Physics II (PH 102)

Electromagnetism (Lecture 8 & 9)

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Feb 2020

Electrostatic Potential Energy of a Point Test Charge in Electric Field

Consider a stationary configuration of source charge distribution.

- Let $\mathbf{E}(\mathbf{r})$ be a pre-existing electric field with potential $V(\mathbf{r})$ at a point P.
- The electrostatic force $\mathbf{F}_{field}$ on a positive test charge Q is

$$\mathbf{F}_{field} = Q\mathbf{E} = -\mathbf{F}_{ext}.$$

Definition

Electrostatic Potential Energy of a test charge Q at point P($\mathbf{r}$) *is equal to the amount of work done by an external agent against the electrostatic field to bring the charge Q from ∞ (or* **ref***) to the point $P(\mathbf{r})$:*

$$U_E(\mathbf{r}) = \int_{\infty}^{\mathbf{r}} \mathbf{F}_{ext} \cdot d\mathbf{r}' = -\int_{\infty}^{\mathbf{r}} \mathbf{F}_{field} \cdot d\mathbf{r}' = -Q\int_{\infty}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{r}' = QV(\mathbf{r}).$$

Note: The ambiguity in the absolute value of U_E at a point like V!

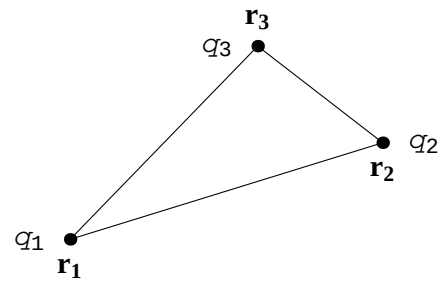
- *Electrostatic Energy difference* between two points **a** and **b** can be unambiguously expressed in terms of the *Potential difference*:

$$U_E(\mathbf{b}) - U_E(\mathbf{a}) = Q[V(\mathbf{b}) - V(\mathbf{a})] = -Q\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}'.$$

Potential Energy due to a system of Point Charges in free Field

Consider bringing in source charges $q_1, q_2, q_3, \cdots, q_n$, one by one from ∞.

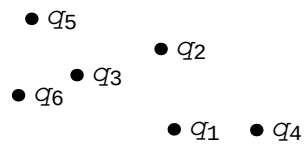
- No work done in placing first charge q_1 at $\mathbf{r}_1$, i.e., $W_1 = 0$
- Total Work done is placing up to the second charge q_2 at $\mathbf{r}_2$:



$$W_2 = W_1 + \delta w_2 = 0 + \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{|\mathbf{r}_1 - \mathbf{r}_2|} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}}$$

- Total Work done in placing up to the third charge q_3 at $\mathbf{r}_3$:

$$W_3 = W_2 + \delta w_3 = W_2 + \frac{q_3}{4\pi\epsilon_0}\left(\frac{q_1}{r_{13}} + \frac{q_2}{r_{23}}\right)$$



- Generalize formula up to n charges:
  ($j \neq i$ Self interactions excluded!)

$$W_n = W_{n-1} + \delta w_n = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{n-1}\sum_{j=2,\, j>i}^{n}\left(\frac{q_i q_j}{r_{ij}}\right) \rightarrow \text{No Double Counting}$$

$$\equiv \frac{1}{8\pi\epsilon_0}\sum_{i=1}^{n}\sum_{j=1,\, j\neq i}^{n}\left(\frac{q_i q_j}{r_{ij}}\right) \rightarrow \text{With Double Counting}$$

Potential Energy of a system of Point Charges in free Field (contd.)

