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MA 101: Mathematics I
Sequence of real numbers
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We know that $\sqrt{2}$ is not a rational number, but we can find rational numbers as close as we wish to $\sqrt{2}$. For instance, the sequence of rationals

$$1.4, 1.41, 1.414, 1.4142, 1.41421, \ldots$$

seem to get closer and closer to $\sqrt{2}$, as their squares indicate:

$$1.96, 1.9881, 1.99396, 1.99996164, 1.9999899241, \ldots.$$

Thus it seems that we can create a square root of 2 by taking a "*limit*" of a sequence of rational numbers. This is how we shall construct the reals.

Definition 1 (Sequence)**.** *A sequence of real numbers or a sequence in* $\mathbb{R}$ *is a mapping* $f : \mathbb{N} \to \mathbb{R}$*. We write* x_n *for* $f(n), n \in \mathbb{N}$ *and it is customary to denote a sequence as* $\langle x_n \rangle$ *or* (x_n) *or* $\{x_n\}$*.*

Example 1. *There are different ways of expressing a sequence. For example:*

(1) Constant sequence: $(a, a, a, \ldots)$, where $a \in \mathbb{R}$

(2) Sequence defined by listing: $(1, 4, 8, 11, 52, \ldots)$

(3) Sequence defined by rule: (x_n), where $x_n = 3n^2$ for all $n \in \mathbb{N}$

(4) Sequence defined recursively: (x_n), where $x_1 = 4$ and $x_{n+1} = 2x_n - 5$ for all $n \in \mathbb{N}$

Convergence: What does it mean?

Think of the examples:

1. $(2, 2, 2, \ldots)$
2. $(\frac{1}{n})$
3. $((-1)^n \frac{1}{n})$
4. $(1, 2, 1, 2, \ldots)$
5. $(\sqrt{n})$
6. $((-1)^n (1 - \frac{1}{n}))$
7. $(n^2 - 1)$

Definition 2 (Convergent sequence)**.** *A sequence* (x_n) *is said to be convergent if there exists* $\ell \in \mathbb{R}$ *such that for every* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *satisfying* $|x_n - \ell| < \varepsilon$ *for all* $n \geq n_0$*. We say that* ℓ *is a limit of* (x_n)*.*

Notation: We write $\lim_{n\to\infty} x_n = \ell$ or $x_n \to \ell$.

Theorem 1. *Limit of a convergent sequence is unique.*

Proof. Let (x_n) be a convergent sequence. Assume that $\lim_{n\to\infty} x_n = \ell_1$ and $\lim_{n\to\infty} x_n = \ell_2$. We claim that $\ell_1 = \ell_2$. To see this, by way of contradiction assume that $\ell_1 \neq \ell_2$. Then $\varepsilon = \frac{|\ell_1-\ell_2|}{3} > 0$. By definition of convergence of a sequence, there are positive integers n_1 and n_2 such that

$$|x_n - \ell_1| < \varepsilon \text{ for all } n \geq n_1 \text{ and } |x_n - \ell_2| < \varepsilon \text{ for all } n \geq n_2.$$

Let $n_0 = \max\{n_1, n_2\}$. Then

$$|x_n - \ell_1| < \varepsilon \text{ and } |x_n - \ell_2| < \varepsilon \text{ for all } n \geq n_0.$$

Using triangle inequality, we have

$$3\varepsilon = |\ell_1 - \ell_2| = |\ell_1 - x_{n_0} + x_{n_0} - \ell_2| \leq |\ell_1 - x_{n_0}| + |x_{n_0} - \ell_2| < 2\varepsilon,$$

which is a contradiction. Hence, we must have $\ell_1 = \ell_2$. $\square$

Example 2. *Using the definition of convergence of a sequence, show that* $\lim_{n\to\infty} \frac{1}{n} = 0$.

Solution. Let $\varepsilon > 0$. By using the Archimedean property, we can find a positive integer n_0 such that $n_0 \cdot \varepsilon > 1$, that is, $\varepsilon > \frac{1}{n_0}$. Now, for all $n \geq n_0$, we have

$$|\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{n_0} < \varepsilon.$$

This proves that $\lim_{n\to\infty} \frac{1}{n} = 0$. $\square$

Example 3. *Consider the sequence (x_n) where $x_n = (-1)^n$. The terms of the sequence are $-1, 1, -1, 1, -1, 1, \ldots$. It is intuitively clear that this sequence does not approach to any real number. Therefore, the sequence does not converge. We now establish this fact by using the definition.*

Solution. By way of contradiction assume that the given sequence converges to ℓ. Then for $\varepsilon = \frac{1}{2}$, there exists a natural number n_0 such that

$$|(-1)^n - \ell| < \frac{1}{2} \text{ for all } n \geq n_0.$$

This gives $|1 - \ell| < \frac{1}{2}$ and $|1 + \ell| < \frac{1}{2}$. Now, using triangle inequality we have

$$2 = |1 + 1| = |1 - \ell + \ell + 1| \leq |1 - \ell| + |1 + \ell| < 1,$$

which is a contradiction. This proves that $((-1)^n)$ does not converge. $\square$

