Department of Mathematics
Indian Institute of Technology Guwahati
MA 101: Mathematics I
Solutions of Tutorial Sheet-6
July-December 2019

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1. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$

   (a) Show that f is Riemann integrable on $[-1, 1]$ and that $\int\limits_{-1}^{1} f(x)\,dx = 0$.

   (b) If $F(x) = \int\limits_{-1}^{x} f(t)\,dt$ for all $x \in [-1, 1]$, then show that $F : [-1, 1] \to \mathbb{R}$ is differentiable, and in particular, $F'(0) = f(0)$, although f is not continuous at 0.

Solution. (a) If $P = \{x_0, x_1, \ldots, x_n\}$ is any partition of $[-1, 1]$, then clearly $m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\} = 0$ and $M_i = \{f(x) : x \in [x_{i-1}, x_i]\} \geq 0$ for $i = 1, 2, \ldots, n$ and so $L(f, P) = 0$ and $U(f, P) \geq 0$. Hence

$$\underline{\int_{-1}^{1}} f(x)\,dx = 0 \quad \text{and} \quad \overline{\int_{-1}^{1}} f(x)\,dx \geq 0.$$

Let $\varepsilon > 0$. There exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\varepsilon}{2}$. We choose u, v and s_k, t_k for $k = 2, 3, \ldots, n_0$ such that $\frac{1}{n_0+1} < u < s_{n_0} < \frac{1}{n_0} < t_{n_0} < \cdots < s_2 < \frac{1}{2} < t_2 < v < 1$ and also $1 - v < \frac{\varepsilon}{2n_0}$ and $t_k - s_k < \frac{\varepsilon}{2n_0}$ for $k = 2, 3, \ldots, n_0$. Then the partition $P_0 = \{-1, 0, u, s_{n_0}, t_{n_0}, \ldots, s_2, t_2, v, 1\}$ of $[-1, 1]$ is such that $U(f, P_0) < \varepsilon$. It follows that $0 \leq \overline{\int\limits_{-1}^{1}} f(x)\,dx \leq U(f, P_0) < \varepsilon$ and so $\overline{\int\limits_{-1}^{1}} f(x)\,dx = 0$. Thus $\underline{\int_{-1}^{1}} f(x)\,dx = \overline{\int\limits_{-1}^{1}} f(x)\,dx = 0$. Therefore f is Riemann integrable on $[-1, 1]$ and

$$\int_{-1}^{1} f(x)\,dx = 0.$$

(b) As above we can see that $F(x) = 0$ for all $x \in [-1, 1]$. Hence F is differentiable and $F'(0) = 0 = f(0)$. However, f is not continuous at 0, because $\frac{1}{n} \to 0$ but $f(\frac{1}{n}) \to 1$ (since $f(\frac{1}{n}) = 1$ for all $n \in \mathbb{N}$). $\square$

2. Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int\limits_{a}^{b} f(x)\,dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.

Solution. If possible, let $f(c) \neq 0$ for some $c \in (a, b)$, so that $f(c) > 0$. Since f is continuous at c, there exists $\delta > 0$ such that $|f(x) - f(c)| < \frac{1}{2}f(c)$ for all

$x \in (c-\delta, c+\delta)$. (We may choose δ such that $(c-\delta, c+\delta) \subset [a,b]$.) This implies that $f(x) > \frac{1}{2}f(c)$ for all $x \in (c-\delta, c+\delta)$. Since $f(x) \geq 0$ on $[a,b]$, so

$$\int_a^b f(x)\,dx \geq \int_{c-\delta}^{c+\delta} f(x)\,dx \geq \frac{1}{2}f(c)\cdot 2\delta > 0,$$

a contradiction. Hence $f(x) = 0$ for all $x \in (a,b)$. Almost similar arguments work if $c = a$ or $c = b$.

We now make the following two remarks.

(a) Equivalently, we have proved that if $f : [a,b] \to \mathbb{R}$ is continuous such that $f(x) \geq 0$ for all $x \in [a,b]$ and $f(c) \neq 0$ for some $c \in [a,b]$, then $\int_a^b f(x)\,dx > 0$.

(b) The above result need not be true if f is assumed to be only Riemann integrable on $[a,b]$. For example, taking $f(0) = 1$ and $f(x) = 0$ for all $x \in (0,1]$, we find that $f : [0,1] \to \mathbb{R}$ is Riemann integrable on $[0,1]$ with $f(x) \geq 0$ for all $x \in [0,1]$ and $\int_0^1 f(x)\,dx = 0$ but $f(0) \neq 0$.

$\square$

3. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$
Examine whether f is Riemann integrable on $[0,1]$.

