PH 101: Physics I

Module 2: Special Theory of Relativity-Basics

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Recap from the last class

Lorentz transformation:

Relating space and time coordinates in frames S and S', where S' is moving with constant velocity $\vec{v}$ along the x-axis.

$$x' = \gamma(x - \beta x_0); \qquad y' = y; \quad z' = z \qquad x_0 = ct \qquad \beta = \frac{v}{c} \qquad \gamma = \sqrt{\frac{1}{1-\beta^2}}$$

$$x'_0 = \gamma(x_0 - \beta x)$$

The invariant interval is

$$ds^2 = c^2\,dt^2 - dx^2 - dy^2 - dz^2$$
$$= c^2\,dt'^2 - dx'^2 - dy'^2 - dz'^2 = ds'^2$$

Light-like interval $(ds^2 = 0)$ remains like light-like in all inertial frames

Time-like interval $(ds^2 > 0)$ remains like time-like in all inertial frames

Space-like interval $(ds^2 < 0)$ remains like space-like in all inertial frames

Lorentz Transformation: A rotation in space-time

Transformation Matrix

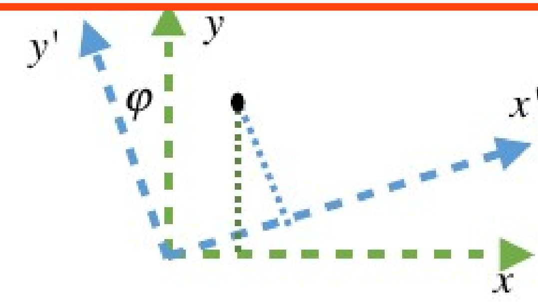
Representing the Lorentz transformations in matrix form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

First, consider Rotation in 2D

$$x' = x\cos\varphi + y\sin\varphi$$

$$y' = -x\sin\varphi + y\cos\varphi$$



In Short notation $X' = R\cdot X$

Can we write the Lorentz transformation in a similar way?

$$x' = \gamma\left(x - \beta x_0\right); \qquad y' = y; \quad z' = z$$

$$x'_0 = \gamma\left(x_0 - \beta x\right)$$

$$\begin{pmatrix} x'_0 \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x \\ y \\ z \end{pmatrix}$$

$X' = L\cdot X$

Noting that $\gamma^2 = \dfrac{1}{1-\beta^2}$

we have $\gamma^2 - (\gamma\beta)^2 = 1$

one may parametrise: $\cosh\vartheta = \gamma, \quad \sinh\vartheta = \gamma\beta$

$$\begin{pmatrix} x'_0 \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\vartheta & -\sinh\vartheta & 0 & 0 \\ -\sinh\vartheta & \cosh\vartheta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x \\ y \\ z \end{pmatrix}$$

Lorentz Transformation: A general form

S' moving in arbitrary direction (with constant velocity)
(General Lorentz transformation)

$$t' = \gamma\left(t - \frac{\vec{r}\cdot\vec{v}}{c^2}\right), \qquad \vec{r}' = r + \left(\frac{\gamma - 1}{v^2}\vec{r}\cdot\vec{v} - \gamma t\right)\vec{v}$$

$$\beta_i = \frac{v_i}{c}$$

$$\gamma = \frac{1}{\sqrt{1 - \dfrac{\vec{v}\cdot\vec{v}}{c^2}}}$$

$$\begin{pmatrix} x'_0 \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & -\gamma\beta_z \\ -\gamma\beta_x & 1+(\gamma-1)\dfrac{\beta_x^2}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_y}{\beta^2} & (\gamma-1)\dfrac{\beta_x\beta_z}{\beta^2} \\ -\gamma\beta_y & (\gamma-1)\dfrac{\beta_x\beta_y}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_y^2}{\beta^2} & (\gamma-1)\dfrac{\beta_y\beta_z}{\beta^2} \\ -\gamma\beta_z & (\gamma-1)\dfrac{\beta_x\beta_z}{\beta^2} & (\gamma-1)\dfrac{\beta_y\beta_z}{\beta^2} & 1+(\gamma-1)\dfrac{\beta_z^2}{\beta^2} \end{pmatrix} \begin{pmatrix} x_0 \\ x \\ y \\ z \end{pmatrix}$$

Lorentz Transformation: Breaking of simultaneity

Simultaneity

Consider two different frames S and S',
with S' moving along along the x direction with speed v with respect to S.

Consider two events with coordinates $\left(t_1, x_1, y_1, z_1\right)$ and $\left(t_2, x_2, y_2, z_2\right)$ in S

$\left(t'_1, x'_1, y'_1, z'_1\right)$ and $\left(t'_2, x'_2, y'_2, z'_2\right)$ in S'

Intervals of the two events be denote by: $dx = x_2 - x_1, \quad dt = t_2 - t_1$ etc.