Example 4. *If $|\alpha| < 1$, then the sequence (α^n) converges to 0.*

Solution. If $\alpha = 0$, then $\alpha^n = 0$ for all $n \in \mathbb{N}$ and so (α^n) converges to 0. Now we assume that $\alpha \neq 0$. Since $|\alpha| < 1$, $\frac{1}{|\alpha|} > 1$ and so $\frac{1}{|\alpha|} = 1 + h$ for some $h > 0$. For all $n \in \mathbb{N}$, we have $(1+h)^n = 1 + nh + \frac{n(n-1)}{2!}h^2 + \cdots + h^n > nh \Rightarrow |\alpha|^n = \frac{1}{(1+h)^n} < \frac{1}{nh}$ for all $n \in \mathbb{N}$. Given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ satisfying $n_0 > \frac{1}{h\varepsilon}$. Then $|\alpha^n - 0| = |\alpha|^n < \frac{1}{n_0 h} < \varepsilon$ for all $n \geq n_0$ and hence (α^n) converges to 0.

Alternative proof: Given $\varepsilon > 0$, we choose $n_0 \in \mathbb{N}$ satisfying $n_0 > \frac{\log \varepsilon}{\log |\alpha|}$. Then for all $n \geq n_0$, we have $|\alpha^n - 0| = |\alpha|^n \leq |\alpha|^{n_0} < \varepsilon$ and hence (α^n) converges to 0. $\square$

Bounded sequence: Given a sequence (x_n), we can ask whether the set $\{x_1, x_2, x_3, \ldots\}$ is bounded or not. If this set is bounded then we call that the sequence (x_n) is bounded. Equivalently, the sequence (x_n) is bounded if there is a positive number M such that $|x_n| \leq M$ for all $n \in \mathbb{N}$. If (x_n) is not bounded then it is said to be unbounded. For example, (a), $((-1)^n)$, $(\frac{1}{n})$ are bounded sequences; whereas (n^2) and $(2\sqrt{n})$ are unbounded sequences.

Theorem 2. *Every convergent sequence is bounded.*

Proof. Let (x_n) be a convergent sequence and let $\lim\limits_{n\to\infty} x_n = \ell$. By taking $\varepsilon = 1$, we find a positive integer n_0 such that $|x_n - \ell| < 1$ for all $n \geq n_0$. Equivalently,

$$\ell - 1 < x_n < \ell + 1 \quad \text{for all} \quad n \geq n_0.$$

Thus, the set $\{x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots\}$ is bounded from below by $\ell - 1$ and bounded from above by $\ell + 1$. Using the triangle inequality, we have

$$|x_n| = |x_n - \ell + \ell| \leq |x_n - \ell| + |\ell| < 1 + |\ell| \quad \text{for all} \quad n \geq n_0.$$

Now, there are only finitely many elements left, namely $x_1, x_2, \ldots, x_{n_0-1}$. Let

$$M = \max\{|x_1|, |x_2|, \ldots, |x_{n_0-1}|, |\ell| + 1\}.$$

Then we have $|x_n| \leq M$ for all $n \geq 1$. Hence, (x_n) is a bounded sequence. $\square$

Remark 1. *From the above theorem, it follows that if a sequence is not bounded then it is not convergent. For example, the sequence $(\sqrt{n})$ is unbounded and hence is not convergent. However, every bounded sequence is not convergent. For example, $((-1)^n)$ is a bounded sequence but it does not converge.*

Limit rules for convergent sequences:

Theorem 3. *Let $x_n \to x$ and $y_n \to y$. Then*

(a) $x_n + y_n \to x + y$.

(b) $\alpha x_n \to \alpha x$ *for all* $\alpha \in \mathbb{R}$.

(c) $|x_n| \to |x|$.

(d) $x_n y_n \to xy$.

(e) $\frac{x_n}{y_n} \to \frac{x}{y}$ *if* $y_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $y \neq 0$.

Example 5. *The sequence* $(\frac{2n^2-3n}{3n^2+5n+3})$ *is convergent with limit* $\frac{2}{3}$.

Solution. We have $\frac{2n^2-3n}{3n^2+5n+3} = \frac{2-\frac{3}{n}}{3+\frac{5}{n}+\frac{3}{n^2}}$ for all $n \in \mathbb{N}$. Since $\frac{1}{n} \to 0$, the limit rules for algebraic operations on sequences imply that the given sequence is convergent with limit $\frac{2-0}{3+0+0} = \frac{2}{3}$. □

Example 6. *The sequence* $(\sqrt{n+1} - \sqrt{n})$ *is convergent with limit* 0.

