1. **A rod of proper length l_0 sits at rest in S frame, lying in the x-y plane at an angle of $\theta = tan^{-1}(3/4)$ with the x-axis. A frame S' moves with velocity $\vec{v} = v\hat{x}$ with respect to S. In S', the rod is angled at 45^o with respect to the x' axis.** [3+5]

   (a) **What is v?**

   (b) **What is the length l' of the rod as measured in S' frame?**

Solution:

(i)We have in S' frame $tan\theta' = \frac{y'}{x'}$, **where** θ' **is the angle made by the rod with** x' **axis.**

As the frame is moving with the velocity $v\hat{x}$, **we have** $y' = y$ **and** $x' = \frac{x}{\gamma_v}$, **where** γ_v **is the Lorentz factor.**

Now, $tan(\theta') = \gamma_v \frac{y}{x} = \gamma_v tan(\theta)$.

Which will give $\gamma_v = (\sqrt{1-\beta^2})^{-1} = \frac{tan(\theta')}{tan(\theta)} = 4/3$.

We get, $\beta = \frac{\sqrt{7}}{4}$. **Therefore,** $v = \frac{\sqrt{7}}{4}c$

(ii)The length measured in S' **would be given by** $l'^2 = y'^2 + x'^2 = y^2 + x^2(1-\beta^2) = l_0^2 - x^2\beta^2 = l_0^2(1 - \cos^2\theta\beta^2)$.

Using,

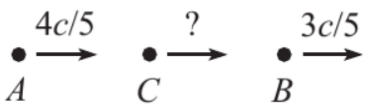
$$\cos^2\theta = \frac{1}{1+tan^2\theta} = \frac{16}{25}.$$

and $\beta^2 = 7/16$. **We get**

$$l'^2 = l_0\left(1 - \frac{7}{25}\right).$$

Finally we get $l' = l_0\frac{3\sqrt{2}}{5}$.

2. **Two observers A and B travel at speed $4c/5$ and $3c/5$, respectively, with respect to the ground, as shown in the Figure. How fast should the observer C travel with respect to the ground so that she sees A and B are approaching towards her at the same speed u? Compute the value of u.** [8]



Solution:

Let C moves at speed v with respect to the ground, and let the speed of A and B measured by C from her frame of reference are $u'_{A,C}$ and $u'_{B,C}$. Velocity of A and B w.r.t ground are $u_{A,g} = 4c/5$ and $u_{A,g} = 3c/5$, respectively.

Now,

$$u'_{A,C} = \frac{u_{A,g} - v}{1 - u_{A,g}v/c^2} = \frac{\frac{4c}{5} - v}{1 - \frac{4}{5c}v}$$

and,

$$u'_{B,C} = \frac{u_{B,g} - v}{1 - u_{B,g}v/c^2} = \frac{\frac{3c}{5} - v}{1 - \frac{3}{5c}v}$$

Since both observers A and B are approaching towards C with the same speed u. Therefore, $u'_{A,C} = -u'_{B,C} = u$

$$\frac{\frac{4c}{5} - v}{1 - \frac{4}{5c}v} = u = \frac{v - \frac{3c}{5}}{1 - \frac{3}{5c}v} \tag{1}$$

Therefore, we have $0 = 35v^2 - 74vc + 35c^2 = (5v - 7c)(7v - 5c)$.

Since v can not be grater than the light velocity. The acceptable solution will be $v = \frac{5c}{7}$

.

Plugging this in Eq.(1) gives $u = c/5$

.

3. **The ground state of a particle in one dimensional harmonic oscillator potential is given as $\psi_0(x) = Ae^{-\alpha x^2/2}$ with energy $E_0 = \frac{1}{2}\hbar\omega$, where $A = (m\omega/\pi\hbar)^{1/4}$, $\alpha = m\omega/\hbar$ and ω is the angular frequency of oscillation.** [4+6]

(a) Compute the uncertainty relation ($\Delta x \Delta p$) of the particle in the state $\psi_0(x)$.

(b) Compute the expectation values of the kinetic energy ($\langle \hat{T} \rangle$) and the potential energy ($\langle \hat{V} \rangle$) of the particle in the state $\psi_0(x)$.

Note: You may consider using the Gaussian integrals given as $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$.

Solution:

(a) Here ψ_0 is even function and hence $|\psi_0|^2$ is also even.

Therefore, $\langle x \rangle = \int x|\psi_0|^2 = 0$.

Hence $\langle p \rangle = md\langle x \rangle/dt = 0$

$$\langle x^2 \rangle = A^2 \int_{-\infty}^{\infty} x^2 |\psi_0|^2 dx = A^2 \int_{-\infty}^{\infty} x^2 e^{-\alpha x^2} dx$$

$$= \frac{\hbar}{2m\omega}$$

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \psi_0 \left(\frac{\hbar}{i} \frac{d}{dx} \right)^2 \psi_0 dx$$

$$= A^2 \hbar^2 \int_{-\infty}^{\infty} e^{-\alpha x^2/2} \frac{d^2}{dx^2} e^{-\alpha x^2/2} = \frac{m\hbar\omega}{2}$$

(b) $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{\hbar}{2m\omega}}$

$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} = \sqrt{\frac{m\hbar\omega}{2}}$

$\Delta x \Delta p = \sqrt{\frac{\hbar}{2m\omega}} \sqrt{\frac{m\hbar\omega}{2}} = \frac{\hbar}{2}$.

