Physics II (PH 102)

Electromagnetism (Lecture 10)

Udit Raha

Indian Institute of Technology Guwahati

Feb 2020

Conductors & Insulators

- Depending on the responses to applied electric fields, materials are classified as CONDUCTORS, INSULATORS & SEMICONDUCTORS
- *Conductors* are materials such as metals, electrolytes (e.g., salt water), graphite, human body, etc., that have a large number of mobile charge carriers, like, electron, ions, etc., that *freely flow* across entire surface of materials and distribute uniformly. They usually have very low resistances.
- A *perfect conductor* is an idealized concept, defined as a material with infinite supply of free charge carriers. They have zero resistances. In electrostatics, metals almost qualify as nearly perfect conductors having an abundance of loosely bound electrons.
- *Insulators* are materials, such as plastic, glass, rubber, etc., that impede the free flow of chages and subsequently charges are seldom distributed evenly across their surfaces. They are substances usually with very high resistances.
- *Semi-conductors* are usually solid materials, such as silicon, germanium, various alloys, etc., that has conductivities intermediate between insulators and most metals, which often can be regulated in a controlled way, e.g., by the addition impurities (doping) or under temperature variations.

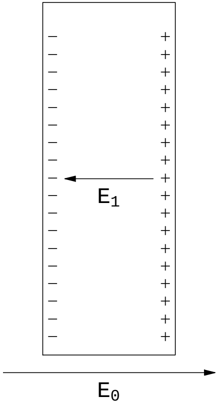
Properties of Conductors

- Static Electric field is always zero inside uncharged conductors.

Example

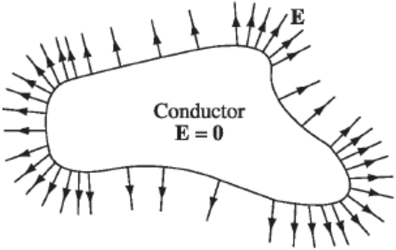
An uncharged conductor is placed in external field $\mathbf{E}_0$.

Positive charges are pushed to right and negative charges to left. These are termed as **induced charges**, which give rise to the **induced Electric field** $\mathbf{E}_1$. The process would continue until $\mathbf{E}_1 = -\mathbf{E}_0$. The net field $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_0 = \mathbf{E}_1 - \mathbf{E}_1 = 0$ inside the conductor.



What happens for the field inside an insulator or a dielectric material ?

Properties of Conductors (contd.)



Gauss's Differential Law: Given that the total field **E** is zero inside the conductor, the volume charge density ρ also vanishes:

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = 0$$

Corollary

- *Any net charge must reside ONLY on the surface of a conductor.*
- *Potential is constant throughout a conductor.*
- *Surface of a conductor represents an equipotential surface.*
- *Electric field lines are directed perpendicular to the surface just outside a conductor.*

Boundary Conditions for $\mathbf{E}$ on the surface of a charged Conductor

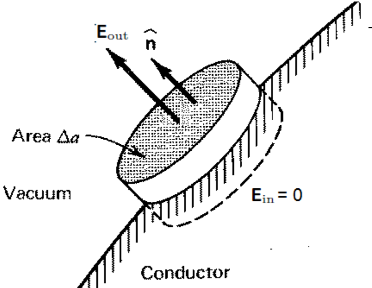
Let $(\hat{\mathbf{n}}, \hat{\mathbf{n}}_{||})$ be the outward (normal, tangent) unit vectors on the smooth conducting interface with charge density $\sigma = \sigma(\mathbf{r}) = const.$ Let $\mathbf{E}_{\rm out}(\mathbf{E}_{\rm in})$ be the outside (inside) fields. The BOUNDARY CONDITIONS yield:

$$\mathbf{E}_{\rm out}(\mathbf{r}) - \mathbf{E}_{\rm in}(\mathbf{r}) = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}\,.$$

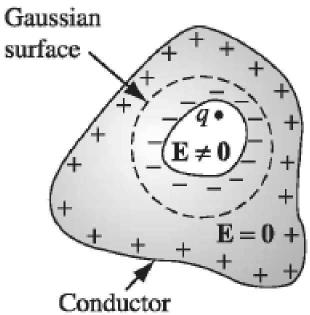
- Electric Field at any point immediately outside a conductor is always constant and points perpendicular to the surface.

