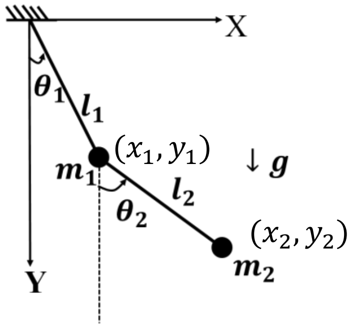
Model solution of Midsem exam problems

Solution of Q1.



Transformation relations:

$$\boxed{\begin{array}{c} x_1 = l_1 \sin\theta_1 \,;\; y_1 = l_1 \cos\theta_1 \\ x_2 = l_1 \sin\theta_1 + l_2 \sin\theta_2 \quad ; \quad y_2 = l_1 \cos\theta_1 + l_2 \cos\theta_2 \end{array}}$$

Kinetic energy (T) and potential energy (U) in Cartesian coordinate

$$T = \frac{1}{2} m_1 \left(\dot{x}_1{}^2 + \dot{y}_1{}^2\right) + \frac{1}{2} m_2 \left(\dot{x}_2{}^2 + \dot{y}_2{}^2\right) \quad ;$$

$$U = -m_1 g y_1 - m_2 g y_2$$

From transformation relations

$$\dot{x}_1 = l_1 \cos\theta_1 \,\dot{\theta}_1 \qquad ; \qquad \dot{y}_1 = -l_1 \sin\theta_1 \dot{\theta}_1$$

$$\dot{x}_2 = l_1 \cos\theta_1 \,\dot{\theta}_1 + l_2 \cos\theta_2 \,\dot{\theta}_2 \qquad ; \; \dot{y}_2 = -l_1 \sin\theta_1 \dot{\theta}_1 - l_2 \sin\theta_2 \dot{\theta}_2$$

Thus,

$$T = \frac{1}{2} m_1 \left[\left(l_1 \cos\theta_1 \,\dot{\theta}_1\right)^2 + \left(-l_1 \sin\theta_1 \dot{\theta}_1\right)^2\right] + \frac{1}{2} m_2 \left[\left(l_1 \cos\theta_1 \,\dot{\theta}_1 + l_2 \cos\theta_2 \,\dot{\theta}_2\right)^2 + \left(-l_1 \sin\theta_1 \dot{\theta}_1 - l_2 \sin\theta_2 \dot{\theta}_2\right)^2\right]$$

$$\boxed{T = \frac{1}{2} m_1 l_1{}^2 \dot{\theta}_1{}^2 + \frac{1}{2} m_2 \left\{l_1{}^2 \dot{\theta}_1{}^2 + l_2{}^2 \dot{\theta}_2{}^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right\}}$$

$$\boxed{U = -m_1 g l_1 \cos\theta_1 - m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2)}$$

Lagrangian(L) of the system,

$$L = T - U$$

$$\boxed{\begin{array}{l} L = \frac{1}{2} m_1 l_1{}^2 \dot{\theta}_1{}^2 + \frac{1}{2} m_2 \left\{l_1{}^2 \dot{\theta}_1{}^2 + l_2{}^2 \dot{\theta}_2{}^2 + 2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right\} + m_1 g l_1 \cos\theta_1 \\ \qquad + m_2 g (l_1 \cos\theta_1 + l_2 \cos\theta_2) \end{array}}$$

Lagrangian equations are

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = 0 \ldots\ldots\ldots (1); \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} = 0 \ldots\ldots (2)}$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2) l_1{}^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \;\; ; \quad \frac{\partial L}{\partial \theta_1} = -m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \, sin(\theta_1 - \theta_2) - (m_1 + m_2) g l_1 \sin\theta_1$$

$$\frac{d}{dt}\left\{(m_1 + m_2) l_1{}^2 \dot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right\} + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \, sin(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0$$

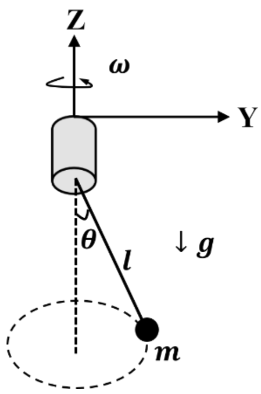
$$\boxed{(m_1 + m_2) l_1{}^2 \ddot{\theta}_1 + m_2 l_1 l_2 \dot{\theta}_2{}^2 \, sin(\theta_1 - \theta_2) + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1 - \theta_2) + (m_1 + m_2) g l_1 \sin\theta_1 = 0}$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = m_2 {l_2}^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2) \;\; ; \qquad \frac{\partial L}{\partial \theta_2} = \; m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \, sin(\theta_1 - \theta_2) - m_2 g l_2 \sin\theta_2$$

$$\frac{d}{dt}\left\{m_2 {l_2}^2 \dot{\theta}_2 + m_2 l_1 l_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)\right\} - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \, sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0$$

$$\boxed{m_2 {l_2}^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1 - \theta_2) - m_2 l_1 l_2 {\dot{\theta}_1}^2 \, sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 = 0}$$

