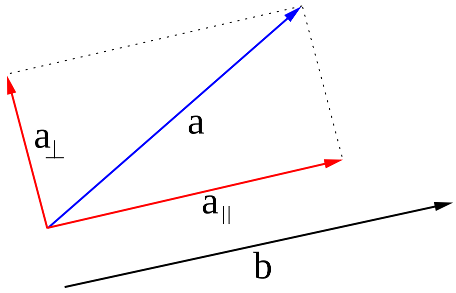
MERELY WRITING THE FINAL ANSWER WITHOUT SHOWING YOUR WORK WILL NOT FETCH YOU ANY MARK.



[2pnts.] 1. Given $\mathbf{a} =< 4, -5, 3 >$ and $\mathbf{b} =< 2, 1, -2 >$, express $\mathbf{a}$ as the sum of a vector $\mathbf{a}_{\|}$ parallel to $\mathbf{b}$ and a vector $\mathbf{a}_{\perp}$ perpendicular to $\mathbf{b}$. Verify that $\mathbf{a}_{\perp} \perp \mathbf{b}$. (**HINT:** See the adjacent figure.)

Soln.:

From the figure, we see that

$$\mathbf{a}_{\|} = \text{comp}_{\mathbf{b}}\mathbf{a}\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\mathbf{b} = \frac{8-5-6}{9}\mathbf{b} = \left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right\rangle$$

and

$$\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\|} = \left\langle \frac{14}{3}, -\frac{14}{3}, \frac{7}{3}\right\rangle$$

Clearly $\mathbf{a}_{\perp}\cdot\mathbf{b} = \frac{28}{3} - \frac{14}{3} - \frac{14}{3} = 0$. Hence the verification.

Aliter: We have $\mathbf{a}_{\|} = \lambda\mathbf{b}$ for some $\lambda \neq 0$ and $\mathbf{a}_{\perp} = \mathbf{a} - \mathbf{a}_{\|} = \mathbf{a} - \lambda\mathbf{b}$.

Now $\mathbf{a}_{\perp}\cdot\mathbf{b} = 0$ or $(\mathbf{a} - \lambda\mathbf{b})\cdot\mathbf{b} = 0$ yields $\lambda = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2} = \frac{8-5-6}{9} = -\frac{1}{3}$. The remaining part is similar to the above answer.

[2pnts.] 2. Find the point on the curve $\mathbf{r}(t) = 5\sin t\hat{\mathbf{i}} + 5\cos t\hat{\mathbf{j}} + 12t\hat{\mathbf{k}}$ at a distance 26π units along the curve from its origin $(t = 0)$ in the direction of increasing arc-length.

Soln.:

The arc length along the curve from its origin to the point corresponding to $t = T$ is given by $\int_0^T |\mathbf{r}'(t)|dt = \int_0^T \left(\sqrt{25\cos^2 t + 25\sin^2 t + 144}\right) dt = 13T$

Thus $13T = 26\pi$ or $T = 2\pi$.

Hence the required point is $(5\sin(2\pi), 5\cos(2\pi), 24\pi)$, i.e., $(0, 5, 24\pi)$.

[1pnts.] 3. Find the unit tangent vector of the curve $\mathbf{r}(t)$ at $(0, 1, 0)$, where $\mathbf{r}(t) = \langle\cos(t), \sin(t), t\cos(t)\rangle, \quad 0 \leq t \leq 2\pi$.

Soln.:

$(0,1,0)$ of the curve $\mathbf{r}(t)$ corresponds to $t = \frac{\pi}{2}$.
Since $\mathbf{r}'\left(\frac{\pi}{2}\right) = \left\langle -1, 0, -\frac{\pi}{2}\right\rangle$, the unit tangent vector of $\mathbf{r}(t)$ at $(0,1,0)$ is
$= \left\langle \frac{-1}{\sqrt{1+(\frac{\pi}{2})^2}}, 0, \frac{-\frac{\pi}{2}}{\sqrt{1+(\frac{\pi}{2})^2}} \right\rangle.$

[3pnts.] 4. Let

$$f(x,y) = \begin{cases} \frac{x^2y^3}{(x^2+y^2)^2} & \text{if} \quad x^2+y^2 \neq 0 \\ 0 & \text{if} \quad x^2+y^2 = 0. \end{cases}$$

Check whether f is differentiable at $(0,0)$.

