PH 102, Electromagnetism,

Post Mid Semester

Lecture 10

Electromagnetic Waves

in vacuum

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Electromagnetic Waves

One dimensional waves : The Wave Equation

Wave???

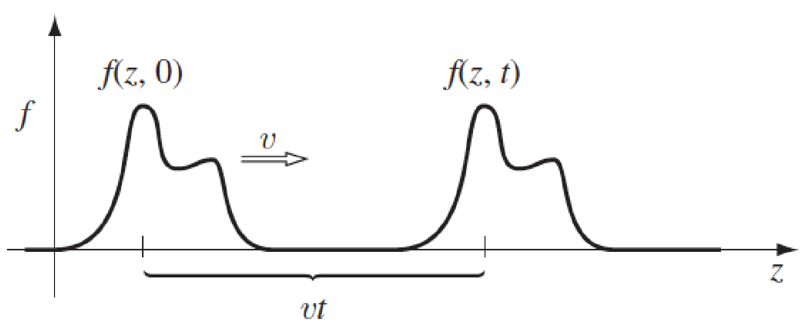
Disturbance of a continuous medium: propagating with a fixed shape, constant velocity.

fixed shape: Dispersive medium, different frequencies @ different speed
Standing waves do not propagate !

We will stick to the simple case.

Mathematical description:

Initial shape $g(z) \equiv f(z,0)$ and at time t, displacement $f(z,t)$



Displacement @ z & t, is the same as the displacement @ vt to the left (@ z – vt), back at time t = 0:

$$f(z, t) = f(z - vt, 0) = g(z - vt)$$

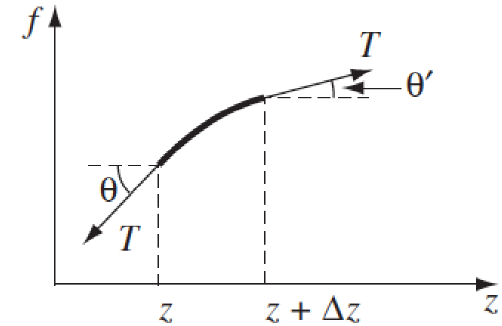
only in the special combination of 'z – vt'.

wave: $f_1(z,t) = Ae^{-b(z-vt)^2}$, $f_2(z,t) = A\sin[b(z-vt)]$, $f_3(z,t) = \dfrac{A}{b(z-vt)^2+1}$

Not wave: $f_4(z,t) = Ae^{-b(bz^2+vt)}$, and $f_5(z,t) = A\sin(bz)\cos(bvt)^3$,

Electromagnetic Waves

<u>Wave motion of a stretched string :</u>



It follows from Newton's 2nd law.

Very long string, tension T, displaced from equilibrium:

Net transverse force on the segment between z and z + Δ z,

$$\Delta F = T\sin\theta' - T\sin\theta,$$

θ′ : angle the string makes with the z -direction at point z + Δ z,

θ : corresponding angle at point z .

For small distortions the angles are small enough to replace sine by tangent

$$\Delta F \cong T(\tan\theta' - \tan\theta) = T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \cong T\frac{\partial^2 f}{\partial z^2}\Delta z = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2},$$

Newton's 2nd law

μ: mass per unit length.

Thus, $$\frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2} \Longrightarrow \boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2},}$$ Wave equation

$$v = \sqrt{\frac{T}{\mu}}.$$

Electromagnetic Waves

Wave Equation :

$$\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2},$$

Admits solutions of the form : $f(z, t) = g(z - vt) = g(u)$,

(z & t dependence in the form $u \equiv z - vt$).

$$\frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du}, \qquad \frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{dg}{du}\right) = -v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 g}{du^2},$$

$$\frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}, \qquad \frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2},$$

$$\boxed{\frac{d^2 g}{du^2} = \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}.}$$

- Such functions like $g(u)$ represents waves propagating in the +z direction with speed v.
- $g(u)$ can be any differentiable function, provided the disturbance propagates without changing its shape.

