PH 102: Physics II

Lecture 10 (Post midsem, Spring 2020)

Debasish Borah
IIT Guwahati

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Division
Lec 1	05-03-2020	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 2	11-03-2020	Application of Ampere's Law, Magnetic Vector Potential	5.3, 5.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 1	12-03-2020	Lec 1		
Tut 2	17-03-2020	Lec 2		
Lec 3	18-03-2020	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic materials, magnetization	5.4, 6.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 4	19-03-2020	Field of a magnetized object, Boundary conditions	6.2, 6.3, 6.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 3	24-03-2020	Lec 3, 4		
Lec 5	25-03-2020	Ohm's law, motional emf, electromotive force	7.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 6	26-03-2020	Faraday's law, Lenz's law, Self & Mutual inductance, Energy stored in magnetic field	7.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 4	31-03-2020	Lec 5, 6		
Lec 7	01-04-2020	Maxwell's equations	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 8	02-04-2020	Discussions, Problem solving	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
	07-04-2020	Quiz II		
Lec 9	08-04-2020	Continuity equation, Poynting theorem	8.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 10	16-04-2020	Wave solution of Maxwell's equations, polarisation	9.1, 9.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 5	21-04-2020	Lec 9, 10		
Lec 11	22-04-2020	Electromagnetic waves in matter, reflection & transmission: normal incidence	9.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 12	23-04-2020	Reflection & transmission: oblique incidence	9.3, 9.4	I, II (4-4:55 pm) III, IV (11-11:55

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

				am)
Tut 6	28-4-2020	Lec 11, 12		
Lec 13	29-04-2020	Relativity and electromagnetism: Galilean & special relativity	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 14	30-04-2020	Discussions, problem solving	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)

Waves

A wave is a disturbance of a continuous medium that propagates with a fixed shape at constant velocity.

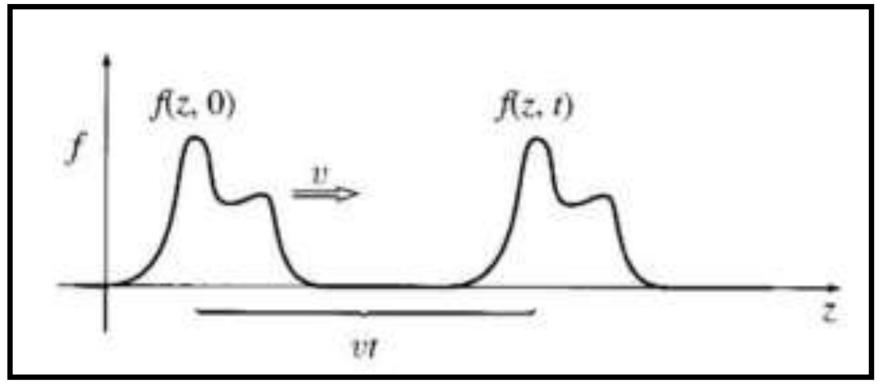


Figure 9.1, Introduction to Electrodynamics, D J Griffiths

Let f(z,t) be the displacement of the string at the point z, at time t, which is moving towards right with velocity v.

Given the initial shape g(z)=f(z,0), what is the subsequent form f(z,t)?

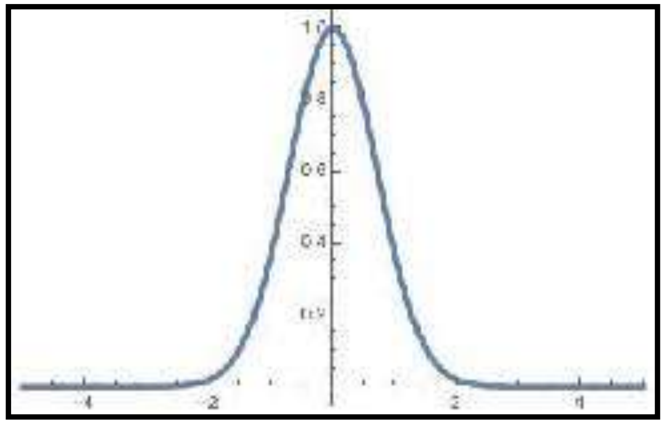
According to the above definition, the displacement at (z, t) should be same as at a distance vt to the left back at time t=0:

$$f(z,t) = f(z-vt,0) = g(z-vt)$$

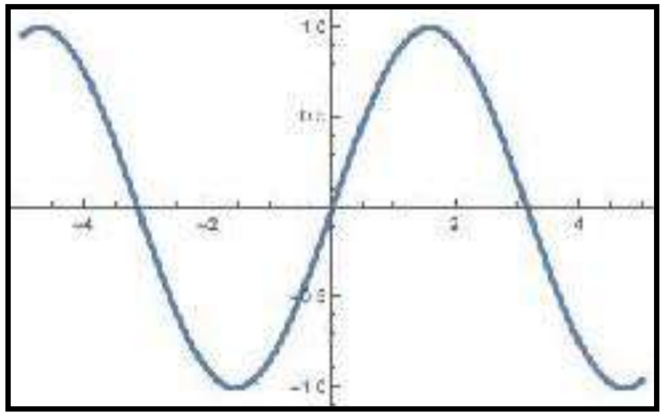
f(z,t) depends upon the combination z-vt

Since the displacement f(z,t) depends only on the combination z-vt, one can define different types of waves with different shapes:

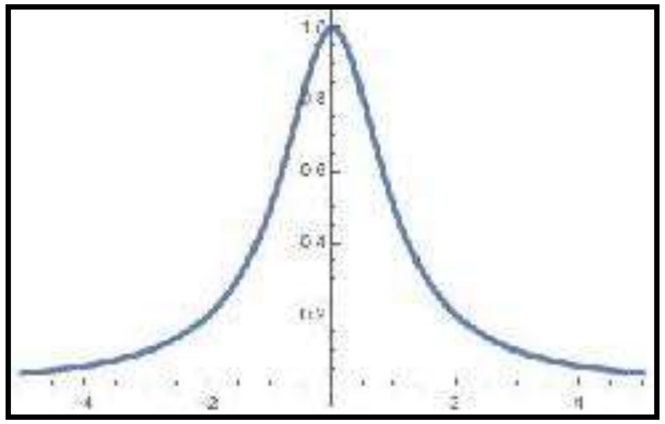
$$f_1(z,t) = Ae^{-b(z-vt)^2}$$



$$f_2(z,t) = A\sin\left[b(z-vt)\right]$$



$$f_3(z,t) = \frac{A}{b(z-vt)^2+1}$$



On the other hand, the following functions do not represent waves:

