Department of Mathematics
Indian Institute of Technology Guwahati
MA 101: Mathematics I
Solutions of Tutorial Sheet-5
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1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ [x] & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$
Determine all the points of $\mathbb{R}$ where f is continuous.

Solution. Let $x \in \mathbb{R} \setminus \mathbb{Q}$. Then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to x$. So $f(r_n) = r_n \to x \neq [x] = f(x)$. Hence f is not continuous at x.
Again, let $y \in \mathbb{Q}$. Then there exists a sequence (t_n) in $\mathbb{R} \setminus \mathbb{Q}$ such that $t_n < y$ for all $n \in \mathbb{N}$ and $t_n \to y$. For each $n \in \mathbb{N}$, $f(t_n) = \begin{cases} [t_n] \le y - 1 & \text{if } y \in \mathbb{Z}, \\ [t_n] \le [y] < y & \text{if } y \notin \mathbb{Z}. \end{cases}$
In either case $f(t_n) \not\to f(y) = y$. Hence f is not continuous at y. Therefore f is not continuous at any point of $\mathbb{R}$. □

2. Let $f : [0, 1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Show that

(a) there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{2}$.

(b) there exist $x_1, x_2 \in [0, 1]$ such that $f(x_1) = f(x_2)$ and $x_1 - x_2 = \frac{1}{3}$.

Solution. (a) Let $g(x) = f(x + \frac{1}{2}) - f(x)$ for all $x \in [0, \frac{1}{2}]$. Since f is continuous, $g : [0, \frac{1}{2}] \to \mathbb{R}$ is continuous. Also $g(0) = f(\frac{1}{2}) - f(0)$ and $g(\frac{1}{2}) = f(1) - f(\frac{1}{2}) = -g(0)$, since $f(0) = f(1)$. If $g(0) = 0$, then we can take $x_1 = \frac{1}{2}$ and $x_2 = 0$. Otherwise, $g(\frac{1}{2})$ and $g(0)$ are of opposite signs and hence by the intermediate value property of continuous functions, there exists $c \in (0, \frac{1}{2})$ such that $g(c) = 0$, *i.e.* $f(c + \frac{1}{2}) = f(c)$. We take $x_1 = c + \frac{1}{2}$ and $x_2 = c$.

(b) Let $g(x) = f(x + \frac{1}{3}) - f(x)$ for all $x \in [0, \frac{2}{3}]$. Since f is continuous, $g : [0, \frac{2}{3}] \to \mathbb{R}$ is continuous. Also $g(0) + g(\frac{1}{3}) + g(\frac{2}{3}) = f(1) - f(0) = 0$. If at least one of $g(0)$, $g(\frac{1}{3})$ and $g(\frac{2}{3})$ is 0, then the result follows immediately. Otherwise, at least two of $g(0)$, $g(\frac{1}{3})$ and $g(\frac{2}{3})$ are of opposite signs and hence by the intermediate value property of continuous functions, there exists $c \in (0, \frac{2}{3})$ such that $g(c) = 0$, *i.e.* $f(c + \frac{1}{3}) = f(c)$. We take $x_1 = c + \frac{1}{3}$ and $x_2 = c$. □

3. Let p be an odd degree polynomial with real coefficients in one real variable. If $g : \mathbb{R} \to \mathbb{R}$ is a bounded continuous function, then show that there exists $x_0 \in \mathbb{R}$ such that $p(x_0) = g(x_0)$.

In particular, this shows that

(a) every odd degree polynomial with real coefficients in one real variable has at least one real zero.

(b) the equation $x^9 - 4x^6 + x^5 + \frac{1}{1+x^2} = \sin 3x + 17$ has at least one real root.

(c) the range of every odd degree polynomial with real coefficients in one real variable is $\mathbb{R}$.

