# ME101: Engineering Mechanics (3 1 0 8)

2019-20 (II Semester)

# LECTURE: 14 - 15

# Contents

# Introduction

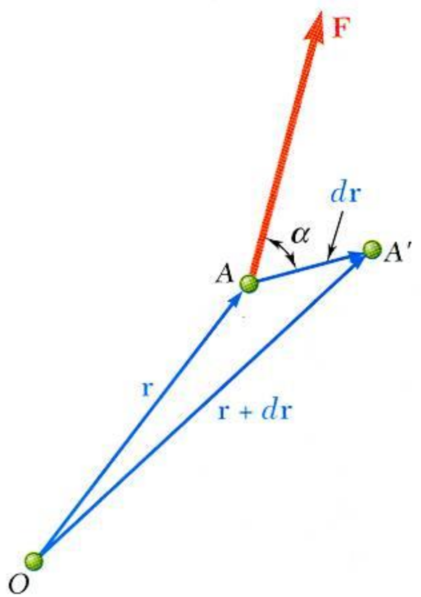
- *Principle of virtual work* - if a particle, rigid body, or system of rigid bodies which is in equilibrium under various forces is given an arbitrary *virtual displacement*, the net work done by the external forces during that displacement is zero.

- The principle of virtual work is particularly useful when applied to the solution of problems involving the equilibrium of machines or mechanisms consisting of several connected members.

- If a particle, rigid body, or system of rigid bodies is in equilibrium, then the derivative of its potential energy with respect to a variable defining its position is zero.

- The stability of an equilibrium position can be determined from the second derivative of the potential energy with respect to the position variable.

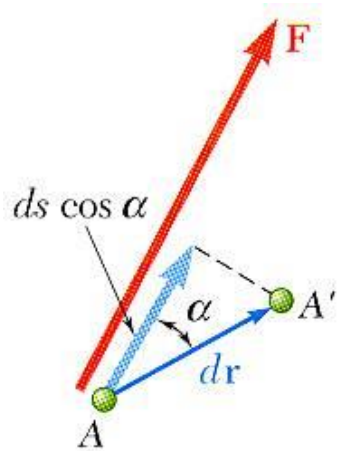
# Work of a Force

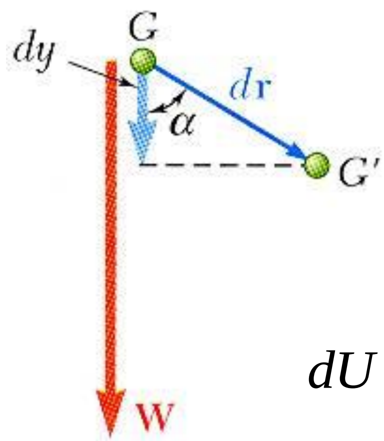



$$dU = \vec{F} \cdot d\vec{r}$$ = work of the force $\vec{F}$ corresponding to the displacement $d\vec{r}$

$$dU = F\,ds\cos\alpha$$

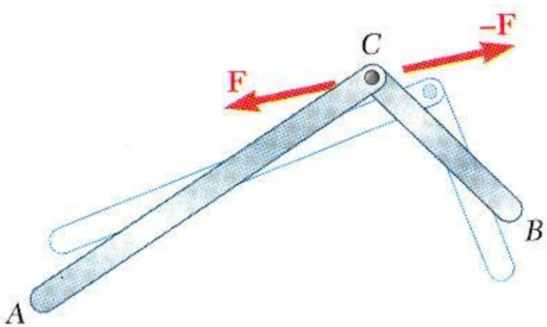
$\alpha = 0,\ dU = +F\,ds$ $\qquad \alpha = \pi,\ dU = -F\,ds$ $\qquad \alpha = \frac{\pi}{2},\ dU = 0$

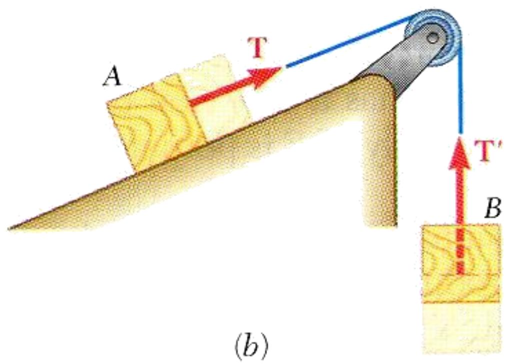



$$dU = W dy$$

# Work of a Force

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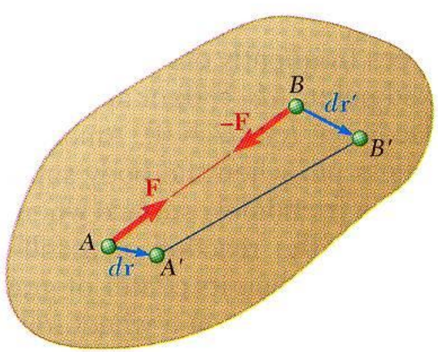


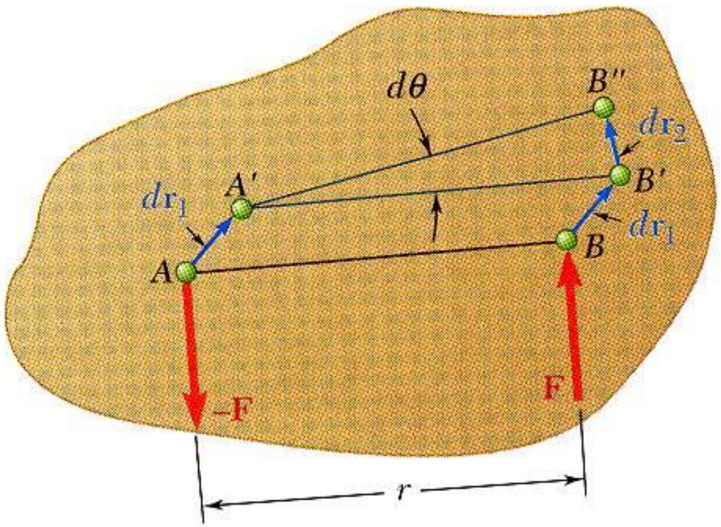

Forces which do no work:

