PH101

Lecture 9

Lagrange's equation,
Generalized forces

Recap: D'Alembert's principle of virtual work

- Started from Newton's law for a particle (say i_{th} particle) of system of particles

$$m_i\ddot{\vec{r}}_i = Total\ force = \vec{F}_{ie} + \vec{f}_{ic}$$

$$\text{I,e } \vec{F}_e + \vec{f}_{ic} - m_i\ddot{\vec{r}}_i = 0$$

- Then have taken dot product with **arbitrary virtual displacement** and summed over all particles , to remove the contribution from constrain forces

$$\sum_{i=1}^{N}(\vec{F}_{ie} - m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i + \sum_{i=1}^{N}\vec{f}_{ic}\cdot\delta\vec{r}_i = 0$$

Since total virtual work done by the all the constraint forces is zero, I,e

$$\sum_{i=1}^{N}\vec{f}_{ic}\cdot\delta\vec{r}_i = 0$$

$$\sum_{i=1}^{N}(\vec{F}_{ie} - m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i = 0$$

D'Alembert's principle of Virtual work

$\vec{F}_{ie}$ →Applied force on i_{th} particle

Recap: D'Alembert's principle of virtual work

$$\sum_{i=1}^{N} (\vec{F}_{ie} - m_i\ddot{\vec{r}}_i) \cdot \delta\vec{r}_i = 0$$

Does not necessarily means that individual terms of the summation are zero as $\vec{r}_i$ **are not independent**, they are connected by constrain relation

Want to express this relation in such a way where all the terms in the summation becomes individually zero.

How to do?

Converting this relationship in terms of generalized coordinates

Quick recap of basic mathematics

$If\ u_1\ \delta x_1 + u_2\ \delta x_2 = 0$; does this always mean $u_1 = 0$ and $u_2 = 0$?

If x_1 and x_2 are independent then $u_1 = 0$ and $u_2 = 0$ for all possible variation of x_1 and x_2,
If x_1 and x_2 are independent then you can vary one without changing other. If you fix x_1 and vary x_2, and still the relation is always giving zero, then only possibility is u_1 and u_2 must be zero.
If x_1 and x_2 are not independent, changing one will change the other.

Generalization:

$$\boldsymbol{If,} \qquad \sum \boldsymbol{u_i\ \delta\ x_i = 0,}$$

$then\ all\ u_i$ will be individually zero for all possible variation of the x_i only when x_i are **independent to each other.**

Lagrange's equation from D'Alembert's principle

- **D'Alembert's principle,**

$$\boldsymbol{\sum_{i=1}^{N}(\vec{F}_{ie} - m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i = 0}$$

Constraint forces are out of the game!

Btw, no need of additional subscript, we shall simply write $\boldsymbol{\vec{F}_i}$ instead of $\vec{F}_{ie}$

But How to express this relation so that individual terms in the summation are zero?

Switch to generalized coordinate system as they are independent

Let's take the 1st term

$$\sum_i \vec{F}_i\cdot\delta\vec{r}_i = \sum_i \vec{F}_i\cdot\sum_{j=1}^{n}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j = \sum_{j=1}^{n}\left(\sum_i \vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\right)\delta q_j = \sum_{j=1}^{n} Q_j\delta q_j$$

$$Q_j = \sum_i \vec{F}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}$$ → **Generalized force**

- Dimensions of Q_j **is not** always of force!
- Dimensions of $Q_j\delta q_j$ **is** always of work!

Lagrange's equation from D'Alembert's principle

- **2^{nd} Term =** $$\sum_i m_i \ddot{\vec{r}}_i \cdot \delta \vec{r}_i = \sum_i m_i \ddot{\vec{r}}_i \cdot \sum_{j=1}^{n} \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j = \sum_{i,j} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j$$

- **Bit of rearrangement in derivatives**

$$\ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \frac{d\dot{\vec{r}}}{dt} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}\right) - \dot{\vec{r}}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

Time and coordinate derivative can be exchanged

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{d\vec{r}_i}{dt}\right) = \left(\frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right)$$

$$= \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_i}\right) - \dot{\vec{r}}_i \cdot \left(\frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right)$$

$$\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

Dot cancellation

$$= \frac{d}{dt}\left(\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_i}\right) - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{\vec{r}}_i \cdot \dot{\vec{r}}_i\right)$$

$$= \frac{d}{dt}\left\{\frac{d}{\partial \dot{q}_j}\left(\frac{1}{2}\dot{r}_i^2\right)\right\} - \frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{r}_i^{\,2}\right)$$

