PH101

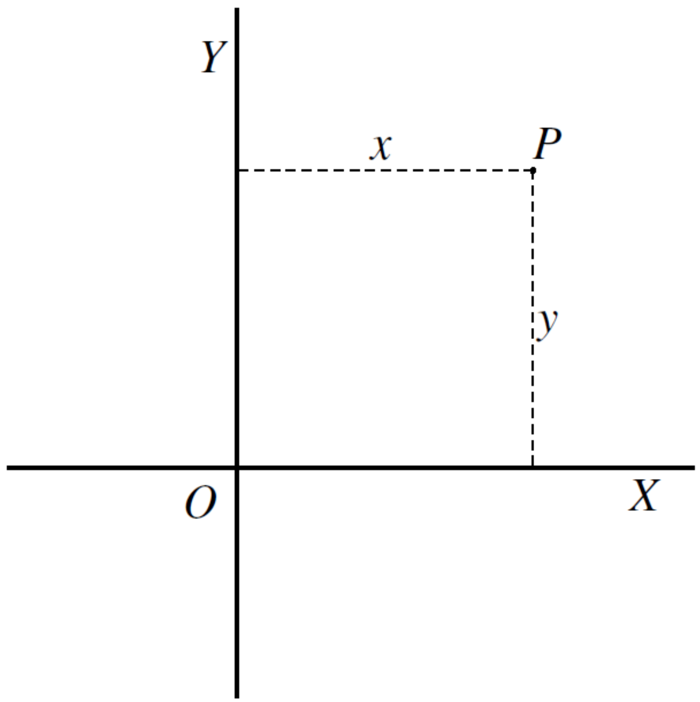
Lecture 2

Coordinate systems

http://www.iitg.ac.in/phy/ph101_2019.php

Cartesian Coordinate

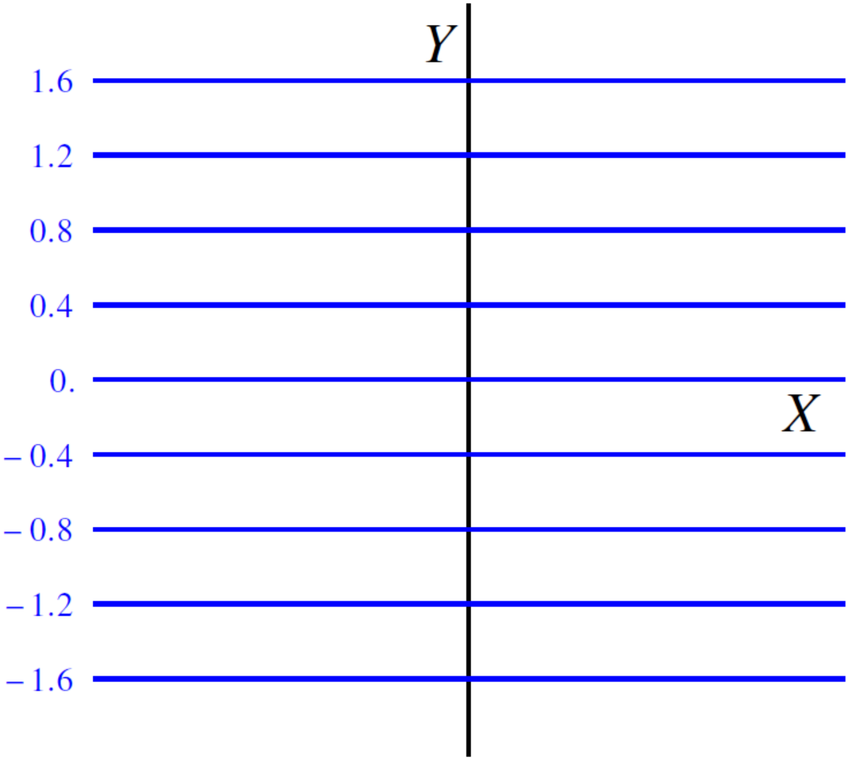
In Cartesian coordinate system each point is uniquely specified in a plane by a pair of numerical coordinates, measured in the same **unit of length**.



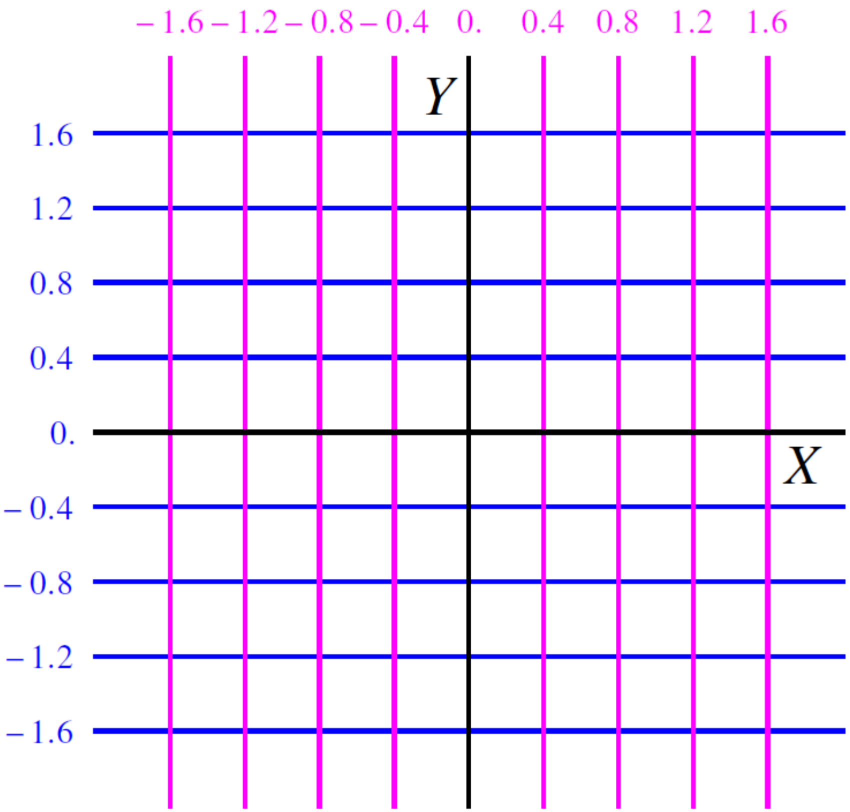
Each point P is identified with its unique x and y coord: $P = P(x, y)$.
Ranges: x, $y \in (-\infty, \infty)$.

