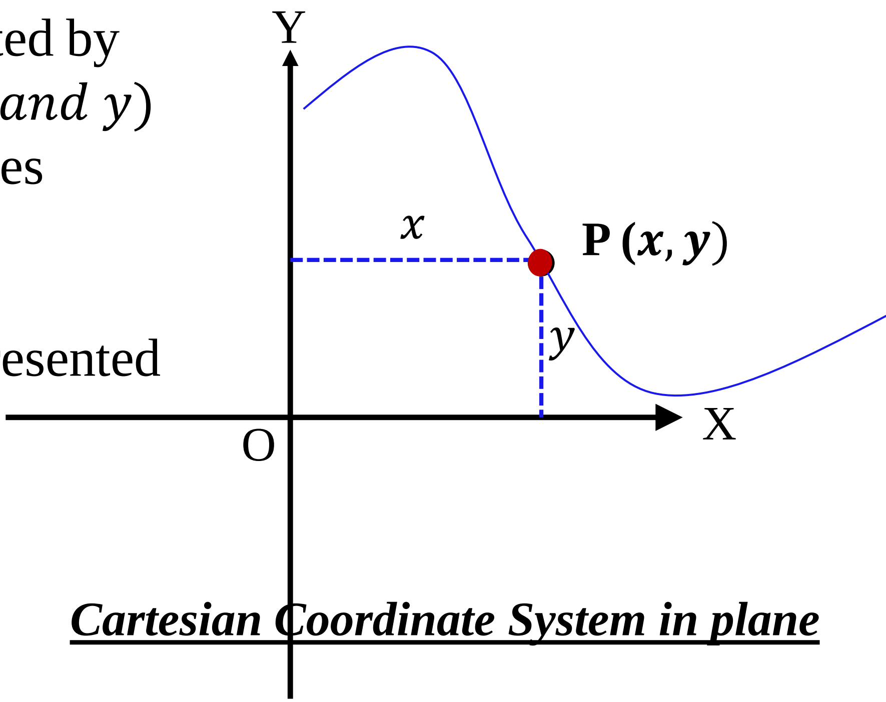
PH101

Lecture 2

Coordinate systems

Cartesian coordinate System in plane

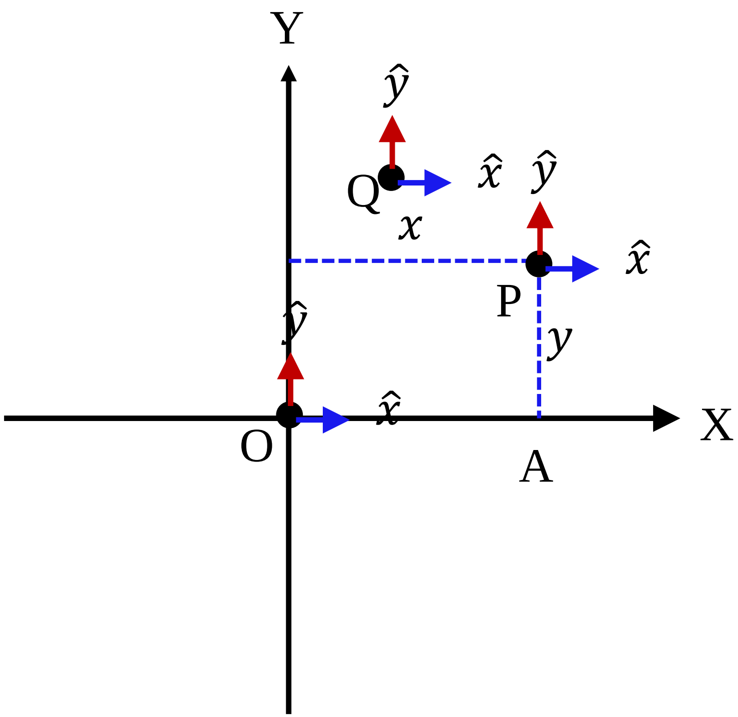
- Instantaneous position P can be represented by measuring its perpendicular distances (x and y) from two perpendicularly intersecting lines (called Axes) passing through the origin.
- In Cartesian coordinate position P is represented by (x, y).



<u>Cartesian Coordinate System in plane</u>

All points in the plane are referred with respect to same set of axes X and Y.

