

Line Integrals in Space



We now suppose that C is a smooth space curve given by the parametric equations

$$x = x(t) \quad y = y(t) \quad z = z(t) \quad a \leq t \leq b$$

or by a vector equation $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$. If f is a function of three variables that is continuous on some region containing C , then we define the **line integral of f along C** (with respect to arc length) in a manner similar to that for plane curves:

$$\int_C f(x, y, z) \, ds = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i^*, y_i^*, z_i^*) \Delta s_i$$

We evaluate it using a formula similar to Formula 3:

$$\boxed{9} \quad \int_C f(x, y, z) \, ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

An Example

V EXAMPLE 5 Evaluate $\int_C y \sin z \, ds$, where C is the circular helix given by the equations $x = \cos t$, $y = \sin t$, $z = t$, $0 \leq t \leq 2\pi$. (See Figure 9.)

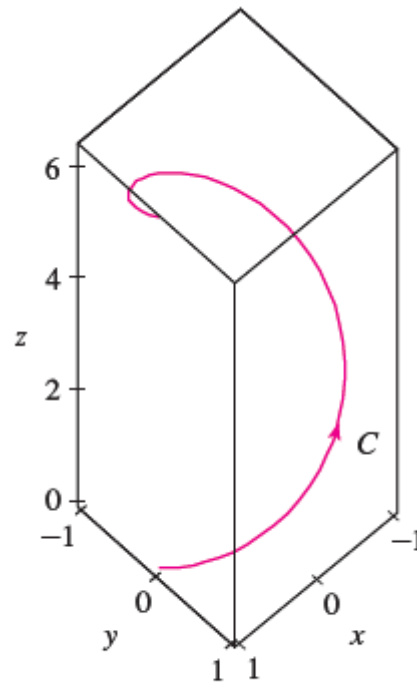


FIGURE 9

Line Integrals of Vector Fields

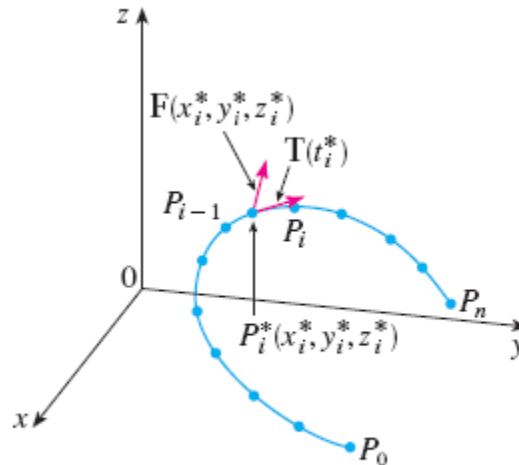


FIGURE 11

13 Definition Let \mathbf{F} be a continuous vector field defined on a smooth curve C given by a vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of \mathbf{F} along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_C \mathbf{F} \cdot \mathbf{T} ds$$

where $\mathbf{T}(x, y, z)$ is the unit tangent vector at the point (x, y, z) on C .

An Example

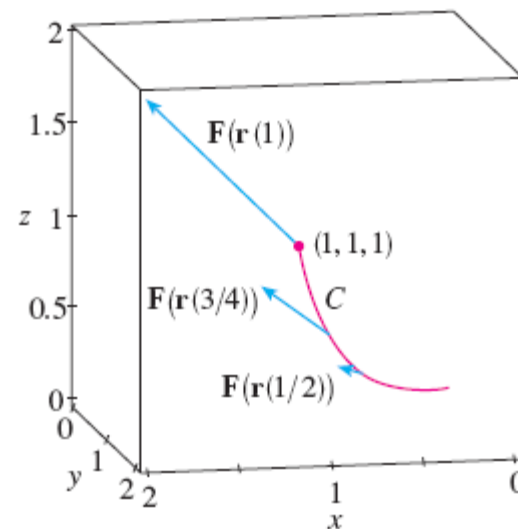


FIGURE 13

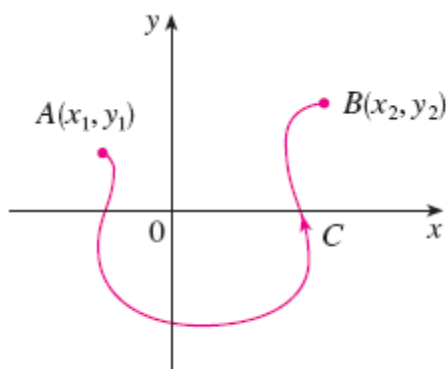
EXAMPLE 8 Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$ and C is the twisted cubic given by