- The Electrostatic Potential Energy is equal to the total work done W_n to assemble the configuration of n point charges at $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$:

$$\begin{aligned} U_E(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n) \equiv W_n &= \frac{1}{8\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \left(\frac{q_i q_j}{r_{ij}} \right) \\ &= \frac{1}{2} \sum_{i=1}^{n} q_i \left(\sum_{j=1, j\neq i}^{n} \frac{1}{4\pi\epsilon_0} \frac{q_j}{r_{ij}} \right) \\ &= \frac{1}{2} \sum_{i=1}^{n} q_i V(\mathbf{r}_i) \equiv \frac{1}{2} \sum_{i=1}^{n} q_i V_i = \frac{1}{2} \sum_{i=1}^{n} u_{Ei} \end{aligned}$$

- $V_i \equiv V(\mathbf{r}_i)$: Potential at the i^{th} charge site $\mathbf{r}_i$ due to other $n-1$ charges.
- The result is independent of the order/sequence in which the charges are assembled at the respective EXACT locations, $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$.

  $$U_E(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n) = U_E(\mathbf{r}_2, \mathbf{r}_n, \cdots, \mathbf{r}_1) = \cdots = U_E\left[\text{Permute}(\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n)\right]$$

- The result OBVIOUSLY, depends on the respective locations of the charges. Thus, U_E is called the CONFIGURATION ENERGY.

Configuration Energy of Point Charges in free space

- Superposition Principle is invalid (1/2 factor avoids **double counting**!):

$$U_E = \frac{1}{2}\sum_{i=1}^{n} u_{Ei}$$

Recall:

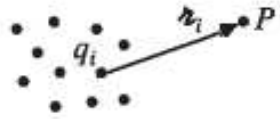
- While defining U_E, we said "*No work done in placing first charge* q_1 at $\mathbf{r}_1$."
- All terms $i = j$ were absent in U_E.

- SELF-ENERGIES of the individual point charges were excluded!

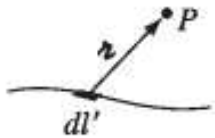
Definition

SELF-ENERGY: This is the amount of energy needed to fabricate or build-up the individual point charges by bringing their respective differential amounts of constituent charges from ∞ *to the specific locations.*

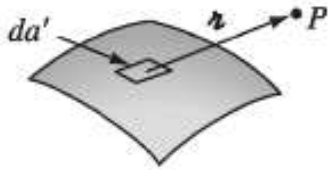
Electrostatic Potential Energy of General Distribution of Source Charges



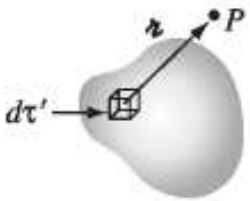
(a) Discrete charges



(b) Line charge, λ



(c) Surface charge, σ



(d) Volume charge, ρ

Localized charge distributions:

- Volume $\mathcal{V}$ with volume charge density $\rho(\mathbf{r})$
- Surface S with surface charge density $\sigma(\mathbf{r})$
- Curve Γ with linear charge density $\lambda(\mathbf{r})$
- Discrete point charges q_i at $\mathbf{r}_i'$
- Potential $V(\mathbf{r})$
- **Configuration Energy** of system of charges:

$$U_E \approx \frac{1}{2}\iiint_{\mathcal{V}} \rho(\mathbf{r}')V(\mathbf{r}')\,d\tau' + \frac{1}{2}\iint_{S} \sigma(\mathbf{r}')V(\mathbf{r}')\,da' + \frac{1}{2}\int_{\Gamma} \lambda(\mathbf{r}')V(\mathbf{r}')\,dl' + \frac{1}{2}\sum_{i=1}^{n} q_i V(\mathbf{r}_i')$$

- Note: Self-energies of continuous distributions are included but not for discrete point charges!
- What else is missing here?
  INTERACTION ENERGIES
- Superposition Principle is invalid in general!

