PH 102, Electromagnetism,

Post Mid Semester

Lecture 2

Magnetostatics:

Application of Ampere’s law
and
Magnetic vector potential.

D. J. Griffiths: 5.3.3-5.4

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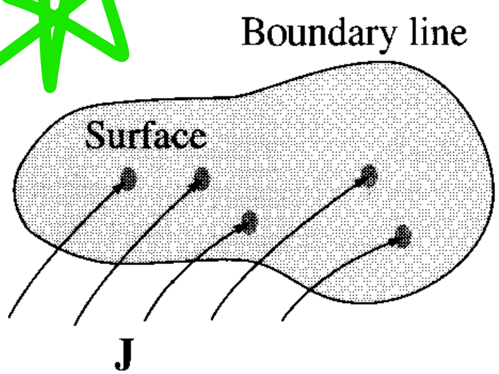
Applications of Ampere's Law

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J},}$$

Differential form

Integral form using Stokes' theorem

$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a}.$$



$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\mathrm{enc}}.}$$

Right hand rule for the direction of positive current

$$\begin{cases} \text{Electrostatics :} & \text{Coulomb} \quad \rightarrow \quad \text{Gauss,} \\ \text{Magnetostatics :} & \text{Biot} - \text{Savart} \quad \rightarrow \quad \text{Ampère.} \end{cases}$$

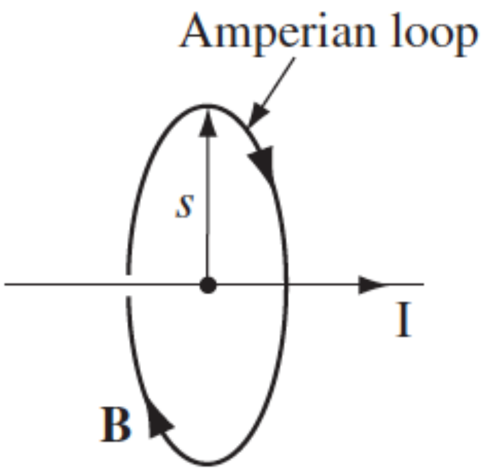
Applications of Ampere's Law

Example 5.7, D J. G

Magnetic field at a distance '***s***' from a long straight wire, carrying steady current '***I***'

$$\oint \mathbf{B} \cdot d\mathbf{l} = B \oint dl = B 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I,$$

B is circumferential and magnitude is constant around Amperian loop



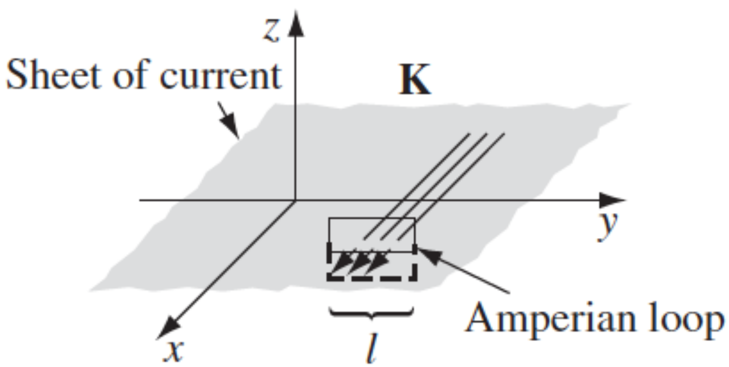
$$B = \frac{\mu_0 I}{2\pi s}.$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}},$$

Applications of Ampere's Law

Example 5.8, D J. G

Magnetic field of an infinite uniform surface current '*K*' flowing over the *xy* plane



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{K}\times\hat{r}}{r^2}da$$

B perpendicular to ***K***,
No ***x*** component of ***B***

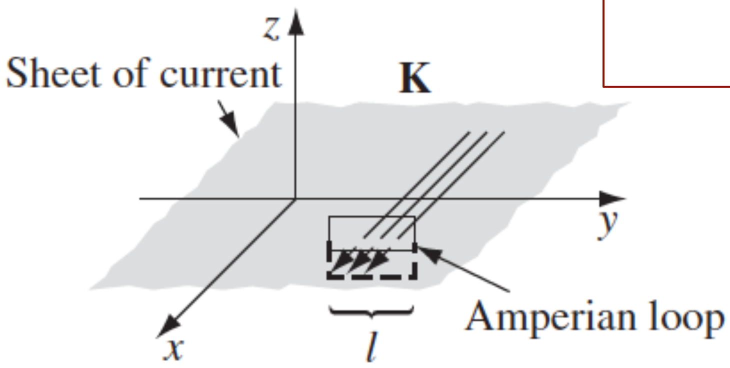
Vertical contribution to ***B*** from filament at **+*y***
canceled by corresponding filament at **– *y*,**
No ***z*** component of ***B*.**
Only **y** component

$$\oint \mathbf{B}\cdot d\mathbf{l} = 2Bl = \mu_0 I_{\text{enc}} = \mu_0 K l,$$

Applications of Ampere's Law

Example 5.8, D J. G

Magnetic field of an infinite uniform surface current '*K*' flowing over the *xy* plane



