DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
Even Semester of the Academic Year 2011-2012
MA 102 Mathematics II
Problem Sheet 3: Critical points, maxima and minima, Lagrange's multipliers and double integrals.
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1. Find the local maximum and minimum values and saddle point(s) of the functions:

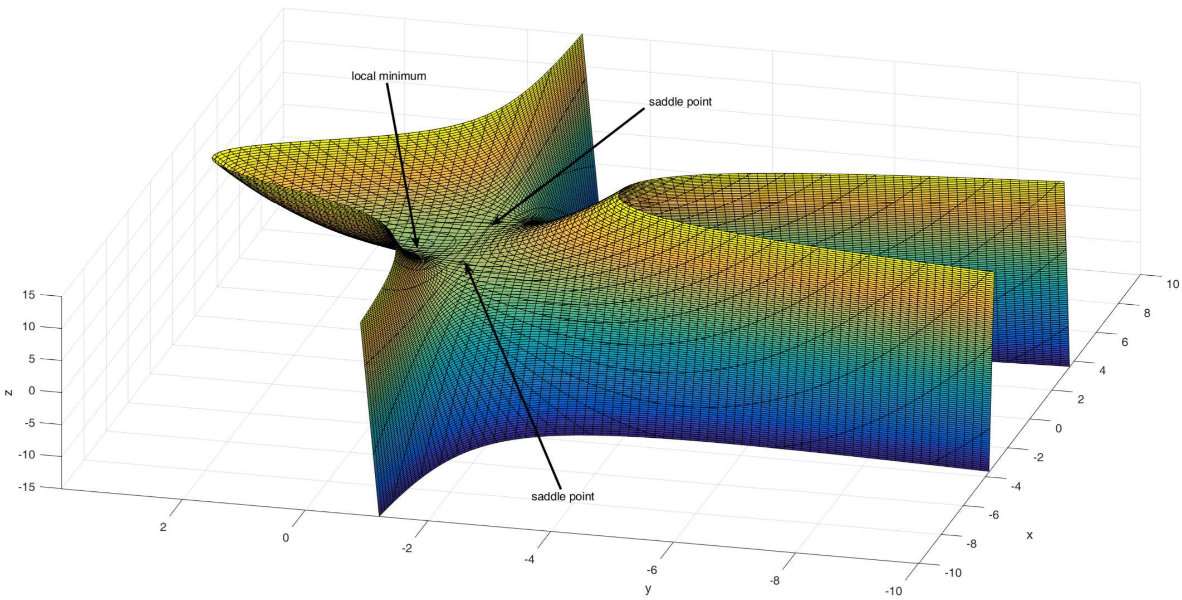
   (a) $f(x, y) = x^2 + y^2 + x^2y + 4$

   (b) $f(x, y) = 4xy - x^4 - y^4$

   (c) $f(x, y) = \sin x \cosh y$

   (d) $f(x, y) = x + 2y + \dfrac{4}{x} - y^2.$

   Solution: (a) Since $f(x, y) = x^2 + y^2 + x^2y + 4$,
   $f_x(x, y) = 2x + 2xy$ and $f_y(x, y) = 2y + x^2$.
   The critical points are given by the following equations:
   $f_x(x, y) = f_y(x, y) = 0$, which gives $x(y + 1) = 0$ and $2y = -x^2$,
   $\Rightarrow (0, 0)$ and $(\sqrt{2}, -1), (-\sqrt{2}, -1)$ are the three critical points.
   $D(x, y) = f_{xx}f_{yy} - f_{xy}^2 = (2 + 2y)(2) - (2x)^2 = 4(1 + y - x^2)$.
   At $(0, 0)$, $D(0, 0) = 4 > 0$, and $f_{xx}(0, 0) = 2 > 0$ hence $(0, 0)$ is a point of local minimum.
   Since $D(\sqrt{2}, -1) = D(-\sqrt{2}, -1) < 0$, $(\sqrt{2}, -1), (-\sqrt{2}, -1)$ are saddle points of f.



(b) Since $f(x,y) = 4xy - x^4 - y^4$,
$f_x(x,y) = 4y - 4x^3$ and $f_y(x,y) = 4x - 4y^3$.
The critical points are given by the following equations:
$f_x(x,y) = f_y(x,y) = 0$, which gives $y = x^3$ and $x = y^3$, or $x = x^9$,
$\Rightarrow x(x^4-1)(x^4+1) = 0. \qquad (1)$
The only real roots to (1) are $x = 0, +1, -1$,
$\Rightarrow$ the critical points are $(0,0), (1,1), (-1,-1)$.
$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (-12x^2)(-12y^2) - 4^2 = 144x^2y^2 - 16$.
At $(0,0)$, $D(0,0) = -16 < 0$, hence $(0,0)$ is a saddle point.
At $(1,1)$ and $(-1,-1)$, $D(1,1) = D(-1,-1) > 0$, and $f_{xx}(1,1) = f_{xx}(-1,-1) = -12$ hence $(1,1)$ and $(-1,-1)$ are both points of local maxima.