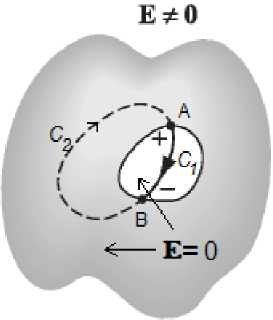
$$E^{\perp}_{\rm out}(\mathbf{r}) - \cancelto{0}{E^{\perp}_{\rm in}(\mathbf{r})} = \frac{\sigma}{\epsilon_0} = const.$$

$$E^{||}_{\rm out}(\mathbf{r}) - \cancelto{0}{E^{||}_{\rm in}(\mathbf{r})} = 0$$



Cavities within Uncharged Conductors





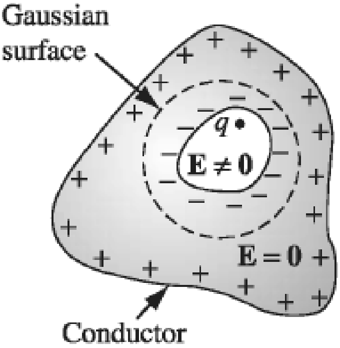
$$\text{FLUX} = \oiint_S \cancelto{0}{\mathbf{E}_{\text{conductor}} \cdot d\mathbf{a}} = \frac{Q_{encl}}{\epsilon_0} = \frac{1}{\epsilon_0}\left(q + q_{\text{induced}}\right)$$

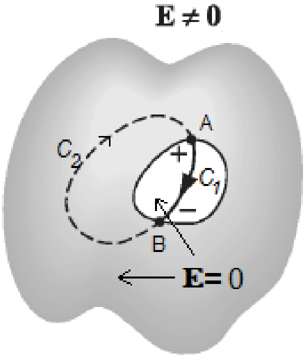
$$\implies q_{\text{induced}} = -q$$

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0 = \int_{C_1} \mathbf{E}_{\text{cavity}} \cdot d\mathbf{l} + \cancelto{0}{\int_{C_2} \mathbf{E}_{\text{conductor}} \cdot d\mathbf{l}} = \int_A^B \mathbf{E}_{\text{cavity}} \cdot d\mathbf{l}$$

$$V(B) = V(A) \Rightarrow \mathbf{E}_{\text{cavity}} = 0$$

Cavities within Uncharged Conductors: Summary

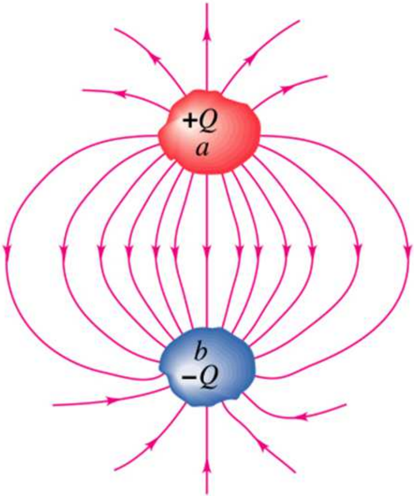




- The induced charges in a conductor have the same magnitude as the total charge enclosed within cavities.
- The Electric field within cavities of conductors are non-zeros ONLY when they enclose some net amount of charge.

Capacitors and Capacitance

- It is a device used to store electrical charge, consisting of a system of two conductors placed in close proximity of each other (suitably shielded from other charged conductors) having equal and opposite charges $\pm Q$.



Capacitors and Capacitance (contd.)

