Physics II (PH 102)

Electromagnetism (Lecture 5)

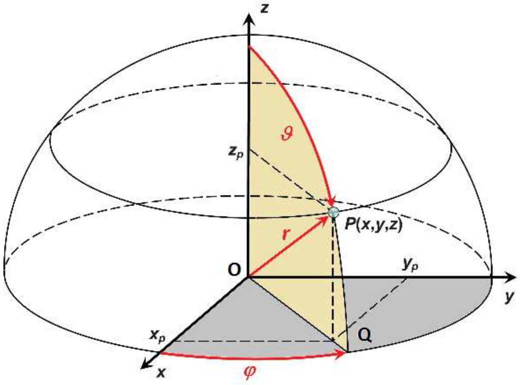
Udit Raha

Indian Institute of Technology Guwahati

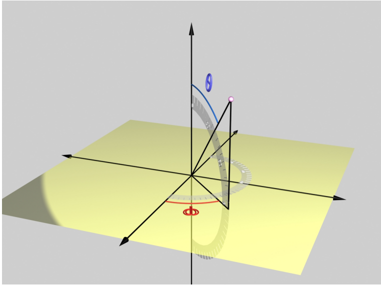
Jan 2020

Spherical-Polar Co-ordinate System: Components

- Position vector of a point P:
  $\mathbf{r}$
- Cartesian coordinates:
  (x, y, z)
- Spherical polar coordinates:
  (r, θ, ϕ)
- Length of $\mathbf{r}$:
  $r = |\mathbf{r}|$ (Radial distance)
- Projection of $\mathbf{r}$ onto XY plane:
  OQ
- Angle between z-axis and $\mathbf{r}$:
  θ (Polar angle/Zenith)
- Angle between x-axis and OQ:
  ϕ (Azimuthal angle)



Spherical-Polar System: 3D Domain



Ranges for Cartesian co-ordinates: $x, y, z \in (-\infty, \infty)$.

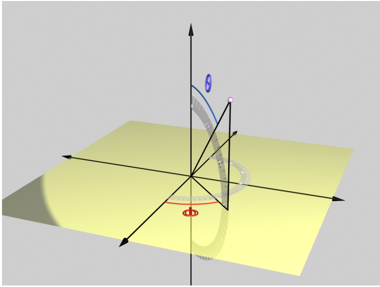
Ranges of Spherical polar co-ordinates:

- Radial co-ordinate (distance): $r \in [0, \infty)$,
- Zenith or Polar co-ordinate: $\theta \in [0, \pi]$
- Azimuthal co-ordinate: $\phi \in [0, 2\pi)$

Note:

- ϕ is undefined for points on z-axis
- θ and ϕ are both undefined at the origin

Spherical-Polar System: Co-ordinate Transformations (Bijective Mappings)



$$
\begin{aligned}
r = r(x,y,z) &= \sqrt{x^2+y^2+z^2} \\
\theta = \theta(x,y,z) &= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \\
\phi = \phi(x,y,z) &= \tan^{-1}\left(\frac{y}{x}\right).
\end{aligned}
$$

$$
\begin{aligned}
x = x(r,\theta,\phi) &= r\sin\theta\cos\phi \\
y = y(r,\theta,\phi) &= r\sin\theta\sin\phi \\
z = z(r,\theta,\phi) &= r\cos\theta
\end{aligned}
$$

Spherical-Polar System: Constant Co-ordinate Surface

Three **Co-ordinate Surfaces** can be obtained by keeping one of the co-ordinates fixed while varying the other two. A point P in 3D space is the intersection of these co-ordinate surfaces.

- r-Constant Surface, $\theta \in [0, \pi]$, $\phi \in [0, 2\pi) \rightarrow$ Sphere
- θ-Constant Surface, $r \in [0, \infty)$, $\phi \in [0, 2\pi) \rightarrow$ Cone
- ϕ-Constant Surface, $\theta \in [0, \pi]$, $r \in [0, \infty) \rightarrow$ Half Plane

Spherical-Polar System: Constant r Surface

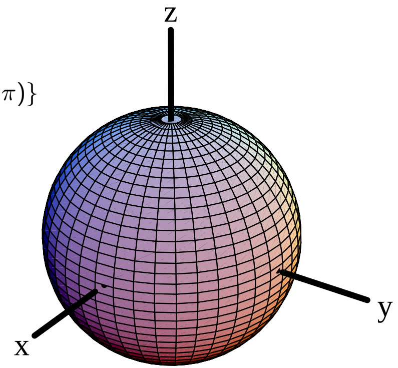
3 Coordinate Surfaces can be obtained by keeping one of the coordinates constant while varying the other two.

$r =$ constant yields a spherical surface.

Let $c = const. > 0$.

$\mathbf{r}(c, \theta, \phi) = \{(c, \theta, \phi) \,|\, \theta \in [0, \pi], \phi \in [0, 2\pi)\}$

which is a sphere of radius c.



Spherical-Polar System: Constant θ Surface

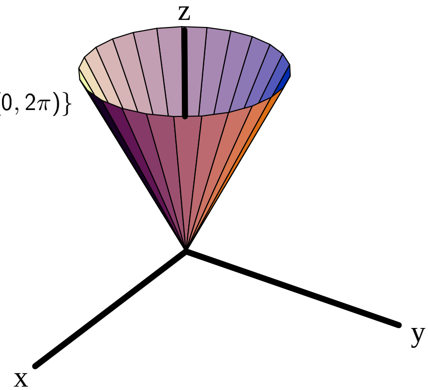
3 Coordinate Surfaces can be obtained by keeping one of the coordinates constant while varying the other two.

$\theta =$ constant yields a conical surface.

Let $\alpha = \mathit{const.} > 0$.

$\mathbf{r}(r, \alpha, \phi) = \{(r, \alpha, \phi) \mid r \in [0, \infty), \phi \in [0, 2\pi)\}$

which is a cone of angle α.



Spherical-Polar System: Constant ϕ Surface

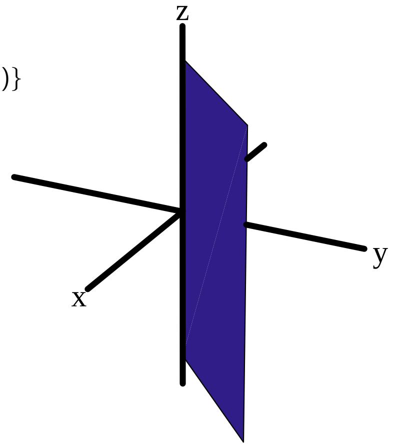
3 Coordinate Surfaces can be obtained by keeping one of the coordinates constant while varying the other two.

