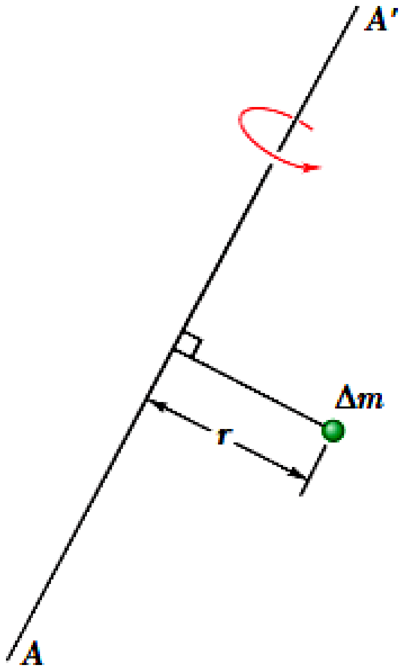
ME101: Engineering Mechanics (3 1 0 8)

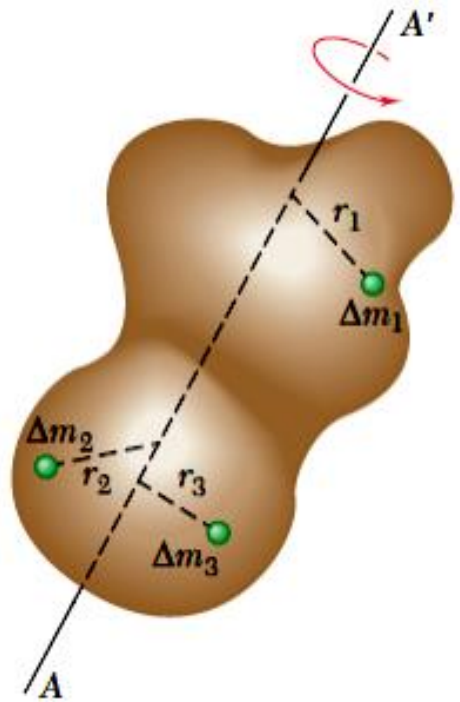
2019-20 (II Semester)

LECTURE: 17 and 18

Mass Moment of Inertia

- Application in rigid body dynamics
  - Measure of distribution of mass of a rigid body w.r.t. the axis *(constant property for that axis)*