- The potential difference between the conductors is proportional to the magnitude of the charge Q carried by the conductors

$$\begin{aligned} \Delta V &= V_+ - V_- = -\int_{(-)}^{(+)} \mathbf{E} \cdot d\mathbf{l} \propto Q \\ &= \left(\frac{1}{C}\right) Q \\ C &= \frac{Q}{\Delta V} = \frac{Q}{V_+ - V_-} \end{aligned}$$

- The purely geometrical coefficient C is called the CAPACITANCE and depends on the shapes, sizes and separation of the two conductors.

The capacitance of a capacitor is a measure of its ability to store charge.

- SI Unit: **Coulomb per Volt** (C/V) or **Farad** (F).
- The Electrostatic energy stored in a capacitor of charge Q and potential difference ΔV:

$$U_E = \frac{1}{2} C (\Delta V)^2 = \frac{1}{2} \frac{Q^2}{\Delta V}.$$

- Spherical capacitor of radius R:

$$C = 4\pi\epsilon_0 R$$

- Parallel plate capacitor with area A and separation d:

$$C = \frac{\epsilon_0 A}{d}$$

- Concentric spherical capacitor with inner radius a and outer radius b:

$$C = 4\pi\epsilon_0 \left(\frac{ab}{b-a}\right)$$

- Coaxial cylindrical capacitor of length L, inner radius a and outer radius b:

$$C = 2\pi\epsilon_0 \frac{L}{\ln\left(\frac{b}{a}\right)}$$

- Effective capacitance of parallel system of capacitors:

$$C = C_1 + C_2 + C_3 + \cdots$$

- Effective capacitance of series connected capacitors:

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots$$

Solving Boundary-Valued Problems: Limitations of Coulomb & Gauss' Laws

- You are so far familar with solving problems in electrostatics where either a system of charges q_i, or charge distributions (λ, σ, ρ) are supplied. Then you are asked to determine both $\mathbf{E}$ or V using formulas derived from *Coulomb's Law*:

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')\,(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^2}\, dv' \\ V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \iiint \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dv' \end{aligned}$$

- For problems with *high degrees of symmetry* you could use *Gauss's Integral Law* to first determine $\mathbf{E}$ and then determine V:

$$\begin{aligned} \oiint \mathbf{E}(\mathbf{r}) \cdot dS &= \frac{1}{\epsilon_0} Q_{\mathrm{encl}} \\ \Longrightarrow \qquad V(\mathbf{r}) &= -\int_{\infty}^{\mathbf{r}} \mathbf{E}(\mathbf{r}) \cdot d\mathbf{l} \end{aligned}$$

- These approaches become useless if the charges or charge densities are *a priori* NOT supplied.

Study of Boundary-Valued Problems

The Problem: *In many practical Electrostatic problems involving a system of conductors ($i = 1, 2, .., n$) in a certain region, the charge distributions may not be a priori known, but instead, either the Potentials V_i, Electric Fields $\mathbf{E}_i$, or the total charges Q_i may be only specified on the surfaces of different conductors. How to determine solutions of $V(\mathbf{r})$, $\mathbf{E}(\mathbf{r})$ or $\rho(\mathbf{r})$ for all space?*

- First require solving for the **Potential** $V(\mathbf{r})$ everywhere by solving either:

$$
\begin{aligned}
\nabla^2 V(\mathbf{r}) &= -\frac{\rho(\mathbf{r})}{\epsilon_0} && \Longrightarrow \text{Poisson's Equation} \\
\nabla^2 V(\mathbf{r}) &= 0 && \Longrightarrow \text{Laplace's Equation}
\end{aligned}
$$