Only **y** component, +y above and –y below the plane from right hand rule

Rectangular Amperian loop, parallel to yz plane

$$\oint \mathbf{B} \cdot d\mathbf{l} = 2Bl = \mu_0 I_{enc} = \mu_0 K l,$$

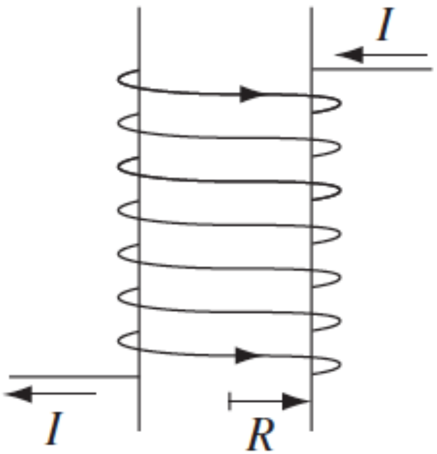
$$\mathbf{B} = \begin{cases} +(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z < 0, \\ -(\mu_0/2)K\,\hat{\mathbf{y}} & \text{for } z > 0. \end{cases}$$

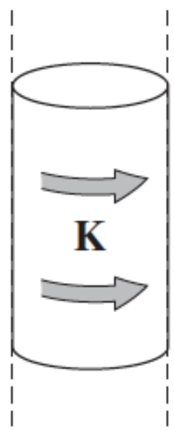
Field is independent of distance from the plane!

Applications of Ampere’s Law

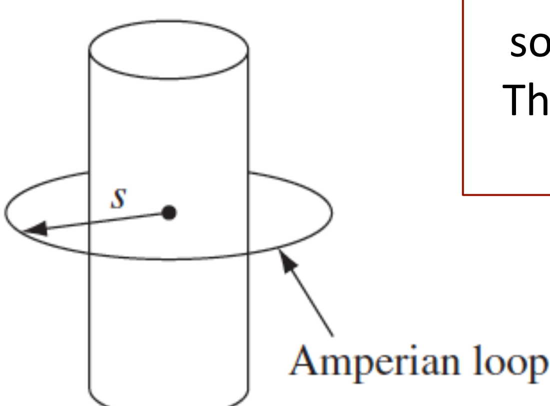
Example 5.9, D J. G

Magnetic field of a very long solenoid (Radius = ***s*** , ‘n’ turns/length) with steady current ‘***I*** ’





$\boldsymbol{B_s}$ will change sign for switching ***I*** or turning solenoid upside down. Hence, no ***radial*** component $\boldsymbol{B_s}$



$\boldsymbol{B_\varphi}$ constant around the solenoid concentric loop and The loop encloses no current, hence

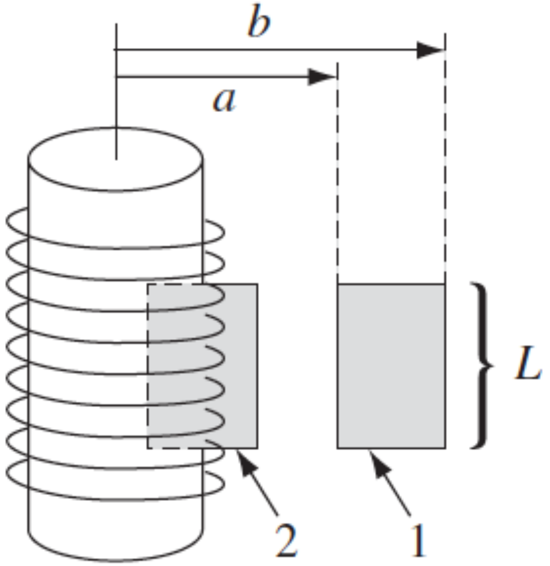
$$\oint \mathbf{B} \cdot d\mathbf{l} = B_\phi(2\pi s) = \mu_0 I_{\text{enc}} = 0,$$

$$B_\varphi = 0$$

Thus ***B*** is parallel to the axis !

Applications of Ampere's Law

From right hand rule
B upward inside solenoid
Downward outside and
zero at far away.



