PH 102: Electromagnetism
Tutorial 2
17/03/20

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1. Two long coaxial solenoids each carry current I, but in opposite directions, as shown in figure 1. The inner solenoid (radius a) has n_1 turns per unit length, and the outer one (radius b) has n_2. Find $\vec{B}$ in each of the three regions: (i) inside the inner solenoid, (ii) between them, and (iii) outside both.

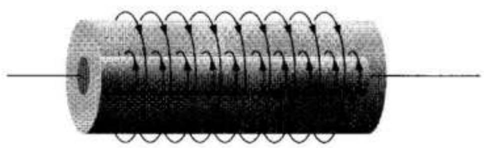

Figure 1: Figure for problem 1.

Solution:

As discussed in the class, the field inside a solenoid is $\mu_0 n I \hat{z}$ and outside it is zero, where the direction of $\hat{z}$ with respect to the current direction is shown in figure 2.
(i) Inside the inner solenoid, both the solenoid will contribute to the field. Since the

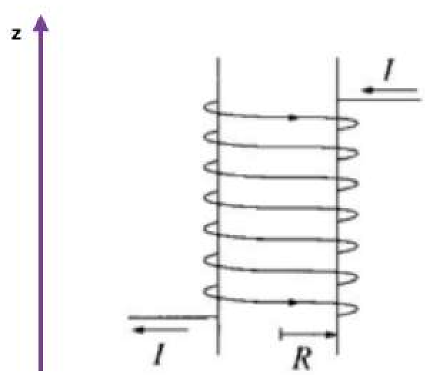

Figure 2: Figure for solution to problem 1.

currents in the two solenoids are in opposite directions, the field due to the inner solenoid points towards left (assumed to be the direction $\hat{z}$) whereas the one due to the outer solenoid points towards right ($-\hat{z}$). Thus the net field is $\vec{B} = \mu_0 I(n_1 - n_2)\hat{z}$.
(ii) Between the two solenoids, only the outer solenoid contributes to the field. Therefore $\vec{B} = -\mu_0 I n_2 \hat{z}$.
(iii) Outside both the solenoids, none of them contribute to the field. So $\vec{B} = 0$.

1. Consider a long current carrying (I) wire of radius R with a volume current density proportional to s^2 and in the $\hat{z}$ direction. Here, s is the distance from the axis of the wire.
(a) Find the proportionality constant.
(b) Find the magnetic field and the vector potential inside ($s < R$) the wire.
(c) Find the vector potential outside ($s > R$) the wire.

Solution:
(a) The volume current density can be written as $\vec{J} = \alpha s^2 \hat{z}$, ($\alpha$ is the proportionality constant). The total current I carried by the wire,

$$I = \int \vec{J} \cdot d\vec{S} = \int_0^R \int_0^{2\pi} \alpha s^2 s ds d\phi = \alpha \frac{2\pi R^4}{4}, \implies \alpha = \frac{2I}{\pi R^4}.$$

(b) Using the Ampere's law,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \implies \vec{B} = \frac{\mu_0 I_{enc}}{2\pi s} \hat{\phi}.$$

For $s < R$,

$$I_{enc} = \int \vec{J} \cdot d\vec{S} = \int_0^s \int_0^{2\pi} \alpha s^2 s ds d\phi = \alpha \frac{2\pi s^4}{4} = \frac{2I}{\pi R^4} \cdot \frac{2\pi s^4}{4} = \frac{I s^4}{R^4}.$$

Hence,

$$\vec{B} = \frac{\mu_0 I s^3}{2\pi R^4} \hat{\phi}$$

The current is along $\hat{z}$, hence the vector potential would be also in the $\hat{z}$ direction and $A_s = A_\phi = 0$.

$$B = (\vec{\nabla} \times \vec{A})_\phi = \left(\frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s} \right) \implies A = A_z = -\int B ds.$$

$$A_z = -\int B \cdot ds = -\frac{\mu_0 I}{2\pi R^4} \int s^3 ds = -\frac{\mu_0 I s^4}{2\pi R^4 4} + \text{Constant} \implies \vec{A} = \left(-\frac{\mu_0 I s^4}{8\pi R^4} + \text{Constant} \right) \hat{z}$$

(c) For $s > R$

$$\vec{B} = \frac{\mu_0 I}{2\pi s} \vec{\phi}$$

$$\vec{A} = -\int B ds \ \hat{z} = -\int \frac{\mu_0 I}{2\pi s} ds \ \hat{z} = \left(-\frac{\mu_0 I}{2\pi} \ln (s/R_0) \right) \hat{z},$$

where R_0 is a constant of integration.