Where, $\dot{r}_i^{\,2} = |\dot{\vec{r}}_i|^2 = \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$

Lagrange's equation from D'Alembert's principle

- **2nd Term =** $$\sum_i m_i\ddot{\vec{r}}_i\cdot\delta\vec{r}_i=\sum_i m_i\ddot{\vec{r}}_i\cdot\sum_{j=1}^{n}\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j=\sum_{i,j} m_i\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$$

$$\ddot{\vec{r}}_i\cdot\frac{\partial\vec{r}_i}{\partial q_j}=\frac{d}{dt}\left\{\frac{d}{\partial\dot{q}_j}\left(\frac{1}{2}\dot{r}_i^2\right)\right\}-\frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{r}_i^{\,2}\right)$$

- **Hence, 2nd term becomes**

$$\sum_i^N m_i\ddot{\vec{r}}_i\cdot\delta\vec{r}_i=\sum_{i,j} m_i\left[\frac{d}{dt}\left\{\frac{d}{\partial\dot{q}_j}\left(\frac{1}{2}\dot{r}_i^{\,2}\right)\right\}-\frac{\partial}{\partial q_j}\left(\frac{1}{2}\dot{r}_i^{\,2}\right)\right]\delta q_j$$

$$=\sum_j\left[\frac{d}{dt}\left\{\frac{d}{\partial\dot{q}_j}\left(\sum_i\frac{1}{2}m_i\dot{r}_i^{\,2}\right)\right\}-\frac{\partial}{\partial q_j}\left(\sum_i\frac{1}{2}m_i\dot{r}_i^{\,2}\right)\right]\delta q_j$$

$$=\sum_j\left\{\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{q}_j}\right)-\frac{\partial T}{\partial q_j}\right\}\delta q_j$$

- **1st term**

$$\sum_i\vec{F}_i\cdot\delta\vec{r}_i=\sum_{j=1}^{n}Q_j\delta q_j$$

Lagrange's equation from D'Alembert's principle

- D'Alembert's principle in generalized coordinates becomes

$$\sum_j \left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} \delta q_j = \sum_j Q_j \delta q_j$$

$$\sum_j \left[\left\{ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} \right\} - Q_j \right] \delta q_j = 0$$

Well, we are very close to Lagrange's equation!

Since generalized coordinates q_j are all **independent to other** and the relation is true for **all possible variation of $\boldsymbol{\delta q_j}$**, thus each term in the summation is individually zero

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j$$

$$-\left(\frac{\partial U_i}{\partial x_i}\hat{\imath} + \frac{\partial U_i}{\partial y_i}\hat{\jmath} + \frac{\partial U_i}{\partial z_i}\hat{k}\right) . \left(\frac{\partial x_i}{\partial q_j}\hat{\imath} + \frac{\partial y_i}{\partial q_j}\hat{\jmath} + \frac{\partial z_i}{\partial q_j}\hat{k}\right)$$

$$= -\left(\frac{\partial U_i}{\partial x_i}\frac{\partial x_i}{\partial q_j} + \frac{\partial U_i}{\partial y_i}\frac{\partial y_i}{\partial q_j} + \frac{\partial U_i}{\partial z_i}\frac{\partial z_i}{\partial q_j}\right)$$

- If all the forces are conservative, then $\vec{F}_i = -\vec{\nabla} U_i$

$$Q_j = \sum_i \left(-\vec{\nabla} U_i\right) \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\sum_i \frac{\partial U_i}{\partial q_j} = -\frac{\partial}{\partial q_j}\sum_i U_i = -\frac{\partial U}{\partial q_j}$$

Total potential

$$U = \sum_i U_j$$

Lagrange's equation from D'Alembert's principle

Hence,

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j = -\frac{\partial U}{\partial q_j}$$

- Assume that U **does not depend on** $\dot{\boldsymbol{q}}_{\boldsymbol{j}}$, then $\frac{\partial \boldsymbol{U}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{j}}} = \mathbf{0}$

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}(T - U)\right\} - \frac{\partial (T - U)}{\partial q_j} = 0$$

$$\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

Where

$$\boldsymbol{L(q_j, \dot{q}_j, t) = T(q_j, \dot{q}_j, t) - U(q_j, t)}$$

We have reached to Lagrange's equation from D'Alembert's principle.

