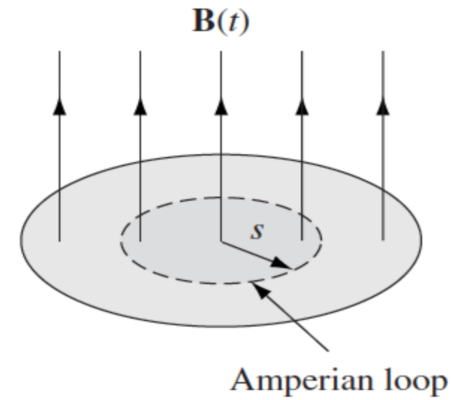


FIGURE 7.25

E points in the circumferential direction like magnetic field inside a long straight wire with uniform current density.

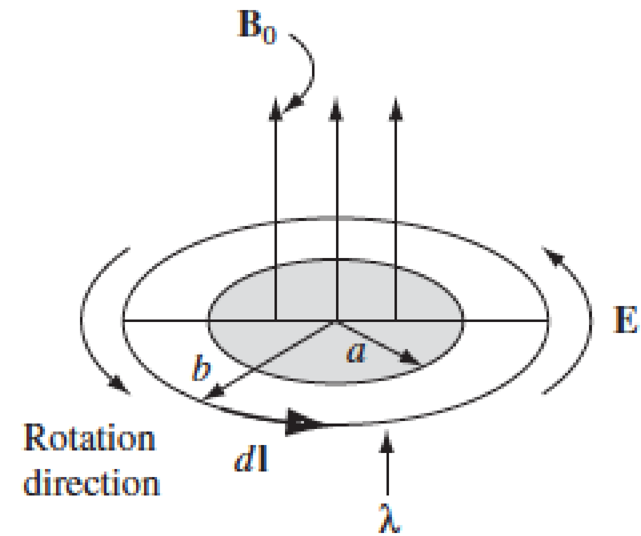
With an Amperian loop of radius *s* and applying Faraday's law.

$$\oint \mathbf{E}\cdot d\mathbf{l} = E(2\pi s) = -\frac{d\Phi}{dt} = -\frac{d}{dt}\left(\pi s^2 B(t)\right) = -\pi s^2 \frac{dB}{dt}.$$

$$\mathbf{E} = -\frac{s}{2}\frac{dB}{dt}\,\hat{\boldsymbol{\phi}}.$$

For increasing B, E runs clockwise as viewed from above.

Example 7.8. A line charge λ is glued onto the rim of a wheel of radius b, which is then suspended horizontally, as shown in Fig. 7.26, so that it is free to rotate (the spokes are made of some nonconducting material—wood, maybe). In the central region, out to radius a, there is a uniform magnetic field $\mathbf{B}_0$, pointing up. Now someone turns the field off. What happens?



Solution:
The electric field due to the changing magnetic field curl around the wheel and turn the wheels. The rotation would be in the direction to restore the upward flux, i.e. Counterclockwise from above.

Using Faraday's law applied to the loop at radius b,

$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi b) = -\frac{d\Phi}{dt} = -\pi a^2 \frac{dB}{dt}, \quad \text{or} \quad \mathbf{E} = -\frac{a^2}{2b}\frac{dB}{dt}\hat{\boldsymbol{\phi}}.$$

FIGURE 7.26

The torque on the segment of dl is $\boldsymbol{r}\times\boldsymbol{F}$ or $b\lambda E\, dl$, then the total torque on the wheel

$$N = b\lambda\left(-\frac{a^2}{2b}\frac{dB}{dt}\right)\oint dl = -b\lambda\pi a^2 \frac{dB}{dt},$$

and the angular momentum imparted to the wheel $\int N dt = -\lambda\pi a^2 b \int_{B_0}^{0} dB = \lambda\pi a^2 b B_0.$

Magneto-statics and Faraday’s Laws

- Electromagnetic induction occurs only when the magnetic fields are changing, and yet we are using apparatus of magneto-statics i.e. Ampere's law, the Biot-Savart law, and the rest.

- Results derived in this way is approximately correct. The error is usually negligible unless the field fluctuates rapidly, or the points of interest are far from the source.

- This regime, in which magneto-static rules can be used to calculate the magnetic field on the right hand side of Faraday's law, is called **Quasistatic.**

Example 7.9. An infinitely long straight wire carries a slowly varying current $I(t)$. Determine the induced electric field, as a function of the distance s from the wire.

