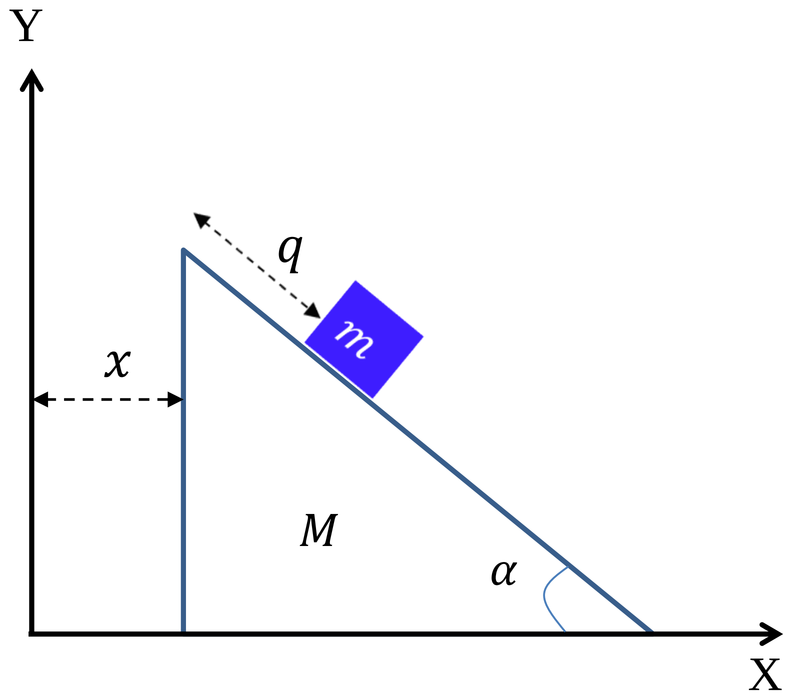
PH101

Lecture 7

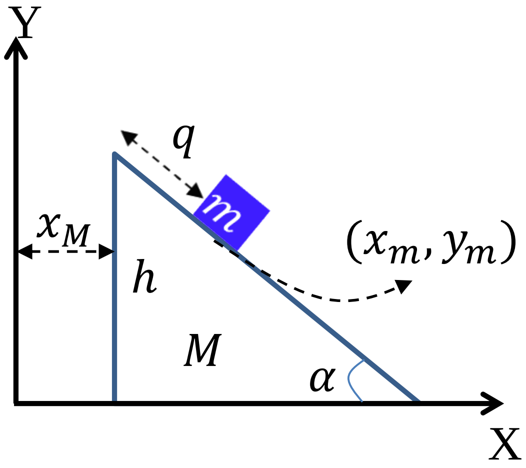
More examples on Lagrange’s equation, **generalized momentum, Cyclic coordinate**, Conservation relation.

Lagrange's equation: Example 3

A block of mass m is sliding on a wedge of mass M. Wedge can slide on the horizontal table. Find the equation of motion.



Lagrange's equation: Example 3



Four constrains: $z_M = 0$; $y_M = 0$; $z_m = 0$;
$$\frac{h - y_m}{x_m - x_M} = \tan\alpha = constant$$

Step-1: *Find the degrees of freedom and choose suitable generalized coordinates*

Two particles system, thus $N = 2$, $no. of\ constrains\ (k) = 4$
By rule, $\boldsymbol{degrees\ of\ freedom = 3 \times 2 - 4 = 2}$
Hence, number of generalized coordinates **must be two**.

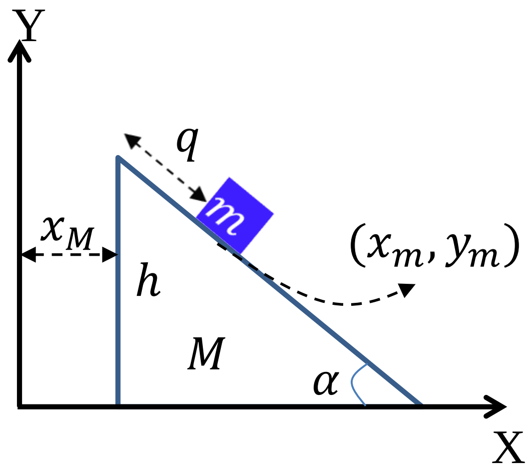
The distance of the wedge from origin ($\boldsymbol{x_M}$) and distance slipped by the block **($\boldsymbol{q}$)** can serve as generalized coordinates of the system.

