Physics II (PH 102)

Electromagnetism (Lecture 11)

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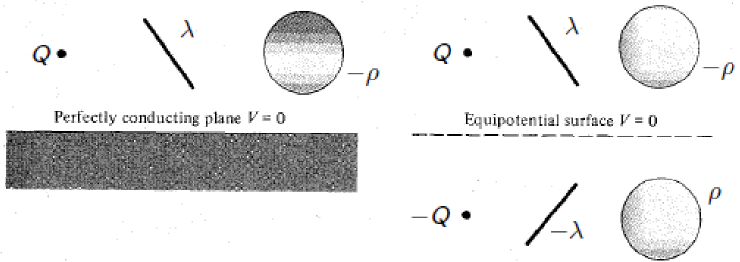
Method of Images: Avoids solving PDEs in Boundary Valued Problems

Method of Images: Invented by Lord Kelvin in 1848, commonly used to determine V, $\mathbf{E}$ and σ (surface charge density) due to static charge configurations in the presence of a system of conductors.

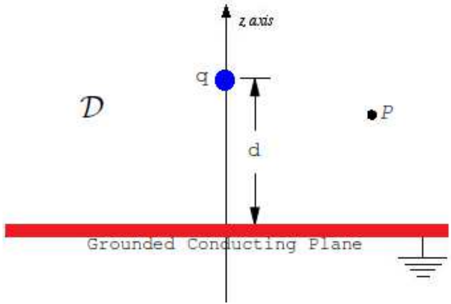
1. Central idea: Map the original hard problem to another easier problem, *but satisfying the same boundary conditions*. Then *Uniqueness Theorem* guarantees the correctness of the solution.
2. Use Fact: All conducting surfaces are represented by equipotentials.
3. Strategy: All Real charge configurations and conducting surfaces are replaced by the same Real charges, equipotential surfaces and some additional **fictitious** charges or charge distributions in the conducting region, called *Image Charges*.



The Classic Image Problem

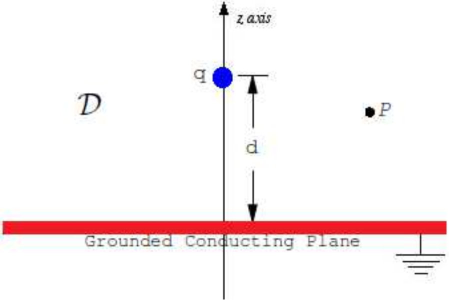
Example

Suppose a point charge q is held at a distance d above a infinite grounded conducting plane. What is the Electrostatic Potential at point P in the non-conducting region $\mathcal{D}$ above the conducting plane?



- The Electrostatic Potential V at point $P \equiv (x, y, z)$ will be due to point charge q and the induced surface charges.
- Problem is, we do not know $\sigma(x, y)$ *a priori* ! How to determine $V(x, y, z)$ without directly knowing $\sigma(x, y)$ on the conducting plane?

Infinite grounded conducting plane



Set up a co-ordinate system with xy-plane as the given infinite conducting plane and q lies on the z-axis:

- Sol. Domain: $\mathcal{D} = \{\mathbf{r} \mid z > 0\}$
- Boundary Surfaces of $\mathcal{D}$:

$$S = \{xy\text{-plane}\} \cup S_{\infty+}$$

- Point Charge density: $\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}_0)$; $\mathbf{r}_0 = (0, 0, d)$

- $V(x, y, z)$ satisfies **Poisson's Equation** $\forall \mathbf{r} \in \mathcal{D}$:

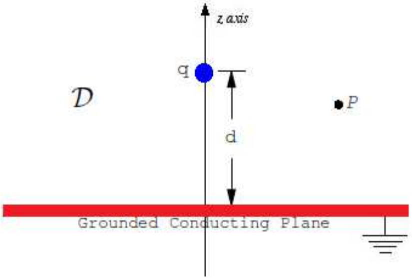
$$\nabla^2 V(\mathbf{r}) = \frac{1}{\epsilon_0}\rho(\mathbf{r})$$

- **Boundary Condition** on V in the original problem:

$$V(\mathbf{S}) = 0, \quad \forall \mathbf{S} \in S.$$

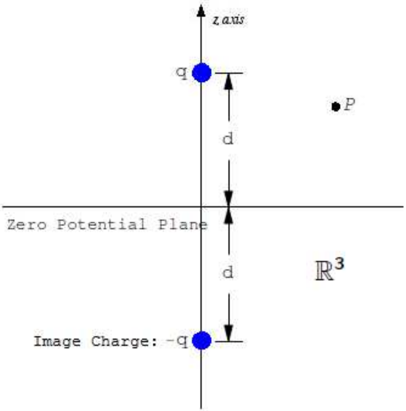
Infinite grounded conducting plane (contd.)