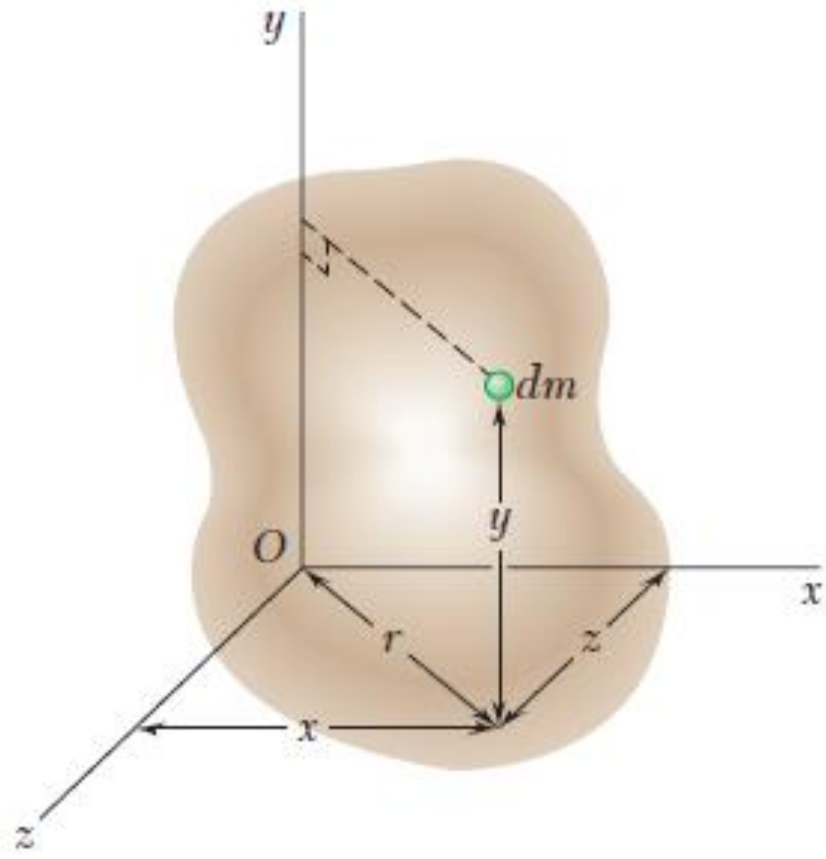
$$I = \int r^2 \, dm$$

r = perpendicular distance of the mass element *dm* from the axis *O-O*

$r^2 \Delta m$ :: measure of the inertia of the system

Mass Moment of Inertia

- About individual coordinate axes



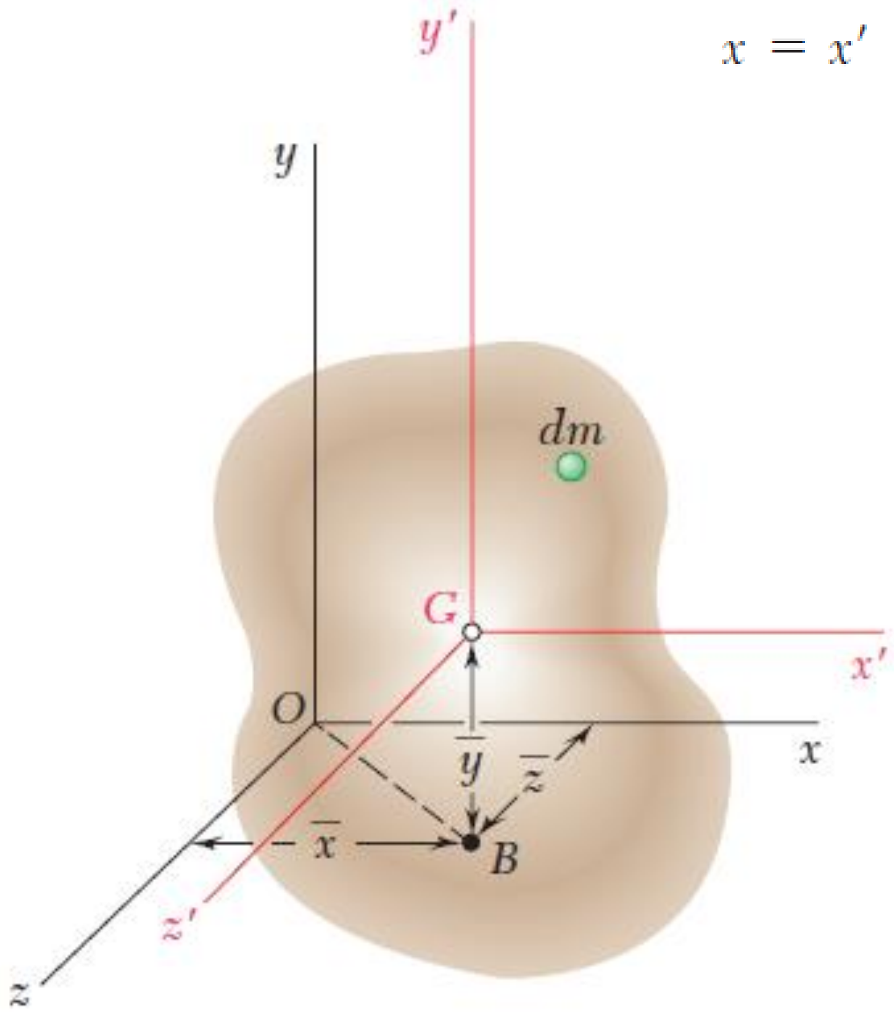
$$I_x = \int (y^2 + z^2)\, dm$$

$$I_y = \int (z^2 + x^2)\, dm$$

$$I_z = \int (x^2 + y^2)\, dm$$

Mass Moment of Inertia

- Parallel Axis Theorem



$$x = x' + \bar{x} \qquad y = y' + \bar{y} \qquad z = z' + \bar{z}$$

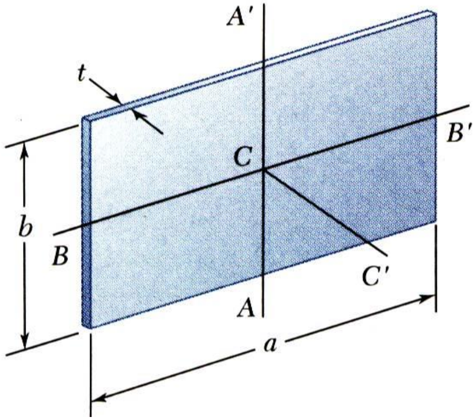
$$I_x = \bar{I}_{x'} + m(\bar{y}^2 + \bar{z}^2)$$

$$I_y = \bar{I}_{y'} + m(\bar{z}^2 + \bar{x}^2)$$

$$I_z = \bar{I}_{z'} + m(\bar{x}^2 + \bar{y}^2)$$

Mass Moment of Inertia

Moments of Inertia of Thin Plates

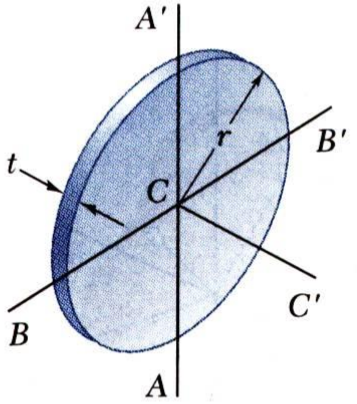


- For the principal centroidal axes on a rectangular plate,

$$I_{AA'} = \rho t\, I_{AA',area} = \rho t\left(\tfrac{1}{12} a^3 b\right) = \tfrac{1}{12} m a^2$$

$$I_{BB'} = \rho t\, I_{BB',area} = \rho t\left(\tfrac{1}{12} a b^3\right) = \tfrac{1}{12} m b^2$$

$$I_{CC'} = I_{AA',mass} + I_{BB',mass} = \tfrac{1}{12} m\left(a^2 + b^2\right)$$