Important point: Only translation of the given rigid bodies are considered, thus for the calculation of degrees of freedom both of them are considered as point particles.

Lagrange's equation: Example 3

Step-2: *Find out transformation relations*

$$\boldsymbol{x_m = x_M + q\cos\alpha \ ;\ y_m = h - q\sin\alpha}$$



Step-3: *Write T and U in Cartesian*

$$\boldsymbol{T = \frac{1}{2}m\left(\dot{x}_m{}^2 + \dot{y}_m{}^2\right) + \frac{1}{2}M\dot{x}_M{}^2\ ;\quad U = mgy_m}$$

Step-4:Convert
T and U in generalized coordinate using transformation

From transformation relations

$$\boldsymbol{T = \frac{1}{2}m[\dot{x}_M{}^2 + \dot{q}^2 + 2\dot{x}_M\dot{q}\cos\alpha] + \frac{1}{2}M\dot{x}_M{}^2;}$$
$$\mathrm{U} = \boldsymbol{mg(h - q\sin\alpha)}$$

$$\dot{x}_m = \dot{x}_M + \dot{q}\cos\alpha$$
$$\dot{y}_m = -\dot{q}\sin\alpha$$

Lagrange's equation: Example 3

Step-5: *Write down Lagrangian*

$$L = T - U$$

$$\boldsymbol{L = \frac{1}{2} m\left[\dot{x}_M{}^2 + \dot{q}^2 + 2\dot{x}_M \dot{q} \cos\alpha\right] + \frac{1}{2} M \dot{x}_M{}^2 - mg(h - q\sin\alpha)}$$

Step-5: *Write down Lagrange's equation for each generalized coordinates (*$\boldsymbol{x_M}$ *and q)*

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_M}\right) - \frac{\partial L}{\partial x_M} = 0 \ldots\ldots(1);} \qquad \boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \ldots\ldots(2)}$$

From eqn. (1)

$$\boldsymbol{\frac{d}{dt}\left[m\dot{x}_M + m\dot{q}\cos\alpha + M\dot{x}_M\right] = 0,}$$

$$\boldsymbol{(m + M)\ddot{x}_M + m\ddot{q}\cos\alpha = 0}$$

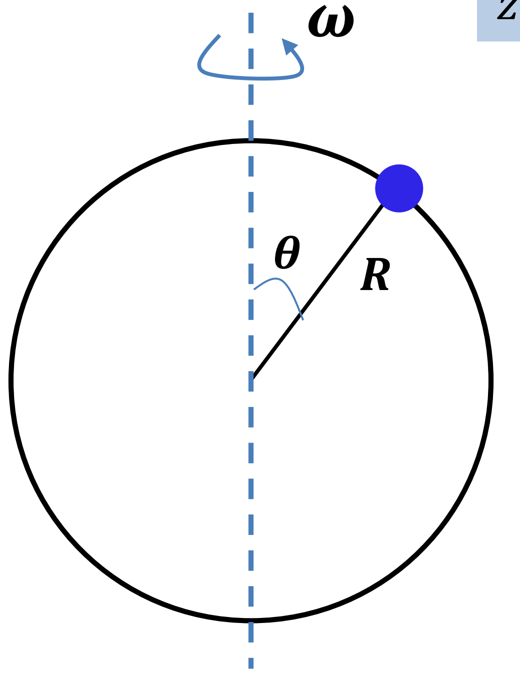
From eqn. (2)

$$\boldsymbol{\frac{d}{dt}\left[m\dot{q} + m\dot{x}_M\cos\alpha\right] - \left[mg\sin\alpha\right] = 0}$$