$$f_4(z,t) = Ae^{-b(bz^2+vt)}, \quad f_5(z,t) = A\sin(bz)\cos(bvt)^3$$

A, b are constants

A stretched string supports wave motion

A long string under tension T if displaced from equilibrium, has a net transverse force on the segment between z and $z + \Delta z$

$$\Delta F = T \sin\theta' - T \sin\theta$$

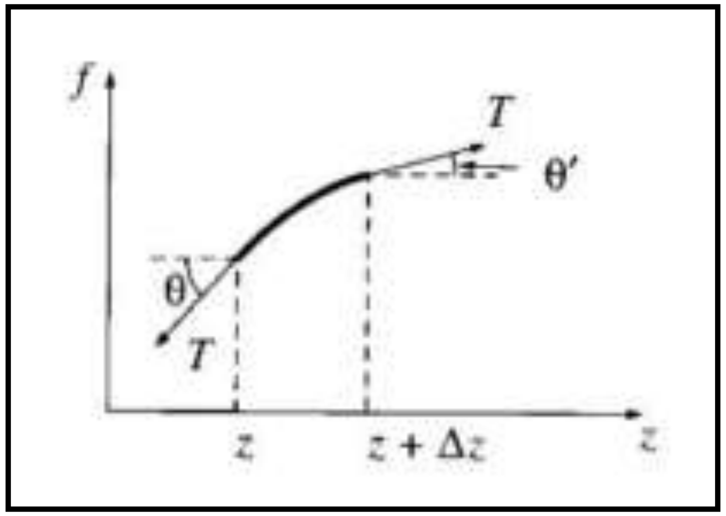


Figure 9.2, Introduction to Electrodynamics, D J Griffiths

For small displacements and hence small θ, θ'

$$\Delta F \approx T(\tan\theta' - \tan\theta)$$

$$= T\left(\left.\frac{\partial f}{\partial z}\right|_{z+\Delta z} - \left.\frac{\partial f}{\partial z}\right|_{z}\right) \approx T\frac{\partial^2 f}{\partial z^2}\Delta z$$

If the string has mass per unit length μ, we can use Newton's law:

$$\Delta F = \mu(\Delta z)\frac{\partial^2 f}{\partial t^2} \implies \frac{\partial^2 f}{\partial z^2} = \frac{\mu}{T}\frac{\partial^2 f}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

$$v = \sqrt{T/\mu}$$

Wave Equation

Speed of Propagation

Wave Equation

The equation $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$ is the classical wave equation.

It admits solutions of the form $f(z,t) = g(z - vt)$

Let $u \equiv z - vt$. We can write:

$$\frac{\partial f}{\partial z} = \frac{dg}{du}\frac{\partial u}{\partial z} = \frac{dg}{du}, \quad \frac{\partial f}{\partial t} = \frac{dg}{du}\frac{\partial u}{\partial t} = -v\frac{dg}{du}$$

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial}{\partial z}\left(\frac{dg}{du}\right) = \frac{d^2 g}{du^2}\frac{\partial u}{\partial z} = \frac{d^2 g}{du^2}$$

$$\frac{\partial^2 f}{\partial t^2} = -v\frac{\partial}{\partial t}\left(\frac{dg}{du}\right) = -v\frac{d^2 g}{du^2}\frac{\partial u}{\partial t} = v^2\frac{d^2 g}{du^2}$$

$$\implies \frac{d^2 g}{du^2} = \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2} \qquad (1)$$

Since f=g(u) satisfies the wave equation, and g(u) represents a wave moving with speed v in the z direction, hence equation (1) represents a wave moving with speed v

Wave Equation

Since the speed of propagation appears as squared in the equation, one can also have another class of solutions simply by changing the sign of velocity:

$$f(z,t) = h(z+vt) \qquad \text{A wave in -z direction}$$

The most general solution of the wave equation is:

$$f(z,t) = g(z-vt) + h(z+vt)$$

Since the wave equation is linear, any linear combination of two solutions is also a solution.

Sinusoidal Waves

$$f(z,t) = A\cos\left[k(z-vt)+\delta\right]$$
$$A \equiv \text{Amplitude}$$
$$[k(z-vt)+\delta] \equiv \text{Phase}$$
$$\delta \equiv \text{Phase constant}$$

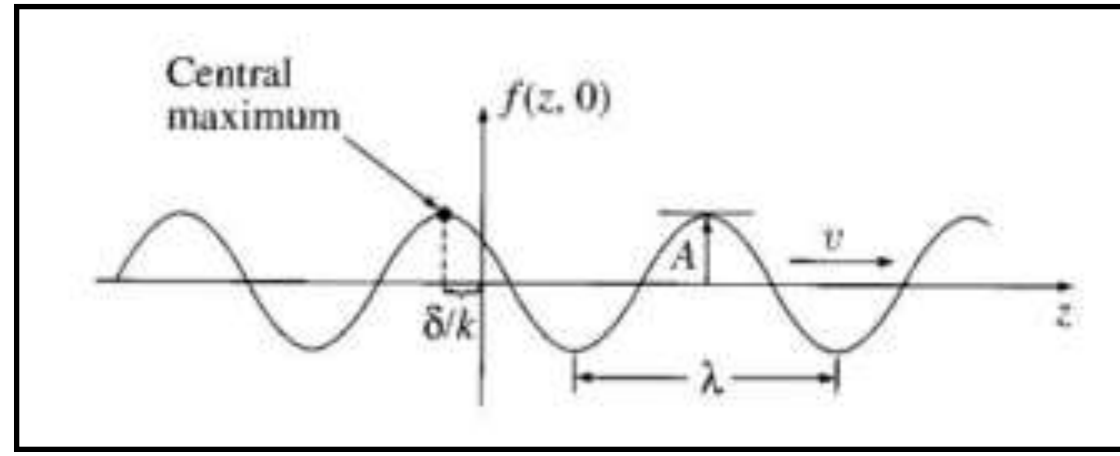


Figure 9.3, Introduction to Electrodynamics, D J Griffiths

Central Maximum: $z = vt - \delta/k$ at which the phase is zero.

If phase constant is zero, the central maximum passes the origin at time t=0. More generally, δ/k is the distance by which the central maximum (and hence the entire wave) is delayed.

k: **Wave number**, related to the wavelength as $\lambda = 2\pi/k$

When z advances by $2\pi/k$, the cosine completes one cycle.