Dot cancelation

$$\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

$$\vec{r}_i = \vec{r}_i(q_1, \ldots q_n, t)$$

$$d\vec{r}_i = \frac{\partial \vec{r}_i}{\partial q_1} dq_1 + \frac{\partial \vec{r}_i}{\partial q_2} dq_2 + \cdots .. + \frac{\partial \vec{r}_i}{\partial q_j} dq_j + \cdots . + \frac{\partial \vec{r}_i}{\partial q_n} dq_n + \frac{\partial \vec{r}_i}{\partial t}$$

$$\dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_1} \dot{q}_1 + \frac{\partial \vec{r}_i}{\partial q_2} \dot{q}_2 + \cdots + \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_j + \cdots + \frac{\partial \vec{r}_i}{\partial q_n} \dot{q}_n + \frac{\partial \vec{r}_i}{\partial t}$$

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$

Taking partial differentiation on both sides w.r.t $\dot{q}_j$

$$\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \vec{r}_i}{\partial q_j}(q_1, \ldots q_n; t)$$

Let's look at the dependency

Partial derivative, so differentiation only to explicit time function.

There is no $\dot{q}_j$(time derivative) dependent term. Thus, can be considered as constant during taking partial differentiation w.r.t. $\dot{q}_j$

Interchange of order of differential operators

$$\boxed{\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_j}\left(\frac{d\vec{r}_i}{dt}\right) = \left(\frac{\partial \dot{\vec{r}}_i}{\partial q_j}\right)}$$

$$\frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \vec{r}_i}{\partial q_j}(q_1, \ldots q_n; t)$$

$$\vec{r}_i = \vec{r}_i(q_1, \ldots q_n; t)$$

$$\dot{\vec{r}}_i = \frac{d\vec{r}_i}{dt} = \frac{\partial \vec{r}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \vec{r}_i}{\partial q_2}\dot{q}_2 + \cdots + \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \cdots + \frac{\partial \vec{r}_i}{\partial q_n}\dot{q}_n + \frac{\partial \vec{r}_i}{\partial t}$$

$$RHS = \frac{\partial \dot{\vec{r}}_i}{\partial q_j} = \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_1}\dot{q}_1 + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_2}\dot{q}_2 + \cdots + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_n}\dot{q}_n + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t}$$

$$LHS = \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right) = \frac{\partial}{\partial q_1}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\frac{dq_1}{dt} + \cdots + \frac{\partial}{\partial q_n}\left(\frac{\partial \vec{r}_i}{\partial q_n}\right)\frac{dq_n}{dt} + \frac{\partial}{\partial t}\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

$$= \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_1}\dot{q}_1 + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_2}\dot{q}_2 + \cdots + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial q_n}\dot{q}_n + \frac{\partial^2 \vec{r}_i}{\partial q_j \partial t} = RHS$$

$$\boxed{\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial^2 U}{\partial y \partial x}}$$

This is true for any x and y !!!
i.e., even if say, $y = t$!!!

Review of the steps we followed

- Started from Newton's law for a particle (say i_{th} particle) of system of particles

$$\boldsymbol{m_i\ddot{\vec{r}}_i = Total\ force = \vec{F}_{ie} + \vec{f}_{ic}}$$

$$\textbf{I,e } \boldsymbol{\vec{F}_e + \vec{f}_{ic} - m_i\ddot{\vec{r}}_i = 0}$$

- Then have taken dot product with arbitrary virtual displacement and summed over all particles , to remove the contribution from constrain forces

This zero

$$\boldsymbol{\sum_{i=1}^{N}(\vec{F}_{ie} - m_i\ddot{\vec{r}}_i)\cdot\delta\vec{r}_i + \sum_{i=1}^{N}\vec{f}_{ic}\cdot\delta\vec{r}_i = 0}$$

- Converted this expression in generalized coordinate system that *"every"* term of this summation is zero to get

$$\boxed{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j}$$

This is a more general expression!