By symmetry of the problem, the constant of integration can be made independent of

ϕ, z.

3. (a) Consider an infinitely long solid cylindrical wire of radius R. Inside the wire, magnetic vector potential is given by $\vec{A} = A_0 r \sin\phi \hat{z}$, A_0 being a constant.
(i) Find the magnetic field inside the wire.
(ii) Find the volume current density inside the wire.
(iii) Find the net current passing through the wire.
(iv) Find the magnetic field outside the wire.

Solution:
(i)
$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{r}\left[\hat{r}\frac{\partial A_z}{\partial \phi} - r\hat{\phi}\frac{\partial A_z}{\partial r}\right] = A_0[\cos\phi\hat{r} - \sin\phi\hat{\phi}]$$

(ii)
$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla} \times \vec{B} = \frac{1}{r}\hat{z}(-A_0 \sin\phi + A_0 \sin\phi) = 0$$

(iii)
$$I = \int \vec{J} \cdot \vec{da} = 0$$

(iv)
$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I = 0 \implies \vec{B} = 0$$

(b) Consider an infinitely long cylinder of radius a carrying a uniform volume current $J_0\hat{z}$, with z-axis coming out of the plane in figure 3. The wire has an off-axis hole of radius b with centre P at a distance d from the axis of the cylinder, passing through the point O. What is the y component of magnetic field at any point within the hole?

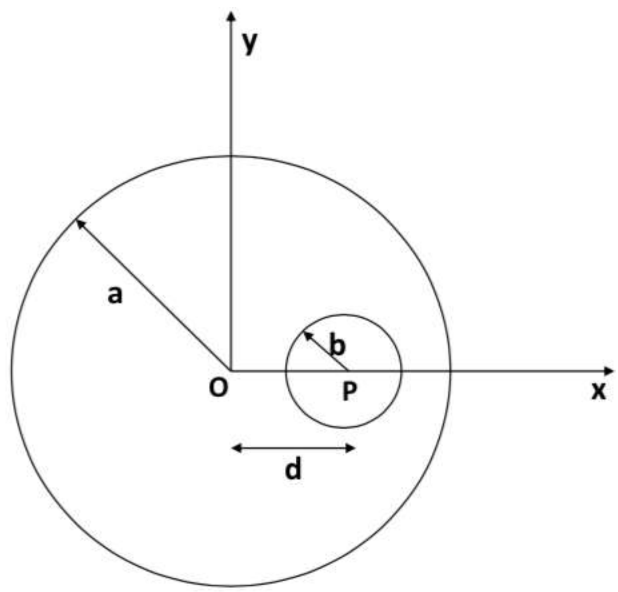

Figure 3: Figure for problem 3 (b)

Solution:

Replace the hole by equal and opposite currents $J_0\hat{z}$ and $-J_0\hat{z}$ and use the principle of linear superposition. For the hole replaced by $J_0\hat{z}$, magnetic field at point P' (due to the entire wire of radius a without any hole) can be found by using Ampere's law as

$$\oint \vec{B_1} \cdot \vec{dl} = \mu_0 I$$

$$B_1(2\pi r) = \mu_0 J_0 \pi r^2$$

$$B_1 = \frac{\mu_0 J_0 r}{2}$$

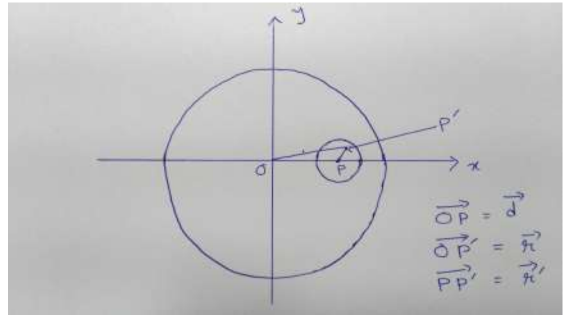

Figure 4: Figure for solution to problem 3 (b). P' is a point inside the hole.