$$x = t \quad y = t^2 \quad z = t^3 \quad 0 \leq t \leq 1$$

The Fundamental Theorem for Line Integrals

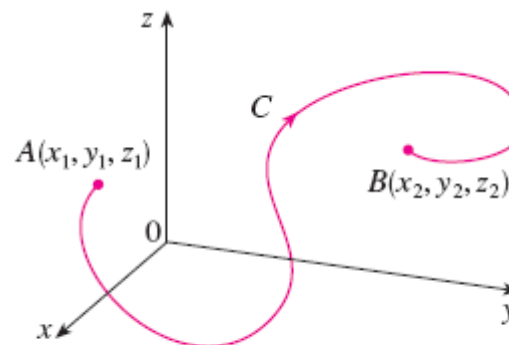
2 Theorem Let C be a smooth curve given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$. Let f be a differentiable function of two or three variables whose gradient vector ∇f is continuous on C . Then

$$\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$



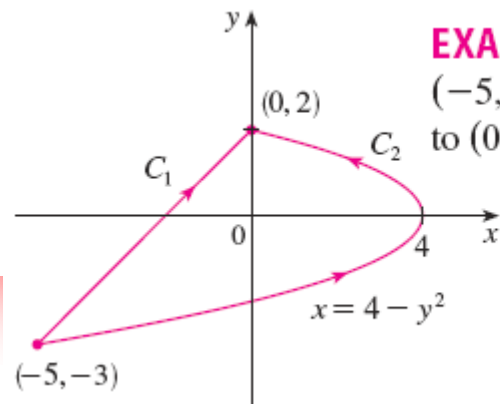
(a)

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2) - f(x_1, y_1)$$



(b)

$$\int_C \nabla f \cdot d\mathbf{r} = f(x_2, y_2, z_2) - f(x_1, y_1, z_1)$$



EXAMPLE 4 Evaluate $\int_C y^2 dx + x dy$, where (a) $C = C_1$ is the line segment from $(-5, -3)$ to $(0, 2)$ and (b) $C = C_2$ is the arc of the parabola $x = 4 - y^2$ from $(-5, -3)$ to $(0, 2)$. (See Figure 7.)

A vector field \mathbf{F} is called a **conservative vector field** if it is the gradient of some scalar function, that is, if there exists a function f such that $\mathbf{F} = \nabla f$. In this situation f is called a **potential function** for \mathbf{F} .

Independence of Path

Suppose C_1 and C_2 are two piecewise-smooth curves (which are called **paths**) that have the same initial point A and terminal point B . We know from Example 4 in Section 16.2 that, in general, $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$. But one implication of Theorem 2 is that

$$\int_{C_1} \nabla f \cdot d\mathbf{r} = \int_{C_2} \nabla f \cdot d\mathbf{r}$$

whenever ∇f is continuous. In other words, the line integral of a *conservative* vector field depends only on the initial point and terminal point of a curve.

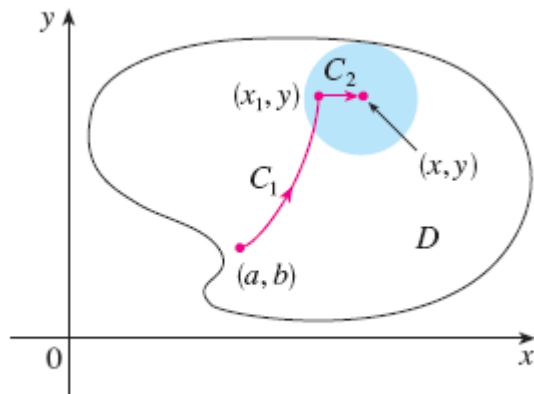
In general, if \mathbf{F} is a continuous vector field with domain D , we say that the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **independent of path** if $\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$ for any two paths C_1 and C_2 in D that have the same initial and terminal points. With this terminology we can say that *line integrals of conservative vector fields are independent of path*.

CLOSED, OPEN, CONNECTED, SIMPLE, SIMPLY CONNECTED (CURVES/REGIONS)



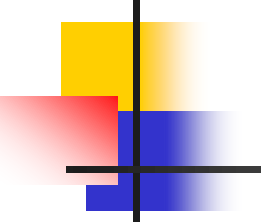
FIGURE 2
A closed curve

A curve is called **closed** if its terminal point coincides with its initial point, that is, $\mathbf{r}(b) = \mathbf{r}(a)$. (See Figure 2.)



