Some commonly used surfaces

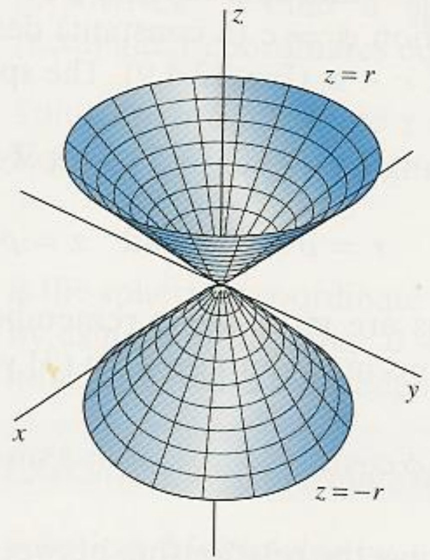


FIGURE 12.4.4 The cone $z^2 = r^2$.

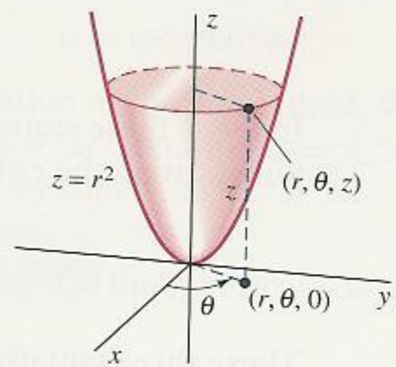


FIGURE 12.4.5 The paraboloid $z = r^2$.

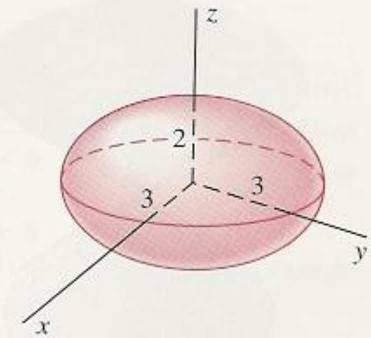
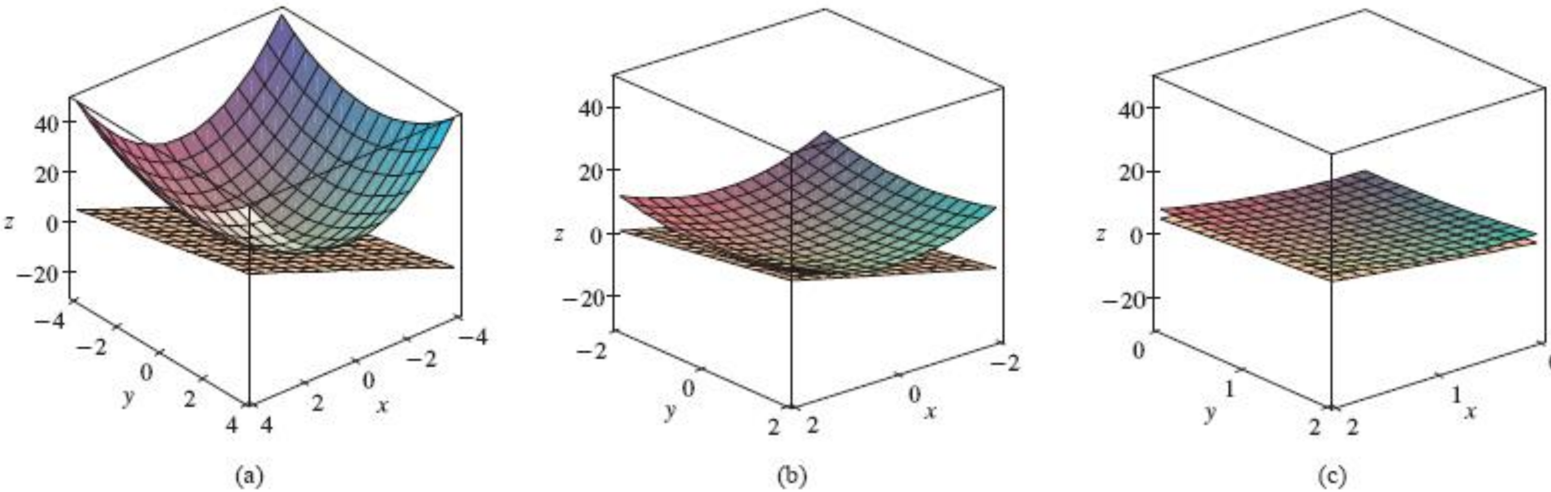


FIGURE 12.4.6 The ellipsoid $\frac{r^2}{9} + \frac{z^2}{4} = 1$.

Linear approximation

V EXAMPLE 1 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point (1, 1, 3).