Amperian loops

2 Amperian loops,

Loop 1: $$\oint \mathbf{B} \cdot d\mathbf{l} = [B(a) - B(b)]L = \mu_0 I_{enc} = 0,$$

$$B(a) = B(b).$$

B = constant and zero for large ***s***
Thus zero everywhere Outside !!

Loop 2: $$\oint \mathbf{B} \cdot d\mathbf{l} = BL = \mu_0 I_{enc} = \mu_0 n I L,$$

$$\mathbf{B} = \begin{cases} \mu_0 n I\,\hat{\mathbf{z}}, & \text{inside the solenoid,} \\ \mathbf{0}, & \text{outside the solenoid.} \end{cases}$$

Strong Uniform field inside !!

Applications of Ampere's Law

Problem 5.13, D.J.G

Problem 5.13 A steady current I flows down a long cylindrical wire of radius a (Fig. 5.40). Find the magnetic field, both inside and outside the wire, if

(a) The current is uniformly distributed over the outside surface of the wire.

(b) The current is distributed in such a way that J is proportional to s, the distance from the axis.

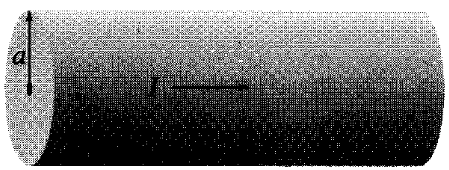
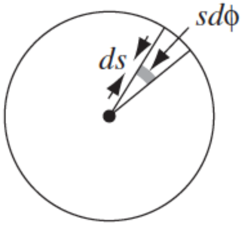


Figure 5.40

(a) $$\oint \mathbf{B} \cdot d\mathbf{l} = B \oint dl = B 2\pi s = \mu_0 I_{\text{enc}} = \mu_0 I,$$

$$\boxed{\mathbf{B} = \begin{cases} 0, & \text{for } s < a; \\ \dfrac{\mu_0 I}{2\pi s}\hat{\phi}, & \text{for } s > a. \end{cases}}$$



(b) $J = k\,s$; current $dI = J\,da_{\perp}$, and $da_{\perp} = s\,ds\,d\phi$

Total current, $$I = \int (ks)(s\,ds\,d\phi) = 2\pi k \int_0^a s^2\,ds = \frac{2\pi k a^3}{3}$$

Thus, $k = 3I / 2\pi a^3$

$$I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a},$$

$I_{enc} = I\, s^3/a^3$ for $s < a$

$I_{enc} = I$ for $s > a$

$$\boxed{\mathbf{B} = \begin{cases} \dfrac{\mu_0 I s^2}{2\pi a^3}\hat{\phi}, & \text{for } s < a; \\ \dfrac{\mu_0 I}{2\pi s}\hat{\phi}, & \text{for } s > a. \end{cases}}$$

Divergence and Curl of B

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J},}$$

$$\boxed{\nabla \cdot \mathbf{B} = 0.}$$

Applications of Ampere's Law

Ampere's Law is useful, only when symmetry enables pulling ***B*** outside integral

Otherwise

The Biot-Savart Law

Magnetic vector potential :

Maxwell's eqs for Electrostatics:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, & \text{(Gauss's law);} \\ \nabla \times \mathbf{E} = \mathbf{0}, & \text{(no name).} \end{cases}$$

Maxwell's eqs for Magnetostatics:

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0\mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

+

Force Law:

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Formulation of Electrostatics and Magnetostatics

Note:
Due to the fundamental constants, electric forces are enormously larger than the magnetic field
Then how do we observe Magnetic effects?

Magnetic vector potential :

Maxwell's eq. for Electrostatics:

$$\begin{cases} \nabla \cdot \mathbf{E} = \dfrac{1}{\epsilon_0}\rho, & \text{(Gauss's law);} \\ \nabla \times \mathbf{E} = \mathbf{0}, & \text{(no name).} \end{cases}$$

Scalar Potential

$$\nabla \times (\nabla f) = 0$$

$$\vec{E} = -\nabla V \qquad V \rightarrow V' + C \qquad \nabla C = 0$$

Maxwell's eq. for Magnetostatics:

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

The Vector Potential ($\mathbf{A}$)

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Let's transform $\quad \mathbf{A} = \mathbf{A}_0 + \nabla \lambda$

Magnetic vector potential :

Maxwell's eq. for Electrostatics:

$$\begin{cases} \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho, & \text{(Gauss's law);} \\ \nabla \times \mathbf{E} = \mathbf{0}, & \text{(no name).} \end{cases}$$

Scalar Potential

$$\nabla \times (\nabla f) = 0$$

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Maxwell's eq. for Magnetostatics:

$$\begin{cases} \nabla \cdot \mathbf{B} = 0, & \text{(no name);} \\ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, & \text{(Ampère's law).} \end{cases}$$

The Vector Potential ($\mathbf{A}$)

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

As, $\nabla \times (\nabla \lambda) = 0$, we can transform $\mathbf{A_0}$ to, $\mathbf{A} = \mathbf{A}_0 + \nabla \lambda$

Magnetic vector potential :

$$\boxed{\mathbf{B} = \nabla \times \mathbf{A}.}$$

Ampere's Law,

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

Almost, Poisson's eq. provided vector potential is Divergence-less

Claim: Always possible to make the vector potential Divergence-less !!