Cartesian Coordinate

Horizontal lines



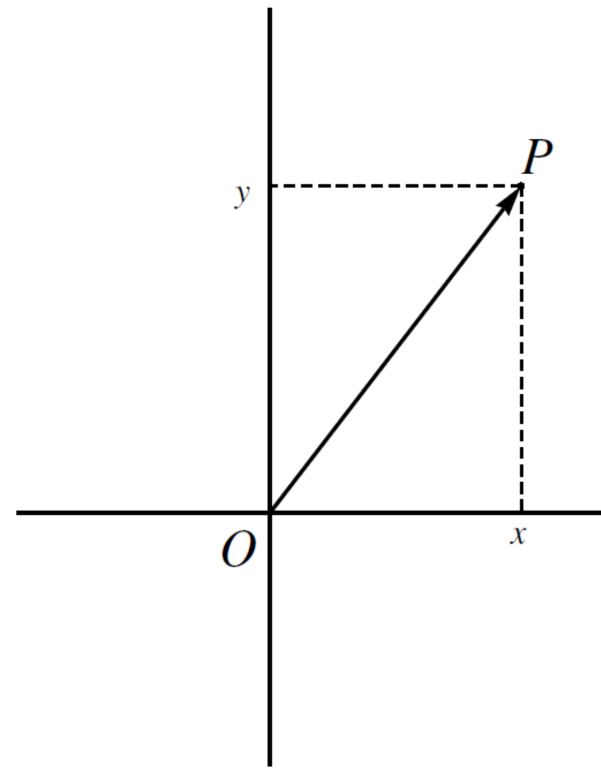
Vertcial lines



Clearly, each point has separate coordinates.

Cartesian Coordinate

- Geometric definition of magnitude and direction of vectors allow us to define operations: addition, subtraction, and multiplication by scalars.
- Often a coordinate system is helpful as it is easier to manipulate the coordinates of a vector, than manipulating its magnitude and direction.
- To determine vector coordinates, the first step is to translate the vector so that its tail is at the origin and head will be at some point $P(x, y)$.



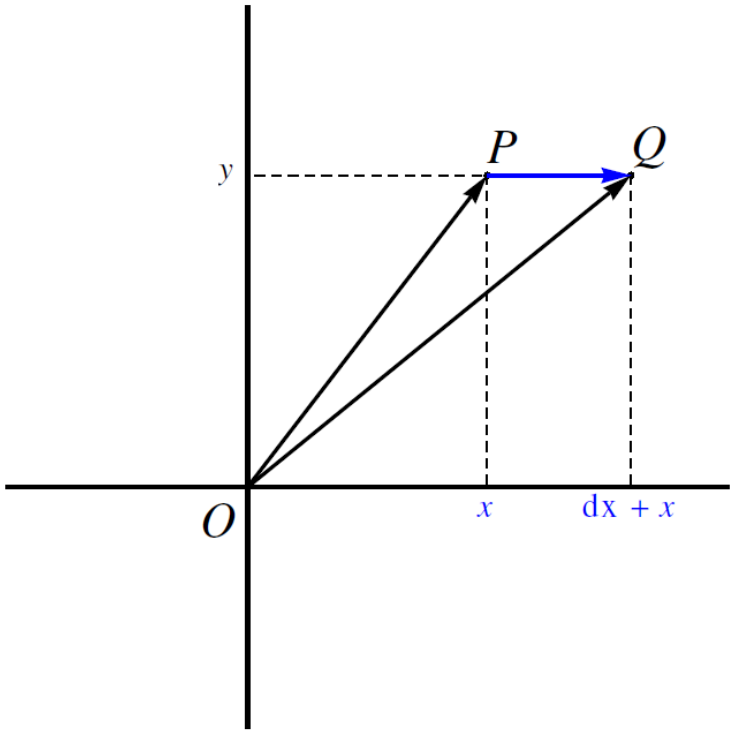
Cartesian Coordinate

- Position vector of P is $\mathbf{r} = \hat{\mathbf{i}}x + \hat{\mathbf{j}}y$.
- Change x only keeping y fixed such that P moved to Q.
- The "direction" of increasing the vector is defined as

$$\mathbf{e}_x = \frac{d\mathbf{r}}{dx}$$

This vector is not necessarily normalized to have unit length, but from it we can always construct the unit vector as

$$\hat{\mathbf{i}} = \frac{\mathbf{e}_x}{|\mathbf{e}_x|} = \frac{1}{|d\mathbf{r}/dx|}\frac{d\mathbf{r}}{dx}$$



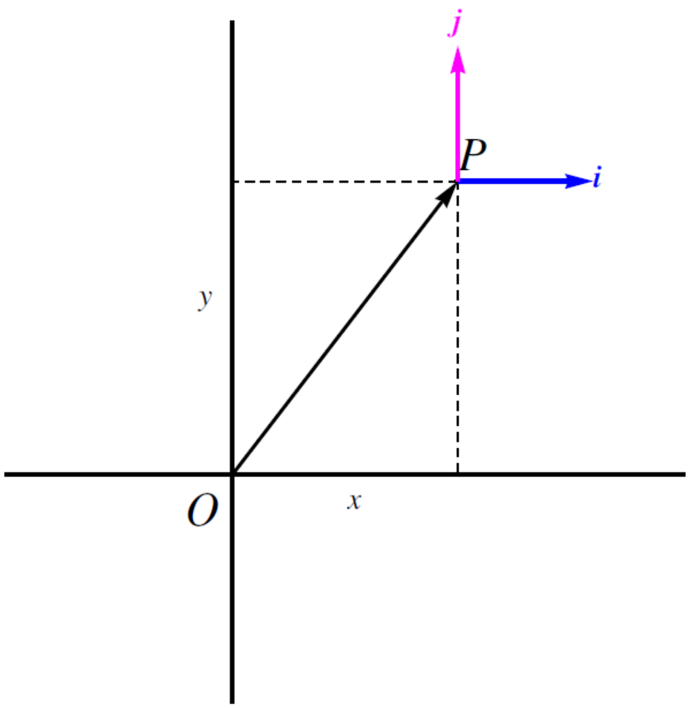
Cartesian Coordinate

Therefore, unit vectors are defined as

$$\hat{\mathbf{i}} = \frac{1}{|d\mathbf{r}/dx|}\frac{d\mathbf{r}}{dx}$$

and

$$\hat{\mathbf{j}} = \frac{1}{|d\mathbf{r}/dy|}\frac{d\mathbf{r}}{dy}$$



Velocity and acceleration in Cartesian

Velocity $\vec{\boldsymbol{v}} = \dfrac{d\vec{\boldsymbol{r}}}{dt}$

Velocity in Cartesian coordinate

$$\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{r}}}{dt} = \frac{d}{dt}\,(x\,\hat{\boldsymbol{x}} + y\,\hat{\boldsymbol{y}})$$

$$= \dot{x}\hat{\boldsymbol{x}} + x\,\frac{d\hat{\boldsymbol{x}}}{dt} + \dot{y}\hat{\boldsymbol{y}} + y\,\frac{d\hat{\boldsymbol{y}}}{dt}$$

$$\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{x}} + \dot{y}\hat{\boldsymbol{y}}$$

Since,

$$\frac{d\hat{\boldsymbol{x}}}{dt} = \frac{d\hat{\boldsymbol{y}}}{dt} = 0$$

Acceleration $\vec{\boldsymbol{a}} = \dfrac{d\vec{\boldsymbol{v}}}{dt} = \ddot{x}\hat{\boldsymbol{x}} + \ddot{y}\hat{\boldsymbol{y}}$