Solution of Q2:



Here, θ serves as generalized coordinates

Kinetic energy in spherical polar coordinate

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\dot{\varphi}^2\right)$$

In this problem, $r = l = constant\ , \varphi = \omega t$; thus $\dot{r} = 0\ and\ \dot{\varphi} = \omega$

$$Thus, T = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\omega^2 sin^2\theta\right)$$

Potential energy, $U = -mgz = -mgl\cos\theta$

Lagrangian (L) of the system, $L = T - U = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\omega^2 sin^2\theta\right) + mgl\cos\theta$

Lagrange's eqn,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$Here, \quad \frac{\partial L}{\partial\dot{\theta}} = ml^2\dot{\theta}; \quad \frac{\partial L}{\partial\theta} = ml^2\omega^2\sin\theta\cos\theta - mgl\sin\theta$$

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) - ml^2\omega^2\sin\theta\cos\theta + mgl\sin\theta = 0$$

$ml^2\ddot{\theta} - ml^2\omega^2\sin\theta\cos\theta + mgl\sin\theta = 0$.....(1)

For $\theta \approx 0,\ \sin\theta \approx \theta\ and\ \cos\theta \approx 1$

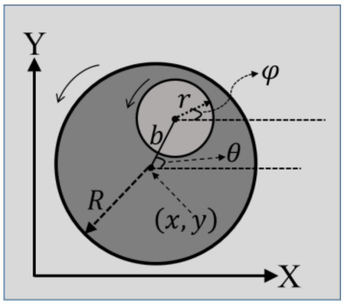
Eqn. 1 becomes

$$ml^2\ddot{\theta} + (mgl - ml^2\omega^2)\theta = 0$$

$$\ddot{\theta} + \left(\frac{g}{l} - \omega^2\right)\theta = 0$$

Angular frequency of oscillation, $\omega_0 = \sqrt{\left(\frac{g}{l} - \omega^2\right)}$

Solution of Q3:



Coordinate of the CM of the bigger disc (x, y)

Coordinate of the CM of smaller disc (x_m, y_m)

CM of bigger disc (x, y), angle of rotation of bigger disc (θ) and angle of rotation of the smaller disc (φ) serves as the generalized coordinates

Transformation equations are

$$x_m = x + b\cos\theta, \quad y_m = y + b\sin\theta$$

$$\dot{x}_m = \dot{x} - b\sin\theta\,\dot{\theta} \;;\; \dot{y}_m = \dot{y} + b\cos\theta\,\dot{\theta}$$

Kinetic energy(T) and potential energy (U) of the system in Cartesian coordinate

$$T = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I_M\dot{\theta}^2 + \frac{1}{2}m\left(\dot{x}_m{}^2 + \dot{y}_m{}^2\right) + \frac{1}{2}I_m\dot{\varphi}^2$$

$$T = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{4}MR^2\dot{\theta}^2 + \frac{1}{2}m\left\{\left(\dot{x} - b\sin\theta\,\dot{\theta}\right)^2 + \left(\dot{y} + b\cos\theta\,\dot{\theta}\,\right)^2\right\} + \frac{1}{4}mr^2\dot{\varphi}^2$$

$$T = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{4}MR^2\dot{\theta}^2 + \frac{1}{2}m\{\dot{x}^2 + \dot{y}^2 + b^2\dot{\theta}^2 - 2b\dot{x}\dot{\theta}\sin\theta + 2b\dot{y}\dot{\theta}\cos\theta\} + \frac{1}{4}mr^2\dot{\varphi}^2$$

$$U = 0$$

Lagrangian of the system

$$L = T - U$$

$$L = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2) + \frac{1}{4}MR^2\dot{\theta}^2 + \frac{1}{2}m\{\dot{x}^2 + \dot{y}^2 + b^2\dot{\theta}^2 - 2b\dot{x}\dot{\theta}\sin\theta + 2b\dot{y}\dot{\theta}\cos\theta\} + \frac{1}{4}mr^2\dot{\varphi}^2$$

Lagrange's equations are

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \ldots (1);\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 .. (2);\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\varphi}}\right) - \frac{\partial L}{\partial \varphi} = 0 \ldots (3);\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \ldots (4)}$$