Sinusoidal Waves

Time period: Time duration in which the vibrations undergo one full cycle: $T = 2\pi/(kv)$

Frequency: Number of oscillations per unit time:

$$\nu = \frac{1}{T} = \frac{kv}{2\pi} = \frac{v}{\lambda}$$

Angular frequency: Number of radians swept out per unit time: $\omega = 2\pi\nu = kv$

$f(z,t) = A\cos\left(kz - \omega t + \delta\right)$ Travelling towards right

$f(z,t) = A\cos\left(kz + \omega t - \delta\right)$ Travelling towards left

Sinusoidal Waves

Complex Notations:

Using Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, one can write the sinusoidal wave as

$$f(z,t) = \mathrm{Re}[Ae^{i(kz-\omega t+\delta)}]$$

Complex wave function: $\tilde{f}(z,t) = \tilde{A}e^{i(kz-\omega t)}$ with a constant amplitude absorbing the phase constant $\tilde{A} = Ae^{i\delta}$

The actual wave function is the real part of $\tilde{f}$

$$f(z,t) = \mathrm{Re}[\tilde{f}(z,t)]$$

Complex wave functions are easier for mathematical calculations compared to sines and cosines.

Any wave can be expressed as a linear combination of sinusoidal ones:

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz-\omega t)} dk$$

where ω is a function of k (*Dispersion Relation*). Here negative values of k imply waves moving in opposite directions while keeping wavelength and frequency always positive.

This general solution can be obtained directly from the wave equation, by separation of variables.

This is similar to the way we obtained general solutions for Laplace's equation while solving boundary value problems in electrostatics.

Example: Standing Wave

Problem 9.2 (Introduction to Electrodynamics, D J Griffiths): Show that the **standing wave** given by $f(z,t) = A\sin(kz)\cos(kvt)$ satisfies the wave equation, and express it as the sum of a wave travelling to the left and a wave travelling to the right.

Solution: $$\frac{\partial f}{\partial z} = Ak\cos(kz)\cos(kvt), \quad \frac{\partial^2 f}{\partial z^2} = -Ak^2\sin(kz)\cos(kvt)$$

$$\frac{\partial f}{\partial t} = -Akv\sin(kz)\sin(kvt), \quad \frac{\partial^2 f}{\partial t^2} = -Ak^2v^2\sin(kz)\cos(kvt)$$

$$\implies \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Using $\sin A\cos B = \dfrac{1}{2}[\sin(A+B) + \sin(A-B)]$ we can write:

$$f(z,t) = \frac{A}{2}[\sin k(z+vt) + \sin k(z-vt)]$$

Such waves do not propagate but simply oscillate in space and time.

Electromagnetic Waves in Vacuum

Maxwell's equations in the absence of any source are:

$$(i)\ \vec{\nabla}\cdot\vec{E}=0,\quad (iii)\ \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t},$$

$$(ii)\ \vec{\nabla}\cdot\vec{B}=0,\quad (iv)\ \vec{\nabla}\times\vec{B}=\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}.$$

These coupled, first order, partial differential equations (PDE) in **E**, **B** can be decoupled by applying curl to (iii), (iv). Taking curl of (iii) and using (iv):

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E})=\vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E}=\vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$=-\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B})=-\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

Electromagnetic Waves in Vacuum

Taking curl of (iv) and using (iii):

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = \vec{\nabla} \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right)$$

$$= \mu_0 \epsilon_0 \frac{\partial}{\partial t} (\vec{\nabla} \times \vec{E}) = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

Using (i) and (ii), we get the following equations:

$$\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}, \quad \nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}.$$

which are decoupled second order PDE's.

Electromagnetic Waves in Vacuum

Thus, in vacuum, each Cartesian component of **E**, **B** satisfies the three dimensional wave equation:

$$\nabla^2 f = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

The speed of propagation of the waves is:

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s}$$

which is precisely the speed of light (c) in vacuum.

Remarks

- Electromagnetic waves travel with the speed of light. Light could be electromagnetic wave!

- Fundamental constants appearing in Coulomb's law and Biot-Savart law (that have nothing to do with light) give us the speed of light c.

- Electromagnetic wave equations won't have emerged if the Maxwell's corrections to Ampere's law was not there.

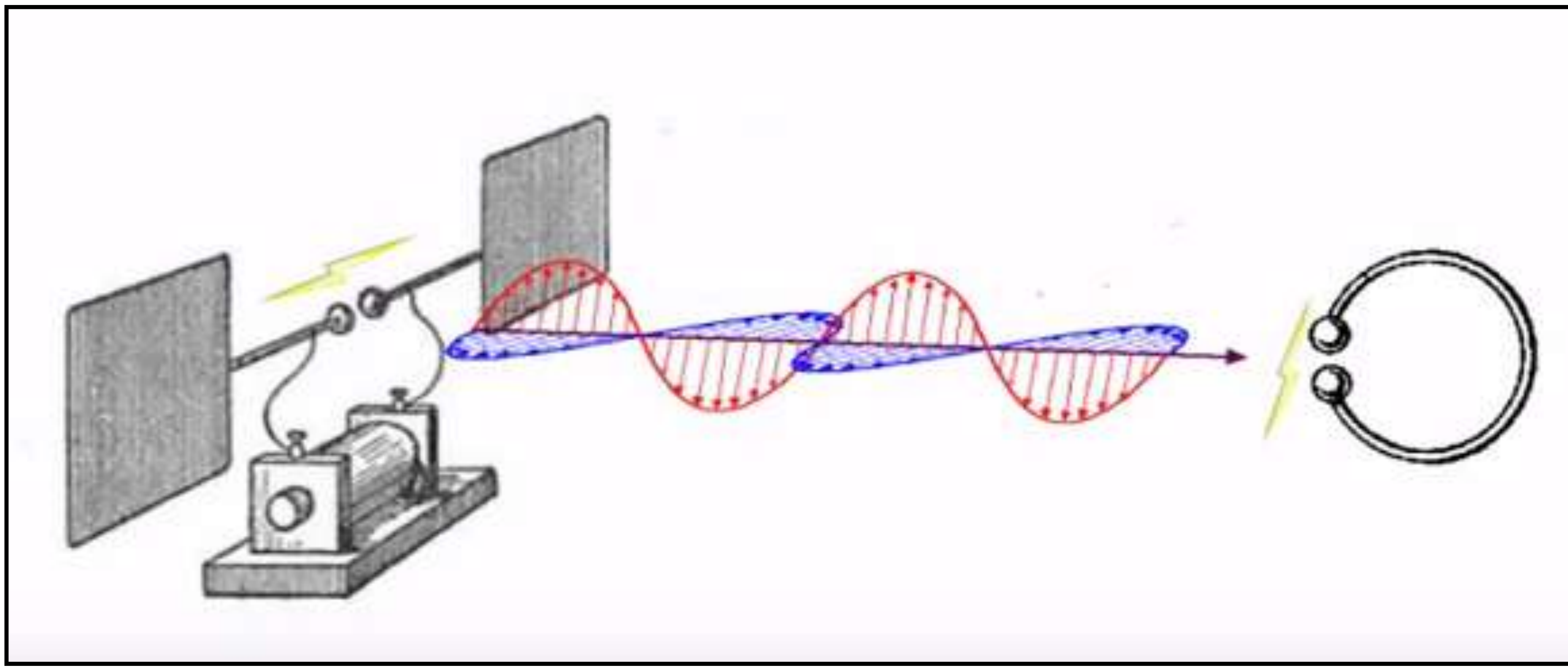
- Maxwell postulated light to be an EM wave, but he died in 1879 before experimental evidence for EM waves could be obtained.