Newton's second law in vector form

$$\vec{\boldsymbol{F}} = F_x\hat{\boldsymbol{x}} + F_y\hat{\boldsymbol{y}} = m\frac{d\vec{\boldsymbol{v}}}{dt} = m(\ddot{x}\hat{\boldsymbol{x}} + \ddot{y}\hat{\boldsymbol{y}})$$

Standard Notations:

$$\frac{dx}{dt} = \dot{x}$$

$$\frac{dr}{dt} = \dot{r}$$

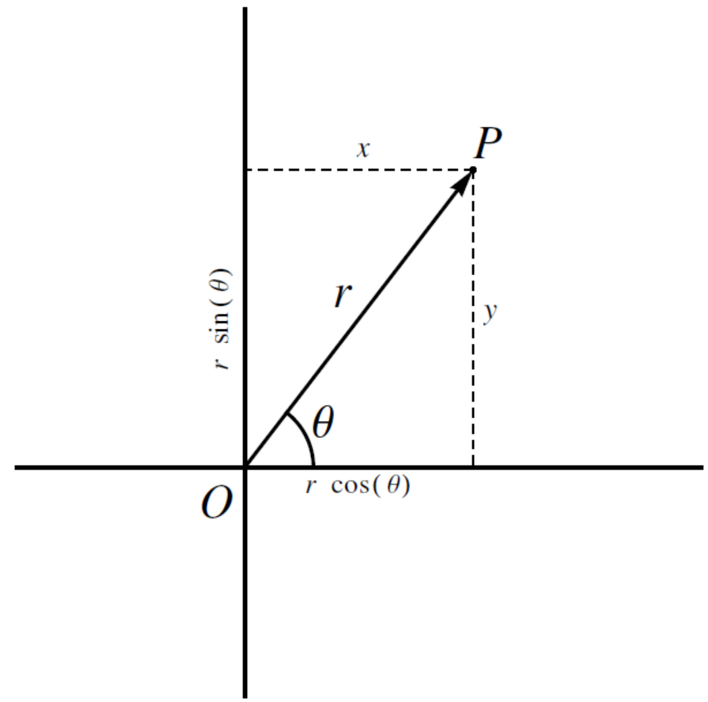
$$\frac{d^2\theta}{dt^2} = \ddot{\theta}$$

Polar Coordinates

Each point $P = (x, y)$ of a plane can also be specified by its distance from the origin, O and the angle that the line OP makes with $x-$axis.

- Transformations:

$$\begin{aligned} r &= +\sqrt{x^2 + y^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$



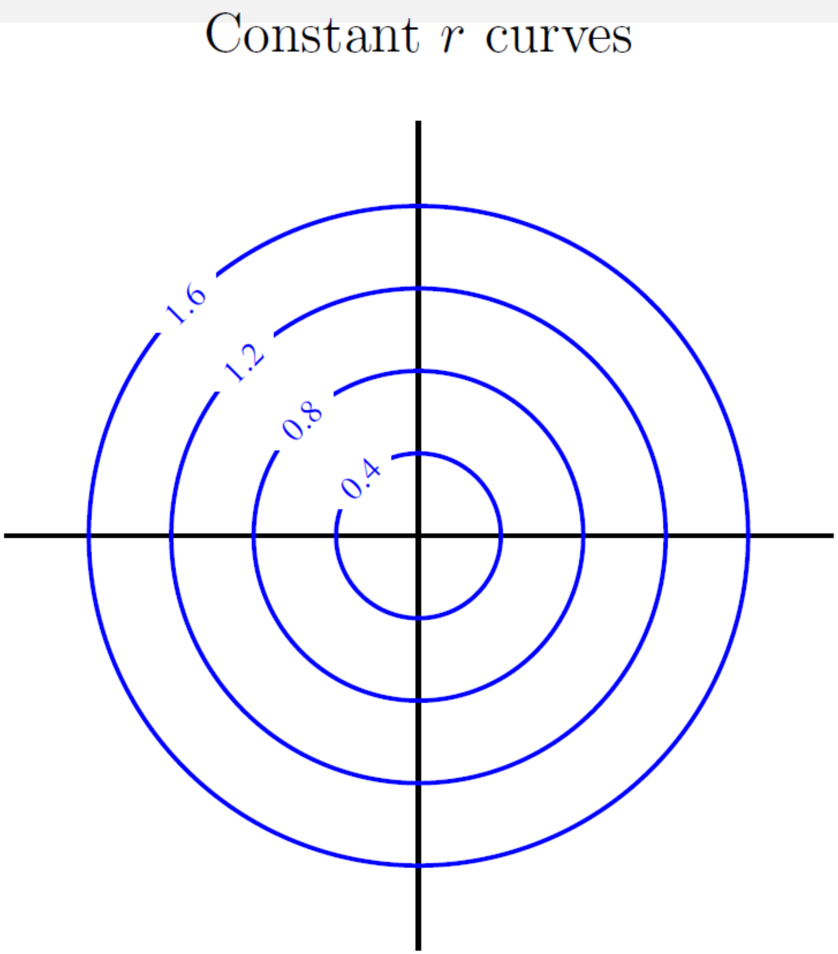
(r, θ) are called Plane Polar Coordinates.

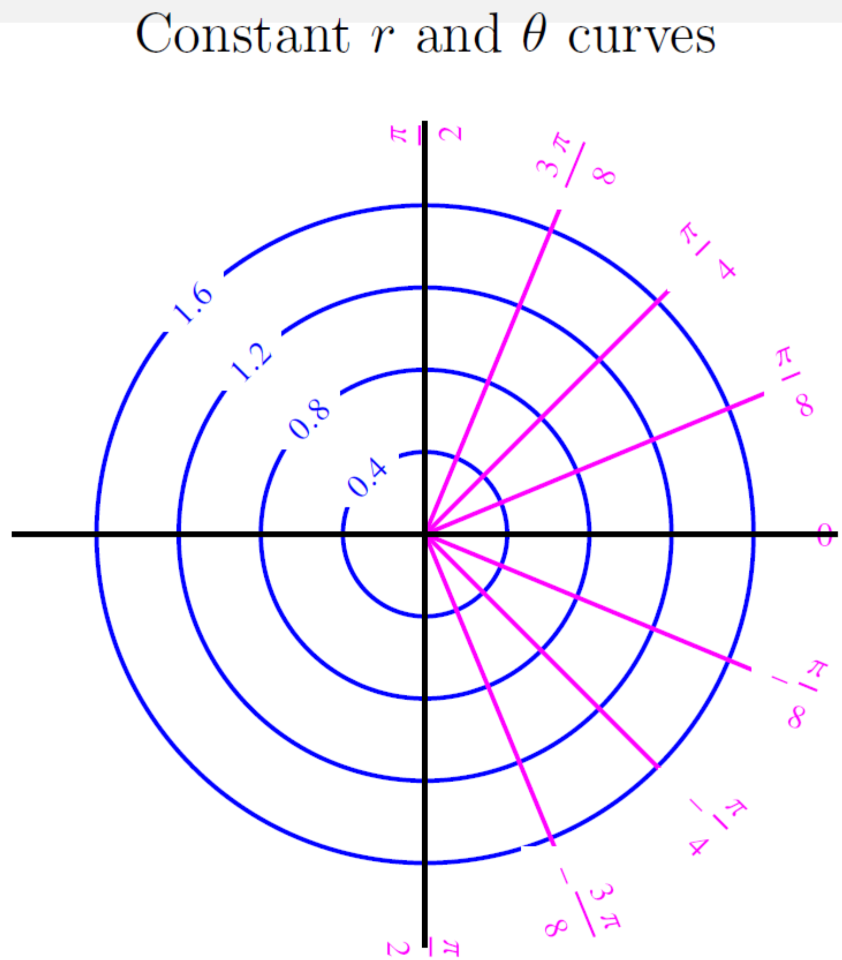
- Inverse Transformations:

$$\begin{aligned} x &= r\ \cos\theta \\ y &= r\ \sin\theta \end{aligned}$$

- Ranges: $r \in (0,\ \infty)$ and $\theta \in (0,\ 2\pi)$.
- $\tan\theta = \tan(\theta + \pi)$ suggests to replace θ by $\theta + \pi$, for $x < 0$, $y < 0$.
- Discontinuity in θ at origin.

Polar Coordinates

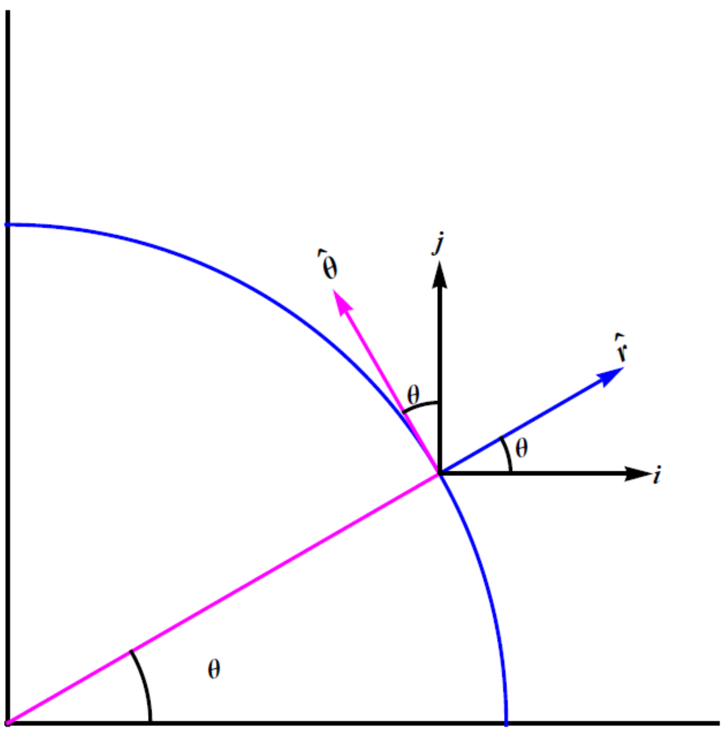




- Unit vectors are perpendicular to constant coordinate curves

Unit Vectors

In Cartesian coordinate, PV of P is $\mathbf{r} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$. We find a set of two unit vectors $\hat{\mathbf{r}}$ and $\hat{\theta}$.



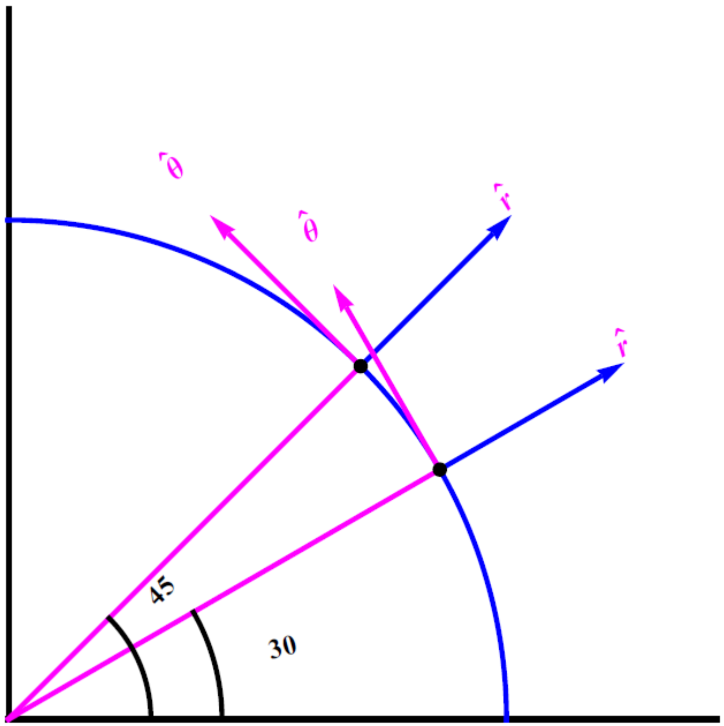
- By Coordinate transformation, $\mathbf{r} = r\cos\theta\hat{\mathbf{i}} + r\sin\theta\hat{\mathbf{j}}$
- Unit vectors are defined as,

$$\begin{aligned}
\hat{\mathbf{r}} &= \frac{1}{|\frac{d\mathbf{r}}{dr}|}\frac{d\mathbf{r}}{dr} = \hat{\mathbf{i}}\cos\theta + \hat{\mathbf{j}}\sin\theta \\
\hat{\theta} &= \frac{1}{|\frac{d\mathbf{r}}{d\theta}|}\frac{d\mathbf{r}}{d\theta} = -\hat{\mathbf{i}}\sin\theta + \hat{\mathbf{j}}\cos\theta
\end{aligned}$$

- And

$$\begin{aligned}
\hat{\mathbf{i}} &= \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta} \\
\hat{\mathbf{j}} &= \sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\theta}
\end{aligned}$$

Unit Vectors



- Unit vectors at a point P depend on θ coordnate of P
- Example: If $P \equiv (1, \pi/6)$, then

$$\hat{\mathbf{r}} = \frac{1}{2}\left(\sqrt{3}\hat{\mathbf{i}} + \hat{\mathbf{j}}\right)$$
$$\hat{\theta} = \frac{1}{2}\left(-\hat{\mathbf{i}} + \sqrt{3}\hat{\mathbf{j}}\right)$$

- Example: Let $P \equiv (1, \pi/4)$, then

$$\hat{\mathbf{r}} = \frac{1}{\sqrt{2}}\left(\hat{\mathbf{i}} + \hat{\mathbf{j}}\right)$$
$$\hat{\theta} = \frac{1}{\sqrt{2}}\left(-\hat{\mathbf{i}} + \hat{\mathbf{j}}\right)$$

- Note that everywhere, $\hat{\mathbf{r}} \perp \hat{\theta}$.