$\phi =$ constant yields a planar surface.

Let $\kappa = \textit{const.} > 0$.

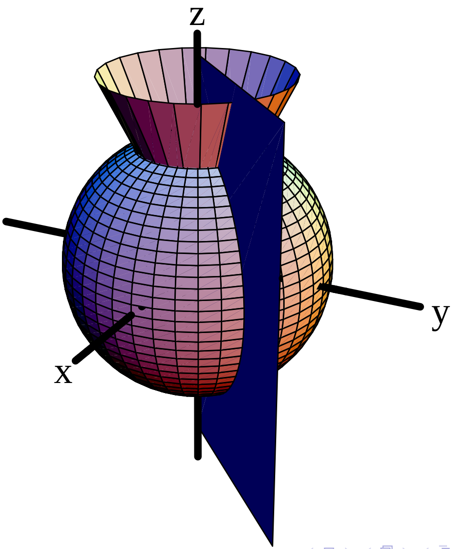
$\mathbf{r}(r, \theta, \kappa) = \{(r, \theta, \kappa) \,|\, \theta \in [0, \pi], r \in [0, \infty)\}$

which is a half plane (only one side of the z-axis) with azimuth κ.



Spherical-Polar System: Intersection of Constant Surfaces

- A point P in 3D space is obtained as an intersection of the 3 const. co-ordinate surfaces
- The intersection of any two co-ordinate surfaces yields a co-ordinate line/axis



Spherical-Polar System: Typical Co-ordinate Curves

Keeping any two co-ordinates fixed and varying the third, we get a co-ordinate curve/line. Let $P\,(r_0, \theta_0, \phi_0)$ be any point in 3D space.

- **r-line**: $r \in [0, \infty)$, $(\theta_0, \phi_0) \to$ fixed

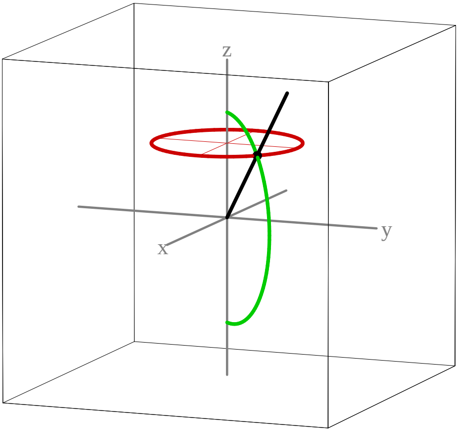
  $$\mathbf{r}(r) = r\,(\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\theta_0)$$

- **θ-curve**: $\theta \in [0, \pi]$, $(r_0, \phi_0) \to$ fixed

  $$\mathbf{r}(\theta) = r_0\,(\sin\theta \cos\phi_0, \sin\theta \sin\phi_0, \cos\theta)$$

- **ϕ-curve**: $\phi \in [0, 2\pi)$, $(r_0, \theta_0) \to$ fixed

  $$\mathbf{r}(\phi) = r_0\,(\sin\theta_0 \cos\phi, \sin\theta_0 \sin\phi, \cos\theta_0)$$



Spherical-Polar System: Unit Vectors & Scale Factors

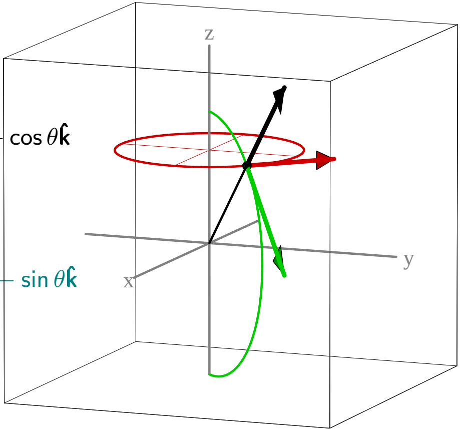
Unit Tangent Vectors to co-ordinate curves at a given point $\mathbf{r} = \mathbf{r}(r, \theta, \phi)$

$$\mathbf{r}(r,\theta,\phi) = r\sin\theta\cos\phi\hat{\mathbf{i}} + r\sin\theta\sin\phi\hat{\mathbf{j}} + r\cos\theta\hat{\mathbf{k}}$$

These vectors are not fixed in space, but depend on angles (θ, ϕ)

(h_r, h_θ, h_ϕ) are the **Scale factors**

$$\begin{aligned}
\mathbf{e}_r(\theta,\phi) &= \frac{\partial \mathbf{r}}{\partial r} \Big/ \left|\frac{\partial \mathbf{r}}{\partial r}\right| = \frac{\partial \mathbf{r}}{\partial r}/h_r \\
&= \sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}} \\
h_r &= 1, \\
\mathbf{e}_\theta(\theta,\phi) &= \frac{\partial \mathbf{r}}{\partial \theta} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = \frac{\partial \mathbf{r}}{\partial \theta}/h_\theta \\
&= \cos\theta\cos\phi\hat{\mathbf{i}} + \cos\theta\sin\phi\hat{\mathbf{j}} - \sin\theta\hat{\mathbf{k}} \\
h_\theta &= r, \\
\mathbf{e}_\phi(\phi) &= \frac{\partial \mathbf{r}}{\partial \phi} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = \frac{\partial \mathbf{r}}{\partial \phi}/h_\phi \\
&= -\sin\phi\hat{\mathbf{i}} + \cos\phi\hat{\mathbf{j}} \\
h_\phi &= r\sin\theta.
\end{aligned}$$



Spherical-Polar System: Orthonormal Basis Vectors

- Orthonormal system of unit vectors:

$$\begin{aligned}
\mathbf{e}_r \cdot \mathbf{e}_r &= 1, & \mathbf{e}_\theta \cdot \mathbf{e}_\theta &= 1, & \mathbf{e}_\phi \cdot \mathbf{e}_\phi &= 1, \\
\mathbf{e}_r \cdot \mathbf{e}_\theta &= 0, & \mathbf{e}_\theta \cdot \mathbf{e}_\phi &= 0, & \mathbf{e}_\phi \cdot \mathbf{e}_r &= 0, \\
\mathbf{e}_r \times \mathbf{e}_\theta &= \mathbf{e}_\phi, & \mathbf{e}_\theta \times \mathbf{e}_\phi &= \mathbf{e}_r, & \mathbf{e}_\phi \times \mathbf{e}_r &= \mathbf{e}_\theta
\end{aligned}$$

