PH 102: Electromagnetism
Tutorial 3
24/03/20

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1. A uniformly charged solid sphere of radius R carries a total charge Q, and is set spinning with angular velocity ω about the z axis.
(a) What is the magnetic dipole moment of the sphere?
(b) Find the magnetic field at a point (r, θ) inside the sphere.
(c) Using the results of (b) find the average magnetic field within the sphere. Hint: Average magnetic field is defined as

$$\vec{B}_{\text{avg}} = \frac{1}{\frac{4}{3}\pi R^3} \int \vec{B} d\tau$$

Compare this result with the result of (a) and show that the average magnetic field is related to the magnetic dipole moment as

$$\vec{B}_{\text{avg}} = \frac{\mu_0}{4\pi} \frac{2\vec{m}}{R^3}$$

Solution:

The vector potential for a charged spinning spherical shell, as discussed in the class, is

$$\vec{A}(\vec{r}) = \begin{cases} \frac{\mu_0 R\sigma}{3}(\vec{\omega} \times \vec{r}) = \frac{\mu_0 R\omega\sigma}{3} r \sin\theta \hat{\phi} & \text{for } r \leq R \\ \frac{\mu_0 R^4 \sigma}{3r^3}(\vec{\omega} \times \vec{r}) = \frac{\mu_0 R^4 \omega\sigma}{3} \frac{\sin\theta}{r^2} \hat{\phi} & \text{for } r \geq R. \end{cases}$$

For $\vec{\omega}$ along the z-axis, the vector potential outside the sphere can be written as

$$\vec{A} = \frac{\mu_0 R^4 \omega\sigma}{3} \frac{\sin\theta}{r^2} \hat{\phi} = \frac{\mu_0}{4\pi} \frac{1}{r^2} \frac{4\pi R^4 \omega\sigma}{3} (-\sin\theta \hat{\theta} \times \hat{r}) \quad (r \geq R)$$

(a) Compering it with the dipole contribution to vector potential $\vec{A} = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$, the dipole moment can be written as

$$\vec{m} = \frac{4\pi R^4 \omega\sigma}{3} \hat{z}$$

To find the dipole moment for a charged spinning solid sphere, we can first consider a shell at a distance r from the centre having a thickness dr. If ρ is the volume charge density of the sphere, then the shell of radius r and thickness dr has surface charge density $\sigma = \rho dr$. The total magnetic dipole moment of the solid sphere can then be found by integrating over all such shells from radius $r = 0$ to $r = R$. This is given by

$$\vec{m} = \frac{4\pi}{3} \omega\rho \hat{z} \int_0^R r^4 dr = \frac{4\pi}{3} \omega\rho \frac{R^5}{5} \hat{z} = \frac{1}{5} Q\omega R^2 \hat{z}$$

where, in the last step, we have used $\rho = \frac{Q}{(4/3)\pi R^3}$.

(b) To find the magnetic field at a point (r, θ) inside the sphere, we first find the vector potential $\vec{A}$. Consider a spherical shell of radius r' and thickness dr' inside the solid sphere of radius R. The surface charge density of this shell is $\sigma = \rho dr'$, where ρ is the volume charge density. The vector potential outside and inside this shell are given by the above expressions. To find the net potential at a radial distance r inside the sphere, we integrate over all possible such shells of infinitesimal thickness inside the radius $r' < r$ and outside the radius $r < r' < R$. The infinitesimal vector potential is

$$d\vec{A}(r,\theta) = \frac{\mu_0 (r')^4 \omega\sigma}{3}\frac{\sin\theta}{r^2}\hat{\phi} + \frac{\mu_0 r'\omega\sigma}{3} r\sin\theta\hat{\phi}$$

. The net vector potential can be found by substituting $\sigma = \rho dr'$ and integrating over r':

$$\vec{A}(r,\theta) = \frac{\mu_0\omega\rho}{3}\frac{\sin\theta}{r^2}\hat{\phi}\int_0^r (r')^4 dr' + \frac{\mu_0\omega\rho}{3} r\sin\theta\hat{\phi}\int_r^R r' dr'$$

$$\implies \vec{A}(r,\theta) = \frac{\mu_0\omega\rho}{3}\sin\theta\left[\frac{1}{r^2}\frac{r^5}{5} + \frac{r}{2}(R^2 - r^2)\right]\hat{\phi} = \frac{\mu_0\omega\rho}{2} r\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\hat{\phi}$$