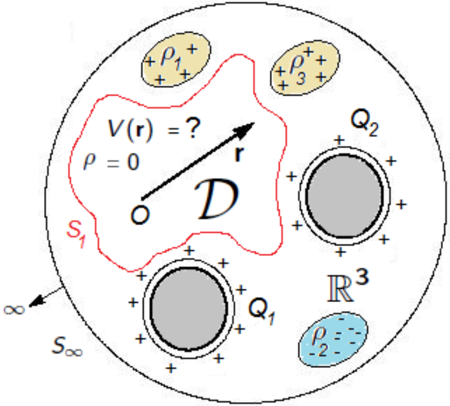
- Using 2 different **Methods of Boundary Conditions**:
  1. Solving via the special technique called "Method of Images"
  2. Rigorously solution of Laplace's Eqs. (2^{nd} order PDEs):

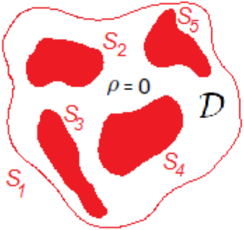
- Especially, we will solve Laplace's Equations in charge free regions in Cartesian System using the method of SEPARATION OF VARIABLES :

$$\nabla^2 V(x,y,z) \equiv \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \Longrightarrow V(x,y,z) = X(x)Y(y)Z(z)$$

Boundary-Valued Problems: Schematic Domain for Solving $\nabla^2 V(\mathbf{r}) = 0$

- Our interest is particularly to obtain a solution to the Potential $V(\mathbf{r})$ in a charge free $\rho(\mathbf{r}) = 0$ region: $\mathbf{r} \in \mathcal{D} \subset \mathbb{R}^3$, bounded by one or multipple closed surfaces $S_1, S_2, S_3 \cdots$





- Note: $\mathcal{D}$ is free of charge distributions, but there may be plenty of charge distributions and charged conductors elsewhere in $\mathbb{R}^3$. In case there are no charges anywhere, the solution becomes trivial, i.e., $V(\mathbf{r}) = 0, const. \; \forall \, \mathbb{R}^3$.

Characteristics of Solutions to Laplace's Equation

- A set of BOUNDARY CONDITIONS ensure UNIQUENESS of solutions.
- The solutions to the Laplace's equations are called HARMONIC FUNCTIONS. They have 2 very special properties:

1. *They attain extrema only at the boundaries of the domain $\mathcal{D}$ of their definition; otherwise they are rather "monotonic" functions without any local maxima or minima at interior points in $\mathcal{D}$.*
2. *Their values at any given interior point P in the domain $\mathcal{D}$ is the average over their values about ANY closed interval around P.*

- **Geometrical Interpretations** :
  1. In 1D, the solution yields the shortest distance between the two given boundary points, i.e., a straight line.
  2. In 2D, the solution minimizes the surface area between given boundary curves.
  3. In 3D, geometrical interpretations of solutions are in general difficult to visualize. The only way to interpret:

For a given set of boundary conditions they yield maximally monotonic solns!

Consequences of Harmonic Nature of Solutions in 1D

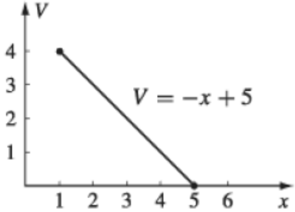
Example

Suppose V depends on only one variable, x.

$$\frac{d^2V}{dx^2} = 0$$

The general solution is $\longrightarrow$ $V(x) = mx + b,$

the boundary conditions $\longrightarrow$ $V = 4$ at $x = 1,$ $V = 0$ at $x = 5.$



→ It contains two undetermined constants *(m* and *b)*

→ *(m* and *b)* are fixed, in any particular case, by the boundary conditions of that problem.