Electromagnetic Waves

$$\boxed{\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2},}$$

<u>Wave Equation :</u>

Apart from g $(z - vt)$, one can figure out another kind of solutions.

v dependence is through v^2, hence v with both signs are allowed solutions.

$$f(z,\ t) = h(z + v\,t)$$

h represents waves propagating in the *negative* z direction with speed v.

The most general solution to the wave equation is the sum of a wave to the right and a wave to the left:

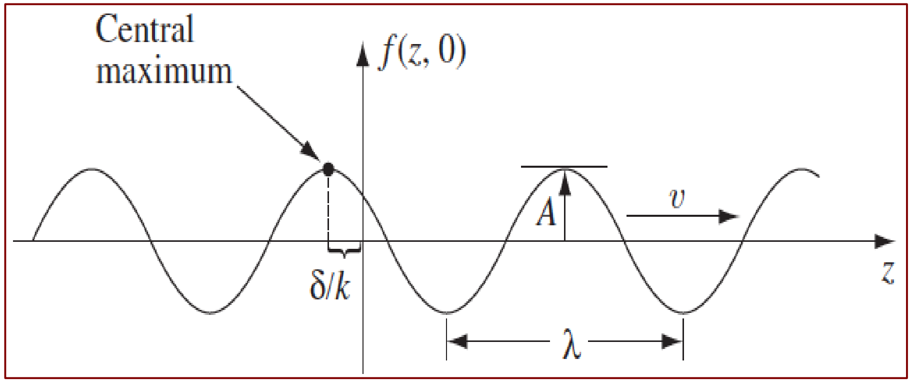
$$f(z, t) = g(z - vt) + h(z + vt).$$

Vibration: Oscillator equation is almost certainly responsible (at least, for small amplitudes).

Something is waving (mechanics or acoustics, optics or oceanography):
The wave equation is bound to be involved.

Electromagnetic Waves

Sinusoidal Waves: $f(z, t) = A\cos[k(z - vt) + \delta]$



A : Amplitude of the wave
(it is +ve & represents the max displacement from equilibrium).

Phase: Argument of the cosine

δ : Phase constant
(add integer multiple of 2π to δ same $f(z,t)$; $0 \le \delta < 2\pi$).

- At $z = vt - \delta/k$, **phase = 0 : Central Maximum.**

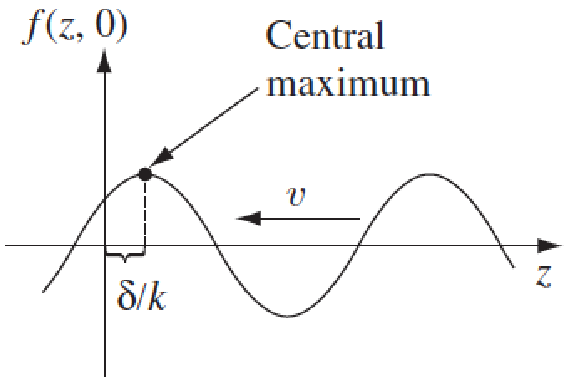
  $\delta = 0$, central maximum @ origin at time $t = 0$.
  δ/k, distance by which the central maximum (thus the entire wave) is "delayed."

- ***k* : wave number: $\lambda = 2\pi/k$.**
  z advances by $2\pi/k$, the cosine executes one complete cycle.

- Time period, $T = 2\pi/kv$, fixed point z, string vibrates up and down, undergoes one full cycle.

- The frequency, $\nu = 1/T = kv/2\pi = v/\lambda$, (number of oscillations per unit time).

Electromagnetic Waves



Sinusoidal Waves: $f(z, t) = A\cos[k(z - vt) + \delta]$

The frequency, $\boldsymbol{\nu = 1/T = kv/2\pi = v/\lambda}$, (number of oscillations per unit time).

Angular frequency $\boldsymbol{\omega = 2\pi\nu = kv}$, (analogous case of uniform circular motion: number of radians per unit time.)

Sinusoidal waves in terms of ω, $\boldsymbol{f(z, t) = A\cos(kz - \omega t + \delta)}$.

Sinusoidal oscillation traveling to the left, $\boldsymbol{f(z, t) = A\cos(kz + \omega t - \delta)}$.