A region D is open, if for every point P in D , there is a disk with center P which entirely lies in D . In addition D is said to be connected if any two points in D can be joined by a path that lies in D .

FIGURE 4



4 Theorem Suppose \mathbf{F} is a vector field that is continuous on an open connected region D . If $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D , then \mathbf{F} is a conservative vector field on D ; that is, there exists a function f such that $\nabla f = \mathbf{F}$.

5 Theorem If $\mathbf{F}(x, y) = P(x, y) \mathbf{i} + Q(x, y) \mathbf{j}$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on a domain D , then throughout D we have

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

6 Theorem Let $\mathbf{F} = P \mathbf{i} + Q \mathbf{j}$ be a vector field on an open simply-connected region D . Suppose that P and Q have continuous first-order derivatives and

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{throughout } D$$

Then \mathbf{F} is conservative.

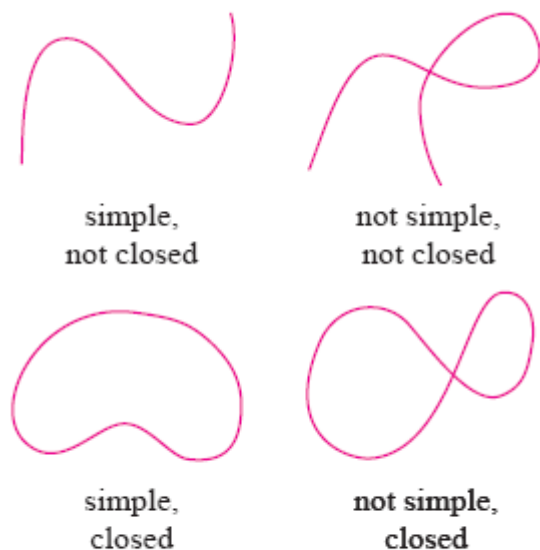


FIGURE 6
Types of curves

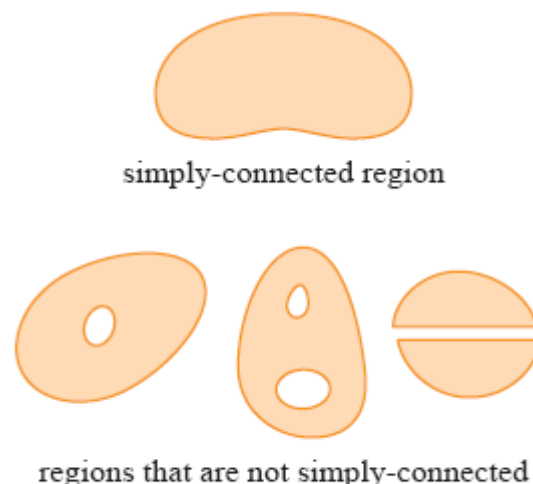


FIGURE 7

we first need the concept of a **simple curve**, which is a curve that does not intersect itself anywhere between its endpoints. [See Figure 6; $\mathbf{r}(a) = \mathbf{r}(b)$ for a simple closed curve, but $\mathbf{r}(t_1) \neq \mathbf{r}(t_2)$ when $a < t_1 < t_2 < b$.]

A **simply-connected region** in the plane is a connected region D such that every simple closed curve in D encloses only points that are in D . Notice from Figure 7 that, intuitively speaking, a simply-connected region contains no hole and cannot consist of two separate pieces.

GREEN'S THEOREM

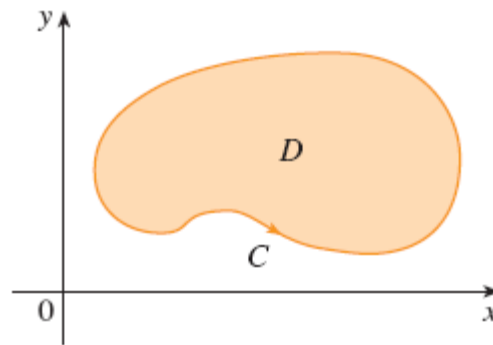
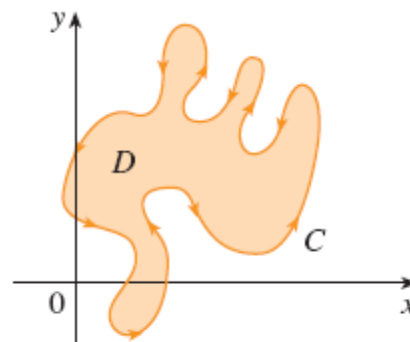
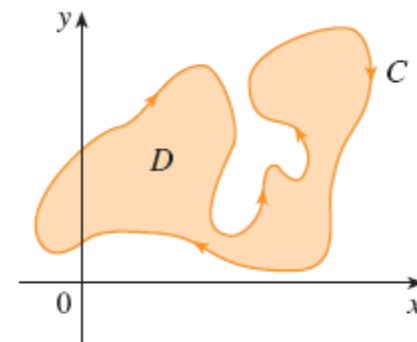