$\langle T \rangle = \frac{\langle p^2 \rangle}{2m} = \frac{\hbar\omega}{4}$

$\langle V \rangle = \frac{1}{2} m\omega^2 \langle x^2 \rangle = \frac{\hbar\omega}{4}$.

4. **Consider the one-dimensional normalised wave functions $\psi_0(x)$ and $\psi_1(x)$ with the properties $\psi_0(-x) = \psi_0(x) = \psi_0^*(x)$ and $\psi_1(x) = N\dfrac{d\psi_0(x)}{dx}$. Consider another wave function $\psi(x) = c_1\psi_0(x) + c_2\psi_1(x)$, with $|c_1|^2 + |c_2|^2 = 1$. N, c_1 and c_2 are known constants.** **[2+6+4]**

   (a) **Check if $\psi_0(x)$ and $\psi_1(x)$ are orthogonal to each other and $\psi(x)$ is normalised.**

   (b) **Compute the expectation values of $\hat{x}$ and $\hat{p}$ in the states ψ_0 and ψ_1 separately.**

   (c) **Compute the expectation value of $\hat{T}$(the kinetic energy operator) in the state ψ_0.**

Solution:

(a) For ψ_0 and ψ_1 to be orthogonal we have to show

$$\int_{-\infty}^{\infty} \psi_0^*(x)\psi_1(x)dx = 0$$

$$\therefore \int_{-\infty}^{\infty} \psi_0(x)\psi_1(x)dx = N\int dx\psi_0^*\frac{d\psi_0}{dx} = N\int dx\psi_0\frac{\psi_0}{dx}$$

$$= \frac{N}{2}\int dx\frac{d\psi_0^2}{dx} = \frac{N}{2}\Big[\psi_0^2(x)\Big]_{-\infty}^{+\infty} = 0.$$

For normalization of $\psi(x)$ we have

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = \int_{-\infty}^{\infty} (c_1\psi_0(x) + c_2\psi_1(x))^*(c_1\psi_0(x) + c_2\psi_1(x))dx$$

$$= \int_{-\infty}^{\infty} (c_1\psi_0c_1\psi_0 + c_1\psi_0c_2\psi_1 + c_2\psi_1c_1\psi_0 + c_2\psi_1c_2\psi_1)dx$$

As ψ_0 and ψ_1 are orthogonal (shown before) and with the given condition $|c_1|^2 + |c_2|^2 = 1$ we have

$$\int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = 1.$$

i.e. $\psi(x)$ is normalised.

(b) Expectation value of $\hat{x}$ vanishes both the state ψ_0 and ψ_1 owing to the oddness of the integrand $x|\psi_0|^2$ and $x|\psi_1|^2$..

$$\langle p\rangle_{\psi_0} = \int_{-\infty}^{\infty} \psi_0^*(x)p\psi_0(x)dx = -i\hbar\int_{-\infty}^{\infty} \psi_0(x)\frac{d\psi_0(x)}{dx}dx$$

$$= \frac{-i\hbar}{N}\int_{-\infty}^{\infty} \psi_0(x)\psi_1(x)dx = 0$$

(As $\psi_1 = N\frac{d\psi_0}{dx}$ and ψ_0 and ψ_1 are orthogonal).

$$\langle p\rangle_{\psi_1} = \int_{-\infty}^{\infty} \psi_1^*(x) p \psi_1(x) dx = -i\hbar \int_{-\infty}^{\infty} \psi_1^*(x) \frac{d\psi_1(x)}{dx} dx$$

$$-i\hbar \frac{N}{N^*} \int_{-\infty}^{\infty} \psi_1(x) \frac{d\psi_1(x)}{dx} dx = -i\hbar \frac{N}{2N^*} \int_{-\infty}^{\infty} \frac{d(\psi_1(x))^2}{dx} dx = -i\hbar \frac{N}{2N^*} \Big[(\psi_1(x))^2 \Big]_{-\infty}^{+\infty} = 0$$

(Owing to the fact that $\psi_1 \to 0$ as $x \to \pm\infty$.)

(c) Expectation value of T in ψ_0 is given as

$$\langle T\rangle = \int_{-\infty}^{\infty} \psi_0 T \psi_0 dx = \int_{-\infty}^{\infty} \psi_0 \frac{p^2}{2m} \psi_0 dx = \frac{-\hbar^2}{2m} \int_{-\infty}^{\infty} \psi_0 \frac{d^2\psi_0}{dx^2} dx$$

.