True Configuration Energy of General Localized Charge Distribution

- Consider the most general localized charge distribution $\rho_{\rm tot}$ in a region $\mathcal{V}$, bounded by a closed surface S :

$$U_E = \frac{1}{2} \iiint_{\mathcal{V}} \rho_{\rm tot}(\mathbf{r}')\, V(\mathbf{r}')\, d\tau'.$$

- Using Gauss's differential law at source: $\rho_{\rm tot}(\mathbf{r}') = \epsilon_0 \nabla' \cdot \mathbf{E}(\mathbf{r}')$,

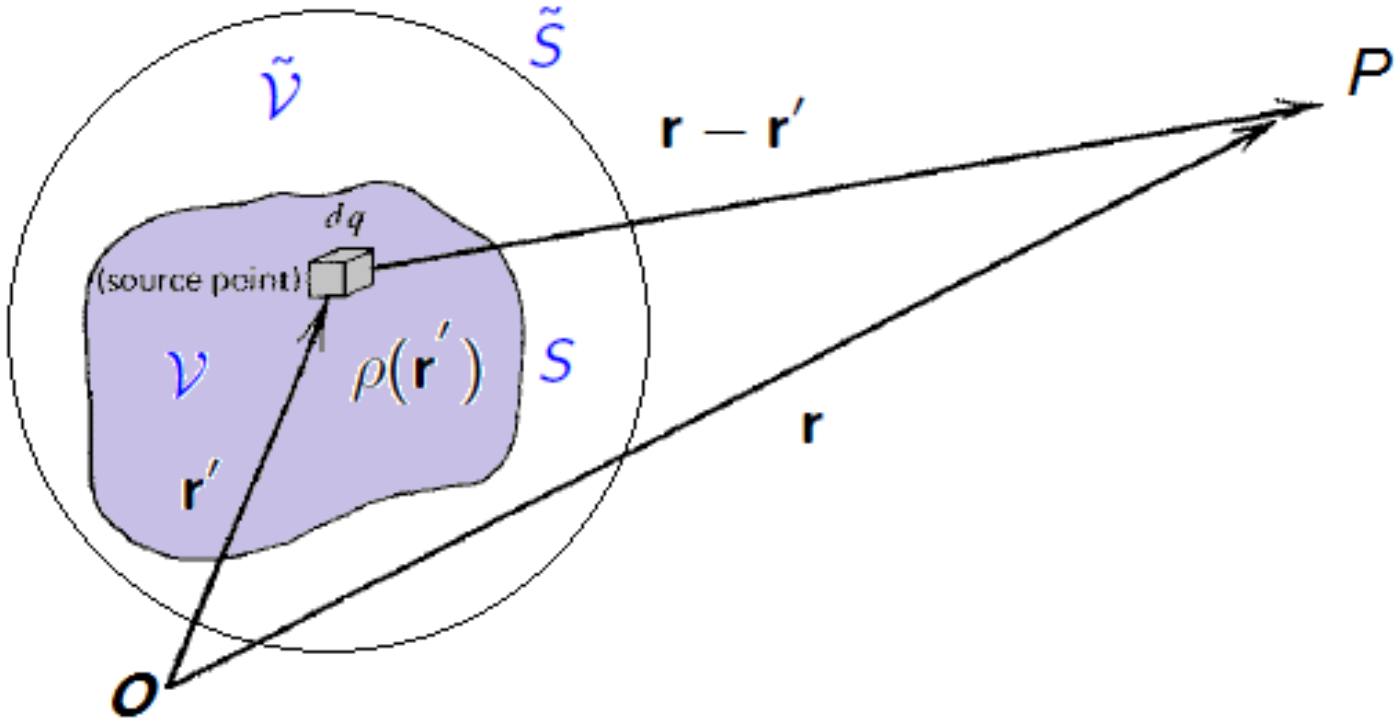
$$\begin{aligned} U_E &= \frac{1}{2} \iiint_{\mathcal{V}} \rho(\mathbf{r}')\, V(\mathbf{r}')\, d\tau' = \frac{\epsilon_0}{2} \iiint_{\mathcal{V}} \left[\nabla' \cdot \mathbf{E}(\mathbf{r}')\right] V(\mathbf{r}')\, d\tau' \\ &= \frac{\epsilon_0}{2} \iiint_{\mathcal{V}} \left[\left|\mathbf{E}(\mathbf{r}')\right|^2 + \nabla' \cdot \left[V(\mathbf{r}')\, \mathbf{E}(\mathbf{r}')\right] \right]_{\rm source} d\tau' \\ &= \frac{\epsilon_0}{2} \iiint_{\mathcal{V}} \left|\mathbf{E}(\mathbf{r}')\right|^2_{\rm source} d\tau' + \frac{\epsilon_0}{2} \oiint_{S} \left[V(\mathbf{r}')\, \mathbf{E}(\mathbf{r}')\right]_{\rm boundary} \cdot d\mathbf{a}'. \end{aligned}$$