Space and time intervals in the two frames are related by

$$dx' = \gamma\left(dx - \beta c\, dt\right); \qquad dy' = dy; \quad dz' = dz$$

$$c\, dt' = \gamma\left(c\, dt - \beta dx\right)$$

where

$$\beta = \frac{v}{c}, \qquad \gamma = \sqrt{\frac{c^2}{c^2 - v^2}}$$

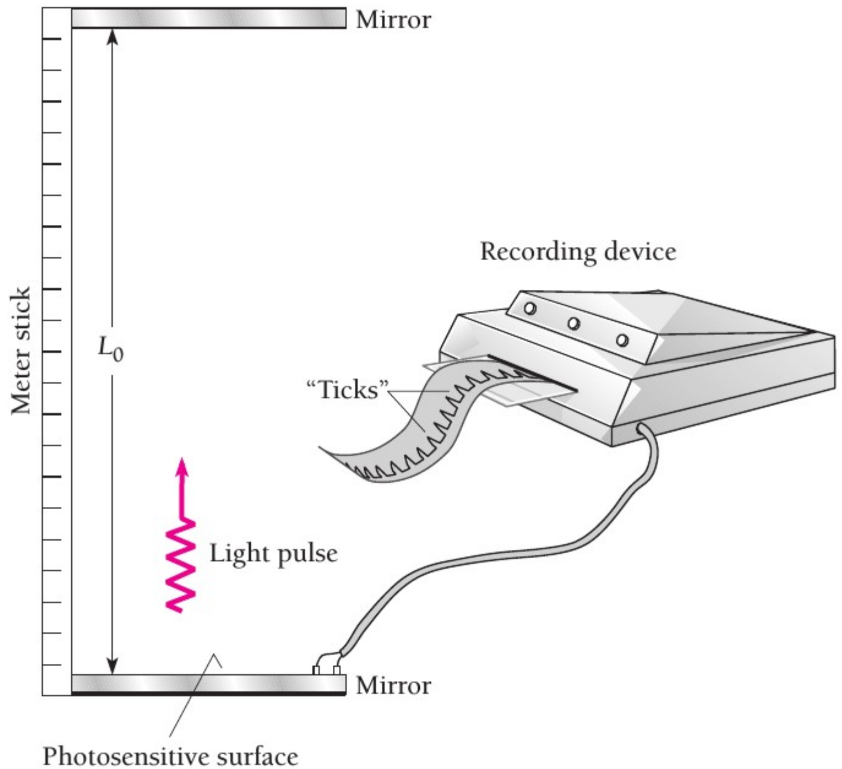
Consider the events as simultaneous in S: $dt = t_2 - t_1 = 0$

The time interval of the same events seen from S': $dt' = \gamma\left(dt - \frac{\beta}{c} dx\right) = -\gamma \frac{\beta}{c} dx$

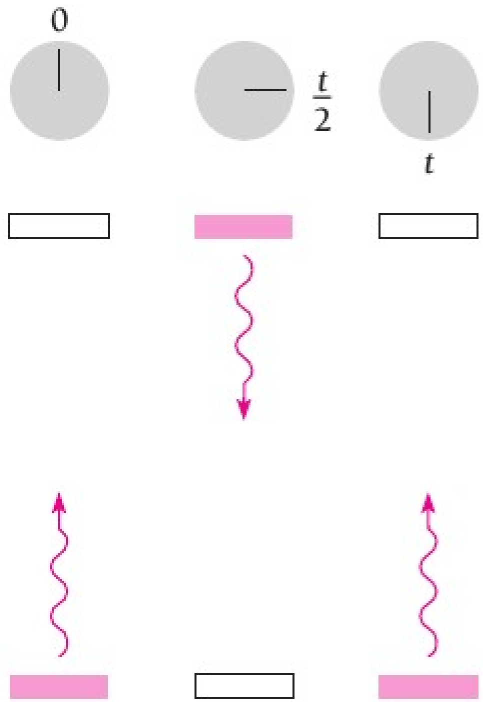
Two events which are simultaneous in one frame are not necessarily simultaneous in another inertial frame.

Time Dilation

We consider the time interval(t_0=proper time) between the events that take place at the same spatial postion.



$$t_0 = \frac{2L_0}{c}$$



Device shows the ticking of an event once the light pulse completes one cycle between two mirrors.

One tick signals one event in the device.

