Department of Mathematics
Indian Institute of Technology Guwahati
MA 101: Mathematics I
Solutions of Tutorial Sheet-3
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1. If $x_n = (-1)^n n^2$ for all $n \in \mathbb{N}$, then examine whether the sequence (x_n) has a convergent subsequence?

   Solution. If possible, let the given sequence have a convergent subsequence $((-1)^{n_k} n_k^2)$. Then $((-1)^{n_k} n_k^2)$ must be bounded. So there exists $M > 0$ such that $|(-1)^{n_k} n_k^2| \leq M$ for all $k \in \mathbb{N} \Rightarrow n_k^2 \leq M$ for all $k \in \mathbb{N}$, which is not possible, since (n_k) is a strictly increasing sequence of positive integers. Therefore the given sequence cannot have any convergent subsequence. $\square$

2. If $x_n = (-1)^n \frac{5n \sin^3 n}{3n-2}$ for all $n \in \mathbb{N}$, then examine whether the sequence (x_n) has a convergent subsequence.

   Solution. Since $|x_n| = \frac{5}{3-\frac{2}{n}} |\sin n|^3 \leq 5$ for all $n \in \mathbb{N}$, the sequence (x_n) is bounded and hence by Bolzano-Weierstrass theorem, (x_n) has a convergent subsequence. $\square$

3. Let $a_1 = 1$ and $a_{n+1} = \left(1 + \frac{(-1)^n}{2^n}\right) a_n$ for all $n \in \mathbb{N}$. Prove that (a_n) is a Cauchy sequence.

   Solution. Using AM-GM inequality,

$$\begin{aligned}
|a_{n+1}| &\leq \left(1+\frac{1}{2^n}\right)\left(1+\frac{1}{2^{n-1}}\right)\cdots\left(1+\frac{1}{2^2}\right)\left(1+\frac{1}{2}\right) \\
&\leq \left(\frac{n+\sum_{k=1}^{n}\frac{1}{2^k}}{n}\right)^n = \left(1+\frac{1}{n}\sum_{k=1}^{n}\frac{1}{2^k}\right)^n < \left(1+\frac{1}{n}\right)^n < 3.
\end{aligned}$$

   Hence, $|a_{n+1} - a_n| = \frac{|a_n|}{2^n} < \frac{3}{2^n}$ for all $n \geq 2$. Now, for $m > n$, we have

$$\begin{aligned}
|a_m - a_n| &\leq |a_m - a_{m-1}| + |a_{m-1} - a_{m-2}| + \cdots + |a_{n+1} - a_n| \\
&< \frac{3}{2^{m-1}} + \frac{3}{2^{m-2}} + \cdots + \frac{3}{2^n} \\
&= \frac{3}{2^n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-1-n}}\right) < \frac{3}{2^{n-1}}.
\end{aligned}$$

   Now, given $\varepsilon > 0$, we can find $n_0 \in \mathbb{N}$ such that $\frac{3}{2^n} < \varepsilon$ for all $n \geq n_0$. This implies that $|a_m - a_n| < \varepsilon$ for all $m > n \geq n_0$. Hence, the given sequence is Cauchy. $\square$

4. Let $x_1 = 1$ and let $x_{n+1} = \frac{1}{x_n+2}$ for all $n \in \mathbb{N}$. Prove that (x_n) is Cauchy and $\lim\limits_{n\to\infty} x_n = \sqrt{2} - 1$.

   Solution. For all $n \in \mathbb{N}$, we have $|x_{n+2} - x_{n+1}| = |\frac{1}{x_{n+1}+2} - \frac{1}{x_n+2}| = \frac{|x_{n+1}-x_n|}{|x_{n+1}+2||x_n+2|}$. Now, $x_1 > 0$ and if we assume that $x_k > 0$ for some $k \in \mathbb{N}$, then $x_{k+1} = \frac{1}{x_k+2} > 0$. Hence by the principle of mathematical induction, $x_n > 0$ for all $n \in \mathbb{N}$. Using this, we get $|x_{n+2} - x_{n+1}| \leq \frac{1}{4}|x_{n+1} - x_n|$ for all $n \in \mathbb{N}$. It follows that (x_n) is a Cauchy sequence in $\mathbb{R}$ and hence (x_n) is convergent. Let $\ell = \lim_{n\to\infty} x_n$. Then $\lim_{n\to\infty} x_{n+1} = \ell$ and since $x_{n+1} = \frac{1}{x_n+2}$ for all $n \in \mathbb{N}$, we get $\ell = \frac{1}{\ell+2} \Rightarrow \ell = -1 \pm \sqrt{2}$. Since $x_n > 0$ for all $n \in \mathbb{N}$, we have $\ell \geq 0$ and so $\ell = \sqrt{2} - 1$. □

5. Given $a, b \in \mathbb{R}$, let $x_1 = a, x_2 = b$ and $x_n = \dfrac{1}{2}(x_{n-1} + x_{n-2})$ for $n \geq 3$. Show that (x_n) is a Cauchy sequence and $\lim x_n = \dfrac{1}{3}(a + 2b)$.