Unit Vectors

These unit vectors are functions of the polar coordinates, only of θ in fact.

$$
\begin{aligned}
\hat{\mathbf{r}} &= \mathbf{i}\cos\theta + \mathbf{j}\sin\theta \\
\hat{\theta} &= -\mathbf{i}\sin\theta + \mathbf{j}\cos\theta
\end{aligned}
$$

$$
\boxed{
\begin{aligned}
\frac{\partial \hat{\mathbf{r}}}{\partial \theta} &= \hat{\theta} \\
\frac{\partial \hat{\theta}}{\partial \theta} &= -\hat{\mathbf{r}}
\end{aligned}
}
$$

Time derivates

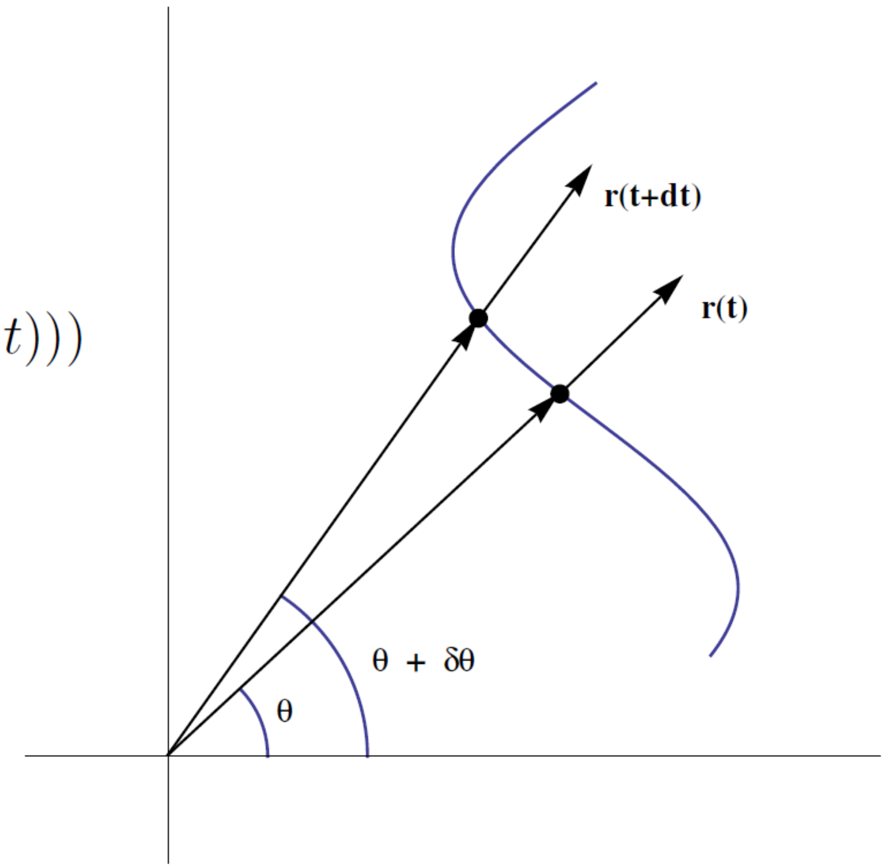
$$
\boxed{
\begin{aligned}
\dot{\hat{\mathbf{r}}} &= \dot{\theta}\hat{\theta} \\
\dot{\hat{\theta}} &= -\dot{\theta}\hat{\mathbf{r}}
\end{aligned}
}
$$

Motion in Polar Coordinates

Suppose a particle is travelling along a trajectory given by $\mathbf{r}(t)$. Now the position vector

$$\begin{aligned}\mathbf{r}(t) &= x(t)\,\hat{\mathbf{i}} + y(t)\,\hat{\mathbf{j}} \\ &= r(t)\left(\hat{\mathbf{i}}\cos(\theta(t)) + \hat{\mathbf{j}}\sin(\theta(t))\right) \\ &= r(t)\,\hat{\mathbf{r}}(\theta(t))\end{aligned}$$

$\mathbf{r}(t)$ depends on θ implicitly through $\hat{\mathbf{r}}$ vector.



We need to know θ to see how $\mathbf{r}$ is oriented. We need to know $\mathbf{r}$ to see how far we are from the origin.

Motion in Polar Coordinates

- Then the velocity vector

$$
\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} \\
&= \frac{d}{dt}\left[r\left(t\right)\hat{\mathbf{r}}\left(\theta\left(t\right)\right)\right] \\
&= \frac{dr}{dt}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} \\
&= \dot{r}\hat{\mathbf{r}}+r\dot{\theta}\hat{\theta}
\end{aligned}
$$

- Radial velocity $= \dot{r}\hat{\mathbf{r}}$
- Tangential velocity $= r\dot{\theta}\hat{\theta}$
- $\dot{\theta}$ is called the angular speed.

Motion in Polar Coordinates

- The Acceleration vector

$$
\begin{aligned}
\mathbf{a} &= \frac{d\mathbf{v}}{dt} \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\frac{d\hat{\mathbf{r}}}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{\mathbf{r}} \\
&= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{\theta}
\end{aligned}
$$

- Radial component: $\ddot{\mathrm{r}}$ is the linear acceleration in radial direction. The term $-\mathrm{r}\dot{\theta}^2\hat{\mathrm{r}}$ is usual centripetal acceleration.
- Tangential component: $\mathrm{r}\ddot{\theta}$ is the linear acceleration in the tangential direction. The term $2\dot{\mathrm{r}}\dot{\theta}\hat{\theta}$ is called Coriolis acceleration.

Newton's law in plane polar coordinate

$$\vec{\boldsymbol{F}} = m\frac{d\vec{\boldsymbol{v}}}{dt}$$

$$\vec{\boldsymbol{F}} = F_r\hat{\boldsymbol{r}} + F_\theta\hat{\boldsymbol{\theta}} = m[(\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\boldsymbol{\theta}}]$$

Force component in **radial** direction: $F_r = m(\ddot{r} - r\dot{\theta}^2)$

Force component in **tangential** direction: $F_\theta = m(2\dot{r}\dot{\theta} + r\ddot{\theta})$

Note: Newton's law in polar coordinates does not follow its **Cartesian form** as,

$$F_r \neq m\ddot{r} \quad \text{or} \qquad F_\theta \neq m\ddot{\theta}$$

Example

Circular Motion

In a circular motion, $r = R = Constant$. Then, $\dot{r} = \ddot{r} = 0$. Thus

$$\begin{aligned} v &= R\dot{\theta}\hat{\theta} \\ a &= -R\dot{\theta}^2\hat{\mathbf{r}} + R\ddot{\theta}\hat{\theta} \end{aligned}$$

