PH101

Lecture 11

Necessary condition for an integral to be stationary

$$\mathbf{I} = \int_{x_1}^{x_2} F(y, y', x)\,\boldsymbol{dx}$$

Here, $y = y(x) \quad and \;\; y' = \frac{dy}{dx}$

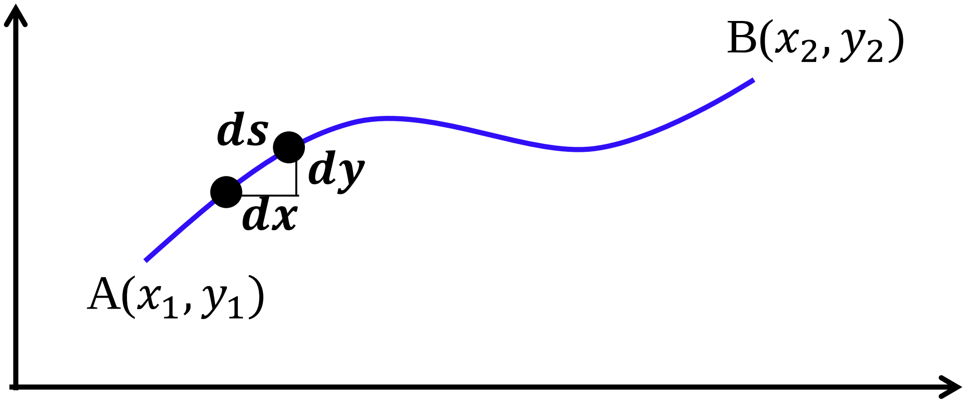
Necessary condition for stationary

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

Solution of this equation will give you desired path [$y = y(x)$], along which the integration is extremum

- ❑ To get stationary condition of any quantity, express the quantity in terms of integral of its infinitesimal value with known integration limit, then use Euler-Lagrange equation

❑ Given two points in a plane, what is the shortest path between them?
You certainly know the answer: **Straight line**.
Let's prove it using variation method



❑ Consider an arbitrary path $y(x)$, elementary length $ds = [(dx)^2 +$

Application of variation principle: Example1

$$\frac{\partial F}{\partial \boldsymbol{y}'} = \frac{\partial}{\partial \boldsymbol{y}'}\left\{\left(1+\boldsymbol{y}'^2\right)^{1/2}\right\} = \boldsymbol{y}'\left(1+\boldsymbol{y}'^2\right)^{-1/2}; \text{ and } \frac{\partial F}{\partial y} = 0$$

Thus

$$\frac{d}{dx}\left\{\boldsymbol{y}'\left(1+\boldsymbol{y}'^2\right)^{-1/2}\right\} = 0$$