- It was Heinrich Hertz who carried out a set of experiments during 1887-89 which lead to their discovery and also to the fact that EM waves have all the properties of light.

Hertz’s Experiments

- Hertz used an oscillator made of polished brass knobs, each connected to an induction coil and separated by a tiny gap over which sparks could leap.
- Hertz reasoned that, if Maxwell's predictions were correct, electromagnetic waves would be transmitted during each series of sparks.
- To confirm this, Hertz made a simple receiver of looped wire. At the ends of the loop were small knobs separated by a tiny gap. The receiver was placed several yards from the oscillator.
- According to theory, if electromagnetic waves were spreading from the oscillator sparks, they would induce a current in the loop that would send sparks across the gap. This occurred when Hertz turned on the oscillator, producing the first transmission and reception of electromagnetic waves.
- Hertz also noted that electrical conductors reflect the waves and that they can be focused by concave reflectors. He found that nonconductors allow most of the waves to pass through. (These properties are well known for visible light in optics).

Schematic of Hertz's Experiment



The Electromagnetic Spectrum

Frequency (Hz)	Type	Wavelength (m)
10^{22}		10^{-13}
10^{21}	gamma rays	10^{-12}
10^{20}		10^{-11}
10^{19}		10^{-10}
10^{18}	x rays	10^{-9}
10^{17}		10^{-8}
10^{16}	ultraviolet	10^{-7}
10^{15}	visible	10^{-6}
10^{14}	infrared	10^{-5}
10^{13}		10^{-4}
10^{12}		10^{-3}
10^{11}		10^{-2}
10^{10}	microwave	10^{-1}
10^{9}		1
10^{8}	TV, FM	10
10^{7}		10^{2}
10^{6}	AM	10^{3}
10^{5}		10^{4}
10^{4}	RF	10^{5}
10^{3}		10^{6}

The Visible Range

Frequency (Hz)	Color	Wavelength (m)
1.0×10^{15}	near ultraviolet	3.0×10^{-7}
7.5×10^{14}	shortest visible blue	4.0×10^{-7}
6.5×10^{14}	blue	4.6×10^{-7}
5.6×10^{14}	green	5.4×10^{-7}
5.1×10^{14}	yellow	5.9×10^{-7}
4.9×10^{14}	orange	6.1×10^{-7}
3.9×10^{14}	longest visible red	7.6×10^{-7}
3.0×10^{14}	near infrared	1.0×10^{-6}

Table 9.1, Introduction to Electrodynamics, D J Griffiths

Monochromatic Plane Waves

Monochromatic waves: Waves with frequency ω . Different frequencies correspond to different *colours* (at least in the visible spectrum).

Plane waves: Waves travelling in one direction, without any dependence on the other two directions.

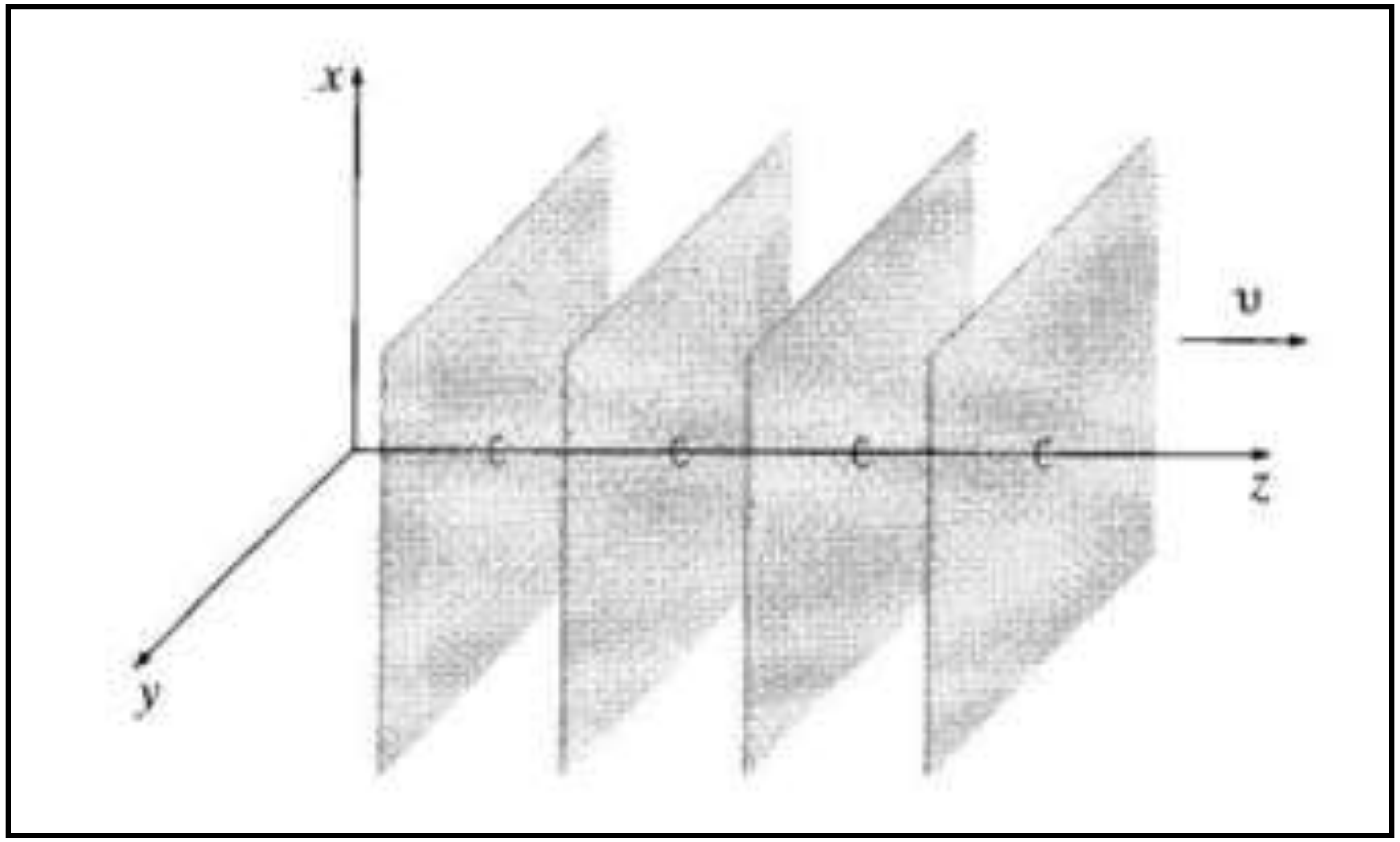
Such solutions to the wave equations can then be written as

