PH 102, Electromagnetism,

Post Mid Semester

Lecture 7.

Electrodynamics

Maxwell's Equations:

D. J. Griffiths: 7.3

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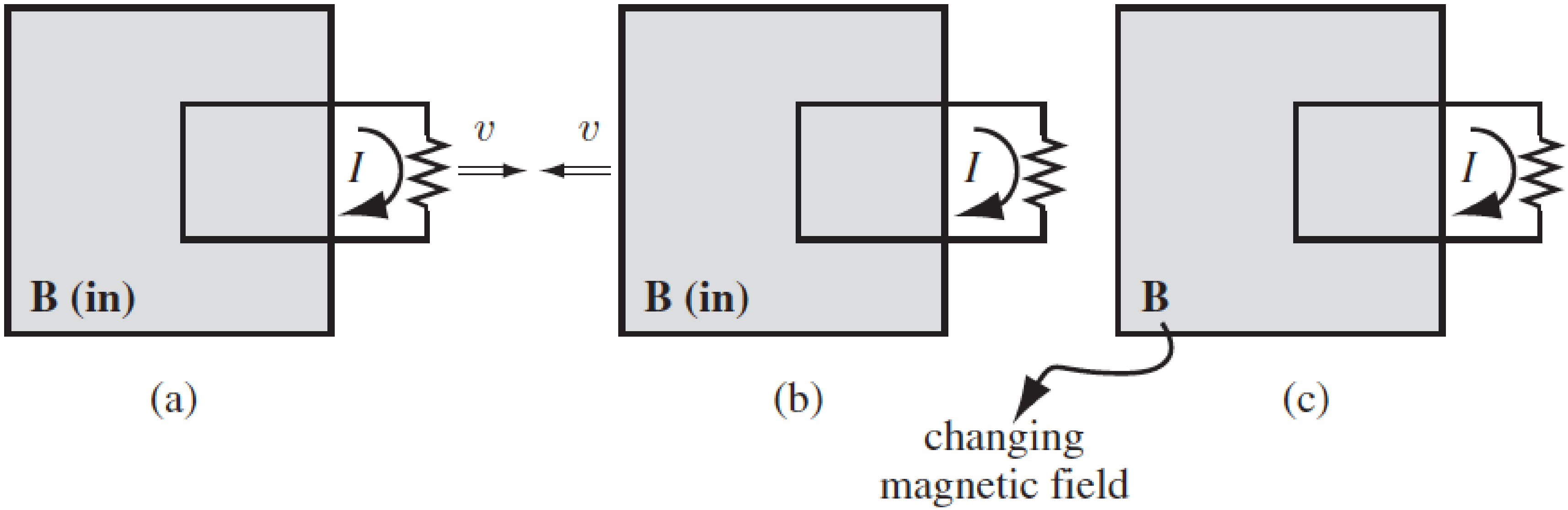
Faraday's Law



(a) Case of motional emf;

(b) 2nd Experiment has the same emf, relative motion of loop and the magnet.

(c) 3rd scenario also an electric field gets generated and gives the same emf $\mathcal{E} = -\frac{d\Phi}{dt}$