. Now the magnetic field can be found by taking the curl of $\vec{A}$:

$$\vec{B} = \vec{\nabla}\times\vec{A} = \frac{\mu_0\omega\rho}{2}\frac{1}{r^2\sin\theta}\left[\hat{r}\frac{\partial}{\partial\theta}\left(r\sin\theta r\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\right) - r\hat{\theta}\frac{\partial}{\partial r}\left(r\sin\theta r\sin\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\right)\right]$$

$$\implies \vec{B} = \frac{\mu_0\omega\rho}{2}\left[\hat{r}2\cos\theta\left(\frac{R^2}{3} - \frac{r^2}{5}\right) - \hat{\theta}\sin\theta 2\left(\frac{R^2}{3} - \frac{2r^2}{5}\right)\right]$$

$$\implies \vec{B} = \mu_0\omega\rho\left[\left(\frac{R^2}{3} - \frac{r^2}{5}\right)\cos\theta\hat{r} - \left(\frac{R^2}{3} - \frac{2r^2}{5}\right)\sin\theta\hat{\theta}\right]$$

Using $\rho = \frac{Q}{(4/3)\pi R^3}$,

$$\vec{B} = \frac{\mu_0\omega Q}{4\pi R}\left[\left(1 - \frac{3r^2}{5R^2}\right)\cos\theta\hat{r} - \left(1 - \frac{6r^2}{5R^2}\right)\sin\theta\hat{\theta}\right]$$

(c) Due to the symmetry of the problem, the average magnetic field will be in the z direction. Therefore, we can take out only the z components of the magnetic field found in part (b). Writing $\hat{r}, \hat{\theta}$ in terms of $\hat{x}, \hat{y}, \hat{z}$ that is $\hat{r} = \sin\theta\cos\phi\hat{x} + \sin\theta\sin\phi\hat{y} + \cos\theta\hat{z}$, $\hat{\theta} = \cos\theta\cos\phi\hat{x} + \cos\theta\sin\phi\hat{y} - \sin\theta\hat{z}$ and considering only the z-component (that is, take the z component of $\hat{r}(\cos\theta)$ and $\hat{\theta}(-\sin\theta)$, we can write down the average magnetic field as

$$B_{\text{avg}} = \frac{1}{\frac{4}{3}\pi R^3}\int B_z d\tau$$

$$\implies B_{\text{avg}} = \frac{\mu_0\omega Q}{4\pi R}\frac{1}{\frac{4}{3}\pi R^3}\int\left[\left(1 - \frac{3r^2}{5R^2}\right)\cos^2\theta + \left(1 - \frac{6r^2}{5R^2}\right)\sin^2\theta\right] r^2\sin\theta dr d\theta d\phi$$

$$\implies B_{\text{avg}} = \frac{3\mu_0\omega Q}{(4\pi R^2)^2} 2\pi \int_0^{\pi} \left[\left(\frac{R^3}{3} - \frac{3}{5}\frac{R^5}{5R^2} \right) \cos^2\theta + \left(\frac{R^3}{3} - \frac{6}{5}\frac{R^5}{5R^2} \right) \sin^2\theta \right] \sin\theta d\theta$$

$$\implies B_{\text{avg}} = \frac{3\mu_0\omega Q}{8\pi R^4} R^3 \int_0^{\pi} \left(\frac{16}{75}\cos^2\theta + \frac{7}{75}\sin^2\theta \right) \sin\theta d\theta = \frac{3\mu_0\omega Q}{8\pi R}\frac{1}{75} \int_0^{\pi} (7+9\cos^2\theta) \sin\theta d\theta$$

$$\implies B_{\text{avg}} = \frac{\mu_0\omega Q}{200\pi R}(-7\cos\theta - 3\cos^3\theta)|_0^{\pi} = \frac{\mu_0\omega Q}{200\pi R}(20) = \frac{\mu_0\omega Q}{10\pi R}$$

Using the expression for magnetic dipole moment obtained in part (a) that is, $\vec{m} = \frac{1}{5}Q\omega R^2\hat{z}$ it is straightforward to show that

$$\frac{\mu_0}{4\pi}\frac{2\vec{m}}{R^3} = \frac{\mu_0\omega Q}{10\pi R}\hat{z} = \vec{B}_{\text{avg}}$$

Therefore, the average magnetic field, over a sphere of radius R, due to steady current within the sphere, is

$$\vec{B}_{\text{avg}} = \frac{\mu_0}{4\pi}\frac{2\vec{m}}{R^3}$$

.