- Now, with the assumptions: i) Forces are conservative, $\vec{F}_i = -\vec{\nabla}U_i$, hence $Q_j = -\frac{\partial U}{\partial q_j}$ and ii) potential does not depend on $\boldsymbol{\dot{q}_j}$, then $\frac{\partial U}{\partial \dot{q}_j} = 0$

We get back our Lagrange's eqn., $\boxed{\boldsymbol{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}}$

Discussion on generalized force

- A system may experience both conservative, non-conservative forces

i,e. $\vec{F}_i = \vec{F}_i^{\,c} + \vec{F}_i^{\,nc}$

- Hence generalized force for the system

$$Q_j = \sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \left(\vec{F}_i^{\,c} + \vec{F}_i^{\,nc}\right) \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \sum_i \vec{F}_i^{\,c} \cdot \frac{\partial \vec{r}_i}{\partial q_j} + \sum_i \vec{F}_i^{\,nc} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

$$Q_j = Q_j^{\,c} + Q_j^{\,nc}$$

$$Q_j^{\,c} = \sum_i \vec{F}_i^{\,c} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

- Generalized force corresponding to conservative part

$$Q_j^{\,nc} = \sum_i \vec{F}_i^{\,nc} \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

- Generalized force corresponding to non-conservative part

Lagrange's equation with both conservative and non-conservative force

- If system may experience both conservative, non-conservative forces

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = Q_j{}^{c} + Q_j{}^{nc}$$

- Generalized force corresponding to conservative force can be derived from potential $Q_j{}^{c} = -\frac{\partial V}{\partial q_j}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} = -\frac{\partial V}{\partial q_j} + Q_j{}^{nc}$$

$$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{q}_j}(T - V)\right\} - \frac{\partial (T - V)}{\partial q_j} = Q_j{}^{nc}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = Q_j{}^{nc}$$

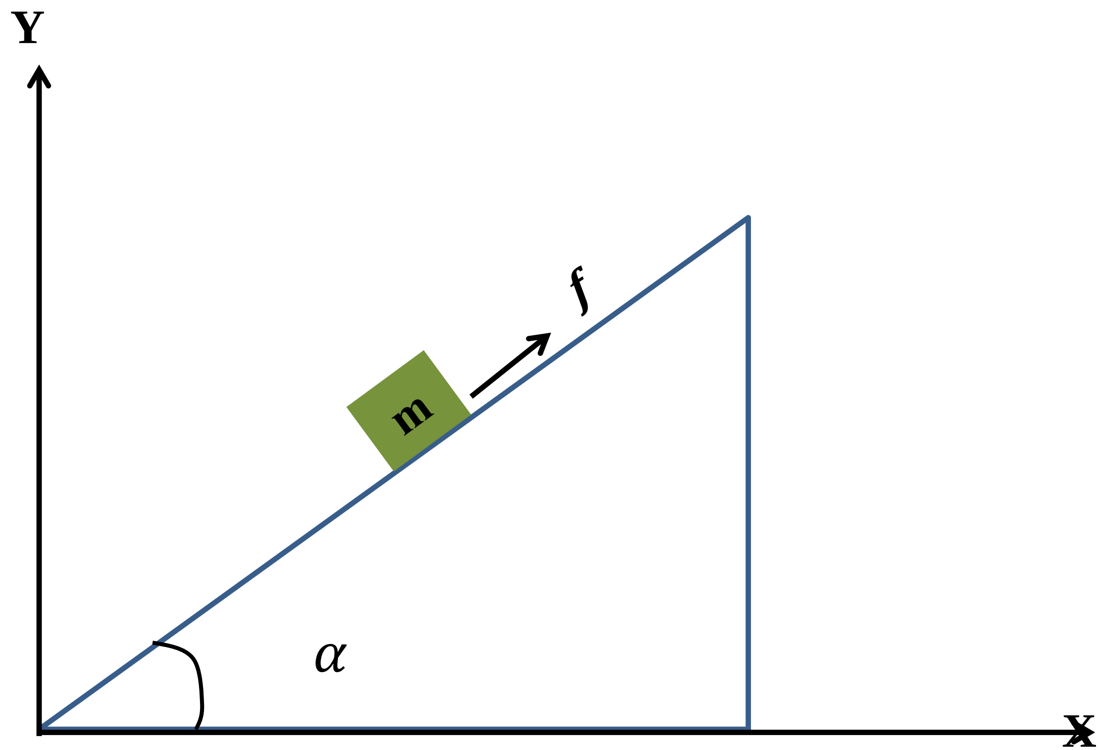
- Assume that $\boldsymbol{V}$ does not depend on $\dot{\boldsymbol{q}}_{\boldsymbol{j}}$, then $\frac{\partial \boldsymbol{V}}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{j}}} = \boldsymbol{0}$