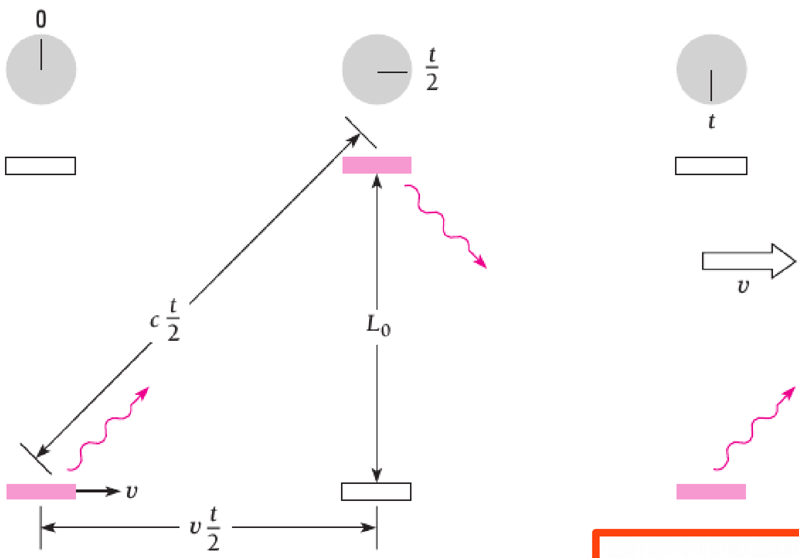
Representation of the event from S frame.

At t=0, a pulse starts from the lower mirror, gets reflected from the upper one at $t=t_0/2$ and is detected at the lower mirror again at time $t=t_0$. (Complete cycle)

Time Dilation: Observation from stationary frame

Event occuring in S'(moving with speed v) and seen from the S frame .

t_0 = time interval on clock at rest relative to an observer = proper time
t = time interval on clock in motion relative to an observer
$\boldsymbol{v}$ = speed of relative motion
c = speed of light



$$\left(\frac{ct}{2}\right)^2 = L_0^2 + \left(\frac{\boldsymbol{v}t}{2}\right)^2$$

$$\frac{t^2}{4}(c^2 - \boldsymbol{v}^2) = L_0^2$$

$$t^2 = \frac{4L_0^2}{c^2 - \boldsymbol{v}^2} = \frac{(2L_0)^2}{c^2(1 - \boldsymbol{v}^2/c^2)}$$

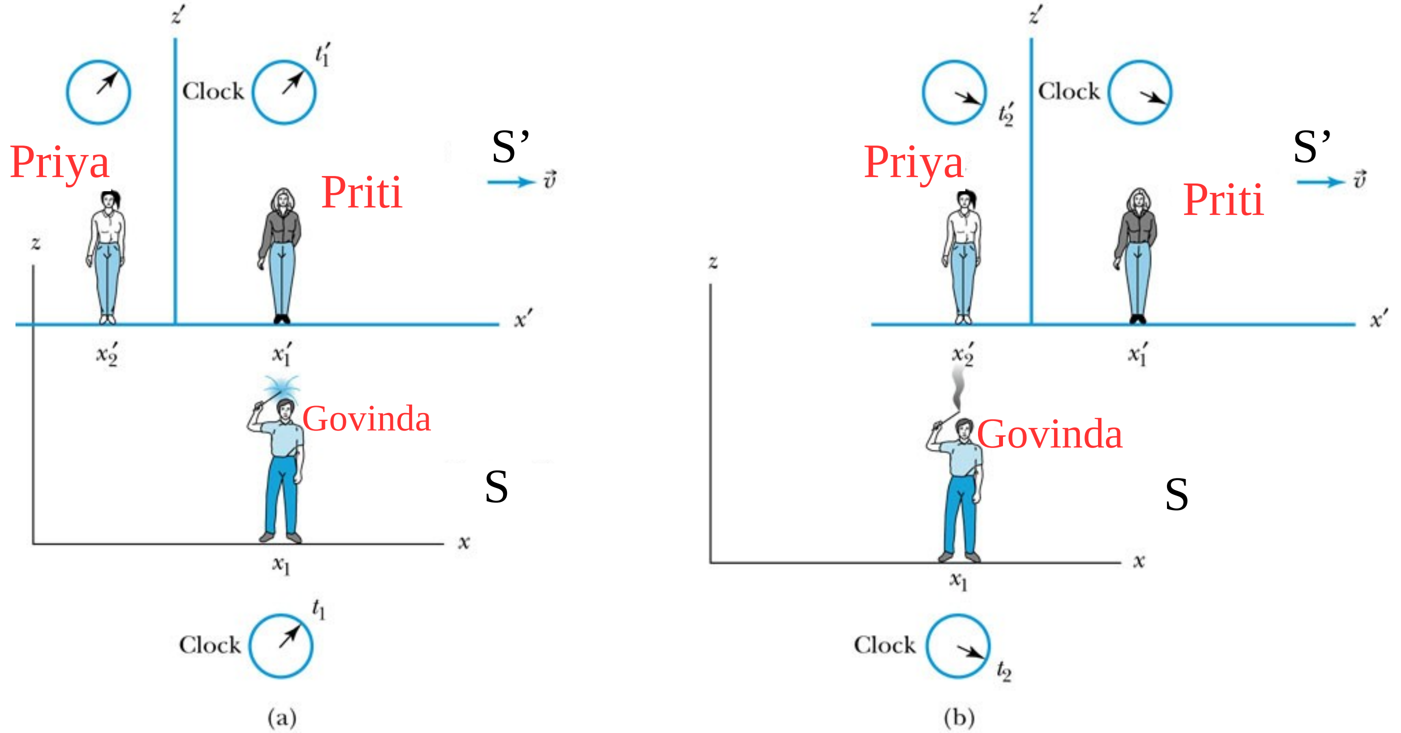
$$t = \frac{2L_0/c}{\sqrt{1 - \boldsymbol{v}^2/c^2}}$$