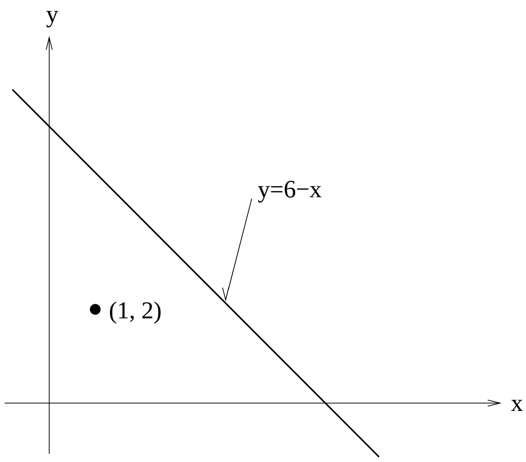
(c) $f_x(x,y) = \cos x \cosh y$ and $f_y(x,y) = \sin x \sinh y$.
The critical points are given by the following equations:
$f_x(x,y) = f_y(x,y) = 0$, which gives $x = \dfrac{(2n+1)\pi}{2}$, $y = 0$ for $n = 0, \pm 1, \pm 2, ...$
$f_{xx}(x,y) = -\sin x \cosh y$, $f_{yx}(x,y) = \cos x \sinh y$, $f_{yy}(x,y) = \sin x \cosh y$.
$D\left(\dfrac{(2n+1)\pi}{2}, 0\right) = f_{xx}(0,0)f_{yy}(0,0) - (f_{xy}(0,0))^2 < 0$, hence $\left(\dfrac{(2n+1)\pi}{2}, 0\right)$ for $n = 0, \pm 1, \pm 2$ are all saddle points of f.

(d) Since $f(x,y) = x + 2y + \dfrac{4}{x} - y^2$,
$f_x(x,y) = 1 - 4x^{-2}$ and $f_y(x,y) = 2 - 2y$, (for $x \neq 0$).
The critical points are given by the following equations:
$f_x(x,y) = f_y(x,y) = 0$, which gives $x = \pm 2$, $y = 1$.
$\Rightarrow$ the critical points are $(2,1), (-2,1)$.
$D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = (\frac{8}{x^3})(-2) - 0$,
$\Rightarrow D(2,1) < 0$ and $D(-2,1) > 0$, and $f_{xx}(-2,1) < 0$.
Hence $(2,1)$ is a saddle point and $(-2,1)$ is a point of local maximum.

2. Find the absolute maximum and minimum values of $f(x,y) = 4xy^2 - x^2y^2 - xy^3$ on the set D where D is the closed triangular region in the xy-plane with vertices $(0,0)$, $(0,6)$ and $(6,0)$.

Solution: Since $f(x,y) = 4xy^2 - x^2y^2 - xy^3$, $f_x(x,y) = 4y^2 - 2xy^2 - y^3$, $f_y(x,y) = 8xy - 2x^2y - 3xy^2$.
The critical points are given by the following equations:
$f_x(x,y) = f_y(x,y) = 0$, which gives $y^2(4 - 2x - y) = 0$ and $xy(8 - 2x - 3y) = 0$.
Hence the critical points are given by the conditions:
(i)$y = 0, x = 0$, $(0,0)$ on the boundary of D
(ii) $y = 0, 8 - 2x = 0$, which gives $(4,0)$ on the boundary of D
(iii)$4 - 2x - y = 0, x = 0$, which gives the point $(0,4)$ on the boundary of D
(iv) $4 - 2x - y = 0, 8 - 2x - 3y = 0$ which gives the point $(1,2)$, which is in the interior of the triangle D
(v) $y = 0, 4 - 2x = 0$ which gives $(2,0)$ which is on the boundary of D.

$f_{xx}(x,y) = -2y^2$, $f_{xy}(x,y) = 8y - 4xy - 3y^2$, and $f_{yy}(x,y) = 8x - 2x^2 - 6xy$.



At $(1,2)$, $D(1,2) = (-8)(8-2-12) - (16-8-12)^2 = 48 - 16 = 32 > 0$ and $f_{xx}(1,2) = -8$, hence $(1,2)$ is a point of local maxima and $f(1,2) = 4$.
On the boundary $y = 0$ of D, $f(x,y) = 0$.
On the boundary $x = 0$, $f(x,y) = 0$.
On the boundary $y = 6-x$, $f(x,y) = 4x(6-x)^2 - x^2(6-x)^2 - x(6-x)^3 = -2x(6-x)^2$.
The critical points of f on the line $y = 6 - x$ is given by:
$f_x(x, 6-x) = 0 = -6(x-2)(x-6)$,
$\Rightarrow x = 2, y = 4$, and $x = 6, y = 0$ are the only critical points of f on this line.
$f_{xx}(x, 6-x) = -6(2x-8)$, hence for $x = 2$, $f_{xx}(2,4) = 24 > 0$,
$\Rightarrow (2,4)$ is a point of local minimum on the line.
For $x = 6$, $f(6,0) = 0$.
The possible candidates for absolute maxima are $(1,2)$ and points on the line $x = 0$ and $y = 0$, but $f(1,2) = 4 > f(x,0) = f(0,y) = 0$, hence $(1,2)$ is a point of absolute maximum.
The possible candidates for absolute minima are $(2,4)$ and points on the line $x = 0$ and $y = 0$, but $f(2,4) = -64 < f(x,0) = f(0,y) = 0$, hence $(2,4)$ is a point of absolute minimum.

3. Check that for the following functions the origin is a critical point; determine whether $f(\mathbf{0})$ is a local minimum value, a local maximum value or neither.

   (a) $f(x,y,z) = 5x^2 + 4y^2 + 7z^2 + 4xy + 2z\sin x + 6y\sin z$

   (b) $f(w,x,y,z) = wx + 2xy + 3yz - w^2 - 2x^2 - 3y^2 - 4z^2$.