Real System is mapped on to the Fictitious System satisfying the same b.c.



Potential $V(x, y, z)$ is ONLY needed in non-conducting region $\mathcal{D}$

The Real System



Potential $V'(x, y, z)$ defined in the whole of $\mathbb{R}^3$

The Fictitious System

Infinite grounded conducting plane (contd.)

Consider the **Fictitious System**:

- Charge distribution: $\rho'(\mathbf{r}) = q\delta^3(\mathbf{r}-\mathbf{r}_0) + (-q)\delta^3(\mathbf{r}+\mathbf{r}_0); \quad \mathbf{r}_0 = (0,0,d)$
- Electrostatic Potential in all of $\mathbb{R}^3$ $\rightarrow$ Trivial to calculate!

$$V'(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} + \frac{(-q)}{\sqrt{x^2+y^2+(z+d)^2}}\right]$$

- $V'(\mathbf{r})$ satisfies **Poisson's Equation** $\forall \mathbf{r} \in \mathbb{R}^3$:

$$\nabla^2 V'(\mathbf{r}) = \frac{1}{\epsilon_0}\rho'(\mathbf{r})$$

- **Boundary Condition**: $V'(\mathbf{S}') = 0, \quad \forall \mathbf{S}' \in S' = \{xy\text{-plane}\} \cup S_\infty$

1. Real charge configuration in the common region $\mathcal{D}$ is identical:

$$\rho'(\mathbf{r})|_{\mathcal{D}} \xrightarrow{z>0} \rho(\mathbf{r})$$

2. Boundary conditions in the common region $\mathcal{D}$ are identical:

$$\nabla^2 V' = \frac{1}{\epsilon_0}\rho' \quad \text{over } \mathbb{R}^3 \quad \xrightarrow{z>0} \quad \nabla^2 V = \frac{1}{\epsilon_0}\rho \quad \text{over } \mathcal{D} \subset \mathbb{R}^3,$$

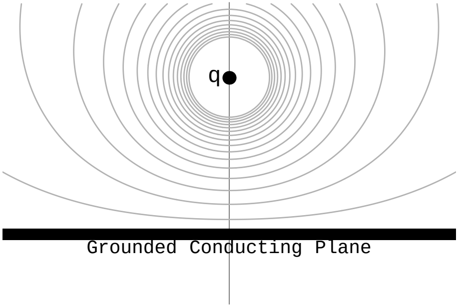
$$V' = 0 \quad \text{on } S' \quad \xrightarrow{z>0} \quad V = 0 \quad \text{on } S \subset S'$$

Infinite grounded conducting plane: Potential

- Uniqueness Theorem guarantees unique solution in $\mathcal{D}$, i,e., $V = V'$
- Thus, we found the solution to the original problem!

Electrostatic Potential in $\mathcal{D}$: $V(x, y, z) = V'(x, y, z \geq 0)$, i.e.,

$$V(x, y, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z+d)^2}}\right]$$

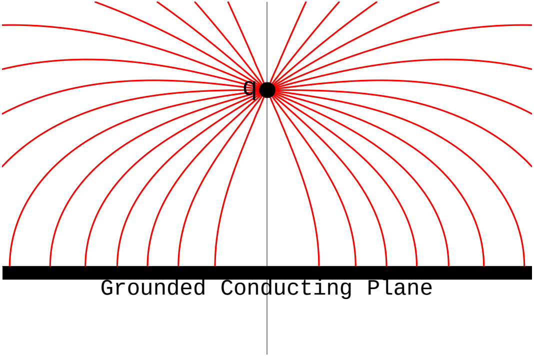


Note: We *needn't bother* that $V = V'$ yields wrong result for $z < 0$!

Infinite grounded conducting plane: Electric Field

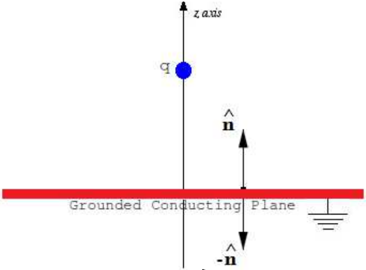
Electrostatic Field in $\mathcal{D}$: $\mathbf{E}(x, y, z) = \mathbf{E}'(x, y, z \geq 0) = -\nabla V(x, y, z)$, i.e.,

$$\mathbf{E}(x, y, z) = \frac{q}{4\pi\epsilon_0}\left[\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z-d)\hat{\mathbf{k}}}{(x^2 + y^2 + (z-d)^2)^{3/2}} - \frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + (z+d)\hat{\mathbf{k}}}{(x^2 + y^2 + (z+d)^2)^{3/2}}\right]$$



Note: Again we *needn't bother* that $\mathbf{E}(\mathbf{r}) = \mathbf{E}'(\mathbf{r})$ yields wrong result for $z < 0$!

