PH 101: Physics I

Module 2: Special Theory of Relativity-Basics

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Review of Important formula in STR

$$x' = \frac{1}{\sqrt{1 - v^2/c^2}}(x - vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \frac{1}{\sqrt{1 - v^2/c^2}}(t - vx/c^2)$$

$$p_x = \frac{1}{\sqrt{1 - v^2/c^2}}\left(p'_x + \frac{E'v}{c^2}\right)$$

$$p_y = p'_y$$

$$p_z = p'_z$$

$$E = \frac{1}{\sqrt{1 - v^2/c^2}}(E' + vp'_x)$$

$$p \equiv \left(p_0 = \frac{E}{c},\ p_x,\ p_y,\ p_z\right)$$

$$u_x' = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} \qquad u_y' = \frac{u_y}{\gamma_v(1 - \frac{u_x v}{c^2})} \qquad u_z' = \frac{u_z}{\gamma_v(1 - \frac{u_x v}{c^2})}$$

Review of Important formula in STR

$$p^2c^2 = \Gamma_u^2 m_0^2 c^4\left(1 - \frac{1}{\Gamma_u^2}\right) = E^2 - m_0^2c^4$$

Relativistic Momentum $\vec{p} = \Gamma_u m_0 \vec{u}$

Total energy of an object of mass m moving with speed u: $E = \Gamma_u m_0 c^2$

Kinetic energy of the object: $K.E. = E - m_0c^2 = (\Gamma_u - 1)m_0c^2$

Practice Problem Set

As seen from Earth, two spaceships A and B are approaching along perpendicular directions. If A is observed by a stationary Earth observer to have velocity $u_y = -0.90c$ and B to have velocity $u_x = 0.90c$, determine the speed of ship A as measured by the pilot of ship B.**Answer:** $u'_x = -0.9c$**,** $u'_y = -0.39c$**,** $u' = 0.98c$

Solution:

We take the S frame to be attached to the Earth and the S' frame to be attached to spaceship B moving with $\beta = 0.90$ along the x-axis. Spaceship A has velocity components $u_x = 0, u_y = 0.90c$ in S.

$u'_y = \dfrac{u_y}{\gamma(1 - u_x v/c^2)}$ give the velocity components of spaceship A in S', from which we have $u'_x = -v = -0.90c, u'_y = u_y/\gamma = -0.39c$ So we have,

$$u' = \sqrt{(u'_x)^2 + (u'_y)^2} = 0.98c$$

Example 1

A free electron is moving with velocity $\vec{u} = \frac{1}{2\sqrt{2}} c\left(\hat{x} + \hat{y}\right)$ as seen by an observer on earth (frame: S). What is its momentum, total energy and kinetic energy? Mass of electron may be taken to be $m_e = 10^{-30}\ kg$

(a) As seen by the observer in S.
(b) As seen by an observer in S', which is moving with velocity $\vec{v} = 0.2c\ \hat{x}$

Example 1

Solution: $\vec{u} = \frac{1}{2\sqrt{2}} c(\hat{x} + \hat{y})$ $\quad |\vec{u}| = 0.5c$ $\quad \gamma_u = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.5^2}} = 1.1547$

$$\vec{p} = \gamma_u m\vec{u} = 1.1547 \times 10^{-30} \times \frac{1}{2\sqrt{2}} c(\hat{x} + \hat{y}) = 1.2247 \times 10^{-22} (\hat{x} + \hat{y}) \ kg\,m/s$$

$$E = \gamma_u mc^2 = 1.1547 \times 10^{-30} \times 9 \times 10^{16} = 1.0392 \times 10^{-13}\ J, \qquad K.E. = (\gamma_u - 1)mc^2 = 0.1392 \times 10^{-13}\ J$$

As seen by S': $\gamma_v = \left(1 - 0.2^2\right)^{-\frac{1}{2}} = 1.0206$

$$E' = \gamma_v (E - \beta c p_x) = 1.0206\left(1.0392 \times 10^{-13} - 0.2 \times 3 \times 10^8 \times 1.2247 \times 10^{-22}\right) J = 0.98566 \times 10^{-13}\ J$$

$$p_x' = \gamma_v \left(p_x - \beta \frac{E}{c}\right) = 1.0206\left(1.2247 \times 10^{-22} - 0.2 \times \frac{1.0392 \times 10^{-13}}{3 \times 10^8}\right) = 0.5429 \times 10^{-22} \quad kgm/s$$

$$p_y' = p_y = 1.2247 \times 10^{-22} \quad kgm/s, \qquad p'_z = p_z = 0$$

$$K.E. = E' - mc^2 = 0.98566 \times 10^{-13} - 10^{-30} \times 9 \times 10^{-16} = 0.08566 \times 10^{-13}\ J$$