$$\boldsymbol{y}'\left(1+\boldsymbol{y}'^2\right)^{-1/2} = constant$$

$$\boldsymbol{y}'^2 = Constant\left(1+\boldsymbol{y}'^2\right),$$

$$\boldsymbol{y}'^2 = Consntant$$

$$y' = \frac{dy}{dx} = constant;$$

$$y(x) = mx + C, Where\ m\ and\ C\ are\ constant$$

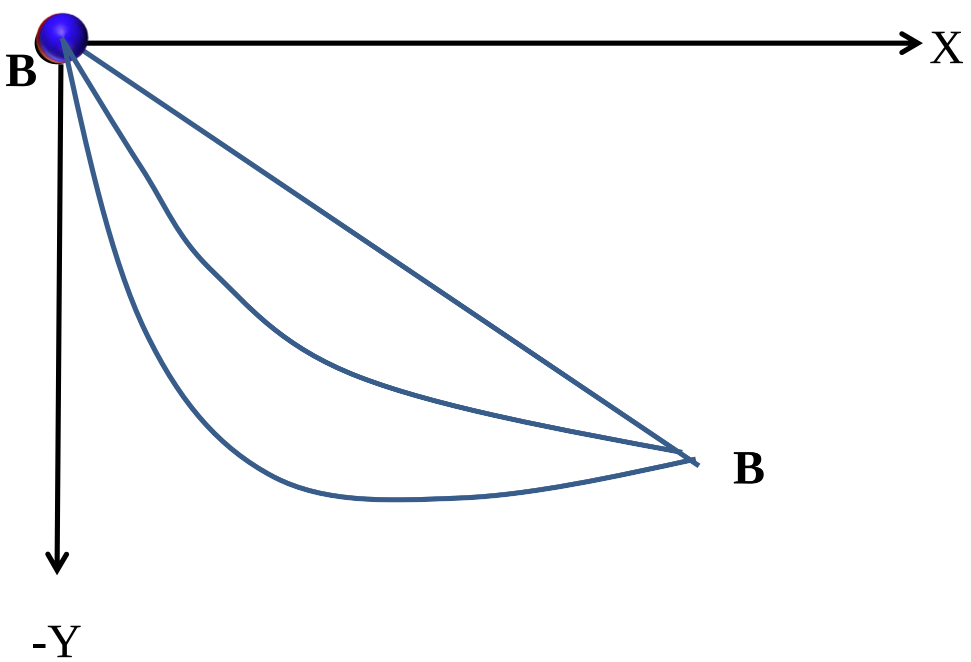
Equation of straight line

- Shortest distance between two points in a plane is straight line.

Jean Bernoulli's challenge! "Brachistochrone"

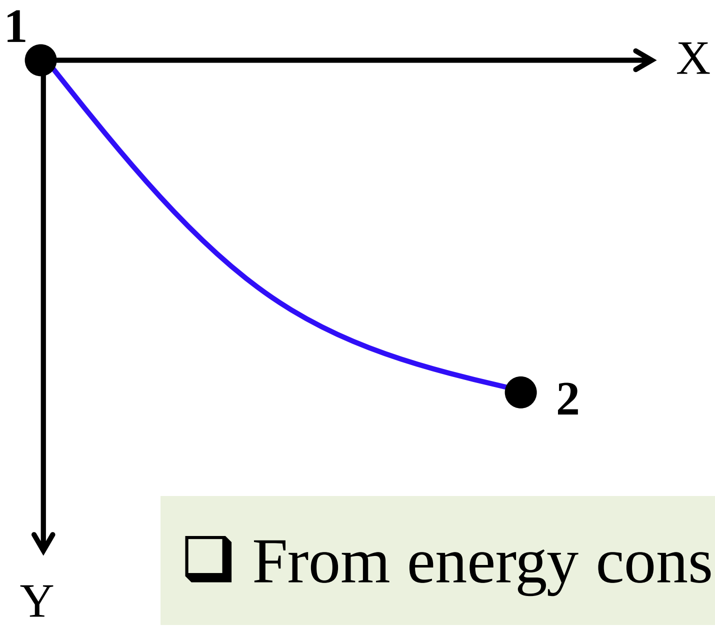
❑ What should be the shape of a stone's trajectory (or, of a roller coaster track) so that released from point A it reaches point B in the shortest possible time? **Brachistochrone problem!** ~1696

Brachisto~ shortest **Chrone** ~ time



Application of variation principle: Example 2

- Given two points 1 and 2, with 1 higher above the ground, in what shape should we build a frictionless roller coaster track so that a car released from point 1 will reach point 2 in the shortest possible time? **Brachistochrone problem**



- Time to travel from 1 to 2

$$\text{Time } (1 \rightarrow 2) = \int_1^2 \frac{ds}{v}$$

$ds \rightarrow Elementary\ path\ length$

$v \rightarrow Instantaneous\ velocity$

- From energy conservation, $\frac{1}{2}mv^2 = mgy\,;\, v = (2gy)^{1/2}$

$$ds = [(dx)^2 + (dy)^2]^{1/2} = \left[\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}\right]^{1/2} ds = \left(1 + x'^2\right)^{1/2} dy$$

$$\boldsymbol{x' = \frac{dx}{dy}}$$

Application of variation principle: Brachistochrone problem

❑ Time to travel from 1 to 2

Time1 → 2)

$$= \int_1^2 \frac{ds}{v} = \int_0^{y_2} \frac{\left(1 + \boldsymbol{x'}^2\right)^{1/2}}{(2gy)^{1/2}} dy = \int_0^{y_2} F\{y, x(y), \boldsymbol{x'}(y)\} dy$$

What about $F\{y(x), y'(x), x\}$?