From equation 1,

$$\frac{\partial L}{\partial \dot{x}} = (M + m)\dot{x} - mb\dot{\theta}\sin\theta = constant; \quad as\ \frac{\partial L}{\partial x} = 0, (i, e.\ \ x\ is\ cyclic)$$

From equation 2

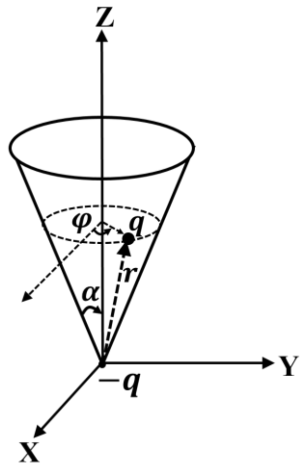
$$\frac{\partial L}{\partial \dot{y}} = (M + m)\dot{y} + mb\dot{\theta}\cos\theta = constant\,; \quad as\ \frac{\partial L}{\partial y} = 0, (i, e.\ \ y\ is\ cyclic)$$

From equation 3

$$\frac{\partial L}{\partial \dot{\varphi}} = \frac{1}{2}mr^2\dot{\varphi} = constant; \quad as\ \frac{\partial L}{\partial \varphi} = 0, (i, e.\ \ \varphi\ is\ cyclic)$$

Question 4:

Kinetic energy of the particle in spherical polar coordinate



$$\boxed{T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 sin^2\theta\dot{\varphi}^2\right)}$$

In the given problem $\theta = \alpha = constant; thus\ \dot{\theta} = 0$

$$\boxed{Hence, T = \frac{1}{2}m(\dot{r}^2 + r^2 sin^2\alpha\ \dot{\varphi}^2)}$$

$$\boxed{Potential\ energy\ (U) = -\frac{q^2}{4\pi\varepsilon_0 r}\quad (ignoring\ gravity)}$$

Lagrangian of the system,

$$\boxed{L = T - U = \frac{1}{2}m(\dot{r}^2 + r^2 sin^2\alpha\ \dot{\varphi}^2) + \frac{q^2}{4\pi\varepsilon_0 r}}$$

Lagrange's equations are

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\varphi}}\right) - \frac{\partial L}{\partial\varphi} = 0 \ldots.(1);\quad \frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right) - \frac{\partial L}{\partial r} = 0 \ldots(2)}$$

From equation 1,

$$\boxed{p_\varphi = \frac{\partial L}{\partial\dot{\varphi}} = mr^2 sin^2\alpha\dot{\varphi} = constant;\quad as\ \frac{\partial L}{\partial\varphi} = 0 \ldots..(3)}$$

From equation 2,

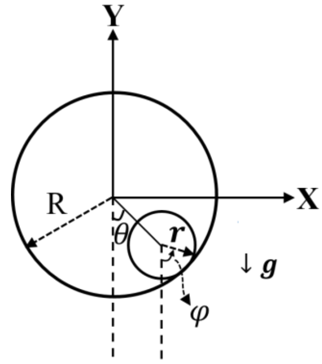
$$\frac{d}{dt}(m\dot{r}) - \left(-\frac{q^2}{4\pi\varepsilon_0 r^2} + mr sin^2\alpha\ \dot{\varphi}^2\right) = 0$$

$$\boxed{m\ddot{r} + \frac{q^2}{4\pi\varepsilon_0 r^2} - mr sin^2\alpha\ \dot{\varphi}^2 = 0 \ldots\ldots(4)}$$

Solution of Question 5:

Coordinate of CM of small cylinder (x, y)

Transformation relations



$$\boxed{\begin{gathered} x = (R-r)\sin\theta\,; \quad y = (R-r)\cos\theta \\ \dot{x} = (R-r)\cos\theta\,\dot{\theta}\,;\ \dot{y} = -(R-r)\sin\theta\,\dot{\theta} \end{gathered}}$$

Rolling without slipping condition

$$\boxed{r\dot{\varphi} = (R-r)\dot{\theta}}$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\varphi}^2$$

$$T = \frac{1}{2}m\{(R-r)^2\dot{\theta}^2 cos^2\theta + (R-r)^2\dot{\theta}^2 sin^2\theta\} + \frac{1}{4}mr^2\dot{\varphi}^2; \quad as\ I = \frac{1}{2}mr^2$$

$$T = \frac{1}{2}m(R-r)^2\dot{\theta}^2 + \frac{1}{4}m(R-r)^2\dot{\theta}^2$$

$$\boxed{T = \frac{3}{4}m(R-r)^2\dot{\theta}^2}$$

$$\boxed{Potential\ energy\ U = -mgy = -mg(R-r)\cos\theta}$$

$$\boxed{Lagrangian\ L = T - U = \frac{3}{4}m(R-r)^2\dot{\theta}^2 + mg(R-r)\cos\theta}$$