Sign of δ : consistent with convention (δ/k distance by which the wave is "delayed", wave is to the left, delay means shift to the right). Cosine being even function,

$$\boldsymbol{f(z, t) = A\cos(kz + \omega t - \delta) = A\cos(-kz - \omega t + \delta)}.$$

Switching the sign of *k*: wave with same amplitude, phase constant, frequency, and wavelength but traveling in the opposite direction.

Electromagnetic Waves

Sinusoidal Waves: $f(z, t) = A \cos[k(z - vt) + \delta]$

Complex notation:

Euler's formula , $e^{i\theta} = \cos\theta + i \sin\theta,$

The sinusoidal wave can be written as, $f(z,t) = \mathrm{Re}\left[Ae^{i(kz-\omega t+\delta)}\right],$

Introduce the complex wave function, $\tilde{f}(z,t) \equiv \tilde{A}e^{i(kz-\omega t)},$

Complex amplitude $\tilde{A} \equiv Ae^{i\delta}$ absorbs the phase constant.

The actual wave function is the real part of : $f(z,t) = \mathrm{Re}[\tilde{f}(z,t)].$

Advantage of the complex notation:

exponentials are easier to manipulate than sines and cosines.

Electromagnetic Waves

Sinusoidal Waves: $f(z, t) = A\cos[k(z - vt) + \delta]$

Complex notation:

Suppose you want to combine two sinusoidal waves:

$$f_3 = f_1 + f_2 = \text{Re}(\tilde{f}_1) + \text{Re}(\tilde{f}_2) = \text{Re}(\tilde{f}_1 + \tilde{f}_2) = \text{Re}(\tilde{f}_3),$$

with $\tilde{f}_3 = \tilde{f}_1 + \tilde{f}_2$. You simply add the corresponding *complex* wave functions, and then take the real part. In particular, if they have the same frequency and wave number,

$$\tilde{f}_3 = \tilde{A}_1 e^{i(kz-\omega t)} + \tilde{A}_2 e^{i(kz-\omega t)} = \tilde{A}_3 e^{i(kz-\omega t)},$$

where

$$\tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2, \quad \text{or} \quad A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2}.$$

In other words, you just add the (complex) amplitudes. The combined wave still has the same frequency and wavelength,

$$f_3(z, t) = A_3 \cos(kz - \omega t + \delta_3),$$

Electromagnetic Waves

Sinusoidal Waves: $f(z, t) = A\cos[k(z - vt) + \delta]$

Problem 9.3. Determine A_3 and δ_3 in terms of A_1, A_2, δ_1, and δ_2.

$$\begin{aligned}(A_3)^2 &= \left(A_3e^{i\delta_3}\right)\left(A_3e^{-i\delta_3}\right) = \left(A_1e^{i\delta_1} + A_2e^{i\delta_2}\right)\left(A_1e^{-i\delta_1} + A_2e^{-i\delta_2}\right)\\ &= (A_1)^2 + (A_2)^2 + A_1A_2\left(e^{i\delta_1}e^{-i\delta_2} + e^{-i\delta_1}e^{i\delta_2}\right) = (A_1)^2 + (A_2)^2 + A_1A_2 2\cos(\delta_1 - \delta_2);\\ A_3 &= \boxed{\sqrt{(A_1)^2 + (A_2)^2 + 2A_1A_2\cos(\delta_1 - \delta_2)}.}\end{aligned}$$

$$\begin{aligned}A_3e^{i\delta_3} &= A_3(\cos\delta_3 + i\sin\delta_3) = A_1(\cos\delta_1 + i\sin\delta_1) + A_2(\cos\delta_2 + i\sin\delta_2)\\ &= (A_1\cos\delta_1 + A_2\cos\delta_2) + i(A_1\sin\delta_1 + A_2\sin\delta_2).\end{aligned}$$

$$\tan\delta_3 = \frac{A_3\sin\delta_3}{A_3\cos\delta_3} = \frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2};$$

$$\delta_3 = \boxed{\tan^{-1}\left(\frac{A_1\sin\delta_1 + A_2\sin\delta_2}{A_1\cos\delta_1 + A_2\cos\delta_2}\right).}$$