$$t = \frac{t_0}{\sqrt{1 - v^2/c^2}} = \gamma t_0$$

$$\gamma > 1$$

Clocks run slower in a moving frame, compared to the frame at rest.

Time Dilation: Observation from moving frame



Govinda's clock is at the same position in system S when the sparkler is lit in (a) and when it goes out in (b). Priti, in the moving system S', is beside the sparkler at (a). Priya then moves into the position where and when the sparkler extinguishes at (b). Thus, Priya, at the new position, measures the time in system S' when the sparkler goes out in (b).

Priya and Priti measure the two times for the sparkler to be lit and to go out in system S' as times t'_1 and t'_2 so that by the Lorentz transformation:

$$t'_2 - t'_1 = \frac{(t_2 - t_1) - (v/c^2)(x_2 - x_1)}{\sqrt{1 - v^2/c^2}}$$

Note here that Govinda records $x_2 - x_1 = 0$ in S with a proper time: $t_0 = t_2 - t_1$ or

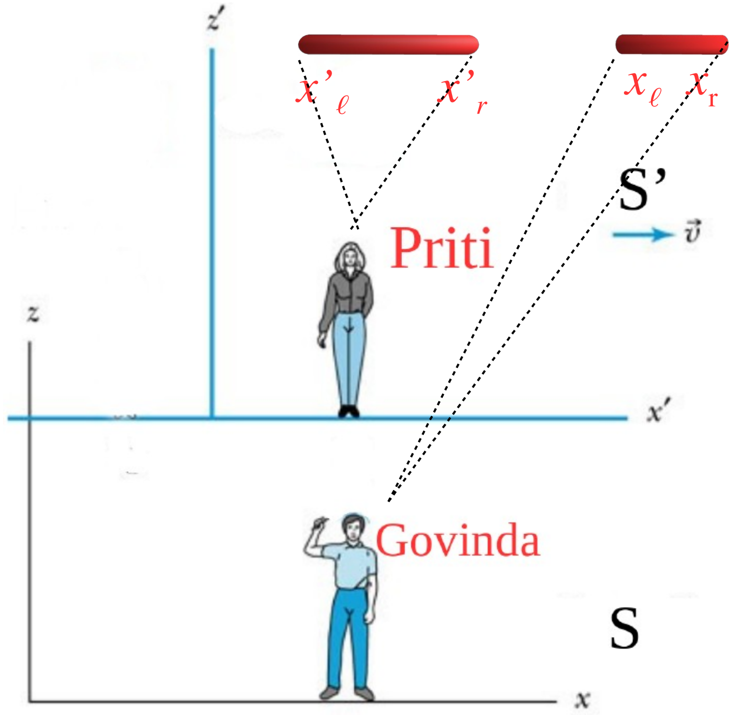
$$t'_2 - t'_1 = \gamma t_0$$

Time interval apears to be longer from the moving frame of Priya and Priti.

Length Contraction

Consider observers from frame S and S' are measuring the length of a stick which is at rest w.r.t. the S'. The length as measured at rest is called the

proper length $L'_0 = x'_r - x'_\ell$.



What Govinda will measure from the S-frame ???

Left and right ends of a stick was measured from S (S') frame with coordinates x_ℓ (x'_ℓ)and x_r (x'_r) respectively.