$$\tilde{\vec{E}}(z,t) = \tilde{\vec{E}}_0 e^{i(kz-\omega t)}, \quad \tilde{\vec{B}}(z,t) = \tilde{\vec{B}}_0 e^{i(kz-\omega t)}.$$

while the physical fields are given by their real parts.

Such solutions can also be derived from the electromagnetic wave equations by using the method of separation of variables.

Monochromatic Plane Waves



Fields are uniform over every plane perpendicular to the direction of propagation

Figure 9.9, Introduction to Electrodynamics, D J Griffiths

Plane Electromagnetic Waves

Assuming the fields to depend upon only one space coordinate say z, the wave equation can be written as

$$\frac{\partial^2 \vec{E}}{\partial z^2} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2}$$

Using separation of variables and considering one Cartesian component of the field at a time: $E = Z(z)T(t)$

Using this in the wave equation: $\frac{1}{Z}\frac{d^2 Z}{dz^2} = \frac{1}{c^2}\frac{1}{T}\frac{d^2 T}{dt^2} = -k^2$

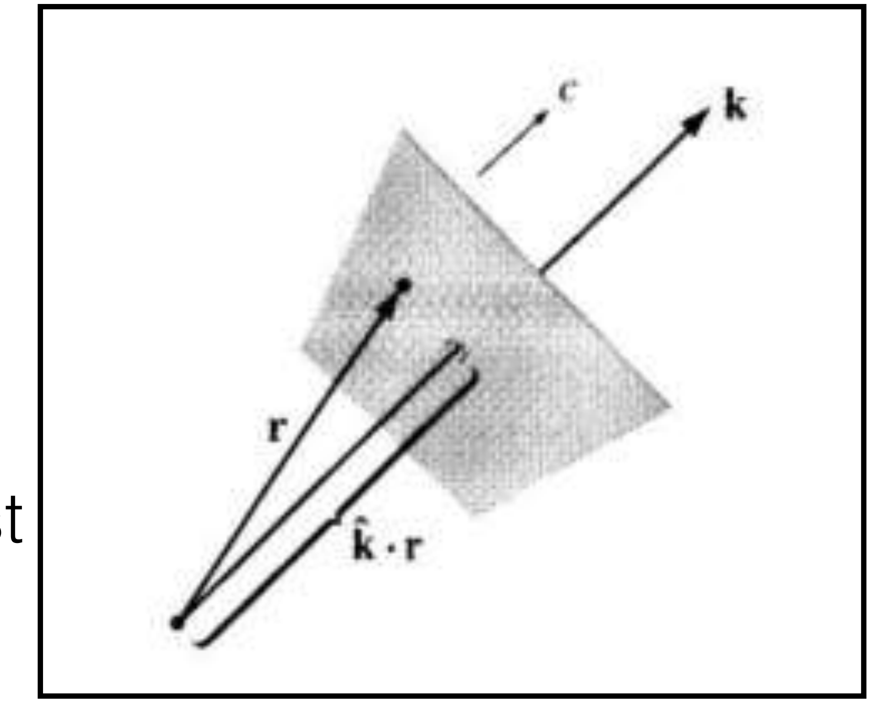
The solutions are:

$$Z(z) = Ae^{\pm ikz}, T(t) = Be^{\pm i\omega t}, \omega = kc$$

The resulting field: $\vec{E}(z,t) = \vec{E}_0 e^{\pm i(kz \pm \omega t)}, E_0 = AB$

In general, the plane waves travelling in any direction, identified as the direction of the wave vector $\vec{k}$ are given by

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \quad \tilde{\vec{B}}(\vec{r},t) = \tilde{\vec{B}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$$



Every solution to Maxwell's equations in vacuum must obey wave equation, **but the converse is not true**.

This is because the Maxwell's equations impose extra restrictions on the wave solutions. Taking divergence of the above solutions and using the first two Maxwell's equations:

$$\vec{k}\cdot\tilde{\vec{E}}_0 = 0, \quad \vec{k}\cdot\tilde{\vec{B}}_0 = 0$$

This means, there are no components of the fields along the direction of propagation.

Electromagnetic waves are transverse!

Figure 9.11, Introduction to Electrodynamics, D J Griffiths

Faraday's equation gives one more restriction:

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \implies \vec{k} \times \tilde{\vec{E}}_0 = \omega \tilde{\vec{B}}_0$$

This implies, $\implies \tilde{\vec{B}}(\vec{r}, t) = \frac{1}{c} \hat{k} \times \tilde{\vec{E}}(\vec{r}, t)$

Notice the smallness of B amplitude compared to that of E

Find the ratio of amplitudes of E and H for plane waves in free space. What is its unit?

E and B are in phase and mutually perpendicular to each other.