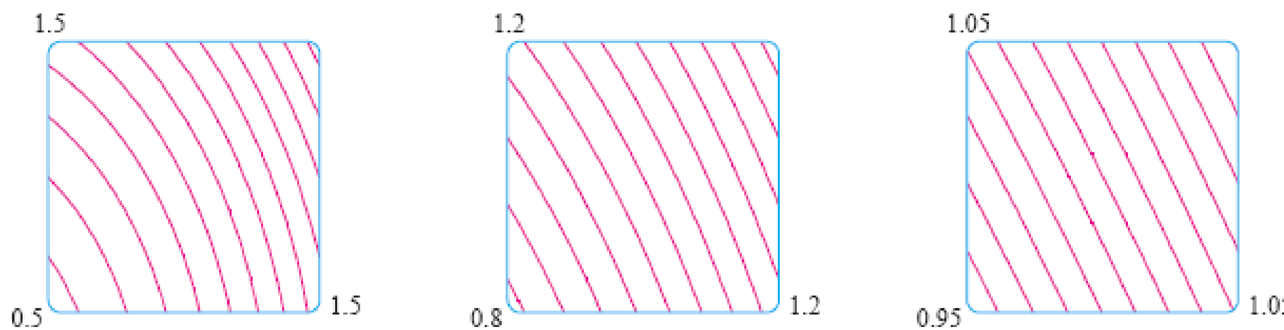


FIGURE 3
Zooming in toward (1, 1)
on a contour map of
$f(x, y) = 2x^2 + y^2$

Equation of the tangent plane to the graph of the function $f(x, y) = 2x^2 + y^2$ at the point $(1, 1, 3)$ is $z = 4x + 2y - 3$.

The linear function of two variables $\boxed{L(x, y) = 4x + 2y - 3}$ is a good approximation to $f(x, y)$ when (x, y) is near $(1, 1)$. The function L is called the *linearization* of f at $(1, 1)$ and the approximation

$$\boxed{f(x, y) \approx 4x + 2y - 3}$$

is called the *linear approximation* or *tangent plane approximation* of f at $(1, 1)$.

$$f(1.1, 0.95) \approx 4(1.1) + 2(0.95) - 3 = 3.3$$
$$f(1.1, 0.95) = 2(1.1)^2 + (0.95)^2 = 3.3225.$$

$$L(2, 3) = 11$$
$$f(2, 3) = 17.$$

What does this suggest?

In general, we know from $\boxed{2}$ that an equation of the tangent plane to the graph of a function f of two variables at the point $(a, b, f(a, b))$ is

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The linear function whose graph is this tangent plane, namely

$$\boxed{3} \qquad L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linearization** of f at (a, b) and the approximation

$$\boxed{4} \qquad f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).

What if f is not continuous at (a, b)?

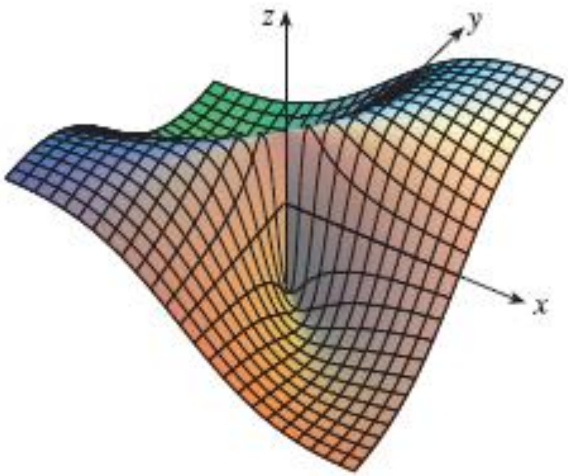


FIGURE 4

$f(x, y) = \dfrac{xy}{x^2 + y^2}$ if $(x, y) \neq (0, 0)$,
$f(0, 0) = 0$

- What is the value of the first order partial derivatives at (0,0)?

- What about the linear approximation thereat?
- What about the value of the function along y=x?

> **7 Definition** If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form
>
> $$\Delta z = f_x(a, b)\,\Delta x + f_y(a, b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$$
>
> where ε_1 and $\varepsilon_2 \to 0$ as $(\Delta x,\ \Delta y) \to (0, 0)$.

where $\quad \Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$

Theorem: If *f* is differentiable, it is continuous

What do the following examples suggest?

Example Show that the function defined by $f(x,y) = (x^{\frac{1}{3}} + y^{\frac{1}{3}})^3$ is continuous and has partial derivatives at the origin but not differentiable thereat.

Example Show that the function defined by

$$f(x,y) = \begin{cases} x^2 \sin(\frac{1}{x}) + y^2 \sin(\frac{1}{y}) & \text{if} \quad xy \neq 0 \\ x^2 \sin(\frac{1}{x}) & \text{if} \quad x \neq 0, \quad y = 0 \\ y^2 \sin(\frac{1}{y}) & \text{if} \quad x = 0, \quad y \neq 0 \\ 0 & \text{if} \quad x = y = 0 \end{cases}$$

is differentiable at the origin, but not continuously differentiable thereat because $f_x(x,y)$ is not continuous there.

Why, only at specific points we calculate partial derivatives from definition?