Changing magnetic field generates an electric field

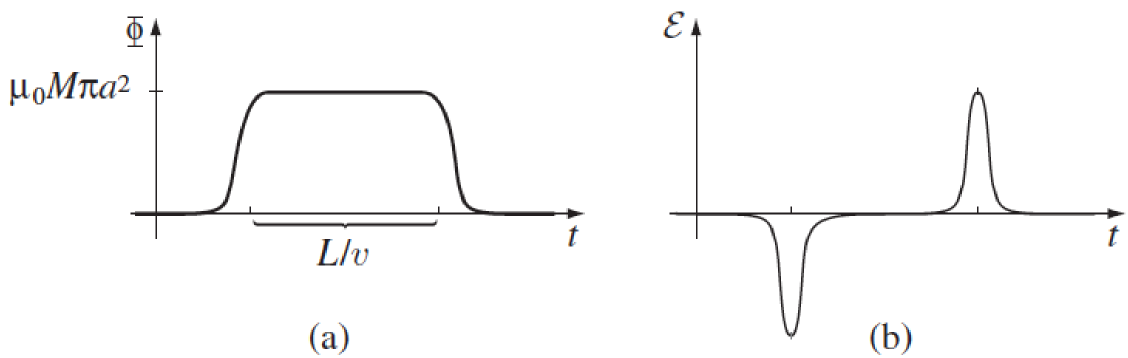
Note: In (a), $\mathbf{v} \times \mathbf{B}$ drives the current $\boldsymbol{I}$ not $\boldsymbol{E}$

In (b) & (c), induced $\boldsymbol{E}$ drives the current $\boldsymbol{I}$

Lenz's Law

The induced current flows in such a direction that the flux it produces tends to cancel the change. We can not quantify the current but can get the directions right.

Nature abhors a change in flux.



The magnet enters the ring, flux increases. The current is clockwise to generate field to the right.

Change in the flux is prevented, not flux

The magnet exits ring, flux drops, counterclockwise current to restore the field.

Induced Electric Field

Two distinct kinds of electric fields:

From electric charges: static case using Coulombs Law
and
changing magnetic fields: Faraday's Law

For pure Faraday Field $\nabla \cdot \mathbf{E} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$

[In magnetostatics, $\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}.$]

The analog to Bio-savart's law

$$\mathbf{E} = -\frac{1}{4\pi} \int \frac{(\partial \mathbf{B}/\partial t) \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int \frac{\mathbf{B} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} d\tau,$$

If symmetry permits, the tricks associated with Ampere's law are permissible.

$$\oint \mathbf{E} \cdot d\mathbf{l} = -\frac{d\Phi}{dt}.$$

Maxwell's Equations

Electrodynamics before Maxwell.

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \text{(Gauss's law)},$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{(no name)},$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(Faraday's law)},$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \qquad \text{(Ampère's law)}.$$

$+$

$$\mathbf{F} = q\{\mathbf{E} + \mathbf{v} \times \mathbf{B}\}.$$

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$$+ \qquad \mathbf{F} = q\{\mathbf{E} + \mathbf{v} \times \mathbf{B}\}.$$

However, there is a vital Inconsistency!!!
Ampere's law may fail for non steady current

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0.$$

From Faraday's law and $\nabla \cdot \mathbf{B} = 0$

$$\nabla \cdot (\nabla \times \mathbf{E}) = \nabla \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla \cdot \mathbf{B}) = 0\,.$$

Maxwell's Equations

Electrodynamics before Maxwell.

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{\epsilon_0}\rho && \text{(Gauss's law)},\\
\nabla \cdot \mathbf{B} &= 0 && \text{(no name)},\\
\nabla \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t} && \text{(Faraday's law)},\\
\nabla \times \mathbf{B} &= \mu_0 \mathbf{J} && \text{(Ampère's law)}.
\end{aligned}$$

$+$

$$\mathbf{F} = q\{\mathbf{E} + \mathbf{v} \times \mathbf{B}\}.$$

However, there is a vital Inconsistency!!!

$$\nabla \cdot (\nabla \times \mathbf{v}) = 0.$$

From Ampere's law

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 (\nabla \cdot \mathbf{J});$$

Only for steady currents, $\nabla \cdot \mathbf{J} = 0.$

Essential, ONLY for magnetostatics!

Maxwell's Equations

Electrodynamics before Maxwell.

However, there is a vital Inconsistency!!!

From Ampere's law

For steady currents,

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 (\nabla \cdot \mathbf{J}); \qquad \nabla \cdot \mathbf{J} = 0.$$

Ampere's law only true for magnetostatics!