Solution. For all $n \in \mathbb{N}$, $\sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{n+1}+\sqrt{n}} = \frac{\frac{1}{\sqrt{n}}}{\sqrt{1+\frac{1}{n}}+1}$. Since $\frac{1}{n} \to 0$, the limit rules for algebraic operations on sequences imply that the given sequence is convergent with limit $\frac{0}{\sqrt{1+0}+1} = 0$. □

Theorem 4 (Sandwich theorem)**.** *Let (x_n), (y_n), (z_n) be sequences such that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$. If both (x_n) and (z_n) converge to the same limit ℓ, then (y_n) also converges to ℓ.*

Proof. Let $\varepsilon > 0$. Since $x_n \to \ell$, so there exists a positive integer n_1 such that $|x_n - \ell| < \varepsilon$ for all $n \geq n_1$. Similarly, as $z_n \to \ell$, so there exists a positive integer n_2 such that $|z_n - \ell| < \varepsilon$ for all $n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$. Then,

$$\ell - \varepsilon < x_n < \ell + \varepsilon \text{ and } \ell - \varepsilon < z_n < \ell + \varepsilon \text{ for all } n \geq n_0.$$

Using the given fact that $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, we obtain

$$\ell - \varepsilon < x_n \leq y_n \leq z_n < \ell + \varepsilon \text{ for all } n \geq n_0.$$

This proves that $|y_n - \ell| < \varepsilon$ for all $n \geq n_0$, and hence $y_n \to \ell$. □

Example 7. *Consider the sequence* $\left(\frac{\cos n}{n}\right)$. *Since* $\frac{-1}{n} \leq \frac{\cos n}{n} \leq \frac{1}{n}$ *for all* $n \in \mathbb{N}$, *so by applying Sandwich theorem we find that* $\lim_{n\to\infty} \frac{\cos n}{n} = 0$.

Example 8. *If* $\alpha > 0$, *then the sequence* $(\alpha^{\frac{1}{n}})$ *converges to* 1.

Solution. We first assume that $\alpha \geq 1$ and let $x_n = \alpha^{\frac{1}{n}} - 1$ for all $n \in \mathbb{N}$. Then $x_n \geq 0$ and $\alpha = (1+x_n)^n = 1 + nx_n + \frac{n(n-1)}{2!}x_n^2 + \cdots + x_n^n > nx_n$ for all $n \in \mathbb{N}$. So $0 \leq x_n < \frac{\alpha}{n}$ for all $n \in \mathbb{N}$. Since $\frac{\alpha}{n} \to 0$, by Sandwich theorem, it follows that $x_n \to 0$. Consequently $\alpha^{\frac{1}{n}} \to 1$. If $\alpha < 1$, then $\frac{1}{\alpha} > 1$ and as proved above, $(\frac{1}{\alpha})^{\frac{1}{n}} \to 1$. It follows that $\alpha^{\frac{1}{n}} \to 1$.

Alternative proof: We first assume that $\alpha \geq 1$. For each $n \in \mathbb{N}$, applying the $A.M. \geq G.M.$ inequality for the numbers $1, ..., 1, \alpha$ (1 is repeated $n-1$ times), we get $1 \leq \alpha^{\frac{1}{n}} \leq 1 + \frac{\alpha-1}{n}$. Since $\frac{\alpha-1}{n} \to 0$, by Sandwich theorem, it follows that $\alpha^{\frac{1}{n}} \to 1$. The case for $\alpha < 1$ is same as given in the above proof. □

Example 9. *The sequence* $(n^{\frac{1}{n}})$ *converges to* 1.

Solution. For all $n \in \mathbb{N}$, let $a_n = n^{\frac{1}{n}} - 1$. Then for all $n \in \mathbb{N}$,

$$n = (1+a_n)^n = 1 + na_n + \frac{n(n-1)}{2!}a_n^2 + \cdots + a_n^n > \frac{n(n-1)}{2!}a_n^2.$$

This implies $0 \leq a_n^2 < \frac{2}{n-1}$ for all $n \geq 2$. Since $\frac{2}{n-1} \to 0$, by Sandwich theorem, it follows that $a_n^2 \to 0$ and so $a_n \to 0$. Consequently $n^{\frac{1}{n}} \to 1$. □

Theorem 5. *Let $r \in \mathbb{R}$. Then there exists a sequence (x_n) of rational numbers such that $\lim\limits_{n\to\infty} x_n = r$.*

Proof. We know that between two real numbers, there is a rational number. For each $n \in \mathbb{N}$, consider the real numbers $r - \frac{1}{n}$ and $r + \frac{1}{n}$. Let x_n be a rational number such that $r - \frac{1}{n} < x_n < r + \frac{1}{n}$. Then (x_n) is a sequence of rational numbers, and by Sandwich theorem (x_n) converges to r. $\square$

Divergent sequence:

Definition 3. *A sequence (x_n) is said to be divergent if it has no limit.*

Example 10. *If (x_n) is unbounded then it is divergent. For example, $(\sqrt{n})$, $(3n^2)$, $((-1)^n n^3)$ are all divergent. We have seen that the sequence $((-1)^n)$ is not convergent, and so it is a divergent sequence although it is bounded.*