Electromagnetic Waves

Sinusoidal Waves: $f(z, t) = A \cos[k(z - vt) + \delta]$

Linear combinations of sinusoidal waves:

Any wave can be expressed as a linear combination of the sinusoidal ones:

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz - \omega t)}\, dk.$$

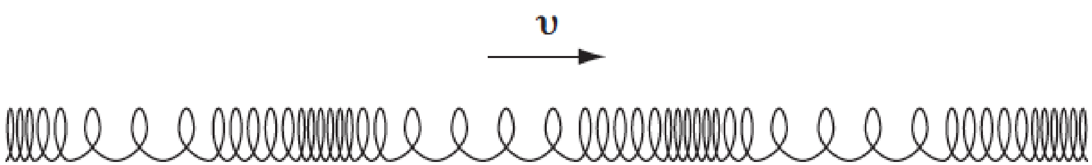
Here, $\omega = \omega(k)$,
and k runs through the -ve values to include waves going in both directions.

Electromagnetic Waves

Polarization :

Transverse wave: displacement perpendicular to the direction of propagation.
(Shaking of string, EM wave)

Longitudinal wave: displacement from equilibrium along the direction of propagation.
(Compression in a slinky, sound wave)



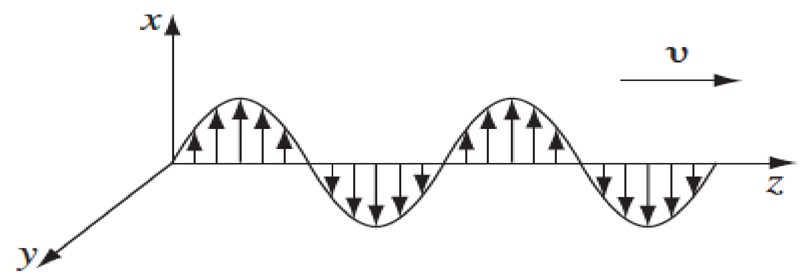
Electromagnetic Waves

Polarization :

Transverse wave: two independent states of polarization

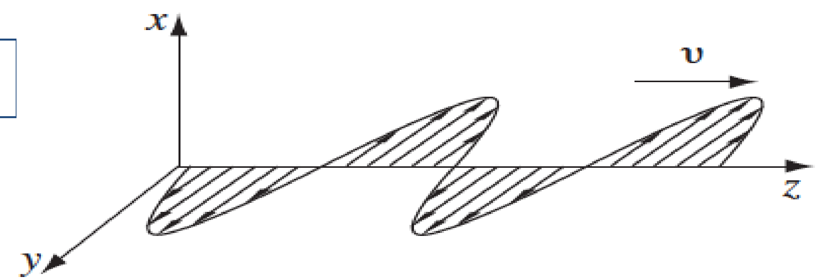
string up-and-down, “vertical” polarization

$$\tilde{\mathbf{f}}_v(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{x}},$$



string left-and-right, “horizontal” polarization

$$\tilde{\mathbf{f}}_h(z,t) = \tilde{A}e^{i(kz-\omega t)}\,\hat{\mathbf{y}},$$



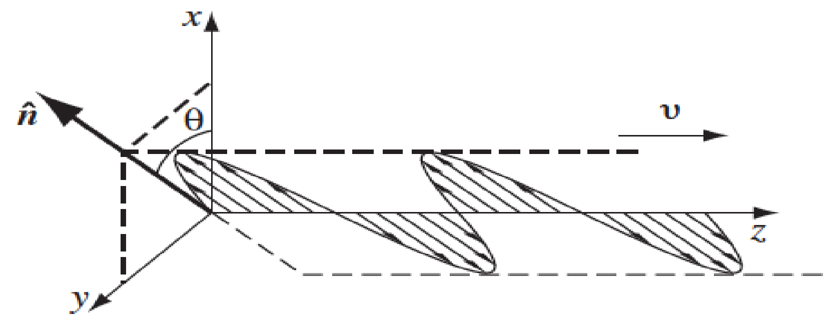
Electromagnetic Waves

Polarization :

Transverse wave: two independent states of polarization

along any other direction in the xy plane

$$\tilde{\mathbf{f}}(z,t) = \tilde{A} e^{i(kz-\omega t)}\,\hat{\mathbf{n}}.$$



Polarization vector $\hat{\mathbf{n}}$ defines the plane of vibration.

