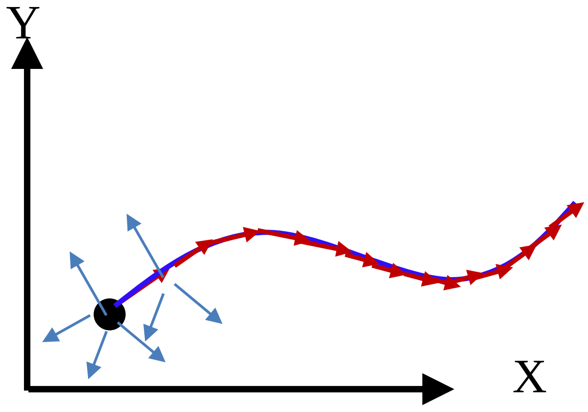
Lecture 12

Principle of least action

How to get trajectory of a particle

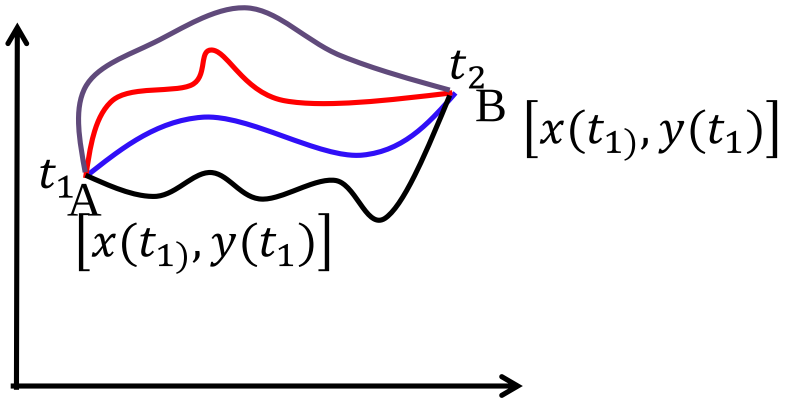
Using Newtons law: **Differential law**

$$\frac{d^2\vec{r}}{dt^2} = \vec{F}$$



This law can choose elementary trajectory at different instances from all possible elementary trajectories, to give final trajectory between time interval t_1 *to* t_2

Using integral method:
A mechanical system will evolve in time in such that action integral is stationary → **Hamilton's Principle of Least Action**



It can choose entire trajectory from all possible trajectories

Principle of Least Action

- A mechanical system will evolve in time in such that action integral is stationary → **Hamilton's Principle of Least Action**

- **Action integral** → $\displaystyle \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt$

- $L(q_j, \dot{q}_j, t)$ →Lagrangian of system of particles

$\displaystyle \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt$ → **Stationary** → $\displaystyle \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt = 0$

- Stationary condition of Action integral

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

→ Lagrange's equation from Variational principle

Principle of Least Action

- ❑ The action integral of a physical system is *stationary* for the actual path

- ❑ Action *I* does not depend on the choice of the coordinates

Three equivalent formulations

- ❑ Newton’s Eqn
- ❑ Lagrange’s Eqn
- ❑ Hamilton’s Principle

- ❑ Hamilton’s Principle is more fundamental

Integration path in Principle of least action

- **Principle of Least Action**→The path of a particle/particles is the one that yields a stationary value of the action