Unit vectors in plane Cartesian



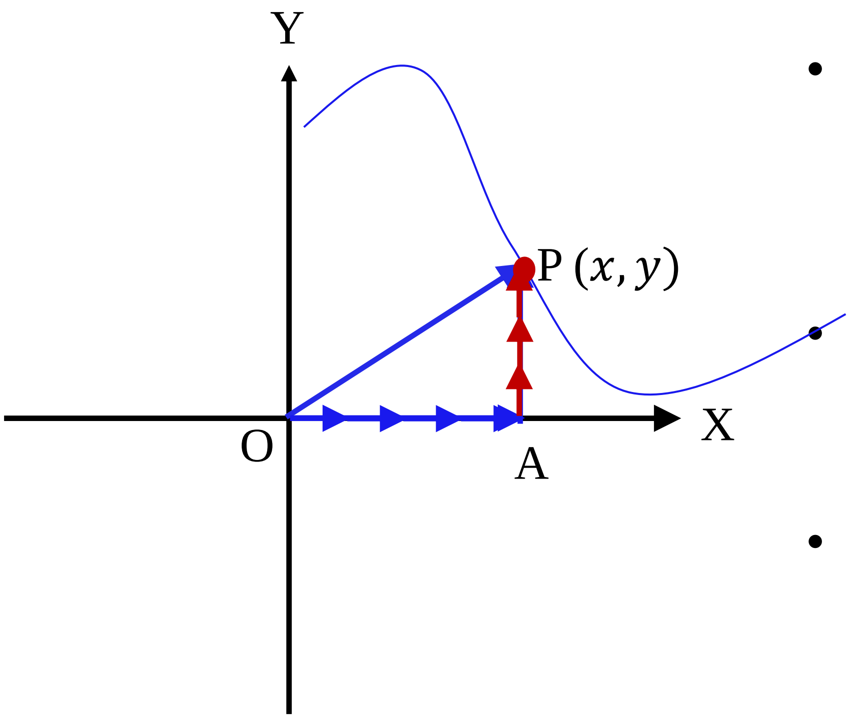
- In Cartesian coordinate system for plane, one can define two unit vectors $\hat{x}$ and $\hat{y}$ at every point in the plane.
- **$\hat{x}$ and $\hat{y}$ are unit vectors in the increasing direction of x and y**, thus they are parallel to coordinate axes X and Y respectively.
- $\hat{x}$ and $\hat{y}$ are orthogonal, and directed in the same direction at every points.

Another way of looking unit vector Cartesian coordinate in plane

$\hat{x}$ is the unit vector perpendicular to $x = constant$ line (surface)

$\hat{y}$ is the unit vector perpendicular to $y = constant$ line (surface)

Position vector in terms of Cartesian components



- Position P can be represented either by vector $\vec{r}$ ($\overrightarrow{OP}$) or through Cartesian coordinate (x, y).

- From vector addition rule

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

- $\overrightarrow{OA}$ = (magnitude of OA) (unit vector along/parallel to X –direction) $= x\,\hat{x}$

- $\overrightarrow{AP}$ = (magnitude of AP) (unit vector along/parallel to Y-direction) $= y\,\hat{y}$

Thus $\boldsymbol{\overrightarrow{OP}} = \boldsymbol{\vec{r}} = \boldsymbol{x\,\hat{x} + y\,\hat{y}}$

We may also use the notation:

$$\hat{x} = \hat{\imath}$$
$$\hat{y} = \hat{\jmath}$$

Velocity and acceleration in Cartesian

Velocity $\vec{v} = \frac{d\vec{r}}{dt}$

Velocity in Cartesian coordinate

$$\boldsymbol{\vec{v}} = \frac{\boldsymbol{d\vec{r}}}{\boldsymbol{dt}} = \frac{\boldsymbol{d}}{\boldsymbol{dt}}\ (\boldsymbol{x}\ \boldsymbol{\hat{x}} + \boldsymbol{y}\ \boldsymbol{\hat{y}})$$

$$= \dot{x}\hat{x} + x\ \frac{d\hat{x}}{dt} + \dot{y}\hat{y} + y\,\frac{d\hat{y}}{dt}$$

$$\vec{v} = \dot{x}\hat{x} + \dot{y}\hat{y}$$

Since,

$$\frac{d\hat{x}}{dt} = \frac{d\hat{y}}{dt} = 0$$

Acceleration $\boldsymbol{\vec{a}} = \frac{\boldsymbol{d\vec{v}}}{\boldsymbol{dt}} = \boldsymbol{\ddot{x}\hat{x}} + \boldsymbol{\ddot{y}\hat{y}}$

Newton's second law in vector form

$$\boxed{\boldsymbol{\vec{F}} = \boldsymbol{F_x\hat{x}} + \boldsymbol{F_y\hat{y}} = \boldsymbol{m}\frac{\boldsymbol{d\vec{v}}}{\boldsymbol{dt}} = \boldsymbol{m}(\boldsymbol{\ddot{x}\hat{x}} + \boldsymbol{\ddot{y}\hat{y}})}$$

