Physics II (PH 102)

Electromagnetism (Lecture 6)

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Charge Distributions: Continuous & Discrete

Continuous charge distributions:

- Linear Charge Density $\lambda(x)$ in 1D

$$\lambda(x) = \lim_{\Delta L \to 0} \frac{\Delta Q}{\Delta L} = \frac{dQ}{dL}$$

- Surface Charge Density $\sigma(\mathbf{r})$ in 2D

$$\sigma(\mathbf{r}) = \lim_{\Delta S \to 0} \frac{\Delta Q}{\Delta S} = \frac{dQ}{dS}$$

- Volume Charge Density $\rho(\mathbf{r})$ in 3D

$$\rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V} = \frac{dQ}{dV}$$

But can we represent discrete point charge distributions or densities?
Using Dirac δ-functions as Volume Charge Densities:

$$\rho(\mathbf{r}) = \sum_i q_i \delta^3(\mathbf{r} - \mathbf{r}_{0i}) = \sum_i q_i \delta(x - x_{0i})\delta(y - y_{0i})\delta(z - z_{0i})$$

Volume Charge Distributions using Dirac δ-functions

All continuous charge distributions in 1D and 2D can ultimately be represented as 3D Volume Densities using Dirac δ-functions.

Examples

Volume charge density due to

- a uniform linear distribution with charge density λ on the x-axis

$$\rho_\lambda(\mathbf{r}) = \lambda\delta^2(y, z) = \lambda\delta(y)\delta(z)$$

- a uniform surface distribution with charge density σ on plane $z = c$

$$\rho_\sigma(\mathbf{r}) = \sigma\delta(z - c)$$

- a uniform surface distribution with charge density σ on a spherical shell of radius R

$$\rho_\sigma(\mathbf{r}) = \sigma\delta(r - R)$$

Continuous Charge Distribution

Example

Let S be a spherical shell of radius R with variable surface charge density, $\sigma(R, \theta, \phi) = \sigma_0 \cos\theta$. Find the total charge using spherical-polar system.

If S_u and S_l denote the upper and lower hemispheres, then its total charge Q is

$$Q = \oiint_S \sigma(\mathbf{r})\, dA = \iint_{S_u} \sigma(\mathbf{r})\, dA + \iint_{S_l} \sigma(\mathbf{r})\, dA \equiv Q_u + Q_l$$

Recall: Elemental area on a spherical surface is $dA = |\mathbf{N}|d\theta\, d\phi = R^2 \sin\theta\, d\theta\, d\phi$

$$\begin{aligned} Q_u &= \iint_{S_u} (\sigma_0 \cos\theta)\, R^2 \sin\theta\, d\theta\, d\phi \\ &= \frac{\sigma_0 R^2}{2} \int_0^{\pi/2} \sin 2\theta\, d\theta \int_0^{2\pi} d\phi \\ &= \frac{\sigma_0 R^2}{2} \left[-\frac{\cos 2\theta}{2} \right]_0^{\pi/2} (2\pi) = \pi \sigma_0 R^2. \end{aligned}$$

However, the integration on $\theta \in [\pi/2, \pi]$ for the lower hemisphere S_l yields $Q_l = -\pi\sigma_0 R^2$. Hence, the total charge on S is $Q = Q_u + Q_l = 0$.

Coulomb's Electrostatic Force Law

Let q_1 and q_2 be two point charges located at $\mathbf{r}_1$ and $\mathbf{r}_2$. Then the electrostatic force exerted on q_2 by q_1 is

$$\mathbf{F}_{21}(\mathbf{r}_2) = k\, q_1 q_2 \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} = -\mathbf{F}_{12}(\mathbf{r}_1)$$

- In SI units, $k = 1/4\pi\epsilon_0$, where ϵ_0 is called **permittivity of free space**

$$\epsilon_0 = 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N\ m}}.$$

- Experiments suggest that this law is valid for a very wide range of distance scales $\sim 10^{-18}$ m to 10^{7} m.

Linear Superposition Principle holds for Coulomb's Law

Force $\mathbf{F}_{AB}$ on a charge, say A, due to another charge, say B, is independent of presence of a third charge, say C. Total force on A is given by $\mathbf{F}_A = \mathbf{F}_{AB} + \mathbf{F}_{AC}$.