Solution:
(a) The conditions $f_x(x,y,z) = 10x + 4y + 2z\cos x = 0$, $f_y(x,y,z) = 8y + 4x + 6\sin z = 0$, $f_z(x,y,z) = 14z + 2\sin x + 6y\cos z = 0$, are satisfied by $(0,0,0)$, hence it is a critical point of f.
$f_{xx}(x,y,z) = 10 - 2z\sin x$, $f_{yy}(x,y,z) = 8$, $f_{zz}(x,y,z) = 14 - 6y\sin z$,
$f_{yx}(x,y,z) = 4$, $f_{zy}(x,y,z) = 6\cos z$, $f_{xz}(x,y,z) = 2\cos x$,
$f_{xx}(0,0,0) = 10$, $f_{yy}(0,0,0) = 8$, $f_{zz}(0,0,0) = 14$,
$f_{yx}(0,0,0) = 4$, $f_{zy}(0,0,0) = 6$, $f_{xz}(0,0,0) = 2$.
Hence the Hessian matrix is given by

$$H = \begin{bmatrix} 10 & 4 & 2 \\ 4 & 8 & 6 \\ 2 & 6 & 14 \end{bmatrix}.$$

Since $\triangle_1 = f_{xx}(0,0,0) = 10 > 0$, $\triangle_2 = f_{xx}(0,0,0)f_{yy}(0,0,0) - (f_{xy}(0,0,0))^2 = 64 > 0$ and $\triangle_3 = |H| = 600 > 0$, hence $(0,0,0)$ is a point of local minimum.

(b) $f(w,x,y,z) = wx + 2xy + 3yz - w^2 - 2x^2 - 3y^2 - 4z^2$

$f_w(x,y,z,w) = x - 2w$, $f_{ww}(x,y,z,w) = -2$, $f_{xw}(x,y,z,w) = 1$, $f_{yw}(x,y,z,w) = 0$, $f_{zw}(x,y,z,w) = 0$.

$f_x(x,y,z,w) = w+2y-4x$, $f_{xx}(x,y,z,w) = -4$, $f_{yx}(x,y,z,w) = 2$, $f_{zx}(x,y,z,w) = 0$

$f_y(x,y,z,w) = 2x + 3z - 6y$, $f_{yy}(x,y,z,w) = -6$, $f_{zy}(x,y,z,w) = 3$

$f_z(x,y,z,w) = 3y - 8z$, $f_{zz}(x,y,z,w) = -8$.

Since $f_w(x,y,z,w) = f_x(x,y,z,w) = f_y(x,y,z,w) = f_z(x,y,z,w) = 0$, at $(0,0,0,0)$ hence it is a critical point of f and the Hessian matrix at $(0,0,0,0)$ is given by

$$H = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 1 & -4 & 2 & 0 \\ 0 & 2 & -6 & 3 \\ 0 & 0 & 3 & -8 \end{bmatrix}.$$

Therefore $\triangle_1 = -2 < 0$, $\triangle_2 = 7 > 0$, $\triangle_3 = -34 < 0$, $\triangle_4 = 337 > 0$.

Since $(-1)^k\triangle_k > 0$ for all $k = 1,2,3,4$, $(0,0,0,0)$ is a local maximum point of f.

4. Find the points on the surface $z^2 = xy + 1$ that are closest to the origin.

Solution: The square of the distance of any point of the given surface from the origin is given by:

$f(x,y) = x^2 + y^2 + z^2 = x^2 + y^2 + xy + 1$

To find the absolute minimum of f.

The critical points are given by the following equations:

$f_x(x,y) = f_y(x,y) = 0$, which gives $2x + y = 0 = 2y + x$, the only solution to this system is $(0,0)$.

$D(0,0) = f_{xx}(0,0)f_{yy}(0,0) - (f_{xy}(0,0))^2 = 2 \times 2 - 1 = 3 > 0$ and $f_{xx}(0,0) = 2 > 0$, hence $(0,0)$ is a point of local minimum.

Since $(0,0)$ is the only critical point so $(0,0)$ is the point of absolute minimum of f.

5. Find the volume of the largest rectangular box with edges parallel to the axes that can be inscribed in the ellipsoid $9x^2 + 36y^2 + 4z^2 = 36$.

Solution: Since the ellipsoid is centered at $(0,0,0)$, clearly the rectangular box of maximum volume should also be centered at the origin and if (x,y,z) gives a corner of the rectangle in the first octant, which touches the ellipsoid then clearly the other corners are $(-x,y,z)$,$(x,-y,z)$,$(x,y,-z)$,$(x,-y,-z)$,$(-x,-y,z)$,$(-x,y,-z)$ and $(-x,-y,-z)$. Then the volume of the rectangular box is given by