Then $\frac{\partial F}{\partial y} \neq 0$

Mathematically complicated

Where,

$$F\{x(y), \boldsymbol{x'}(y), y\} = \frac{\left(1 + \boldsymbol{x'}^2\right)^{1/2}}{(2gy)^{1/2}}$$

❑ Necessary condition for the integral (total time) to be extremum

$$\frac{\boldsymbol{d}}{\boldsymbol{dy}}\left(\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial x'}}\right) - \frac{\boldsymbol{\partial F}}{\boldsymbol{\partial x}} = 0$$

Application of variation principle: Brachistochrone problem

$$\frac{\partial F}{\partial \boldsymbol{x'}} = \frac{\partial}{\partial x'}\left\{\frac{\left(1+\boldsymbol{x'}^2\right)^{1/2}}{(2gy)^{1/2}}\right\} = \frac{\boldsymbol{x'}\left(1+\boldsymbol{x'}^2\right)^{-1/2}}{(2gy)^{1/2}};\ \frac{\partial F}{\partial x} = 0$$

$Thus,$

$$\frac{\boldsymbol{d}}{\boldsymbol{dy}}\left(\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial x'}}\right) - \frac{\boldsymbol{\partial F}}{\boldsymbol{\partial x}} = \frac{d}{dy}\left[\frac{\boldsymbol{x'}\left(1+\boldsymbol{x'}^2\right)^{-1/2}}{(2gy)^{1/2}}\right] = 0;$$

Hence, $\dfrac{\boldsymbol{x'}\left(1+\boldsymbol{x'}^2\right)^{-1/2}}{(2gy)^{1/2}} = Constant;$

$$\frac{\boldsymbol{x'}^2}{y\left(1+\boldsymbol{x'}^2\right)} = Constant = \frac{1}{2a}\text{(say)};$$

$$x'^2 = \frac{y}{2a-y}\ ;\ x' = \sqrt{\frac{y}{2a-y}}\quad ; \frac{dx}{dy} = \sqrt{\frac{y}{2a-y}}$$

$$dx = \sqrt{\frac{y}{2a-y}}\,dy$$

Application of variation principle: Brachistochrone problem

- To solve the integral, substitute $y = a(1-\cos\theta)$ ….. (1)

$$\text{thus } dy = a\sin\theta\, d\theta$$

$$x = \int \sqrt{\frac{a(1-\cos\theta)}{a(1+\cos\theta)}}\; a\sin\theta\, d\theta$$

$$= a\int \sqrt{\frac{(1-\cos\theta)}{(1+\cos\theta)}}\sqrt{(1-\cos\theta)(1+\cos\theta)}\, d\theta$$

$$= a\int (1-\cos\theta)\, d\theta;$$

$$x = a(\theta - \sin\theta) + constant \ldots (2)$$

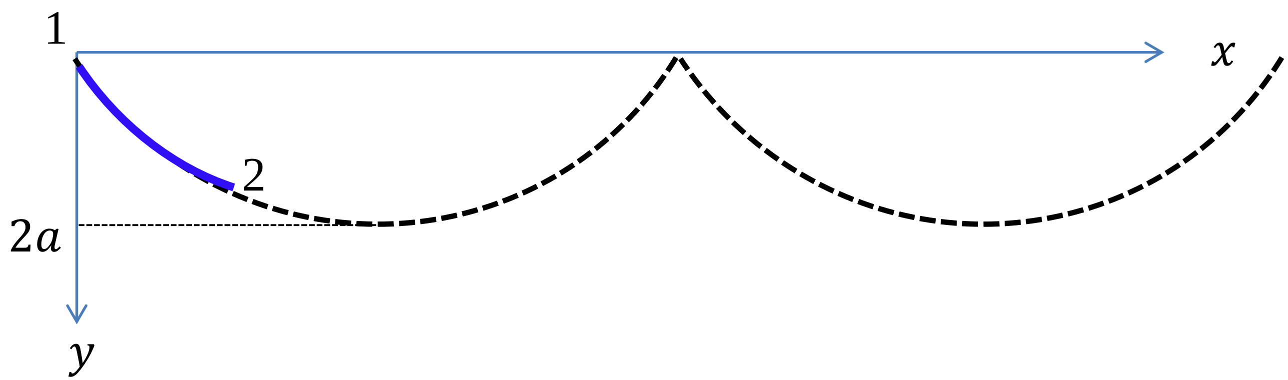
Application of variation principle: Brachistochrone problem