In terms of the polarization angle θ, $\hat{\mathbf{n}} = \cos\theta\,\hat{\mathbf{x}} + \sin\theta\,\hat{\mathbf{y}}.$

Transverse waves : $\hat{\mathbf{n}}$ is perpendicular to the direction of propagation, $\boxed{\hat{\mathbf{n}}\cdot\hat{\mathbf{z}} = 0.}$

Superposition of two waves—one horizontally polarized and the other one vertically:

$$\tilde{\mathbf{f}}(z,t) = (\tilde{A}\cos\theta) e^{i(kz-\omega t)}\,\hat{\mathbf{x}} + (\tilde{A}\sin\theta) e^{i(kz-\omega t)}\,\hat{\mathbf{y}}.$$

Electromagnetic Waves

The Wave Equation for E and B in **Vacuum**:

For no charge or current, Maxwell's equations: **Coupled**, first-order, pde for ***E*** & ***B*** .

$$\text{(i)} \quad \nabla \cdot \mathbf{E} = 0, \quad \text{(iii)} \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t},$$

$$\text{(ii)} \quad \nabla \cdot \mathbf{B} = 0, \quad \text{(iv)} \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}.$$

Decoupling E & B:

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{E}) &= \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \left(-\frac{\partial \mathbf{B}}{\partial t}\right) \\ &= -\frac{\partial}{\partial t}(\nabla \times \mathbf{B}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \end{aligned}$$

and

$$\begin{aligned} \nabla \times (\nabla \times \mathbf{B}) &= \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = \nabla \times \left(\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\right) \\ &= \mu_0 \epsilon_0 \frac{\partial}{\partial t}(\nabla \times \mathbf{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}. \end{aligned}$$

$$\nabla \cdot \mathbf{E} = 0 \text{ and } \nabla \cdot \mathbf{B} = 0,$$

$$\boxed{\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.}$$

Interdependent, one is the other!

Price for decoupling: 2^{nd} order equations.

Electromagnetic Waves

The Wave Equation for E and B in **Vacuum**:

$$\boxed{\nabla^2 \mathbf{E} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2}, \quad \nabla^2 \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{B}}{\partial t^2}.}$$

$$\nabla^2 f = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}. \quad \longrightarrow$$

empty space EM waves
travelling speed v

$$\boxed{v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \text{ m/s},}$$

- 2 static quantities decide speed of em wave!!
  - Exactly same as light speed!!

May be light is an EM wave!!!

No EM wave without Maxwell's term!!

Electromagnetic Waves

Monochromatic Plane Waves :

Sinusoidal waves of frequency ω: different frequencies in the visible range correspond to different colors, such waves are called monochromatic.

The Electromagnetic Spectrum

Frequency (Hz)	Type	Wavelength (m)
10^{22}		10^{-13}
10^{21}	gamma rays	10^{-12}
10^{20}		10^{-11}
10^{19}		10^{-10}
10^{18}	x-rays	10^{-9}
10^{17}		10^{-8}
10^{16}	ultraviolet	10^{-7}
10^{15}	visible	10^{-6}
10^{14}	infrared	10^{-5}
10^{13}		10^{-4}
10^{12}		10^{-3}
10^{11}		10^{-2}
10^{10}	microwave	10^{-1}
10^{9}		1
10^{8}	TV, FM	10
10^{7}		10^{2}
10^{6}	AM	10^{3}
10^{5}		10^{4}
10^{4}	RF	10^{5}
10^{3}		10^{6}