- reaction at a frictionless pin due to rotation of a body around the pin
- reaction at a frictionless surface due to motion of a body along the surface
- weight of a body with cg moving horizontally
- friction force on a wheel moving without slipping

Sum of work done by several forces may be zero:

- bodies connected by a frictionless pin
- bodies connected by an inextensible cord
- internal forces holding together parts of a rigid body

# Work of a Couple

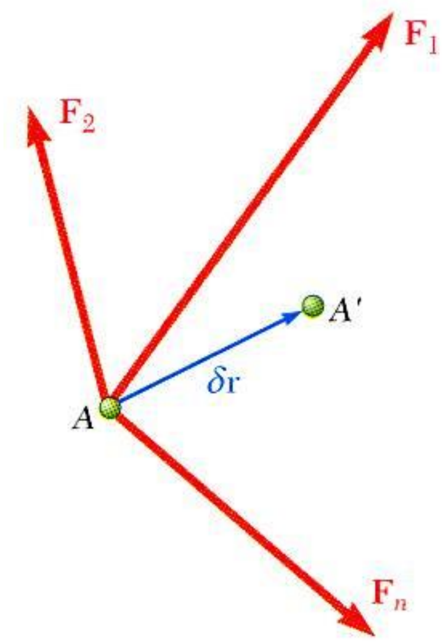



Small displacement of a rigid body:

- translation to *A'B'*
- rotation of *B'* about *A'* to *B''*

$$
\begin{aligned}
W &= -\vec{F} \cdot d\vec{r}_1 + \vec{F} \cdot \left(d\vec{r}_1 + d\vec{r}_2\right) \\
&= \vec{F} \cdot d\vec{r}_2 = F\, ds_2 = F\, r d\theta \\
&= M\, d\theta
\end{aligned}
$$

# Principle of Virtual Work



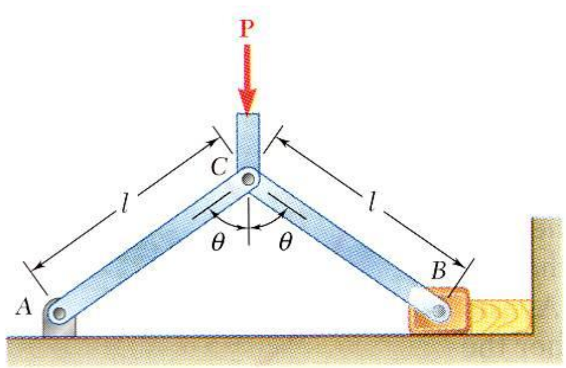

- *Imagine* the small *virtual displacement* of particle which is acted upon by several forces.
- The corresponding *virtual work*,

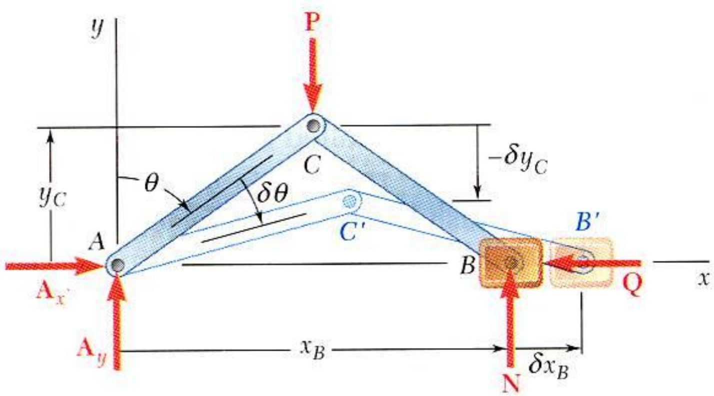
$$\delta U = \vec{F}_1 \cdot \delta\vec{r} + \vec{F}_2 \cdot \delta\vec{r} + \vec{F}_3 \cdot \delta\vec{r} = \left(\vec{F}_1 + \vec{F}_2 + \vec{F}_3\right) \cdot \delta\vec{r}$$
$$= \vec{R} \cdot \delta\vec{r}$$

*Principle of Virtual Work*:

- If a particle is in equilibrium, the total virtual work of forces acting on the particle is zero for any virtual displacement.
- If a rigid body is in equilibrium, the total virtual work of external forces acting on the body is zero for any virtual displacement of the body.
- If a system of connected rigid bodies remains connected during the virtual displacement, only the work of the external forces need be considered.

# Applications of the Principle of Virtual Work




- Wish to determine the force of the vice on the block for a given force *P*.
- Consider the work done by the external forces for a virtual displacement $\delta\theta$. Only the forces *P* and *Q* produce nonzero work.

$$\delta U = 0 = \delta U_Q + \delta U_P = -Q\,\delta x_B - P\,\delta y_C$$

$$x_B = 2l\sin\theta \qquad\qquad y_C = l\cos\theta$$

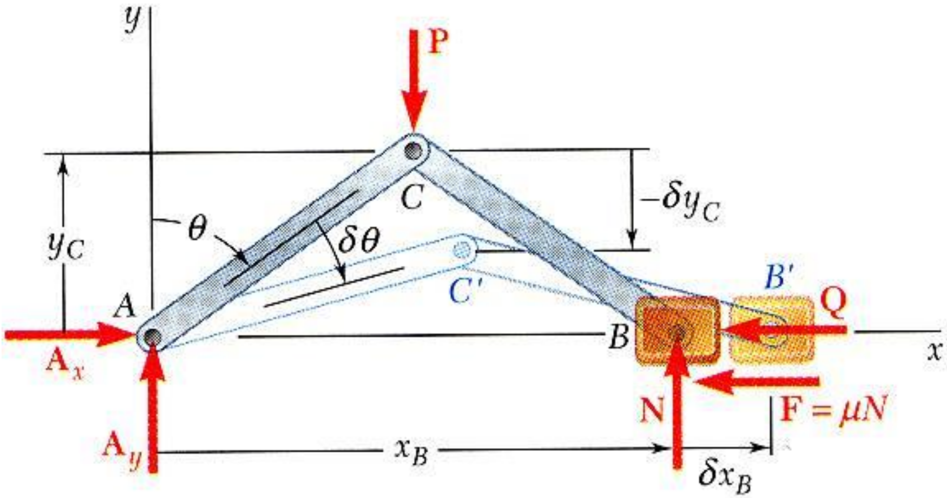
$$\delta x_B = 2l\cos\theta\,\delta\theta \qquad\qquad \delta y_C = -l\sin\theta\,\delta\theta$$