Example 1

Alternatively (in frame S');

$$u'_x = \frac{u_x - v}{1 - \frac{u_x v}{c^2}} = \frac{(0.3536 - 0.2)c}{1 - 0.3536 \times 0.2} = 0.1652c, \quad u'_y = \frac{u_y}{\gamma_v\left(1 - \frac{u_x v}{c^2}\right)} = \frac{0.3536c}{1.0206 \times 0.9293} = 0.3728c, \quad u'_z = 0$$

$$|\vec{u}'| = \sqrt{0.1652^2 + 0.3728^2}\, c = 0.40775c, \qquad \gamma_{u'} = \frac{1}{\sqrt{1 - \frac{u'^2}{c^2}}} = \frac{1}{\sqrt{1 - 0.40775^2}} = 1.095178$$

$$E' = \gamma_{u'} mc^2 = 1.095178 \times 10^{-30} \times 9 \times 10^{16} = 0.98566 \times 10^{-13} \ J,$$

$$p'_x = \gamma_{u'} mu'_x = 1.095178 \times 10^{-30} \times 0.1652 \times 3 \times 10^8 = 0.5429 \times 10^{-22} \quad kgm/s,$$

$$p'_y = \gamma_{u'} mu'_y = 1.095178 \times 10^{-30} \times 0.3728 \times 3 \times 10^8 = 1.2247 \times 10^{-22} \quad kgm/s,$$

Example 2

An object moving with velocity $\vec{u} = 0.2c\,\hat{x}$ has momentum $|\vec{p}| = 3\times10^{-20}\ kg\,m/s$

What is the total energy of the object?

What is the mass of the object?

Solution:

$$|\vec{p}| = p_x = \gamma_u m u_x = \gamma_u m\ 0.2c = 3\times10^{-20} \quad kg\,m/s,$$

$$E = \gamma_u mc^2 = \frac{\vec{p}\cdot\vec{u}}{|\vec{u}|^2}c^2 = \frac{3\times10^{-20}}{0.2}c = 0.45\times10^{-10} \quad J,$$

$$\gamma_u = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-0.2^2}} = 1.0206$$

$$E = \gamma_u mc^2 \quad \Rightarrow \quad m = \frac{E}{\gamma_u c^2} = \frac{0.45\times10^{-10}}{1.0206\times9\times10^{16}} = 0.4899\times10^{-27}\ kg$$

Example 3

A body of rest mass m_0 moving at speed u in the x-direction collides with and sticks to an identical body at rest. What would be the mass and momentum of the final clump?

Solution:

The energy-momentum vector of the moving mass is $\mathbf{P_1} = (\Gamma_u m_0 c \equiv E_1/c,\ \gamma m_0 u,\ 0,\ 0)$, while for the mass at rest it is $\mathbf{P_2} = (m_0 c,\ 0,\ ,0\ ,0)$, where, $\Gamma_u = \dfrac{1}{\sqrt{1-u^2/c^2}}$.

Therefore, the total energy-momentum vector of the final clump is $\mathbf{P} = \mathbf{P_1} + \mathbf{P_2}$.
Therefore, the momentum of the clump is $\gamma m_0 v$.

For the mass we can write ;

$$P^2 = M^2c^2 = (P_1 + P_2)^2 = P_1^2 + 2P_1.P_2 + P_2^2 = m_0^2c^2 + 2\gamma m_0^2 c^2 + m_0^2 c^2 = 2m_0^2c^2(1+\gamma),$$

$$\Rightarrow M = m_0\sqrt{2(1+\gamma)}.$$

Tutorial-7 (Q3)

A spaceship moves away from Earth with speed v and fires a shuttle craft in the forward direction at a speed v relative to the spaceship. The pilot of the shuttle craft launches a probe in the forward direction at speed v relative to the shuttle craft.