Solution. Clearly f is bounded on $[0,1]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be any partition of $[0,1]$. Since between any two distinct real numbers, there exist a rational as well as an irrational number, it follows that $M_i = x_i$ and $m_i = 0$ for $i = 1, \ldots, n$. (Note that M_i cannot be less than x_i, because otherwise we can find a rational number r_i between M_i and x_i and so $f(r_i) = r_i > M_i$, which is not possible.) Hence $L(f,P) = 0$ and

$$U(f,P) = \sum_{i=1}^n x_i(x_i - x_{i-1}) = \sum_{i=1}^n x_i^2 - \sum_{i=1}^n x_i x_{i-1} \geq \frac{1}{2}\sum_{i=1}^n (x_i^2 - x_{i-1}^2) = \frac{1}{2}$$

(since $x_i^2 + x_{i-1}^2 \geq 2x_i x_{i-1}$ for $i = 1, \ldots, n$).
Consequently $\overline{\int_0^1} f(x)\,dx \geq \frac{1}{2}$ and $\underline{\int_0^1} f(x)\,dx = 0$. Since $\overline{\int_0^1} f(x)\,dx \neq \underline{\int_0^1} f(x)\,dx$, f is not Riemann integrable on $[0,1]$. $\square$

4. If $f : [0,1] \to \mathbb{R}$ is Riemann integrable, then find $\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx$.

Solution. Since f is Riemann integrable on $[0,1]$, f is bounded on $[0,1]$. So there exists $M > 0$ such that $|f(x)| \leq M$ for all $x \in [0,1]$. Now

$$\left|\int_0^1 x^n f(x)\,dx\right| \leq \int_0^1 |x^n f(x)|\,dx \leq M\int_0^1 x^n\,dx = \frac{M}{n+1} \to 0 \quad \text{as} \quad n \to \infty.$$

Hence it follows that $\lim_{n\to\infty} \int_0^1 x^n f(x)\,dx = 0$. $\square$

5. If $f : [0, 2\pi] \to \mathbb{R}$ is continuous such that $\int\limits_0^{\frac{\pi}{2}} f(x)\,dx = 0$, then show that there exists $c \in (0, \frac{\pi}{2})$ such that $f(c) = 2\cos 2c$.

Solution. Let $g(x) = \int\limits_0^x f(t)\,dt - \sin 2x$ for all $x \in [0, 2\pi]$. Since $f : [0, 2\pi] \to \mathbb{R}$ is continuous, by the first fundamental theorem of calculus, $g : [0, 2\pi] \to \mathbb{R}$ is differentiable and $g'(x) = f(x) - 2\cos 2x$ for all $x \in [0, 2\pi]$. Also, $g(0) = 0 = g(\frac{\pi}{2})$ (since $\int\limits_0^{\frac{\pi}{2}} f(x)\,dx = 0$). Hence by Rolle's theorem, there exists $c \in (0, \frac{\pi}{2})$ such that $g'(c) = 0$, *i.e.* $f(c) = 2\cos 2c$. □

6. Evaluate the limit: $\lim\limits_{n\to\infty} \left(\frac{1^8+3^8+\cdots+(2n-1)^8}{n^9} \right)$.

Solution. Let $f(x) = 2^8 x^8$ for all $x \in [0, 1]$. Considering the partition $P_n = \{0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n}{n} = 1\}$ of $[0, 1]$ for each $n \in \mathbb{N}$ and observing that $c_i = \frac{2i-1}{2n} = \frac{1}{2}(\frac{i-1}{n} + \frac{i}{n}) \in [\frac{i-1}{n}, \frac{i}{n}]$ for $i = 1, \ldots, n$, we find that

$$S(f, P_n) = \sum_{i=1}^{n} f\left(\frac{2i-1}{2n}\right)\left(\frac{i}{n} - \frac{i-1}{n}\right) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{2i-1}{n}\right)^8.$$

Since $f : [0, 1] \to \mathbb{R}$ is continuous, f is Riemann integrable on $[0, 1]$ and hence $\lim\limits_{n\to\infty} \left(\frac{1^8+3^8+\cdots+(2n-1)^8}{n^9} \right) = \lim\limits_{n\to\infty} S(f, P_n) = \int\limits_0^1 f(x)\,dx = \frac{2^8 x^9}{9}\Big|_{x=0}^1 = \frac{256}{9}$. □

7. Let $f : [0, \infty) \to \mathbb{R}$ be continuous. If $x\sin(\pi x) = \displaystyle\int_0^{x^2} f(t)dt$, find the value of $f(4)$.

Solution. Using 1st Fundamental thm, we have $f(4) = \pi/2$. □

8. Examine whether the integral $\int\limits_0^{\infty} \sin(x^2)\,dx$ is convergent.

Solution. Since the Riemann integral $\int\limits_0^1 \sin(x^2)\,dx$ exists, $\int\limits_0^{\infty} \sin(x^2)\,dx$ is convergent if $\int\limits_1^{\infty} \sin(x^2)\,dx$ is convergent. Let $f(x) = \frac{1}{2x}$ and $g(x) = 2x\sin(x^2)$ for all $x \in [1, \infty)$. Then f is decreasing on $[1, \infty)$ and $\lim\limits_{x\to\infty} f(x) = 0$. Also $\left|\int\limits_1^x g(t)\,dt\right| = |\cos 1 - \cos(x^2)| \le 2$ for all $x \in [1, \infty)$. Hence by Dirichlet's test, $\int\limits_1^{\infty} f(x)g(x)\,dx$, *i.e.* $\int\limits_1^{\infty} \sin(x^2)\,dx$ is convergent. Consequently $\int\limits_0^{\infty} \sin(x^2)\,dx$ is convergent. □