- $y = a(1 - \cos\theta)\ ;\ x = a(\theta - \sin\theta) + constant$

These two equations are parametric equation of the required path.

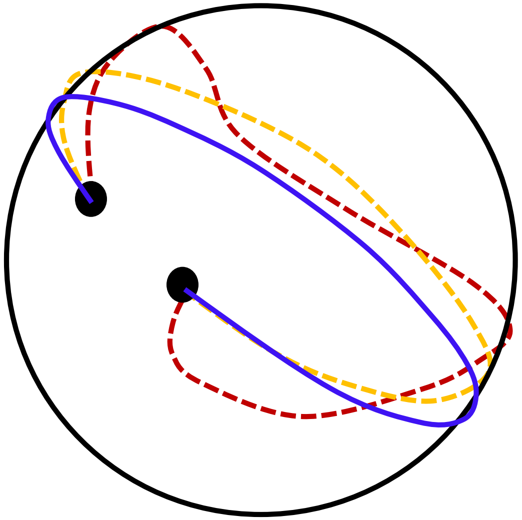
According to choice, initial point $x = y = 0$;

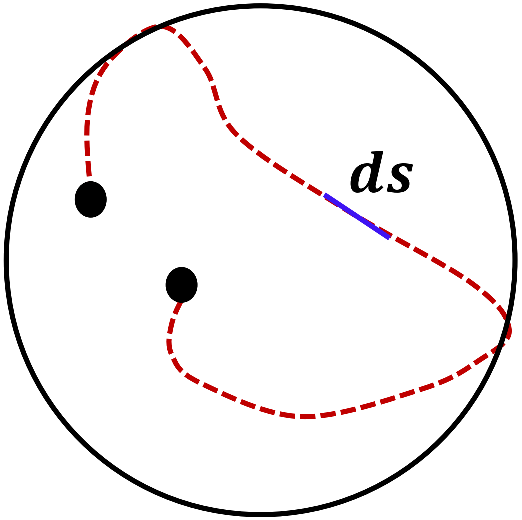
Thus $y = a(1 - \cos\theta)\ ;\ x = a(\theta - \sin\theta)$ required path



- Path is a part of cycloid the curve traced out by a point on the rim of a wheel of radius a, rolling along the underside of the x axis.

Application of variation principle: Shortest path between two points on the surface of a sphere





- ❑ Shortest path is the path along the great circle connecting the two points

- ❑ Elementary length (ds) between two points in spherical polar coordinates

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 sin^2\theta \, d\varphi^2$$

- ❑ On the surface of the sphere,

$$r = R = constant$$

$$\dot{r} = 0$$

$$ds^2 = R^2 d\theta^2 + R^2 sin^2\theta \, d\varphi^2$$

Shortest path between two points on the surface of a sphere

❑ Total length between two points 1&2

$$S = \int_1^2 ds = \int_1^2 \sqrt{R^2 d\theta^2 + R^2 sin^2\theta \, d\varphi^2}$$

$$S = R\int_{\theta_1}^{\theta_2} \sqrt{1 + sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2} \, d\theta$$

❑ You can also express as

$$S = R\int_{\varphi_1}^{\varphi_2} \sqrt{sin^2\theta + \left(\frac{d\theta}{d\varphi}\right)^2} \, d\varphi$$