Lagrange's equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{d}{dt}\left\{\frac{3}{2}m(R-r)^2\dot{\theta}\right\} + mg(R-r)\sin\theta = 0$$

$$\frac{3}{2}m(R-r)^2\ddot{\theta} + mg(R-r)\sin\theta = 0$$

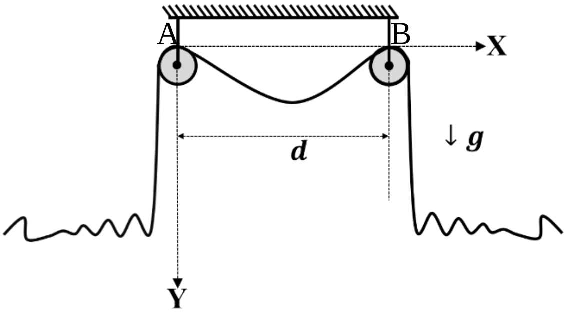
$$\boxed{\ddot{\theta} + \frac{2}{3}\frac{g}{(R-r)}\sin\theta = 0}$$

$$For\ small\ \theta, \quad \ddot{\theta} + \frac{2}{3}\frac{g}{(R-r)}\theta = 0$$

$$Angular\ frequency, \omega = \sqrt{\frac{2}{3}\frac{g}{(R-r)}}$$

$$\boxed{Time\ period\ T = \frac{2\pi}{\omega} = \pi\sqrt{\frac{6(R-r)}{g}}}$$

Solution of Question 6: ***Method-1***



Consider an elementary length ds of the rope in-between two pulley at (x, y) point.

Potential energy of the elementary length $dU = (\rho g\, ds)\, y$

Method 1:

$$Where, x = x(y)$$

$$\boxed{U = \int_A^B \rho g y\, ds = \int_A^B \rho g y \sqrt{dx^2 + dy^2} = \int_A^B \rho g y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \rho g \int_A^B y\sqrt{1 + x'^2}dy = \rho g \int_A^B F(y, x, x')dy}$$

For the integral to be extremum EL equation to be satisfied,

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0;\ \ Now\ in\ this\ case\ \frac{\partial F}{\partial x} = 0$$

$$Thus, \frac{\partial F}{\partial x'} = constant$$

$$y\frac{x'}{\sqrt{1 + x'^2}} = \ C(constant)$$

$$y^2 x'^2 = C^2\left(1 + x'^2\right);\quad x'^2(y^2 - C^2) = C^2$$

$$x' = \pm\sqrt{\frac{C^2}{(y^2 - C^2)}}\qquad ;\ \frac{dy}{\sqrt{y^2 - C^2}} = \pm\frac{dx}{C}$$

$$\boxed{cosh^{-1}\left(\frac{y}{C}\right) = \pm\frac{x}{C} + D;\qquad y = C\ cosh\left(\pm\frac{x}{C} + D\right)}$$

Method 2:

Total potential energy

$$\int_A^B \rho g y\, ds = \int_A^B \rho g y \sqrt{dx^2 + dy^2} = \int_A^B \rho g y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \rho g \int_A^B y\sqrt{1 + y'^2}\, d\,x$$

$$= \rho g \int_A^B F(x, y, y')dx$$

Where, $y = y(x)$

System would be in equilibrium, when potential energy is minimum. F needs to safisfy EL equation.

IF F is independent of free variable x then, from EL equation

$$\boxed{\frac{\partial F}{\partial y'} y' - F = constant}$$

$$y\frac{y'}{\sqrt{1 + y'^2}} y' - y\sqrt{1 + y'^2} = constant = C(say)$$

$$yy'^2 - y(1 + y'^2) = C\sqrt{1 + y'^2}$$

$$y^2 = C^2(1 + y'^2); \qquad y'^2 = \frac{y^2 - C^2}{C^2}$$

$$y' = \pm\frac{1}{C}\sqrt{y^2 - C^2}\ ; \qquad \frac{dy}{\sqrt{y^2 - C^2}} = \pm\frac{dx}{C}$$

$$cosh^{-1}\left(\frac{y}{C}\right) = \pm\frac{x}{C} + D$$

$$\boxed{y = A\,cosh\left(\pm\frac{x}{C} + D\right)}$$