2. Suppose the field inside a large piece of magnetic material is $\vec{B}_0$, so that $\vec{H}_0 = \vec{B}_0/\mu_0 - \vec{M}$.
(a) Now a small spherical cavity is hollowed out of the material (as shown in figure 1). Find the field at the centre of the cavity, in terms of $\vec{B}_0, \vec{M}$. Also find $\vec{H}$ at the centre of the cavity in terms of $\vec{H}_0, \vec{M}$.
(b) Do the same for a long needle-shaped cavity running parallel to $\vec{M}$.
(c) Do the same for a thin wafer-shaped cavity perpendicular to $\vec{M}$.

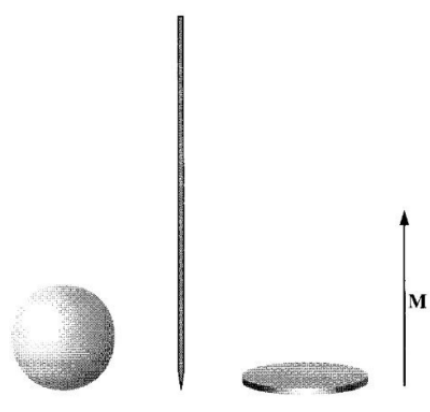
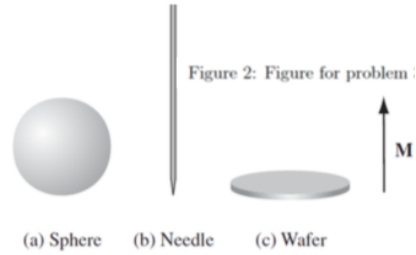


Figure 1: Figure for problem 2.

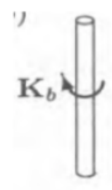
Solution:



(a) D. J. Griffiths, Example 6.1, the field inside a magnetized sphere is (2/3) μ_0**M.** The field after the removal of the sphere is $\mathbf{B} = \mathbf{B}_0$ - (2/3) μ_0**M.**

Thus in the cavity $\mathbf{H} = \mathbf{B}/\mu_0 = [\,\mathbf{B}_0 - (2/3)\ \mu_0\mathbf{M}\,]/\mu_0 = \mathbf{H_0} + \mathbf{M} - (2/3)\,\mathbf{M} = \mathbf{H_0} + 1/3\ \mathbf{M}.$

(b) The field inside a long solenoid is $\mu_0 K = \mu_0 M$. Thus the field of the bound charge on the inside surface of the needle shaped cavity is μ_0M, but *pointing down*. Thus, $\mathbf{B} = \mathbf{B}_0 - \mu_0\mathbf{M}$ and $\mathbf{H} = \mathbf{B}/\mu_0 = [\,\mathbf{B}_0 - \mu_0\mathbf{M}\,]/\mu_0 = \mathbf{H_0} + \mathbf{M} - \mathbf{M} = \mathbf{H_0}.$



(c) For the thin wafer, the bound currents are very small and also far away from the center. Hence, $\mathbf{B} = \mathbf{B}_0$ and the $\mathbf{H} = \mathbf{B}\,/\,\mu_0$
$= \mathbf{B}_0/\mu_0 = \mathbf{H_0} + \mathbf{M}.$

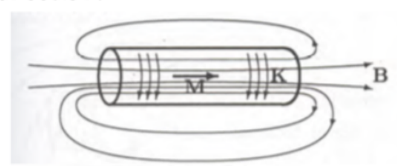
3. A short circular cylinder of radius a and length L carries a "frozen-in" uniform magnetisation $\vec{M}$ parallel to its axis. Find the bound current and sketch the magnetic field of the cylinder: one for $L \gg a$, one for $L \ll a$ and one for $L \approx a$.

Solution: The magnetization being uniform, the volume current, $\mathbf{J}_b = \nabla \times \mathbf{M} = \mathbf{0}$,

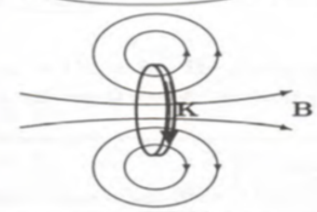
The surface current, $\mathbf{K}_b = \mathbf{M} \times \hat{\mathbf{n}} = M\hat{\boldsymbol{\phi}}$.