$$\boldsymbol{m(\ddot{q} + \ddot{x}_M\ \cos\alpha) - mg\sin\alpha = 0}$$

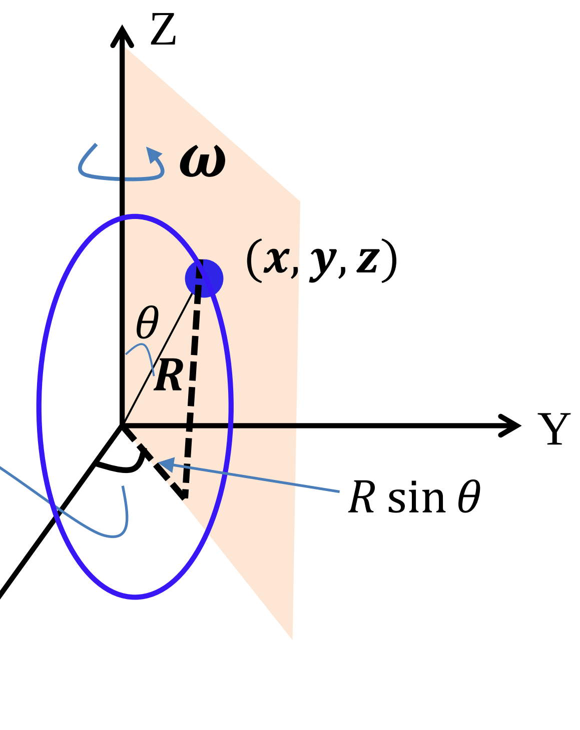
Lagrange's equation: Example 4

Take vertical axis about which the hoop is rotating as z −axis and the plane of the hoop at $\boldsymbol{t = 0}$ as the $\boldsymbol{XZ}$**-plane**.



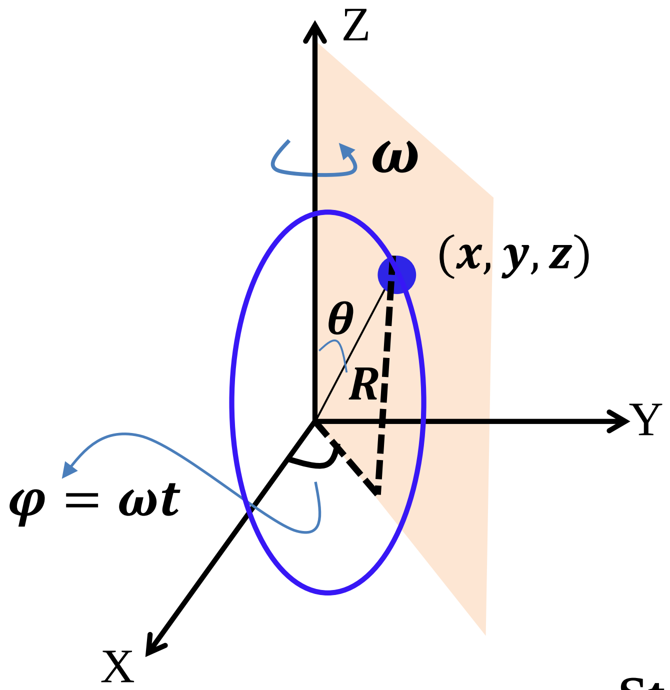
Angle by which plane of the hoop is rotated in time t

$\boldsymbol{\varphi = \omega t}$



A bead of mass $\boldsymbol{m}$ is free to slide along a frictionless hoop of radius $\boldsymbol{R}$. The hoop rotates with constant angular speed $\boldsymbol{\omega}$ around its vertical diameter. Find the equation of motion of the bead.

Lagrange's equation: Example 4



Two holonomic constrain relations

$$x^2 + y^2 + z^2 = R^2$$

$$\dot{\varphi} = \omega, \qquad i, e. \;\; \varphi = \omega t$$

Note:

$\varphi = \omega t$ constrain is equivalent to $\tan^{-1}\frac{y}{x} = \omega t$ in Cartesian system

Step-1: *Find the degrees of freedom and choose suitable generalized coordinates*

One particle system, hence $N = 1$, $no. of \ holonomic \ constrains \ (k) = 2$
By rule, $degrees \ of \ freedom = 3 \times 1 - 2 = 1$
Hence number of generalized coordinates must be **one**.