- Note: $\mathbf{e}_r \to \mathbf{e}_\theta \to \mathbf{e}_\phi$ are in cyclic order

- Cartesian unit vectors $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ are constants in space and do not depend on position, but spherical unit vectors especially depend on angles (θ, ϕ) :

$$\begin{aligned}
\frac{\partial \mathbf{e}_r}{\partial r} &= 0 \quad , & \frac{\partial \mathbf{e}_r}{\partial \theta} &= \mathbf{e}_\theta \quad , & \frac{\partial \mathbf{e}_r}{\partial \phi} &= \sin\theta \mathbf{e}_\phi \\
\frac{\partial \mathbf{e}_\theta}{\partial r} &= 0 \quad , & \frac{\partial \mathbf{e}_\theta}{\partial \theta} &= -\mathbf{e}_r \quad , & \frac{\partial \mathbf{e}_\theta}{\partial \phi} &= \cos\theta \mathbf{e}_\phi \\
\frac{\partial \mathbf{e}_\phi}{\partial r} &= 0 \quad , & \frac{\partial \mathbf{e}_\phi}{\partial \theta} &= 0 \quad , & \frac{\partial \mathbf{e}_\phi}{\partial \phi} &= -\sin\theta \mathbf{e}_r - \cos\theta \mathbf{e}_\theta
\end{aligned}$$

Co-ordinate Transformations: Cartesian $\Longleftrightarrow$ Spherical-Polar

Co-ordinate tranformations from Cartesian $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ to spherical $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ unit vectors:

$$\begin{pmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

Co-ordinate Transformations: Cartesian $\Longleftrightarrow$ Spherical-Polar

Co-ordinate tranformations from Cartesian $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ to spherical $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ unit vectors:

$$\begin{pmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \cos\theta\cos\phi & \cos\theta\sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{pmatrix} \begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix}$$

Inverse transformations from spherical $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$ to Cartesian $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$ unit vectors:

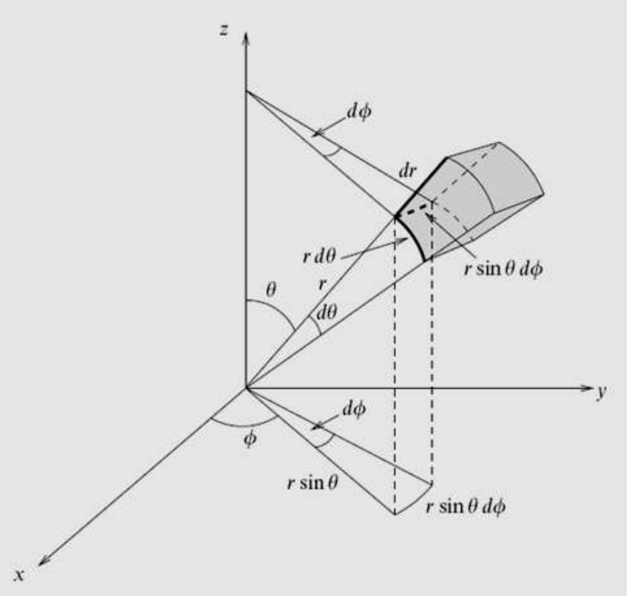
$$\begin{pmatrix} \hat{i} \\ \hat{j} \\ \hat{k} \end{pmatrix} = \begin{pmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{pmatrix} \begin{pmatrix} \hat{e}_r \\ \hat{e}_\theta \\ \hat{e}_\phi \end{pmatrix}$$

Note:

- The above matrices are orthognal matrices where, $M^T = M^{-1}$
- The same tranformation rules as above are applicable for transforming the components of a vector $\mathbf{A}(x, y, z) \equiv \mathbf{A}(r, \theta, \phi)$, i.e.,

$$(A_x,\ A_y,\ A_z) \Longleftrightarrow (A_r,\ A_\theta,\ A_\phi)$$

Spherical-Polar System: Line, Surface and Volume Elements



Spherical-Polar System: Line, Surface and Volume Elements (contd.)

Position vector to any point $P(x, y, z) \equiv (r, \theta, \phi)$ is $\overrightarrow{OP} = \mathbf{r} = \mathbf{r}(r, \theta, \phi)$.

Arc/Line Elements:

$$\begin{aligned} d\mathbf{r} = d\mathbf{r}(r,\theta,\phi) &= \left(\frac{\partial \mathbf{r}}{\partial r}\right) dr + \left(\frac{\partial \mathbf{r}}{\partial \theta}\right) d\theta + \left(\frac{\partial \mathbf{r}}{\partial \phi}\right) d\phi \\ &= (h_r \mathbf{e}_r)\, dr + (h_\theta \mathbf{e}_\theta)\, d\theta + (h_\phi \mathbf{e}_\phi)\, d\phi \\ &= 1\mathbf{e}_r\, dr + r\, \mathbf{e}_\theta\, d\theta + r \sin\theta\, \mathbf{e}_\phi\, d\phi \\ &\equiv \mathbf{e}_r ds_r + \mathbf{e}_\theta ds_\theta + \mathbf{e}_\phi ds_\phi \end{aligned}$$

$$ds_r = dr, \quad ds_\theta = r d\theta, \quad ds_\phi = r \sin\theta d\phi$$

Surface Elements:

Surface	Shape	Unit Normal	Elemental Area $d\mathbf{S}$
$r =$ const.	Sphere	$\mathbf{e}_r \equiv \hat{\mathbf{r}}$	$(\mathbf{e}_\theta \times \mathbf{e}_\phi)\, ds_\theta ds_\phi = r^2 \sin\theta\, d\theta\, d\phi\, \mathbf{e}_r$
$\theta =$ const.	Cone	$\mathbf{e}_\theta \equiv \hat{\theta}$	$(\mathbf{e}_\phi \times \mathbf{e}_r)\, ds_r ds_\phi = r \sin\theta\, dr\, d\phi\, \mathbf{e}_\theta$
$\phi =$ const.	Half Plane	$\mathbf{e}_\phi \equiv \hat{\phi}$	$(\mathbf{e}_r \times \mathbf{e}_\theta)\, ds_r ds_\theta = r\, dr\, d\theta\, \mathbf{e}_\phi$

Volume Element: With **Jacobian** $J = h_r h_\theta h_\phi = r^2 \sin\theta$

$$dV = ds_r ds_\theta ds_\phi = J dr\, d\theta\, d\phi = r^2 \sin\theta\, dr\, d\theta\, d\phi$$