Then EL equation would be

$$\frac{\boldsymbol{d}}{\boldsymbol{d\varphi}}\left(\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial \theta'}}\right) - \frac{\boldsymbol{\partial F}}{\boldsymbol{\partial \theta}} = 0$$

Mathematically difficult due to non-zero $\frac{\partial F}{\partial \theta}$

$$F\{\theta, \varphi(\theta), \varphi'(\theta)\} = \sqrt{1 + sin^2\theta \left(\frac{d\varphi}{d\theta}\right)^2} = \sqrt{1 + sin^2\theta \varphi'^2}$$

$$\varphi' = \frac{d\varphi}{d\theta}$$

❑ Necessary condition for the integral (total time) to be extremum

$$\frac{\boldsymbol{d}}{\boldsymbol{d\theta}}\left(\frac{\boldsymbol{\partial F}}{\boldsymbol{\partial \varphi'}}\right) - \frac{\boldsymbol{\partial F}}{\boldsymbol{\partial \varphi}} = 0$$

Shortest path between two points on the surface of a sphere

$$F = \sqrt{1 + sin^2\theta\, \varphi'^2}$$

$$\frac{\partial F}{\partial \varphi} = 0$$

$$\frac{\partial F}{\partial \varphi'} = \frac{sin^2\theta \boldsymbol{\varphi}'}{\sqrt{1 + sin^2\theta\, \varphi'^2}}$$

$$\frac{d}{d\theta}\left(\frac{sin^2\theta \boldsymbol{\varphi}'}{\sqrt{1 + sin^2\theta\, \varphi'^2}}\right) = \mathbf{0};\ \frac{sin^2\theta \boldsymbol{\varphi}'}{\sqrt{1 + sin^2\theta\, \varphi'^2}} = \boldsymbol{constant} = \boldsymbol{k}$$

$$sin^4\theta \varphi'^2 = k^2\left(1 + sin^2\theta\, \varphi'^2\right);\ \varphi' = \pm\frac{k\, csc^2\theta}{\sqrt{1 - k^2 csc^2\theta}}$$

$$\varphi' = \pm\frac{k\, csc^2\theta}{\sqrt{1 - k^2(1 + cot^2\theta)}} = \pm\frac{k\, csc^2\theta}{\sqrt{1 - k^2 - k^2 cot^2\theta}}$$

$$\varphi' = \frac{\pm k}{\sqrt{1 - k^2}}\frac{csc^2\theta}{\sqrt{1 - \frac{k^2}{1 - k^2}cot^2\theta}}$$

Shortest path between two points on the surface of a sphere

$$d\varphi = \alpha \frac{csc^2\theta\, d\theta}{\sqrt{1-\alpha^2 cot^2\theta}}$$

$$d\varphi = -\frac{dq}{\sqrt{1-q^2}};\ \int d\varphi = \int -\frac{dq}{\sqrt{1-q^2}}$$

$$\varphi = -\sin^{-1} q + \beta;\ q = \sin(\beta - \varphi)$$

$$\boldsymbol{\alpha\, cot\theta = \sin(\beta - \varphi)} \ldots\ldots..[\mathbf{1}]$$

Let $\alpha = \frac{\pm k}{\sqrt{1-k^2}}$ and $q = \alpha \cot\theta$

$$dq = -\alpha\, csc^2\theta\, d\theta$$

$\beta \rightarrow Integration\ constant$

To understand the meaning of $\boldsymbol{equation\ 1}$, multiply both sides by R

$$\alpha\, R\, cot\theta = R \sin(\beta - \varphi)$$

$$\alpha R\, cos\theta = R\, sin\theta\, (sin\beta cos\varphi\ - cos\beta sin\varphi)$$