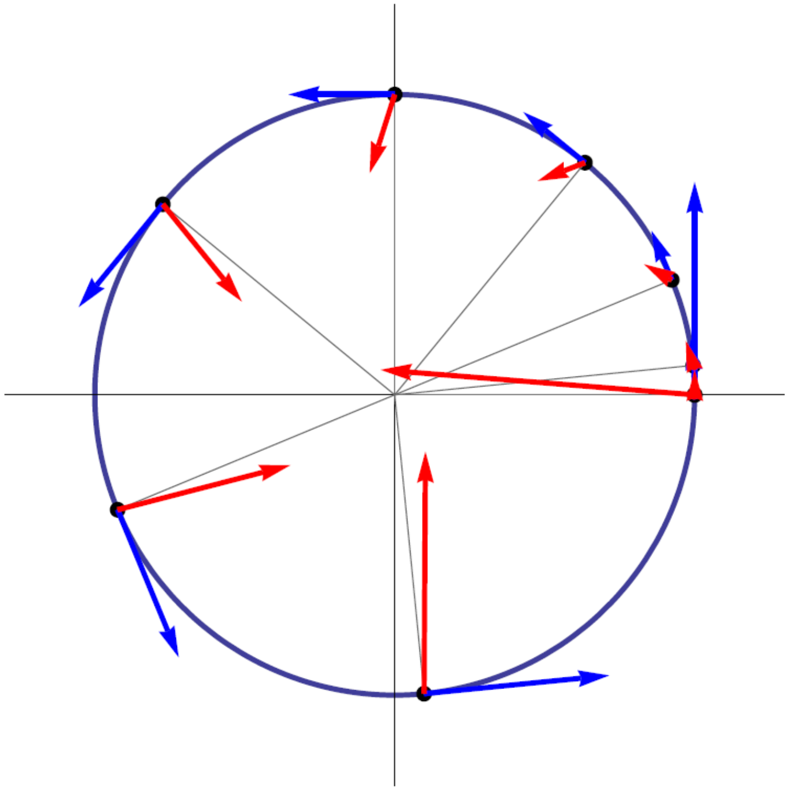
Consider a special case of uniform circular motion: $\ddot{\theta} = 0$.

- Speed is constant, and velocity is tangential
- Acceleration is radial (central)

Example

Circular Motion

Consider a case of a non-uniform circular motion in which
$\ddot{\theta} = \alpha = \text{Constant}$



Here,

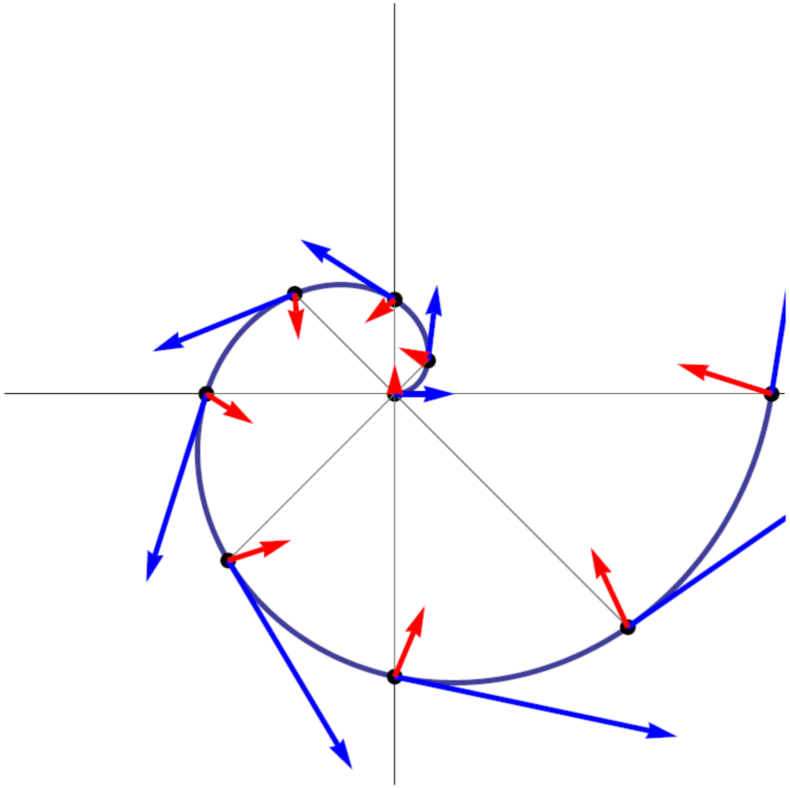
$$
\begin{aligned}
v &= R\dot{\theta}\hat{\theta} = R\alpha t\hat{\theta} \\
a &= -R\dot{\theta}^2\hat{\mathbf{r}} + R\ddot{\theta}\hat{\theta} \\
&= -R\alpha^2 t^2\hat{r} + R\alpha\hat{\theta}
\end{aligned}
$$

Example

Spiral Motion

Consider a particle moving on a spiral given by r $= a\theta$ with a uniform angular speed ω. Then $\dot{r} = a\dot{\theta} = a\omega$.

- $\mathbf{v} = a\omega\hat{\mathbf{r}} + a\omega^2 t\hat{\theta}$and $\mathbf{a} = -a\omega^3 t\hat{\mathbf{r}} + 2a\omega^2\hat{\theta}$



Example

Central Acceleration

- Central Accelerations

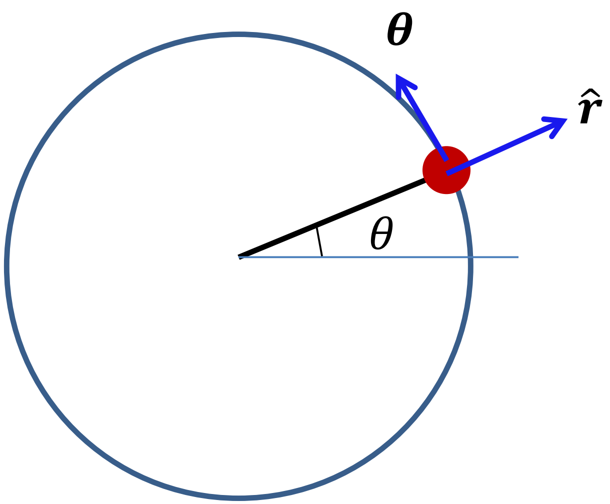
When the acceleration of a particle points to the origin at all times and is only function of distance of the particle from the origin, the acceleration is called central acceleration.

$$a = f(r)\hat{\mathbf{r}}$$

- Examples are $1/r^2$ (Gravitational and Electrostatic), $1/r^6$ (Van-dar-Waals), kr (Spring) etc.
- Angular momentum of the particle remains constant.

$$\begin{aligned} r\ddot{\theta} + 2\dot{r}\dot{\theta} &= \frac{d}{dt}(r^2\dot{\theta}) = 0 \\ r^2\dot{\theta} &= \text{constant during motion.} \end{aligned}$$

Choice of proper coordinate system makes analysis easier



Equation of trajectory in polar coordinate

$$r = R = constant$$

$$\theta = \omega t + \frac{1}{2}\alpha t^2$$

Motion in non-uniform circular trajectory with $\dot{\theta} = \omega + \alpha t$ where ω and α are constants.