Definition 4. *A sequence (x_n) is said to approach infinity or diverges to infinity if for any real number $M > 0$, there is a positive integer n_0 such that $a_n \geq M$ for all $n \geq n_0$. Similarly, (x_n) is said to approach $-\infty$ or diverges to $-\infty$ if for any real number $M > 0$, there is a positive integer n_0 such that $a_n \leq -M$ for all $n \geq n_0$.*

Remark 2. *Let (x_n) and (y_n) be two sequences of real numbers.*

- *If (x_n) and (y_n) both diverge to ∞, then the sequences $(x_n + y_n)$ and $(x_n y_n)$ also diverge to ∞.*
- *If (x_n) diverges to ∞ and (y_n) converges, then $(x_n + y_n)$ diverges to ∞.*

Monotone sequence:

Definition 5. *A sequence (x_n) is said to be increasing if $x_{n+1} \geq x_n$ for all $n \in \mathbb{N}$. Similarly, (x_n) is said to be decreasing if $x_{n+1} \leq x_n$ for all $n \in \mathbb{N}$. We say that (x_n) is monotonic if it is either increasing or decreasing.*

Example 11. *The sequence $(1 - \frac{1}{n})$ is increasing.*

Solution. For all $n \in \mathbb{N}$, $\frac{1}{n+1} < \frac{1}{n}$ and so $1 - \frac{1}{n+1} > 1 - \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore the given sequence is increasing. $\square$

Example 12. *The sequence $(n + \frac{1}{n})$ is increasing.*

Solution. For all $n \in \mathbb{N}$, $(n + 1 + \frac{1}{n+1}) - (n + \frac{1}{n}) = 1 - \frac{1}{n(n+1)} > 0 \Rightarrow n + 1 + \frac{1}{n+1} > n + \frac{1}{n}$ for all $n \in \mathbb{N}$. Therefore the given sequence is increasing. $\square$

Example 13. *The sequence $(\cos \frac{n\pi}{3})$ is not monotonic.*

Solution. Since $\cos \frac{\pi}{3} = \frac{1}{2}$, $\cos \frac{3\pi}{3} = -1$ and $\cos \frac{6\pi}{3} = 1$, we have $\cos \frac{\pi}{3} > \cos \frac{3\pi}{3} < \cos \frac{6\pi}{3}$ and hence the given sequence is neither increasing nor decreasing. Consequently the given sequence is not monotonic. $\square$

Theorem 6. *If (x_n) is increasing and not bounded above then (x_n) diverges to ∞. If (x_n) is decreasing and not bounded below then (x_n) diverges to $-\infty$.*

Theorem 7 (Monotone convergence theorem). *Let (x_n) be a sequence of real numbers.*

(a) *If (x_n) is increasing and bounded above then (x_n) converges to* $\sup\{x_n : n \in \mathbb{N}\}$.

(b) *If (x_n) is decreasing and bounded below then (x_n) converges to* $\inf\{x_n : n \in \mathbb{N}\}$.

(c) *A monotonic sequence converges if and only if it is bounded.*

Proof. We only prove (a). Since (x_n) is increasing, so it is bounded below by x_1. Also, (x_n) is bounded above and hence (x_n) is bounded. Let $s = \sup\{x_n : n \in \mathbb{N}\}$. We claim that $x_n \to s$. To prove this, let $\varepsilon > 0$. Then $s - \varepsilon$ is not an upper bound and so there exists some n_0 such that $s - \varepsilon < x_{n_0}$. Since (x_n) is increasing so $x_n \geq x_{n_0}$ for all $n \geq n_0$. Therefore

$$s - \varepsilon < x_{n_0} \leq x_n \leq s < s + \varepsilon \quad \text{for all} \quad n \geq n_0.$$

This proves that $x_n \to s$. □

Example 14. *Let $x_1 = 1$ and $x_{n+1} = \frac{1}{3}(x_n + 1)$ for all $n \in \mathbb{N}$. Then the sequence (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = \frac{1}{2}$.*

Solution. For all $n \in \mathbb{N}$, we have $x_{n+1} - x_n = \frac{1}{3}(1 - 2x_n)$. Also, $x_1 > \frac{1}{2}$ and if we assume that $x_k > \frac{1}{2}$ for some $k \in \mathbb{N}$, then $x_{k+1} = \frac{1}{3}(x_k + 1) > \frac{1}{3}(\frac{1}{2} + 1) = \frac{1}{2}$. Hence by the principle of mathematical induction, $x_n > \frac{1}{2}$ for all $n \in \mathbb{N}$. So (x_n) is bounded below. Again, from above, we get $x_{n+1} - x_n < 0$ for all $n \in \mathbb{N} \Rightarrow x_{n+1} < x_n$ for all $n \in \mathbb{N} \Rightarrow (x_n)$ is decreasing. Therefore (x_n) is convergent. Let $\ell = \lim\limits_{n\to\infty} x_n$. Then $\lim\limits_{n\to\infty} x_{n+1} = \ell$ and since $x_{n+1} = \frac{1}{3}(x_n + 1)$ for all $n \in \mathbb{N}$, we get $\ell = \frac{1}{3}(\ell + 1) \Rightarrow \ell = \frac{1}{2}$. □