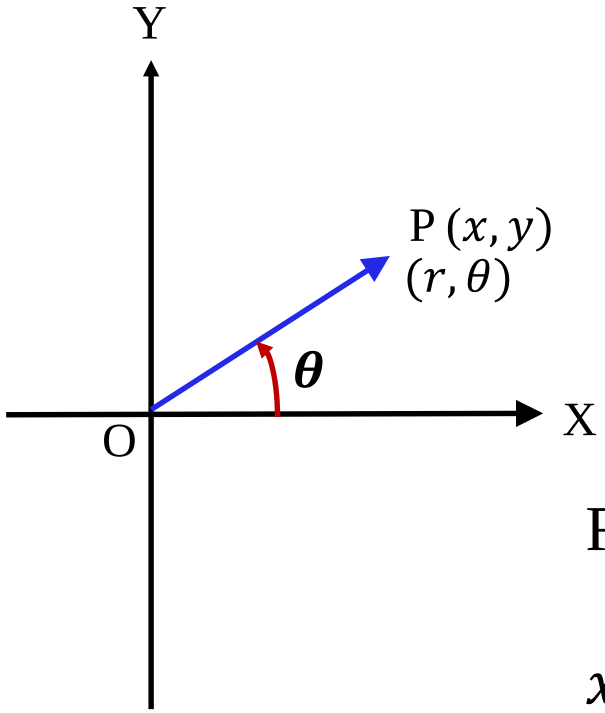
Standard Notations:

$$\frac{dx}{dt} = \dot{x}$$

$$\frac{dr}{dt} = \dot{r}$$

$$\frac{d^2\theta}{dt^2} = \ddot{\theta}$$

Plane polar coordinate



Each point P (x, y) on the plane can also be represented by its distance (r) from the origin O and the angle (θ) OP makes with X-axis.

Relationship with Cartesian coordinates

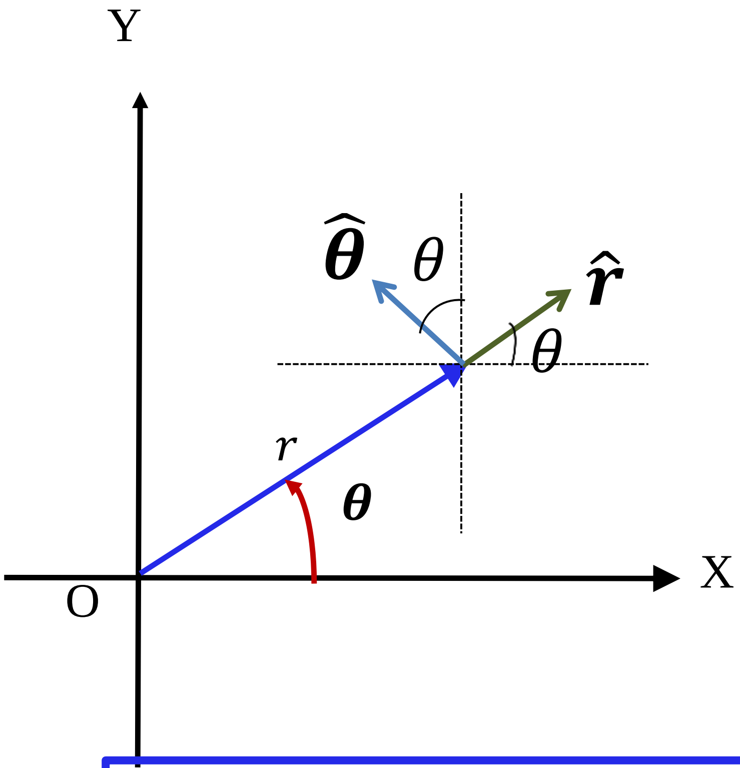
$$x = r\ \cos\theta \quad \& \quad y = r\ \sin\theta$$

Thus ,

$$r = \left(x^2 + y^2\right)^{1/2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

Unit vector in plane polar coordinate



- For plane polar unit vectors:

  $$\hat{\boldsymbol{r}} \quad \text{and} \quad \hat{\boldsymbol{\theta}}$$

  associated to **each point** in the plane.

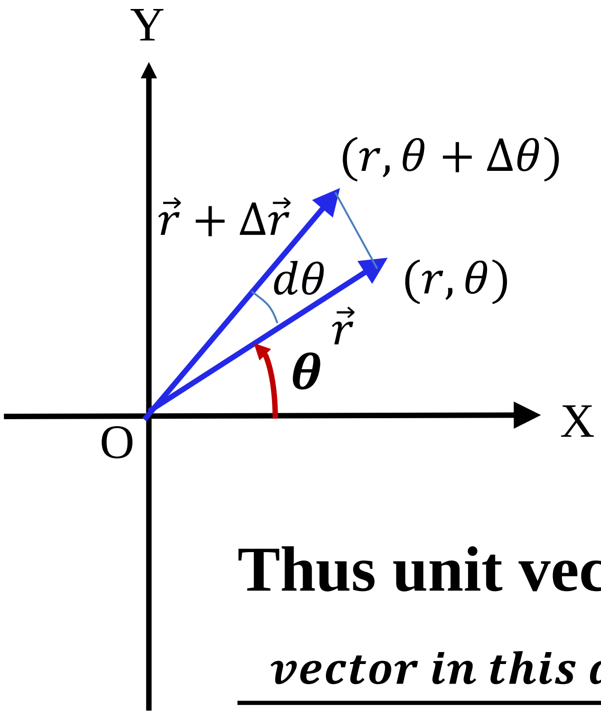
- $\hat{r}$ and $\hat{\theta}$ are unit vector along increasing direction of coordinate r and θ.