Differential Operators In Spherical Coordinates

$\Phi(\mathbf{r})$ be a differentiable scalar field, and $\mathbf{A}(\mathbf{r})$, a differentiable vector field, then

- Gradient:

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi$$

- Divergence:

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(A_\theta\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

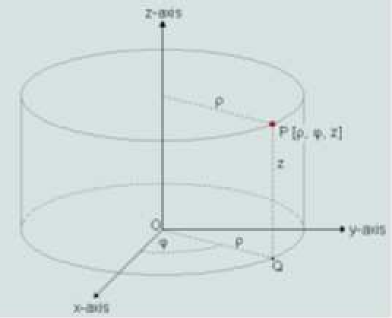
- Curl:

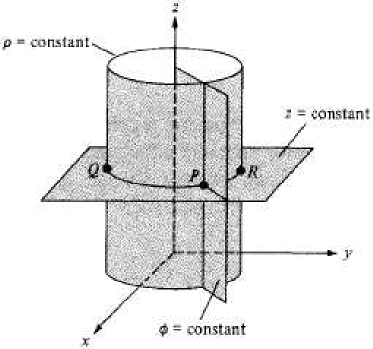
$$\begin{aligned}\nabla\times\mathbf{A} \quad &= \quad \frac{1}{r\sin\theta}\left[\frac{\partial(A_\phi\sin\theta)}{\partial\theta} - \frac{\partial A_\theta}{\partial\phi}\right]\mathbf{e}_r \\ &\quad + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial(rA_\phi)}{\partial r}\right]\mathbf{e}_\theta + \frac{1}{r}\left[\frac{\partial(rA_\theta)}{\partial r} - \frac{\partial A_r}{\partial\theta}\right]\mathbf{e}_\phi\end{aligned}$$

- Laplacian:

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Cylindrical Co-ordinates: Co-ordinate Surfaces & Axes





Transformation: Cartisian (x, y, z) to Cylinderical (ρ, ϕ, z)

$$x = \rho Cos\phi, \quad y = \rho Sin\phi, \quad z = z$$

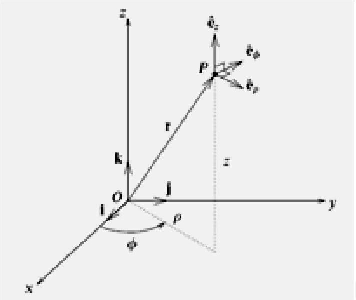
where $\rho \geq 0, \quad 0 \leq \phi \leq 2\pi, \quad -\infty \leq z \leq \infty$

$$\rho = \sqrt{x^2 + y^2}, \qquad \phi = \tan^{-1}\frac{y}{x}, \qquad z = z$$

The position vector of P can be written as

$$\vec{r} = \rho\ Cos\phi\ \hat{e}_x + \rho\ Sin\phi\ \hat{e}_y + z\ \hat{e}_z$$

Cylindrical Co-ordinates: Unit Vectors & Scale Factors



$$\hat{e}_\rho \cdot \hat{e}_\rho = 1, \quad \hat{e}_\phi \cdot \hat{e}_\phi = 1, \quad \hat{e}_z \cdot \hat{e}_z = 1,$$
$$\hat{e}_\rho \cdot \hat{e}_\phi = 0, \quad \hat{e}_\phi \cdot \hat{e}_z = 0, \quad \hat{e}_z \cdot \hat{e}_\rho = 0,$$
$$\hat{e}_\rho \times \hat{e}_\phi = \hat{e}_z, \; \hat{e}_\phi \times \hat{e}_z = \hat{e}_\rho, \; \hat{e}_z \times \hat{e}_\rho = \hat{e}_\phi$$

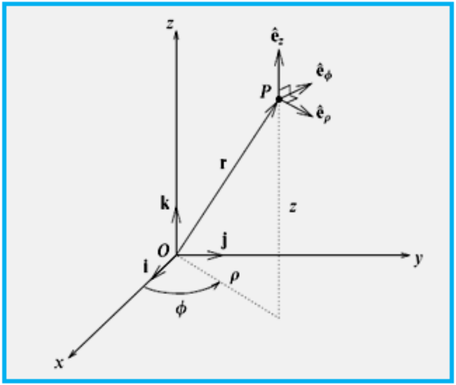
$$\hat{e}_\rho = \frac{\partial \vec{r}}{\partial \rho} \bigg/ \left| \frac{\partial \vec{r}}{\partial \rho} \right| = \frac{\partial \vec{r}}{\partial \rho} \bigg/ h_\rho = \cos\phi\, \hat{e}_x + \sin\phi \hat{e}_y; \qquad h_\rho = 1$$

$$\hat{e}_\phi = \frac{\partial \vec{r}}{\partial \phi} \bigg/ \left| \frac{\partial \vec{r}}{\partial \phi} \right| = \frac{\partial \vec{r}}{\partial \phi} \bigg/ h_\phi = -\sin\phi\, \hat{e}_x + \cos\phi\, \hat{e}_y; \qquad h_\phi = \rho$$

$$\hat{e}_z = \frac{\partial \vec{r}}{\partial z} \bigg/ \left| \frac{\partial \vec{r}}{\partial z} \right| = \frac{\partial \vec{r}}{\partial z} \bigg/ h_z = \hat{e}_z; \qquad h_z = 1$$

Note: $\mathbf{e}_\rho \rightarrow \mathbf{e}_\phi \rightarrow \mathbf{e}_z$ are in cyclic order

Cylindrical Co-ordinates: Unit Vectors



- Cartesian unit vectors are constants and do not depend on position, but cylindrical unit vectors do:

$$\frac{\partial \mathbf{e}_\rho}{\partial \rho} = 0 \quad , \quad \frac{\partial \mathbf{e}_\rho}{\partial \phi} = \mathbf{e}_\phi \quad , \quad \frac{\partial \mathbf{e}_\rho}{\partial z} = 0$$

$$\frac{\partial \mathbf{e}_\phi}{\partial \rho} = 0 \quad , \quad \frac{\partial \mathbf{e}_\phi}{\partial \phi} = -\mathbf{e}_r \quad , \quad \frac{\partial \mathbf{e}_\phi}{\partial z} = 0$$

$$\frac{\partial \mathbf{e}_z}{\partial \rho} = 0 \quad , \quad \frac{\partial \mathbf{e}_z}{\partial \phi} = 0 \quad , \quad \frac{\partial \mathbf{e}_z}{\partial z} = 0$$