$$L = T - V$$

Problems with generalized force

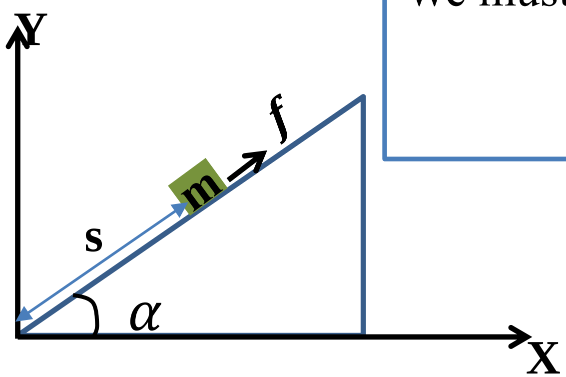
Lagrangian with non-conservative forces

A block of mass m is sliding down along the plane shown. The frictional force is $\vec{f}$ acting on the particle in the opposite direction. Obtain the Lagrange's equation of motion



Frictional force $\vec{f}$ is not conservative, thus can not be derived from scalar potential. How to incorporate this friction (non-conservative force) in the problem?

Lagrangian with non-conservative forces: Example 1



We must use the form of Lagrange's equation including Generalized force

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_j} = Q_j^{\,nc}$$

Thus all the steps we followed for conservative forces will be the same , additionally you need to calculate $Q_j^{\,nc}$

Step-1: *Find the degrees of freedom and choose suitable generalized coordinates*

Two constrain relations are $z = 0$, $y = x \tan\alpha$
Thus degrees of freedom (n)=3$\times 1 - 2 = 1$
$\boldsymbol{s}$ can serve as generalized coordinate.

Step-2: *Find out transformation relations*

$$x = s\ \cos\alpha;\ y = s\ \sin\alpha$$

Step-3: *Write T and U in Cartesian*

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2); \qquad U = mgy$$

Step-4:Convert *T and U in generalized coordinate*

$$T = \frac{1}{2}m\dot{s}^2\ ; \qquad U = mgs\ \sin\alpha$$

Step-5: *Write down Lagrangian*

$$L = T - U$$

$$L = \frac{1}{2}m\dot{s}^2 - mgs\ \sin\alpha$$

Lagrangian with non-conservative forces: Example 1

Step-6: *Find **generalized force** corresponding to non-conservative forces*

$$Q_j{}^{nc} = \sum_i \vec{F}_i{}^{nc} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \vec{F}^{nc} \cdot \frac{\partial \vec{r}}{\partial q} = \vec{f} \cdot \frac{\partial \vec{r}}{\partial s}$$

In the given problem only one non-conservative force $\vec{F}_i{}^{nc} = \vec{f}$

Now, $\vec{f} = f\hat{s} = f(\hat{x}\cos\alpha + \hat{y}\sin\alpha)$

$\vec{r} = (\hat{x}x + \hat{y}y) = (\hat{x}\, s\cos\alpha + \hat{y}\, s\sin\alpha)$; *Generalazied corodinate* $q \rightarrow s$

$$Q_j{}^{nc} = \vec{f} \cdot \frac{\partial \vec{r}}{\partial s}$$
$$= f(\hat{x}\cos\alpha + \hat{y}\sin\alpha) \cdot (\hat{x}\cos\alpha + \hat{y}\sin\alpha)$$
$$= f$$

Since, $\frac{\partial \vec{r}}{\partial s} = (\hat{x}\cos\alpha + \hat{y}\sin\alpha)$

Step-7: *Write down Lagrange's equation for each generalized coordinates*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_j} = Q_j{}^{nc} \quad ; \textit{in the given problem } q_j \rightarrow s$$