Stationary

$$\int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)dt$$

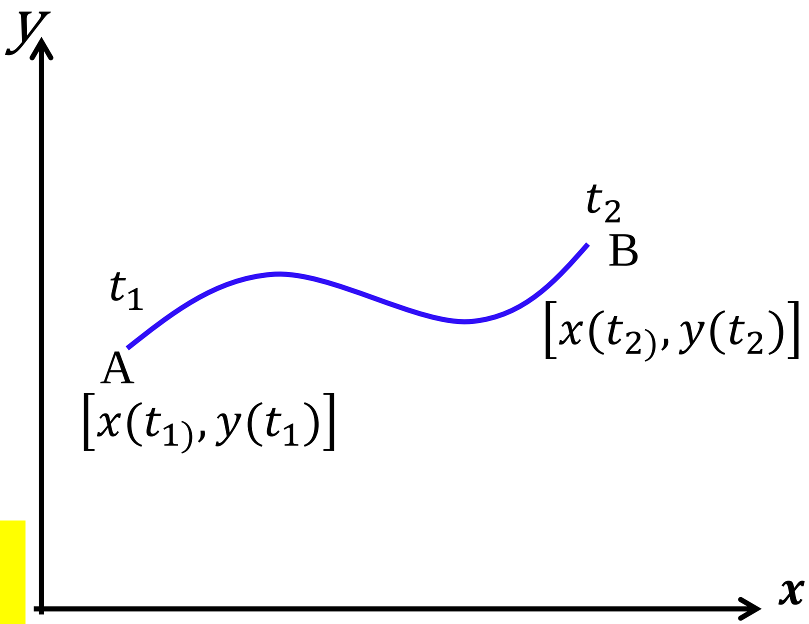
- **Which integration path it refer to ?**
  → ***It will be clear from some example***

Ex 1: A single particle is moving in XY plane without any additional constrain. Generalized coordinates are (x, y). Lagrangian of the system

$$L = L(x, y, \dot{x}, \dot{y}, t)$$

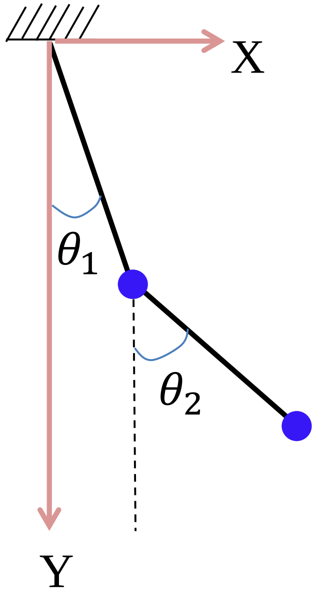
$$\int_{t_1}^{t_2} L(x, y, \dot{x}, \dot{y}, t)\, dt$$

Stationary path is the trajectory in XY plane



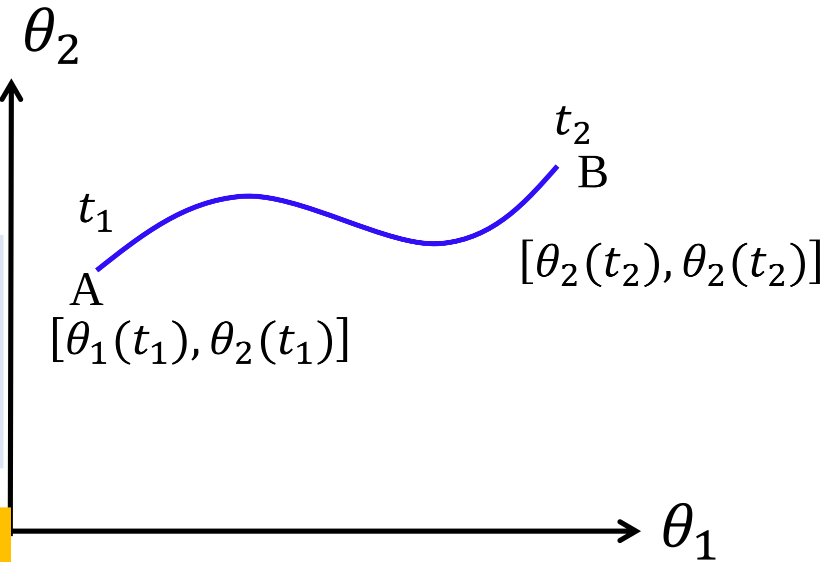
Integration path in Principle of least action

Ex 2: Double pendulum



- Every instant, (θ_1, θ_2) can represent a point in 2D plane if two axes are chosen as θ_1 and θ_2.