$$\implies \vec{B_1} = \frac{\mu_0 J_0}{2}(r\hat{\phi})$$

Using $\hat{\phi} = -\sin\phi\hat{x} + \cos\phi\hat{y}$, we can write

$$r\hat{\phi} = -r\sin\phi\hat{x} + r\cos\phi\hat{y} = -y\hat{x} + x\hat{y}$$

Therefore $\vec{B_1} = -\frac{\mu_0 J_0}{2}(y\hat{x} - x\hat{y})$. Now let us consider the hole to be replaced by a wire with uniform volume current $-J_0\hat{z}$. For this wire (of radius b) alone, one can find magnetic field at point P' using Ampere's law as

$$\vec{B_2} = -\frac{\mu_0 J_0}{2} r'\hat{\phi}'$$

Similar to before, for the coordinate system with origin at P, we can write $r'\hat{\phi}' = -y'\hat{x} + x'\hat{y}$. Now, with respect to the point O, $y' = y, x' = x - d$. Therefore

$$\vec{B_2} = \frac{\mu_0 J_0}{2}[y\hat{x} - (x-d)\hat{y}]$$

The net field at any point inside the hole is

$$\vec{B} = \vec{B_1} + \vec{B_2} = \frac{\mu_0 J_0 d}{2}\hat{y}$$

whose y component is $\mu_0 J_0 d/2$.

(c) Consider an infinitely long solid cylinder of radius R to be made up of a dielectric material and to be made permanently polarised so that the polarisation is everywhere radially outward, with a magnitude proportional to the distance from the axis of the cylinder $\vec{P} = \frac{1}{2}P_0 r\hat{r}$, P_0 being a constant. If the cylinder is rotated with a constant angular velocity ω about its axis (which coincides with z-axis) without change in P, what is the magnetic field on the axis of the cylinder?

Solution:
Angular velocity $\vec{\omega} = \omega\hat{z}$. Volume current density $\vec{J} = \rho\vec{v} = \rho\vec{\omega} \times \vec{r}$. Volume charge density $\rho = -\vec{\nabla} \cdot \vec{P}$, $\vec{P} = \frac{1}{2}P_0 r\hat{r}$. Using cylindrical coordinate

$$\rho = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{P_0 r}{2}\right) = -P_0$$

$$\implies \vec{J} = \rho\vec{\omega} \times \vec{r} = -P_0\omega r\hat{\phi}$$

Surface charge density $\sigma = \vec{P} \cdot \hat{n} = \hat{r} \cdot \frac{P_0\vec{r}}{2}|_{r=R} = \frac{P_0 R}{2}$. Surface current density $\vec{K} = \sigma\vec{v} = \sigma\omega \times \vec{r}|_{r=R} = \frac{P_0\omega R^2}{2}\hat{\phi}$. Using Biot-Savart law, magnetic field at a point on the axis of the cylinder can be found as

$$\vec{B} = \frac{\mu_0}{4\pi}\left[\int \frac{\vec{J}(\vec{r'}) \times \vec{r'}}{(r')^3}d\tau' + \int \frac{\vec{K}(\vec{r'}) \times \vec{r'}}{(r')^3}da'\right]$$

Now

$$\int \frac{\vec{J}(\vec{r'}) \times \vec{r'}}{(r')^3}d\tau' = \int \frac{-P_0\omega r\hat{\phi} \times (r\hat{r} + z\hat{z})}{(r^2+z^2)^{3/2}}rdrd\phi dz$$

$$\implies \int \frac{\vec{J}(\vec{r'}) \times \vec{r'}}{(r')^3}d\tau' = P_0\omega\left[\int \frac{r^3drd\phi dz}{(r^2+z^2)^{3/2}}\hat{z} - \int \frac{r^2drd\phi dz}{(r^2+z^2)^{3/2}}\hat{r}\right.$$

The second integral vanishes by symmetry of the infinitely long cylinder. The first integral can be found by replacing $z = r\tan\beta$:

$$P_0\omega\int_0^{2\pi}d\phi\int_0^R rdr\int_{-\pi/2}^{\pi/2}\cos\beta d\beta\hat{z} = 2\pi P_0\omega R^2\hat{z} = \vec{I_1}$$

Similarly, the surface integral can be found as

$$\int \frac{\vec{K}(\vec{r'}) \times \vec{r'}}{(r')^3}da' = \int \frac{P_0\omega R^2\hat{\phi} \times (R\hat{r} + z\hat{z})}{2(R^2+z^2)^{3/2}}da' = -\frac{P_0}{2}\omega R^3\int \frac{Rd\phi dz\hat{z}}{(R^2+z^2)^{3/2}}\hat{z} = \vec{I_2}$$

where the term proportional to $\hat{r}$ vanishes by symmetry, as before. Using $z = \tan\beta$, the above integral can be found as

$$\vec{I_2} = -2\pi P_0\omega R^2\hat{z}$$

Adding the contributions from volume and surface currents, the net magnetic field, therefore, is $\vec{B} = 0$.

Note: Students may try to solve this problem using Ampere's law also.