Infinite grounded conducting plane: Surface Charge Density



- Normal to conductor: $\pm\hat{\mathbf{n}} = \pm\hat{\mathbf{k}}$
- Induced surface charge density $E_\perp = \sigma/\epsilon_0$:

$$\begin{aligned}
\sigma(x,y) &= -\epsilon_0 \left.\frac{\partial V(x,y,z)}{\partial n}\right|_{z=0} \\
&\equiv \epsilon_0 \mathbf{E}(x,y,z=0)\cdot\hat{\mathbf{k}} \\
\sigma(x,y) &= -\frac{q}{2\pi}\frac{d}{(x^2+y^2+d^2)^{3/2}}
\end{aligned}$$

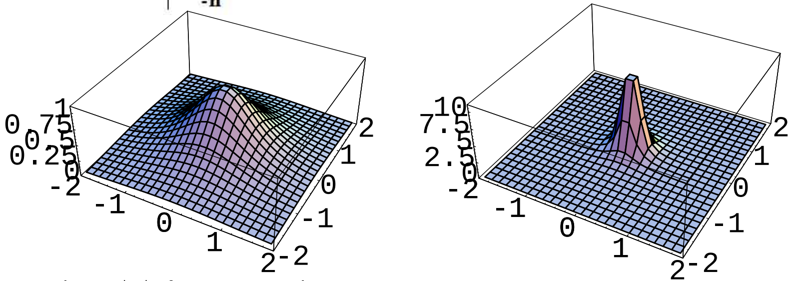


Figure shows $|\sigma|$, for $d = 1$ and $d = 0.1$

$\Longrightarrow$ Maximum induced charge density is right below the point charge

Infinite grounded conducting plane: Total Induced Charge

- Total induced surface charge: (with $dA = dx\, dy = s\, ds\, d\phi$)

$$
\begin{aligned}
Q_{\text{induced}} = \iint\limits_{xy\text{-plane}} \sigma(x,y)\, dA \quad &= \quad -\int\limits_{-\infty}^{\infty} dx \int\limits_{-\infty}^{\infty} dy\, \frac{qd}{2\pi} \frac{1}{\left(x^2+y^2+d^2\right)^{3/2}} \\
&= \quad -\frac{qd}{2\pi} \int_0^{\infty} \frac{s\, ds}{\left(s^2+d^2\right)^{3/2}} \int_0^{2\pi} d\phi \\
&= \quad -\frac{qd}{2\pi} \left[-\frac{1}{\left(s^2+d^2\right)^{1/2}} \right]_{s=0}^{s=\infty} (2\pi) \\
Q_{\text{induced}} \quad &= \quad -q
\end{aligned}
$$

Infinite grounded conducting plane: Force on Real Charge q

Electric Field at $P(x, y, z)$ in $\mathcal{D}$, that we have already calculated:

$$\mathbf{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\left[q\left(\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+(z-d)\hat{\mathbf{k}}}{(x^2+y^2+(z-d)^2)^{3/2}}\right)+(-q)\left(\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+(z+d)\hat{\mathbf{k}}}{(x^2+y^2+(z+d)^2)^{3/2}}\right)\right]$$

- The first (second) term is the field due to the *Real Charge q (Image Charge $-q$)*.
- Electric Field $\to$ *Induced charges* $\equiv$ Electric Field $\to$ *Image charge*:

$$\mathbf{E}_{\rm induced}(x,y,z) \equiv \mathbf{E}_{\rm image}(x,y,z) = \frac{(-q)}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}}+y\hat{\mathbf{j}}+(z+d)\hat{\mathbf{k}}}{(x^2+y^2+(z+d)^2)^{3/2}}$$

- Force on q due to conducting plane $\equiv$ Force on q due to Image charge:

$$\begin{aligned}\mathbf{F}_q &= q\mathbf{E}_{\rm induced}(0,0,d) \equiv q\mathbf{E}_{\rm image}(0,0,d)\\ &= -\frac{q^2\hat{\mathbf{z}}}{4\pi\epsilon_0(2d)^2} \to \text{Attractive force}\end{aligned}$$

***QUESTION:** Is there any difference in calculated physical quantities in non-conducting region $\mathcal{D}$, between those obtained from the *Fictitious System* (charge-image) and those from the *Real System* (charge-conductor)?

Infinite grounded conducting plane: Electrostatic Energy??