(i)Determine the speed of the shuttle craft relative to the Earth.
(ii) Determine the speed of the probe relative to the Earth.

Tutorial-7 (Q3)

Solution:

We consider the S frame to be attached to the Earth and the S' frame to be attached to the spaceship moving with v along the $x-$axis. The shuttle craft has speed $u'_x = v$. Therefore,

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}. \tag{1}$$

Gives the speed $u_x = \frac{2v}{1+\beta^2}$, where $\beta = v/c$.

Now consider S' frame is attached to the shuttle craft moving with speed,

$v' = \frac{2v}{1+\beta^2}$ with respect to the S-frame (Earth)

along the $x-$ axis. The probe has a speed $u'_x = v$ in S'. Its speed u_x in the S frame would be given by Eq. (1) with v replaced by v' from which we get

$$u_x = \left(\frac{3+\beta^2}{1+3\beta^2} v\right) \tag{2}$$

It follows from Eq.(2) that $u_x \to 3v$ when $\beta \ll 1$ and $u_x \to c$ when $\beta \to 1$

Tutorial-6 (Q5)

An observer in frame S who lives on the x-axis sees a flash of red light at $x = 1210m$. After $4.96\mu s$, he sees flash of blue light at $x = 480m$. Use subscripts R and B to label the coordinates of the events related to the red and blue light respectively.
(i) Now suppose there is an observer in S' which is moving with a velocity 'v' with respect to the S frame watches these events. Compute the velocity v for the situation when the observer in S' records both the events occurring at the same place?
(ii) Which event occurs first according to S' and what is the measured time interval between these flashes?

Tutorial-6 (Q5)

Solution:

(i)As the observer from S' finds $x'_R = x'_B$, where x'_R and x'_B are the x-coordinate of the R and B events, respectively.

Therefore we have, $x_R - vt_R = x_B - vt_B$

$$\Rightarrow v = \frac{x_R - x_B}{t_R - t_B} = \frac{1210m - 480m}{0 - 4.96 \times 10^{-6}s} = -1.47 \times 10^8 m/s$$

(ii) Note that the order of the events must be the same in every frame for light-like event, i.e. $\Delta x = c\Delta t$. This can be explicitly shown as follows.

$$(t'_R - t'_B) = \gamma\left[(t_R - t_B) - \frac{v}{c^2}(x_R - x_B)\right]$$

$$= \gamma\left[(t_R - t_B) - \frac{v}{c^2}c(t_R - t_B)\right] = \gamma(t_R - t_B)\left[1 - \frac{v}{c}\right]$$

Since $v \leq c$, $t'_R - t'_B$ and $t_R - t_B$ will have same sign.

Therefore the observer in S' sees event R happen before event B. To find the time interval in S' , we can use the invariant spacetime interval and use the fact that the events occur at the same place in S' :

$$(c\Delta t')^2 - (\Delta x')^2 = (c\Delta t')^2 = (c\Delta t)^2 - (\Delta x)^2$$

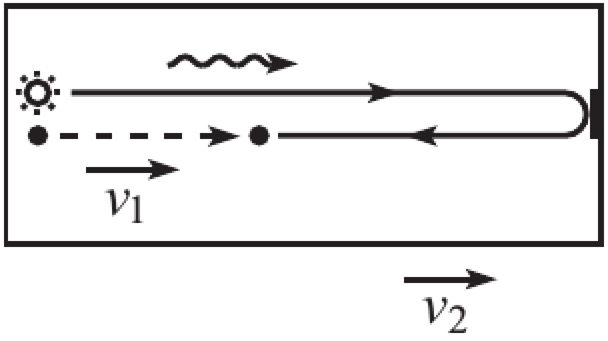
Therefore,

$$\Delta t' = \sqrt{(\Delta t)^2 - \left(\frac{\Delta x}{c}\right)^2} = \sqrt{(4.96 \times 10^6 s)^2 - \left(\frac{1210\ m\ -\ 480\ m}{3 \times 10^8\ m/s}\right)^2} = 4.32\mu s.$$

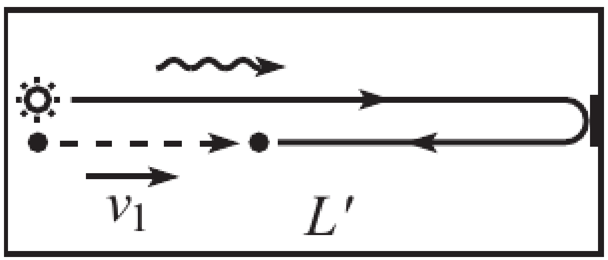
Example 2

A ball moves at speed v_1 with respect to a train. The train moves at speed v_2 with respect to the ground. What is the speed of the ball with respect to the ground?