Call attention to two features of this result

1. V(x) is the average of V(x + a) and V(x - a), for any a:

$$\boxed{V(x) = \frac{1}{2}[V(x+a) + V(x-a)]}$$

➔ Laplace's equation is a kind of averaging instruction

2. Laplace's equation tolerates no local maxima or minima

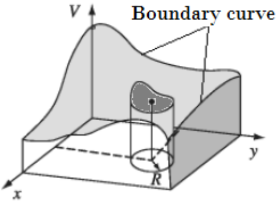
➔ Extreme values of *V* must occur at the end points

Consequences of Harmonic Nature of Solutions in 2D

Example

If V depends on two variables,

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$



This is no longer an *ordinary* differential equation

→ it is **a *partial* differential equation**

→ general solution to this equation **doesn't contain just two arbitrary constants** despite the fact that it's a second order equation.

→ the boundary conditions ⟶ $V(\mathbf{r}(t_b)) = f(t_b)$; $t_b \in \mathbb{R} \rightarrow$ variable parameter , specified at all points on the boundary curve.

Nevertheless, possible to deduce certain properties common to all solutions.

1. The value of *V* at a point *(x, y)* is the average of those *around* the point.

→ If you draw a circle of any radius R about the point (x, y), the average value of V on the circle is equal to the value at the center:

$$\boxed{V(x, y) = \frac{1}{2\pi R} \oint_{\text{circle}} V \, dl}$$

2. V has no local maxima or minima; all extrema occur at the boundaries.

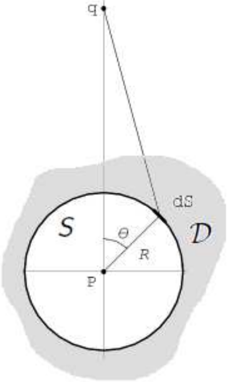
→ Laplace's equation picks the most **featureless function** possible, consistent with the boundary conditions:

→ Laplace's equation picks no hills, no valleys, **just the smoothest surface available.**

Harmonic Nature of Solution to Laplace's Equation in 3D

Example

Find the average potential over an imaginary constructed spherical surface S of radius R with center P, located in a charge free domain $\mathcal{D}$ at a distance $d \gg R$ from a far away point charge q located outside $\mathcal{D}$.



- Electrostatic Potential at P: $V(P) = \frac{q}{4\pi\epsilon_0 d}$
- Average Potential on surface S ($dS = R^2 \sin\theta \, d\theta \, d\phi$):

$$
\begin{aligned}
\langle V \rangle_S &= \frac{1}{4\pi R^2} \oiint_S V(\mathbf{r}(R,\theta,\phi))\, dS \\
&= \frac{1}{4\pi R^2} \frac{q}{4\pi\epsilon_0} \int_{\phi=0}^{\phi=2\pi} \int_{\theta=0}^{\theta=\pi} \frac{R^2 \sin\theta \, d\theta \, d\phi}{\sqrt{R^2 + d^2 - 2Rd\cos\theta}} \\
&= \frac{q}{4\pi\epsilon_0} \frac{1}{2dR} \sqrt{R^2 + d^2 - 2Rd\cos\theta}\,\Big|_0^{\pi} \\
&= \frac{q}{4\pi\epsilon_0 d} = V(P)!
\end{aligned}
$$

Potential at the point P due to the distant point charge is same as the average Potential over the spherical surface S centered at P. By Superposition Principle, the same is true for any collection of distant point charges.

Harmonic nature of Electrostatic Potential in 3D

Theorem

Let $\mathcal{D}$ be a charge free region in $\mathbb{R}^3$ with point P located within this domain. If S is any spherical surface in $\mathcal{D}$ centered at P, then the net Electrostatic Potential at P due to any collection of distant point charges located outside $\mathcal{D}$ is equal to the average of the net Electrostatic Potential over S.

Proof.