The Visible Range

Frequency (Hz)	Color	Wavelength (m)
1.0×10^{15}	near ultraviolet	3.0×10^{-7}
7.5×10^{14}	shortest visible blue	4.0×10^{-7}
6.5×10^{14}	blue	4.6×10^{-7}
5.6×10^{14}	green	5.4×10^{-7}
5.1×10^{14}	yellow	5.9×10^{-7}
4.9×10^{14}	orange	6.1×10^{-7}
3.9×10^{14}	longest visible red	7.6×10^{-7}
3.0×10^{14}	near infrared	1.0×10^{-6}

Electromagnetic Waves

Monochromatic Plane Waves :

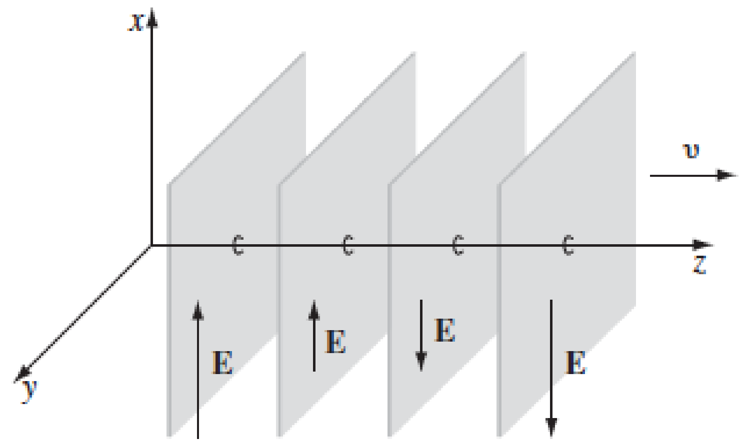
Sinusoidal waves of frequency ω: different frequencies in the visible range correspond to different colors, such waves are called monochromatic.

Waves traveling in the *z* direction, no *x* or *y* dependence: Plane waves

The fields are uniform over every plane perpendicular to the direction of propagation.

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)},$$

where $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are the (complex) amplitudes



Extra constraints on $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$

$$\nabla \cdot \mathbf{E} = 0 \text{ and } \nabla \cdot \mathbf{B} = 0, \quad \text{implies} \quad (\tilde{E}_0)_z = (\tilde{B}_0)_z = 0.$$

E and B is perpendicular to the propagation direction.

EM plane wave can not have a longitudinal component.

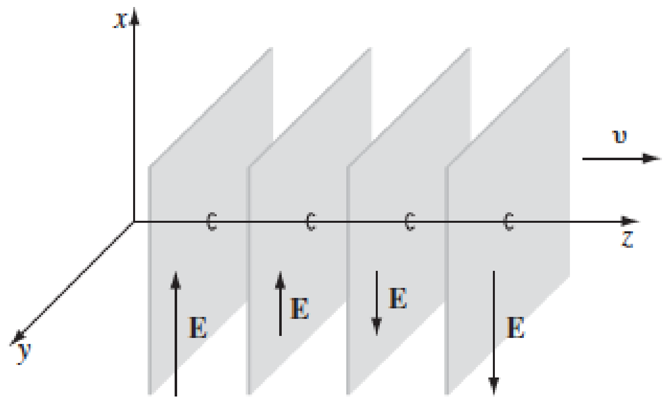
Thus EM waves are purely transverse!

Electromagnetic Waves

Monochromatic Plane Waves :

$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)},$$

where $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are the (complex) amplitudes



Extra constraints on $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$

Faraday's law, $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t},$

$$\vec{\nabla} \times \vec{\tilde{E}} = -\frac{\partial \tilde{B}_x}{\partial t}\hat{x} - \frac{\partial \tilde{B}_y}{\partial t}\hat{y} - \frac{\partial \tilde{B}_z}{\partial t}\hat{z}$$

$$\vec{\tilde{E}} = \left(E_{0x}\hat{x} + E_{0y}\hat{y}\right)e^{i(kz-\omega t)}\, e^{i\delta}$$

$$-k(\tilde{E}_0)_y = \omega(\tilde{B}_0)_x, \quad k(\tilde{E}_0)_x = \omega(\tilde{B}_0)_y,$$