$$\boxed{\begin{aligned} \hat{\boldsymbol{r}} &= \cos\boldsymbol{\theta}\ \hat{\boldsymbol{x}} + \sin\boldsymbol{\theta}\ \hat{\boldsymbol{y}} \\ \hat{\boldsymbol{\theta}} &= -\sin\boldsymbol{\theta}\ \hat{\boldsymbol{x}} + \cos\boldsymbol{\theta}\ \hat{\boldsymbol{y}} \end{aligned}}$$

$\hat{r}$ and $\hat{\theta}$ are **orthogonal** $\hat{\boldsymbol{r}}\cdot\hat{\boldsymbol{\theta}}=\mathbf{0}$

Their directions depend on location.

Unit vector in plane polar coordinate



In the limit $\Delta\boldsymbol{\theta} \to 0$, keeping $\boldsymbol{r}$ constant, $\frac{\Delta\vec{r}}{\Delta\theta}$ (I,e $\frac{\partial\vec{r}}{\partial\theta}$) gives a vector in the increasing direction of '$\boldsymbol{\theta}$'

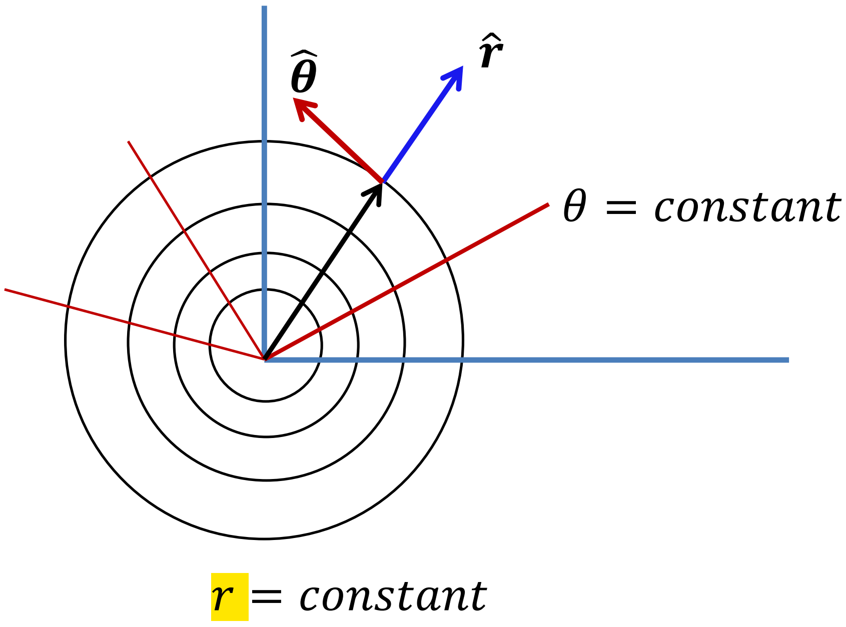
Thus unit vector in the $\boldsymbol{\theta}$ direction is $\widehat{\boldsymbol{\theta}} = \dfrac{\textit{vector in this direction}}{\textit{magnitude of that vector}} = \dfrac{\frac{\partial\vec{r}}{\partial\theta}}{\left|\frac{\partial\vec{r}}{\partial\theta}\right|}$

Similarly $\hat{\boldsymbol{r}} = \dfrac{\textit{vector in this direction}}{\textit{magnitude of that vector}} = \dfrac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|}$

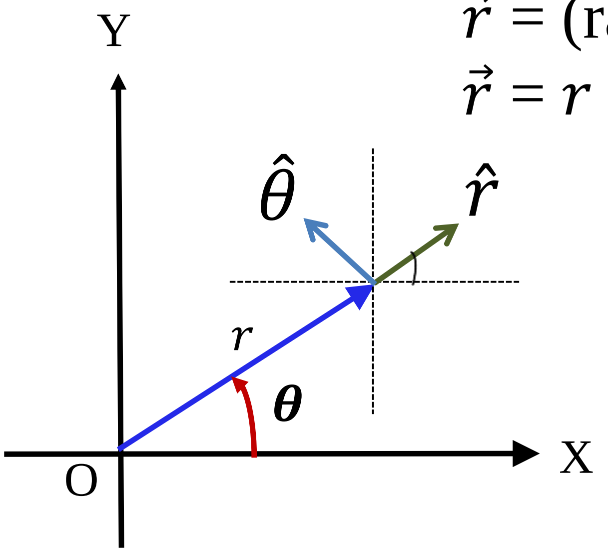
Unit vector in plane polar coordinate

The unit vectors in the polar coordinate can also be viewed in another way. $\hat{r}$ is the unit vector perpendicular to $r = constant$ surface and points in the increasing direction of r.