Consider the collection of distant point charges $q_1, q_2 \ldots, q_n$, all placed outside $\mathcal{D}$, and $V_1, V_2, \ldots, V_n$ be the potentials at $P \in \mathcal{D}$ due to these charges. Then, we know

$$V_i(P) = \langle V_i \rangle_S = \frac{1}{\mathrm{Area}} \oiint_S V_i(\mathbf{r} \in S)\, dS$$

So, the net Potential at P by **Superposition Principle** is

$$\begin{aligned}
V_{\mathrm{net}}(P) &= \sum_{i=1}^{n} V_i(P) \\
&= \sum_{i=1}^{n} \langle V_i \rangle_S = \frac{1}{\mathrm{Area}} \oiint_S \left[\sum_{i=1}^{n} V_i(\mathbf{r} \in S) \right] dS \\
&= \frac{1}{\mathrm{Area}} \oiint_S \left[V_{\mathrm{net}}(\mathbf{r} \in S) \right] dS = \langle V_{\mathrm{net}} \rangle_S
\end{aligned}$$

Harmonic nature of Electrostatic Potential in 3D (contd.)

Corollary

(1) If $\mathcal{D}$ be a charge free region $\subset \mathbb{R}^3$, then there CAN NOT be any local maxima or minima in the solution to the Electrostatic Potential anywhere INTERIOR within $\mathcal{D}$. All extremities must occur at the BOUNDARIES of $\mathcal{D}$.

Corollary

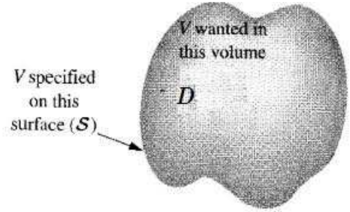
(2) Let $\mathcal{D}$ be a charge free region $\subset \mathbb{R}^3$ such that the Electrostatic Potential is constant on all its boundaries. Then the MONOTONIC nature of the solution demands that the Electrostatic Potential has the same CONSTANT value throughout $\mathcal{D}$.

Note:

- If the charge free region $\mathcal{D} \equiv \mathbb{R}^3$, then the boundary condition, $V = const.\ \forall \mathbf{r} \in S_\infty$, implies trivial solution $V = const.$ everywhere in $\mathbb{R}^3$.
- For a region with charge distributions, V satifies the Poisson's Equation, which **does not** guarantee a HARMONIC (monotonic) solution.

Uniqueness Theorem

Solutions to PDEs like, the Laplace's or Poisson's Equations, can be obtained in a variety of different ways in general. Moreover there are infinite number of solutions depending on different boundary conditions. Fortunately, the so-called **Uniqueness Theorem** *guarantees the solution to be unique regardless of the methodology used provided there exists a* unique set of boundary conditions.



Theorem

First Uniqueness Theorem: *Let $\mathcal{D} \in \mathbb{R}^3$ be a region free of charge with a smooth boundary surface S and $\alpha : S \to \mathbb{R}$ be any arbitrary smooth function defined on the boundary points $\mathbf{S} \in S$. Then the Laplace's equation,*

$$\nabla^2 V(\mathbf{r}) = 0 \qquad \textit{over } \mathcal{D},$$

$$\text{given the b.c. that, } V(\mathbf{S}) = \alpha(\mathbf{S}) \quad \textit{on } S\,,$$

always guarantees a unique solution.

First Uniqueness Theorem

Proof.

First, assume that two solutions $V_1(\mathbf{r})$ and $V_2(\mathbf{r})$ to the Laplace's equations in the "simple" charge fee region $\mathcal{D}$, assuming the <u>same</u> boundary condition that,

$$V_1(\mathbf{S}) = V_2(\mathbf{S}) = \alpha(\mathbf{S}) \qquad \forall \mathbf{S} \in S$$

for an arbitrary smooth function α on the "simple" boundary surface S, i.e.,

$$\begin{aligned} \nabla^2 V_1(\mathbf{r}) &= 0 \qquad \forall \mathbf{r} \in \mathcal{D}, \\ \nabla^2 V_2(\mathbf{r}) &= 0 \qquad \forall \mathbf{r} \in D. \end{aligned}$$