Example 15. *The sequence $((1 + 1/n)^n)$ is convergent.*

Solution. Let $a_n = (1 + 1/n)^n$. Then

$$\begin{aligned} a_n &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1}{n}\right)^k = 1 + \frac{n}{1} \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 2 \cdot 1}{n!} \cdot \frac{1}{n^n} \\ &= 1 + 1 + \frac{1}{2!}\left(1 - \frac{1}{n}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} a_{n+1} = 1 + 1 &+ \frac{1}{2!}\left(1 - \frac{1}{n+1}\right) + \cdots + \frac{1}{n!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{n-1}{n+1}\right) \\ &+ \frac{1}{(n+1)!}\left(1 - \frac{1}{n+1}\right)\left(1 - \frac{2}{n+1}\right)\cdots\left(1 - \frac{n}{n+1}\right). \end{aligned}$$

Note that the expression for a_n contains $n+1$ terms, while that for a_{n+1} contains $n+2$ terms. Moreover, each term appearing in a_n is less than or equal to the corresponding term in a_{n+1}, and a_{n+1} has one more positive term. Therefore, we have

$$2 \leq a_1 < a_2 < \cdots < a_n < a_{n+1} < \cdots,$$

so that the sequence (a_n) is increasing. For $n > 1$, we have

$$2 < a_n < 1 + 1 + \left(\frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}\right) = 2 + \left(1 - \frac{1}{2^{n-1}}\right) < 3.$$

By Monotone convergence theorem, the sequence (a_n) converges to a real number that lies between 2 and 3. We define the number e to be the limit of this sequence. □

Theorem 8. *Let $x_n \neq 0$ for all $n \in \mathbb{N}$ and let $L = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right|$ exist.*

(a) *If $L < 1$, then $x_n \to 0$.*

(b) *If $L > 1$, then (x_n) is divergent.*

Proof. Note that $L \geq 0$. Let r be such that $L < r < 1$. Let $\varepsilon = r - L$. Then $\varepsilon > 0$, and hence there is some $n_0 \in \mathbb{N}$ such that

$$0 \leq \frac{|x_{n+1}|}{|x_n|} < L + \varepsilon = r \quad \text{for all} \quad n \geq n_0.$$

Using the above inequality for $n_0, n_0 + 1, \cdots, n$, we have

$$\begin{aligned} 0 &\leq \frac{|x_{n_0+1}|}{|x_{n_0}|} \frac{|x_{n_0+2}|}{|x_{n_0+1}|} \cdots \frac{|x_{n+1}|}{|x_n|} < r^{n-n_0+1} \\ \Rightarrow 0 &\leq |x_{n+1}| < \left(\frac{|x_{n_0}|}{r^{n_0-1}}\right) \cdot r^n \end{aligned}$$

Since $0 < r < 1$, so $r^n \to 0$. Now using Sandwich theorem, we have $x_n \to 0$. This completes the proof of (a).

Suppose that $L > 1$. Let r be such that $1 < r < L$. Let $\varepsilon = L - r$. Since $\varepsilon > 0$, there is some $n_0 \in \mathbb{N}$ such that

$$r = L - \varepsilon < \frac{|x_{n+1}|}{|x_n|} \quad \text{for all} \quad n \geq n_0.$$

This yields (as shown before)

$$|x_{n+1}| > \left(\frac{|x_{n_0}|}{r^{n_0-1}}\right) \cdot r^n$$

Since $r > 1$, so r^n diverges to infinity and hence (x_n) also diverges. □

Remark 3. *If $L = \lim_{n\to\infty} \left|\frac{x_{n+1}}{x_n}\right| = 1$, then (x_n) may converge or diverge. For example, the sequence $((-1)^n)$ diverges and $L = 1$. For any nonzero constant sequence, $L = 1$ and constant sequences are convergent.*

Example 16. *If $\alpha \in \mathbb{R}$, then the sequence $(\frac{\alpha^n}{n!})$ is convergent.*

Solution. Let $x_n = \frac{\alpha^n}{n!}$ for all $n \in \mathbb{N}$. If $\alpha = 0$, then $x_n = 0$ for all $n \in \mathbb{N}$ and so (x_n) converges to 0. If $\alpha \neq 0$, then $\lim_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim_{n\to\infty} \frac{|\alpha|}{n+1} = 0 < 1$ and so (x_n) converges to 0. □

Example 17. *The sequence $(\frac{2^n}{n^4})$ is not convergent.*

Solution. If $x_n = \frac{2^n}{n^4}$ for all $n \in \mathbb{N}$, then $\lim_{n\to\infty} |\frac{x_{n+1}}{x_n}| = \lim_{n\to\infty} \frac{2}{(1+\frac{1}{n})^4} = 2 > 1$. Therefore the sequence (x_n) is not convergent. □

Subsequence:

Definition 6 (Subsequence)**.** *Let (x_n) be a sequence in $\mathbb{R}$. If (n_k) is a sequence of positive integers such that $n_1 < n_2 < n_3 < \cdots$, then (x_{n_k}) is called a subsequence of (x_n).*

Example 18. *Think of some divergent sequences and their convergent subsequences.*

Theorem 9. *If a sequence (x_n) converges to ℓ, then every subsequence of (x_n) must converge to ℓ.*

Remark 4. *From the above theorem, we have the following:*

- *If (x_n) has a subsequence (x_{n_k}) such that $x_{n_k} \not\to \ell$, then $x_n \not\to \ell$.*
- *If (x_n) has two subsequences converging to two different limits, then (x_n) cannot be convergent.*

Example 19. *If $x_n = (-1)^n(1 - \frac{1}{n})$ for all $n \in \mathbb{N}$, then (x_n) is not convergent.*

Solution. We have $x_{2n-1} = (-1)^{2n-1}(1 - \frac{1}{2n-1}) = \frac{1}{2n-1} - 1 \to -1$.
Also $x_{2n} = (-1)^{2n}(1 - \frac{1}{2n}) = 1 - \frac{1}{2n} \to 1 \neq -1$. Hence, (x_n) is not convergent. □

Example 20. *Let (x_n) be a sequence in $\mathbb{R}$. Then (x_{2n}) and (x_{2n-1}) are two subsequences of (x_n). Suppose that $x_{2n} \to \ell \in \mathbb{R}$ and $x_{2n-1} \to \ell$. Then $x_n \to \ell$.*

Solution. Let $\varepsilon > 0$. Since $x_{2n} \to \ell$ and $x_{2n-1} \to \ell$, there exist $n_1, n_2 \in \mathbb{N}$ such that $|x_{2n} - \ell| < \varepsilon$ for all $n \geq n_1$ and $|x_{2n-1} - \ell| < \varepsilon$ for all $n \geq n_2$. Taking $n_0 = \max\{2n_1, 2n_2 - 1\} \in \mathbb{N}$, we find that $|x_n - \ell| < \varepsilon$ for all $n \geq n_0$. Hence $x_n \to \ell$. □

Example 21. *The sequence $(1, \frac{1}{2}, 1, \frac{2}{3}, 1, \frac{3}{4}, ...)$ converges to 1.*

Solution. If (x_n) denotes the given sequence, then $x_{2n} = \frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1$ and $x_{2n-1} = 1 \to 1$. Therefore (x_n) converges to 1. □

Theorem 10. *Every sequence of real numbers has a monotone subsequence.*

Proof. Let (x_n) be a sequence of real numbers. A term x_p is called a **peak** in (x_n) if $x_p > x_m$ for all $m > p$. That is, a peak in (x_n) is a term which is greater than all the succeeding terms. Let $\mathcal{P}$ be the set of all the peaks of (x_n). We now consider the following two cases:

- $\mathcal{P}$ is finite: Note that in this case $\mathcal{P}$ can be empty also. Let $p_1 < p_2 < \cdots < p_\ell$ so that $x_{p_1}, x_{p_2}, \ldots, x_{p_\ell}$ are the only peaks of (x_n). Let $n_1 > p_\ell$. Then x_{n_1} is not a peak. Hence there is some $n_2 \in \mathbb{N}$ such that $n_2 > n_1$ and $x_{n_1} \leq x_{n_2}$. Again, since $n_2 > p_\ell$ so x_{n_2} is not a peak. Hence, there is some $n_3 > n_2$ such that $x_{n_2} \leq x_{n_3}$. In this way, using the principle of mathematical induction, we have an increasing sequence $n_1 < n_2 < n_3 < \cdots < n_k < \cdots$ in $\mathbb{N}$ such that $x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \cdots < x_{n_k} < \cdots$. This proves that (x_{n_k}) is an increasing subsequence of (x_n).
- $\mathcal{P}$ is infinite: In this case, we have $p_1 < p_2 < \cdots < p_k < \cdots$ so that $x_{p_1}, x_{p_2}, \ldots, x_{p_k}, \ldots$ are the peaks of (x_n). Clearly, $x_{p_1} > x_{p_2} > \ldots > x_{p_k} > \ldots$. Hence, (x_{p_k}) is a decreasing sequence of (x_n).