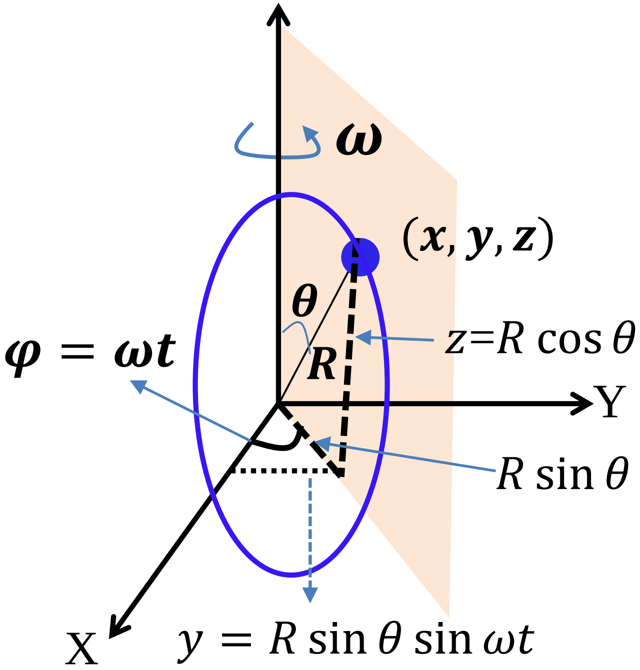
Angle '$\boldsymbol{\theta}$', which the particle makes with rotation axis (***z*-axis**) can serve as generalized coordinates.

Lagrange's equation: Example 4

Step-2: *Find out transformation relations*

$$x = R\sin\theta\cos\omega t\ ; y = R\sin\theta\sin\omega t\ ; z = R\cos\theta$$

Note: All the constrain relations have entered in the problem via these relationship



Step-3: *Write T and U in Cartesian*

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2); \quad U = mgz$$

Step-4:Convert
T and U in generalized coordinate using transformation

$$T = \frac{1}{2}m[R^2\dot{\theta}^2 + R^2\omega^2 sin^2\theta];$$
U= $mgR\cos\theta$

From transformation relations

$$\dot{x} = R\cos\theta\cos\omega t\ \dot{\theta} - R\,\omega\sin\theta\sin\omega t$$
$$\dot{y} = R\cos\theta\sin\omega t\ \dot{\theta} + R\omega\sin\theta\cos\omega t$$
$$\dot{z} = -R\sin\theta\ \dot{\theta}$$

Example 4: continue

Step-5: *Write down Lagrangian*

$$L = T - U$$

$$L = \frac{1}{2} m\left[R^2\dot{\theta}^2 + R^2\omega^2 sin^2\theta\right] - mgR\cos\theta$$

Step-5: *Write down Lagrange's equation for each generalized coordinates*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{d}{dt}\left[mR^2\dot{\theta}\right] - \left[mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta\right] = 0$$

$$mR^2\ddot{\theta} - \left[mR^2\omega^2\sin\theta\cos\theta + mgR\sin\theta\right] = 0$$

Example 4: Kinetic energy relooked

In this problem of rotation, kinetic energy can be directly written in generalized coordinates **without using transformation relations** because one can easily identify that generalized coordinate $\boldsymbol{\theta}$ is the coordinate of spherical polar coordinate system (r, θ, φ), and in this case $\boldsymbol{r = R = constant, \dot{\varphi} = \omega}$

Velocity in spherical polar coordinate

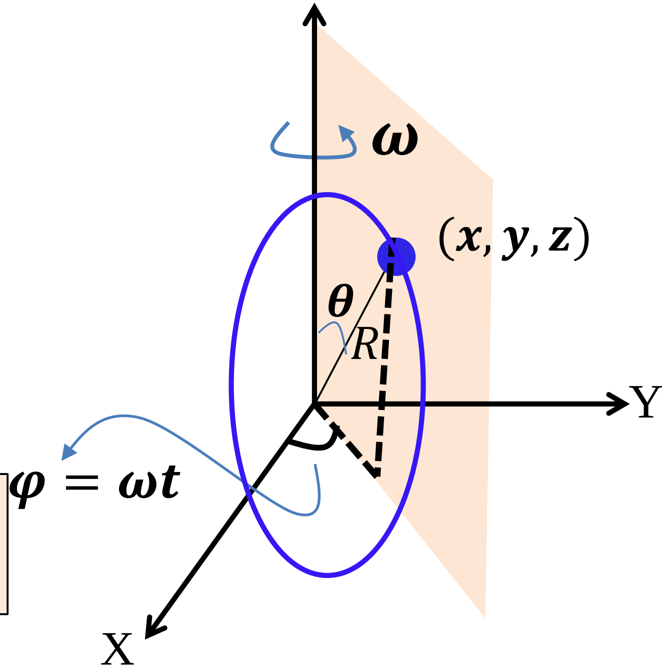
$$\vec{\boldsymbol{v}} = \dot{\boldsymbol{r}}\hat{\boldsymbol{r}} + \boldsymbol{r}\dot{\boldsymbol{\theta}}\,\widehat{\boldsymbol{\theta}} + \boldsymbol{r}\sin\boldsymbol{\theta}\,\dot{\boldsymbol{\varphi}}\widehat{\boldsymbol{\varphi}}$$