Magnetic vector potential :

$$\mathbf{B} = \nabla \times \mathbf{A}.$$

Curl of ***A*** is specified

$$\mathbf{A} = \mathbf{A}_0 + \nabla\lambda$$

This gives us freedom in choosing *A* to eliminate Div. A = 0

Proof : Assume Div. $\mathbf{A_0} \neq \mathbf{0}$, thus new divergence is $\nabla \cdot \mathbf{A} = \nabla \cdot \mathbf{A}_0 + \nabla^2\lambda.$

λ is such that,

$$\nabla \cdot \mathbf{A} = 0. \longrightarrow \nabla^2\lambda = -\nabla \cdot \mathbf{A}_0.$$

$$\lambda = \frac{1}{4\pi}\int \frac{\nabla \cdot \mathbf{A}_0}{\mathscr{r}}\, d\tau'.$$

provided Div. $\mathbf{A_0} = \mathbf{0}$ @ infinity.

not applicable to non zero current at ∞

Poisson's eq.

$$\nabla^2 V = -\frac{\rho}{\epsilon_0},$$

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho}{\mathscr{r}}\, d\tau'$$

ρ= 0 @ infinity

Magnetic vector potential :

Ampere's Law,

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

Choose
Div. A = 0

Ampere's Law: 3 Poisson's eqs.

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \, d\tau'.$$

Assuming ***J*** goes to zero at infinity

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\mathscr{r}} \, dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\mathscr{r}} \, d\mathbf{l}'; \quad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\mathscr{r}} \, da'.$$

Assuming current zero at infinity

Can we write a scalar potential for ***B*** ?

$$\mathbf{B} = -\nabla U,$$

Magnetic vector potential :

Ampere's Law,

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

Choose
Div. A = 0

Ampere's Law: 3 Poisson's eqs.

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\, d\tau'.$$

Assuming ***J*** goes to zero at infinity

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\mathscr{r}}\, dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\mathscr{r}}\, d\mathbf{l}'; \quad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\mathscr{r}}\, da'.$$

Assuming current zero at infinity

Can we write a scalar potential for ***B*** ?

$$\mathbf{B} = -\nabla U,$$

Only if ***J*** = 0

Magnetic vector potential :

Ampere's Law,

$$\nabla \times \mathbf{B} = \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu_0 \mathbf{J}.$$

Choose
Div. A = 0

Ampere's Law: 3 Poisson's eqs.

$$\boxed{\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J}.}$$

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}}\, d\tau'.$$

Assuming ***J*** goes to zero at infinity

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{I}}{\mathscr{r}}\, dl' = \frac{\mu_0 I}{4\pi} \int \frac{1}{\mathscr{r}}\, d\mathbf{l}'; \quad \mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\mathscr{r}}\, da'.$$

Assuming current zero at infinity

Note:

- We define ***A*** as potential, but F_{mag} doesn't work !
- What is the direction of ***A*** ? Direction of ***J***

Magnetic vector potential :

Example 5.11. A spherical shell of radius R, carrying a uniform surface charge σ, is set spinning at angular velocity $\boldsymbol{\omega}$. Find the vector potential it produces at point $\mathbf{r}$ (Fig. 5.45).