$$0 = -2Ql\cos\theta\,\delta\theta + Pl\sin\theta\,\delta\theta$$

$$Q = \tfrac{1}{2}P\tan\theta$$

- If the virtual displacement is consistent with the constraints imposed by supports and connections, only the work of loads, applied forces, and friction forces need be considered.

# Real Machines. Mechanical Efficiency




- For an ideal machine without friction, the output work is equal to the input work.
- When the effect of friction is considered, the output work is reduced.

$$\delta U = -Q\delta x_B - P\delta y_C - F\delta x_B = 0$$

$$0 = -2Ql\cos\theta\delta\theta + Pl\sin\theta\delta\theta - \mu Pl\cos\theta\delta\theta$$

$$Q = \tfrac{1}{2}P(\tan\theta - \mu)$$

$$\eta = \text{mechanical efficiency}$$

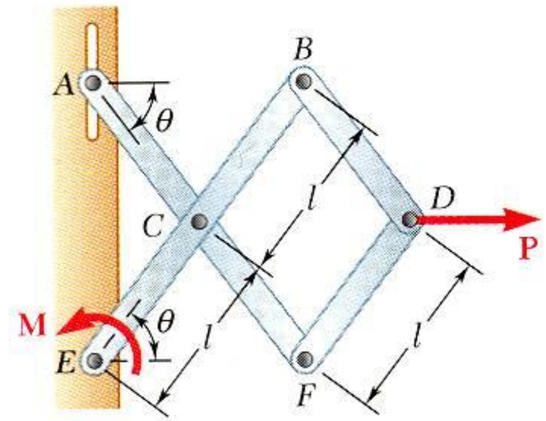
$$= \frac{\text{output work of actual machine}}{\text{output work of ideal machine}}$$

$$\eta = \frac{\text{output work}}{\text{input work}}$$

$$= \frac{2Ql\cos\theta\delta\theta}{Pl\sin\theta\delta\theta}$$

$$= 1 - \mu\cot\theta$$

# Sample Problem




Determine the magnitude of the couple $M$ required to maintain the equilibrium of the mechanism.

SOLUTION:

- Apply the principle of virtual work

$$\delta U = 0 = \delta U_M + \delta U_P$$

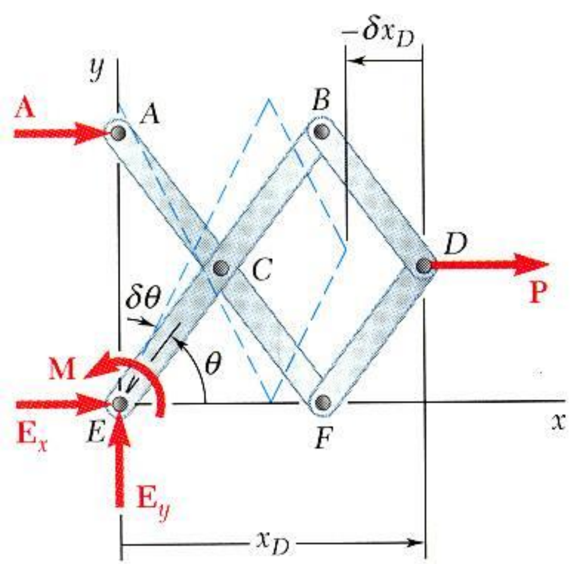
$$0 = M\delta\theta + P\delta x_D$$

$$x_D = 3l\cos\theta$$

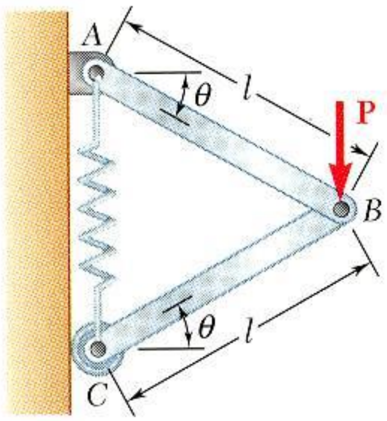
$$\delta x_D = -3l\sin\theta\delta\theta$$

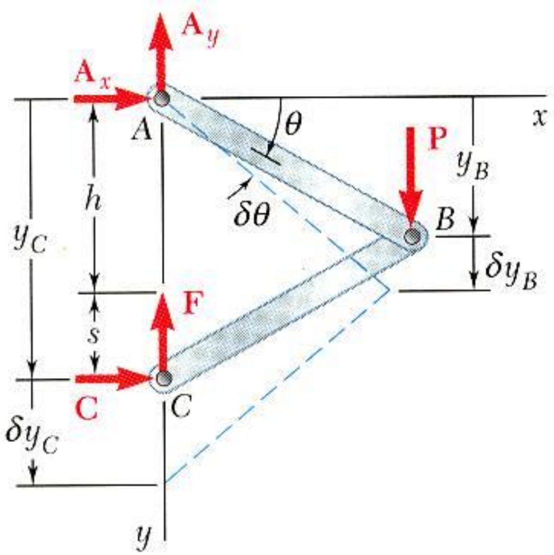
$$0 = M\delta\theta + P(-3l\sin\theta\delta\theta)$$

$$\boxed{M = 3Pl\sin\theta}$$



# Sample Problem




Determine the expressions for $\theta$ and for the tension in the spring which correspond to the equilibrium position of the spring. The unstretched length of the spring is *h* and the constant of the spring is k. Neglect the weight of the mechanism.