$$\alpha z = \sin\beta\ x - cos\beta\ y$$

$$\boldsymbol{sin\,\beta\ x - cos\beta\ y - \alpha z = 0}$$

Equation of a plane passing through origin

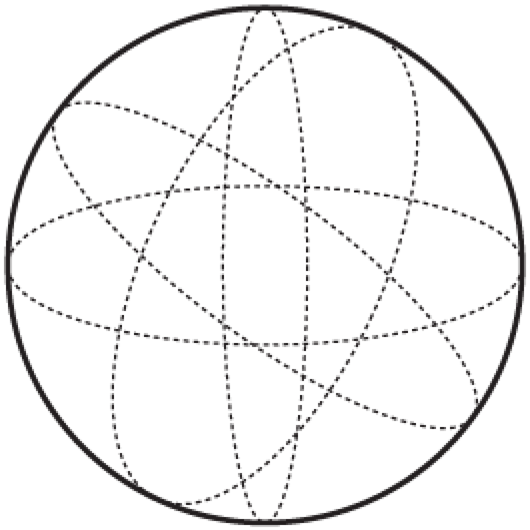
Equation of a plane passing through (x_0, y_0, z_0)

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Shortest path between two points on the surface of a sphere

$$\sin\beta\ x - \cos\beta\ y - \alpha z = 0$$

This plane which passes through the origin slices through the sphere in great circles



Thus solution of Euler-Lagrange's equation are great circle routes

Shortest path between two points on the surface of a sphere must lie on this the great circle passing through those points.

Steps to be followed to get the equation of extremum path

Step-1: Identify the quantity (say, I) for which maxima and minima (extremum) condition to be determined.
Then express $I\ as$ integration of its elementary value, I,e,
$Total\ quantiy\ (I) = \int Elementary\ value\ of\ the\ quantity\ (dI)$; I,e $I = \int dI$

Step 2: Find connection between dI and elementary path length (ds)
(for example $dI = \gamma\ dS$), and put in the expression of $I.\ I = \int \gamma\ dS$

Step 3: Express dS in suitable coordinate system, (initial and final point of integration is known in terms of that particular coordinate system)
Example: $In\ cartesian\ ds = \sqrt{dx^2 + dy^2}$, if integration limit is known in terms $(x_1, x_2)\ or\ (y_1, y_2)$
In polar, $ds = \sqrt{dr^2 + r^2 d\theta^2}$, if integration limit is known in terms $(\theta_1, \theta_2)\ or\ (r_1, r_2)$
In spherical polar, $ds = \sqrt{dr^2 + r^2 d\theta^2 + r^2 sin^2\theta\ d\varphi^2}$, if integration limit is known in terms of $(\theta_1, \theta_2)\ or (\varphi_1, \varphi_2)\ or (r_1, r_2)$

Step 4: Expressing $\gamma\ dS$ in appropriate coordinate, express it as $\mathbf{I} = \int_{\boldsymbol{u_1}}^{\boldsymbol{u_2}} F(y, y', u)\boldsymbol{du}$
Out of variables in ds, take that particular coordinate (y) as dependent variable for which $\frac{\partial F}{\partial y} = 0$. Then use EL equation

Principle of Least Action

❑ $L(q_j, \dot{q}_j, t)$ →Lagrangian of system of particles

❑ **Action integral** → $\boldsymbol{\int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt}$

❑ A mechanical system will evolve in time in such that action integral is stationary → **Hamilton's Principle of Least Action**

$\boldsymbol{\int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt}$ → **Stationary** → $\boldsymbol{\delta \int_{t_1}^{t_2} L(q_j, \dot{q}_j, t)\,dt = 0}$

❑ Stationary condition of Action integral

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0$$

→ ❑ Lagrange's equation from Variational principle

Summery

$$\mathbf{I} = \int_{x_1}^{x_2} F(y, \dot{y}, x)\,\boldsymbol{dx}$$

Necessary condition for stationary

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) - \frac{\partial F}{\partial y} = 0$$

- To get stationary condition of any quantity, express the quantity in terms of integral of its infinitesimal value with known integration limit, then use Euler-Lagrange equation

- **Action Integral:**

$$\mathbf{I} = \int_{t_1}^{t_2} L(t, q, \dot{q})\,dt$$

- **Principle of Least Action**→ A mechanical system will evolve in time in such that action integral is stationary

Question please