- Easily generalize to several *source charges* $q_1, q_2, q_3 \cdots$ in which case the total force on a *test charge* is

$$\mathbf{F}_{\mathrm{Total}} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \cdots = \sum_i \mathbf{F}_i$$

Fact

The superposition principle is a consequence of the Coulomb's force law bearing a linear dependence on each source charge, i.e., $F_{\mathrm{test}} \propto q_{\mathrm{source}}$

Example

Would superposition principle hold, e.g., with a quadratic dependence of Coulomb's Law on each source charge, i.e., $F_{\mathrm{test}} \propto q^2_{\mathrm{source}}$? ***NO***

Consider a situation with two source charges q_1 & q_2 *located at the same point. Then, the net force* F_{Total} *on a test charge due to the combined source charge* $(q_1 + q_2)$ *would not be equal to the sum of the individual forces,* $F_1 \propto q_1^2$ *and* $F_2 \propto q_2^2$ *, since* $F_{\mathrm{Total}} \propto (q_1 + q_2)^2 \neq q_1^2 + q_1^2 \implies F_{\mathrm{Total}} \neq F_1 + F_2$

Electric Field due to Point-like Source Charges

If there are several discrete point source charges, q_i ($i = 1, \ldots, n$), at locations $\mathbf{r}'_i$, then net Electric field at $\mathbf{r}$ is defined as

$$\mathbf{E}_{\mathrm{Total}}(\mathbf{r}) = \mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r}) + \cdots + \mathbf{E}_n(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i\,(\mathbf{r} - \mathbf{r}'_i)}{|\mathbf{r} - \mathbf{r}'_i|^3}$$

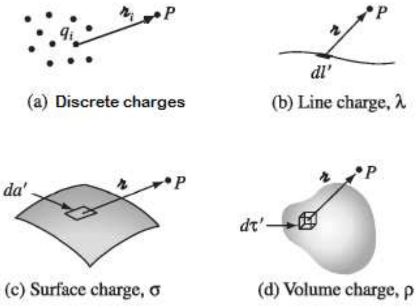
- Its unit is measured in Newton/Coulomb $(\mathrm{N/C})$
- Electric field is a vector quantity.
- Linear Superposition Principle holds for electric fields.
- Total electric or Coulomb force on a test charge Q_{test} at $\mathbf{r}$ is

$$\mathbf{F}_{\mathrm{Total}}(\mathbf{r}) = Q_{\mathrm{test}}\,\mathbf{E}_{\mathrm{Total}}(\mathbf{r}).$$

Electric Field: Discrete (Point-like) & Continuous Distributions

For the most general source charge distribution, with volume charge density ρ, surface charge density σ, linear charge density λ, as well as discrete point charges, the electric field at a point $P(\mathbf{r})$ by virtue of the Superposition Principle has the expression

$$\mathbf{E}_P(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n}\frac{q_i\ (\mathbf{r}-\mathbf{r}_i')}{|\mathbf{r}-\mathbf{r}_i'|^3} + \frac{1}{4\pi\epsilon_0}\int_C \frac{\lambda(\mathbf{r}')\ (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,dl'$$
$$+ \frac{1}{4\pi\epsilon_0}\iint_S \frac{\sigma(\mathbf{r}')\ (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,da' + \frac{1}{4\pi\epsilon_0}\iiint_V \frac{\rho(\mathbf{r}')\ (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\,d\tau'$$



(a) Discrete charges

(b) Line charge, λ

(c) Surface charge, σ

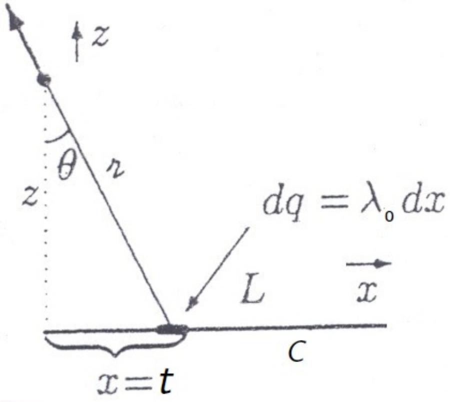
(d) Volume charge, ρ

Electric Field due to a Linear Distribution

Example

Consider the straight line segment $C : \mathbf{r}'(t) = (t, 0, 0)$; $x = t \in [0, L]$ along the x-axis with uniform linear charge density λ_0. Calculate the Electric field at the target point $\mathbf{r} = (0, 0, z)$, assuming $z \gg L$.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_C \frac{\lambda(\mathbf{r}')\,(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dl'$$