Now $r = R$ =constant, hence $\dot{\boldsymbol{r}} = \boldsymbol{0}$; $\dot{\boldsymbol{\varphi}} = \boldsymbol{\omega}$

Thus for this problem

$$\vec{\boldsymbol{v}} = \boldsymbol{R}\dot{\boldsymbol{\theta}}\widehat{\boldsymbol{\theta}} + \boldsymbol{R}\,\boldsymbol{\omega}\sin\boldsymbol{\theta}\,\widehat{\boldsymbol{\varphi}}$$

$$T = \frac{1}{2}mv^2 = \frac{1}{2}m\vec{v}.\vec{v} = \frac{1}{2}m[R^2\dot{\theta}^2 + R^2\omega^2 sin^2\theta]$$



Basically the transformation relations from Cartesian to Spherical polar (SP) have already been accounted in getting the relation for velocity in SP.

Concept of generalized momentum

Momentum corresponding to generalized coordinates

Generalized momentum

Lagrangian of a free particle

$$L = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$$

Thus, $\frac{\partial L}{\partial \dot{x}} = m\dot{x}$; now $m\dot{x} \rightarrow x$ component of **linear momentum** (p_x)

$p_x = m\dot{x} = \frac{\partial L}{\partial \dot{x}}$; Similarly, $p_y = \frac{\partial L}{\partial \dot{y}}$ and $p_z = \frac{\partial L}{\partial \dot{z}}$

Lagrangian of a freely rotating wheel with moment of inertia I is

$$L = \frac{1}{2}I\dot{\theta}^2$$

And $\frac{\partial L}{\partial \dot{\theta}} = I\dot{\theta}$ →**Angular momentum**

In both the examples, momentum is the derivative (partial) of the Lagrangian with respect to velocity (linear/angular)

By carrying this idea, we can define *generalized momentum associated with generalized coordinate* q_j by

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Also known as
conjugate momentum/canonical momentum

Generalized momentum

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial}{\partial \dot{q}_j}(T - U) = \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial U}{\partial \dot{q}_j}$$

Generalized momentum does not arise totally from kinetic energy term but also from generalized velocity dependent potential energy term

Most of the cases, U is function of position only and does not contain any velocity($\dot{q}_j$, generalized velocity) dependent term (so $\frac{\partial U}{\partial \dot{q}_j} = 0,$),

in those cases $p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j}$

But there are systems, for which U is velocity dependent ($\dot{q}_j$, generalized velocity), $\frac{\partial U}{\partial \dot{q}_j} \neq 0$

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial U}{\partial \dot{q}_j}$$

Example: Charge particle in electromagnetic field

Generalized momentum for velocity dependent potential

Example: Motion of particle of charge $+\boldsymbol{e}$ and mass $\boldsymbol{m}$ in electromagnetic field (electric field $\vec{E}$ and magnetic field $\vec{B}$).

Electric scalar potential corresponding to $\vec{E}$ is φ (where $\vec{\boldsymbol{E}} = -\vec{\boldsymbol{\nabla}}\boldsymbol{\varphi}$)
Magnetic vector potential corresponding to magnetic field $\vec{B}$ is $\vec{A}$ (where $\vec{\boldsymbol{B}} = \vec{\boldsymbol{\nabla}} \times \vec{\boldsymbol{A}}$)

Potential energy of the moving charge in electromagnetic field

$$U = e\varphi - e(\vec{v}.\vec{A}) \qquad = e\varphi - e(\hat{x}\dot{x} + \hat{y}\dot{y} + \hat{z}\dot{z}).(\hat{x}A_x + \hat{y}A_y + \hat{z}A_z)$$
$$= e\varphi - e(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)$$

where φ and $\vec{A}$ are scalar and vector potential of EM field.

$$L = T - U = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - [e\varphi - \mathrm{e}(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z)]$$

Generalized momentum corresponding to $\boldsymbol{x}$; $\boldsymbol{p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x} + eA_x}$

Similarly, $p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y} + eA_y$; $p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} + eA_z$

All the three generalized momentums can be written in a single vector form

$$\vec{p} = \hat{x}p_x + \hat{y}p_y + \hat{z}p_z = m\vec{v} + e\vec{A}$$

Arise from kinetic energy term

Arise from velocity dependent potential energy term

Generalized momentum: A few points

Generalized velocity is the rate of charge of generalized coordinate $\dot{q}_j = \frac{dq_j}{dt}$

Generalized momentum is not the mass multiplied by generalized velocity.