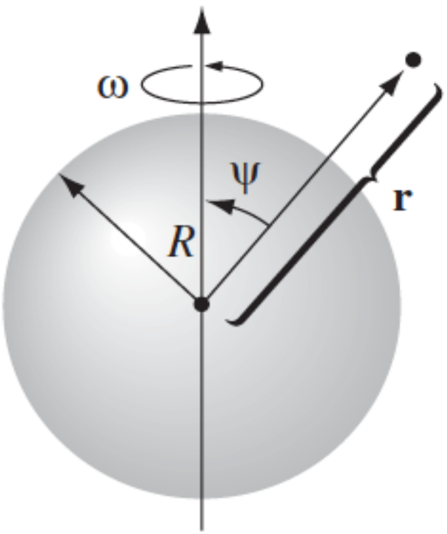


FIGURE 5.45

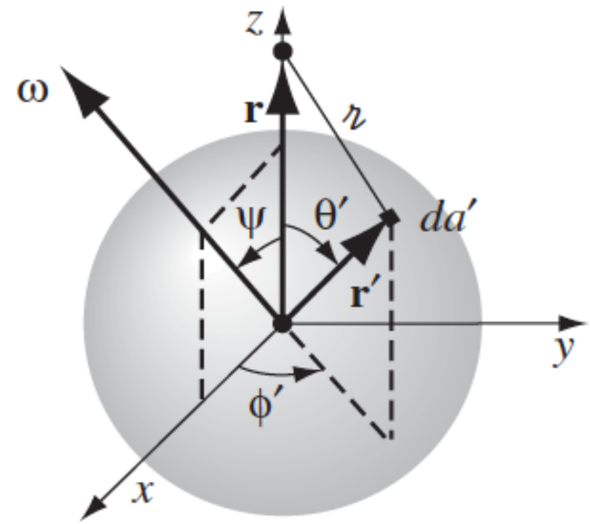


FIGURE 5.46

let $\mathbf{r}$ lie on the z axis

$\boldsymbol{\omega}$ is tilted at an angle ψ

orient the x axis

$\boldsymbol{\omega}$ lies in the xz plane

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\eta}\, da'.$$

$$\mathbf{K} = \sigma \mathbf{v},\ \eta = \sqrt{R^2 + r^2 - 2Rr\cos\theta'},$$

$$da' = R^2 \sin\theta'\, d\theta'\, d\phi'$$

Magnetic vector potential :

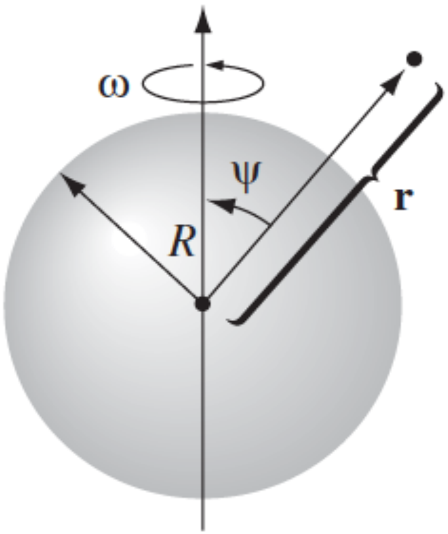


FIGURE 5.45

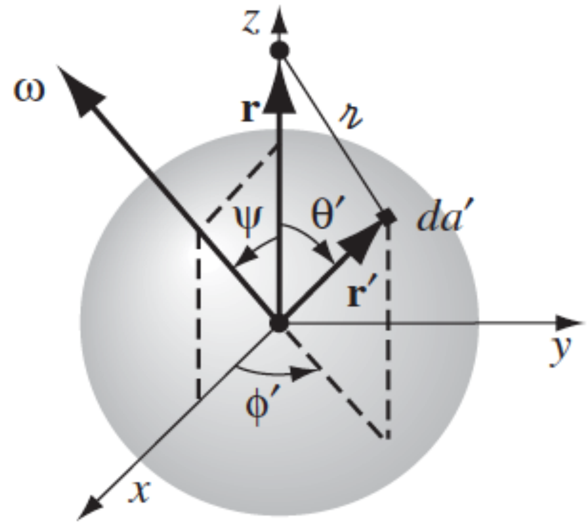


FIGURE 5.46

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}}{\mathscr{r}}\, da'.$$

$$\boxed{\mathbf{K} = \sigma \mathbf{v}}$$

velocity of a point $\mathbf{r}'$ in a rotating rigid body

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega \sin\psi & 0 & \omega\cos\psi \\ R\sin\theta'\cos\phi' & R\sin\theta'\sin\phi' & R\cos\theta' \end{vmatrix}$$

$$= R\omega\Big[-(\cos\psi\sin\theta'\sin\phi')\,\hat{\mathbf{x}} + (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta')\,\hat{\mathbf{y}}$$

$$+ (\sin\psi\sin\theta'\sin\phi')\,\hat{\mathbf{z}}\Big].$$

Magnetic vector potential :