- Hence all sets of points (θ_1, θ_2) in the time interval $t_1 < t < t_2$ can represent a line in which time remains a parameter



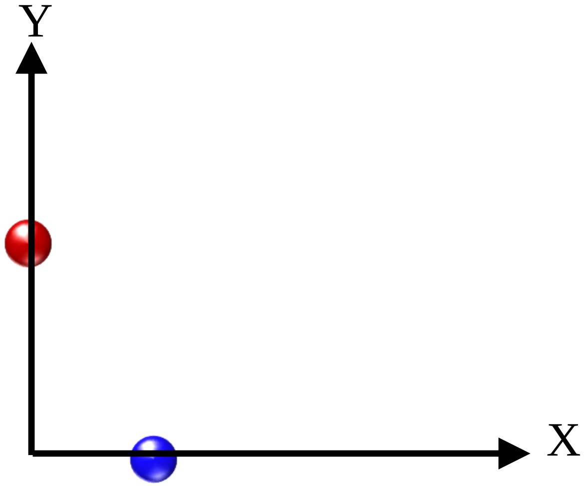
- Each point (θ_1, θ_2) represent the configuration of the system at a particular instant→Point in configuration plane/space

$$\int_{t_1}^{t_2} \boldsymbol{L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2, t)\ dt}$$

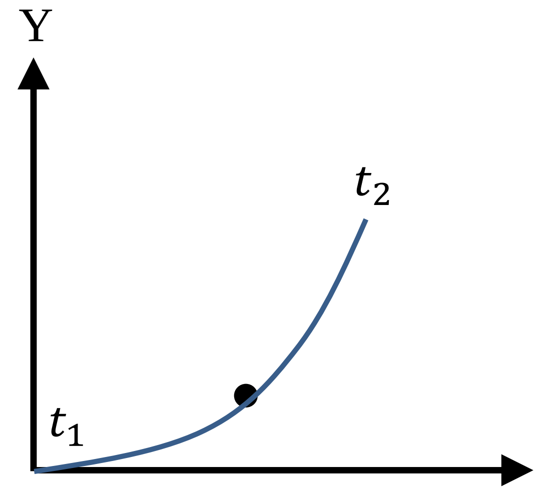
Stationary path in the (θ_1, θ_2) configuration space

Note: This curve is not the trajectory of any particles.