Using Ampere's Law:
Since this is an infinite cylinder, magnetic field can also be calculated using Ampere's law in much easier way compared to using Biot-Savart law discussed above. Similar to the derivation of magnetic field for infinite solenoid discussed in the class, here also, one can show that magnetic field outside is zero, by taking Amperian loop 1 shown in figure 5.

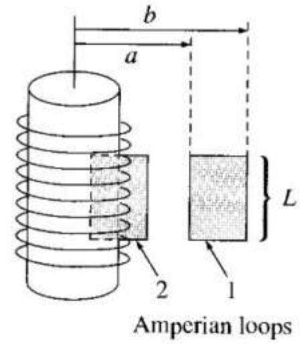

Figure 5: Figure for solution to problem 3 (c)

Now, take another Amperian loop (loop 2 in figure 5 where one vertical end is outside the cylinder (where B=0) while the other vertical end coincides with the axis of the cylinder. Using Ampere's law

$$\oint \vec{B} \cdot d\vec{l} = BL = \mu_0 I_{\text{enc}}$$

Here, both volume and surface current will contribute to $I_{\text{enc}} = I_{\text{volume}} + I_{\text{surface}}$. Current passing through loop 2 due to volume current is

$$I_{\text{volume}} = -P_0 \omega L \int_0^R r dr = -P_0 \omega L \frac{R^2}{2}$$

where negative sign reflects the fact that volume current is in $-\hat{\phi}$ direction and hence the current $P_0 \omega L \frac{R^2}{2}$ is coming out of the loop. Similarly, current passing through the loop due to surface current of the cylinder is

$$I_{\text{surface}} = KL = P_0 \omega L \frac{R^2}{2}$$

which is going into the loop. Therefore, net current passing through the loop 2 is $I_{\text{enc}} = 0$. Therefore magnetic field at the axis $B = 0$.

(d) Consider an infinitely long hollow cylinder of radius R carrying uniform surface current $\vec{K} = K\hat{z}$ with z-axis coinciding with the axis of the cylinder. Find out the magnetic field inside the cylinder $\vec{B}_{\text{inside}}$ and outside the cylinder $\vec{B}_{\text{outside}}$.

Solution:
Using Ampere's law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{\text{enc}}$$

Outside the cylinder

$$B(2\pi r) = \mu_0 K(2\pi R) \implies B = \frac{\mu_0 K R}{r}$$

$$\vec{B} = \frac{\mu_0 K R}{r}\hat{\phi}$$

Inside the cylinder, $I_{\text{enc}} = 0$. Hence $\vec{B}_{\text{inside}} = 0$.

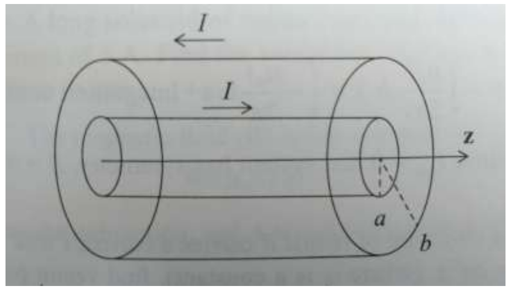

Figure 6: Figure for tutorial problem 4.

4. A long hollow coaxial wire has inner radius a and outer radius b. Uniform current I flows along its inner surface and return through the outer surface as shown in figure 6. Find vector potential at a distance s from its axis.

Solution:
Using Stoke's theorem, we can write,

$$\oint \vec{A}\cdot d\vec{l} = \int (\vec{\nabla}\times\vec{A})\cdot d\vec{S} = \int \vec{B}\cdot d\vec{S},$$

and using the Ampere's law,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{enc}} \implies \vec{B} = \frac{\mu_0 I_{\text{enc}}}{2\pi s}\hat{\phi}$$

The current is along $\hat{z}$, hence the vector potential would be also in the $\hat{z}$ direction and $A_s = A_\phi = 0$.

$$\boxed{B = (\vec{\nabla}\times\vec{A})_\phi} = \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial s}\right) \implies A = A_z = -\int B\, ds.$$

For $s < a$, $I_{\text{enc}} = 0$ hence $B = 0$ and $A = \text{constant}$.

For $a < s < b$, $I_{\text{enc}} = I$ thus $\vec{A} = -\int B ds\ \hat{z} = -\int \frac{\mu_0 I}{2\pi s} ds\ \hat{z} = \left(-\frac{\mu_0 I}{2\pi}\ln\left(s/R_0\right)\right)\hat{z}$
where R_0 is a constant of integration, independent of ϕ, z.
For $s > b$, $I_{\text{enc}} = 0$ so again $B = 0$ and $A = \text{constant}$.