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{\imath} \, da'. \qquad \mathbf{K} = \sigma \mathbf{v} \qquad da' = R^2 \sin\theta' \, d\theta' \, d\phi'$$

$$\mathbf{v} = \boldsymbol{\omega} \times \mathbf{r}' = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \omega \sin\psi & 0 & \omega \cos\psi \\ R \sin\theta' \cos\phi' & R \sin\theta' \sin\phi' & R \cos\theta' \end{vmatrix}$$

$$= R\omega \left[-(\cos\psi \sin\theta' \underline{\sin\phi'}) \, \hat{\mathbf{x}} + (\cos\psi \sin\theta' \underline{\cos\phi'} - \underline{\sin\psi \cos\theta'}) \, \hat{\mathbf{y}} \right.$$

$$\left. + (\sin\psi \sin\theta' \underline{\sin\phi'}) \, \hat{\mathbf{z}} \right].$$

$$\boxed{\int_0^{2\pi} \sin\phi' \, d\phi' = \int_0^{2\pi} \cos\phi' \, d\phi' = 0,}$$

Therefore,

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2} \left(\int_0^{\pi} \frac{\cos\theta' \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} \, d\theta' \right) \hat{\mathbf{y}}.$$

Magnetic vector potential :

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{\mathbf{K}}{r} \, da'.$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2} \left(\underbrace{\int_0^{\pi} \frac{\cos\theta' \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}} \, d\theta'} \right) \hat{\mathbf{y}}.$$

Letting $u \equiv \cos\theta'$,

$$\int_{-1}^{+1} \frac{u}{\sqrt{R^2 + r^2 - 2Rru}} \, du = -\frac{(R^2 + r^2 + Rru)}{3R^2 r^2} \sqrt{R^2 + r^2 - 2Rru} \, \Bigg|_{-1}^{+1}$$

$$= -\frac{1}{3R^2 r^2} \left[(R^2 + r^2 + Rr)|R - r| - (R^2 + r^2 - Rr)(R + r) \right].$$

$$= \begin{cases} (2r/3R^2) & (r \le R), \\ (2R/3r^2) & (r \ge R). \end{cases}$$

$$y = \sqrt{R^2 + r^2 - 2Rru};$$

$$\int_{-1}^{+1} \frac{u}{(R^2 + r^2 - 2Rru)^{1/2}} du$$

$$= \frac{1}{2R^2 r^2} \int_{\sqrt{R^2 + r^2 + 2Rr}}^{\sqrt{R^2 + r^2 - 2Rr}} [y^2 - (R^2 + r^2)] dy$$

Magnetic vector potential :

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}}{\mathscr{r}}\, da'.$$

$$\begin{cases} (2r/3R^2) & (r \leq R), \\ (2R/3r^2) & (r \geq R). \end{cases}$$

$$\mathbf{A}(\mathbf{r}) = -\frac{\mu_0 R^3 \sigma \omega \sin\psi}{2}\left(\int_0^{\pi} \frac{\cos\theta' \sin\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}\, d\theta'\right)\hat{\mathbf{y}}.$$

$$\boxed{(\boldsymbol{\omega} \times \mathbf{r}) = -\omega r \sin\psi\, \hat{\mathbf{y}}}$$

$$= \begin{cases} \dfrac{\mu_0 R\sigma}{3}(\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{inside} \text{ the sphere,} \\ \dfrac{\mu_0 R^4 \sigma}{3r^3}(\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{outside} \text{ the sphere.} \end{cases}$$

revert to the “natural” coordinates

Magnetic vector potential :

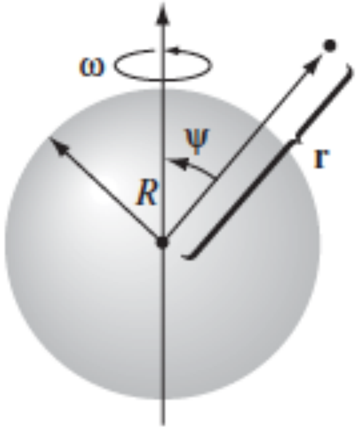


FIGURE 5.45

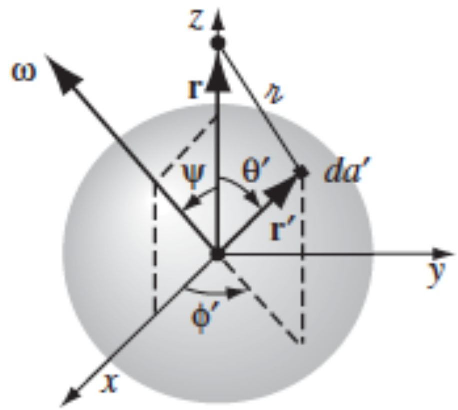


FIGURE 5.46

revert to the "natural" coordinates

$\boldsymbol{\omega}$ coincides with the z axis

$\mathbf{r}$ is at (r, θ, ϕ)

$$\mathbf{A}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0 R \sigma}{3}(\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{inside} \text{ the sphere,} \\ \dfrac{\mu_0 R^4 \sigma}{3r^3}(\boldsymbol{\omega} \times \mathbf{r}), & \text{for points } \textit{outside} \text{ the sphere.} \end{cases}$$

$$\mathbf{A}(r, \theta, \phi) = \begin{cases} \dfrac{\mu_0 R \omega \sigma}{3} r \sin\theta \, \hat{\boldsymbol{\phi}}, & (r \le R), \\ \dfrac{\mu_0 R^4 \omega \sigma}{3} \dfrac{\sin\theta}{r^2} \hat{\boldsymbol{\phi}}, & (r \ge R). \end{cases}$$