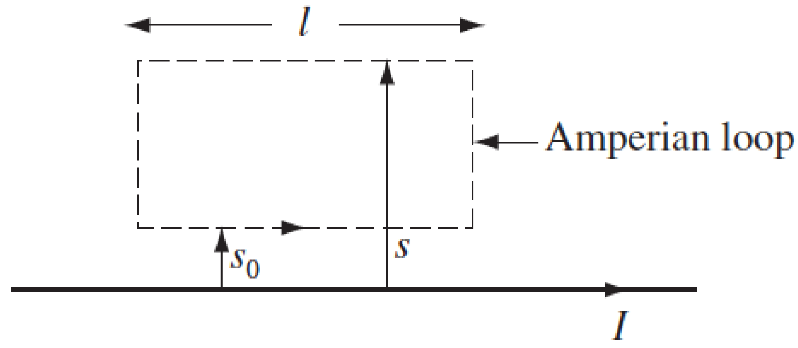
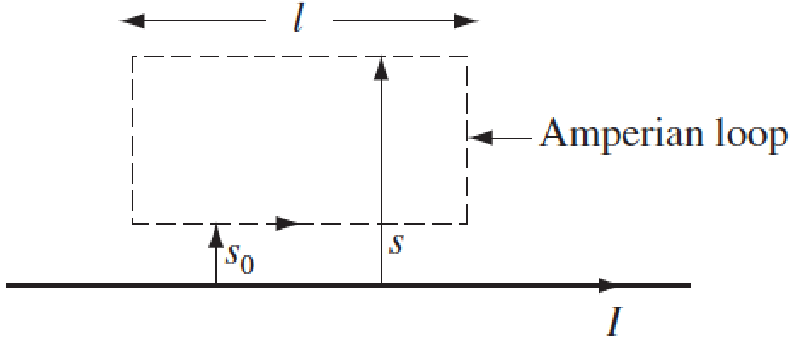


FIGURE 7.27

In the quasistatic approximation, the magnetic field is $(\mu_0 I/2\pi s)$ circling around the wire.

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}. \qquad \mathbf{\Phi} = \oint \vec{B} \cdot \overrightarrow{da}$$



$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}. \qquad \boldsymbol{\Phi} = \oint \vec{B} \cdot \overrightarrow{da}$$

$$\oint \vec{E} \cdot \overrightarrow{dl} = E(s_0)l - E(s)l = -\frac{d}{dt}\oint \vec{B} \cdot \overrightarrow{da} = -\frac{d}{dt}\oint \frac{\mu_0 I}{2\pi s} \cdot \overrightarrow{da}$$

$$= -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}\int_{s_0}^{s} \frac{1}{s'} ds'$$

$$= -\frac{\mu_0 l}{2\pi}\frac{dI}{dt}[\ln s - \ln s_0]$$

$$\boldsymbol{E}(\boldsymbol{s}) = \left[\frac{\mu_0}{2\pi}\frac{dI}{dt}\ln s + K\right]\hat{\boldsymbol{z}}$$

Here constant K is independent of s, it might still be a function of t.

However, $\boldsymbol{E}$ blows up at large s.

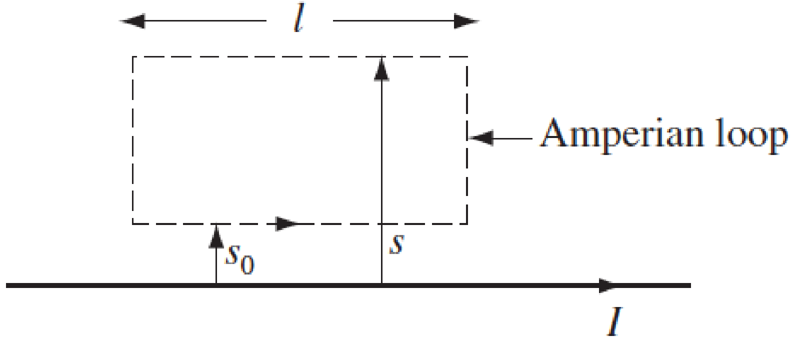


FIGURE 7.27

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}. \qquad \mathbf{\Phi} = \oint \vec{B} \cdot \overrightarrow{da}$$