Electrodynamics: Ampere's law bound to fail for non steady current

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}.$$

and

$-\frac{\partial \mathbf{B}}{\partial t}$ is connected to ***E***

$\frac{\partial \mathbf{E}}{\partial t}$ ***???***

Maxwell's Equations

Electrodynamics before Maxwell.

However, there is a vital Inconsistency!!!

From Ampere's law

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0(\nabla \cdot \mathbf{J});$$

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and

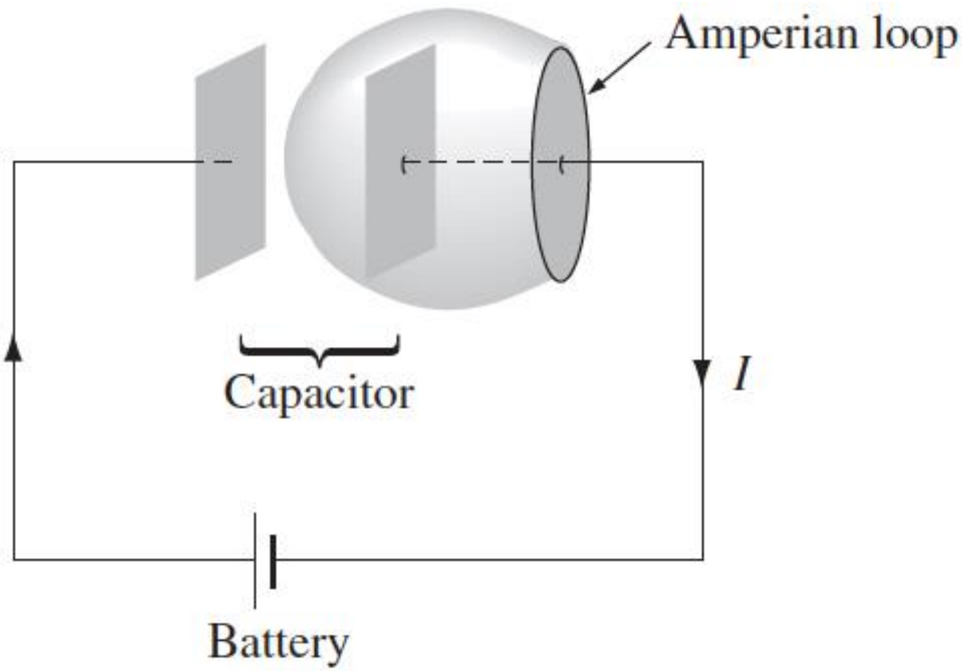
$-\frac{\partial \mathbf{B}}{\partial t}$ is connected to ***E***

$\frac{\partial \mathbf{E}}{\partial t}$ ***???***

Non steady current?? Charging of capacitor

Maxwell's Equations

Example of non-steady currents: Capacitor, charges are piling on the plates.
Steady currents can not flow into a capacitor forever
but a varying current can, as the capacitor charges and discharges!



$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{enc}.$$

- I_{enc}: Current piercing a surface with loop for the boundary.
- Surface lies on the plane of the loop, $I_{enc} = I$.
- Balloon shaped surface, $I_{enc} = 0$.
- Physics cannot depend on your choice of surface!!

How to fix it!!

Introduce Displacement current

Maxwell's Equations

Fixing Ampere's law by displacement current

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial t}(\epsilon_0 \nabla \cdot \mathbf{E}) = -\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right)$$

[using , $\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho$]

Thus, the following modified form can rescue the $\nabla \cdot (\nabla \times \mathbf{v}) = 0$.

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

$$\nabla \cdot (\nabla \times \mathbf{B}) = \mu_0 (\nabla \cdot \mathbf{J}) + \mu_0 \nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) = \mu_0 \left(\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}\right) = 0.$$

Also rescue the continuity equation!!

Comments:

1) Magnetostatics can not detect this as ***E*** is constant .