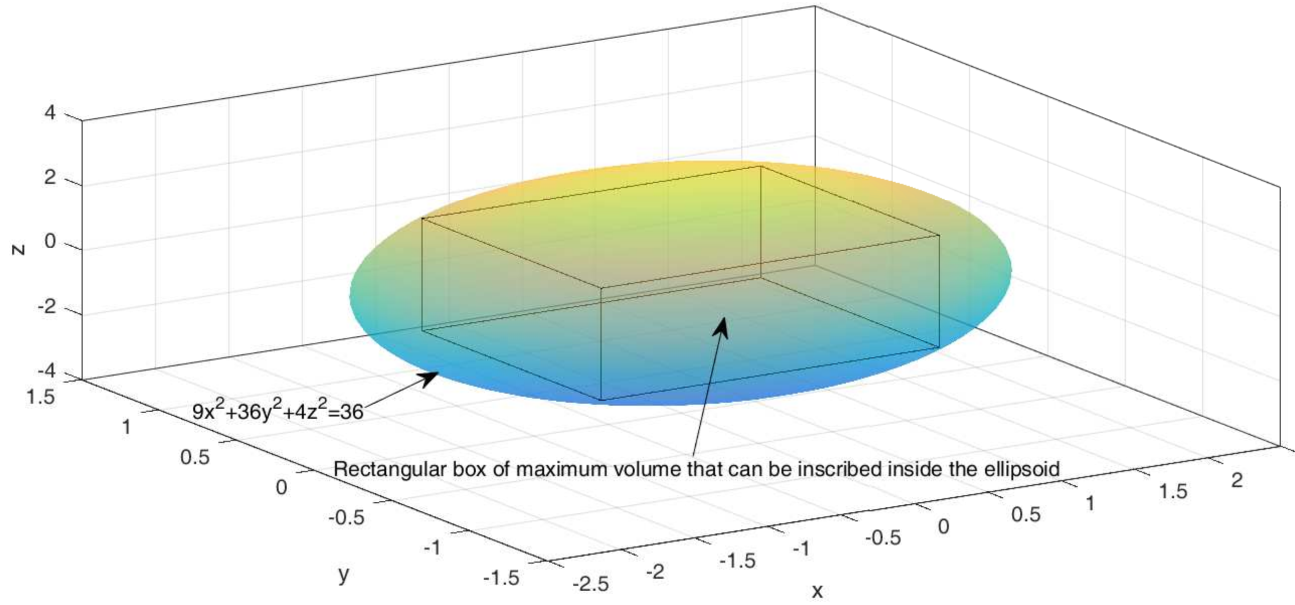
$V = 8xyz$ and the point (x,y,z) should satisfy the condition $9x^2 + 36y^2 + 4z^2 = 36$.

So the problem is to $\max 8xyz$ or $\max cx^2y^2z^2$ $(c > 0)$

subject to $4z^2 = 36 - 9x^2 - 36y^2$.

or $\max x^2y^2(36 - 9x^2 - 36y^2)$.

The critical points are given by the following conditions:



$f_x(x,y) = (2x)y^2(36-9x^2-36y^2)+x^2y^2(-18x) = 0$ or $xy^2(2-x^2-2y^2) = 0$ and $f_y(x,y) = (2y)x^2(36-9x^2-36y^2)+x^2y^2(-72y) = 0$ or $yx^2(4-x^2-8y^2) = 0$.
The critical points satisfy either of the following conditions:
(i) $x = 0$
(ii) $y = 0$
(iii) $x \neq 0$, $y \neq 0$, $2-x^2-2y^2 = 0$ and $4-x^2-8y^2 = 0$, which gives $y = \pm\frac{1}{\sqrt{3}}$ and $x = \pm\frac{2}{\sqrt{3}}$.
Since for critical points of the type (i), (ii), $V = 0$,
we need to check for solutions of type (iii) if it maximizes V.
(**) Since $D(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2 > 0$, and $f_{xx}(x,y) < 0$ at $(\pm\frac{2}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$ these points give the (x,y) coordinates of the corner points of the inscribed rectangular solid of maximum volume.
Since the z coordinates of the solid are $\pm\sqrt{3}$, and the volume of the required solid is $8 \times \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{3}} \times \sqrt{3}$.

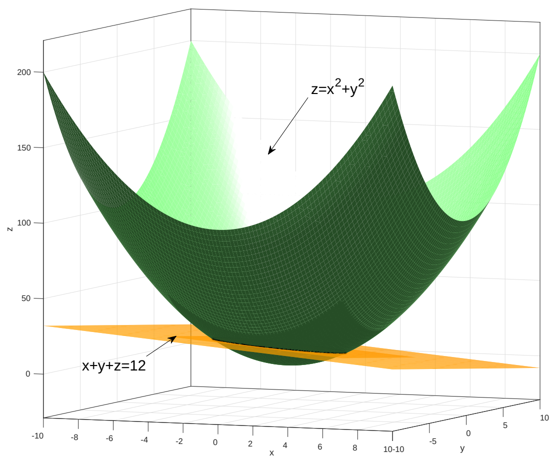
(**) **Aliter:** Since the existence of such a solid is guaranteed (maximizing a continuous function $x^2y^2z^2$ in a closed bounded region) so the critical points other than those given by $x = 0$, $y = 0$ (which gives the minimum value of $x^2y^2z^2$) must necessarily maximize $x^2y^2z^2$.

Aliter: Using lagrange's multipliers to solve the above problem we get:
$\max x^2y^2z^2$
subject to $g(x,y,z) = 4z^2 - 36 + 9x^2 + 36y^2 = 0$.
Let λ such that
$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$
$\Rightarrow (2xy^2z^2, 2yx^2z^2, 2zy^2x^2) = \lambda(18x, 72y, 8z) \qquad (1)$

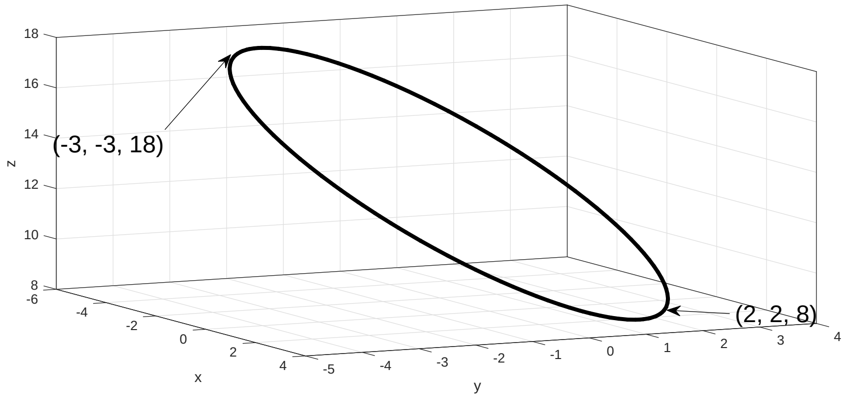
$\Rightarrow$ either $x = 0, y = 0$ or $z = 0$.
or none of $x = 0, y = 0$ $z = 0$ is satisfied, then $\lambda \neq 0$ and (1) implies:
$y^2z^2 = 9\lambda$, $x^2z^2 = 36\lambda$ and $y^2x^2 = 4\lambda$,
$\Rightarrow x^4 = 16\lambda$ or $x^2 = 4\sqrt{\lambda}$, which implies,
$y^2 = \sqrt{\lambda}$, $z^2 = 9\sqrt{\lambda}$, which when substituted in $g(x, y, z) = 0$ gives $\sqrt{\lambda} = \frac{1}{3}$.
Hence $x = \pm\frac{2}{\sqrt{3}}$, $\quad y = \pm\frac{1}{\sqrt{3}}$, $z = \pm\sqrt{3}$.