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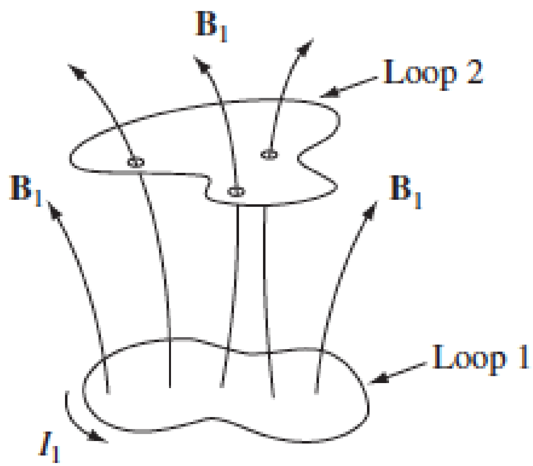
Here constant K is independent of s, it might still be a function of t.

However, $\boldsymbol{E}$ blows up at large s.

Breaking of quasistatic approximation:

If τ is the time it takes I to change substantially, $s << c\tau$

Inductance



Under quasistatic approximation

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi} I_1 \oint \frac{d\mathbf{l}_1 \times \hat{\boldsymbol{\imath}}}{\imath^2}$$

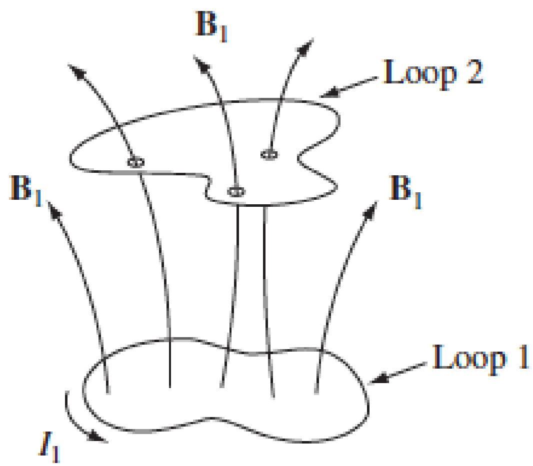
$\mathbf{B}_1$ is proportional to $\mathbf{I}_1$ hence flux through loop2 is also proportional to $\mathbf{I}_1$

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2.$$

$$\Phi_2 = M_{21} I_1,$$

$\mathbf{M_{21}}$: Mutual Inductance

Inductance



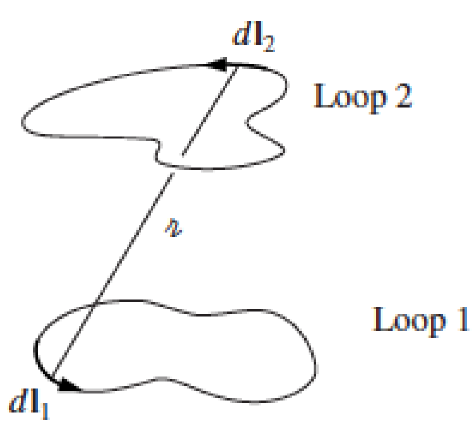
$$\Phi_2 = M_{21} I_1,$$

M_{21}: Mutual Inductance

$$\Phi_2 = \int \mathbf{B}_1 \cdot d\mathbf{a}_2 = \int (\nabla \times \mathbf{A}_1) \cdot d\mathbf{a}_2 = \oint \mathbf{A}_1 \cdot d\mathbf{l}_2.$$

$$= \frac{\mu_0 I_1}{4\pi} \oint \left(\oint \frac{d\mathbf{l}_1}{\mathscr{r}} \right) \cdot d\mathbf{l}_2.$$

$$\boxed{\mathbf{A}_1 = \frac{\mu_0 I_1}{4\pi} \oint \frac{d\mathbf{l}_1}{\mathscr{r}},}$$

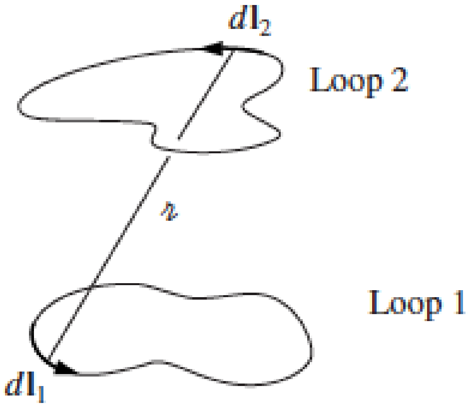


$$\boxed{M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}.}$$

Neumann formula
for mutual inductance

Inductance

M_{21}: Mutual Inductance



$$\Phi_2 = M_{21} I_1,$$

$$M_{21} = \frac{\mu_0}{4\pi} \oint \oint \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{\mathscr{r}}.$$