SOLUTION:

- Apply the principle of virtual work

$$\delta U = \delta U_B + \delta U_F = 0$$

$$0 = P\delta y_B - F\delta y_C$$

$$y_B = l\sin\theta \qquad y_C = 2l\sin\theta \qquad F = ks$$

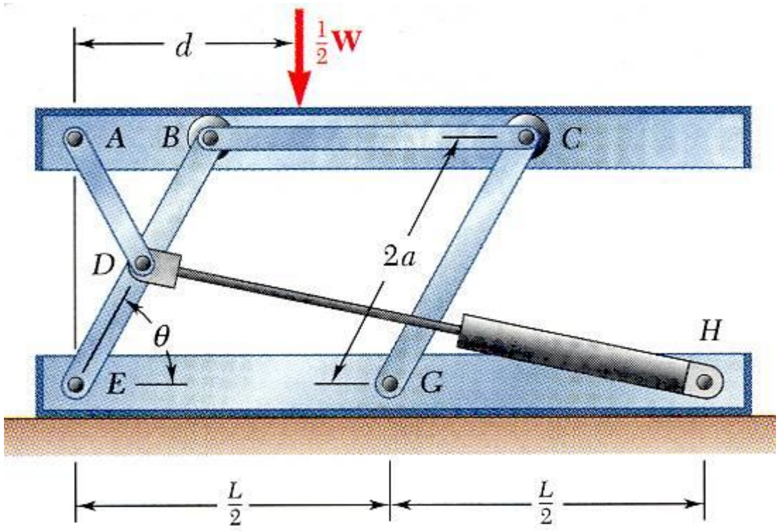
$$\delta y_B = l\cos\theta\delta\theta \qquad \delta y_C = 2l\cos\theta\delta\theta \qquad = k(y_C - h)$$

$$= k(2l\sin\theta - h)$$

$$0 = P(l\cos\theta\delta\theta) - k(2l\sin\theta - h)(2l\cos\theta\delta\theta)$$

$$\boxed{\sin\theta = \frac{P + 2kh}{4kl} \qquad F = \tfrac{1}{2}P}$$

# Sample Problem

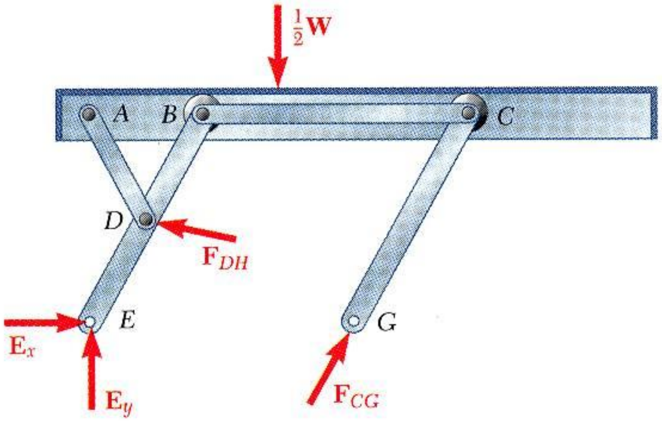



A hydraulic lift table consisting of two identical linkages and hydraulic cylinders is used to raise a 1000-kg crate. Members *EDB* and *CG* are each of length 2*a* and member *AD* is pinned to the midpoint of *EDB*.

Determine the force exerted by each cylinder in raising the crate for $\theta = 60^{o}$, $a = 0.70$ m, and $L = 3.20$ m.

SOLUTION:

- Create a free-body diagram for the platform and linkage.

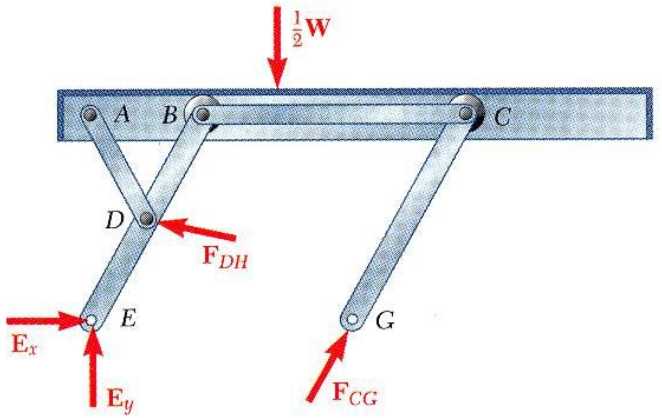


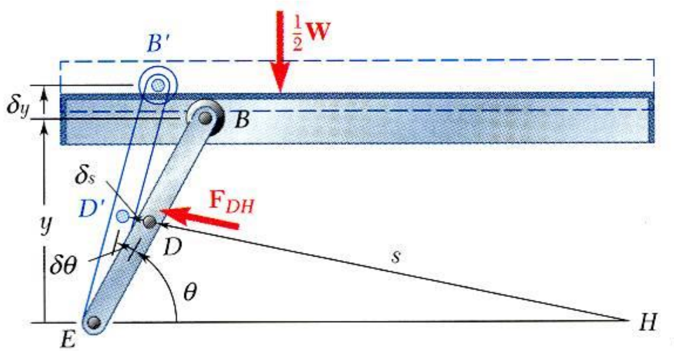

- Apply the principle of virtual work for a virtual displacement $\delta\theta$ recognizing that only the weight and hydraulic cylinder do work.