6. The plane $x + y + z = 12$ cuts the paraboloid $z = x^2 + y^2$ in an ellipse. Find the highest and lowest points on this ellipse.

Solution: The required problem is to Maximize or Minimize $f(x, y, z) = z$ subject to $g(x, y, z) = x + y + z - 12 = 0$ and $h(x, y, z) = x^2 + y^2 - z = 0$.
Using Lagrange's method, let λ and μ be such that
$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \mu \nabla h(x, y, z)$
$\Rightarrow (0, 0, 1) = \lambda(1, 1, 1) + \mu(2x, 2y, -1)$
$\Rightarrow \lambda + 2\mu x = 0, \lambda + 2\mu y = 0$ and $\lambda - \mu = 1$, which gives $2\mu(x - y) = 0$.
Since $\mu = 0$ implies $\lambda = 0$ which contradicts $\lambda - \mu = 1$, hence $x = y$.
When we substitute $y = x$ in the equation for the plane and the paraboloid we get $12 - 2x = z = 2x^2$, which gives $(x - 2)(x + 3) = 0$,
so the critical points are $(2, 2, 8)$, the lowest point and $(-3, -3, 18)$, the highest point.



(a)



(b)

7. Use Lagrange multipliers to find the global minimum and maximum values of the functions subject to the given constraint(s)

(a) $f(x, y) = 4x + 6y$; $x^2 + y^2 = 13$

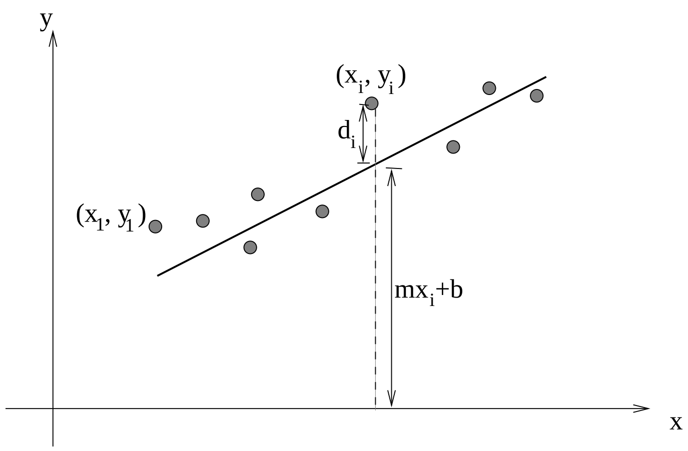
(b) $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$; $x_1^2 + x_2^2 + \cdots + x_n^2 = 1$

(c) $f(x, y) = e^{-xy}$; $x^2 + 4y^2 \leq 1$.

Solution:
(a) Maximize or Minimize $f(x, y) = 4x + 6y$ subject to $g(x, y) = x^2 + y^2 - 13 = 0$.
Let λ be such that
$\nabla f(x, y) = \lambda \nabla g(x, y)$
$\Rightarrow (4, 6) = \lambda(2x, 2y) \Rightarrow \lambda x = 2, \lambda y = 3$
$\Rightarrow 2y = 3x$, and $g(x, \frac{3}{2}x) = 13x^2 - 13 \times 4 = 0$ implies $x = 2$, $y = 3$, or $x = -2$, $y = -3$.
Since $(2, 3)$ and $(-2, -3)$ are the only two solutions of the above equations: $f(2, 3) = 26$ and $f(-2, -3) = -26$ gives the absolute minimum and maximum values of f.

(b) Maximize or Minimize $f(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n$ subject to $g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 - 1 = 0$.
Let λ be such that
$\nabla f(x_1, x_2, \ldots, x_n) = \lambda \nabla g(x_1, x_2, \ldots, x_n)$.
$(1, 1, \ldots, 1) = \lambda(2x_1, 2x_2, \ldots, 2x_n)$
$\Rightarrow x_1 = x_2 = \cdots = x_n$, and $g(x_1, x_1, \ldots, x_1) = nx_i^2 - 1 = 0$ gives
$x_i = \frac{1}{\sqrt{n}}$ for all $i = 1, 2, \ldots, n$ or $x_i = -\frac{1}{\sqrt{n}}$ for all $i = 1, 2, \ldots, n$.
The absolute maximum is attained at $(\frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}})$ and the absolute minimum is attained at $(-\frac{1}{\sqrt{n}}, -\frac{1}{\sqrt{n}}, \ldots, -\frac{1}{\sqrt{n}})$.