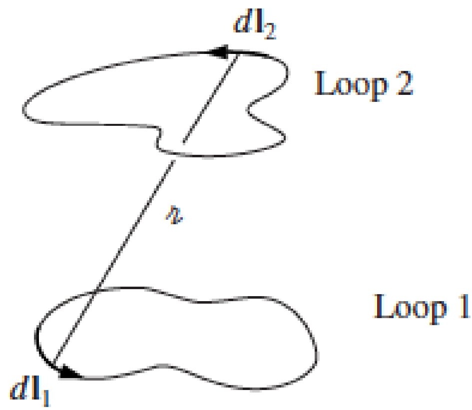
Neumann formula
for mutual inductance

- Mutual Inductance **M_{21}** is a purely geometric quantity
- Switching loops 1 and 2 keeps **M_{21}** same, i.e. **M_{21} = M_{12}** = **M**

D.J.G

This is an astonishing conclusion: *Whatever the shapes and positions of the loops, the flux through 2 when we run a current I around 1 is identical to the flux through 1 when we send the same current I around 2.*

Inductance

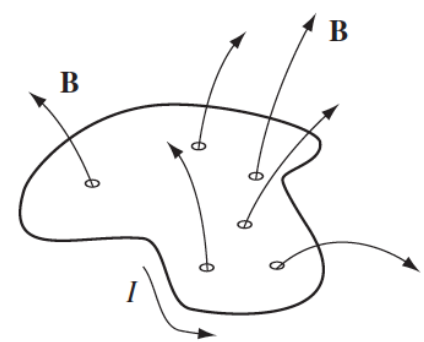


emf in loop 2, due to change of current in loop 1

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M\frac{dI_1}{dt}.$$

!! Current change in loop1, current flow in loop2!!
No wires connecting them!!

Self Inductance:



Changing current will induce an emf in the source loop as well

$$\Phi = LI.$$

self inductance L also depends on the geometry of the loop!

$$\mathcal{E} = -L\frac{dI}{dt}.$$

Back emf,
-ve sign, Lenz's Law

L: unit is henries (H), volt-second per ampere

Example 7.10. A short solenoid (length l and radius a, with n_1 turns per unit length) lies on the axis of a very long solenoid (radius b, n_2 turns per unit length) as shown in Fig. 7.32. Current I flows in the short solenoid. What is the flux through the long solenoid?

Short Solenoid, complicated flux
Equality of Mutual Inductance!

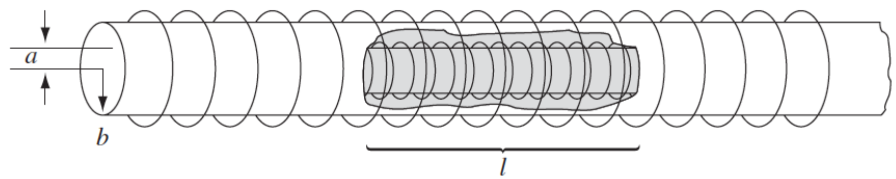


FIGURE 7.32

Let us, run the current I through the *outer* solenoid, and calculate the flux through the *inner* one. The field inside the long solenoid is constant:

The field inside the long solenoid, $B = \mu_0 n_2 I$

Flux through the single loop of the short solenoid, $B\pi a^2 = \mu_0 n_2 I \pi a^2$.

Total Flux through the short solenoid, $\Phi = \mu_0 \pi a^2 n_1 n_2 l I$.

The mutual inductance, $M = \mu_0 \pi a^2 n_1 n_2 l$

Energy in Magnetic field

Work against the *back emf* to get the current going. It is fixed and *recoverable!*

Work done on a unit charge against the back emf in one trip around the circuit is $-\mathcal{E}$.