Co-ordinate Transformations: Cartesian $\Longleftrightarrow$ Cylindrical

Tranformations from Cartesian $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}) \equiv (\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ to cylindrical $(\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z)$:

$$\begin{pmatrix} \hat{e}_\rho \\ \hat{e}_\phi \\ \hat{e}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix}$$

Inverse transformations from cylindrical $(\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z)$ to Cartesian $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$:

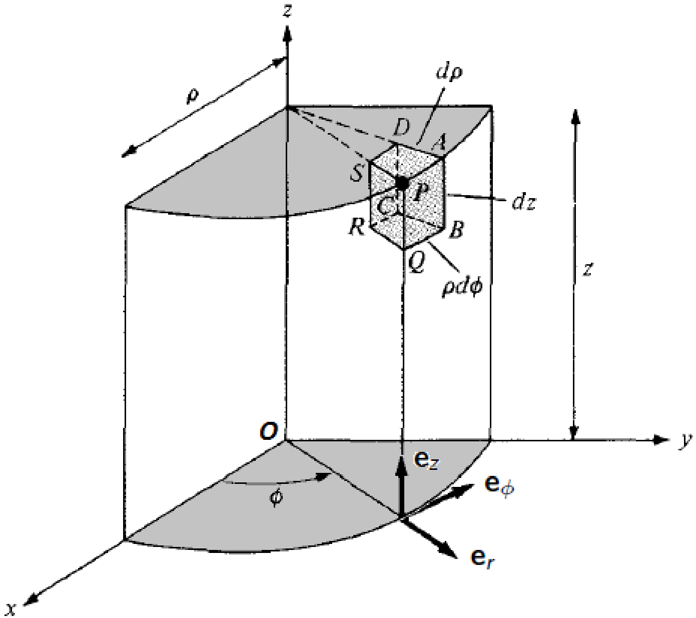
$$\begin{pmatrix} \hat{e}_x \\ \hat{e}_y \\ \hat{e}_z \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{e}_\rho \\ \hat{e}_\phi \\ \hat{e}_z \end{pmatrix}$$

Note:

- The above matrices are orthognal matrices where, $M^T = M^{-1}$
- that the same tranformation rules as above are applicable for transforming the components of a vector $\mathbf{A}(x, y, z) \equiv \mathbf{A}(\rho, \phi, z)$, i.e.,

$$(A_x,\ A_y,\ A_z) \Longleftrightarrow (A_\rho,\ A_\phi,\ A_z)$$

Co-ordinate System: Line, Surface and Volume Elements



Cylindrical System: Line, Surface and Volume Elements

Position vector to any point $P = (x, y, x) \equiv (\rho, \phi, z)$ is $\overrightarrow{OP} = \mathbf{r} = \mathbf{r}(\rho, \phi, z)$.

Arc/Line Elements:

$$\begin{aligned} d\mathbf{r} = d\mathbf{r}(\rho, \phi, z) &= \frac{\partial \mathbf{r}}{\partial \rho}\, d\rho + \frac{\partial \mathbf{r}}{\partial \phi}\, d\phi + \frac{\partial \mathbf{r}}{\partial z}\, dz \\ &= h_\rho \mathbf{e}_\rho\, d\rho + h_\phi \mathbf{e}_\phi\, d\phi + h_z \mathbf{e}_z\, dz \\ &= \mathbf{e}_\rho\, d\rho + \mathbf{e}_\phi\, \rho\, d\phi + \mathbf{e}_\mathbf{z}\, dz \\ &\equiv \mathbf{e}_\rho\, ds_\rho + \mathbf{e}_\phi\, ds_\phi + \mathbf{e}_\mathbf{z}\, ds_z \end{aligned}$$

$$ds_\rho = d\rho, \quad ds_\phi = \rho d\phi, \quad ds_z = dz$$

Surface Elements:

Surface	Shape	Unit Normal	Elemental Area $d\mathbf{S}$
$\rho = \text{const.}$	Cylinder	$\mathbf{e}_\rho \equiv \hat{\rho}$	$(\mathbf{e}_\phi \times \mathbf{e}_z)\, ds_\phi ds_z = \rho\, d\phi\, dz\, \mathbf{e}_\rho$
$\phi = \text{const.}$	Half Plane	$\mathbf{e}_\phi \equiv \hat{\phi}$	$(\mathbf{e}_z \times \mathbf{e}_\rho)\, ds_\rho ds_z = d\rho\, dz\, \mathbf{e}_\phi$
$z = \text{const.}$	Plane	$\mathbf{e}_z \equiv \hat{\mathbf{k}}$	$(\mathbf{e}_\rho \times \mathbf{e}_\phi)\, ds_\rho ds_\phi = \rho\, d\rho\, d\phi\, \mathbf{e}_z$

Volume Element: With **Jacobian** $J = h_\rho h_\phi h_z = \rho$

$$dV = ds_\rho ds_\phi ds_z = J\, d\rho\, d\phi\, dz = \rho\, d\rho\, d\phi\, dz$$

Differential Operators In Cylindrical Coordinates

$\Phi(\mathbf{r})$ be a differentiable scalar field, and $\mathbf{A}(\mathbf{r})$, a differentiable vector field, then

- Gradient:

$$\nabla\Phi = \frac{\partial\Phi}{\partial\rho}\mathbf{e}_\rho + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\mathbf{e}_\phi + \frac{\partial\Phi}{\partial z}\mathbf{e}_z$$

- Divergence:

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

- Curl:

$$\begin{aligned}\nabla\times\mathbf{A} &= \left[\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{e}_\rho \\ &\quad + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right]\mathbf{e}_\phi + \frac{1}{\rho}\left[\frac{\partial(\rho A_\phi)}{\partial\rho} - \frac{\partial A_\rho}{\partial\phi}\right]\mathbf{e}_z\end{aligned}$$

- Laplacian:

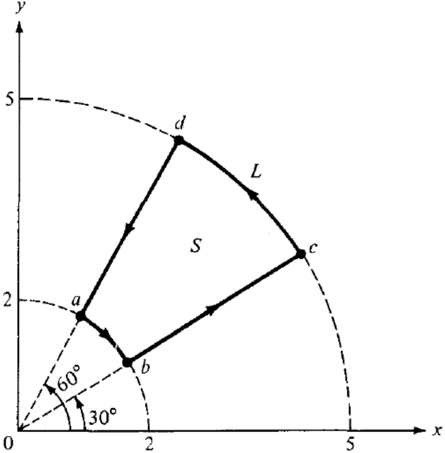
$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} + \frac{\partial^2\Phi}{\partial z^2}$$

Verification of Stokes' Theorem in Cylindrical (ρ, ϕ, z) System

Example

Note: Unit vector symbols $(\mathbf{a}_\rho, \mathbf{a}_\phi, \mathbf{a}_z) \equiv (\mathbf{e}_\rho, \mathbf{e}_\phi, \mathbf{e}_z)$ is used in the book on Electrodynamics by *Sadiku*. This example is taken from there.