The velocity components are

$$v_r = \dot{r} = 0; \; v_\theta = r\dot{\theta} = R(\omega + \alpha t) = v$$

Acceleration components are

$$a_r = \ddot{r} - r\dot{\theta}^2 = -R(\omega + \alpha t)^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = R\alpha = a_t$$

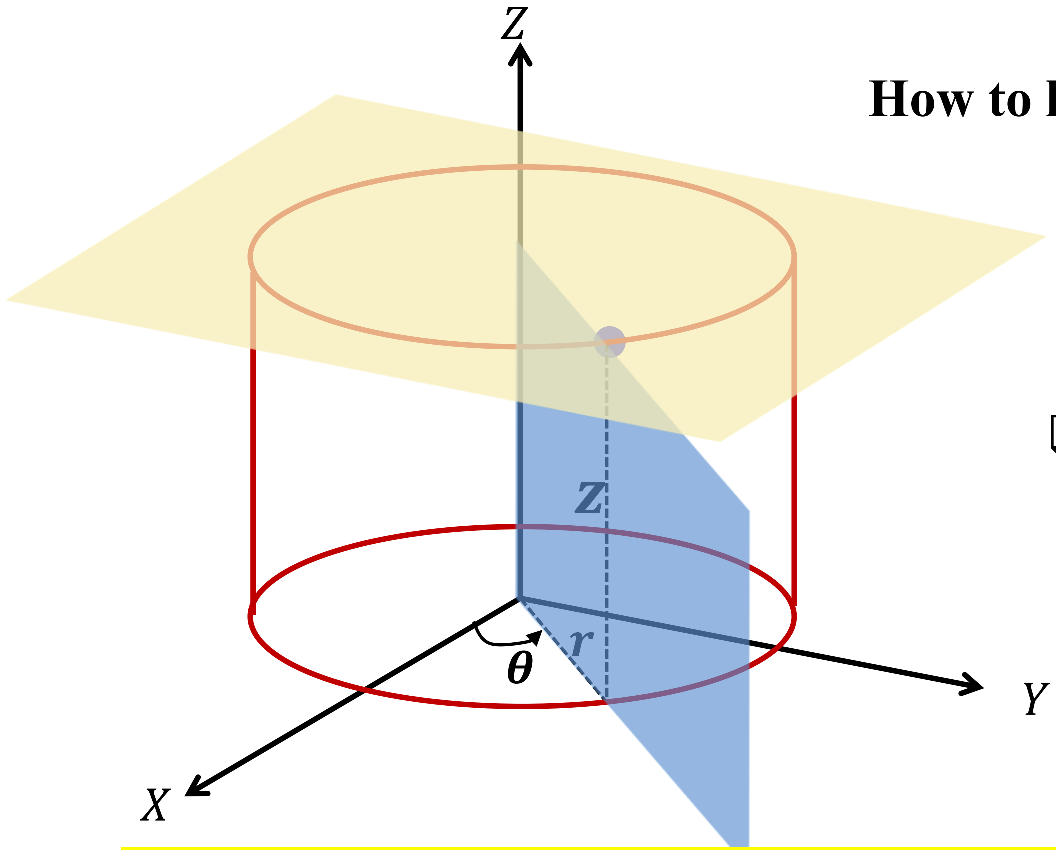
Equation of trajectory in Cartesian coordinate

$x = R\cos\left(\omega t + \frac{1}{2}\alpha t^2\right)$; $y = R\sin\left(\omega t + \frac{1}{2}\alpha t^2\right)$;

velocity components are

$$v_x = -R(\omega + \alpha t)\sin\left(\omega t + \frac{1}{2}\alpha t^2\right); \; v_y = R(\omega + \alpha t)\cos\left(\omega t + \frac{1}{2}\alpha t^2\right)$$

Cylindrical coordinate system



How to locate a point 'P' in space ?

- ❑ z-Height from the XY plane
- ❑ (r, θ) Coordinate of the foot of the point in XY plane

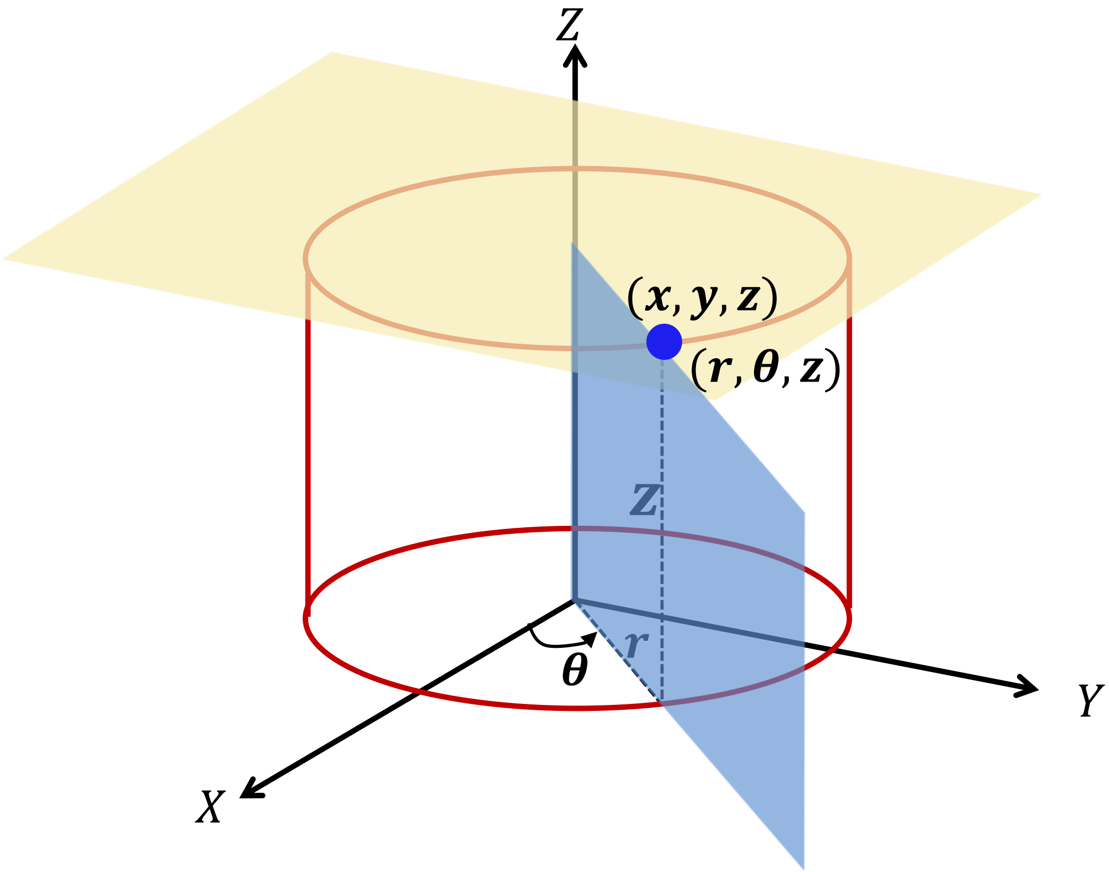
❑ $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{z})$ coordinates system is known as cylindrical coordinate system

Why the name cylindrical?

❑ Point 'P' is the intersection of three surfaces: A plane **z = constant**, a cylindrical surface $\boldsymbol{r = constant}$ and a half plane containing z-axis with $\boldsymbol{\theta}$ = **constant.**

Coordinate transformation: Cartesian to cylindrical

Transformation equation is very similar to polar coordinate with additional *z*-coordinate.