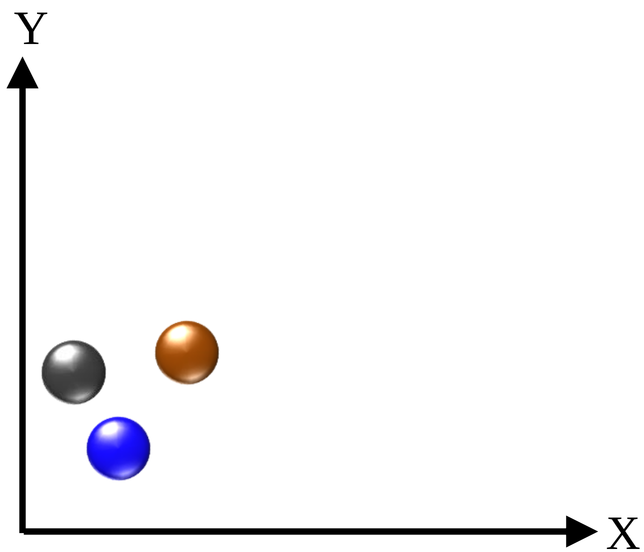
Real space vs configuration space



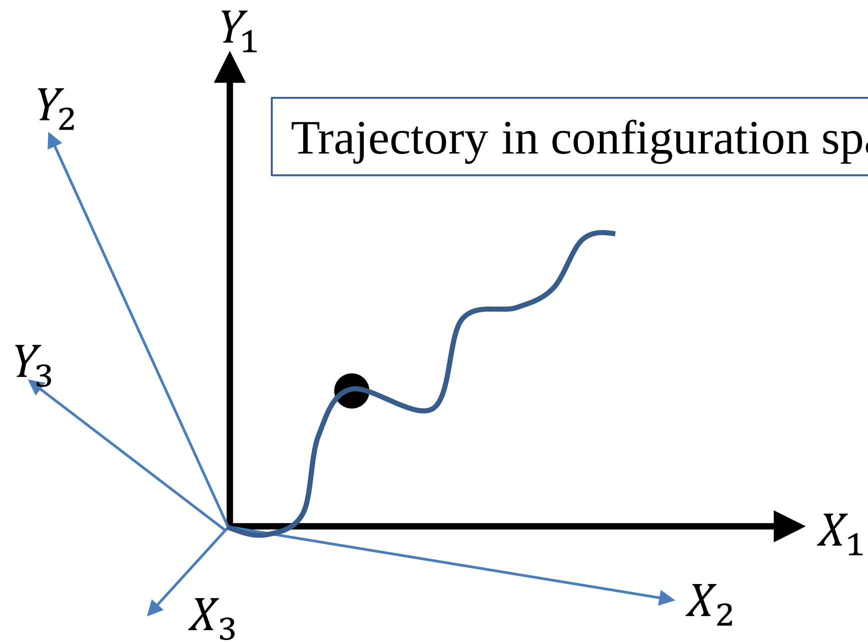
Trajectory in real space



Trajectory in configuration space



Trajectory in real space



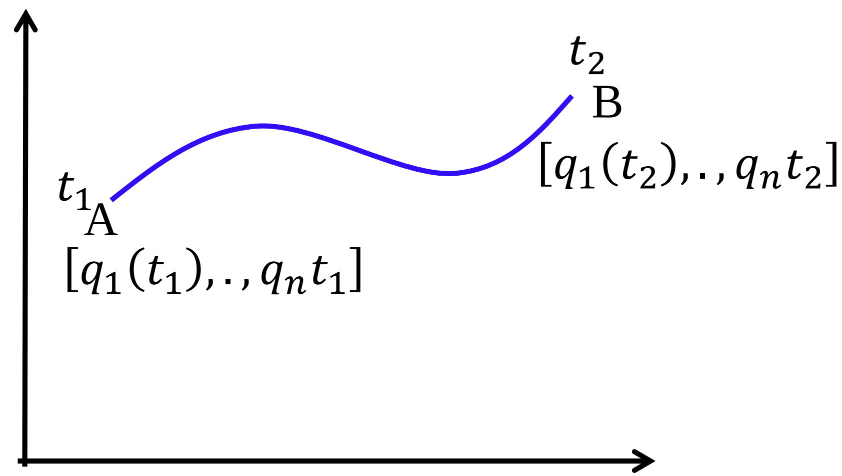
Trajectory in configuration space

Integration path in Principle of least action: Configuration space

System with n generalized coordinates $q_1, q_2, \ldots.., q_n$

- Impossible to draw 'n' perpendicular coordinates/ n-dimensional space, but possible to imagine

- Each point $(q_1, q_2, \ldots.., q_n)$ at a particular instance represent system configuration at that instance in the configuration space.

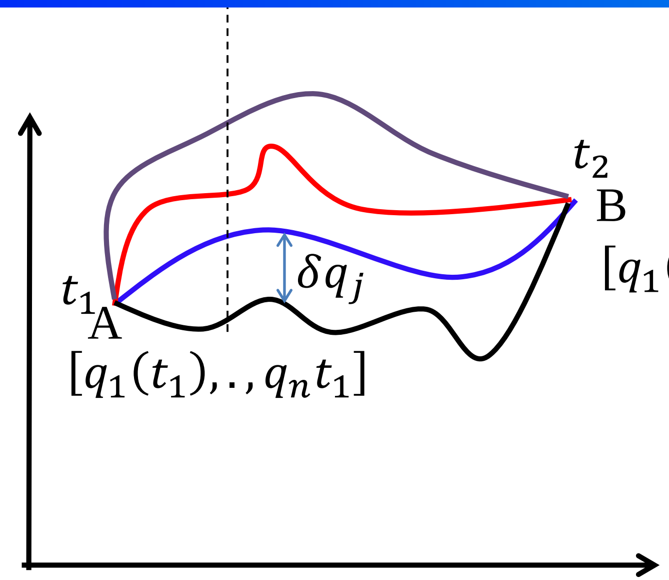


$$\int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)dt \longrightarrow \textbf{Stationary}$$

- Out of all possible paths in the configuration space a mechanical system chooses that path for which action in stationary.