- In the last step, we applied the Gauss' Divergence Theorem to obtain the surface integral over the bounding surface S.

Recall Identity: $\nabla \cdot (V\,\mathbf{E}) = V\,(\nabla \cdot \mathbf{E}) + (\nabla V) \cdot \mathbf{E} = V\,(\nabla \cdot \mathbf{E}) - |\mathbf{E}|^2$

- Let us EXTEND the integration over a very large SPHERICAL volume $\tilde{\mathcal{V}}$, with bounding surface $\tilde{S}$, enclosing the LOCALIZED distribution $\mathcal{V}$:

$$U_E = \frac{1}{2}\iiint_{\mathcal{V}} \rho_{\rm tot}(\mathbf{r}')V(\mathbf{r}')\,d\tau' = \frac{1}{2}\iiint_{\tilde{\mathcal{V}}} \rho_{\rm tot}(\mathbf{r}')V(\mathbf{r}')\,d\tau'$$



- For <u>localized</u> distribution at a distant point P, $V(\mathbf{r}) \propto 1/r$, $|\mathbf{E}(\mathbf{r})| \propto 1/r^2$
- For points $\mathbf{r}'$ on this very large spherical surface $\tilde{S}$,

$$\oiint_{\tilde{S}} [V(\mathbf{r}')\,\mathbf{E}(\mathbf{r}')]_{\rm boundary}\cdot d\mathbf{a}' \propto \left(\frac{1}{r'}\cdot\frac{1}{r'^2}\cdot r'^2\right)_{\tilde{S}} \sim \#\left(\frac{1}{r'}\right)_{\tilde{S}} \to 0,\ r' \to \infty.$$

True Configuration Energy of General Charge Distribution (contd.)

- Extending to include **ALL SPACE**: $\tilde{\mathcal{V}} \to \mathcal{V}_\infty \equiv \mathbb{R}^3$ and $\tilde{S} \to S_\infty$, then the surface integral vanishes!

$$
\begin{aligned}
U_E &= \frac{1}{2} \iiint_{\mathcal{V}} \rho_{\text{tot}}(\mathbf{r}')V(\mathbf{r}')\,d\tau' = \frac{1}{2} \iiint_{\tilde{\mathcal{V}}} \rho_{\text{tot}}(\mathbf{r}')V(\mathbf{r}')\,d\tau' \\
&= \frac{\epsilon_0}{2} \iiint_{\tilde{\mathcal{V}}\to\mathcal{V}_\infty} |\mathbf{E}|^2_{\text{source}}\,d\tau' + \frac{\epsilon_0}{2} \oiint_{\tilde{S}\to S_\infty} (V\,\mathbf{E})_{\text{boundary}} \cdot d\mathbf{a}' \quad \to 0 \\
&= \frac{\epsilon_0}{2} \iiint_{\text{All Space}} |\mathbf{E}(\mathbf{r}')|^2_{\text{source}}\,d\tau'
\end{aligned}
$$

- **All inclusive formula**: SELF-ENERGIES and INTERACTION ENERGIES of ALL localized charge distributions.