Next, define function $V_3 = V_1 - V_2$, then V_3 also satisfies Laplace's equation:

$$\nabla^2 V_3 = \nabla^2 V_1 - \nabla^2 V_2 = 0 \quad , \quad \text{over } \mathcal{D},$$

$$\text{such that, } V_3(\mathbf{S}) = V_1(\mathbf{S}) - V_2(\mathbf{S}) = \alpha(\mathbf{S}) - \alpha(\mathbf{S}) = 0 \quad , \quad \text{on } S.$$

Use Corollaries (1) & (2): *Solutions to Laplace's equation tolerates no local extrema within the region $\mathcal{D}$; all extrema must occur only at the boundary S.*

Since, $\nabla^2 V_3 = 0$ over $\mathcal{D}$, and $V_3 = 0$ on S, it must imply $V_3 = 0 \ \forall \mathbf{r} \in \mathcal{D}$.

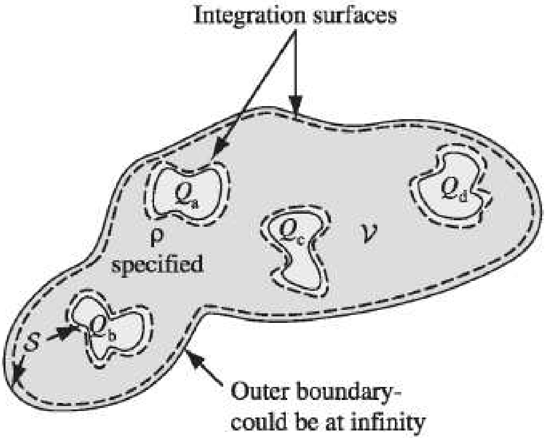
$$\hookrightarrow V_1(\mathbf{r}) = V_2(\mathbf{r}) \qquad \forall \mathbf{r} \in \mathcal{D},$$

i.e., both solutions are identical ! $\square$

Second Uniqueness Theorem: Charged regions and Conductors

Theorem

Second Uniqueness Theorem: *If $\mathcal{V} \subset \mathbb{R}^3$ be a region surrounded by a system of charged conductors and filled with a specified charge density ρ, then the Electric Field $\mathbf{E}$ is uniquely determined by specifying the total charge on each conductor.*



Second Uniqueness Theorem

Proof.

- First, assume two solutions, $\mathbf{E}_1(\mathbf{r}) = -\nabla V_1(\mathbf{r})$ and $\mathbf{E}_2(\mathbf{r}) = -\nabla V_1(\mathbf{r})$ in the space between the conductors in region $\mathcal{V}$, satisfying the differential form of Gauss's law:

$$\nabla \cdot \mathbf{E}_1(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0} \qquad \& \qquad \nabla \cdot \mathbf{E}_2(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}.$$

- For surfaces $S_1, ..., S_n$, enclosing the conductors with charges $Q_1, ..., Q_n$ (integral form of Gauss's law):

$$\oiint_{S_i} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{Q_i}{\epsilon_0} \qquad \& \qquad \oiint_{S_i} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{Q_i}{\epsilon_0}.$$

- Likewise, for the outer boundary $\mathcal{S}$ of $\mathcal{V}$ enclosing total charge $Q_{\rm tot}$:

$$\oiint_{\mathcal{S}} \mathbf{E}_1 \cdot d\mathbf{a} = \frac{Q_{\rm tot}}{\epsilon_0} \qquad \& \qquad \oiint_{\mathcal{S}} \mathbf{E}_2 \cdot d\mathbf{a} = \frac{Q_{\rm tot}}{\epsilon_0}.$$

- Next we define a new vector field $\mathbf{E}_3$ in $\mathcal{V}$:

$$\begin{aligned} \mathbf{E}_3(\mathbf{r}) &= \mathbf{E}_1(\mathbf{r}) - \mathbf{E}_2(\mathbf{r}) \\ -\nabla V_3(\mathbf{r}) &= -\nabla(V_1 - V_2), \quad \forall \mathbf{r} \in \mathcal{V} \end{aligned}$$

Proof. (contd.)