Alternative proof of Lagrange's equation from Principle of least action

- Out of all possible path in the interval $t_1 < t < t_2$, right path [say blue one] is that one for which the variation of action integral in the nearby path is zero.



$$\boldsymbol{\delta I = \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)dt = 0}$$

- Variation from one path to another (due to variation in *all* q_j and hence $\dot{q}_j$) at same instance $\delta t = 0$.

$$\boldsymbol{\delta I = \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)dt = \int_{t_1}^{t_2} \delta L(q_j, \dot{q}_j, t)dt + \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\delta(dt) = 0}$$

$= 0$

Alternative proof of Lagrange's equation from Principle of least action

$$\delta L = \sum_j \left(\frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\delta \dot{q}_j + \frac{\partial L}{\partial t}\delta t \right)$$

This term is zero as $\delta t = 0$

$$\delta \dot{q}_j = \delta \frac{dq_j}{dt} = \frac{d}{dt}(\delta q_j)$$

$$\delta L = \sum_j \left\{ \frac{\partial L}{\partial q_j}\delta q_j + \frac{\partial L}{\partial \dot{q}_j}\frac{d}{dt}(\delta q_j) \right\}$$

$$\delta L = \sum_j \left\{ \frac{\partial L}{\partial q_j}\delta q_j + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\delta q_j\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j \right\}$$

$$\delta L = \sum_j \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\delta q_j\right) - \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right]\delta q_j \right\}$$

$$\delta I = \int_{t_1}^{t_2} \delta L dt = \int_{t_1}^{t_2} \sum_j \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\delta q_j\right) - \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right]\delta q_j \right\} dt = 0$$

Alternative proof of Lagrange's equation from Principle of least action

$$\delta I = \int_{t_1}^{t_2} \delta L dt = \sum_j \int_{t_1}^{t_2} \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\delta q_j\right) - \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right]\delta q_j \right\} dt = 0$$

$$\delta I = \sum_j \left\{ \frac{\partial L}{\partial \dot{q}_j}\delta q_j \Big|_1^2 - \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right]\delta q_j dt \right\} = 0$$

First term vanishes because all curves pass through the fixed end points.

$$\sum_j \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j}\right]\delta q_j dt = 0$$

True for any possible variation of δq_j

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

Lagrange's equation

A quick review of previous class

- $L = L(q_1, \ldots q_n, \dot{q}_1, \ldots . \dot{q}_n, t)$

- Using the chain rule of partial differentiation

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \sum_j \frac{\partial L}{\partial q_j} \dot{q}_j + \frac{\partial L}{\partial t}$$

Using this equation means we have taken into consideration that Action is stationary ,$\boldsymbol{\delta I = \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t) dt = 0}$

- Using Lagrange's eqn.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

$$\frac{dL}{dt} = \sum_j \frac{\partial L}{\partial \dot{q}_j} \ddot{q}_j + \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \dot{q}_j + \frac{\partial L}{\partial t}$$

$$\frac{dL}{dt} = \sum_j \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \dot{q}_j\right) + \frac{\partial L}{\partial t}$$

$$\frac{d}{dt}\left(\sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L\right) + \frac{\partial L}{\partial t} = 0$$

Note, If L does not have explicit time dependence, i, e $\frac{\partial L}{\partial t} = 0$

$$\sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L = Constant$$

An addition to stationary condition of Action,$\boldsymbol{\delta I = \delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t) dt = 0}$

Constant equation for general function

- **Principle of least action**: Action I= $\int_{t_1}^{t_2} \boldsymbol{L(q_j, \dot{q}_j, t)dt}$ is stationary

Necessary condition of stationary action

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

- In addition to stationary condition, if L does not have explicit time dependence, i,e $L = L(q_1, \ldots q_n, \dot{q}_1, \ldots . \dot{q}_n)$

$$\sum_j \frac{\partial L}{\partial \dot{q}_j} \dot{q}_j - L = Constant$$

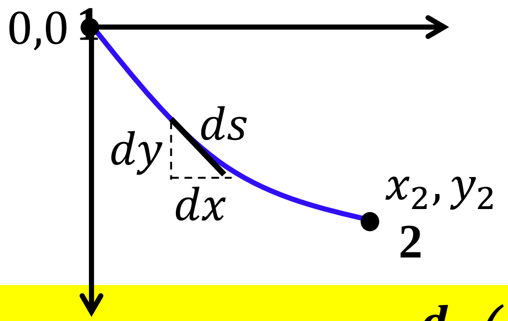
- Necessary condition for I $= \int_{x_1}^{x_2} F(x, y, y')dx$ to be stationary