If $\mathbf{A} = \rho \cos\phi \, \mathbf{a}_\rho + \sin\phi \, \mathbf{a}_\phi$, evaluate $\oint \mathbf{A} \cdot d\mathbf{l}$ around the path Confirm this using Stokes's theorem.



Note: Line/Arc element is $d\mathbf{r} \equiv d\mathbf{l} = \mathbf{a}_\rho \, d\rho + \mathbf{a}_\phi \, \rho \, d\phi + \mathbf{a}_z \, dz$; $\quad dz = 0$

Solution:

Let
$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \left[\int_a^b + \int_b^c + \int_c^d + \int_d^a \right] \mathbf{A} \cdot d\mathbf{l}$$

where path L has been divided into segments ab, bc, cd, and da as in Figure.

Along ab, $\rho = 2$ and $d\mathbf{l} = \rho \, d\phi \, \mathbf{a}_\phi$. Hence,

$$\int_a^b \mathbf{A} \cdot d\mathbf{l} = \int_{\phi=60^\circ}^{30^\circ} \rho \sin\phi \, d\phi = 2(-\cos\phi) \Big|_{60^\circ}^{30^\circ} = -(\sqrt{3} - 1)$$

Along bc, $\phi = 30^\circ$ and $d\mathbf{l} = d\rho \, \mathbf{a}_\rho$. Hence,

$$\int_b^c \mathbf{A} \cdot d\mathbf{l} = \int_{\rho=2}^{5} \rho \cos\phi \, d\rho = \cos 30^\circ \frac{\rho^2}{2} \Big|_2^5 = \frac{21\sqrt{3}}{4}$$

Along cd, $\rho = 5$ and $d\mathbf{l} = \rho \, d\phi \, \mathbf{a}_\phi$. Hence,

$$\int_c^d \mathbf{A} \cdot d\mathbf{l} = \int_{\phi=30^\circ}^{60^\circ} \rho \sin\phi \, d\phi = 5(-\cos\phi) \Big|_{30^\circ}^{60^\circ} = \frac{5}{2}(\sqrt{3} - 1)$$

Along da, $\phi = 60°$ and $d\mathbf{l} = d\rho\, \mathbf{a}_\rho$. Hence,

$$\int_d^a \mathbf{A} \cdot d\mathbf{l} = \int_{\rho=5}^{2} \rho \cos\phi \, d\rho = \cos 60° \frac{\rho^2}{2}\Bigg|_5^2 = -\frac{21}{4}$$

Putting all these together results in

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = -\sqrt{3} + 1 + \frac{21\sqrt{3}}{4} + \frac{5\sqrt{3}}{2} - \frac{5}{2} - \frac{21}{4}$$
$$= \frac{27}{4}(\sqrt{3} - 1) = 4.941$$

Using Stokes's theorem (because L is a closed path)

$$\oint_L \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$$

But $d\mathbf{S} = \rho\, d\phi\, d\rho\, \mathbf{a}_z$ and

$$\nabla \times \mathbf{A} = \mathbf{a}_\rho\left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right] + \mathbf{a}_\phi\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] + \mathbf{a}_z \frac{1}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right]$$
$$= (0 - 0)\mathbf{a}_\rho + (0 - 0)\mathbf{a}_\phi + \frac{1}{\rho}(1 + \rho)\sin\phi\, \mathbf{a}_z$$

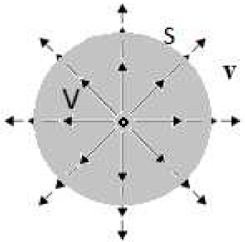
$$\iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \int_{\phi=30^\circ}^{60^\circ} \int_{\rho=2}^{5} \frac{1}{\rho}(1+\rho)\sin\phi \, \rho \, d\rho \, d\phi$$

$$= \int_{30^\circ}^{60^\circ} \sin\phi \, d\phi \int_2^5 (1+\rho) d\rho$$

$$= -\cos\phi \Big|_{30^\circ}^{60^\circ} \left(\rho + \frac{\rho^2}{2}\right) \Big|_2^5$$

$$= \frac{27}{4}(\sqrt{3} - 1) = 4.941$$

Divergence of Inverse Square Vector Field: The Delta Function



Surface element of the enclosed sphere

$$dA = R^2 \sin\theta \, d\theta \, d\phi$$

The vector function

$$\mathbf{v} = \frac{1}{r^2}\hat{r}$$

Divergence $\nabla\cdot$ of the above function

$$\vec{\nabla}\cdot\vec{\mathbf{v}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = 0$$

$$\Downarrow$$

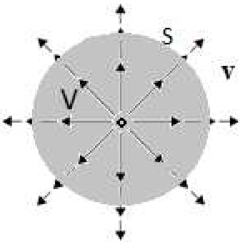
$$\iiint_V \vec{\nabla}\cdot\vec{\mathbf{v}}\, d\tau = 0\,!!$$

$$\iiint_V \vec{\nabla}\cdot\vec{\mathbf{v}}\, d\tau = \oiint_S \vec{\mathbf{v}}\cdot\hat{r}\, dA$$

$$= \oiint_S \left(\frac{1}{R^2}\hat{r}\right)\cdot\left(R^2 \sin\theta\, d\theta\, d\phi\, \hat{r}\right)$$

$$= \int_0^{\pi} \sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi$$

Divergence of Inverse Square Vector Field: The Dirac-Delta



Surface element of the enclosed sphere

$$dA = R^2\, sin\theta\, d\theta\, d\phi$$

The vector function

$$\mathbf{v} = \frac{1}{r^2}\hat{r}$$

Divergence $\nabla\cdot$ of the above function

$$\vec{\nabla}\cdot\vec{\mathbf{v}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}(1) = \cancel{0}$$

$$\Downarrow$$

$$\cancel{\iiint_V \vec{\nabla}\cdot\vec{\mathbf{v}}\, d\tau = 0\,!!}$$

$$\iiint_V \vec{\nabla}\cdot\vec{\mathbf{v}}\, d\tau = \oiint_S \vec{\mathbf{v}}\cdot\hat{r}\, dA$$

$$= \oiint_S \left(\frac{1}{R^2}\hat{r}\right)\cdot\left(R^2\, Sin\theta\, d\theta\, d\phi\, \hat{r}\right)$$

$$= \int_0^{\pi} Sin\theta\, d\theta \int_0^{2\pi} d\phi = 4\pi$$

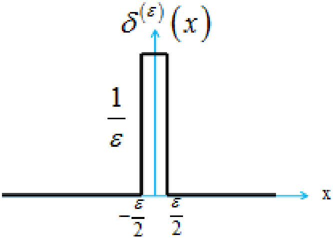
Divergence of Inverse Square Vector Field: The Dirac-Delta (contd.)