If **E** points in the x direction, then **B** points in y direction for a plane wave moving in z direction:

$$\tilde{\vec{E}}(z, t) = \tilde{E}_0 e^{i(kz - \omega t)} \hat{x}, \tilde{\vec{B}}(z, t) = \frac{1}{c} \tilde{E}_0 e^{i(kz - \omega t)} \hat{y}$$

Taking real parts:

$$\vec{E}(z, t) = E_0 \cos(kz - \omega t + \delta) \hat{x}$$

$$\vec{B}(z, t) = \frac{1}{c} E_0 \cos(kz - \omega t + \delta) \hat{y}$$

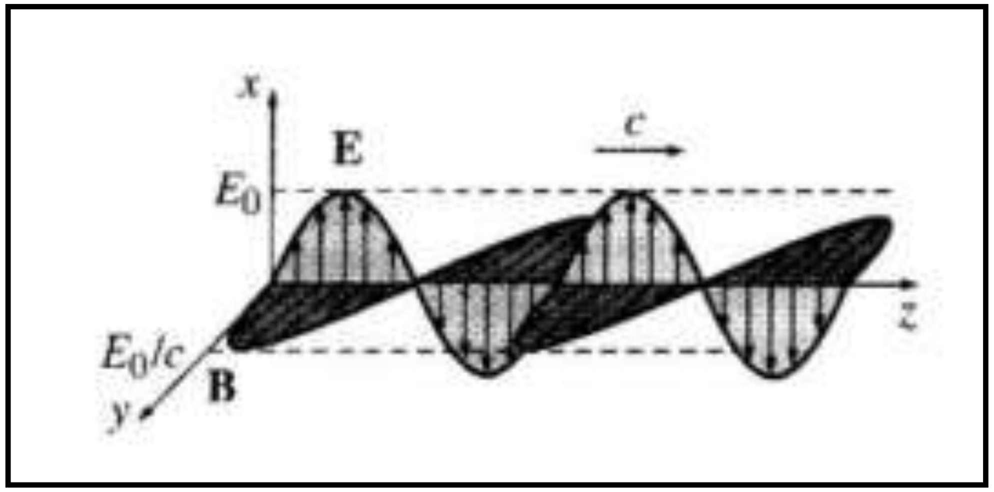


Figure 9.10, Introduction to Electrodynamics, D J Griffiths

Polarisation

For a plane electromagnetic wave moving in a direction given by $\vec{k}$, the plane perpendicular to the direction of propagation can be characterised by two unit vectors $\hat{e}_1, \hat{e}_2$

A general solution for the electric field is

$$\vec{E}(\vec{r},t) = \mathrm{Re}[(\tilde{E}_{01}\hat{e}_1 + \tilde{E}_{02}\hat{e}_2)e^{i(\vec{k}\cdot\vec{r}-\omega t)}]$$

One can write

$$\tilde{E}_{01} = E_0\cos\theta e^{i\delta_1}, \quad \tilde{E}_{02} = E_0\sin\theta e^{i\delta_2}, \quad E_0^2 = |\tilde{E}_{01}|^2 + |\tilde{E}_{02}|^2$$

Therefore,

$$\vec{E}(\vec{r},t) = E_0[\cos\theta\cos(\vec{k}\cdot\vec{r}-\omega t+\delta_1)\hat{e}_1 + \sin\theta\cos(\vec{k}\cdot\vec{r}-\omega t+\delta_2)\hat{e}_2]$$

Polarisation

Let us denote $\vec{k} \cdot \vec{r} - \omega t \equiv \alpha$

If the phases are equal i.e., $\delta_1 = \delta_2 = \delta$ the field can be written as $\vec{E}(\vec{r}, t) = E_0(\cos\theta \hat{e}_1 + \sin\theta \hat{e}_2) \cos(\alpha + \delta)$

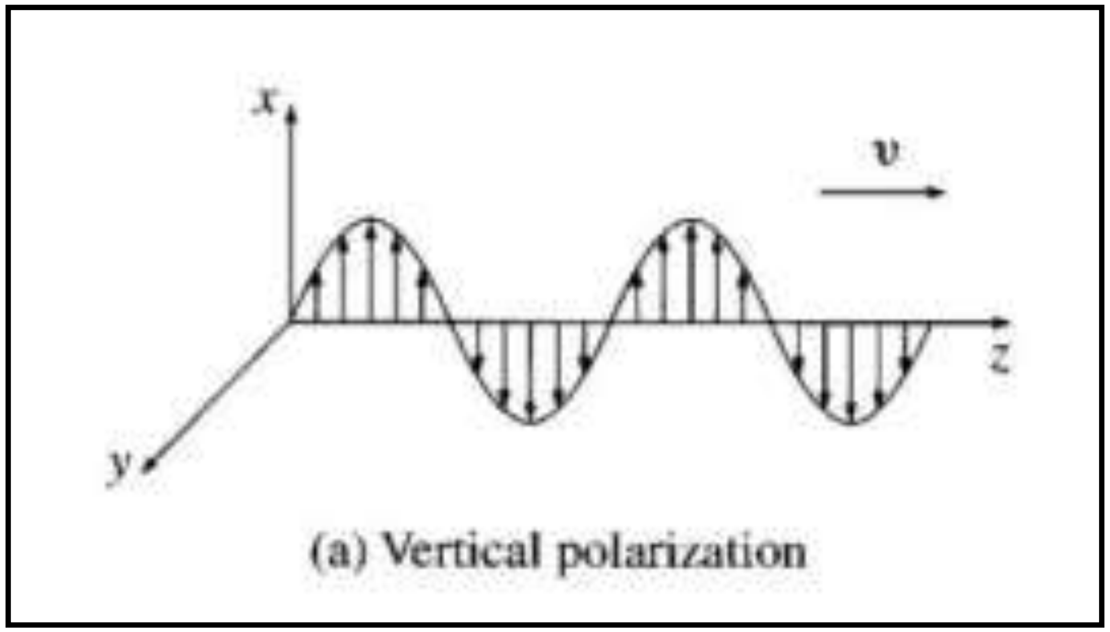
The direction of **E** (and **B**) remain fixed in this case. Such waves are called **linearly polarised**.