Let the ball be thrown from the back of the train. At the same instant, a photon is released next to it (as shown in the figure). The photon heads to the front of the train, bounces off a mirror, heads back, and eventually runs into the ball. In both the frame of the train and the frame of the ground, calculate the fraction of the way along the train where the meeting occurs, and then equate these fractions.

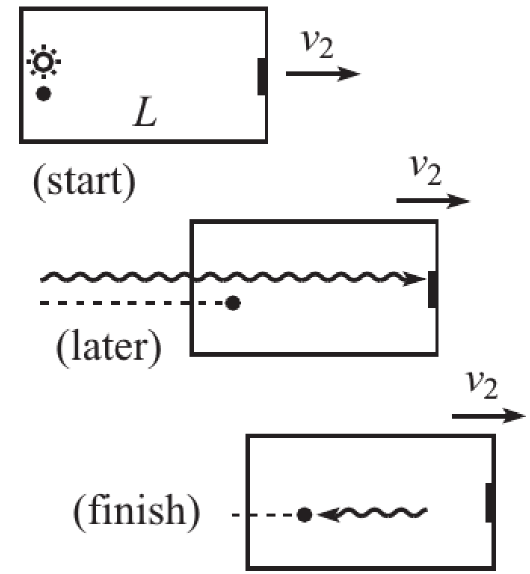


Example 2



(frame S')

(frame S)

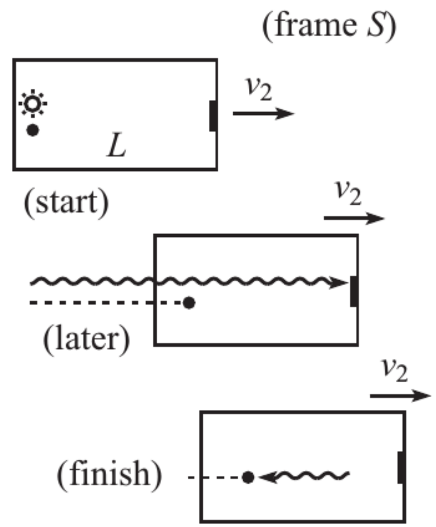


TRAIN FRAME: Let the train have length L' in the train frame. Let's first find the time at which the photon meets the ball (see Figure ’). From the figure, we see that the sum of the distances traveled by the ball and the photon, which is $v_1 t' + ct'$, must equal twice the length of the train, which is $2L'$. The time of the meeting is therefore $t' = \frac{2L'}{c + v_1}$.

The distance the ball has traveled is then $v_1 t' = 2v_1 L'/(c + v_1)$, and the desired fraction F' is

$$\boxed{F' = \frac{2v_1}{c + v_1}}$$

Example 2



Ground frame: Let the speed of the ball with respect to the ground be v, and let the train have length L in the ground frame (L equals L'/γ, but we're not going to use this). Again, let's first find the time at which the photon meets the ball Light takes a time $L/(c-v_2)$ to reach the mirror, because the mirror is receding at speed v_2. At this time, the light has traveled a distance $cL/(c-v_2)$. From the figure, we see that we can use the same reasoning as in the train-frame case, but now with the sum of the distances traveled by the ball and the photon, which is $vt+ct$, equalling $2cL/(c-v_2)$. The time of the meeting is therefore

$$\boxed{t = \frac{2cL}{(c-v_2)(c+v)}\,.}$$

The relative speed of the ball and the back of the train (as viewed in the ground frame) is $v-v_2$, so the distance between the ball and the back of the train at this time is $2(v-v_2)cL/[(c-v_2)(c+v)]$. The desired fraction F is therefore

$$\boxed{F = \frac{2(v-v_2)c}{(c-v_2)(c+v)}}$$

We shall continue in the next class