Similarly, $\hat{\theta}$ is the unit vector perpendicular to $\theta = constant$ surface (i,e tangential to $r = constant$) and points in the increasing direction of θ.



Unit vector in plane polar coordinate



$\vec{r}$ = (radial distance) (unit vector along the vector)

$\vec{r} = r\,\hat{r}$

Unit vectors in polar coordinate are function of θ only.

$$\frac{\boldsymbol{\partial \hat{r}}}{\boldsymbol{\partial \theta}} = \frac{\partial}{\partial \theta}(\hat{x}\cos\theta + \hat{y}\sin\theta) = -\hat{x}\sin\theta + \hat{y}\cos\theta = \boldsymbol{\hat{\theta}}$$

$$\frac{\boldsymbol{\partial \hat{\theta}}}{\boldsymbol{\partial \theta}} = \frac{\partial}{\partial \theta}(-\hat{x}\sin\theta + \hat{y}\cos\theta) = -(\hat{x}\cos\theta + \hat{y}\sin\theta) = -\boldsymbol{\hat{r}}$$

Velocity in plane polar coordinate

Velocity $\vec{v} = \frac{d\vec{r}}{dt}$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}\,(r\,\hat{r}) \quad = \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\hat{r} + r\frac{\partial\hat{r}}{\partial\theta}\frac{d\theta}{dt}$$

Since,

$$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta}$$

$$\boxed{\vec{\boldsymbol{v}} = \dot{\boldsymbol{r}}\hat{\boldsymbol{r}} + \boldsymbol{r}\dot{\boldsymbol{\theta}}\hat{\boldsymbol{\theta}}}$$

Radial component $\dot{\boldsymbol{r}}$ and

tangential/transverse component $\boldsymbol{r}\dot{\boldsymbol{\theta}}$

Acceleration in plane polar coordinate

$$\vec{a} = \frac{d\vec{v}}{dt}$$

Note: $\frac{\partial \hat{r}}{\partial \theta} = \hat{\theta}$ & $\frac{\partial \hat{\theta}}{\partial \theta} = -\hat{r}$

$$= \frac{d}{dt}(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta}) = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\frac{\partial \hat{r}}{\partial \theta}\frac{d\theta}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{\partial \hat{\theta}}{\partial \theta}\frac{d\theta}{dt}$$

$$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$

$$\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}}$$

Radial component of acceleration**:** $\ddot{r} - r\dot{\theta}^2$

(Note: $-r\dot{\theta}^2$ is the familiar ***Centripetal*** *contribution*!)

Tangential component: $2\dot{r}\dot{\theta} + r\ddot{\theta}$

(Note: $2\dot{r}\dot{\theta}$ is called the ***Coriolis*** contribution!)

Newton's law in plane polar coordinate

$$\vec{F} = m\frac{d\vec{v}}{dt}$$

$$\vec{F} = F_r\hat{r} + F_\theta\hat{\theta} = m[(\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}]$$

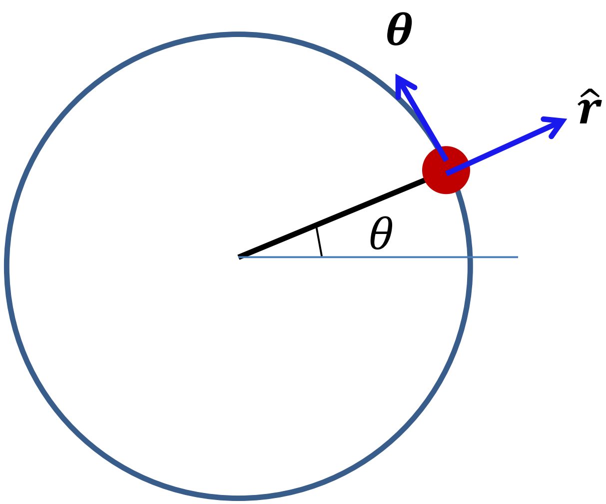
Newton's law for **radial** direction: $\boldsymbol{F_r} = \mathbf{m}(\ddot{\boldsymbol{r}} - \boldsymbol{r}\dot{\boldsymbol{\theta}}^2)$

Newton's law for **tangential** direction: $\boldsymbol{F_\theta} = \boldsymbol{m}(2\dot{\boldsymbol{r}}\dot{\boldsymbol{\theta}} + \boldsymbol{r}\ddot{\boldsymbol{\theta}})$

Note: Newton's law in polar coordinates does not follow its **Cartesian form** as,

$\boldsymbol{F_r} \neq \boldsymbol{m\ddot{r}}$ or $\boldsymbol{F}_\theta \neq \boldsymbol{m}\ddot{\theta}$