FIGURE 1

Green's Theorem gives the relationship between a line integral around a simple closed curve C and a double integral over the plane region D bounded by C . (See Figure 1. We assume that D consists of all points inside C as well as all points on C .) In stating Green's Theorem we use the convention that the **positive orientation** of a simple closed curve C refers to a single *counterclockwise* traversal of C . Thus if C is given by the vector function $\mathbf{r}(t)$, $a \leq t \leq b$, then the region D is always on the left as the point $\mathbf{r}(t)$ traverses C . (See Figure 2.)

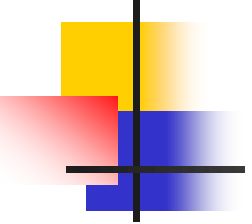


(a) Positive orientation



(b) Negative orientation

FIGURE 2



Green's Theorem Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and let D be the region bounded by C . If P and Q have continuous partial derivatives on an open region that contains D , then

$$\int_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Proof of Green's Theorem for Simple Regions (Type I and II)

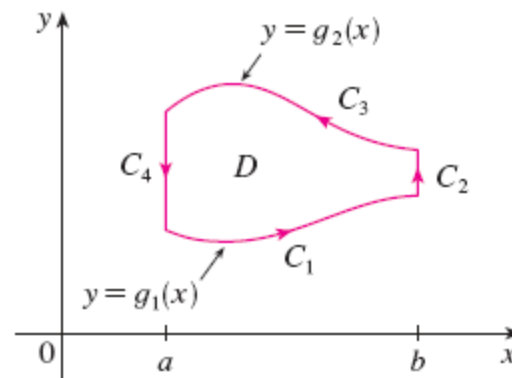


FIGURE 3

An Example

EXAMPLE 1 Evaluate $\int_C x^4 dx + xy dy$, where C is the triangular curve consisting of the line segments from $(0, 0)$ to $(1, 0)$, from $(1, 0)$ to $(0, 1)$, and from $(0, 1)$ to $(0, 0)$.

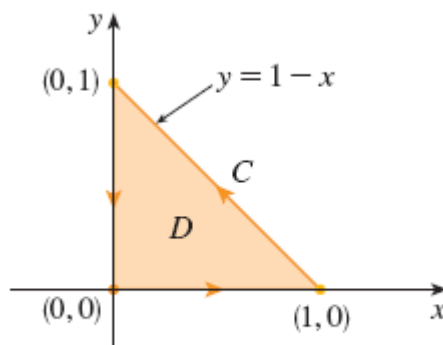
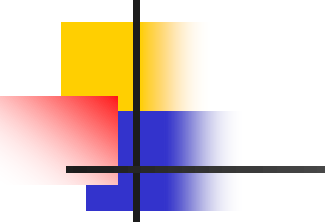


FIGURE 4

V EXAMPLE 2 Evaluate $\oint_C (3y - e^{\sin x}) dx + (7x + \sqrt{y^4 + 1}) dy$, where C is the circle $x^2 + y^2 = 9$.

Application of Green's Theorem



Since the area of D is $\iint_D 1 \, dA$, we wish to choose P and Q so that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1$$

There are several possibilities:

$$P(x, y) = 0 \qquad P(x, y) = -y \qquad P(x, y) = -\frac{1}{2}y$$

$$Q(x, y) = x \qquad Q(x, y) = 0 \qquad Q(x, y) = \frac{1}{2}x$$

Then Green's Theorem gives the following formulas for the area of D :

5

$$A = \oint_C x \, dy = -\oint_C y \, dx = \frac{1}{2} \oint_C x \, dy - y \, dx$$

EXAMPLE 3 Find the area enclosed by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.