(c) Maximize or Minimize $f(x, y) = e^{-xy}$ subject to $x^2 + 4y^2 \leq 1$.
The conditions for critical points inside the ellipse $x^2 + 4y^2 = 1$ are given by:
$f_x(x, y) = 0$ and $f_y(x, y) = 0$ which gives $-ye^{-xy} = 0$ and $-xe^{-xy} = 0$,
$\Rightarrow x = 0$ and $y = 0$, which lies inside the ellipse.
$D(0, 0) = f_{xx}(0, 0)f_{yy}(0, 0) - f_{xy}(0, 0)^2 = -e^0 = -1 < 0$, hence $(0, 0)$ is a saddle point of f.
For points on the boundary of the ellipse $x^2 + 4y^2 = 1$, we use Lagrange's method.
Let λ be such that
$\nabla f(x, y) = \lambda \nabla g(x, y)$, where $g(x, y) = x^2 + 4y^2 - 1$.
$\Rightarrow (-ye^{-xy}, -xe^{-xy}) = \lambda(2x, 8y) \qquad (1)$
If $x = 0$ then $y = 0$, however $(0, 0)$ does not satisfy $x^2 + 4y^2 = 1$ (also $(0, 0)$ is a saddle point as noted earlier).
If $x \neq 0$ then $y \neq 0$ and we get $\frac{y}{2x} = \frac{x}{8y}$ or $x^2 = 4y^2$, which gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$.
Hence the four possible solutions of (1) are $(x, y) = (\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}})$.



Since $f(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}) = f(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}) = e^{-\frac{1}{4}} < f(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}}) = f(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}}) = e^{\frac{1}{4}}$.
$(\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}})$ and $(-\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}})$ are absolute minima of f and
$(-\frac{1}{\sqrt{2}}, \frac{1}{2\sqrt{2}})$ and $(\frac{1}{\sqrt{2}}, -\frac{1}{2\sqrt{2}})$ are absolute maxima of f.

8. Suppose that a scientist has reason to believe that two quantities x and y are related linearly, that is, $y = mx + b$, at least approximately for some values of m and b. The scientist performs an experiment and collects data in the form of points (x_1, y_1), (x_2, y_2), $\ldots$, (x_n, y_n), and then plots these points. The points don't exactly lie on a straight line, so the scientist wants to find constants m and b such that the line $y = mx + b$ "fits" the points as well as possible. Let $d_i = y_i - (mx_i + b)$ be the vertical deviation of the point (x_i, y_i) from the line. The **method of least squares** determines m and b so as to minimize $\sum_{i=1}^{n} d_i^2$, the sum of the squares of these deviations. Show that according to this method, the line of best fit is obtained when

$$m\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i$$

$$m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i.$$

Solution: To find m and b such that $f(m, b) = \sum_{i=1}^{n} d_i^2$ is minimum, we have to find the critical points of f
$f_m(m, b) = 0$ gives $\sum_{i=1}^{n} d_i(-x_i) = 0$ (1)
and $f_b(m, b) = 0$ gives $\sum_{i=1}^{n} d_i = 0$ (2)
From (2) we get

$$m\sum_{i=1}^{n} x_i + bn = \sum_{i=1}^{n} y_i.$$

From (1) we get

$$m\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i y_i.$$

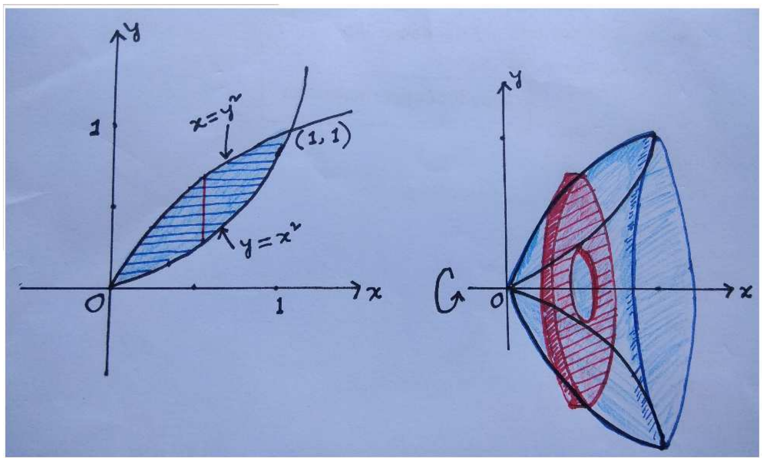
9. Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line.

(a) $y = x^2$, $y^2 = x$; about x-axis

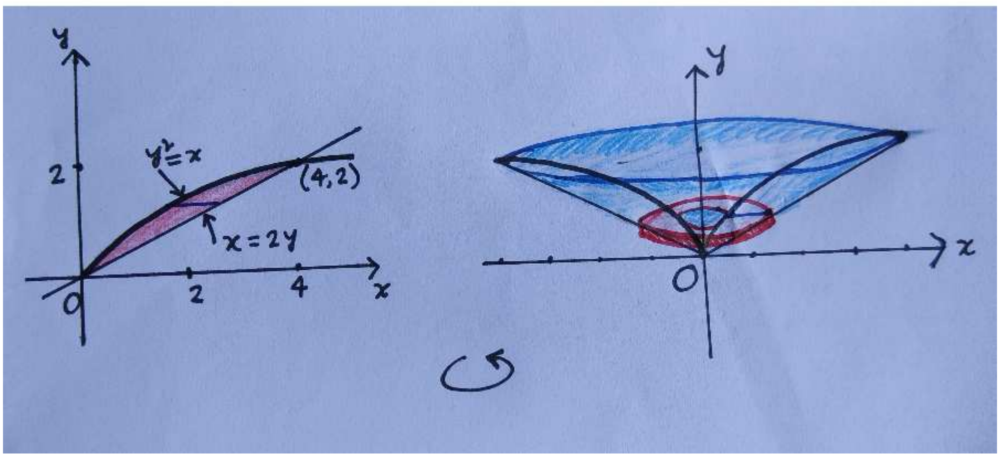
(b) $y^2 = x$, $x = 2y$; about y-axis

(c) $y = x$, $y = x^2$; about the line $x = -1$.