$\boldsymbol{\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y}}$ =0

- In addition to stationary, If F does not have explicit dependence on x, i,e $F = F(y, y')$

$$\frac{\partial F}{\partial y'} y' - F = Constant$$

Brachistochrone problem revisited



$$ds = [(dx)^2 + (dy)^2]^{1/2}$$

$$= \left[\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}\right]^{1/2} dy = \left(1 + \boldsymbol{x'}^2\right)^{1/2} dy$$

Method 1: Using $\boldsymbol{\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x}} = 0$

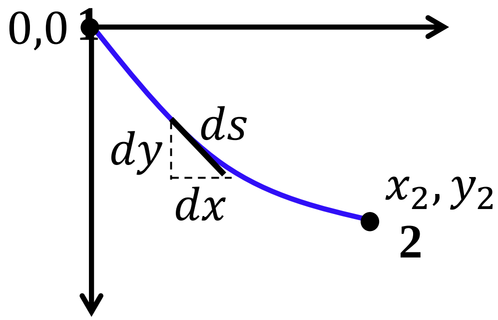
- Time of travel $= \int dt = \int_1^2 \frac{ds}{v} = \int_0^{y_2} \frac{\left(1+\boldsymbol{x'}^2\right)^{1/2}}{(2gy)^{1/2}} dy;$

$$F = F(y, x, x') = \frac{\left(1 + \boldsymbol{x'}^2\right)^{1/2}}{(2gy)^{1/2}}$$

$\boldsymbol{\frac{\partial F}{\partial x} = 0}$; $\quad \frac{\partial F}{\partial \boldsymbol{x'}} = \frac{\partial}{\partial x'}\left\{\frac{\left(1+\boldsymbol{x'}^2\right)^{1/2}}{(2gy)^{1/2}}\right\} = \frac{\boldsymbol{x'}\left(1+\boldsymbol{x'}^2\right)^{-1/2}}{(2gy)^{1/2}}$

$$dx = \sqrt{\frac{y}{2a - y}}\, dy$$

Brachistochrone problem revisited



$$ds = [(dx)^2 + (dy)^2]^{1/2}$$

$$= \left[\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}\right]^{1/2} dx = \left(1 + \boldsymbol{y'}^2\right)^{1/2} dx$$

Method 2: Using $\sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L = Constant$

- Time of travel $= \int_1^2 \frac{ds}{v} = \int_0^{x_2} \frac{\left(1+\boldsymbol{y'}^2\right)^{1/2}}{(2gy)^{1/2}} dx; F = F(x, y, y') = \frac{\left(1+\boldsymbol{y'}^2\right)^{1/2}}{(2gy)^{1/2}}$
- F is independent of independent variable x, thus, $\frac{\partial F}{\partial y'} y' - F = Constant$
- $\frac{\partial F}{\partial y'} = \frac{y'\left(1+\boldsymbol{y'}^2\right)^{-1/2}}{(2gy)^{1/2}}; \frac{y'^2\left(1+\boldsymbol{y'}^2\right)^{-1/2}}{(2gy)^{1/2}} - \frac{y'\left(1+\boldsymbol{y'}^2\right)^{-1/2}}{(2gy)^{1/2}} = Constant$

$$dx = \sqrt{\frac{y}{2a - y}}\, dy$$

Summery

- **Action Integral: I =** $\int_{t_1}^{t_2} L(t,q,\dot{q})dt$

- **Principle of Least Action**→The path of a particle/particles in configuration space is the one that yields a stationary value of the action

Hamiltonian of a system of particles

$$H(q_j, p_j, t) = \sum_j p_j \dot{q}_j - L$$

- Necessary condition for I = $\int_{x_1}^{x_2} F(x, y, y')dx$ to be stationary

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

- In addition to stationary, If F does not have explicit dependence on x, i,e $F = F(y, y')$

$$\frac{\partial F}{\partial y'} y' - F = Constant$$