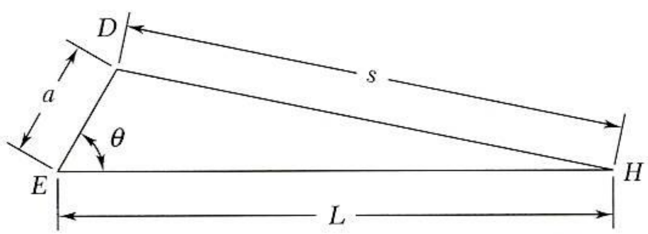
$$\delta U = 0 = \delta Q_W + \delta Q_{F_{DH}}$$

- Based on the geometry, substitute expressions for the virtual displacements and solve for the force in the hydraulic cylinder.

# Sample Problem




SOLUTION:

- Create a free-body diagram for the platform.
- Apply the principle of virtual work for a virtual displacement $\delta\theta$

$$\delta U = 0 = \delta Q_W + \delta Q_{F_{DH}}$$

$$0 = -\tfrac{1}{2} W \delta y + F_{DH} \delta s$$

- Based on the geometry, substitute expressions for the virtual displacements and solve for the force in the hydraulic cylinder.

$$y = 2a \sin\theta$$

$$\delta y = 2a \cos\theta \, \delta\theta$$

$$s^2 = a^2 + L^2 - 2aL \cos\theta$$

$$2s\delta s = -2aL(-\sin\theta)\delta\theta$$

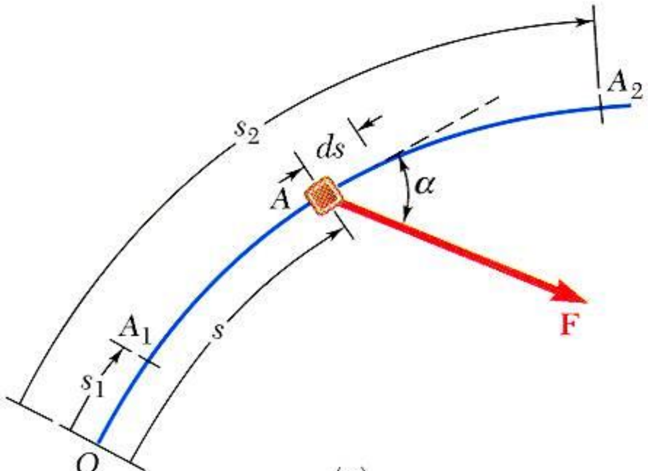
$$\delta s = \frac{aL\sin\theta}{s} \delta\theta$$

$$0 = \left(-\tfrac{1}{2} W\right) 2a \cos\theta \, \delta\theta + F_{DH} \frac{aL\sin\theta}{s} \delta\theta$$

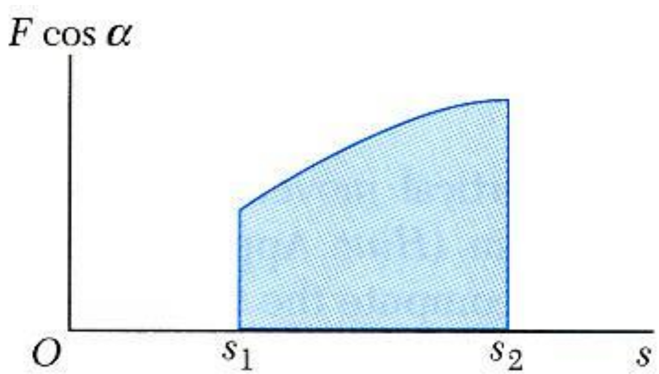
$$F_{DH} = W \frac{s}{L} \cot\theta$$

$$\boxed{F_{DH} = 5.15\,\text{kN}}$$

# Work of a Force During a Finite Displacement



(a)



(b)

- Work of a force corresponding to an infinitesimal displacement,

$$dU = \vec{F} \cdot d\vec{r}$$
$$= F\,ds\cos\alpha$$

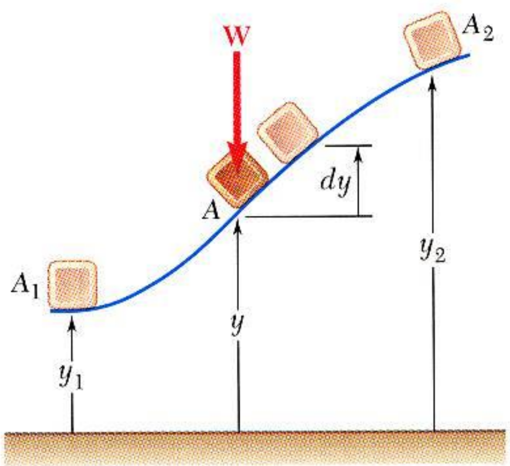
- Work of a force corresponding to a finite displacement,

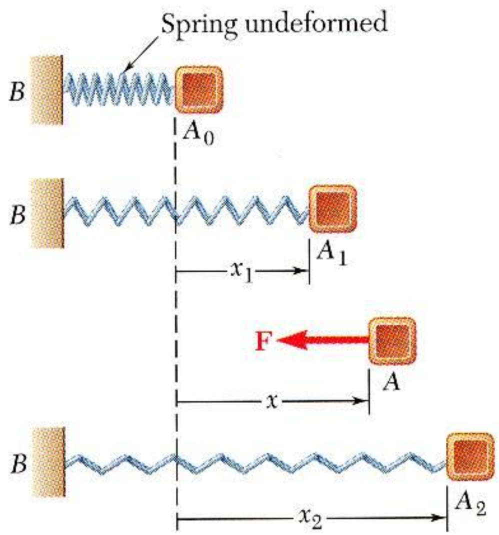
$$U_{1\to 2} = \int_{s_1}^{s_2} (F\cos\alpha)\,ds$$