Definition (in V_n) $\longrightarrow$ A function $f(\mathbf{x})$ is said to be differentiable at a point $\mathbf{a}$ if

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{f(\mathbf{a}+\mathbf{h}) - f(\mathbf{a}) - \mathbf{C}\cdot\mathbf{h}}{|\mathbf{h}|} = 0$$

8 Theorem If the partial derivatives f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

Example: Show that the function f, where

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0 \end{cases}$$

is differentiable at the origin.

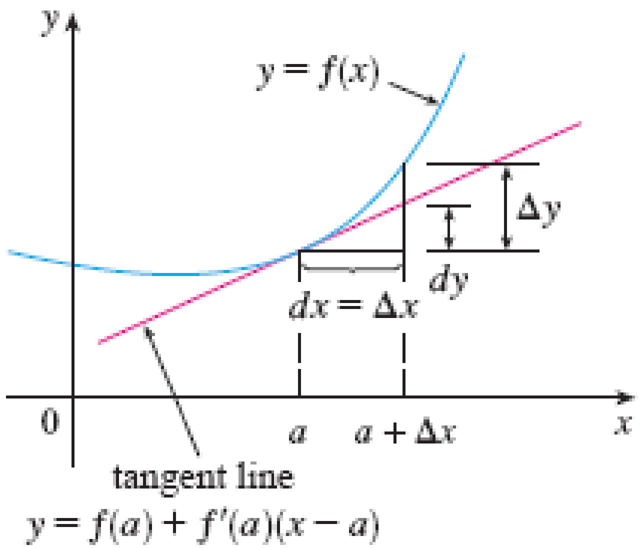
Example: Consider the function f, where

$$f(x,y) = \begin{cases} x^2 \sin\frac{1}{x} + y^2 \sin\frac{1}{y} & \text{if } xy \neq 0 \\ x^2 \sin\frac{1}{x} & \text{if } x \neq 0, \quad y = 0 \\ y^2 \sin\frac{1}{y} & \text{if } x = 0, \quad y \neq 0 \\ 0, & \text{if } x = y = 0 \end{cases}$$

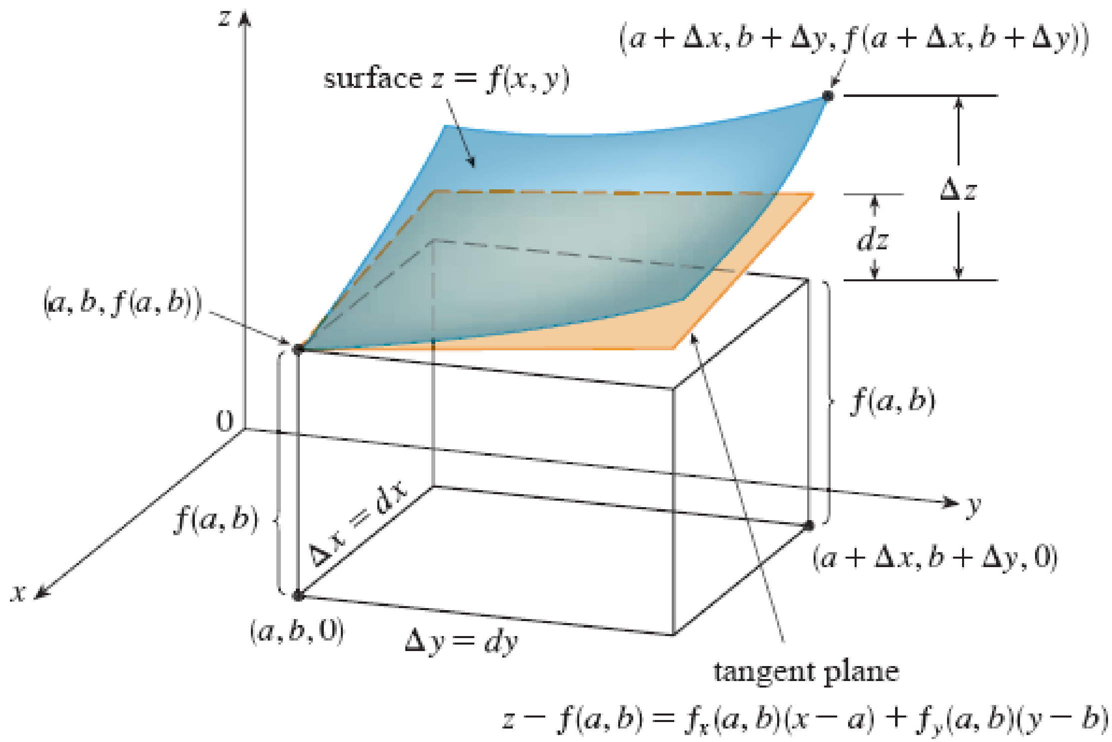
Is the function continuously differentiable at the origin? What about its differentiability thereat?

- If f is continuously differentiable, then f is differentiable
- If f is differentiable, then all partial derivatives of f exist
- If f is differentiable, it is continuous

Differential: Function of one variable



Differential: Function of two variables



$$\boxed{dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy}$$