Example: charged particle in EM field $\vec{p} = m\vec{v} + e\vec{A}$

$$\cancel{p_j = m\dot{q}_j}$$

In specific cases, this relation may be true but it is not the general case.

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Definition of **generalized momentum**

Unit/dimension of the generalized momentum depends on generalized coordinate under consideration. For example, if q_j represents translation (dimension of linear displacement) then p_j has the dimension of linear momentum. If q_j represents an angle then the dimension of p_j is that of angular momentum.

Generalized definition of momentum allows to consider non-mechanical systems, for example EM field

Cyclic coordinates

If a particular coordinate does not appear in the Lagrangian, it is called ‘**Cyclic**’ or ‘**Ignorable**’ coordinate.

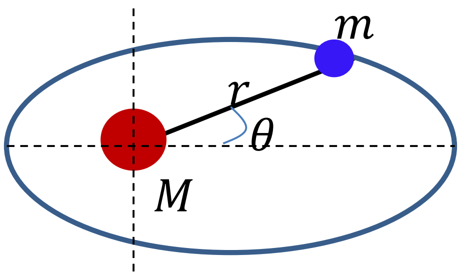
Example 1: Lagrangian of a point mass in gravitational field

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

Since neither x nor y appear in the Lagrangian, they are cyclic.

Example 2: Lagrangian for a planet of mass m orbiting around the sun (mass M) $L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}$

Since θ does not appear in the Lagrangian, it is cyclic coordinate.



Cyclic coordinates and conservation of conjugate momentum

- If there is no explicit dependence of L on generalized coordinate q_j, then

$$\frac{\partial L}{\partial q_j} = 0$$

Thus Lagrange's equation corresponding to cyclic coordinate become

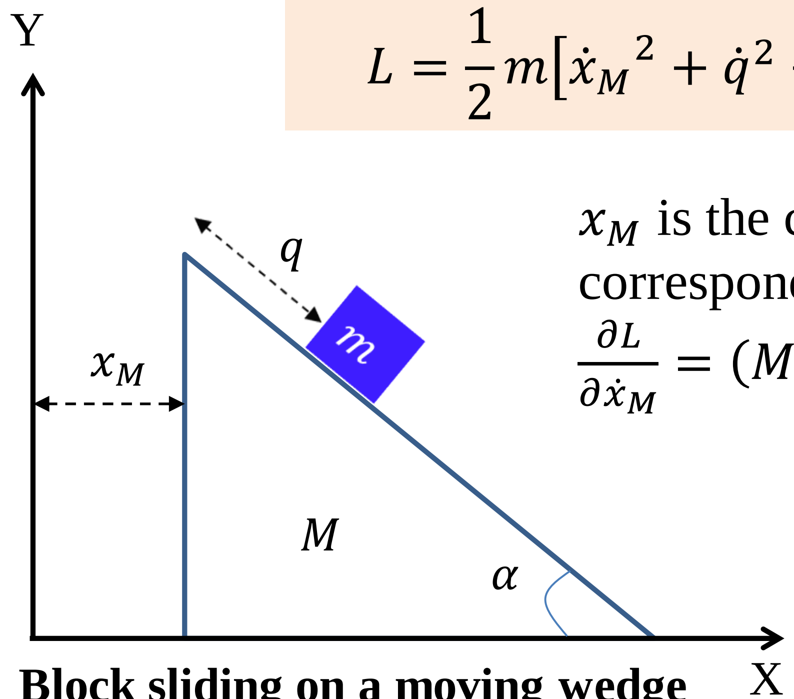
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0$$

Hence, $\frac{\partial L}{\partial \dot{q}_j} = \boldsymbol{p_j}$ **=constant**

Generalized momentum conjugate to a cyclic coordinate is a constant

Conservation of generalized momentum: Examples 3

$$L = \frac{1}{2} m\left[\dot{x}_M{}^2 + \dot{q}^2 + 2\dot{x}_M \dot{q}\cos\alpha\right] + \frac{1}{2} M \dot{x}_M{}^2 - mg(h - q\sin\alpha)$$