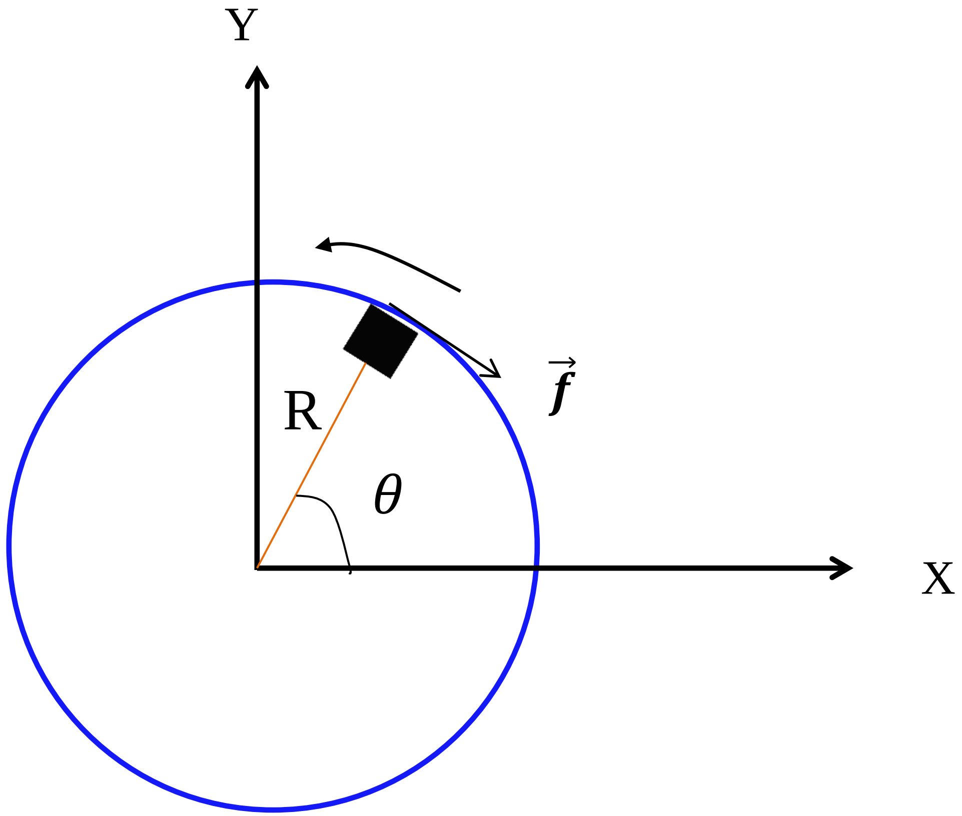
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = Q_s{}^{nc}$$

$$\frac{d}{dt}(m\dot{s}) - (-mg\ \sin\alpha) = f$$

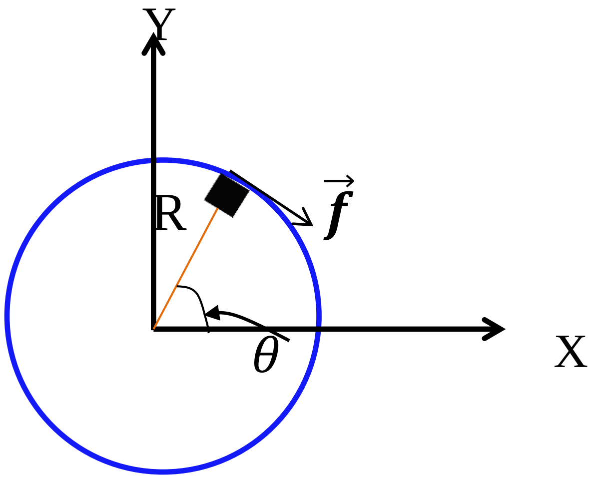
$$m\ddot{s} + mg\ \sin\alpha = f$$

Lagrangian with non-conservative forces: Example 2

A block of mass m is rotating in a circular orbit along the inner surface of ring as shown below. The frictional force is $\vec{f}$ acting on the particle in the opposite direction of its instantaneous velocity. Obtain the Lagrange's equation of motion



Lagrangian with non-conservative forces: Example 2



Step-1: *Find the degrees of freedom and choose suitable generalized coordinates*

Two constrain relations are $z = 0,\ x^2 + y^2 = R^2$
Thus degrees of freedom (n)=$3\times 1 - 2 = 1$
θ can serve as generalized coordinate.

Step-2: *Find out transformation relations*

$$x = R\cos\theta \quad ; y = R\ \sin\theta$$

Step-3: *Write T and U in Cartesian*

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2); \qquad U = mgy$$

Step-4:Convert
T and U in generalized coordinate

$$T = \frac{1}{2}mR^2\dot{\theta}^2 \quad ; U = mgR\sin\theta$$

Step-5: *Write down Lagrangian*

$$L = T - U$$

$$L = \frac{1}{2}mR^2\dot{\theta}^2 - mgR\sin\theta$$

Lagrangian with non-conservative forces: Example 2

Step-6: *Find **generalized force** corresponding to non-conservative forces*

$$Q_j{}^{nc} = \sum_i \vec{F}_i{}^{nc} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \vec{F}^{nc} \cdot \frac{\partial \vec{r}}{\partial q} = \vec{f} \cdot \frac{\partial \vec{r}}{\partial \theta}$$