Choice of proper coordinate system makes analysis easier



Motion in circular trajectory

Equation of trajectory in polar coordinate

$$r = R = constat$$

$$\theta = \omega t + \frac{1}{2}\alpha t^2$$

The velocity components are

$$v_r = \dot{r} = 0; \; v_\theta = r\dot{\theta} = R(\omega + \alpha t) = v$$

Acceleration components are

$$a_r = \ddot{r} - r\dot{\theta}^2 = -R(\omega + \alpha t)^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = R\alpha = a_t$$

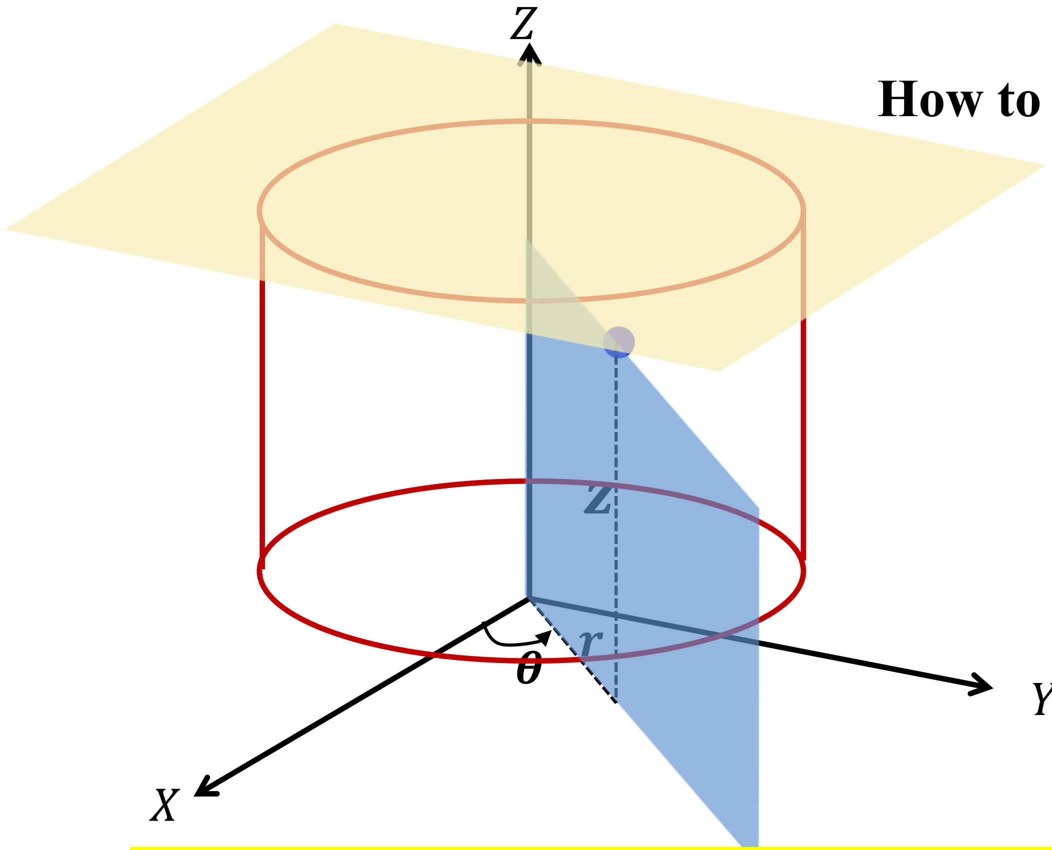
Equation of trajectory in cartesian coordinate

$x = R\cos\left(\omega t + \frac{1}{2}\alpha t^2\right)$; $y = R\sin\left(\omega t + \frac{1}{2}\alpha t^2\right)$;

velocity components are

$$v_x = -R(\omega + \alpha t)\sin\left(\omega t + \frac{1}{2}\alpha t^2\right); v_y = R(\omega + \alpha t)\cos\left(\omega t + \frac{1}{2}\alpha t^2\right)$$

Cylindrical coordinate system



How to locate a point 'P' in space ?

- ❑ $\boldsymbol{z}$-Height from the XY plane
- ❑ (r, θ) Coordinate of the foot of the point in XY plane