Here, **M** is in the z direction and surface vector **n** is in the radial direction.

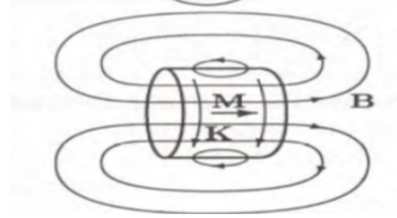
L >> a, This situation is similar to a long solenoid with the surface current flowing in the azimuth direction and uniform field inside.



L << a, This situation is similar to a physical dipole.



L ≈ a, This is the intermediate case of the above two.



4. (a) Find the magnetic dipole moment of a spherical shell, of radius R, carrying a uniform surface charge σ which is set to spin at angular velocity $\vec{\omega}$. (b) Consider a

charge of $3pC$ being distributed over a sphere of radius $1cm$ and having a uniform surface charge density σ. If this sphere is rotated about its diameter with angular velocity $\omega = 10^6$ radians per second, find the magnetic dipole moment of the sphere.

Solution:
(a) Consider the elemental ring of radius $R\sin\theta$, thickness $Rd\theta$ so that the charge on its surface is $dq = \sigma(2\pi R\sin\theta)Rd\theta$. Time period of the spinning sphere is $dt = 2\pi/\omega$. Therefore, the current in the ring is $I = dq/dt = \sigma\omega R^2\sin\theta d\theta$. The area of the ring is $\pi(R\sin\theta)^2$. The magnetic dipole moment of the ring is therefore, $dm = Ia = (\sigma\omega R^2\sin\theta d\theta)\pi(R\sin\theta)^2$.

The total magnetic dipole moment of the shell can be found by integrating over θ:

$$m = \sigma\omega\pi R^4\int_0^\pi \sin^3\theta d\theta = (4/3)\sigma\omega\pi R^4$$

$$\implies \vec{m} = \frac{4\pi}{3}\sigma\omega R^4\hat{z}.$$

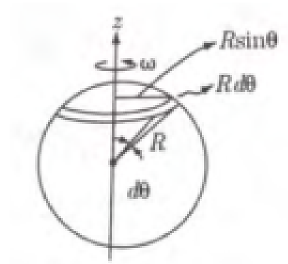


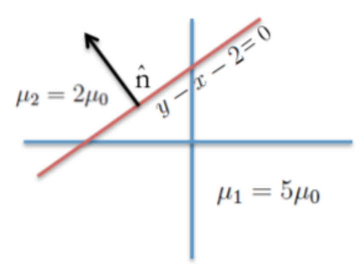
Figure 2: Solution to problem 4.

(b) For charge Q spread uniformly over the sphere, we can replace $\sigma = \frac{Q}{4\pi R^2}$. Using it in the expression for dipole moment found in part (a), we get

$$\vec{m} = \frac{Q\omega R^2}{3}\hat{z} = \frac{3\times10^{-12}\times10^6\times10^{-4}}{3}\hat{z}\ \mathrm{Am}^2 = 10^{-10}\hat{z}\ \mathrm{Am}^2. \quad (+\mathbf{1})$$

.

5. Given that $\vec{H_1} = -2\hat{i} + 6\hat{j} + 4\hat{k}$ A/m in the region $y - x - 2 \leq 0$, where $\mu_1 = 5\mu_0$. Calculate
(a) $\vec{M_1}$ and $\vec{B_1}$.
(b) $\vec{M_2}$ and $\vec{B_2}$ in the region $y - x - 2 \geq 0$, where $\mu_2 = 2\mu_0$.



Solution: $\hat{\mathbf{n}} = (\hat{j} - \hat{i})/\sqrt{2}$

(a) $\mathbf{M_1} = \chi_m \mathbf{H_1} = (\mu_{r1} - 1)\mathbf{H_1} = 4\mathbf{H_1}$ and $\mathbf{B_1} = \mu_1\mathbf{H_1} = 5\mu_0\mathbf{H_1}$

(b) $\mathbf{H_{1n}} = (\mathbf{H_1}.\hat{\mathbf{n}})\hat{\mathbf{n}} = 4(-\hat{i} + \hat{j})$, $\mathbf{H_{1t}} = \mathbf{H_1} - \mathbf{H_{1n}} = 2(\hat{i} + \hat{j} + \hat{k})$.