Solution. Let $f(x) = p(x) - g(x)$ for all $x \in \mathbb{R}$. Since both p and g are continuous, $f : \mathbb{R} \to \mathbb{R}$ is continuous. Since g is bounded, there exists $M > 0$ such that $|g(x)| \le M$ for all $x \in \mathbb{R}$. Let $p(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$ for all $x \in \mathbb{R}$, where $a_i \in \mathbb{R}$ for $i = 0, 1, \dots, n$, $n \in \mathbb{N}$ is odd and $a_0 \neq 0$.
So $p(x) = a_0x^n(1 + \frac{a_1}{a_0} \cdot \frac{1}{x} + \cdots + \frac{a_{n-1}}{a_0} \cdot \frac{1}{x^{n-1}} + \frac{a_n}{a_0} \cdot \frac{1}{x^n})$ for all $x(\neq 0) \in \mathbb{R}$. We assume that $a_0 > 0$. (The case $a_0 < 0$ is almost similar.) Then $\lim\limits_{x\to\infty} p(x) = \infty$ and $\lim\limits_{x\to-\infty} p(x) = -\infty$ (since n is odd). So there exist $x_1 > 0$ and $x_2 < 0$ such that $p(x_1) > M$ and $p(x_2) < -M$. Hence $f(x_1) > 0$ and $f(x_2) < 0$. By the intermediate value property of continuous functions, there exists $x_0 \in (x_2, x_1)$ such that $f(x_0) = 0$, *i.e.* $p(x_0) = g(x_0)$.

For (a), we take $g(x) = 0$ for all $x \in \mathbb{R}$. For (b), we take $p(x) = x^9 - 4x^6 + x^5 - 17$ and $g(x) = \sin 3x - \frac{1}{1+x^2}$ for all $x \in \mathbb{R}$ and we note that $|g(x)| \le 2$ for all $x \in \mathbb{R}$. For (c), given $y \in \mathbb{R}$, we take $g(x) = y$ for all $x \in \mathbb{R}$. □

4. Does there exist a continuous function from $(0, 1]$ onto $\mathbb{R}$? Justify.

Solution. If $f(x) = \frac{1}{x} \sin \frac{1}{x}$ for all $x \in (0, 1]$, then $f : (0, 1] \to \mathbb{R}$ is continuous and $f(\frac{2}{(4n+1)\pi}) = 2n\pi + \frac{\pi}{2}$, $f(\frac{2}{(4n+3)\pi}) = -2n\pi - \frac{3\pi}{2}$ for all $n \in \mathbb{N}$. For each $y \in \mathbb{R}$, we can find $n \in \mathbb{N}$ such that $-2n\pi - \frac{3\pi}{2} < y < 2n\pi + \frac{\pi}{2}$ and hence by the intermediate value property of continuous functions, there exists $x \in \mathbb{R}$ such that $f(x) = y$. Thus $f : (0, 1] \to \mathbb{R}$ is onto. Therefore there exists such a function. □

5. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable on $(-\delta, \delta)$ for some $\delta > 0$ and let $f''(0)$ exist. If $f(\frac{1}{n}) = 0$ for all $n \in \mathbb{N}$, then find $f'(0)$ and $f''(0)$.

Solution. Since f is continuous at 0 and since $\frac{1}{n} \to 0$, we have $f(0) = \lim\limits_{n\to\infty} f(\frac{1}{n}) = 0$. Also, since $f'(0)$ exists (in $\mathbb{R}$) and since $\frac{1}{n} \to 0$, we have $f'(0) = \lim\limits_{n\to\infty} \frac{f(\frac{1}{n})-f(0)}{1/n} = 0$. Again, we can choose $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \delta$. By Rolle's theorem, for each $n \ge n_0$, there exists $x_n \in (\frac{1}{n+1}, \frac{1}{n})$ such that $f'(x_n) = 0$. By sandwich theorem, $x_n \to 0$. Since $f''(0)$ exists, we have $f''(0) = \lim\limits_{n\to\infty} \frac{f'(x_n)-f'(0)}{x_n} = 0$. □

6. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable such that $f(0) = f(1) = 0$ and $f'(0) > 0$, $f'(1) > 0$. Show that there exist $c_1, c_2 \in (0, 1)$ with $c_1 \neq c_2$ such that $f'(c_1) = f'(c_2) = 0$.

Solution. Since $f'(0) > 0$, there exists $\delta_1 \in (0, \frac{1}{2})$ such that $f(x) > f(0) = 0$ for all $x \in (0, \delta_1)$. Also, since $f'(1) > 0$, there exists $\delta_2 \in (0, \frac{1}{2})$ such that $f(x) < f(1) = 0$ for all $x \in (1 - \delta_2, 1)$. By the intermediate value property of continuous functions, there exists $c \in (\frac{\delta_1}{2}, 1 - \frac{\delta_2}{2})$ such that $f(c) = 0$. Now, by Rolle's theorem, there exists $c_1 \in (0, c)$ and $c_2 \in (c, 1)$ such that $f'(c_1) = f'(c_2) = 0$. □

7. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f''(c)$ exists, where $c \in \mathbb{R}$. Show that $\lim\limits_{h\to0} \frac{f(c+h)-2f(c)+f(c-h)}{h^2} = f''(c)$. Give an example of an $f : \mathbb{R} \to \mathbb{R}$ and a point $c \in \mathbb{R}$ for which $f''(c)$ does not exist but the above limit exists.