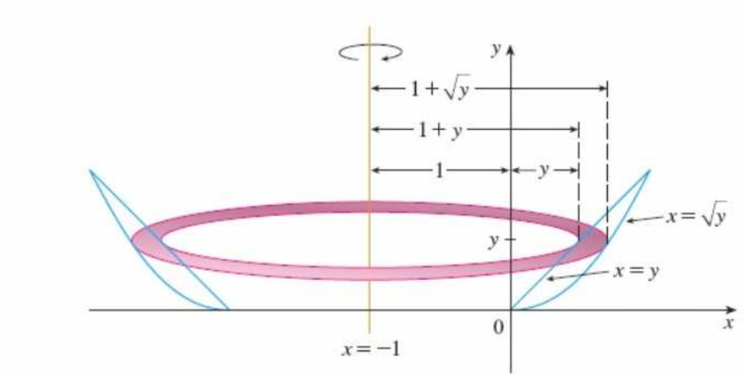
Solution:
(a) For a given x the area of a cross section of the solid obtained is given by
$A(x) = \pi(\sqrt{x})^2 - \pi(x^2)^2$.
Hence the volume of the solid is given by
$V = \int_0^1 A(x)dx = \frac{3\pi}{10}$.



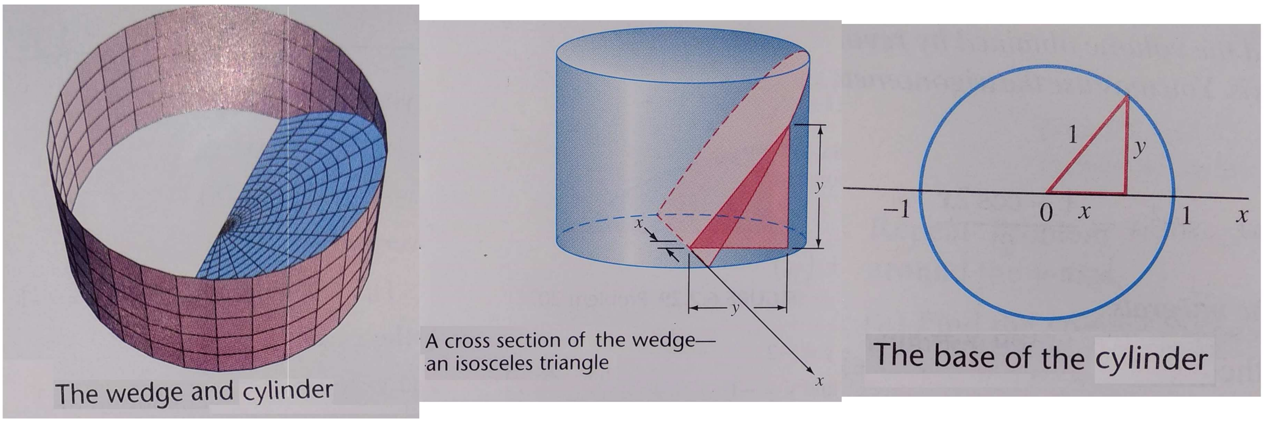
(b) For a given y the area of a cross section of the solid obtained is given by
$A(y) = \pi(2y)^2 - \pi(y^2)^2$.
Hence the volume of the solid is given by
$V = \int_0^2 A(y)dy = \frac{64\pi}{15}$.



(c) For a given y the area of a cross section of the solid obtained is given by
$A(y) = \pi(\sqrt{y}+1)^2 - \pi(y+1)^2$.
Hence the volume of the solid is given by
$V = \int_0^1 A(y)dy = \frac{\pi}{2}$.



10. Find the volume of the wedge that is cut from a circular cylinder with unit radius and unit height by a plane that passes through a diameter of the base of the cylinder and through a point on the circumference of its top.



Solution: We show the cylinder and the wedge in the figure above.

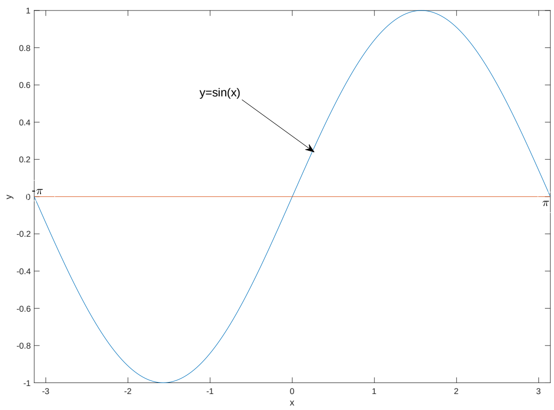
> To form such a wedge, you may fill a cylindrical glass with water and then drink slowly, tipping the bottom up as you drink, until the half of the bottom of the glass is exposed; the remaining water forms the edge.