In frame S, Govinda measures the stick length : $L = x_r - x_\ell$

From frame S'(moving with velocity v w.r.t S), Priti measures:

$$L'_0 = x'_r - x'_\ell$$

From Lorentz transformations we have:

$$x'_r - x'_\ell = \frac{(x_r - x_\ell) - v(t_r - t_\ell)}{\sqrt{1 - v^2/c^2}}$$

Where both ends of the stick must be measured simultaneously, i.e, $t_r = t_\ell$.

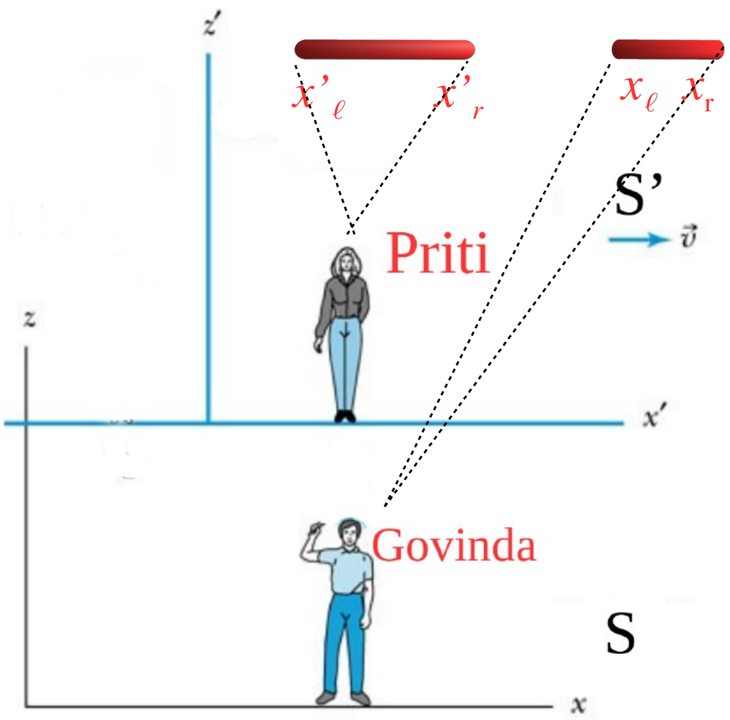
From Priti's frame the proper length is $L'_0 = x'_r - x'_\ell$