Solution. Since $f''(c)$ exists, there exists $\delta > 0$ such that $f'(x)$ exists for each $x \in (c - \delta, c + \delta)$. Hence by L'Hôpital's rule, $\lim\limits_{h\to0} \frac{f(c+h)-2f(c)+f(c-h)}{h^2} = \lim\limits_{h\to0} \frac{f'(c+h)-f'(c-h)}{2h}$, provided the second limit exists. Now

$$\begin{aligned}\lim_{h\to 0}\frac{f'(c+h)-f'(c-h)}{2h} &= \frac{1}{2}\left[\lim_{h\to 0}\frac{f'(c+h)-f'(c)}{h}+\lim_{h\to 0}\frac{f'(c-h)-f'(c)}{-h}\right]\\ &= \frac{1}{2}[f''(c)+f''(c)] = f''(c).\end{aligned}$$

Hence $\lim\limits_{h\to 0}\frac{f(c+h)-2f(c)+f(c-h)}{h^2} = f''(c)$.

If $f(x) = \begin{cases} 1 & \text{if } x>0,\\ 0 & \text{if } x=0,\\ -1 & \text{if } x<0,\end{cases}$ then $f:\mathbb{R}\to\mathbb{R}$ is not continuous at 0 and hence $f''(0)$ does not exist, but $\lim\limits_{h\to 0}\frac{f(0+h)-2f(0)+f(0-h)}{h^2} = 0$, since $f(h)+f(-h)=0$ for all $h(\neq 0)\in\mathbb{R}$. □

8. Prove that, for $x>0$,

$$|\log(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)| < \frac{x^{n+1}}{n+1}.$$

Proof. Easy. Show that the remainder term $R_n(x)$ satisfies $|R_n(x)| < \frac{x^{n+1}}{n+1}$. □

9. Test the convergence of the power series: $\sum\limits_{n=1}^{\infty} a_n x^n$, where $a_n = \begin{cases} 2^{-n} & \text{if } n \text{ is even,}\\ 3^{-n} & \text{if } n \text{ is odd.}\end{cases}$

Solution. Clearly $\beta = \limsup \sqrt[n]{|a_n|} = \frac{1}{2}$. Hence the power series converges (absolutely) for all $x\in(-2,2)$. When $x=\pm 2$, the n-th term $a_n x^n$ does not converge to zero. Therefore, the series converges on $(-2,2)$ only. □

10. Prove that the Maclaurin series for $\cos x$ converges to $\cos x$ for all $x\in\mathbb{R}$.

Proof. If $f(x)=\cos x$ for all $x\in\mathbb{R}$, then $f:\mathbb{R}\to\mathbb{R}$ is infinitely differentiable and $f^{(2n-1)}(x) = (-1)^n \sin x$, $f^{(2n)}(x) = (-1)^n\cos x$ for all $x\in\mathbb{R}$ and for all $n\in\mathbb{N}$. Hence the Maclaurin series for $\cos x$ is the series $1+\sum\limits_{n=1}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$, where $x\in\mathbb{R}$. For $x=0$, the Maclaurin series of $\cos x$ becomes $1-0+0-\cdots$, which clearly converges to $\cos 0 = 1$. Let $x(\neq 0)\in\mathbb{R}$. The remainder term in the Taylor expansion of $\sin x$ about the point 0 is given by $R_n(x) = \frac{x^{n+1}}{(n+1)!}f^{(n+1)}(c_n)$, where c_n lies between 0 and x. Since $|\sin c_n|\leq 1$ and $|\cos c_n|\leq 1$, we get $|R_n(x)|\leq\frac{|x|^{n+1}}{(n+1)!}$. Also, since $\lim\limits_{n\to\infty}\frac{|x|^{n+2}}{(n+2)!}\cdot\frac{(n+1)!}{|x|^{n+1}} = \lim\limits_{n\to\infty}\frac{|x|}{n+2} = 0 < 1$, we get $\lim\limits_{n\to\infty}\frac{|x|^{n+1}}{(n+1)!} = 0$ and hence it follows that $\lim\limits_{n\to\infty} R_n(x) = 0$. Therefore the Maclaurin series of $\cos x$ converges to $\cos x$.

□