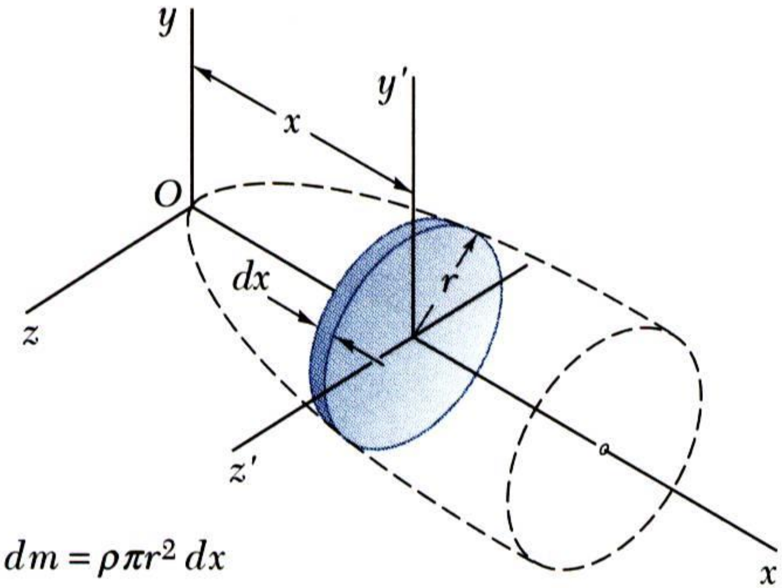


- For centroidal axes on a circular plate,

$$I_{AA'} = I_{BB'} = \rho t\, I_{AA',area} = \rho t\left(\tfrac{1}{4}\pi r^4\right) = \tfrac{1}{4} m r^2$$

Mass Moment of Inertia

Moments of Inertia of a 3D Body by Integration



$$dm = \rho \pi r^2 \, dx$$

$$dI_x = \frac{1}{2} r^2 \, dm$$

$$dI_y = dI_{y'} + x^2 \, dm = \left(\frac{1}{4} r^2 + x^2\right) dm$$

$$dI_z = dI_{z'} + x^2 \, dm = \left(\frac{1}{4} r^2 + x^2\right) dm$$

- Moment of inertia of a homogeneous body is obtained from double or triple integrations of the form

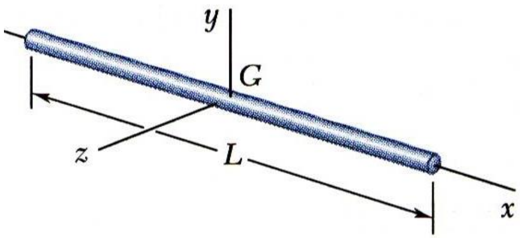
$$I = \rho \int r^2 dV$$

- For bodies with two planes of symmetry, the moment of inertia may be obtained from a single integration by choosing thin slabs perpendicular to the planes of symmetry for *dm*.

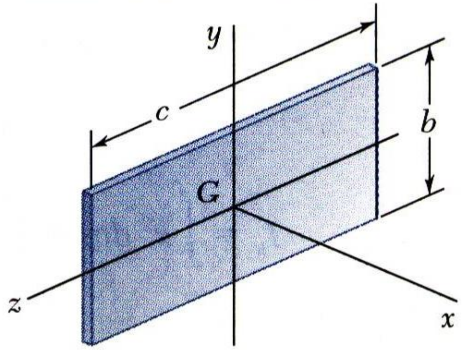
- The moment of inertia with respect to a particular axis for a composite body may be obtained by adding the moments of inertia with respect to the same axis of the components.

Mass Moment of Inertia

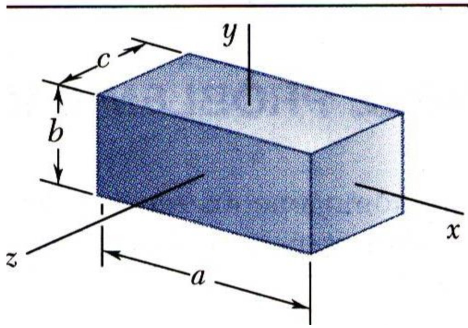
MI of some common geometric shapes



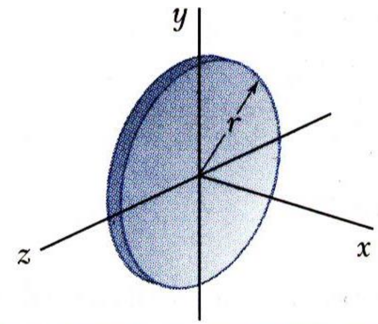
$$I_y = I_z = \frac{1}{12} mL^2$$



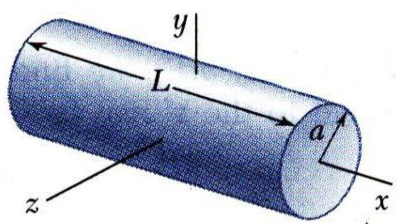
$$I_x = \frac{1}{12} m(b^2 + c^2)$$
$$I_y = \frac{1}{12} mc^2$$
$$I_z = \frac{1}{12} mb^2$$



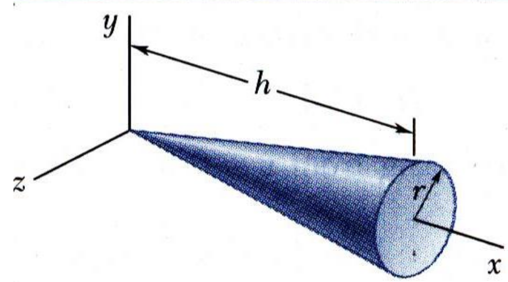
$$I_x = \frac{1}{12} m(b^2 + c^2)$$
$$I_y = \frac{1}{12} m(c^2 + a^2)$$
$$I_z = \frac{1}{12} m(a^2 + b^2)$$