and Govinda's measured length is $L = x_r - x_\ell$

Length Contraction

So Govinda measures the moving length as L given by

$$L'_0 = \frac{L}{\sqrt{1 - v^2/c^2}} = \gamma L$$

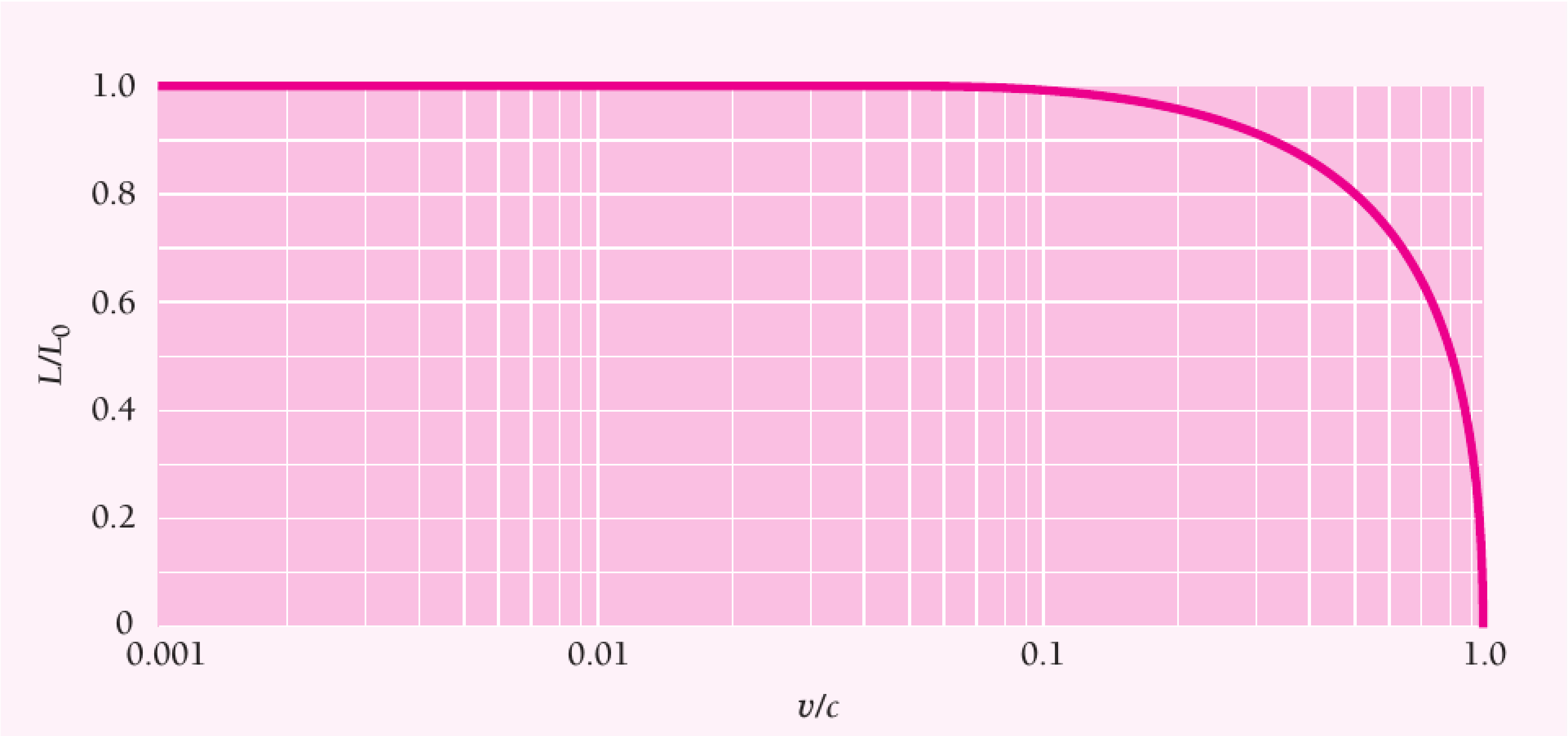


However, the proper length remains invariant for Priti and Govinda,
i.e. $L'_0 = L_0$. So we have;

$$L = L_0\sqrt{1 - v^2/c^2} = \frac{L_0}{\gamma}$$

and $L_0 > L$, i.e. the moving stick shrinks. Hence length contraction !!

Length Contraction



At v=c, the length will shrink to zero. Photon moves at velocity c and is a point particle!!

Length contraction: Imagine a rod is placed on a table inside a spaceship S' moving relative to earth S with speed v.

The astronaut at rest relative to the rod wishes to measure the length of the rod. He could do one of two things.

i) Note down the location of the left end of the rod and right end of the rod at the same time according to the astronaut. This is guaranteed to give the correct length since the rod is not moving relative to the astronaut.

ii) Since the rod is not moving relative to the astronaut, there is another way to measure the length of the rod. He marks the location of the left end of the rod. Then he leaves the room goes to bed wakes up the next day, comes back to the table where the rod is still there and then locates the right end of the rod. The distance between the two marks is STILL THE CORRECT length of the rod since the rod was not moving when the astronaut was asleep.

iii) However for someone looking at the rod through a telescope on earth it is imperative that the location of the left end and right end of the rod be located AT THE SAME TIME according to the person watching the rod through the telescope. The person on earth does not have the luxury of marking the location of the left end of the rod, then going to bed, waking up and then marking the right end of the rod. This is because when the person on earth was asleep, the spaceship would have travelled a great distance and the right end of the rod will be really far away compared to the location of the left end of the rod noted by this observer the previous day.

Hence for length contraction:

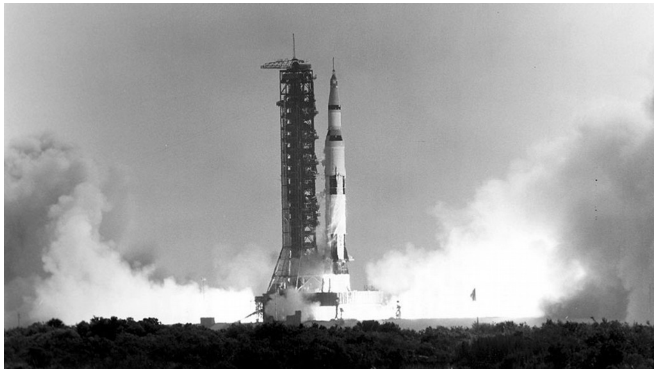
$X'_R - X'_L$ is the correct length of the rod EVEN IF t'_R is not equal to t'_L.
(since the rod is at rest in the spaceship according to S')

But $X_R - X_L$ is NOT the correct length of the rod as seen by S UNLESS $t_R = t_L$ (since the rod is in a moving spaceship according to S). Note that this length is less that the "proper" length measured by S'.