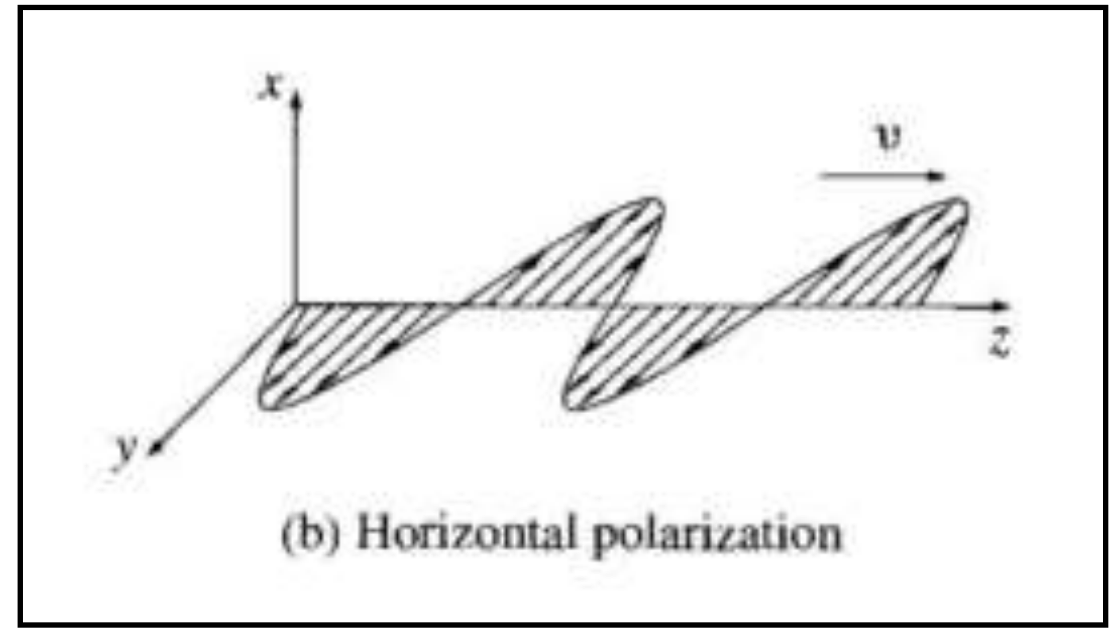
Polarisation vector: $\hat{n} = (\cos\theta \hat{e}_1 + \sin\theta \hat{e}_2)$

In this case, the wave can be considered as a superposition of two waves, one horizontally placed, and other vertically.

$$\vec{E}(\vec{r}, t) = (E_0 \cos\theta) \cos(\alpha + \delta)\hat{e}_1 + (E_0 \sin\theta) \cos(\alpha + \delta)\hat{e}_2$$

