DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
Odd Semester of the Academic Year 2019-2020
MA 101 Mathematics I
Problem Sheet 2: Partial derivatives, tangent and normals, gradient, directional derivatives and chain rules etc.
Instructors: Dr. J. C. Kalita,

---

1. Let $f(x,y) = \begin{cases} \frac{x^2-xy}{x+y} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$

   (a) Find $f_x(0,0)$, $f_y(0,0)$.

   (b) Find $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$, and check whether it is equal to $f_x(0,0)$.

2. Let $f(x,y) = \sqrt{x^2+y^2}$.

   (a) Find $f_x(x,y)$ and $f_y(x,y)$ for $(x,y) \neq (0,0)$.

   (b) Show that $f_x(0,0)$ and $f_y(0,0)$ does not exist.

3. Let

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$$

   (a) Calculate $f_x(x,y)$ and $f_y(x,y)$ at all points where $(x,y) \neq (0,0)$.

   (b) Compute all first and second order partial derivatives at $(0,0)$ if they exist.

   (c) Show that f is discontinuous at $(0,0)$.

4. Find the equation of the tangent plane to the surface $z = \sqrt{4-x^2-2y^2}$ at the point $(1,-1,1)$.

5. It is geometrically evident that every plane tangent to the cone $z^2 = x^2+y^2$ pass through the origin. Show this by the method of calculus.

6. Find the equations of the tangent plane and normal line to the given surface at the specified point

   (a) $x^2+y^2-z^2-2xy+4xz = 4, \quad (1,0,1)$.

   (b) $z+1 = xe^y \cos z, \quad (1,0,0)$.

7. Suppose you need to know the equation of the tangent plane to a surface S at the point $P = (2,1,3)$. You don't have an equation for S, but you know that the curves

$$\mathbf{r}_1(t) = \left\langle 2+3t, 1-t^2, 3-4t+t^2 \right\rangle$$

$$\mathbf{r}_2(t) = \left\langle 1+t^2, 2t^3-1, 2t+1 \right\rangle$$

   both lie on S. Find an equation of the tangent plane at P.

8. Show that the sum of the x-, y-, and z-intercepts of any tangent plane (at any point of the surface wherever it is defined) to the surface $\sqrt{x} + \sqrt{y} + \sqrt{z} = \sqrt{c}$ is a constant.

9. If $z = f(x, y) = x^2 + 3xy - y^2$,

   (a) write the expression for the differential dz at (x, y, z);

   (b) if x changes from 2 to 2.05 and y changes from 3 to 2.96, compare the values of Δz and dz.

10. Find the directional derivative of the function at the given point in the direction of the vector $\mathbf{v}$.

    (a) $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$, $(1, 2, -2)$ $\mathbf{v} = \langle -6, 6, -3 \rangle$

    (b) $g(x, y, z) = x \tan^{-1}\left(\frac{y}{z}\right)$, $(1, 2, -2)$, $\mathbf{v} = \mathbf{i} + \mathbf{j} - \mathbf{k}$.

11. Find the directional derivatives of the scalar field $f(x, y) = x^3 - 3xy$ along the parabola $y = x^2 - x + 2$ at the point (1,2).

12. Let $f(x, y) = \dfrac{x}{|x|}\sqrt{x^2 + y^2}$ if $x \neq 0$ and $f(x, y) = 0$ if $x = 0$. Show that f is continuous at $(0, 0)$ and the directional derivatives exist thereat, but it is not differentiable at $(0, 0)$.

13. Show that the following functions are differentiable at the respective points mentioned below:

    (a) Let
    $$f(x, y) = \begin{cases} \frac{x}{x+y} & \text{if} \quad x + y \neq 0 \\ 0 & \text{if} \quad x + y = 0. \end{cases}$$
    Show that f is differentiable at $(2, 1)$ but not differentiable at $(0, 0)$.

    (b) Show that $f(x, y) = \sqrt{x + e^y}$ is differentiable at $(3, 0)$, where x, y is such that $x + e^y \geq 0$.

14. Show that the following function is differentiable throughout $\mathbf{R}^2$ and find the maximum rate of change of $f(x, y) = 6 - 3x^2 - y^2$ at the point (1,2) and the direction in which it occurs.

15. If R is the total resistance of three resistors, connected in parallel, with resistances R_1, R_2, R_3, then
    $$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$
    The resistances are measured in ohms as $R_1 = 100\Omega$, $R_2 = 100\Omega$ and $R_3 = 200\Omega$. R_1 and R_2 are increasing at $1\Omega/s$ whereas R_3 is decreasing at $2\Omega/s$. Is R increasing or decreasing at that instant? At what rate?

16. Assume that $w = f(x, y)$, $x = r\cos\theta$ and $y = r\sin\theta$. Assuming the existence of all the required first and second order partial derivatives of w with respect to x, y, r and θ, show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}.$$

17. Suppose $w = f(u)$ where $u = \dfrac{x^2 - y^2}{x^2 + y^2}$. Assuming the existence of all the required first order partial derivatives of w and u show that $xw_x + yw_y = 0$.

18. **Implicit differentiation:** If $\phi(x, y, z) = 0$ defines z as an implicit function of x and y in a region R of the xy-plane, assuming the existence of all the required partial derivatives prove that $\dfrac{\partial z}{\partial x} = -\dfrac{\phi_x}{\phi_z}$ and $\dfrac{\partial z}{\partial y} = -\dfrac{\phi_y}{\phi_z}$, where $\phi_z \neq 0$. Hence find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ when $x^{\frac{2}{3}} + y^{\frac{2}{3}} + z^{\frac{2}{3}} = 1$.

19. Suppose that $w = \dfrac{1}{r} f\left(t - \dfrac{r}{a}\right)$ and that $r = \sqrt{x^2 + y^2 + z^2}$. Assuming the existence of all the required second order partial derivatives, show that

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = \frac{1}{a^2}\frac{\partial^2 w}{\partial t^2}.$$

20. A function f is called **homogeneous of degree** n if it satisfies the equation $f(tx, ty) = t^n f(x, y)$ for all t, where n is a positive integer and f has continuous second order partial derivatives.

    (a) Verify that $f(x, y) = x^3 - 2xy^2 + 5y^3$ is homogeneous of degree 3.

    (b) Show that if f is homogeneous of degree n, then

    i. $x\dfrac{\partial f}{\partial x} + y\dfrac{\partial f}{\partial y} = nf(x, y)$

    ii. $x^2\dfrac{\partial^2 f}{\partial x^2} + 2xy\dfrac{\partial^2}{\partial x \partial y} + y^2\dfrac{\partial^2 f}{\partial y^2} = n(n-1)f(x, y)$

    iii. $f_x(tx, ty) = t^{n-1} f_x(x, y)$.