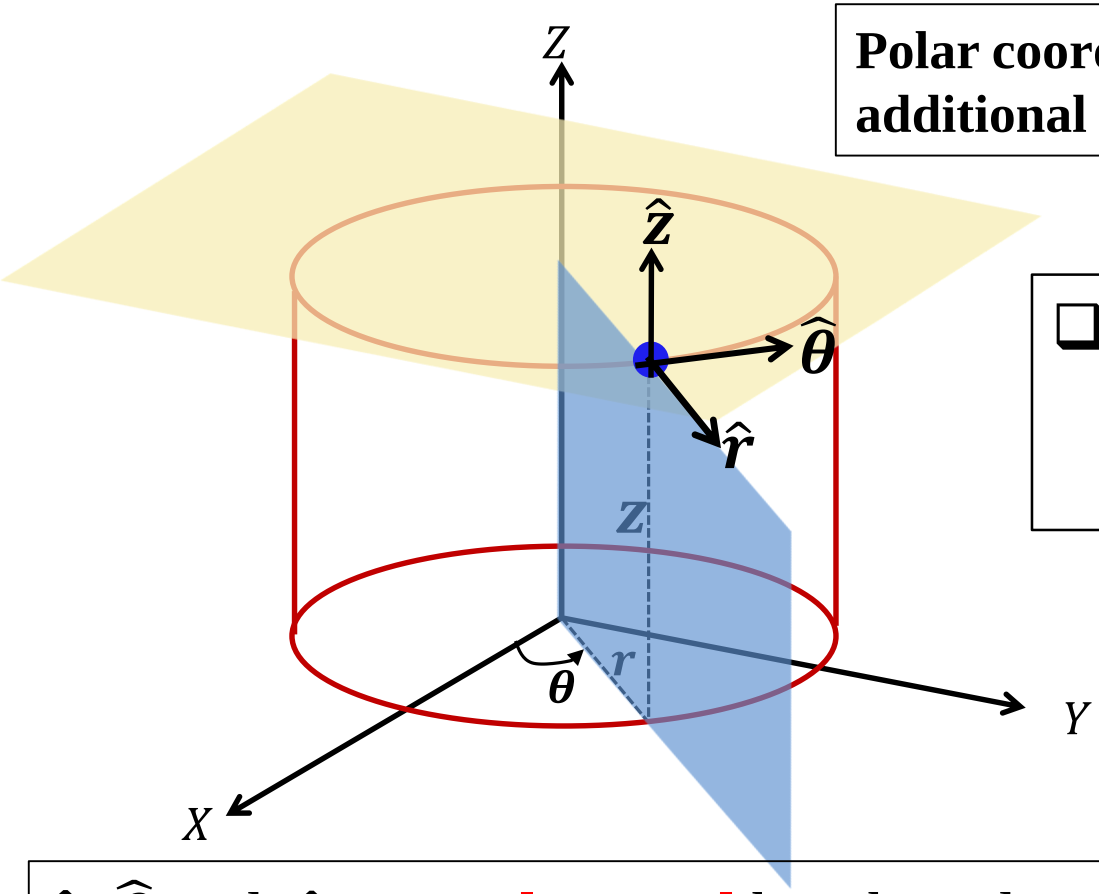
$$\boldsymbol{x = r\cos\theta}$$
$$\boldsymbol{y = r\sin\theta}$$
$$\boldsymbol{z = z}$$

Reverse transformation

$$\boldsymbol{r = \left(x^2 + y^2\right)^{1/2}}$$
$$\boldsymbol{\theta = \tan^{-1}\frac{y}{x}}$$
$$\boldsymbol{z = z}$$

Note: Instead of (r, θ) many books use notation (ρ, φ).

Unit vectors in cylindrical coordinate system



Polar coordinate unit vectors $(\hat{\boldsymbol{r}}, \widehat{\boldsymbol{\theta}})$ + additional unit vector in the $\boldsymbol{z}$ − direction.

- $\hat{\boldsymbol{r}}, \widehat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{z}}$ are unit vectors along increasing direction of coordinates r, θ and z.

$$\hat{\boldsymbol{r}} = \cos\theta\ \widehat{\boldsymbol{x}} + \sin\theta\ \widehat{\boldsymbol{y}}$$
$$\widehat{\boldsymbol{\theta}} = -\sin\theta\ \widehat{\boldsymbol{x}} + \cos\theta\ \widehat{\boldsymbol{y}}$$
$$\hat{\boldsymbol{z}} = \hat{\boldsymbol{z}}$$

$\hat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{z}}$ are **orthogonal** but their directions depend on location.

$\hat{\boldsymbol{r}}$, $\widehat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{z}}$ are **perpendicular** to surfaces $\boldsymbol{r = constant}$; $\boldsymbol{\theta}$ **= constant** and $\boldsymbol{z}$ **= constant** respectively.

Position, Velocity, Acceleration, Newton's law in cylindrical coordinate system

Vector components are very similar to polar coordinate + $z -$ component

Position vector → $\vec{\boldsymbol{r}} = r\hat{\boldsymbol{r}} + z\hat{\mathbf{z}}$

Velocity → $\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\widehat{\boldsymbol{\theta}} + \dot{z}\hat{\mathbf{z}}$

Acceleration → $\vec{\boldsymbol{a}}=(\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta}\ )\widehat{\boldsymbol{\theta}} + \ddot{z}\hat{\mathbf{z}}$

Newton's law →

$$\vec{\boldsymbol{F}} = F_r\hat{\boldsymbol{r}} + F_\theta\widehat{\boldsymbol{\theta}} + F_z\hat{\mathbf{z}}$$

$$= \ m[(\ddot{r} - r\dot{\theta}^2)\hat{\boldsymbol{r}} + (2\dot{r}\dot{\theta} + r\ddot{\theta}\ )\widehat{\boldsymbol{\theta}} + \ddot{z}\hat{\mathbf{z}}]$$

Summery

- A point in plane can be represented by Cartesian coordinate P(x, y) or polar coordinate P(r, θ). A point in space can be represented by (x, y, z) or (r, θ, z) or (r, θ, φ).

- Coordinate transformation relation between *Cartesian* and *cylindrical coordinate* is given by

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

- For plane polar coordinate, transformation relation is $x = r\cos\theta ; y = r\sin\theta$

- Unit vector in plane polar coordinate:

$$\hat{\boldsymbol{r}} = \cos\theta\ \hat{\boldsymbol{x}} + \sin\theta\ \hat{\boldsymbol{y}}\ \ ; \hat{\boldsymbol{\theta}} = -\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}}$$

- Unit vectors in cylindrical coordinate**:**

$$\hat{\boldsymbol{r}} = \cos\theta\ \hat{\boldsymbol{x}} + \sin\theta\ \hat{\boldsymbol{y}}\ \ ; \hat{\boldsymbol{\theta}} = -\sin\theta\ \hat{\boldsymbol{x}} + \cos\theta\ \hat{\boldsymbol{y}}, z = z$$

- Form of Newton’s law is different in different coordinate systems.

Questions please