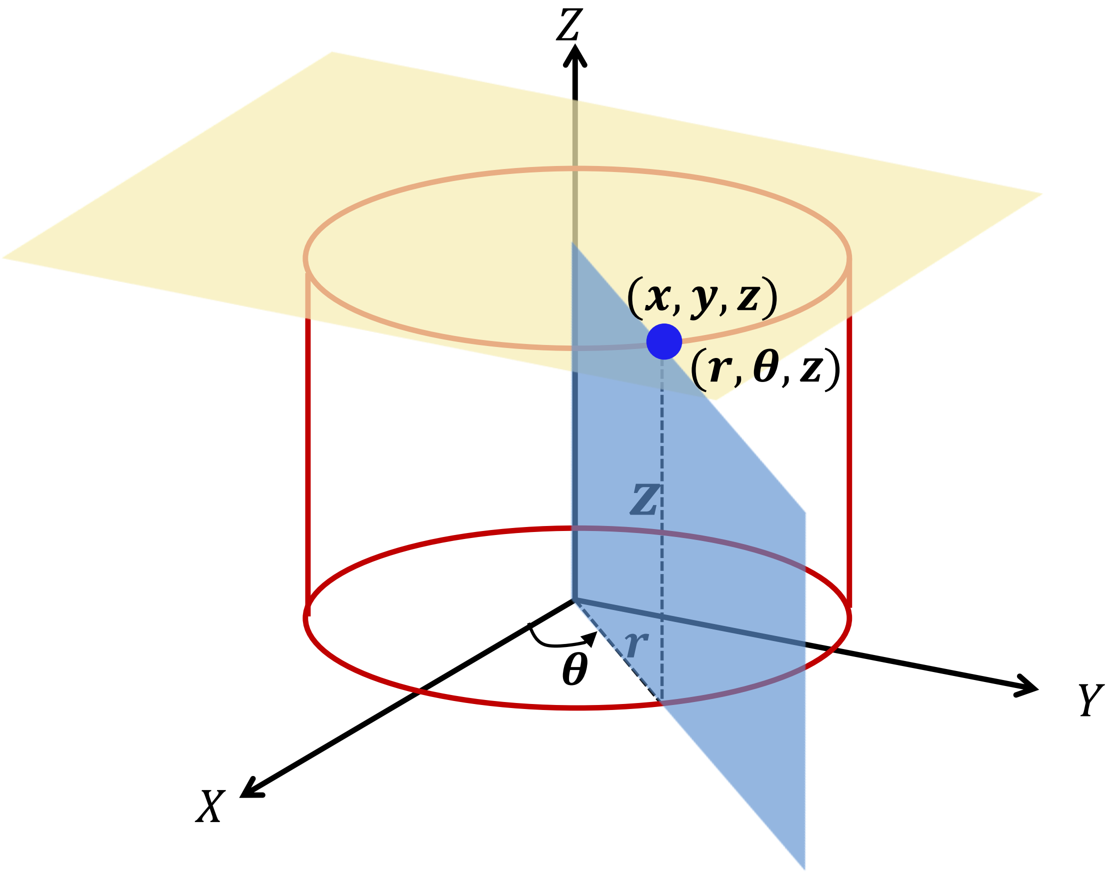
❑ $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{z})$ coordinates system is known as cylindrical coordinate system

Why the name cylindrical?

❑ Point 'P' is the intersection of three surfaces: A cylindrical surface $\boldsymbol{r} = \boldsymbol{constant}$; A half plane containing z-axis with $\boldsymbol{\theta}$**=constant** and a plane $\boldsymbol{z}$**=constant**.

Coordinate transformation: Cartesian to cylindrical

Transformation equation is very similar to polar coordinate with additional *z*-coordinate.



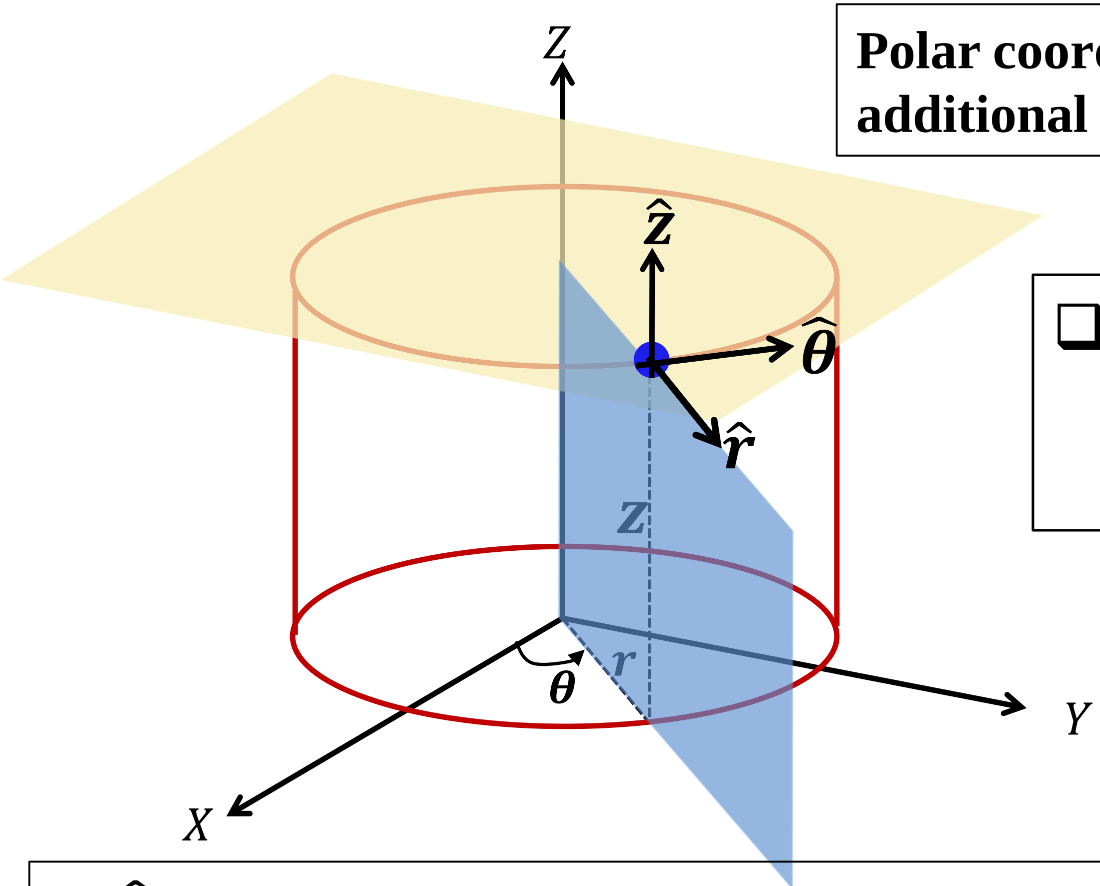
$$\boldsymbol{x = r\cos\theta}$$
$$\boldsymbol{y = r\sin\theta}$$
$$\boldsymbol{z = z}$$