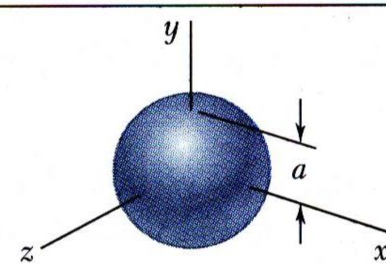
$$I_x = \frac{1}{2} mr^2$$
$$I_y = I_z = \frac{1}{4} mr^2$$



$$I_x = \frac{1}{2} ma^2$$
$$I_y = I_z = \frac{1}{12} m(3a^2 + L^2)$$

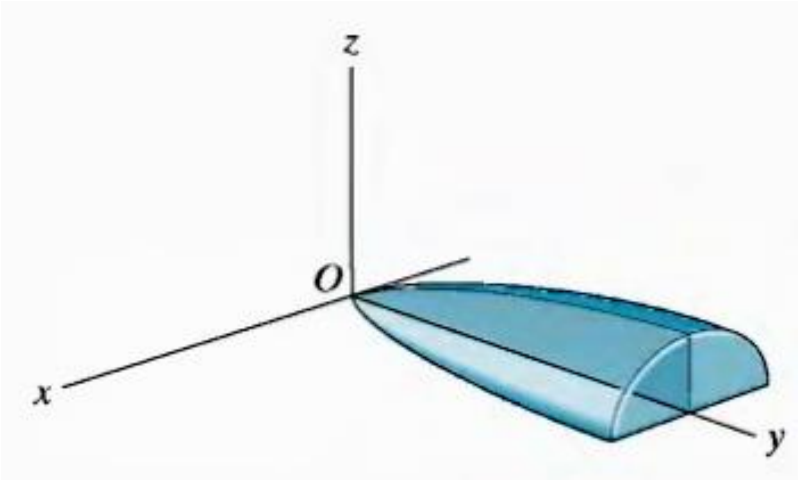


$$I_x = \frac{3}{10} ma^2$$
$$I_y = I_z = \frac{3}{5} m(\frac{1}{4} a^2 + h^2)$$



$$I_x = I_y = I_z = \frac{2}{5} ma^2$$

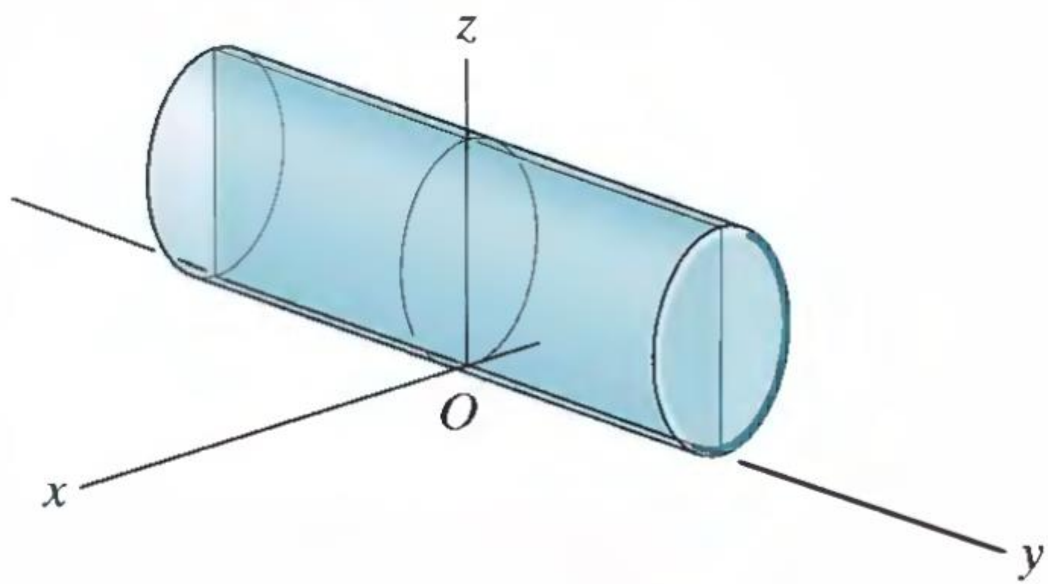
Product of Inertia



y–z plane is a plane of symmetry, hence $I_{xy} = I_{xz} = 0$

I_{yz} will be positive since all the elements of mass are located using only +ve y and z coord