In the given problem only one non-conservative force

$$\vec{F}_i{}^{nc} = \vec{f}$$

And

$$generalazied\ corodinate \quad q \rightarrow \theta$$

Now, $\vec{f} = f\hat{\theta} = f(-\hat{x}\sin\theta + \hat{y}\cos\theta)$

and, $\vec{r} = (\hat{x}\, R\cos\theta + \hat{y}\, R\sin\theta)$

$$Thus, \frac{\partial \vec{r}}{\partial \theta} = (-\hat{x}\, R\sin\theta + \hat{y}\, R\cos\theta)$$

$$Q_j{}^{nc} = \vec{f} \cdot \frac{\partial \vec{r}}{\partial \theta} = f(-\hat{x}\sin\theta + \hat{y}\cos\theta) \cdot (-\hat{x}\, R\sin\theta + \hat{y}\, R\cos\theta) = f$$

Step-7: *Write down Lagrange's equation for each generalized coordinates*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_j} = Q_j{}^{nc} \quad ; in\ the\ given\ problem\ q_j \rightarrow \theta$$

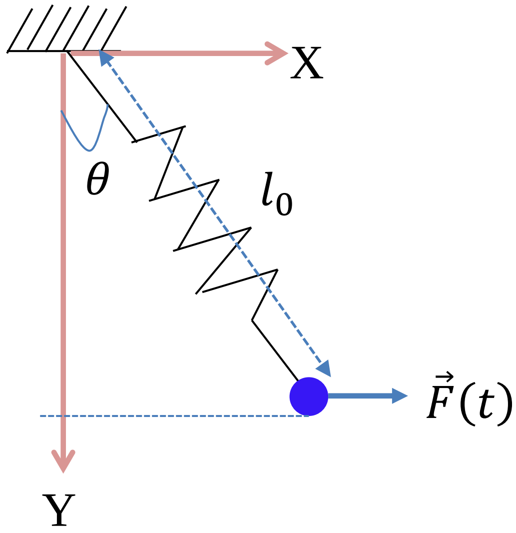
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_s{}^{nc}$$

$$\frac{d}{dt}\left(mR^2\dot{\theta}\right) - (-mgR\cos\theta\ ) = f$$

$$mR^2\ddot{\theta} + mgR\ \cos\theta = f$$

Lagrangian with non-conservative forces: Example 3

A particle of mass m is connected to the ceiling through a spring (upstretched length is $l_0 and\ spring\ constant$ C) and it is acted by a non-conservative force $\vec{F}(t)$ acting in the x-direction as shown below.

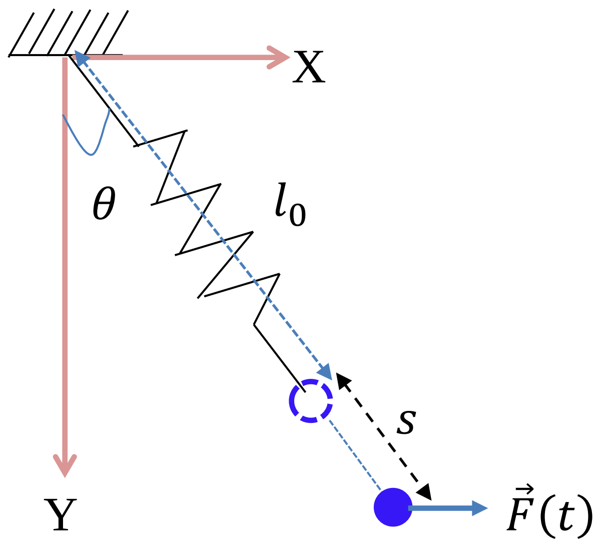


Conservative forces: (i) Wight of the particle
(ii) Restoring force due to spring
(can include within Lagrangian through their corresponding potential)

Non-conservative force: $\vec{F}(t)$

There is no potential corresponding to this force, thus needs to be included in Lagrangian formalism through generalized force

Lagrangian with non-conservative forces: Example 3



Step-1: *Find the degrees of freedom and choose suitable generalized coordinates*

$$One\ Constrain\ equation, z = 0$$

Degree's of freedom =3-1=2

Generalized coordinates:(s, θ)

$s \rightarrow$ Stretching from equilibrium

Step-2: *Find out transformation relations*

$$x = (s + l_0)\sin\theta \quad ; y = (s + l_0)\cos\theta$$

Step-3: *Write T and U in Cartesian*

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2); \qquad U = -mgy + \frac{1}{2}Cs^2$$