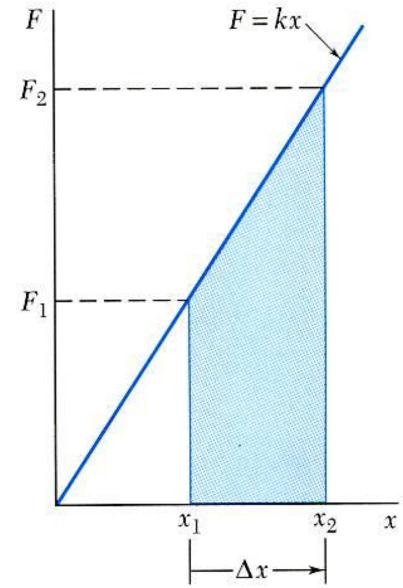
- Similarly, for the work of a couple,

$$dU = M d\theta$$
$$U_{1\to 2} = M(\theta_2 - \theta_1)$$

# Work of a Force During a Finite Displacement




Work of a weight,

$$dU = -W dy$$

$$U_{1\to2} = -\int_{y_1}^{y_2} W dy$$

$$= W y_1 - W y_2$$

$$= -W \Delta y$$

Work of a spring,

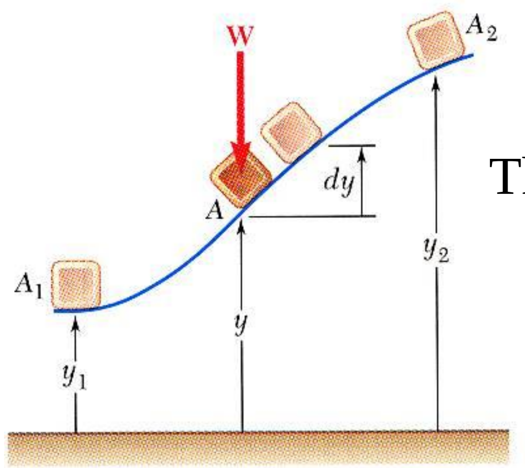
$$dU = -F dx = -(kx) dx$$

$$U_{1\to2} = -\int_{x_1}^{x_2} kx\, dx$$

$$= \tfrac{1}{2} k x_1^2 - \tfrac{1}{2} k x_2^2$$

$$U_{1\to2} = -\tfrac{1}{2}(F_1 + F_2)\Delta x$$

# Potential Energy



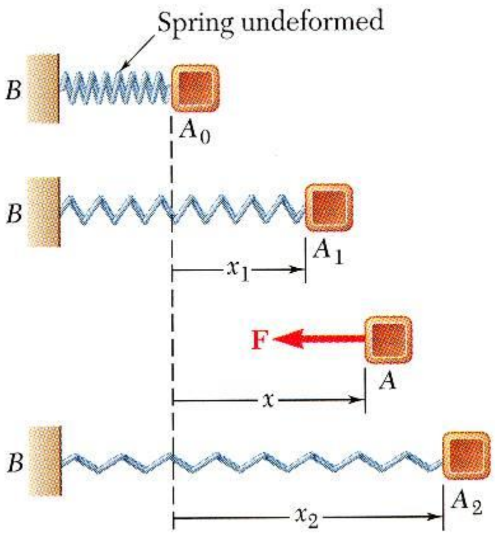

- Work of a weight

$$U_{1\to 2} = Wy_1 - Wy_2$$

The work is independent of path and depends only on initial and final locations

$$Wy = V_g = $$ *potential energy* of the body with $\vec{W}$ respect to the *force of gravity*

$$U_{1\to 2} = \left(V_g\right)_1 - \left(V_g\right)_2$$




- Work of a spring,

$$U_{1\to 2} = \tfrac{1}{2}kx_1^2 - \tfrac{1}{2}kx_2^2$$
$$= \left(V_e\right)_1 - \left(V_e\right)_2$$

$V_e =$ *potential energy* of the body with respect to the *elastic force* $\vec{F}$

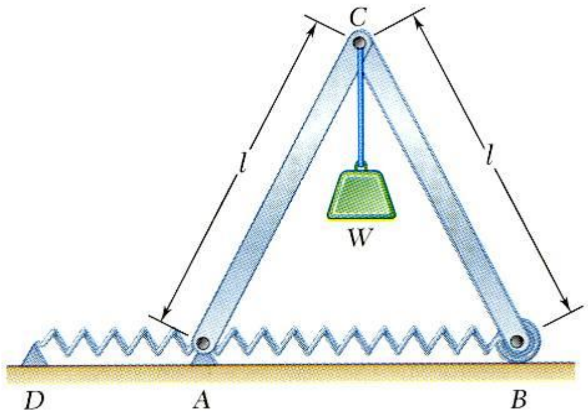
# Potential Energy

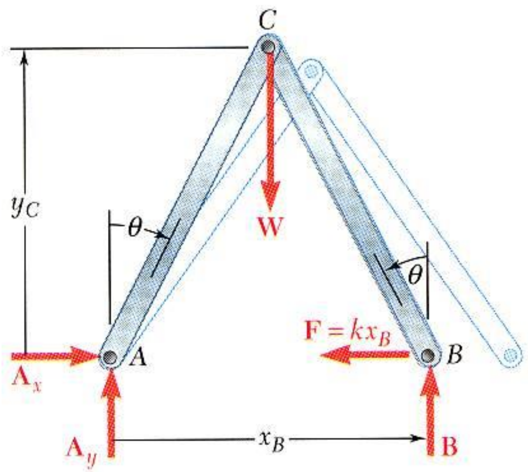
- When the differential work is a force is given by an exact differential,

$$
\begin{aligned}
dU &= -dV \\
U_{1\to 2} &= V_1 - V_2 \\
&= \textit{negative of change in potential energy}
\end{aligned}
$$

- Forces for which the work can be calculated from a change in potential energy are *conservative forces*.