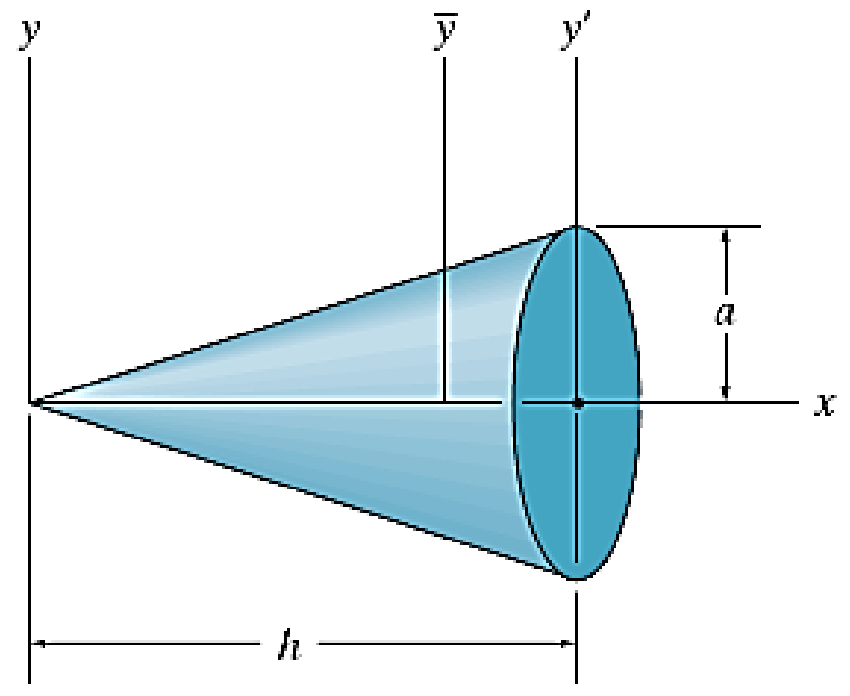
Product of Inertia



cylinder, with the coordinate axes located as
shown in Fig. , the x–z and y–z planes are both planes of symmetry
Thus, $I_{xy} = I_{yz} = I_{zx} = 0$.

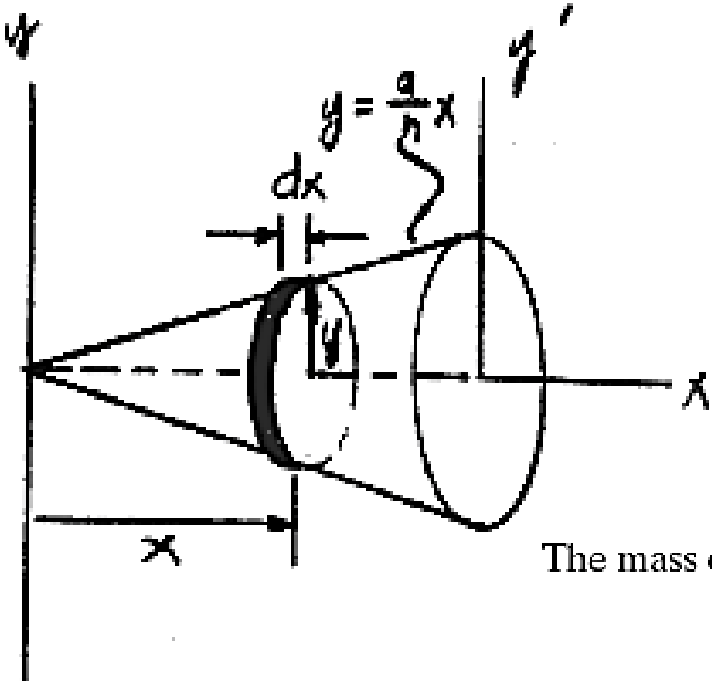
Example-1: Mass Moment of Inertia

Determine the moment of inertia of the cone with respect to a vertical $\bar{y}$ axis passing through the cone's center of mass. What is the moment of inertia about a parallel axis y' that passes through the diameter of the base of the cone? The cone has a mass m.



Example-1: Mass Moment of Inertia

Solution



The mass of the differential element is $dm = \rho dV = \rho(\pi y^2)\,dx = \dfrac{\rho\pi a^2}{h^2}x^2 dx.$

$$dI_y = \frac{1}{4}dmy^2 + dmx^2$$

$$= \frac{1}{4}\left[\frac{\rho\pi a^2}{h^2}x^2\,dx\right]\left(\frac{a}{h}x\right)^2 + \left(\frac{\rho\pi a^2}{h^2}x^2\right)x^2\,dx$$

$$= \frac{\rho\pi a^2}{4h^4}(4h^2 + a^2)\,x^4\,dx$$

Example-1: Mass Moment of Inertia

$$I_y = \int dI_y = \frac{\rho\pi a^2}{4h^4}(4h^2 + a^2)\int_0^h x^4 dx = \frac{\rho\pi a^2 h}{20}(4h^2 + a^2)$$

$$m = \int_m dm = \frac{\rho\pi a^2}{h^2}\int_0^h x^2\, dx = \frac{\rho\pi a^2 h}{3}$$

$$I_y = \frac{3m}{20}(4h^2 + a^2)$$

Example-1: Mass Moment of Inertia

Using the parallel axis theorem:

$$I_y = I_{\bar{y}} + md^2$$

$$\frac{3m}{20}(4h^2 + a^2) = I_{\bar{y}} + m\left(\frac{3h}{4}\right)^2$$

$$I_{\bar{y}} = \frac{3m}{80}(h^2 + 4a^2)$$

$$I_{y'} = I_{\bar{y}} + md^2$$

$$= \frac{3m}{80}(h^2 + 4a^2) + m\left(\frac{h}{4}\right)^2$$

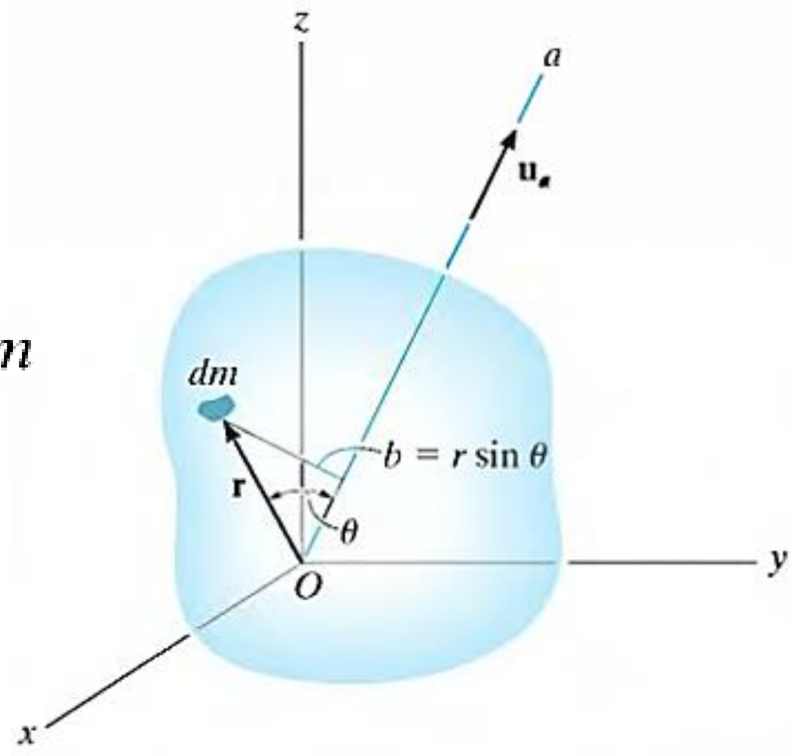
$$= \frac{m}{20}(2h^2 + 3a^2)$$

Mass Moment of Inertia

About an arbitrary axis

$$I_{Oa} = \int b^2\, dm$$

b is the *perpendicular distance* from dm



$$I_{Oa} = \int_m |(\mathbf{u}_a \times \mathbf{r})|^2 dm = \int_m (\mathbf{u}_a \times \mathbf{r}) \cdot (\mathbf{u}_a \times \mathbf{r}) dm$$

$\mathbf{u}_a = u_x\mathbf{i} + u_y\mathbf{j} + u_z\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$,

$$\mathbf{u}_a \times \mathbf{r} = (u_y z - u_z y)\mathbf{i} + (u_z x - u_x z)\mathbf{j} + (u_x y - u_y x)\mathbf{k}$$

$$I_{Oa} = \int_m [(u_y z - u_z y)^2 + (u_z x - u_x z)^2 + (u_x y - u_y x)^2] dm$$

$$= u_x^2 \int_m (y^2 + z^2) dm + u_y^2 \int_m (z^2 + x^2) dm + u_z^2 \int_m (x^2 + y^2)\, dm$$

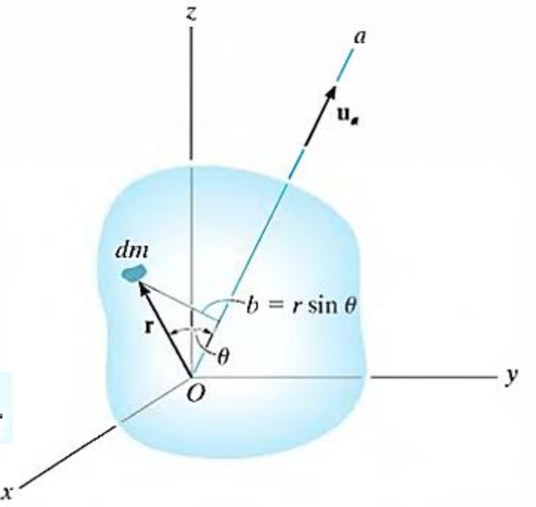
Mass Moment of Inertia

About an arbitrary axis

$$I_{Oa} = \int_m [(u_y z - u_z y)^2 + (u_z x - u_x z)^2 + (u_x y - u_y x)^2] dm$$

$$= u_x^2 \int_m (y^2 + z^2) dm + u_y^2 \int_m (z^2 + x^2) dm + u_z^2 \int_m (x^2 + y^2)\, dm$$

$$I_{Oa} = I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_xu_y - 2I_{yz}u_yu_z - 2I_{zx}u_zu_x$$

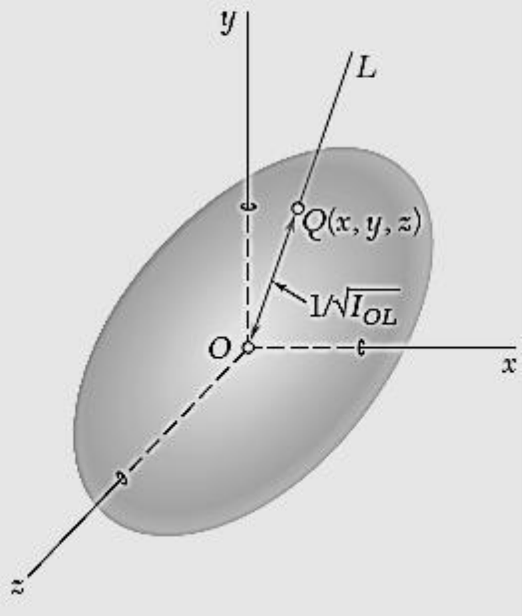


Ellipsoid of inertia

Let us assume that the moment of inertia of the body considered in the preceding section has been determined with respect to a large number of axes *OL* *passing* through the fixed point *O* and that a point *Q* has been plotted on each axis *OL* at a distance $OQ = \frac{1}{(OL)^{0.5}}$ from O