□

Theorem 11 (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence in $\mathbb{R}$ has a convergent subsequence.*

Proof. Let (x_n) be a sequence of real numbers. By the previous theorem, (x_n) has a monotone subsequence, say (x_{n_k}). Since (x_n) is bounded, so (x_{n_k}) is also bounded. By the Monotone convergence theorem, (x_{n_k}) is convergent. □

Cauchy sequence:

Definition 7 (Cauchy sequence)**.** *A sequence (x_n) is called a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \geq n_0$.*

Theorem 12. *Every Cauchy sequence is bounded.*

Theorem 13 (Cauchy's criterion for convergence)**.** *A sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence.*

Proof. Let (x_n) be convergent and $x_n \to \ell$. Let $\varepsilon > 0$. Then there is some $n_0 \in \mathbb{N}$ such that $|x_n - \ell| < \varepsilon/2$ for all $n, m \geq n_0$. Now,

$$|x_n - x_m| = |x_n - \ell + \ell - x_m| \leq |x_n - \ell| + |x_m - \ell| < \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ for all } m, n \geq n_0.$$

Hence, (x_n) is a Cauchy sequence. Conversely, suppose that (x_n) is a Cauchy sequence. Then (x_n) is bounded, and by Bolzano Weierstrass theorem, (x_n) has a convergent subsequence, say (x_{n_k}). Suppose that $x_{n_k} \to \ell$. We claim that $x_n \to \ell$. To prove this, let $\varepsilon > 0$. Since (x_n) is Cauchy, so there is some $n_0 \in \mathbb{N}$ such that $|x_n - x_m| < \varepsilon/2$ for all $n, m \geq n_0$. Also, since $x_{n_k} \to \ell$, there is $k_0 \in \mathbb{N}$ such that $|x_{n_k} - \ell| < \varepsilon/2$ for all $k \geq k_0$. Let $j = \max\{k_0, n_0\}$. Since $n_j \geq j$, so $n_j \geq n_0$. Also, $j \geq k_0$. Therefore

$$|x_n - \ell| \leq |x_n - x_{n_j}| + |x_{n_j} - \ell| < \varepsilon/2 + \varepsilon/2 = \varepsilon \text{ for all } n \geq n_0.$$

Hence the sequence (x_n) is convergent. □

Example 22. *Let (x_n) satisfy either of the following conditions:*

(a) $|x_{n+1} - x_n| \leq \alpha^n$ *for all* $n \in \mathbb{N}$

(b) $|x_{n+2} - x_{n+1}| \leq \alpha |x_{n+1} - x_n|$ *for all* $n \in \mathbb{N}$,

where $0 < \alpha < 1$. Then (x_n) is a Cauchy sequence.

Solution. (a) For all $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned} |x_m - x_n| &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\ &\leq \alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1} \\ &= \frac{\alpha^n}{1-\alpha}(1 - \alpha^{m-n}) \\ &< \frac{\alpha^n}{1-\alpha} \end{aligned}$$

Since $0 < \alpha < 1$, $\alpha^n \to 0$ and so given any $\varepsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such that $\frac{\alpha^{n_0}}{1-\alpha} < \varepsilon$. Hence for all $m, n \geq n_0$, we have $|x_m - x_n| < \frac{\alpha^{n_0}}{1-\alpha} < \varepsilon$. Therefore (x_n) is a Cauchy sequence.

(b) For all $m, n \in \mathbb{N}$ with $m > n$, we have

$$\begin{aligned}
|x_m - x_n| &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\
&\leq (\alpha^{n-1} + \alpha^n + \cdots + \alpha^{m-2})|x_2 - x_1| \\
&= \frac{\alpha^{n-1}}{1-\alpha}(1 - \alpha^{m-n})|x_2 - x_1| \\
&\leq \frac{\alpha^{n-1}}{1-\alpha}|x_2 - x_1|
\end{aligned}$$

Since $0 < \alpha < 1$, $\alpha^{n-1} \to 0$ and so given any $\varepsilon > 0$, we can choose $n_0 \in \mathbb{N}$ such that $\frac{\alpha^{n_0-1}}{1-\alpha}|x_2 - x_1| < \varepsilon$. Hence for all $m, n \geq n_0$, we have $|x_m - x_n| \leq \frac{\alpha^{n_0-1}}{1-\alpha}|x_2 - x_1| < \varepsilon$. Therefore (x_n) is a Cauchy sequence. $\square$

Example 23. *Let (x_n) be a sequence defined as $x_1 = 1$ and $x_{n+1} = 1 + \frac{1}{x_n}$ for $n \in \mathbb{N}$. Then $x_{n+1}x_n = 1 + x_n > 2$. Now,*

$$|x_{n+2} - x_{n+1}| = |\frac{x_{n+1} - x_n}{x_{n+1}x_n}| < \frac{1}{2}|x_{n+1} - x_n|.$$

Hence, (x_n) is a Cauchy sequence.

Limit superior and limit inferior:

Let (x_n) be a bounded sequence. Let $y_1 = \sup\{x_1, x_2, \ldots\}$, $y_2 = \sup\{x_2, x_3, \ldots\}$, and so on. That is, for $n \in \mathbb{N}$,

$$y_n = \sup\{x_n, x_{n+1}, \ldots\} = \sup_{k \geq n} x_k.$$

Let A and B be two nonempty subsets of $\mathbb{R}$ such that $A \subseteq B$. Then clearly, $\sup(A) \leq \sup(B)$ and $\inf(B) \leq \inf(A)$. Hence,

$$y_1 \geq y_2 \geq y_3 \geq \cdots.$$

Since (x_n) is bounded so the sequence (y_n) is bounded below. By Monotone convergence theorem, (y_n) is convergent and converges to the infimum of $\{y_1, y_2, \ldots\}$. The limit of the sequence (y_n) is called the limit superior of the sequence (x_n), and is denoted by $\limsup x_n$. Thus,

$$\limsup x_n := \lim_{n\to\infty} y_n = \inf_n \sup_{k \geq n} x_k.$$

Similarly, let $z_1 = \inf\{x_1, x_2, \ldots\}$, $z_2 = \inf\{x_2, x_3, \ldots\}$, and so on. That is, for $n \in \mathbb{N}$,

$$z_n = \inf\{x_n, x_{n+1}, \ldots\} = \inf_{k \geq n} x_k.$$

We have $z_1 \leq z_2 \leq z_3 \leq \cdots$. Since (x_n) is bounded so the sequence (z_n) is bounded above. By Monotone convergence theorem, (z_n) is convergent and converges to the supremum of $\{z_1, z_2, \ldots\}$. The limit of the sequence (z_n) is called the limit inferior of the sequence (x_n), and is denoted by $\liminf x_n$. Thus,

$$\liminf x_n := \lim_{n\to\infty} z_n = \sup_n \inf_{k \geq n} x_k.$$

Example 24. *Consider the sequence (x_n), where $x_n = (-1)^n$. Clearly, for any n, $y_n = \sup\{x_n, x_{n+1}, \ldots\} = 1$ and $z_n = \inf\{x_n, x_{n+1}, \ldots\} = -1$. Hence, $\limsup x_n = 1$ and $\liminf x_n = -1$.*

Example 25. *Consider the sequence (x_n), where $x_n = \frac{1}{n}$. Clearly, for any n, $y_n = \sup\{\frac{1}{k} : k \geq n\} = \frac{1}{n}$ and $z_n = \inf\{\frac{1}{k} : k \geq n\} = 0$. Hence, $\limsup x_n = \lim\limits_{n\to\infty} y_n = \lim\limits_{n\to\infty} \frac{1}{n} = 0$ and $\liminf x_n = 0$.*

Remark 5. *Suppose that $|x_n| < M$ for $n \in \mathbb{N}$. Then $-M \leq z_n \leq y_n \leq M$ for all n. Hence,*

$$-M \leq \liminf x_n \leq \limsup x_n \leq M.$$

Theorem 14. *Let (a_n) and (b_n) be two bounded sequences.*

1. $\liminf a_n \leq \limsup a_n$.
2. *If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $\limsup a_n \leq \limsup b_n$ and $\liminf a_n \leq \liminf b_n$.*
3. $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$ *and* $\liminf(a_n + b_n) \geq \liminf a_n + \liminf b_n$.

Theorem 15. *If (a_n) is a convergent sequence, then*

$$\liminf a_n = \lim_{n\to\infty} a_n = \limsup a_n.$$

Proof. Let $\ell = \lim a_n$. Let $\varepsilon > 0$. Then there is some $n_0 \in \mathbb{N}$ such that

$$\ell - \varepsilon/2 < a_n < \ell + \varepsilon/2 \quad \text{for all} \quad n \geq n_0.$$

Let $y_n = \sup\{a_k : k \geq n\}$ and $z_n = \inf\{a_k : k \geq n\}$. Then, for all $n \geq n_0$, we have

$$\ell - \varepsilon < \ell - \varepsilon/2 \leq z_n \leq a_n \leq y_n \leq \ell + \varepsilon/2 < \ell + \varepsilon.$$

Hence, $\limsup a_n = \lim y_n = \ell$ and $\liminf a_n = \lim z_n = \ell$. □

Theorem 16. *Let (a_n) be a bounded sequence. If $\limsup a_n = \liminf a_n$, then (a_n) is convergent and $\lim\limits_{n\to\infty} a_n = \limsup a_n$.*

Proof. Let $\limsup a_n = \liminf a_n = \ell$. Let $y_n = \sup\{a_k : k \geq n\}$ and $z_n = \inf\{a_k : k \geq n\}$. Then we have $\limsup a_n = \lim y_n = \ell$ and $\liminf a_n = \lim z_n = \ell$. Let $\varepsilon > 0$. Then there are positive integers n_1 and n_2 such that

$$\ell - \varepsilon < y_n < \ell + \varepsilon \quad \text{for all} \quad n \geq n_1 \quad \text{and} \quad \ell - \varepsilon < z_n < \ell + \varepsilon \quad \text{for all} \quad n \geq n_2.$$

Let $n_0 = \max\{n_1, n_2\}$. Then,

$$\ell - \varepsilon < z_n \leq a_n \leq y_n < \ell + \varepsilon \quad \text{for all} \quad n \geq n_0.$$

Hence, $\lim a_n = \ell$.
Alternative proof: Equivalently, we can directly apply Sandwich Theorem. We have $z_n \leq a_n \leq y_n$ for all n. Since $z_n \to \ell$ and $y_n \to \ell$, by using Sandwich Theorem, we have $a_n \to \ell$. □