- It is true that $\nabla \cdot \mathbf{v} = 0$ everywhere, except at $\mathbf{r} = 0$
- The source of the problem is that $\nabla \cdot \mathbf{v} \neq 0$ at $\mathbf{r} = 0$, where the divergence blows up!
- To ensure validity of the Volume Integral and the Divergence Theorem we must assign a functional form of $\nabla \cdot \mathbf{v}$, $\forall \mathbf{r}$, and termed as the **3-dim Dirac-Delta Function**:

$$\frac{\nabla \cdot \mathbf{v}}{4\pi} \equiv \delta^3(\mathbf{r}) = \begin{cases} 0 & \text{if} \quad \mathbf{r} \neq 0 \\ \infty & \text{if} \quad \mathbf{r} = 0 \end{cases} \qquad \Longleftrightarrow \iiint_V \delta^3(\mathbf{r}) d\tau = 1$$

This bizarre property of δ-function that it vanishes everywhere except at the origin $\mathbf{r} = 0$, and yet its integral over ANY volume enclosing the origin has a finite value (i.e.,4π), makes this "function" different from standard functions and can rather be termed as a "distribution" or a "generalized function".

The Delta Step Function in 1D



$$\delta^{(\varepsilon)}(x) = \begin{cases} \frac{1}{\varepsilon} & for \quad -\frac{\varepsilon}{2} < x < \frac{\varepsilon}{2} \\ 0 & for \quad |x| > \frac{\varepsilon}{2} \end{cases}$$

f(x) is arbitrary function and well defined at x=0

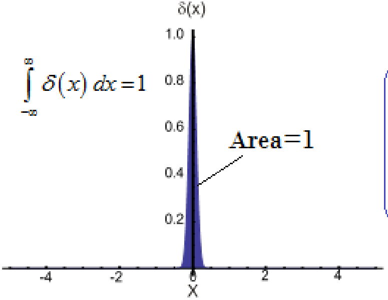
$$\int_{-\infty}^{\infty} f(x)\,\delta^{(\varepsilon)}(x)\,dx = \int_{-\varepsilon/2}^{\varepsilon/2} f(x)\,\delta^{(\varepsilon)}(x)\,dx$$

$$\cong f(0) \int_{-\varepsilon/2}^{\varepsilon/2} \delta^{(\varepsilon)}(x)\,dx \cong f(0)$$

The smaller ε , the better the approximation. Therefore, at the limit of ε→0, we define the Dirac delta function as

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

The Dirac-Delta Function in 1D [$\delta^{(\varepsilon\to 0)}(x)$]



Definition

$$\delta(x) = \begin{cases} 0 & if \quad x \neq 0 \\ \infty & if \quad x = 0 \end{cases}$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x)\,dx = f(0)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-a)\,dx = f(a)$$

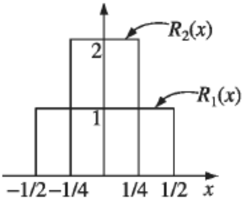
Functions which approach the δ-function as a limit of a sequence as $\varepsilon \to 0$

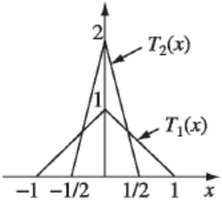
$$\frac{1}{2\varepsilon} e^{-\frac{|x|}{\varepsilon}}, \quad \frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}, \quad \frac{1}{\varepsilon\sqrt{\pi}} e^{-\frac{x^2}{\varepsilon^2}},$$

$$\frac{1}{\pi}\frac{Sin\left(\frac{x}{\varepsilon}\right)}{x}, \quad \frac{\varepsilon}{\pi}\frac{Sin^2\left(\frac{x}{\varepsilon}\right)}{x^2}$$

Note: There is no unique way in defining the Dirac δ-function !

The Dirac-Delta Function: As the Limit of a Sequence of Fucnctons





Technically, $\delta(x)$ is not a function at all, since its value is not finite at $x = 0$; in the mathematical literature it is known as a **generalized function**, or **distribution**. It is, if you like, the *limit* of a *sequence* of functions, such as rectangles $R_n(x)$, of height n and width $1/n$, or isosceles triangles $T_n(x)$, of height n and base $2/n$

$$R_1(x),\ R_2(x),\ R_3(x), \cdots,\ \lim_{n\to\infty} R_n(x) \quad \to \quad \delta(x)$$
$$T_1(x),\ T_2(x),\ T_3(x), \cdots,\ \lim_{n\to\infty} T_n(x) \quad \to \quad \delta(x)$$

Facts about definition of Dirac δ-function in 1D: A Summary

- Infinitely high and vanishingly thin spike, with the total area under the curve being unity.
- Different from STANDARD FUNCTIONS, since any standard function that is equal to zero everywhere and ∞ at a single point must have total integral zero.
- GENERALIZED FUNCTION or a DISTRIBUTION which can be obtained in the "limiting sequence" of an infinitely many functions.
- POINT DENSITY FUNCTION: Physically, its represents density of an idealized point mass, charge, etc., $\lambda = M, Q, \cdots$ located at, say, $x = c$, i.e,

$$\lambda\delta(x-c) = \begin{Bmatrix} 0, & \text{if} \quad x \neq c \\ \infty, & \text{if} \quad x = c \end{Bmatrix} \quad \text{with} \quad \int_{-\infty}^{\infty} \lambda\delta(x-c)dx = \lambda$$

- ONLY makes sense when used *under an integral sign*. When convoluted with a well-defined test function $f(x)$, the delta function "picks out" the value of a function at the location of the δ-function:

$$\int_{-\infty}^{\infty} f(x)\,\delta(x-c)\,dx = \int_{-\infty}^{\infty} f(c)\,\delta(x-c)\,dx = f(c)$$

Properties of Dirac δ-function in 1D (Prove them!)