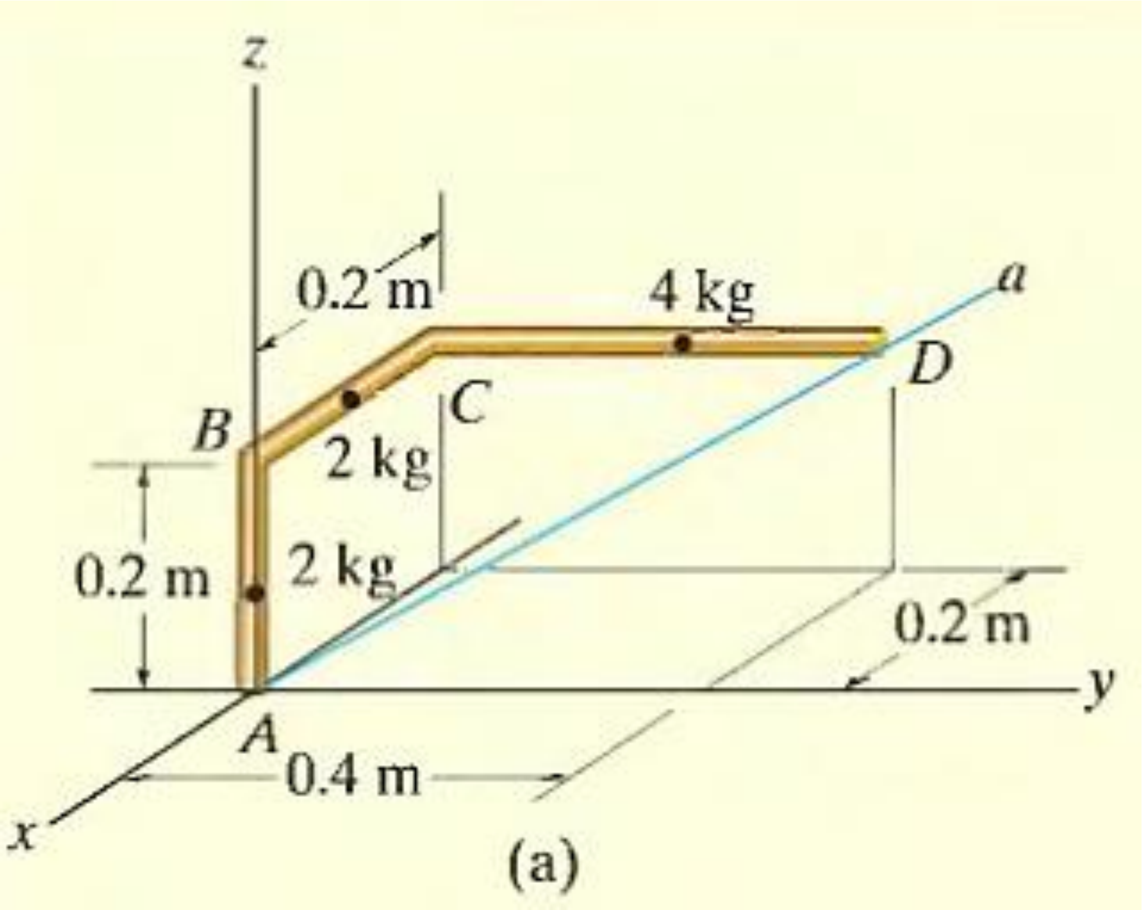
The locus of the point Q is called the ellipsoid if inertia