Block sliding on a moving wedge

x_M is the cyclic coordinate, thus generalized momentum corresponding to x_M is

$$\frac{\partial L}{\partial \dot{x}_M} = (M+m)\dot{x}_M + m\dot{q}\cos\alpha = \text{Constant}$$

Note: $(M+m)\dot{x}_M + m\dot{q}\cos\alpha$ is not the momentum of any particular object in the system. This is the generalized momentum of the entire system corresponding to generalized coordinate x_M

Cyclic coordinates, symmetries and conservation laws

Cyclic coordinate (q_j)

Corresponding generalized (canonical) momentum ($p_j = \frac{\partial L}{\partial \dot{q}_j}$) conserved

L is independent of the particular cyclic coordinate (q_j).
Any change in the cyclic coordinate has no effect on L.

System is symmetric under changes in that particular coordinate.

Each symmetry in a coordinate gives rise to a **conserved canonical momentum**

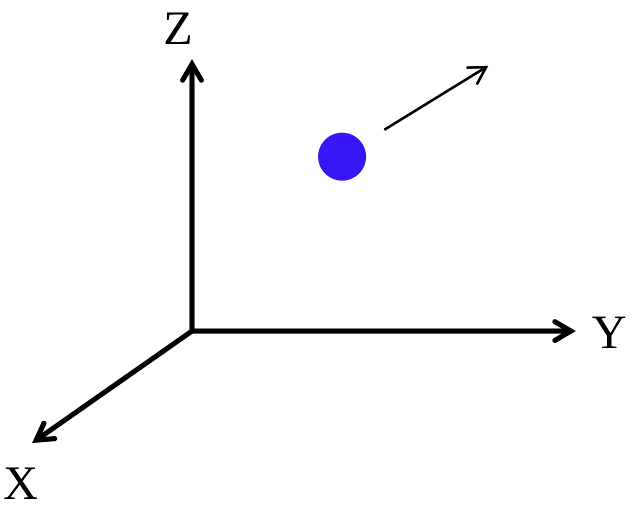
Example: For planetary motion

$$L = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}$$

L is independent of rotation angle θ(cyclic coordinate), the system has rotational symmetry, the system remains the same after change in θ.
As θ is cyclic corresponding generalized (canonical) momentum $p_\theta = mr^2\dot{\theta}$=Constant; which is nothing but angular momentum.

Conclusion: Conservation of angular momentum is related to rotational symmetry of the system.

Cyclic coordinates, symmetries and conservation laws



Lagrangian of a point mass moving in general direction under gravitational field.

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

x and y are cyclic coordinates, thus any change in either x or y will not change the Lagrangian.

In other words the system has translation symmetry in x and y coordinates.

$p_x = m\dot{x}$ and $p_y = m\dot{y}$ both are constant of motion as x and y are cyclic. The generalized momentum in this case are the linear momentum.

Conclusion: Conservation of linear momentum is associated with translational symmetry of the system.

Symmetry and conservation laws

Translational symmetry → **Conservation of linear momentum**

Rotational symmetry → **Conservation of angular momentum**

? → **Conservation of energy**

If L does not explicitly depend on time, then energy of the system is conserved, provided potential energy is velocity independent.

If L does not have explicit time dependence, change in time does not cause any change in Lagrangian i,e, $\frac{\partial L}{\partial t} = 0$,

$E = T + V = constnat$ **(looking for proof !)**

Translational symmetry in time → **Conservation of energy**

Summery

Generalized momentum/canonical momentum/conjugate momentum

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

Generalized momentum in general is not the mass multiplied by generalized velocity,

If a particular coordinate does not appear in the Lagrangian, it is called 'Cyclic' or 'Ignorable' coordinate.

Generalized momentum corresponding to a cyclic coordinate is a constant of motion

Presence of cyclic coordinate (i,e. that coordinate will appear in L) also means the system is symmetric under change in that coordinate

Translational symmetry $\Rightarrow$ **Conservation of linear momentum**

Rotational symmetry $\Rightarrow$ **Conservation of angular momentum**

Translational symmetry in time $\Rightarrow$ **Conservation of energy**