Magnetic vector potential :

field inside this spherical shell

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{2\mu_0 R\omega\sigma}{3}(\cos\theta\,\hat{\mathbf{r}} - \sin\theta\,\hat{\boldsymbol{\theta}})$$

$$= \frac{2}{3}\mu_0\sigma R\omega\,\hat{\mathbf{z}} = \frac{2}{3}\mu_0\sigma R\boldsymbol{\omega}.$$

$$\boxed{\begin{aligned}\vec{B} &= \nabla \times \vec{A}\\ &= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\; A)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rA)\hat{\theta}\end{aligned}}$$

Uniform!! ✓

Magnetic vector potential :

Example 5.12. Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

$\boxed{\text{current itself extends to infinity}}$

$$\cancel{\mathbf{A} = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}}{\mathscr{r}}\, da'.}$$

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int (\nabla \times \mathbf{A}) \cdot d\mathbf{a} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi,$$

$\boxed{\Phi \text{ is the flux of } \mathbf{B} \text{ through the loop}}$

Ampère's law in integral form

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.$$

$\mathbf{B} \rightarrow \mathbf{A}$ and $\mu_0 I_{\text{enc}} \rightarrow \Phi$

Magnetic vector potential :

Example 5.12. Find the vector potential of an infinite solenoid with n turns per unit length, radius R, and current I.

Inside:

$$\oint \mathbf{A} \cdot d\mathbf{l} = A(2\pi s) = \int \mathbf{B} \cdot d\mathbf{a} = \mu_0 n I (\pi s^2), \quad \mathbf{A} = \frac{\mu_0 n I}{2} s\,\hat{\boldsymbol{\phi}}, \quad \text{for } s \le R.$$

A is 'circumferential', ***s*** is inside the solenoid

Check! If

$$\nabla \times \mathbf{A} = \mathbf{B}$$

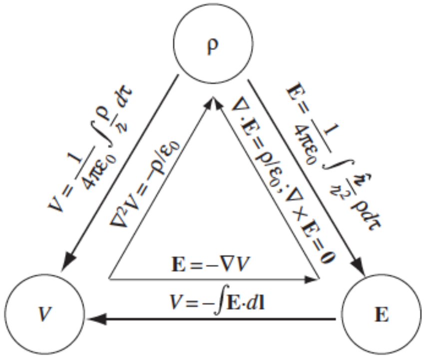
$$\nabla \cdot \mathbf{A} = 0$$

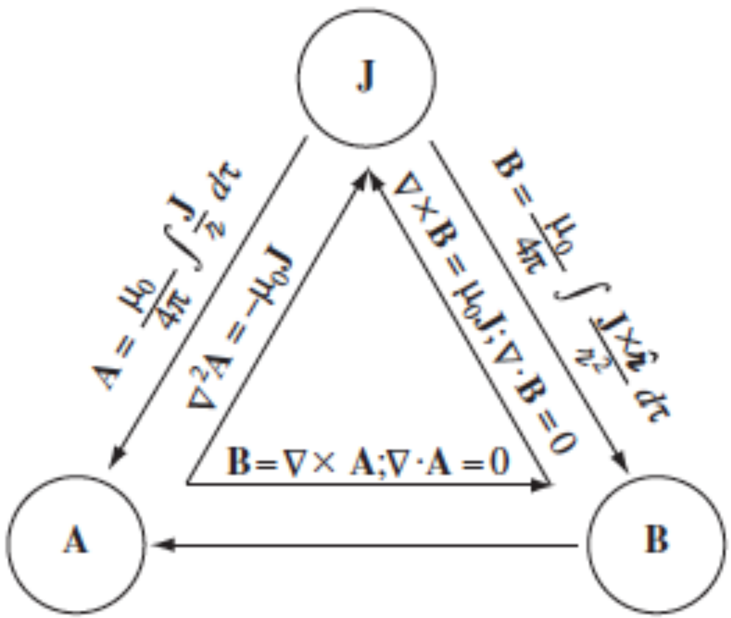
Outside:

$$\int \mathbf{B} \cdot d\mathbf{a} = \mu_0 n I (\pi R^2), \quad \mathbf{A} = \frac{\mu_0 n I}{2} \frac{R^2}{s}\,\hat{\boldsymbol{\phi}}, \quad \text{for } s \ge R.$$

field only extends out to R.