2) Difficult to find/detect in ordinary EM experiments, Faraday did not see any effect.

3) **Changing Electric field also induces magnetic field!!**

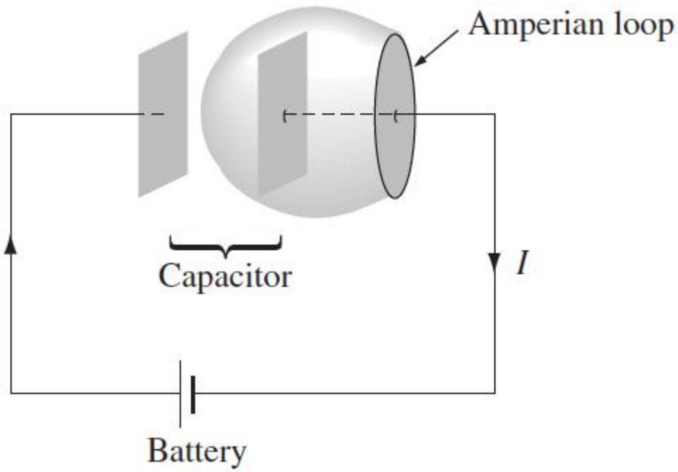
Wait for the EM waves discussion!

Maxwell's Equations

Dispalcement Current

$$\mathbf{J}_d \equiv \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

(Current! as added to $\boldsymbol{J}$)



Electric field between the plates,

Thus, $$E = \frac{1}{\epsilon_0}\sigma = \frac{1}{\epsilon_0}\frac{Q}{A},$$

[Q charge on plate with are A]

$$\frac{\partial E}{\partial t} = \frac{1}{\epsilon_0 A}\frac{dQ}{dt} = \frac{1}{\epsilon_0 A} I$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} + \mu_0 \epsilon_0 \int \left(\frac{\partial \mathbf{E}}{\partial t}\right) \cdot d\mathbf{a}$$

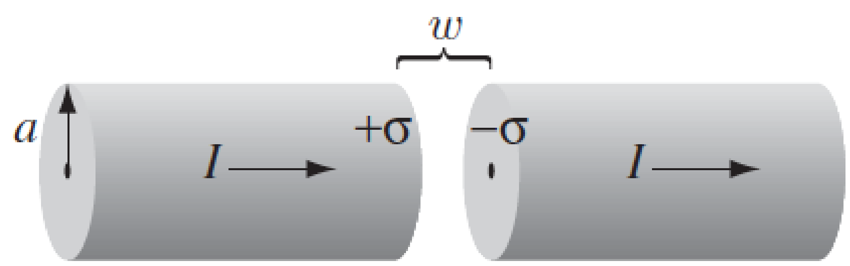
New surface term!!

- Surface lies on the plane of the loop, $E = 0$ and $I_{\text{enc}} = I$; Conduction current
- Balloon shaped surface, $I_{\text{enc}} = 0$, but $\int (\partial \mathbf{E}/\partial t) \cdot d\mathbf{a} = I/\epsilon_0$;

Displacement current

Maxwell's Equations

Problem 7.34 A fat wire, radius a, carries a constant current I, uniformly distributed over its cross section. A narrow gap in the wire, of width $w \ll a$, forms a parallel-plate capacitor, as shown in Fig. 7.45. Find the magnetic field in the gap, at a distance $s < a$ from the axis.



Displacement current,

$$\vec{J}_d = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} = \frac{\vec{I}}{A} = \frac{I}{A}\hat{z}$$

For an Amperian loop (s < a),

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{denc}$$

$$B \times 2\pi s = \mu_0 \frac{I}{\pi a^2} \pi s^2$$

Thus the magnetic field in the gap is,

$$\vec{B} = \frac{\mu_0 I s^2}{2\pi a^2}\hat{\phi}$$

Maxwell's Equations

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$+ \quad \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$, can be derived from the above equations.