Soln.:
$f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = 0.$
Similarly $f_y(0,0) = 0$.
f is differentiable at $(0,0)$ if

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(h_1,h_2) - f(0,0) - f_x(0,0)h_1 - f_y(0,0)h_2}{\sqrt{h_1^2+h_2^2}} = 0.$$

$$\frac{f(h_1,h_2) - f(0,0) - f_x(0,0)h_1 - f_y(0,0)h_2}{\sqrt{h_1^2+h_2^2}} = \frac{h_1^2h_2^3}{(h_1^2+h_2^2)^{\frac{5}{2}}}.$$

Consider $(h_1,h_2) \to (0,0)$ along a line L with slope m, $m \neq 0$,
then $\lim_{(h_1,h_2)\to(0,0)} \dfrac{h_1^2h_2^3}{(h_1^2+h_2^2)^{\frac{5}{2}}}$ along L
$= \lim_{h_1\to 0} \dfrac{m^3h_1}{(1+m^2)^{\frac{5}{2}}|h_1|}$, which does not exist (or $\neq 0$),
hence f is not differentiable at $(0,0)$.

Aliter: $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = 0.$
Similarly $f_y(0,0) = 0$.
f is differentiable at $(0,0)$, if (and only if)
$(\triangle z =) f(\triangle x, \triangle y) - f(0,0) = f_x(0,0)\triangle x + f_y(0,0)\triangle y + \epsilon_1\triangle x + \epsilon_2 \triangle y$
where $\epsilon_1, \epsilon_2 \to 0$ whenever $(\triangle x, \triangle y) \to (0,0)$.
Since $f(0,0) = f_x(0,0) = f_y(0,0) = 0$, f is differentiable at $(0,0)$ if

$$\triangle z = \frac{(\triangle x)^2(\triangle y)^3}{((\triangle x)^2 + (\triangle y)^2)^2} = \epsilon_1 \triangle x + \epsilon_2 \triangle y$$

for some ϵ_1, ϵ_2 such that $\epsilon_1, \epsilon_2 \to 0$ whenever $(\triangle x, \triangle y) \to (0,0)$.

Consider $(\triangle x, \triangle y) \to (0,0)$ along a line L with slope m, $(\triangle y = m\triangle x)$, $m \neq 0$.
If f is differentiable at $(0,0)$ then $\triangle z = \epsilon \triangle x$ for some $\epsilon (= \epsilon_1 + m\epsilon_2)$ such that $\epsilon \to 0$ whenever $\triangle x \to 0$.
Since $\triangle z = \dfrac{(\triangle x)^2(\triangle y)^3}{((\triangle x)^2 + (\triangle y)^2)^2} = \triangle x \dfrac{m^3(\triangle x)^4}{(1+m^2)^2(\triangle x)^4}$, along L

$$\epsilon = \frac{m^3(\triangle x)^4}{(1+m^2)^2(\triangle x)^4} = \frac{m^3}{(1+m^2)^2}.$$

Since ϵ does not tend to 0 as $\triangle x \to 0$, f is not differentiable at $(0,0)$.

[2pnts.] 5. Let $f(x,y) = (x-1)^2 + (y-2)^2$. Given $\epsilon > 0$, find a $\delta > 0$ (explicitly in terms of ϵ) such that, $\sqrt{(x-3)^2 + (y-1)^2} < \delta$ implies $|f(x,y) - f(3,1)| < \epsilon$.

Soln.:

$|f(x,y) - f(3,1)| = \left|(x-1)^2 + (y-2)^2 - 2^2 - 1^2\right|$
$= \left|(x-3)^2 + (y-1)^2 + 4(x-3) - 2(y-1)\right|$
$\leq \left((x-3)^2 + (y-1)^2\right) + 4\,|(x-3)| + 2\,|(y-1)|$
$\leq \left((x-3)^2 + (y-1)^2\right) + 4\sqrt{(x-3)^2 + (y-1)^2} + 2\sqrt{(x-3)^2 + (y-1)^2}$ (**)
$\leq 7\sqrt{(x-3)^2 + (y-1)^2}$ if (x,y) is such that $\sqrt{(x-3)^2 + (y-1)^2} \leq 1$.
Hence for any δ such that, $0 < \delta \leq \min\{1, \frac{\epsilon}{7}\}$,
$\sqrt{(x-3)^2 + (y-1)^2} < \delta$ implies $|f(x,y) - f(3,1)| < \epsilon$.