   Solution. We have $|x_{n+2} - x_{n+1}| = \frac{1}{2}|x_{n+1} - x_n|$ for $n \in \mathbb{N}$. Hence (x_n) is Cauchy. Let $x_n \to \ell$. (Note that if we try to find the value of ℓ using the recurrence relation, we get $\ell = \ell$). We have $x_{n+1} - x_n = -\frac{1}{2}(x_n - x_{n-1}) = \cdots = \left(-\frac{1}{2}\right)^{n-1}(x_2 - x_1)$ for all $n \geq 1$. This yields

$$\begin{aligned}
x_{n+1} - x_1 &= (x_{n+1} - x_n) + \cdots + (x_2 - x_1) \\
&= \left(-\frac{1}{2}\right)^{n-1}(x_2 - x_1) + \left(-\frac{1}{2}\right)^{n-2}(x_2 - x_1) + \cdots + (x_2 - x_1) \\
&= \left[\left(-\frac{1}{2}\right)^{n-1} + \left(-\frac{1}{2}\right)^{n-2} + \cdots + 1\right](x_2 - x_1) \\
&= \frac{2}{3}\left[1 - \left(-\frac{1}{2}\right)^{n}\right](x_2 - x_1).
\end{aligned}$$

   Since $x_n \to \ell$, so $\ell - a = \frac{2}{3}(b - a)$. This gives $\ell = \frac{1}{3}(a + 2b)$. □

6. Let $x_n = (-1)^n\left(1 + \frac{1}{n}\right)$, $n \geq 1$. Find $\limsup x_n$ and $\liminf x_n$.

   Proof. We have

$$\begin{aligned}
y_k = \sup\{x_n : n \geq k\} &= \sup\{(-1)^n\left(1 + \frac{1}{n}\right) : n \geq k\} \\
&= \begin{cases} 1 + \frac{1}{k+1} & \text{if } k \text{ is odd;} \\ 1 + \frac{1}{k} & \text{if } k \text{ is even.} \end{cases}
\end{aligned}$$

   Hence, $\limsup x_n = \lim_{k\to\infty} y_k = 1$. Similarly,

$$\begin{aligned}
z_k = \inf\{x_n : n \geq k\} &= \inf\{(-1)^n\left(1 + \frac{1}{n}\right) : n \geq k\} \\
&= \begin{cases} -1 - \frac{1}{k+1} & \text{if } k \text{ is even;} \\ -1 - \frac{1}{k} & \text{if } k \text{ is odd.} \end{cases}
\end{aligned}$$

   Hence, $\liminf x_n = \lim_{k\to\infty} z_k = -1$. □

7. Let $x_n = (1 + 1/n)^n$ and $y_n = \displaystyle\sum_{k=0}^{n} \frac{1}{k!}$. Prove that $\lim x_n = \lim y_n$.

Solution. We know that both (x_n) and (y_n) are convergent sequences. Now,

$$\begin{aligned}
x_n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k &= 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 2 \cdot 1}{n!} \cdot \frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\
&\leq 1 + 1 + \frac{1}{2!} + \cdots + \frac{1}{n!} = y_n.
\end{aligned}$$

Thus, $x_n \leq y_n$ for all n, and hence $\lim x_n \leq \lim y_n$. Now, let m be a fixed positive integer. Then, for $n \geq m$ we have

$$\begin{aligned}
x_n = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k &\geq \sum_{k=0}^{m} \binom{n}{k} \left(\frac{1}{n}\right)^k \\
&= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{m!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{m-1}{n}\right).
\end{aligned}$$

This yields $\lim x_n \geq \displaystyle\sum_{k=0}^{m} \frac{1}{k!}$. Thus, $\lim x_n \geq y_m$ for all m. Hence $\lim x_n \geq \lim y_n$. This proves that $\lim x_n = \lim y_n$. $\square$

8. Examine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$ is convergent.

Solution. Let $x_n = \frac{1}{n^{1+\frac{1}{n}}}$ and let $y_n = \frac{1}{n}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = 1 \neq 0$. Since $\sum_{n=1}^{\infty} y_n$ is not convergent, by the limit comparison test, $\sum_{n=1}^{\infty} x_n$ is also not convergent. $\square$

9. Examine whether the following series are convergent.

(a) $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

Solution. Taking $x_n = \frac{n!}{n^n}$ for all $n \in \mathbb{N}$, we find that $\lim\limits_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim\limits_{n\to\infty} (\frac{n}{n+1})^n = \lim\limits_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$. Hence by the ratio test, the given series is convergent. $\square$

(b) $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{1}{n}$

Solution. Since $0 \leq \frac{1}{n} \sin \frac{1}{n} \leq \frac{1}{n^2}$ for all $n \in \mathbb{N}$ and since the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by comparison test, the given series is convergent. $\square$

10. Let $x_n > 0$ for all $n \in \mathbb{N}$. Show that the series $\sum_{n=1}^{\infty} x_n$ converges iff the series $\sum_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converges.

Solution. We have $0 < \frac{x_n}{1+x_n} < x_n$ for all $n \in \mathbb{N}$. Hence by comparison test, $\sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converges if $\sum\limits_{n=1}^{\infty} x_n$ converges.

Conversely, let $\sum\limits_{n=1}^{\infty} \frac{x_n}{1+x_n}$ converge. Then $\frac{x_n}{1+x_n} \to 0$ and so there exists $n_0 \in \mathbb{N}$ such that $\frac{x_n}{1+x_n} < \frac{1}{2}$ for all $n \geq n_0$. This implies that $x_n < 1$ for all $n \geq n_0$, *i.e.* $1 + x_n < 2$ for all $n \geq n_0$ and so $x_n < \frac{2x_n}{1+x_n}$ for all $n \geq n_0$. By comparison test, we conclude that $\sum\limits_{n=1}^{\infty} x_n$ converges. $\square$