$$I_x x^2 + I_y y^2 + I_z z^2 - 2I_{xy}xy - 2I_{yz}yz - 2I_{zx}zx = 1$$



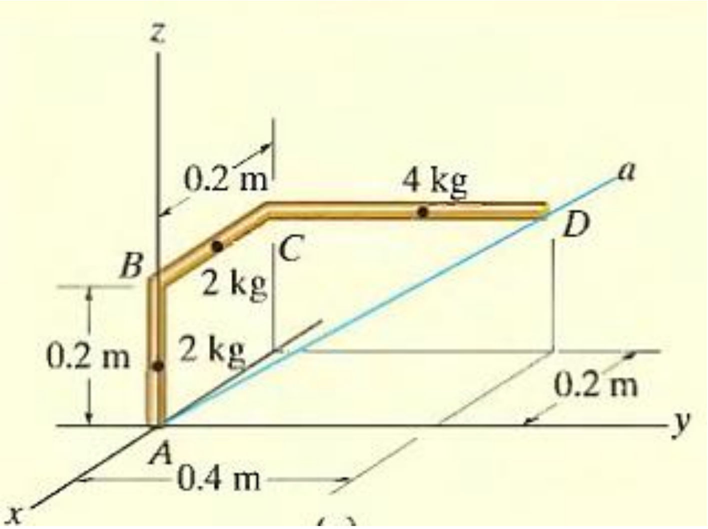
Mass Moment of Inertia

Example



Mass Moment of Inertia

Example



First find the moment of inertia w.r.t x, y & z axis

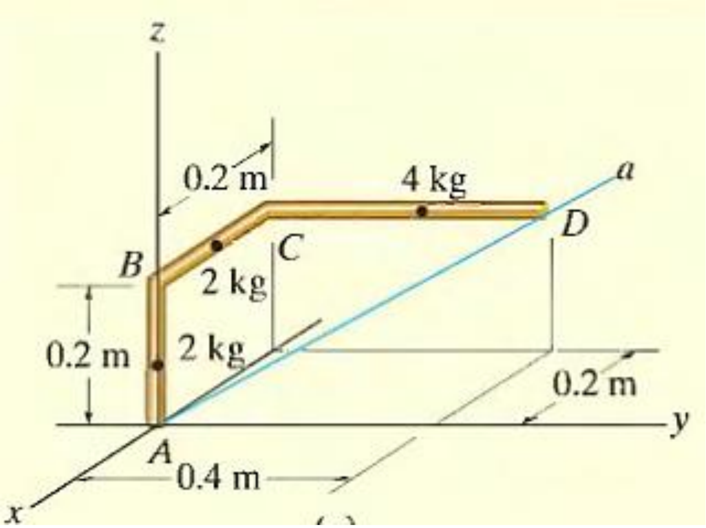
$$I_{xx} = \left[\tfrac{1}{12}(2)(0.2)^2 + 2(0.1)^2\right] + \left[0 + 2(0.2)^2\right] + \left[\tfrac{1}{12}(4)(0.4)^2 + 4((0.2)^2 + (0.2)^2)\right] = 0.480\ \text{kg}\cdot\text{m}^2$$

$$I_{yy} = \left[\tfrac{1}{12}(2)(0.2)^2 + 2(0.1)^2\right] + \left[\tfrac{1}{12}(2)(0.2)^2 + 2((-0.1)^2 + (0.2)^2)\right] + \left[0 + 4((-0.2)^2 + (0.2)^2)\right] = 0.453\ \text{kg}\cdot\text{m}^2$$

$$I_{zz} = [0 + 0] + \left[\tfrac{1}{12}(2)(0.2)^2 + 2(-0.1)^2\right] + \left[\tfrac{1}{12}(4)(0.4)^2 + 4((-0.2)^2 + (0.2)^2)\right] = 0.400\ \text{kg}\cdot\text{m}^2$$

Mass Moment of Inertia

Example



Find the product of inertia w.r.t x, y & z axis

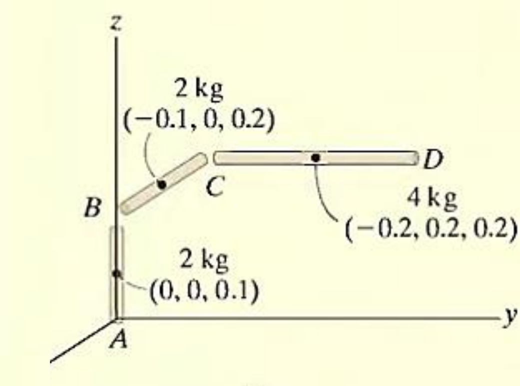
$$I_{xy} = [0 + 0] + [0 + 0] + [0 + 4(-0.2)(0.2)] = -0.160 \text{ kg} \cdot \text{m}^2$$

$$I_{yz} = [0 + 0] + [0 + 0] + [0 + 4(0.2)(0.2)] = 0.160 \text{ kg} \cdot \text{m}^2$$

$$I_{zx} = [0 + 0] + [0 + 2(0.2)(-0.1)] + [0 + 4(0.2)(-0.2)] = -0.200 \text{ kg} \cdot \text{m}^2$$

Mass Moment of Inertia

Example



The Aa axis is defined by the unit vector

$$\mathbf{u}_{Aa} = \frac{\mathbf{r}_D}{r_D} = \frac{-0.2\mathbf{i} + 0.4\mathbf{j} + 0.2\mathbf{k}}{\sqrt{(-0.2)^2 + (0.4)^2 + (0.2)^2}} = -0.408\mathbf{i} + 0.816\mathbf{j} + 0.408\mathbf{k}$$

Thus,

$$u_x = -0.408 \qquad u_y = 0.816 \qquad u_z = 0.408$$

Substituting these results into Eq. 21–5 yields

$$\begin{aligned}
I_{Aa} &= I_{xx}u_x^2 + I_{yy}u_y^2 + I_{zz}u_z^2 - 2I_{xy}u_xu_y - 2I_{yz}u_yu_z - 2I_{zx}u_zu_x \\
&= 0.480(-0.408)^2 + (0.453)(0.816)^2 + 0.400(0.408)^2 \\
&\qquad - 2(-0.160)(-0.408)(0.816) - 2(0.160)(0.816)(0.408) \\
&\qquad - 2(-0.200)(0.408)(-0.408) \\
&= 0.169\ \text{kg}\cdot\text{m}^2
\end{aligned}$$

Ans.