Now consider the term $\frac{d}{dx}\left(\psi_0 \frac{d\psi_0}{dx}\right) = \frac{d\psi_0}{dx}\frac{d\psi_0}{dx} + \psi_0 \frac{d^2\psi_0}{dx^2}$

$$\Rightarrow \psi_0 \frac{d^2\psi_0}{dx^2} = \frac{d}{dx}\left(\psi_0 \frac{d\psi_0}{dx}\right) - \frac{d\psi_0}{dx}\frac{d\psi_0}{dx}$$

$$\Rightarrow \psi_0 \frac{d^2\psi_0}{dx^2} = \frac{1}{N}\frac{d}{dx}\left(\psi_0\psi_1\right) - \frac{1}{N^2}\psi_1\psi_1$$

Substituting this in the expression for $\langle T\rangle$ we have ,

$$\langle T\rangle = \frac{-\hbar^2}{2mN} \int_{-\infty}^{\infty} \frac{d}{dx}\left(\psi_0\psi_1\right) dx + \frac{\hbar^2}{2mN^2} \int_{-\infty}^{\infty} \psi_1\psi_1 dx$$

$$= \frac{-\hbar^2}{2mN} \Big[\psi_0\psi_1\Big]_{-\infty}^{\infty} + \frac{\hbar^2}{2mN^2}$$

The first term vanishes as $\psi_0,\ \psi_1 \to 0$ as $x \to \pm\infty$ we have

$$\langle T\rangle = \tfrac{\hbar^2}{2mN^2}$$

5. **A particle in the infinite square well potential of width 'a' has its initial wave function which is a linear combination of the first two stationary eigenstates, namely the ground state ($\psi_1(x)$) and the first excited state ($\psi_2(x)$) given by:** [1+4+3]

$$\Psi(x,0) = A[\psi_1(x) + \psi_2(x)].$$

(a) **Find the value of A.**

(b) **Find $\Psi(x,t)$ and $|\Psi(x,t)|^2$. Express the latter in terms of *sinusoidal* functions of time, eliminating the exponentials with the help of formula: $e^{i\theta} = \cos\theta + i\sin\theta$.**

(c) **Compute the expectation value of the Hamiltonian operator $< \hat{H} >$ at $t = 0$ and at t in the state $\Psi(x,t)$.**

(a)

$$|\Psi|^2 = \Psi^2\Psi = |A|^2(\psi_1^* + \psi_2^*)(\psi_1 + \psi_2) = |A|^2[\psi_1^*\psi_1 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + \psi_2^*\psi_2].$$

$$1 = \int |\Psi|^2 dx = |A|^2 \int [|\psi_1|^2 + \psi_1^*\psi_2 + \psi_2^*\psi_1 + |\psi_2|^2]dx = 2|A|^2 \Rightarrow \boxed{A = 1/\sqrt{2}.}$$

(b)

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left[\psi_1 e^{-iE_1 t/\hbar} + \psi_2 e^{-iE_2 t/\hbar}\right] \quad \left(\text{but } \frac{E_n}{\hbar} = n^2\omega\right)$$

$$= \frac{1}{\sqrt{2}}\sqrt{\frac{2}{a}}\left[\sin\left(\frac{\pi}{a}x\right)e^{-i\omega t} + \sin\left(\frac{2\pi}{a}x\right)e^{-i4\omega t}\right] = \boxed{\frac{1}{\sqrt{a}}e^{-i\omega t}\left[\sin\left(\frac{\pi}{a}x\right) + \sin\left(\frac{2\pi}{a}x\right)e^{-3i\omega t}\right].}$$

$$|\Psi(x,t)|^2 = \frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\left(e^{-3i\omega t} + e^{3i\omega t}\right) + \sin^2\left(\frac{2\pi}{a}x\right)\right]$$

$$= \boxed{\frac{1}{a}\left[\sin^2\left(\frac{\pi}{a}x\right) + \sin^2\left(\frac{2\pi}{a}x\right) + 2\sin\left(\frac{\pi}{a}x\right)\sin\left(\frac{2\pi}{a}x\right)\cos(3\omega t)\right].}$$

(c) You could get either $\boxed{E_1 = \pi^2\hbar^2/2ma^2}$ or $\boxed{E_2 = 2\pi^2\hbar^2/ma^2,}$ with equal probability $\boxed{P_1 = P_2 = 1/2.}$

So $\langle H \rangle = \boxed{\frac{1}{2}(E_1 + E_2) = \frac{5\pi^2\hbar^2}{4ma^2};}$ it's the *average* of E_1 and E_2.

6. **Hydrogen atoms in states of high quantum number are known as the Rydberg atoms.** **[2+2]**

   (a) **Find the quantum number of the Bohr orbit in a hydrogen atom whose radius is 0.01 mm.**

   (b) **What is the energy of a hydrogen atom in this state ?**

Solution:

(a) **From the radius formula for Bohr atom we have** $r_n = n^2 a_0$, **where** $a_0 = 5.29 \times 10^{-11} m$ **is the Bohr radius .**

Hence the quantum number of the Bohr orbit of H-atom whose radius is $0.01\ mm$ **is ;**

$$n = \sqrt{\frac{r_n}{a_0}} = \sqrt{\frac{1.0 \times 10^{-5}\ m}{5.29 \times 10^{-11} m}} \sim 435$$

(b) **From the energy forumula we know that** $E_n = \dfrac{E_1}{n^2} = \dfrac{-13.6\ eV}{(435)^2} = -7.19 \times 10^{-5}\ eV$.