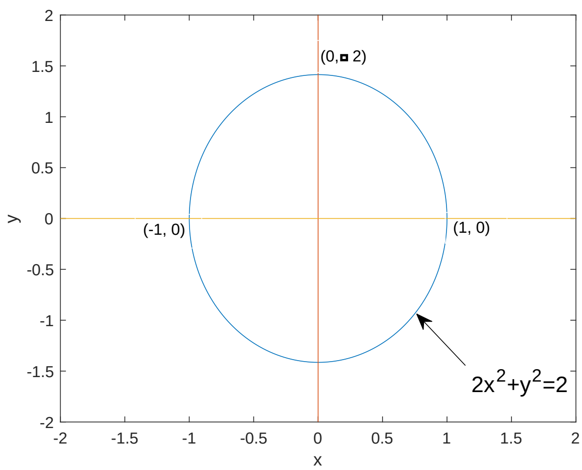
Without loss of generality let us assume the plane to cut the base of the cylinder along the x axis, with the centre of the cylinder at the origin (see the second figure in the sequence).
Then for a fixed x the cross section of the wedge is an isosceles triangle, since the plane intersects the base of the cylinder at a fixed angle, and at $x = 0$, the height and the base (given by y) are both equal to 1 (see the third figure in the sequence).

The area of each of these triangles is given by:
$A(x) = \frac{1}{2}y \times y = \frac{1}{2}(1 - x^2)$.
Hence the required volume is
$V = \int_{-1}^{1} A(x)dx = \int_{-1}^{1} \frac{1}{2}(1 - x^2)dx = \frac{2}{3}$.

11. Prove that the length of one arch of the sine curve $y = \sin x$ is equal to half the circumference of the ellipse $2x^2 + y^2 = 2$.

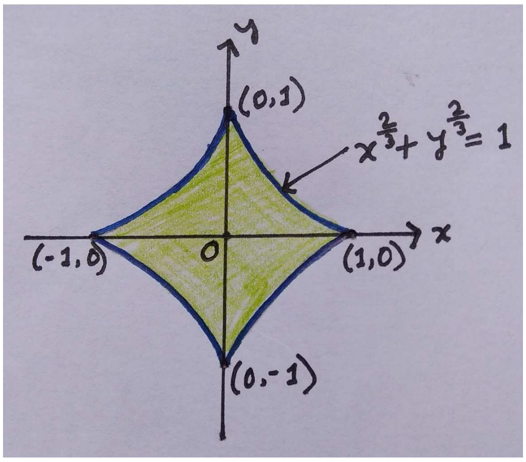
Solution: The length of one arch of the sine curve is given by:
$$L_1 = \int_0^{\pi} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$$
$$= \int_0^{\pi} \sqrt{1 + (\cos x)^2} dx.$$





The length of half the circumference of the ellipse is given by:
$$L_2 = \int_{-1}^{+1} \sqrt{1 + \left(\frac{2x}{\sqrt{2 - 2x^2}}\right)^2} dx$$
$$= \int_{-1}^{+1} \sqrt{1 + \left(\frac{2x}{\sqrt{2 - 2x^2}}\right)^2} dx$$
$$= \int_{-1}^{+1} \sqrt{\frac{1 + x^2}{1 - x^2}} dx.$$
Take $x = \cos t$ then the value of the above integral is equal to:
$$\int_{\pi}^{0} \sqrt{\frac{1 + (\cos t)^2}{(\sin t)^2}} (-\sin t)dt$$
$$= \int_0^{\pi} \sqrt{1 + (\cos t)^2} dt,$$
hence $L_1 = L_2$.

12. Find the total length of the asteroid given by $x^{\frac{2}{3}} + y^{\frac{2}{3}} = 1$ and then find the area of the surface generated by revolving the asteroid around the y-axis.



Solution: By implicit differentiation:

$$\frac{2}{3}x^{-\frac{1}{3}} + \frac{2}{3}y^{-\frac{1}{3}}\frac{dy}{dx} = 0, \text{ provided } x \neq 0 \text{ and } y \neq 0.$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{y}{x}\right)^{\frac{1}{3}} \qquad \left(\frac{dx}{dy} = -\left(\frac{x}{y}\right)^{\frac{1}{3}}\right).$$

The required length of the asteroid is given by:

$$L = 4\int_0^1 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \qquad \left(= 4\int_0^1 \sqrt{1+\left(\frac{dx}{dy}\right)^2}\,dy\right)$$

$$L = 4\int_0^1 \sqrt{1+\left(-\left(\frac{y}{x}\right)^{\frac{1}{3}}\right)^2}\,dx$$

$$= 4\int_0^1 x^{-\frac{1}{3}}dx = 6.$$

As the asteroid is symmetric about the origin we can assume $y \geq 0$, then the required surface area is given by:

$$S = 2\int_0^1 2\pi x ds = 4\pi\int_0^1 x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$

$$= 4\pi\int_0^1 x\sqrt{1+\left(-\left(\frac{y}{x}\right)^{\frac{1}{3}}\right)^2}\,dx$$

$$= 4\pi\int_0^1 x \times x^{-\frac{1}{3}}dx$$

$$= 4\pi\int_0^1 x^{\frac{2}{3}}dx$$

$$= 4\pi \times \frac{3}{5}\left[x^{\frac{5}{3}}\right]_0^1 = \frac{12\pi}{5}.$$

Aliter: $S = 2\int_0^1 2\pi x ds = 4\pi \int_0^1 (1-y^{\frac{2}{3}})^{\frac{3}{2}} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

$= 4\pi \int_0^1 (1-y^{\frac{2}{3}})^{\frac{3}{2}} y^{-\frac{1}{3}} dy$

$= 4\pi \int_{\frac{\pi}{2}}^0 (\sin\theta)^3(-3\cos\theta\sin\theta)d\theta \quad$ (by taking $y^{\frac{2}{3}} = \cos^2\theta$).

$= 12\pi \int_0^{\frac{\pi}{2}} (\sin\theta)^4(\cos\theta)d\theta = \dfrac{12\pi}{5}.$