9. Determine all real values of p for which the integral $\int\limits_0^{\infty} \frac{x^{p-1}}{1+x}\,dx$ is convergent.

Solution. The given integral is convergent if and only if both the integrals $\int\limits_0^1 \frac{x^{p-1}}{1+x}\,dx$ and $\int\limits_1^\infty \frac{x^{p-1}}{1+x}\,dx$ are convergent. If $p \geq 1$, then $\int\limits_0^1 \frac{x^{p-1}}{1+x}\,dx$ exists as a Riemann integral. For $p < 1$, since $\lim\limits_{x\to 0+} \frac{x^{p-1}}{1+x} \cdot x^{1-p} = 1 \neq 0$, by the limit comparison test, $\int\limits_0^1 \frac{x^{p-1}}{1+x}\,dx$ converges if and only if $\int\limits_0^1 \frac{1}{x^{1-p}}\,dx$ converges. We know that $\int\limits_0^1 \frac{1}{x^{1-p}}\,dx$ converges if and only if $1-p < 1$, *i.e.* if and only if $p > 0$. Hence $\int\limits_0^1 \frac{x^{p-1}}{1+x}\,dx$ converges if and only if $p > 0$. Again, since $\lim\limits_{x\to\infty} \frac{x^{p-1}}{1+x} \cdot x^{2-p} = \lim\limits_{x\to\infty} \frac{x}{1+x} = 1 \neq 0$, by the limit comparison test, $\int\limits_1^\infty \frac{x^{p-1}}{1+x}\,dx$ converges if and only if $\int\limits_1^\infty \frac{1}{x^{2-p}}\,dx$ converges. We know that $\int\limits_1^\infty \frac{1}{x^{2-p}}\,dx$ converges if and only if $2-p > 1$, *i.e.* if and only if $p < 1$. Hence $\int\limits_1^\infty \frac{x^{p-1}}{1+x}\,dx$ converges if and only if $p < 1$. Therefore the given integral is convergent if and only if $0 < p < 1$. $\square$

10. Determine all real values of p for which the integral $\int\limits_0^\infty \frac{e^{-x}-1}{x^p}\,dx$ is convergent.

Proof. The given integral is convergent if and only if both $\int\limits_0^1 \frac{1-e^{-x}}{x^p}\,dx$ and $\int\limits_1^\infty \frac{1-e^{-x}}{x^p}\,dx$ are convergent. If $p \leq 0$, then $\int\limits_0^1 \frac{1-e^{-x}}{x^p}\,dx$ exists as a Riemann integral. For $p > 0$, since $\lim\limits_{x\to 0+} (\frac{1-e^{-x}}{x^p} \cdot x^{p-1}) = \lim\limits_{x\to 0+} (e^{-x} \cdot \frac{e^x-1}{x}) = 1 \neq 0$, by the limit comparison test, $\int\limits_0^1 \frac{1-e^{-x}}{x^p}\,dx$ converges if and only if $\int\limits_0^1 \frac{1}{x^{p-1}}\,dx$ converges. We know that $\int\limits_0^1 \frac{1}{x^{p-1}}\,dx$ converges if and only if $p-1 < 1$, *i.e.* if and only if $p < 2$. Hence $\int\limits_0^1 \frac{1-e^{-x}}{x^p}\,dx$ converges if and only if $p < 2$. Again, since $\lim\limits_{x\to\infty} (\frac{1-e^{-x}}{x^p} \cdot x^p) = \lim\limits_{x\to\infty} (1-e^{-x}) = 1 \neq 0$, by the limit comparison test, $\int\limits_1^\infty \frac{1-e^{-x}}{x^p}\,dx$ converges if and only if $\int\limits_1^\infty \frac{1}{x^p}\,dx$ converges. We know that $\int\limits_1^\infty \frac{1}{x^p}\,dx$ converges if and only if $p > 1$. Hence $\int\limits_1^\infty \frac{1-e^{-x}}{x^p}\,dx$ converges if and only if $p > 1$. Therefore the given integral is convergent if and only if $1 < p < 2$. $\square$

11. Examine whether the improper integral $\int\limits_{-\infty}^\infty te^{-t^2}\,dt$ is convergent.

Proof. Since $\lim\limits_{x\to\infty} \int\limits_0^x te^{-t^2}\,dt = -\frac{1}{2} \lim\limits_{x\to\infty} e^{-t^2}|_0^x = \frac{1}{2} \lim\limits_{x\to\infty} (1-e^{-x^2}) = \frac{1}{2}$, $\int\limits_0^\infty te^{-t^2}\,dt$ is convergent.
Again, since $\lim\limits_{x\to-\infty} \int\limits_x^0 te^{-t^2}\,dt = -\frac{1}{2} \lim\limits_{x\to-\infty} e^{-t^2}|_x^0 = \frac{1}{2} \lim\limits_{x\to-\infty} (e^{-x^2}-1) = -\frac{1}{2}$, $\int\limits_{-\infty}^0 te^{-t^2}\,dt$ is convergent. Therefore the given integral is convergent. $\square$