- Configuration energy of *Real System:*
  Work done by external agent to assemble the **charge-conductor system** is

$$W_1^{\text{ext}}(\text{Real}) = -\int_{z=\infty}^{z=d} \mathbf{F}_q(z)\cdot \mathbf{dz} = \int_{\infty}^{d} \frac{q^2\,dz}{4\pi\epsilon_0(2z)^2} = -\frac{1}{2}\left(\frac{q^2}{8\pi\epsilon_0 d}\right)$$

- Configuration energy of *Fictitious System:*
  Work done by external agent to assemble the **charge-image system** is

$$W_2^{\text{ext}}(\text{Fictitious}) = -\frac{q^2}{4\pi\epsilon_0(2d)} = -\frac{q^2}{8\pi\epsilon_0 d}!$$

$\Longrightarrow$ *It takes only half the amount of energy to assemble the Real System!!*

- Intuitive way of understanding this difference is to use the **integral formula**:

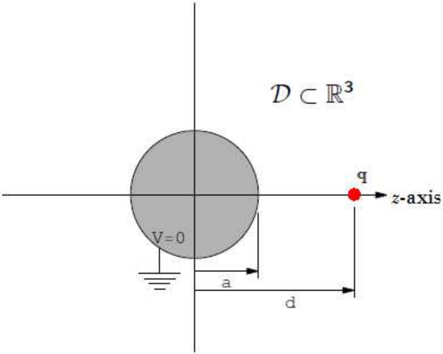
$$U_E(\text{Fictitious}) = \frac{\epsilon_0}{2}\iiint\limits_{\mathbb{R}^3} E^2\,dv = 2\cdot\frac{\epsilon_0}{2}\iiint\limits_{\mathcal{D}} E^2\,dv = 2U_E(\text{Real}).$$

$\Longrightarrow$ *The true domain of integration $\mathcal{D}$ is only half the domain $\mathbb{R}^3$ for the Fictitious System.*

Another classic image problem: Grounded Conducting Sphere

Example

Consider a grounded conducting sphere of radius a and a charge q held at a distance of d from the center. What is the potential in region $\mathcal{D}$ outside the conducting sphere?



Potential $V(x, y, z)$ is needed in $\mathcal{D}$

The Real System

- Set up co-ordinate system with z-axis along the line joining the center and q
- Domain: $\mathcal{D} \subset \mathbb{R}^3 = \{\mathbf{r} \mid r > a\}$
- Surface/s: $S = \{\mathbf{r} \mid r = a\} \cup S_\infty$
- Point charge density:

$$\rho(\mathbf{r}) = q\delta^3\left(\mathbf{r} - d\hat{\mathbf{k}}\right)$$

- $V(\mathbf{r})$ satisfies **Poisson's Equation**:

$$\nabla^2 V(\mathbf{r}) = \frac{1}{\epsilon_0}\rho(\mathbf{r}), \quad \forall \mathbf{r} \in \mathcal{D}$$

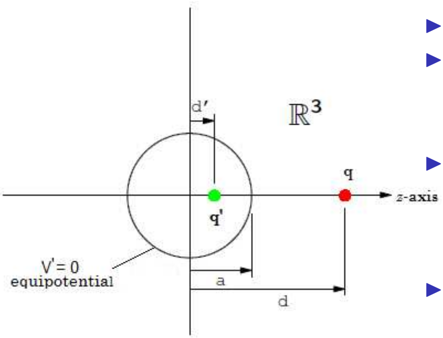
- **Boundary Condition** for Potential:

$$V(\mathbf{S}) = 0, \quad \forall \mathbf{S} \in S$$

Grounded Conducting Sphere

Replace Real System with Fictitious System: *Real charge q, Image charge q' & Equipotential surface* $\Longrightarrow$ $V'(\mathbf{r})$ in $\mathbb{R}^3$ is identical to $V(\mathbf{r})$ in $\mathcal{D}$.

Note: You should never put the Image charge in $\mathcal{D}$ where you want to calculate the potential. It should not mater if V' yields the wrong answer outside $\mathcal{D}$!



Potential $V'(x, y, z)$ defined in $\mathbb{R}^3$

The Fictitious System

- Location of q' is $(0, 0, d')$ with $d' < a$
- Point charge densities:

$$\rho'(\mathbf{r}) = \left[q\delta^3(\mathbf{r} - d\hat{\mathbf{k}}) + q'\delta^3(\mathbf{r} - d'\hat{\mathbf{k}})\right]\Big|_{\mathcal{D}} \xrightarrow{r>a} \rho(\mathbf{r})$$

- V'&V satisfy Poisson's Eqns. in $\mathbb{R}^3$ & $\mathcal{D}$:

$$\nabla^2 V' = \frac{\rho'}{\epsilon_0} \xrightarrow{r>a} \nabla^2 V = \frac{\rho}{\epsilon_0}$$

- V'&V satisfy BC in $\mathbb{R}^3$ & $\mathcal{D}$:

$$V'(a, \theta, \phi) = V(a, \theta, \phi) = 0$$
$$V'(\mathbf{S}) = V(\mathbf{S}) = 0 \quad .$$

If such a q' and d' can be found, then we have nailed the problem!