- ***E*** and ***B*** are not due to change in ***B*** and ***E*.**
- Rather fields are attributable to ρ and **J**.

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0}\rho \qquad \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = \mathbf{0},$$

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J},$$

$$+ \quad \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

Fields affect charges!

Maxwell's Equations

Magnetic Charge!!

In absence of ρ and $\boldsymbol{J}$

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$\boldsymbol{E} \Leftrightarrow \boldsymbol{B}$

$\boldsymbol{B} \Leftrightarrow -\mu_o \epsilon_o \boldsymbol{E}$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}, \qquad \nabla \cdot \mathbf{B} = 0$$

ρ and $\boldsymbol{J}$ spoil the symmetry!!

Way of restoring symmetry is the following

$$\text{(i)} \quad \nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \rho_e, \qquad \text{(iii)} \quad \nabla \times \mathbf{E} = -\mu_0 \mathbf{J}_m - \frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(ii)} \quad \nabla \cdot \mathbf{B} = \mu_0 \rho_m, \qquad \text{(iv)} \quad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}_e + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

ρ_e and ρ_m are electric and magnetic charge. J_e and J_m are the corresponding currents

$$\nabla \cdot \mathbf{J}_e = -\frac{\partial \rho_e}{\partial t} \quad \text{and} \quad \nabla \cdot \mathbf{J}_m = -\frac{\partial \rho_m}{\partial t}$$

Maxwell's Equations

Magnetic Charge!!

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$$\nabla \cdot \mathbf{E} = 0,$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$\boldsymbol{E} \Leftrightarrow \boldsymbol{B}$

$\boldsymbol{B} \Leftrightarrow -\mu_o \epsilon_o \boldsymbol{E}$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\nabla \cdot \mathbf{B} = 0$$

ρ and $\boldsymbol{J}$ spoil the symmetry!!

Maxwell's equation points towards magnetic charge, however it was never found!

ρ_m and J_m are zero everywhere! ***B*** is not of equal footing of ***E*,** no stationary source for ***B***!

Remember, no monopole term in magnetic multipole expansion!

Only current loops no north or south poles!

Maxwell's Equations

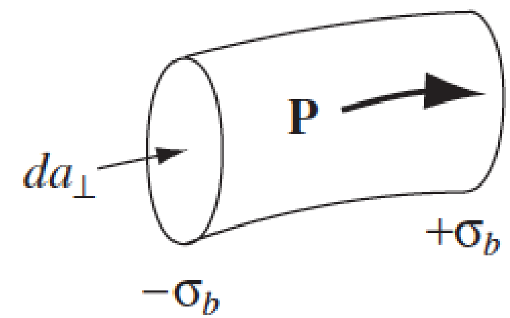
Maxwell's Equations in Matter : Electric and Magnetic Polarization

$$\rho_b = -\nabla \cdot \mathbf{P} \qquad \mathbf{J}_b = \nabla \times \mathbf{M}$$

electric/magnetic polarization produces bound charge density/current

Polarization current: P changes with time

$$dI = \frac{\partial \sigma_b}{\partial t} da_\perp = \frac{\partial P}{\partial t} da_\perp$$



Polarization current, $\mathbf{J}_p = \dfrac{\partial \mathbf{P}}{\partial t}$

$\mathbf{J_b}$ connected to magnetization but $\mathbf{J_p}$ connected to change of electric polarization

$$\nabla \cdot \mathbf{J}_p = \nabla \cdot \frac{\partial \mathbf{P}}{\partial t} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{P}) = -\frac{\partial \rho_b}{\partial t}$$

Continuity Equation for $\mathbf{J}_p$

Maxwell's Equations

Maxwell's Equations in Matter : Electric and Magnetic Polarization

$$\rho_b = -\nabla \cdot \mathbf{P} \qquad \mathbf{J}_b = \nabla \times \mathbf{M}$$

electric/magnetic polarization produces bound charge density/current

Total charge density:

$$\rho = \rho_f + \rho_b = \rho_f - \nabla \cdot \mathbf{P}$$

Total current density:

$$\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \nabla \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}$$