Example: Time dilation

A spacecraft is moving relative to the earth. An observer on the earth finds that, between 1 P.M. and 2 P.M. according to her clock, 3601 s elapse on the spacecraft's clock. What is the spacecraft's speed relative to the earth?

Here $t_0 = 3600$ s is the proper time interval on the earth and $t = 3601$ s is the time interval in the moving frame as measured from the earth. We proceed as follows:



$$t = \frac{t_0}{\sqrt{1 - \boldsymbol{v}^2/c^2}}$$

$$1 - \frac{\boldsymbol{v}^2}{c^2} = \left(\frac{t_0}{t}\right)^2$$

$$\boldsymbol{v} = c\sqrt{1 - \left(\frac{t_0}{t}\right)^2} = (2.998 \times 10^8 \text{ m/s})\sqrt{1 - \left(\frac{3600 \text{ s}}{3601 \text{ s}}\right)^2}$$

$$= 7.1 \times 10^6 \text{ m/s}$$

Today's spacecraft are much slower than this. For instance, the highest speed of the Apollo 11 spacecraft that went to the moon was only 10,840 m/s, and its clocks differed from those on the earth by less than one part in 10^9. Most of the experiments that have confirmed time dilation made use of unstable nuclei and elementary particles which readily attain speeds not far from that of light.

Example: Time dilation

A microwave oven in a spaceship moving at speed 0.95c relative to the earth is set to heat a frozen pizza for 3 minutes. To an observer on the earth, for how long does the microwave oven heat the pizza?

Let S' be the reference frame on the spaceship and S be the reference frame on the earth.

Event **E1**: Start of the microwave oven

Event **E2:** Microwave oven switches off

The two events happen at the same spatial location in S' (let us set this to be the origin).

$$t'_1 = 0 = x'_1 = y'_1 = z'_1$$

$$t'_2 = 3m,\ x'_2 = y'_2 = z'_2 = 0$$

We also synchronise the coordinates of the two frames so that E1 happens as time $t_1 = 0$ and at the origin of the reference frames in both the cases.

$$t_1 = 0 = x_1 = y_1 = z_1$$

How does E2 look, when seen from the two frames?

	As seen from S':	As seen from S:
$E_1:$	$(0,0,0,0)$	$(0,0,0,0)$
$E_2:$	$(t'_2 = 3m, 0, 0, 0)$	$(t_2, x_2, 0, 0)$

$$t = \gamma(t' + \beta x') \qquad x = \gamma(x' + \beta t')$$

$$\beta = \frac{v}{c} = 0.95, \qquad \gamma = \sqrt{\frac{c^2}{c^2 - v^2}} = 3.20256$$

$$t_2 = \gamma(t'_2 + \beta x'_2) = \gamma t'_2 = 3.20256 \times 3\ m$$

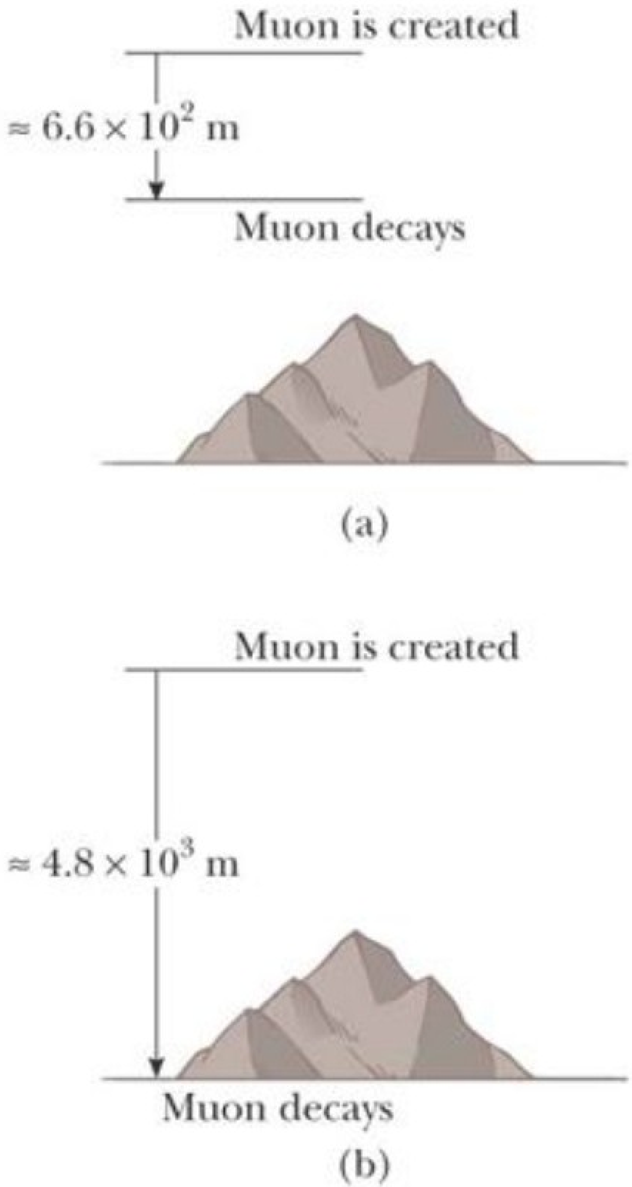
$$\Rightarrow t_2 = 9.60768\ m$$

The pizza heats up for more than 9 minutes !!