Reverse transformation

$$\boldsymbol{r = \left(x^2 + y^2\right)^{1/2}}$$
$$\boldsymbol{\theta = \tan^{-1}\frac{y}{x}}$$
$$\boldsymbol{z = z}$$

Note: Instead of (r, θ) many books use notation (ρ, φ).

Unit vectors in cylindrical coordinate system



Polar coordinate unit vectors ($\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$) + additional unit vector in the z −direction.

- $\hat{\boldsymbol{r}}, \hat{\boldsymbol{\theta}}$ **and** $\hat{\boldsymbol{z}}$ are unit vectors along increasing direction of coordinates $\boldsymbol{r}, \boldsymbol{\theta}$ **and** $\boldsymbol{z}$.

$$\hat{\boldsymbol{r}} = \cos\boldsymbol{\theta}\ \hat{\boldsymbol{x}} + \sin\boldsymbol{\theta}\ \hat{\boldsymbol{y}}$$

$$\hat{\boldsymbol{\theta}} = -\sin\boldsymbol{\theta}\ \hat{\boldsymbol{x}} + \cos\boldsymbol{\theta}\ \hat{\boldsymbol{y}}$$

$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$$

$\hat{r}$, $\hat{\theta}$ and $\hat{\boldsymbol{z}}$ are **orthogonal** but their directions depend on location.

$\hat{r}$, $\hat{\theta}$ and $\hat{\boldsymbol{z}}$ are **perpendicular** to surfaces $\boldsymbol{r = constant}$; $\boldsymbol{\theta}$**=constant** and $\boldsymbol{z}$**=constant**. Respectively.

Position, Velocity, Acceleration, Newton's law in cylindrical coordinate system

Vector components are very similar to polar coordinate+ z −component

Position vector $\vec{r} = r\hat{r} + \boldsymbol{z}\hat{\mathbf{z}}$

Velocity $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + \dot{\boldsymbol{z}}\hat{\mathbf{z}}$

Acceleration $\vec{a}=(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{\boldsymbol{z}}\hat{\mathbf{z}}$

Newton's law

$$\vec{F} = F_r\hat{r} + F_\theta\hat{\theta} + \boldsymbol{F_z}\hat{\mathbf{z}}$$
$$= m[(\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta} + \ddot{\boldsymbol{z}}\hat{\mathbf{z}}]$$

Summery

- A point in plane can be represented by Cartesian coordinate P(x, y) or polar coordinate P(r, θ). A point in space can be represented by (x, y, z) or (r, θ, z) or (r, θ, φ).

- Coordinate transformation relation between *Cartesian* and *cylindrical coordinate* is given by

$$\boldsymbol{x = r\cos\theta}$$
$$\boldsymbol{y = r\sin\theta}$$
$$\boldsymbol{z = z}$$

- For plane polar coordinate, transformation relation is $x = r\cos\theta; y = r\sin\theta$

- Unit vector in plane polar coordinate: $\boldsymbol{\hat{r} = \cos\theta\ \hat{x} + \sin\theta\ \hat{y}\ \ ;\ \hat{\theta} = -\sin\theta\ \hat{x} + \cos\theta\ \hat{y}}$

- Unit vectors in cylindrical coordinate**:** $\boldsymbol{\hat{r} = \cos\theta\ \hat{x} + \sin\theta\ \hat{y}\ \ ;\ \hat{\theta} = -\sin\theta\ \hat{x} + \cos\theta\ \hat{y}}$ $\boldsymbol{, z = z}$

- Form of Newton's law is different in different coordinate systems.

Questions please