Magnetostatic Boundary Conditions :



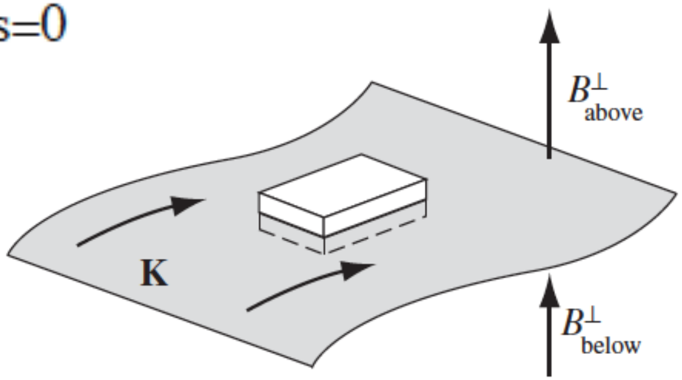


Magnetostatic Boundary Conditions :

$\boxed{\nabla \cdot \mathbf{B} = 0.}$ Integral form $\oint \mathbf{B} \cdot d\mathbf{a} = 0,$

$$\vec{B}_1 \cdot \hat{n}_1 \delta A_1 + \vec{B}_2 \cdot \hat{n}_2 \delta A_2 + \text{contribution from walls} = 0$$

$$\hat{n}_{12} \cdot \left(\vec{B}_2 - \vec{B}_1 \right) = 0$$

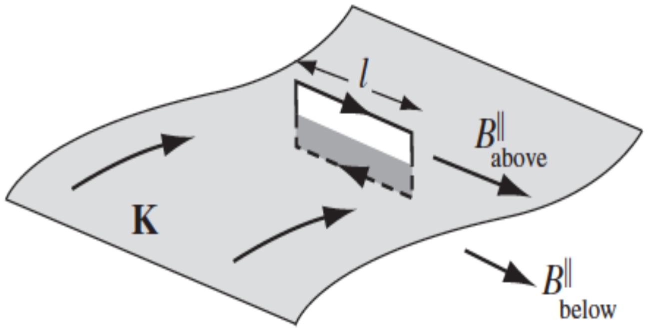


For the thin pillbox, $B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}.$

Magnetostatic Boundary Conditions :

$$B^{\perp}_{\text{above}} = B^{\perp}_{\text{below}}.$$

For the tangential component:



$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J},}$ Integral form $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}}.$

amperian loop
perpendicular to the current.

$$\oint \mathbf{B} \cdot d\mathbf{l} = \left(B^{\|}_{\text{above}} - B^{\|}_{\text{below}}\right) l = \mu_0 I_{\text{enc}} = \mu_0 K l,$$

$$B^{\|}_{\text{above}} - B^{\|}_{\text{below}} = \mu_0 K.$$

B that is parallel to the surface perpendicular to the current discontinuous in the amount $\mu_0 K$.

Summarized formula $\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}}),$

Magnetostatic Boundary Conditions :

Boundary conditions for *A* :

$$\mathbf{A}_{\text{above}} = \mathbf{A}_{\text{below}},$$

Similar to scalar potential

$\nabla \cdot \mathbf{A} = 0$ *normal* component is continuous

$\nabla \times \mathbf{A} = \mathbf{B},$

$$\oint \mathbf{A} \cdot d\mathbf{l} = \int \mathbf{B} \cdot d\mathbf{a} = \Phi,$$

tangential components are continuous

discontinuity of **B**:

$$\frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}.$$

$$(\nabla \times \vec{A})_{above} - (\nabla \times \vec{A})_{below} = \mu_0 (\vec{K} \times \hat{n})$$

Post Mid Semester

Lecture 2.

Magnetic vector potential

$$\mathbf{B} = \nabla \times \mathbf{A}. \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}')}{\mathscr{r}} \, d\tau'. \qquad \mathbf{A} = \mathbf{A}_0 + \nabla \lambda$$

Magnetostatic Boundary Conditions

$$\mathbf{B}_{\text{above}} - \mathbf{B}_{\text{below}} = \mu_0 (\mathbf{K} \times \hat{\mathbf{n}}), \qquad \frac{\partial \mathbf{A}_{\text{above}}}{\partial n} - \frac{\partial \mathbf{A}_{\text{below}}}{\partial n} = -\mu_0 \mathbf{K}.$$