- Target/Field point $\mathbf{r} = (0, 0, z)$, Source Point $\mathbf{r}' = (t, 0, 0)$
- $\mathbf{r} - \mathbf{r}'(t) = (-t, 0, z)$
- $|\mathbf{r} - \mathbf{r}'(t)| = \sqrt{t^2 + z^2}$
- Constant linear density, $\lambda(\mathbf{r}'(t)) = \lambda_0$
- Line element, $dl' = dx = dt$

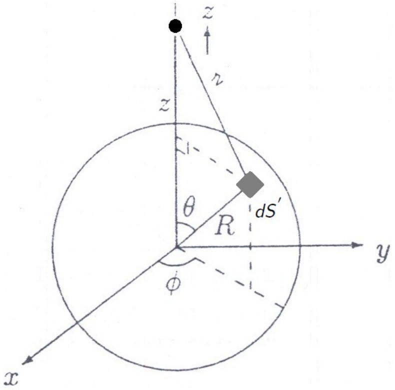
$$
\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_C \frac{\lambda(\mathbf{r}')\,(\mathbf{r} - \mathbf{r}'(t))}{|\mathbf{r} - \mathbf{r}'(t)|^3}\, dl' = \frac{\lambda_0}{4\pi\epsilon_0} \int_0^L \frac{\left(-t\hat{\mathbf{i}} + z\hat{\mathbf{k}}\right)}{(t^2 + z^2)^{3/2}}\, dt \\
&= \frac{\lambda_0}{4\pi\epsilon_0 z} \left[\left(-1 + \frac{z}{\sqrt{z^2 + L^2}} \right) \hat{\mathbf{i}} + \left(\frac{L}{\sqrt{z^2 + L^2}} \right) \hat{\mathbf{k}} \right] \\
&= \frac{\lambda_0}{4\pi\epsilon_0 z} \left[\left(-1 + \frac{1}{\sqrt{1 + \left(\frac{L}{z}\right)^2}} \right) \hat{\mathbf{i}} + \left(\frac{L}{z\sqrt{1 + \left(\frac{L}{z}\right)^2}} \right) \hat{\mathbf{k}} \right] \\
&= \frac{\lambda_0}{4\pi\epsilon_0 z} \left[\left(-1 + 1 - \frac{L^2}{2z^2} + \cdots \right) \hat{\mathbf{i}} + \left(\frac{L}{z} \right) \left(1 - \frac{1}{2} \left(\frac{L}{z} \right)^2 + \cdots \right) \hat{\mathbf{k}} \right] \\
&= \frac{\lambda_0}{4\pi\epsilon_0 z} \left[-\frac{L^2}{2z^2} \hat{\mathbf{i}} + \frac{L}{z} \hat{\mathbf{k}} \right] + \ldots \cancel{\mathcal{O}(L^3)}, \text{ for } \quad z \gg L
\end{aligned}
$$

Electric Field due to a Surface Distribution

Example

Consider a spherical conducting shell of radius R with uniform surface charge density σ_0. Calculate the Electric field at the target point $\mathbf{r} = (0, 0, z)$.

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oiint_S \frac{\sigma(\mathbf{r}')\,(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dS'$$



- Target/Field Point $\mathbf{r} = (0, 0, z)$, Source Point $\mathbf{r}' = (R, \theta, \phi)$ of dS'
- Parametric form $\mathbf{r}'(\theta, \phi) = R(\sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}})$
- $\mathbf{r} - \mathbf{r}'(\theta, \phi) = -R(\sin\theta\cos\phi)\hat{\mathbf{i}} - R(\sin\theta\sin\phi)\hat{\mathbf{j}} + (z - R\cos\theta)\,\hat{\mathbf{k}}$
- $|\mathbf{r} - \mathbf{r}'(\theta, \phi)| = \sqrt{R^2 + z^2 - 2R\,z\cos\theta}$
- Elemental surface area at $\mathbf{r}'$: $dS' = R^2\sin\theta d\theta d\phi$