# Potential Energy and Equilibrium




- When the potential energy of a system is known, the principle of virtual work becomes

$$\delta U = 0 = -\delta V = -\frac{dV}{d\theta}\delta\theta$$

$$0 = \frac{dV}{d\theta}$$

- For the structure shown,

$$V = V_e + V_g = \tfrac{1}{2}kx_B^2 + Wy_C$$

$$= \tfrac{1}{2}k(2l\sin\theta)^2 + W(l\cos\theta)$$

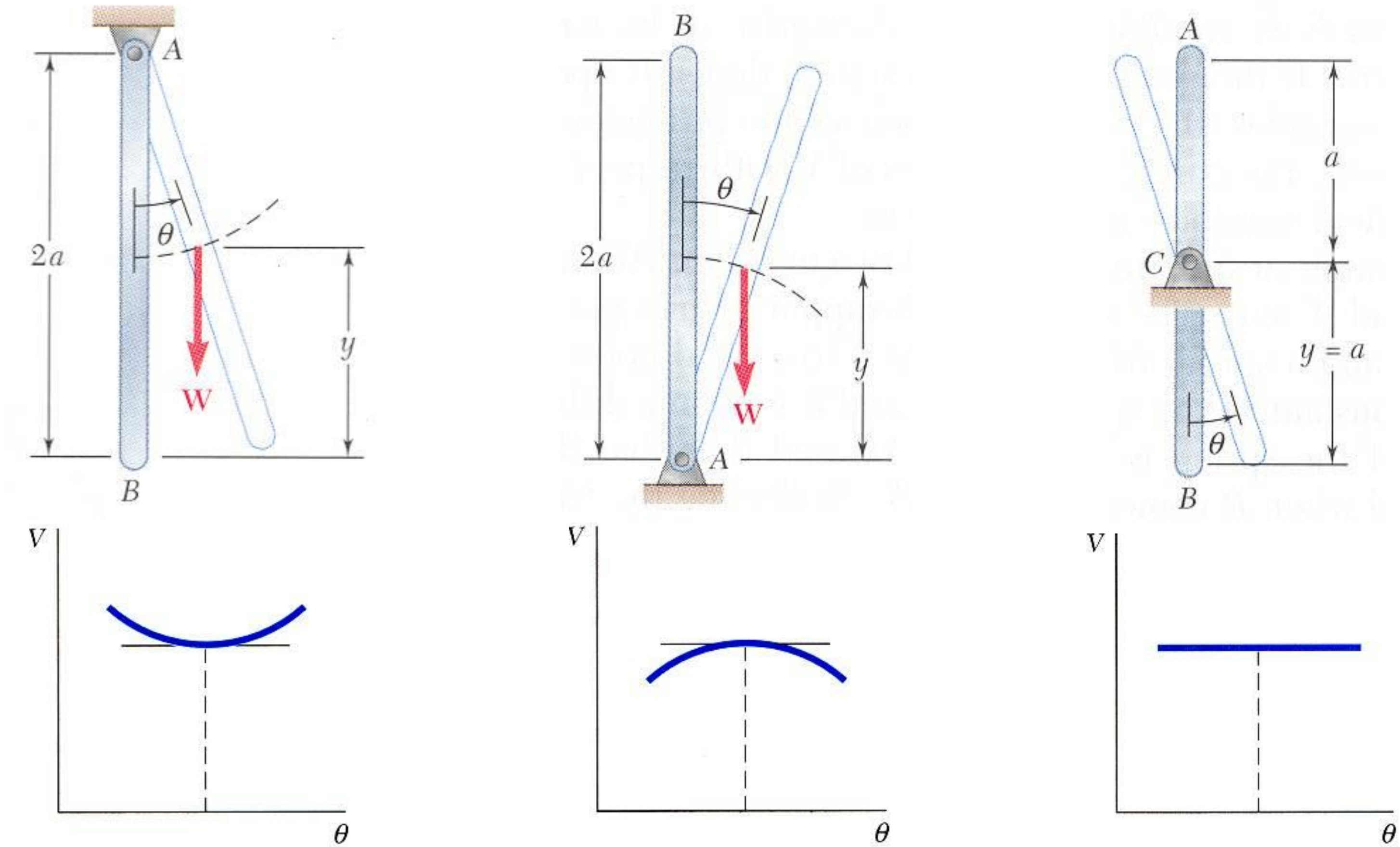
- At the position of equilibrium,

$$\frac{dV}{d\theta} = 0 = l\sin\theta(4kl\cos\theta - W)$$

indicating two positions of equilibrium.

# Stability of Equilibrium

$$\frac{dV}{d\theta} = 0$$

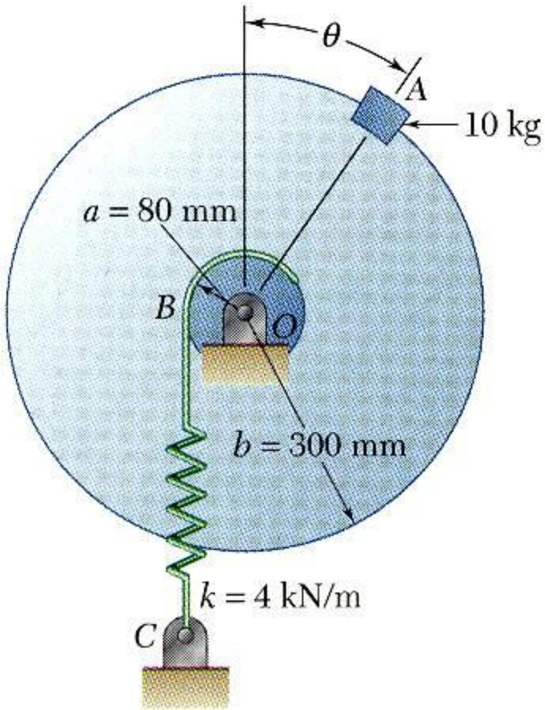



(a) Stable equilibrium (b) Unstable equilibrium (c) Neutral equilibrium

$$\frac{d^2V}{d\theta^2} > 0$$

Must examine higher order derivatives.

# Sample Problem




Knowing that the spring *BC* is unstretched when $\theta = 0$, determine the position or positions of equilibrium and state whether the equilibrium is stable, unstable, or neutral.