Time Dilation verification: Muon Decays

- Muons are unstable particles that have the same charge as an electron, but a mass 207 times more than an electron
- Muons have a half-life of Δt_p = 2.2 μs when measured in a reference frame at rest with respect to them (a)
- Relative to an observer on the Earth, muons should have a lifetime of

  $\gamma\,\Delta t_p$ (b)
- A CERN experiment measured lifetimes in agreement with the predictions of relativity



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Example: Length Contraction

Imagine an astronaut on the same spaceship (that we considered earlier today, moving with a speed 0.95c) is also an athlete and is practicing long jump. He jumps a distance of 6m in the direction of motion of the spaceship as measured by his friend, who is also on the spaceship. To an observer on the earth, how far does this athlete appear to jump?

After the jump the observer on earth measures the starting location E_1 and the finish location E_2 at the same time (according to him). It is important to measure the starting location and ending location at the same time when these locations are moving along with the spaceship to get the distance between them. But on the spaceship it is not important when you note down the starting and ending location you will still get the correct distance.

$$x_1 = y_1 = z_1 = t_1 = 0 \,; \qquad x_2 = ?\,, y_2 = z_2 = 0\,; \; t_2 = 0$$

$$x_1' = y_1' = z_1' = 0; \qquad x_2' = 6m,\; y_2' = z_2' = 0$$

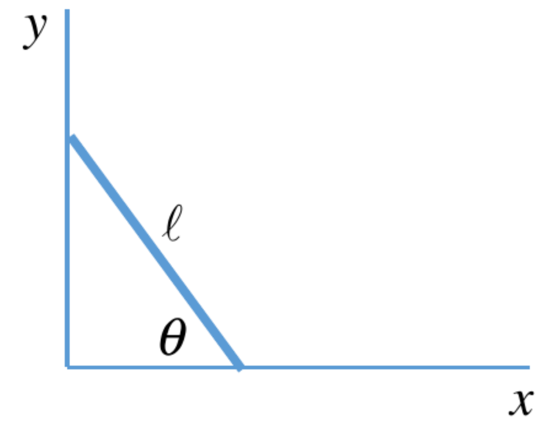
$$x'_1 = \gamma\,(x_1 - \mathrm{v}\,t_1)\,; \qquad x'_2 = \gamma\,(x_2 - \mathrm{v}\,t_2)$$

$$6\,m = \gamma\,(x_2 - \mathrm{v} \times 0)\;; \qquad x_2 = \frac{6\,m}{3.20256} = 1.8735\,m$$

The observer on earth is not at all impressed by this jump whereas the athlete on the spaceship is celebrating his longest ever jump!

Example: Length Contraction

A long thin rod is kept on slanted on a wall so as to make an angle θ with the horizontal, as measured in the rest frame of the rod (on earth). The length of the rode as measured in this frame is ℓ. What is the angle, as measured from a frame moving with speed 0.7c along the horizontal x-axis (see figure)?



The two ends of the rod are to be measured simultaneously in the moving frame, S'.

Left end $(t'_L, x'_L, y'_L, 0)$

Right end $(t'_R, x'_R, y'_R, 0)$ with $t'_L = t'_R = t'$

As measured in the rest frame of the rod, (S):

Left end $(t_L, x_L, y_L, 0)$ $\quad x_L = \gamma\left(x'_L - \beta c t'\right)$ $\quad x_R - x_L = \gamma\left(x'_R - x'_L\right)$

Right end $(t_R, x_R, y_R, 0)$ $\quad x_R = \gamma\left(x'_R - \beta c t'\right)$ $\quad y_R - y_L = y'_R - y'_L$

$$\tan\theta' = \frac{y'_R - y'_L}{x'_R - x'_L} = \frac{\gamma\left(y_R - y_L\right)}{\left(x_R - x_L\right)} = \gamma \tan\theta$$

The angle will increase, as the perpendicular component remain the same, and the horizontal component undergoes length contraction.

We shall continue in the next class