1. **Convolution:** $f(x)\delta(x-a) = f(a)\delta(x-a)\,,\ a \in \mathbb{R}$
2. **Even function:** $\delta(-x) = \delta(x) \equiv \delta(|x|)$
3. **Scaling:** $\delta(ax) = \frac{1}{|a|}\delta(x)\,,\ a \in \mathbb{R}$
4. **Product:** $\delta(x-y)\delta(x-z) = \delta(z-y)\delta(x-z) = \delta(x-y)\delta(y-z)$
5. **Derivative:** $x\delta'(x) = -\delta(x)$
6. **Derivative is an Odd function:** $\delta'(-x) = -\delta'(x)$

Note: All the above properties must be understood under the integral sign, i.e., if $f(x)$ is well-defined test function then, e.g., (3) must be interpretted as:

$$\int_{-\infty}^{\infty} f(x)\,\delta(ax)\,dx = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{|a|}\,\delta(x)\right]\,dx$$

Proof of (6): Using integration by parts and the property (2),

$$\int_{-\infty}^{\infty} f(x)\,\delta'(x)\,dx = f(x)\delta(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\,\delta(x)\,dx = -f'(0)$$

$$\int_{-\infty}^{\infty} f(x)\,\delta'(-x)\,dx = f(x)\delta(-x)\Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} f'(x)\,\delta(-x)\,dx$$

$$= \int_{-\infty}^{\infty} f'(x)\,\delta(x)\,dx = f'(0)$$

$$=> \quad \delta'(-x) = -\delta'(x)$$

The 3D Dirac δ-function in Cartesian System (Note: $d^3r \equiv dV \equiv d\tau$)

$$\iiint_V f(\mathbf{r})\,\delta^3(\mathbf{r})\,d^3r = f(0) \quad ; \quad \text{V => All space}$$

$$\delta^3(x,y,z) = \delta^3(\mathbf{r}) = \begin{cases} 0 & \text{if} \quad x^2+y^2+z^2 \neq 0 \\ \infty & \text{if} \quad x^2+y^2+z^2 = 0 \end{cases}$$

$$\iiint_V \delta^3(\mathbf{r})\,d^3r = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta^3(x,y,z)\;dx\,dy\,dz = 1$$

More generally,

$$\iiint_V f(\mathbf{r})\,\delta^3(\mathbf{r}-\mathbf{r}_0)\,d^3r = f(\mathbf{r}_0)$$

$\delta^3(\mathbf{r}-\mathbf{r}_0)$ can be split into a product of three one dimensional functions

$$\delta^3(\mathbf{r}-\mathbf{r}_0) = \delta(x-x_0)\,\delta(y-y_0)\,\delta(z-z_0)$$

The 3D Dirac δ-function in Curvilinear Co-ordinates (q_1, q_2, q_3)

In general curvilinear co-ordinates with $\mathbf{r} = \mathbf{r}(q_1, q_2, q_3)$, the tranformation from Cartesian form, i.e.,

$$\delta^3(\mathbf{r} - \mathbf{r}_0) \propto \delta(q_1 - q_1^0)\delta(q_2 - q_2^0)\delta(q_3 - q_3^0)$$

is given as:

$$\boxed{\delta^3(\mathbf{r} - \mathbf{r}_0) = \frac{\delta^3(q_1 - q_1^0, q_2 - q_2^0, q_3 - q_3^0)}{J} = \frac{\delta(q_1 - q_1^0)\delta(q_2 - q_2^0)\delta(q_3 - q_3^0)}{h_1 h_2 h_3}}$$

where $\mathbf{r}_0 \equiv \mathbf{r}_0(q_1^0, q_2^0, q_3^0)$ and h_1, h_2, h_3 are the scale factors.

- **Spherical-Polar System** with $\mathbf{r}_0 \equiv \mathbf{r}_0(r_0, \theta_0, \phi_0)$ and scale factors $h_r = 1, h_\theta = r, h_\phi = r \sin\theta$:

$$\delta^3(\mathbf{r} - \mathbf{r}_0) = \frac{\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0)}{r^2 \sin\theta}$$

- **Cylindrical System** with $\mathbf{r}_0 \equiv \mathbf{r}_0(\rho_0, \phi_0, z_0)$ and scale factors $h_\rho = 1, h_\phi = \rho, h_z = 1$:

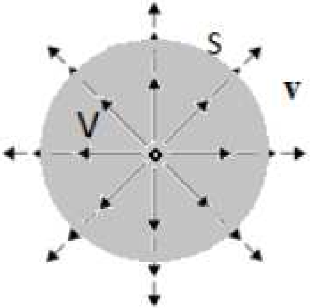
$$\delta^3(\mathbf{r} - \mathbf{r}_0) = \frac{\delta(\rho - \rho_0)\delta(\phi - \phi_0)\delta(z - z_0)}{\rho}$$

Revisiting $\nabla \cdot (\hat{r}/r^2)$

We define :

$$\boxed{\vec{\nabla} \cdot \vec{\mathbf{v}} = 4\pi\, \delta^{(3)}(\vec{r})}$$

$$d^3r \;=\; r^2\, Sin\theta\, dr\, d\theta\, d\phi$$



Then,

$$\iiint\limits_{V} \vec{\nabla} \cdot \vec{\mathbf{v}}\, d^3r \;=\; \iiint\limits_{V} 4\pi\, \delta^{(3)}(\vec{r})\, d^3r$$

$$= 4\pi \int_0^R \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} \frac{\delta(r)\,\delta(\theta)\,\delta(\phi)}{r^2\, Sin\theta} \left(r^2\, Sin\theta\, dr\, d\theta\, d\phi\right)$$

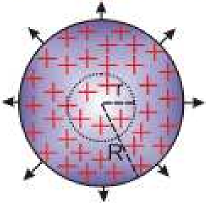
$$= 4\pi$$

Application of the 3D δ-function

Example

In spherical coordinates, a charge Q uniformly distributed over a spherical shell of radius R. Find the three dimensional charge density $\rho(\mathrm{r})$ by using Dirac delta functions.

Solution:



Here the 3D charge density reduces to a 1D charge density along r

Let $\rho(\mathbf{r}) = f\,Q\,\delta(r-R)$, where f is to be determined

$$Q = \int \rho(\mathbf{r})\,dv = \int_0^R \int_{\theta=0}^{\pi} \int_{\phi=0}^{2\pi} f\,Q\,\delta(r-R)\left(r^2 \sin\theta\, dr\, d\theta\, d\phi\right)$$

$$= \int_0^R f\,Q\,\delta(r-R)\,4\pi r^2\,dr$$

$$= 4\pi R^2 f\,Q$$

$$\boxed{\rho(\mathbf{r}) = \frac{Q\,\delta(r-R)}{4\pi R^2}}$$