- $\mathbf{E}_3$ must satisfy:

$$\nabla\cdot\mathbf{E}_3(\mathbf{r}) = 0,\ \forall\mathbf{r}\in\mathcal{V} \qquad \& \qquad \underset{\text{all surfaces}}{\oiint} \mathbf{E}_3\cdot d\mathbf{a} = 0, \quad \forall\mathbf{S}\in\{\mathcal{S}\cup S_1\cup\ldots\cup S_n\}\,.$$

- Now, on the surface of each conductor, $S_1, \ldots, S_n$:

$$V_1(\mathbf{S}_i) = C_{1i}(const.) \quad \& \quad V_2(\mathbf{S}_i) = C_{2i}(const.)\,, \quad \mathbf{S}_i\in S_i$$

$$\Rightarrow V_3(\mathbf{S}_i) = V_1(\mathbf{S}_i) - V_2(\mathbf{S}_i) = C_{3i} \quad \Longrightarrow \quad \text{again a constant.}$$

- If we extend $\mathcal{V}$ to include **ALL SPACE:** $\mathcal{V}\to\mathbb{R}^3$ and $\mathcal{S}\to\mathcal{S}_\infty$, then

$$V_3(\mathcal{S}) \to V_3(\mathcal{S}_\infty) = 0 \Longrightarrow \quad \text{also a constant.}$$

- Thus, we conclude that V_3 is constant for ALL surfaces in $\mathcal{V}\to\mathbb{R}^3$, i.e.,

$$V_3(\mathbf{S}) \to consts.\ \forall\mathbf{S}\in\{\mathcal{S}\cup S_1\cup\ldots\cup S_n\}$$

Proof. (contd.)

- **Recall Identity:**
  $\nabla \cdot (V_3\, \mathbf{E}_3) = V_3\, (\nabla \cdot \mathbf{E}_3) + (\nabla V_3) \cdot \mathbf{E}_3 = V_3\, \cancelto{0}{(\nabla \cdot \mathbf{E}_3)} - |\mathbf{E}_3|^2 = -E_3^2$
- Applying Gauss's Divergence Theorem to the region $\mathcal{V} \to \mathbb{R}^3$:

$$\iiint_{\mathbb{R}^3} \nabla \cdot [V_3(\mathbf{r})\, \mathbf{E}_3(\mathbf{r})]\; d\tau = \oiint_{\text{all surfaces}} [V_3(\mathbf{S})\, \mathbf{E}_3(\mathbf{S})] \cdot d\mathbf{a} = \sum_i C_{3i} \cancelto{0}{\oiint_{S_i} \mathbf{E}_3 \cdot d\mathbf{a}} = 0$$

$$\Longrightarrow \iiint_{\mathbb{R}^3} \nabla \cdot [V_3(\mathbf{r})\, \mathbf{E}_3(\mathbf{r})]\; d\tau = -\iiint_{\mathbb{R}^3} E_3^2(\mathbf{r})\, d\tau = 0$$

- Since the integrand $E_3^2 > 0 \Longrightarrow E_3 = 0$.
- Consequently,

$$\mathbf{E}_1 = \mathbf{E}_2, \; \forall \mathbf{r} \in \mathbb{R}^3$$

.

Uniqueness Theorem: Application

Example

Show that the potential is *constant* inside an enclosure completely surrounded by conducting material, provided there is no charge within the enclosure

→ potential on the cavity wall is some constant, V_0

→ potential inside is a function that satisfies Laplace's equation and has the constant value V_0 at the boundary

$$\nabla^2 V(\mathbf{r}) = 0 \quad \text{inside cavity;}$$
$$V(\mathbf{r}) = V_0 \quad \text{on cavity wall}$$

→ $V = V_0$ everywhere inside enclosure

→ uniqueness theorem guarantees that this is the *only* solution

→ It follows that the *field* inside an empty cavity is zero.