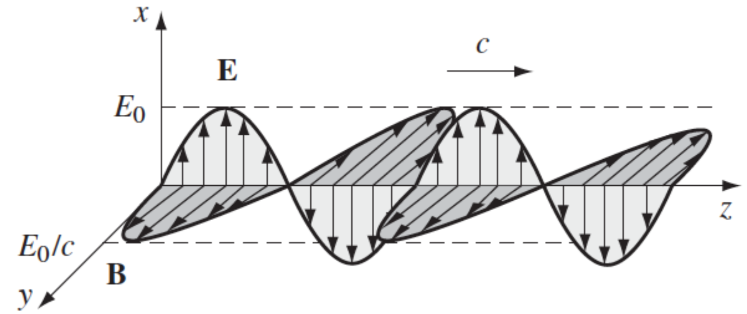
$$\boxed{\tilde{\mathbf{B}}_0 = \frac{k}{\omega}(\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_0).}$$

***E* & *B* are in phase**
and
mutually perpendicular.

The real amplitudes, $\displaystyle B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0.$

Electromagnetic Waves

Monochromatic Plane Waves :



$$\tilde{\mathbf{E}}(z,t) = \tilde{\mathbf{E}}_0 e^{i(kz-\omega t)}, \quad \tilde{\mathbf{B}}(z,t) = \tilde{\mathbf{B}}_0 e^{i(kz-\omega t)},$$

where $\tilde{\mathbf{E}}_0$ and $\tilde{\mathbf{B}}_0$ are the (complex) amplitudes

Now, if ***E*** points in the x direction then B points in the y direction ($\tilde{\mathbf{B}}_0 = \frac{k}{\omega}(\hat{\mathbf{z}} \times \tilde{\mathbf{E}}_0).$)

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}\,\hat{\mathbf{x}}, \quad \tilde{\mathbf{B}}(z,t) = \frac{1}{c}\tilde{E}_0 e^{i(kz-\omega t)}\,\hat{\mathbf{y}},$$

or

taking the real part,

$$\mathbf{E}(z,t) = E_0 \cos(kz - \omega t + \delta)\,\hat{\mathbf{x}}, \quad \mathbf{B}(z,t) = \frac{1}{c}E_0 \cos(kz - \omega t + \delta)\,\hat{\mathbf{y}}.$$

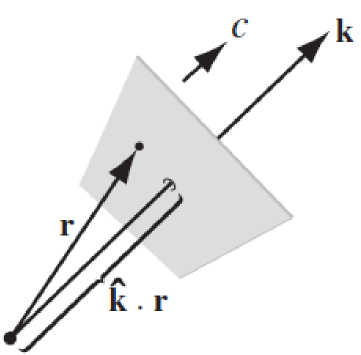
Monochromatic plane wave! (Direction of ***E*** specify polarization direction)

Electromagnetic Waves

Monochromatic Plane Waves :

Generalize to monochromatic plane waves traveling in an arbitrary direction.
(nothing special about the z direction)

Introducing the wave vector ***k*** (pointing in the direction of propagation)
The scalar product ***k.r*** is the appropriate generalization of *kz* so,



$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\,\hat{\mathbf{n}},$$

$$\tilde{\mathbf{B}}(\mathbf{r}, t) = \frac{1}{c}\tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}(\hat{\mathbf{k}} \times \hat{\mathbf{n}}) = \frac{1}{c}\hat{\mathbf{k}} \times \tilde{\mathbf{E}},$$

$\hat{\mathbf{n}}$ polarization vector

E being transverse,

$$\hat{\mathbf{n}} \cdot \hat{\mathbf{k}} = 0.$$

The real ***E*** & ***B*** fields in a monochromatic plane wave with propagation vector ***k*** and polarization $\hat{\mathbf{n}}$ are

$$\mathbf{E}(\mathbf{r}, t) = E_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)\,\hat{\mathbf{n}},$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{1}{c}E_0 \cos(\mathbf{k}\cdot\mathbf{r} - \omega t + \delta)(\hat{\mathbf{k}} \times \hat{\mathbf{n}}).$$

Demonstrations of EM wave radiation, polarization

https://www.youtube.com/watch?v=j2gOh39IyPM&t=305s

https://www.youtube.com/watch?v=4xF1Fq2wB1I