(-sign for work done by me)

Thus total work done per unit time is, $\dfrac{dW}{dt} = -\mathcal{E} I = L I \dfrac{dI}{dt}.$

Current goes from zero to *I* over time (integrating the above expression),

the total work done,

$$\boxed{W = \frac{1}{2} L I^2.}$$

(only geometry of the loop 'L'
and
final current I)

Energy in Magnetic field

W in terms of the surface and volume currents:

Flux through the loop is '*L I*' also $\Phi = \int \mathbf{B} \cdot d\mathbf{a} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \oint \mathbf{A} \cdot d\mathbf{l},$

Thus, $LI = \oint \mathbf{A} \cdot d\mathbf{l},$ and therefore $W = \frac{1}{2} I \oint \mathbf{A} \cdot d\mathbf{l} = \frac{1}{2} \oint (\mathbf{A} \cdot \mathbf{I})\, dl.$

Generalized to volume currents

$$W = \frac{1}{2} \int_{\mathcal{V}} (\mathbf{A} \cdot \mathbf{J})\, d\tau.$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$$

$$= \frac{1}{2\mu_0} \int \mathbf{A} \cdot (\nabla \times \mathbf{B})\, d\tau.$$

[$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$

$\mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot \mathbf{B} - \nabla \cdot (\mathbf{A} \times \mathbf{B}).$]

$$= \frac{1}{2\mu_0} \left[\int B^2\, d\tau - \int \nabla \cdot (\mathbf{A} \times \mathbf{B})\, d\tau \right]$$

$$= \frac{1}{2\mu_0} \left[\int_{\mathcal{V}} B^2\, d\tau - \oint_{\mathcal{S}} (\mathbf{A} \times \mathbf{B}) \cdot d\mathbf{a} \right],$$

$\mathcal{S}$ is the surface bounding the volume $\mathcal{V}$.

the surface term is zero if *S* goes to infinity

Energy in Magnetic field

$$W = \frac{1}{2\mu_0} \int_{\text{all space}} B^2 \, d\tau .$$

In view of this result, we say the energy is "stored in the magnetic field," in the amount ($B^2/2\mu_o$) per unit volume.

In the light of this, it is extraordinary how similar the magnetic energy formulas are to their electrostatic counterparts:

$$W_{\text{elec}} = \frac{1}{2} \int (V\rho) \, d\tau = \frac{\epsilon_0}{2} \int E^2 \, d\tau,$$

$$W_{\text{mag}} = \frac{1}{2} \int (\mathbf{A} \cdot \mathbf{J}) \, d\tau = \frac{1}{2\mu_0} \int B^2 \, d\tau .$$

Energy in Magnetic field

Example 7.13. A long coaxial cable carries current I (the current flows down the surface of the inner cylinder, radius a, and back along the outer cylinder, radius b) as shown in Fig. 7.40. Find the magnetic energy stored in a section of length l.

field between the cylinders is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}.$$

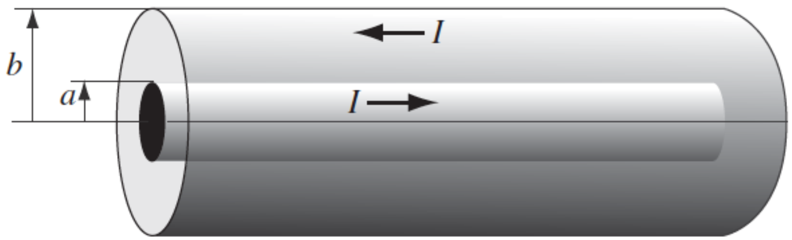


FIGURE 7.40

Elsewhere, the field is zero.

energy per unit volume is $\frac{1}{2\mu_0}\left(\frac{\mu_0 I}{2\pi s}\right)^2 = \frac{\mu_0 I^2}{8\pi^2 s^2}$. $\boxed{=(B^2/2\mu_o)}$

Energy dW, in a cylindrical shell of length l, radius s, and thickness ds, $\left(\frac{\mu_0 I^2}{8\pi^2 s^2}\right) 2\pi l s\, ds = \frac{\mu_0 I^2 l}{4\pi}\left(\frac{ds}{s}\right)$.

Integrating dW,

$$W = \frac{\mu_0 I^2 l}{4\pi}\ln\left(\frac{b}{a}\right).$$

However, $\boxed{W = LI^2/2}$

Comparing both Expressions of 'W'

$$\boxed{L = \frac{\mu_0 l}{2\pi}\ln\left(\frac{b}{a}\right).}$$