$$
\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\oiint_S \frac{\sigma(\mathbf{r}')\ (\mathbf{r} - \mathbf{r}'(\theta, \phi))}{|\mathbf{r} - \mathbf{r}'(\theta, \phi)|^3}dS' \\
&= \frac{\sigma_0}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\frac{R^2\sin\theta d\theta d\phi}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}} \\
&\qquad \times\left[-R\sin\theta\cos\phi\hat{\mathbf{i}} - R\sin\theta\sin\phi\hat{\mathbf{j}} + (z - R\cos\theta)\,\hat{\mathbf{k}}\right] \\
&= \frac{\sigma_0}{4\pi\epsilon_0}2\pi R^2\int_0^{\pi}\frac{(z - R\cos\theta)\sin\theta d\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}\,\hat{\mathbf{k}} = \frac{(4\pi R^2)\sigma_0}{4\pi\epsilon_0\, z^2}\,\hat{\mathbf{k}}
\end{aligned}
$$

Divergence of Electric Field due to a Point Charge

Suppose a point (source) charge of magnitude q located at $\mathbf{r}' = (x', y', z')$. Then the volume charge density at any target point $\mathbf{r} = (x, y, z)$ can be expressed as $\rho(\mathbf{r}) = q\delta^3(\mathbf{r} - \mathbf{r}')$ and the Electric field at $\mathbf{r} \in \mathbb{R}^3$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

Recall: Divergence of *Inverse Square* field is $\nabla \cdot \frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta^3(\mathbf{r})$

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0} \left[\nabla \cdot \left(\frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \right) \right] = \frac{q}{4\pi\epsilon_0} \left[4\pi\delta^3(\mathbf{r} - \mathbf{r}') \right] = \frac{1}{\epsilon_0} \left[q\delta^3(\mathbf{r} - \mathbf{r}') \right]$$

Differential form of Gauss's Law:

$$\boxed{\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}}$$

Also,

$$\iiint_V \left[\nabla \cdot \mathbf{E}(\mathbf{r}) \right] dV = \frac{1}{\epsilon_0} \iiint_V \rho(\mathbf{r})\, d^3r = \frac{1}{\epsilon_0} \iiint_V q\delta^3(\mathbf{r} - \mathbf{r}')\, d^3r = \begin{cases} \frac{1}{\epsilon_0} q & \text{if} \quad \mathbf{r}' \in V \\ 0 & \text{if} \quad \mathbf{r}' \notin V \end{cases}$$

Divergence of Electric Field due to Continuous Volume Distribution

Now we extend the result to arbitrary charge distribution with volume density ρ

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \iiint\limits_V \frac{\rho(\mathbf{r}')\ (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3r'.$$

Divergence with respect to which variable, $\mathbf{r}$ or $\mathbf{r}'$, i.e., $\nabla\cdot$ or $\nabla'\cdot$?

Here we are interested in divergence at "Target", i.e., with respect to variable $\mathbf{r}$:

$$\begin{aligned}
\nabla\cdot\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\nabla\cdot\iiint\limits_V \frac{\rho(\mathbf{r}')\ (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} d^3r' \\
&= \frac{1}{4\pi\epsilon_0}\iiint\limits_V \rho(\mathbf{r}') \left(\nabla\cdot\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\right) d^3r' \\
&= \frac{1}{4\pi\epsilon_0}\iiint\limits_V \rho(\mathbf{r}')\ (4\pi\delta^3(\mathbf{r}-\mathbf{r}'))\ d^3r' \\
\nabla\cdot\mathbf{E}(\mathbf{r}) &= \frac{1}{\epsilon_0}\iiint\limits_V \rho(\mathbf{r}')\,\delta^3(\mathbf{r}'-\mathbf{r})d^3r' = \frac{1}{\epsilon_0}\rho(\mathbf{r})
\end{aligned}$$

Application of Gauss's Differential Law

Example

Find the corresponding charge density for the Electric field in space given by

$$\mathbf{E}(\mathbf{r}) = Ae^{-\lambda r}(1+\lambda r)\frac{\hat{\mathbf{r}}}{r^2}$$

where A and λ are constants.

Use Gauss's differential formula:

$$\rho(\mathbf{r}) = \epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r})$$

Use Spherical-polar Co-ordinates:

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \cancelto{0}{\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta)} + \cancelto{0}{\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(E_\phi)} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(Ae^{-\lambda r}(1+\lambda r)\right) = -A\frac{\lambda^2}{r}e^{-\lambda r}
\end{aligned}$$

Then,

$$\rho(\mathbf{r}) = \epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r}) = -\epsilon_0 A\frac{\lambda^2}{r}e^{-\lambda r}$$