Polarisation





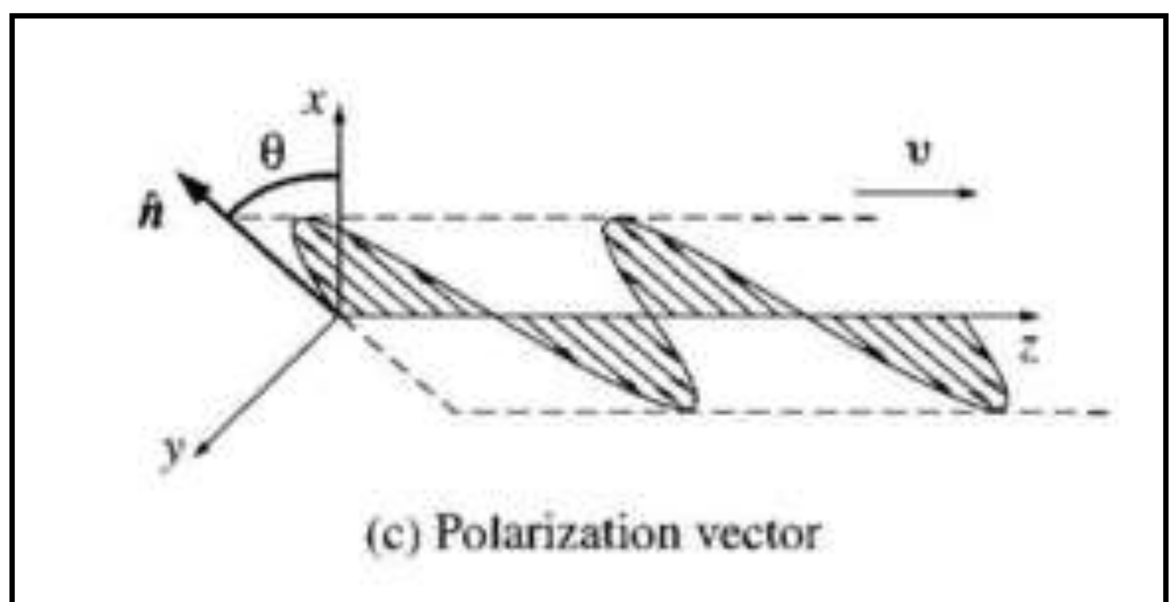
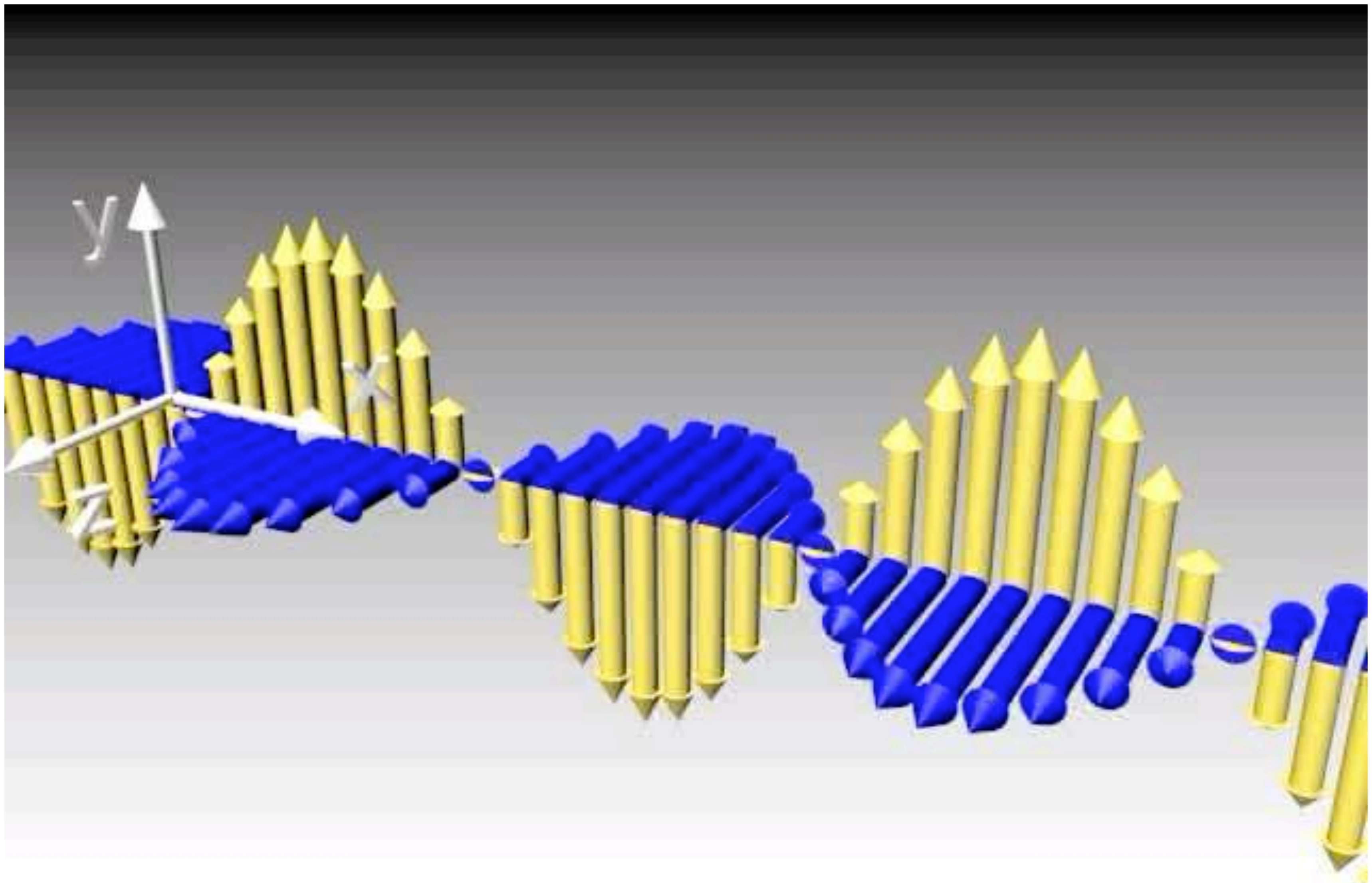
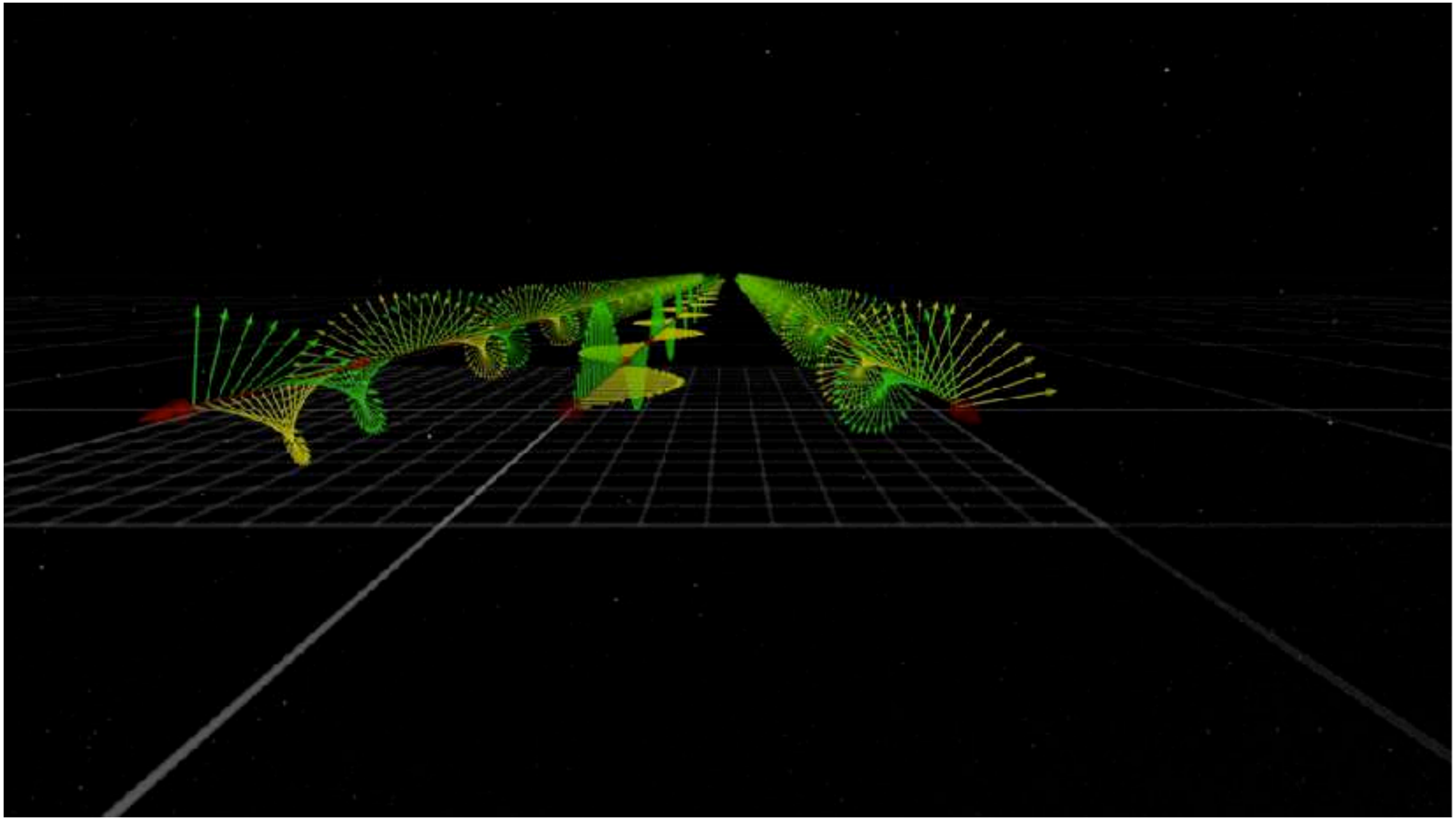


Figure 9.8, Introduction to Electrodynamics, D J Griffiths



Credit: MIT

Polarisation



Electromagnetic waves of different polarizations: Right circular polarization (upper/right); Linear polarization (middle); and Left circular polarization (lower/left). Yellow arrows are the electric field, green arrows are the magnetic field.

Credit: NASA

EM Waves: Summary

- The wave is transverse; both **E** and **B** fields are perpendicular to the direction of propagation which points in the direction of $\vec{E} \times \vec{B}$

- The **E** and **B** fields are perpendicular to each other, hence their dot product vanishes $\vec{E} \cdot \vec{B} = 0$

- The speed of propagation in vacuum is equal to the speed of light c.

- The ratio of the magnitudes (amplitudes) of the fields is 1/c where c is the speed of propagation in a vacuum.

- Electromagnetic waves obey superposition principle, as the wave equations are linear.