Second term is already expressed in terms of generalized coordinates

Step-4:Convert

T and U in generalized coordinate

$$T = \frac{1}{2}m[\dot{s}^2 + (s + l_0)^2\dot{\theta}^2; \quad U = -mg(s + l_0)\cos\theta + \frac{1}{2}Cs^2$$

Check that

$$\dot{x} = \dot{s}\sin\theta + (s + l_0)\cos\theta\,\dot{\theta}$$
$$\dot{y} = \dot{s}\cos\theta - (s + l_0)\sin\theta\,\dot{\theta}$$

Lagrangian with non-conservative forces: Example 3

Step-5: *Write down Lagrangian*

$$L = T - U$$

$$L = \frac{1}{2} m[\dot{s}^2 + (s + l_0)^2 \dot{\theta}^2] + mg(s + l_0)\cos\theta - \frac{1}{2} C s^2$$

Step-6: *Find* ***generalized force*** *corresponding to non-conservative forces*

$$Q_j{}^{nc} = \sum_i \vec{F}_i{}^{nc} \cdot \frac{\partial \vec{r}_i}{\partial q_j} = \vec{F}^{nc} \cdot \frac{\partial \vec{r}}{\partial q_j}$$

In the given problem only one non-conservative force

$$\vec{F}_i{}^{nc} = \vec{F} = F\hat{x}$$

And *generalazied corodinates* $q_1 \to s;\ \ q_2 \to \theta$

Two generalized forces corresponding to two generalized coordinates

$$Q_s{}^{nc} = \vec{F} \cdot \frac{\partial \vec{r}}{\partial s}$$

$$Q_\theta{}^{nc} = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \theta}$$

$$Now, \vec{r} = \hat{x}x + \hat{y}y$$
$$= \hat{x}(s + l_0)\sin\theta + \hat{y}(s + l_0)\cos\theta$$

$$\frac{\partial \vec{r}}{\partial s} = \hat{x}\sin\theta + \hat{y}\cos\theta$$

$$\frac{\partial \vec{r}}{\partial \theta} = \hat{x}(s + l_0)\cos\theta - \hat{y}(s + l_0)\sin\theta$$

$$Hence\ , Q_s{}^{nc} = \vec{F} \cdot \frac{\partial \vec{r}}{\partial s}$$
$$= F\hat{x} \cdot (\hat{x}\sin\theta + \hat{y}\cos\theta)$$
$$= F\sin\theta$$

$$Hence\ , Q_\theta{}^{nc} = \vec{F} \cdot \frac{\partial \vec{r}}{\partial \theta}$$
$$= F\hat{x} \cdot [\hat{x}(s + l_0)\cos\theta$$
$$- \hat{y}(s + l_0)\sin\theta] = F\,(s + l_0)\cos\theta$$

Lagrangian with non-conservative forces: Example 3

Step-7: *Write down Lagrange's equation for each generalized coordinates*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_j} = Q_j{}^{nc}$$

$$\mathit{In\ the\ given\ problem\ generalazied\ corodinates}$$
$$q_1 \to s; \quad q_2 \to \theta$$
$$\mathit{And,}$$
$$L = \frac{1}{2}m[\dot{s}^2 + (s + l_0)^2\dot{\theta}^2] + mg(s + l_0)\cos\theta - \frac{1}{2}Cs^2$$

$$\mathit{Eqn.\,corresponding\ to\ s}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{s}}\right) - \frac{\partial L}{\partial s} = Q_s{}^{nc}$$
$$\frac{d}{dt}(m\dot{s}) - [m(s + l_0)\dot{\theta}^2 + mg\cos\theta - Cs] = F\sin\theta$$
$$m\ddot{s} - (m\dot{\theta}^2 - C)s + (mg\cos\theta + ml_0\dot{\theta}^2) = F\sin\theta$$

$$\mathit{Eqn.\,corresponding\ to\ \theta}$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = Q_\theta{}^{nc}$$
$$\frac{d}{dt}\left[m(s + l_0)^2\dot{\theta}\right] - [-mg(s + l_0)\sin\theta] = F\,(s + l_0)\cos\theta$$
$$m(s + l_0)^2\ddot{\theta} + 2m(s + l_0)\dot{s}\dot{\theta} + mg(s + l_0)\sin\theta = F(s + l_0)\cos\theta$$

QUESTIONS PLEASE