Maxwell's Equations

Total charge density:

$$\boxed{\rho = \rho_f + \rho_b = \rho_f - \boldsymbol{\nabla} \cdot \mathbf{P}}$$

Total current density:

$$\boxed{\mathbf{J} = \mathbf{J}_f + \mathbf{J}_b + \mathbf{J}_p = \mathbf{J}_f + \boldsymbol{\nabla} \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t}}$$

Gauss's Law:

$$\boldsymbol{\nabla} \cdot \mathbf{E} = \frac{1}{\epsilon_0}(\rho_f - \boldsymbol{\nabla} \cdot \mathbf{P}) \Rightarrow \boxed{\boldsymbol{\nabla} \cdot \mathbf{D} = \rho_f}$$

[where $\mathbf{D} = \epsilon_o \mathbf{E} + \mathbf{P}$]

Ampere's Law:

$$\boldsymbol{\nabla} \times \mathbf{B} = \mu_0 \left(\mathbf{J}_f + \boldsymbol{\nabla} \times \mathbf{M} + \frac{\partial \mathbf{P}}{\partial t} \right) + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \Rightarrow \boxed{\boldsymbol{\nabla} \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}}$$

[where $\mathbf{H} = (\mathbf{B}/\mu_o) - \mathbf{M}$]

Maxwell’s Equations

Maxwell’s Equations in Matter :

$$\nabla \cdot \mathbf{D} = \rho_f, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.$$

Maxwell’s Equations

Maxwell’s Equations in Matter :

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.$$

For linear media, $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$, and $\mathbf{M} = \chi_m \mathbf{H}$,

$$\mathbf{D} = \epsilon \mathbf{E}, \text{ and } \mathbf{H} = \frac{1}{\mu}\mathbf{B}$$

[where,

$\epsilon \equiv \epsilon_0(1+\chi_e)$

$\mu \equiv \mu_0(1+\chi_m)$]

Here, **D** is the electric displacement and $\mathbf{J}_d$ is the displacement current

$$\mathbf{J}_d \equiv \frac{\partial \mathbf{D}}{\partial t}$$

Maxwell's Equations

Maxwell's Equations: Differential form,

$$\nabla \cdot \mathbf{D} = \rho_f, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t}.$$

Maxwell's Equations: integral form,

over closed surface S	any surface S bounded by closed loop P
$\oint_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}}$	$\oint_{\mathcal{P}} \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a}$
$\oint_{\mathcal{S}} \mathbf{B} \cdot d\mathbf{a} = 0$	$\oint_{\mathcal{P}} \mathbf{H} \cdot d\mathbf{l} = I_{f_{\text{enc}}} + \frac{d}{dt} \int_{\mathcal{S}} \mathbf{D} \cdot d\mathbf{a}$

Maxwell's Equations

Boundary Conditions: General Conditions

$$D_1^{\perp} - D_2^{\perp} = \sigma_f \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0}$$

$$B_1^{\perp} - B_2^{\perp} = 0 \qquad \mathbf{H}_1^{\parallel} - \mathbf{H}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}$$

For linear media: Boundary conditions in terms of **E** and **B** alone,

$$\epsilon_1 E_1^{\perp} - \epsilon_2 E_2^{\perp} = \sigma_f, \qquad \mathbf{E}_1^{\parallel} - \mathbf{E}_2^{\parallel} = \mathbf{0},$$

$$B_1^{\perp} - B_2^{\perp} = 0, \qquad \frac{1}{\mu_1}\mathbf{B}_1^{\parallel} - \frac{1}{\mu_2}\mathbf{B}_2^{\parallel} = \mathbf{K}_f \times \hat{\mathbf{n}}.$$

See use in Theory of reflection and refraction!