5. Show that for uniform magnetic field $\vec{B}$, the magnetic vector potential can be written as $\vec{A} = -\frac{1}{2}(\vec{r}\times\vec{B})$. Is this result unique, or are there other functions with the same properties?

Solution:

$$\vec{\nabla}\times\vec{A} = -\frac{1}{2}\vec{\nabla}\times(\vec{r}\times\vec{B}) = -\frac{1}{2}[(\vec{B}\cdot\vec{\nabla})\vec{r} - (\vec{r}\cdot\vec{\nabla})\vec{B} + \vec{r}(\vec{\nabla}\cdot\vec{B}) - \vec{B}(\vec{\nabla}\cdot\vec{r})]$$

Here $(\vec{r}\cdot\vec{\nabla})\vec{B} = 0$, as $\vec{B}$ is uniform and $\vec{r}(\vec{\nabla}\cdot\vec{B}) = 0, \vec{B}(\vec{\nabla}\cdot\vec{r}) = 3\vec{B}$. The remaining term can be written as

$$(\vec{B}\cdot\vec{\nabla})\vec{r} = \left(B_x\frac{\partial}{\partial x} + B_y\frac{\partial}{\partial y} + B_z\frac{\partial}{\partial z}\right)(x\hat{x} + y\hat{y} + z\hat{z}) = \vec{B}$$

Therefore, $\vec{\nabla}\times\vec{A} = -\frac{1}{2}(\vec{B} - 3\vec{B}) = \vec{B}$. Also,

$$\vec{\nabla}\cdot\vec{A} = -\frac{1}{2}\vec{\nabla}\cdot(\vec{r}\times\vec{B}) = -\frac{1}{2}[\vec{B}\cdot(\vec{\nabla}\times\vec{r}) - \vec{r}\cdot(\vec{\nabla}\times\vec{B})] = 0$$

using the fact that $\vec{B}$ is uniform and $\vec{\nabla}\times\vec{r} = 0$.

This result is not unique, we can always change $\vec{A}\to\vec{A}+\vec{C}$ where $\vec{C}$ is a constant vector. It can also be changed to $\vec{A}\to\vec{A}+\vec{\nabla}\lambda$ where λ is a scalar function which obeys Laplace's equation $\nabla^2\lambda = 0$. Note that we are considering Coulomb gauge $\vec{\nabla}\cdot\vec{A} = 0$ which the given expression for vector potential satisfies.

Alternate Solution:

(a) Let us take the direction of uniform magnetic field (B_0) to be along the z-axis. Using $\vec{B} = \vec{\nabla}\times\vec{A}$, we can write

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = 0, \quad B_y = \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} = 0, B_z = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} = B_0.$$

Although the vector potential is not unique, there exists an interesting solution of the above equations:

$$A_x = -\frac{1}{2}yB_0, \quad A_y = \frac{1}{2}xB_0, \quad A_z = 0.$$

Here $\vec{B}$ is along the z-axis and $\vec{A}$ is zero along z-axis. Also, the x-component of $\vec{A}$ is proportional to $-y$ and the y-component is proportional to $+x$, indicating that $\vec{A}$ must be at right angles to the vector from the z-axis. The magnitude of $\vec{A}$ is proportional to $\sqrt{x^2+y^2}$, the distance from the z-axis, denoted by r'. For any uniform field $\vec{B}$ the vector potential, therefore, can be written as

$$\vec{A} = \frac{1}{2}(\vec{B}\times\vec{r}) = -\frac{1}{2}(\vec{r}\times\vec{B}).$$

(b) Another way to find $\vec{A}$ is to take its line integral around a closed circular loop and then use the Stoke's theorem:

$$\oint\vec{A}\cdot d\vec{l} = \int(\vec{\nabla}\times\vec{A})\cdot d\vec{a} = \int\vec{B}\cdot\hat{n}da.$$

For uniform field chosen to be perpendicular to the plane of the loop, the right hand side of the above equation is $\pi r^2 B$. Choosing the origin to be on the axis of symmetry, so that we can take $\vec{A}$ as circumferential and a function of r only, then the line integral is

$$\oint \vec{A} \cdot d\vec{l} = A(2\pi r) = \pi r^2 B \implies A = \frac{Br}{2}.$$

Since $\vec{B}$ is perpendicular to the area of the loop and $\vec{A}$ is circumferential, it is straightforward to write $\vec{A} = \frac{1}{2}(\vec{B} \times \vec{r}) = -\frac{1}{2}(\vec{r} \times \vec{B})$.