Using the boundary conditions, $\mathbf{B_{2n}} = \mathbf{B_{1n}}$ and $\mu_2\mathbf{H_{2n}} = \mu_1\mathbf{H_{1n}} \Rightarrow 2\mu_0\mathbf{H_{2n}} = 5\mu_0\mathbf{H_{1n}}$

Thus, $\mathbf{H_{2n}} = 10(-\hat{i} + \hat{j})$ and $\mathbf{H_{2t}} = \mathbf{H_{1t}} = 2(\hat{i} + \hat{j} + \hat{k})$, $\mathbf{H_2} = \mathbf{H_{2n}} + \mathbf{H_{2t}} = -8\hat{i} + 12\hat{j} + 4\hat{k}$

and $\mathbf{B_2} = \mu_2\mathbf{H_2}$

6. A circular loop of radius a is at a distance D above a tiny magnetic dipole of infinitesimal area dS carrying a current I_1, as shown in figure 3. Assume current through the circular loop $I_2 = 0$, for the time being. Also, the distance D and loop radius a are related as $D = \sqrt{3}a$. Write your final answers only in terms of I_1, dS, a and fundamental constants.

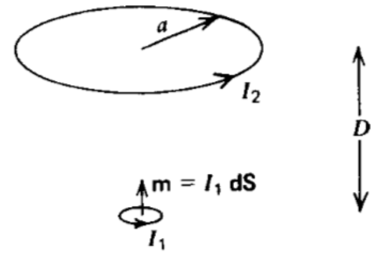


Figure 3: Figure for problem 6

(i) What is the vector potential due to the dipole at all points on the circular loop.

(ii) Consider the loop to be carrying a current $I_2 \neq 0$. The relation between D and a remains same as before $D = \sqrt{3}a$. What is the magnetic field due to I_2 at the position of the tiny dipole? What is the force on the magnetic dipole? What is the torque on the magnetic dipole?

Solution:

(i) Vector potential due to a magnetic dipole is given as

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m} \times \hat{r}}{r^2}$$

$$\implies \vec{A} = \frac{\mu_0}{4\pi}\frac{I_1 dS \sin\theta}{r^2}\hat{\phi}$$

Here, $r^2 = a^2 + D^2, \sin\theta = \frac{a}{r} = \frac{a}{(a^2+D^2)^{1/2}}$. Also, using $D = \sqrt{3}a$, one can find

$$\vec{A} = \frac{\mu_0}{4\pi}\frac{I_1 dSa}{8a^3}\hat{\phi} = \frac{\mu_0}{32\pi}\frac{I_1 dS}{a^2}\hat{\phi}$$

(ii) Magnetic field at the axis of a current carrying loop of radius a and current I is

$$\vec{B} = \frac{\mu_0 I}{2}\frac{a^2}{(a^2+z^2)^{3/2}}\hat{z}$$

Here, $I = I_2, z = D = \sqrt{3}a$. Therefore,

$$\vec{B} = \frac{\mu_0 I_2 a^2}{2(a^2+D^2)^{3/2}}\hat{z} = \frac{\mu_0 I_2}{16a}\hat{z}$$

Force on the (infinitesimal) magnetic dipole $\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B})$. Here $\vec{m} = I_1 dS\hat{z}, \vec{B} = \frac{\mu_0 I_2 a^2}{2(a^2+z^2)^{3/2}}\hat{z}$. Therefore,

$$\vec{m}\cdot\vec{B} = \frac{\mu_0 I_1 I_2 dSa^2}{2(a^2+z^2)^{3/2}}$$

$$\implies \vec{\nabla}(\vec{m}\cdot\vec{B}) = \frac{\mu_0 I_1 I_2 dSa^2}{2}(-\frac{3}{2})\frac{2z}{(a^2+z^2)^{5/2}}\hat{z}$$

At $z = -D$, the force is, therefore

$$\vec{F} = \frac{3\mu_0}{2} I_1 I_2 dSa^2 \frac{D}{(a^2+D^2)^{5/2}}\hat{z}$$

Using $D = \sqrt{3}a$,

$$\vec{F} = \frac{3\mu_0}{2} I_1 I_2 dSa^2 \frac{\sqrt{3}a}{32a^5}\hat{z} = \frac{3\sqrt{3}\mu_0}{64} I_1 I_2 \frac{dS}{a^2}\hat{z}$$

Torque on the dipole is $\vec{\tau} = \vec{m}\times\vec{B} = 0$.