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3. The ground state of a particle in one dimensional harmonic oscillator potential is given as $\psi_0(x) = Ae^{-\alpha x^2/2}$ with energy $E_0 = \frac{1}{2}\hbar\omega$, where $A = (m\omega/\pi\hbar)^{1/4}$, $\alpha = m\omega/\hbar$ and ω is the angular frequency of oscillation. [4+6]

(a) Compute the uncertainty relation ($\Delta x \Delta p$) of the particle in the state $\psi_0(x)$.

(b) Compute the expectation values of the kinetic energy ($\langle \hat{T} \rangle$) and the potential energy ($\langle \hat{V} \rangle$) of the particle in the state $\psi_0(x)$.

Note: You may consider using the Gaussian integrals given as $\int_{-\infty}^{\infty} e^{-ax^2} dx = \sqrt{\frac{\pi}{a}}$.

$$\langle x \rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\, x\, \psi_0(x)\, dx = \int_{-\infty}^{\infty} \psi_0^2(x) \cdot x\, dx$$

$$= \int_{-\infty}^{\infty} A^2 e^{-\alpha x^2} x\, dx = 0 \quad \text{(odd integrand)}$$

$\frac{1}{2}$

$$\hat{p}\,\psi_0 = -i\hbar \frac{\partial \psi_0(x)}{\partial x} = -i\hbar A e^{-\frac{\alpha x^2}{2}} (-\alpha x)$$

$$= i\hbar \alpha A x e^{-\alpha x^2/2}$$

$$\therefore \langle \hat{p} \rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\, \hat{p}\, \psi_0(x)\, dx = \int_{-\infty}^{\infty} A e^{-\frac{\alpha x^2}{2}} \cdot i\hbar \alpha A x e^{-\frac{\alpha x^2}{2}} dx$$

$$= A^2 i\hbar\alpha \int_{-\infty}^{\infty} x e^{-\alpha x^2} dx = 0 \ldots \text{(odd integrand)}$$

$\frac{1}{2}$

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\, x^2 \psi_0(x)\, dx = \int_{\infty}^{\infty} \psi_0^2(x) \cdot x^2 dx$$

$$= \int_{-\infty}^{\infty} A^2 e^{-\alpha x^2} \cdot x^2 dx$$

Now using $\left[\int_{-\infty}^{\infty} e^{-\alpha x^2} \cdot x^2 dx = \frac{1}{2\alpha}\sqrt{\pi/\alpha} \right]$ — (1)

$$\langle x^2 \rangle = A^2 \cdot \frac{1}{2\alpha}\sqrt{\frac{\pi}{\alpha}} = \left(\frac{m\omega}{\pi\hbar}\right)^{1/2} \cdot \frac{\hbar}{2m\omega} \sqrt{\frac{\pi\hbar}{m\omega}}$$

$$\therefore \boxed{\langle x^2 \rangle = \frac{\hbar}{2m\omega}}$$

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$$\therefore \Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2} = \sqrt{\langle x^2\rangle} = \sqrt{\frac{\hbar}{2m\omega}} \quad \text{—— (2)}$$

$$V = \frac{1}{2}kx^2 = \frac{1}{2}m\omega^2 x^2$$

$$\therefore \langle \hat{V}\rangle = \frac{1}{2}k\langle x^2\rangle = \frac{1}{2}m\omega^2 \frac{\hbar}{2m\omega}$$

$$\boxed{\langle \hat{V}\rangle = \frac{1}{4}\hbar\omega} \quad \text{—— (3)}$$

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$$\langle \hat{p}^2\rangle = \int_{-\infty}^{\infty} \psi_0^*(x)\,\hat{p}^2\,\psi_0(x)\,dx$$

$$\hat{p}^2\psi_0(x) = -\hbar^2\frac{\partial^2\psi_0(x)}{\partial x^2} = A\alpha\hbar^2 e^{-\alpha x^2/2}(1-\alpha x^2)$$

$$\therefore \langle \hat{p}^2\rangle = \int_{-\infty}^{\infty} A^2 e^{-\alpha x^2}\cdot\alpha\hbar^2(1-\alpha x^2)\,dx$$

$$= A^2\alpha\hbar^2\int_{-\infty}^{\infty} e^{-\alpha x^2}dx - A^2\alpha^2\hbar^2\int_{-\infty}^{\infty} x^2 e^{-\alpha x^2}dx$$

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$$= A^2\cdot\alpha\cdot\hbar^2\cdot\sqrt{\pi/\alpha} - A^2\alpha^2\hbar^2\cdot\frac{1}{2\alpha}\cdot\sqrt{\frac{\pi}{\alpha}}$$

$$= \frac{A^2\alpha\hbar^2}{2}\sqrt{\pi/\alpha} = \frac{m\omega\hbar}{2}$$

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$$\therefore \langle \hat{p}^2\rangle = \frac{m\omega\hbar}{2}$$

$$KE = \frac{p^2}{2m} \quad \therefore \boxed{\langle \hat{T}\rangle = \frac{\langle p^2\rangle}{2m} = \frac{1}{4}\hbar\omega}$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\langle p^2\rangle} = \sqrt{\frac{m\omega\hbar}{2}} \quad \text{—— (4)}$$

From (2) and (4), $\Delta x\cdot\Delta p = \sqrt{\frac{\hbar}{2m\omega}}\cdot\sqrt{\frac{m\omega\hbar}{2}} = \boxed{\frac{\hbar}{2}}$

which clearly satisfies uncertainty principle

Ans: a) $\Delta x\cdot\Delta p = \frac{\hbar}{2}$

b) $\langle \hat{T}\rangle = \langle \hat{V}\rangle = \frac{1}{4}\hbar\omega$

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