SOLUTION:

- Derive an expression for the total potential energy of the system.

$$V = V_e + V_g$$

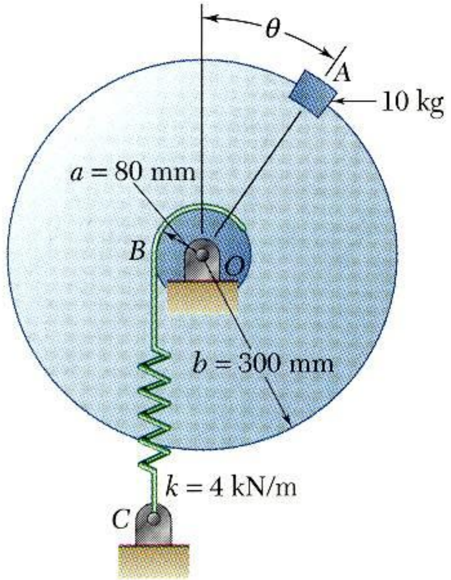
- Determine the positions of equilibrium by setting the derivative of the potential energy to zero.

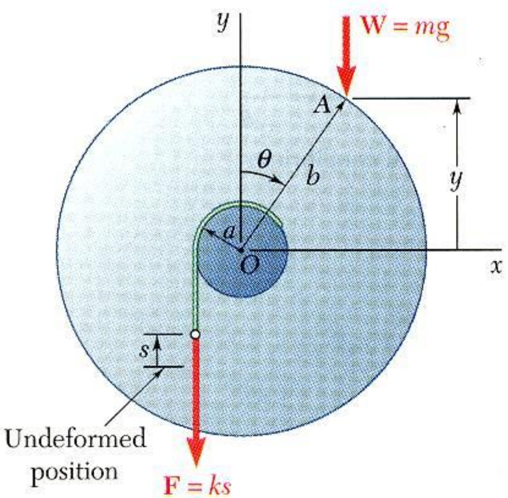
$$\frac{dV}{d\theta} = 0$$

- Evaluate the stability of the equilibrium positions by determining the sign of the second derivative of the potential energy.

$$\frac{d^2V}{d\theta^2} \overset{?}{><} 0$$

# Sample Problem




SOLUTION:

- Derive an expression for the total potential energy of the system.

$$V = V_e + V_g$$
$$= \tfrac{1}{2} ks^2 + mgy$$
$$= \tfrac{1}{2} k(a\theta)^2 + mg(b\cos\theta)$$

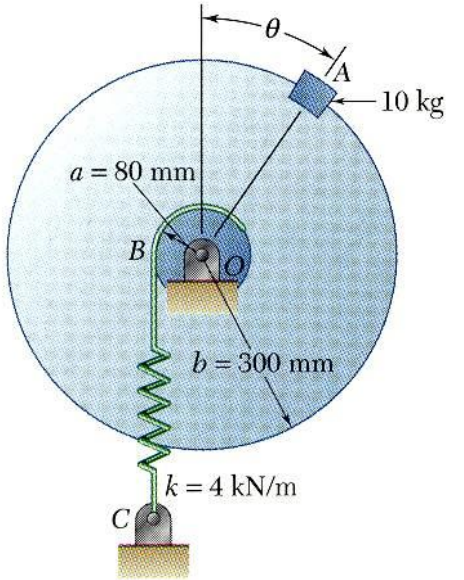
- Determine the positions of equilibrium by setting the derivative of the potential energy to zero.

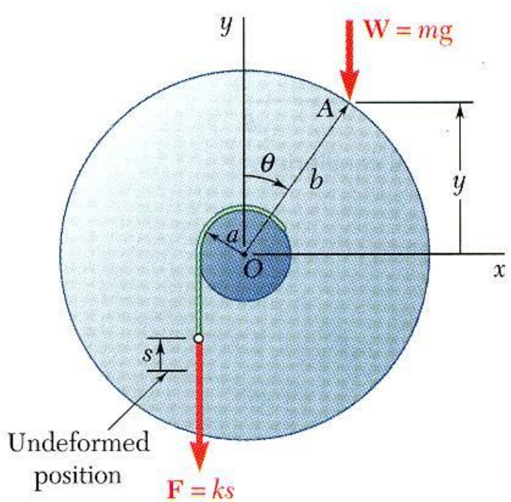
$$\frac{dV}{d\theta} = 0 = ka^2\theta - mgb\sin\theta$$

$$\sin\theta = \frac{ka^2}{mgb}\theta = \frac{(4\,\mathrm{kN/m})(0.08\,\mathrm{m})^2}{(10\,\mathrm{kg})(9.81\,\mathrm{m/s^2})(0.3\,\mathrm{m})}\theta$$
$$= 0.8699\,\theta$$

$$\boxed{\theta = 0 \qquad \theta = 0.902\,\mathrm{rad} = 51.7^\circ}$$

# Sample Problem




- Evaluate the stability of the equilibrium positions by determining the sign of the second derivative of the potential energy.

$$V = \tfrac{1}{2}k(a\theta)^2 + mg(b\cos\theta)$$

$$\frac{dV}{d\theta} = 0 = ka^2\theta - mgb\sin\theta$$

$$\theta = 0$$

$$\theta = 0.902\,\text{rad} = 51.7°$$

$$\frac{d^2V}{d\theta^2} = ka^2 - mgb\cos\theta$$

$$= (4\,\text{kN/m})(0.08\text{m})^2 - (10\text{kg})\left(9.81\,\text{m/s}^2\right)(0.3\text{m})\cos\theta$$

$$= 25.6 - 29.43\cos\theta$$

at $\theta = 0$: $\dfrac{d^2V}{d\theta^2} = -3.83 < 0$ $\boxed{\text{unstable}}$

at $\theta = 51.7°$: $\dfrac{d^2V}{d\theta^2} = +7.36 > 0$ $\boxed{\text{stable}}$