In Summary: 2 Prescriptions to determine the Configuration Energy of a Localized distribution (depending on convenience) in a given problem

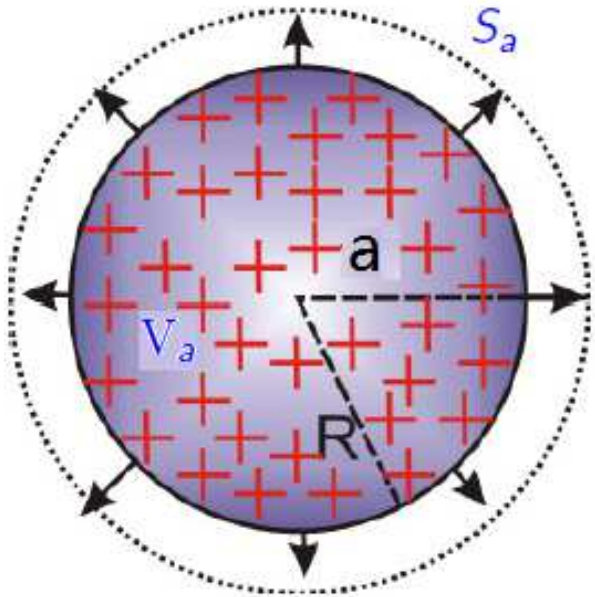
$$
U_E = \frac{1}{2} \iiint_{\mathcal{V}\to\text{original vol.}} \rho_{\text{tot}}(\mathbf{r})V(\mathbf{r})\,d\tau = \frac{\epsilon_0}{2} \iiint_{\text{All Space}} |\mathbf{E}(\mathbf{r})|^2\,d\tau
$$

Configuration Energy of a Charged Sphere

Example

Determine the total configuration/self energy of a uniformly charged solid sphere V_R of radius R and charge q.

- **AUGMENTED VOLUME:** Consider a very large concentric spherical volume V_a of radius $a \gg R$, with its bounding surface S_a, enclosing the original charged sphere V_R .



$$U_E = \frac{1}{2} \iiint_{V_R \to V_a} \rho(\mathbf{r})\, V(\mathbf{r})\, d\tau = \frac{\epsilon_0}{2} \iiint_{V_a} |\mathbf{E}(\mathbf{r})|^2 \, d\tau + \frac{\epsilon_0}{2} \oiint_{S_a} V(\mathbf{r})\, \mathbf{E}(\mathbf{r}) \cdot d\mathbf{a}$$

- Electric fields due to **original** charged sphere V_R (by Using Gauss's Law):

$$\mathbf{E}(\mathbf{r}) = \begin{cases} \frac{1}{4\pi\epsilon_0}\frac{qr}{R^3}\hat{\mathbf{r}} & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}} & r > R \end{cases}$$

- Potentials due to **original** charged sphere V_R (using $V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l}$):

$$V(\mathbf{r}) = \begin{cases} \frac{q}{8\pi\epsilon_0 R}\left(3 - \frac{r^2}{R^2}\right) & r < R \\ \frac{1}{4\pi\epsilon_0}\frac{q}{r} & r > R \end{cases}$$

- Volume integral over the **augmented** volume V_a of radius $a \gg R$:

$$\frac{\epsilon_0}{2}\iiint_{\mathrm{V}_a} |\mathbf{E}(\mathbf{r})|^2 d\tau = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \left\{\int_0^R \left(\frac{r^2}{R^6}\right) r^2 dr + \int_R^a \left(\frac{1}{r^4}\right) r^2 dr\right\} \int_0^{4\pi} d\Omega$$

$$= \frac{q^2}{8\pi\epsilon_0}\left\{\left(\frac{1}{5R}\right)_{r\le R} + \left(\frac{1}{R} - \frac{1}{a}\right)_{R<r\le a}\right\}$$

- Surface integral over the **augmented** sphere S_a of radius $a \gg R$:

$$\frac{\epsilon_0}{2}\oiint_{S_a} [V(a\hat{\mathbf{r}})\,\mathbf{E}(a\hat{\mathbf{r}})]\cdot d\mathbf{a} = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0 a}\right)\left(\frac{q}{4\pi\epsilon_0 a^2}\right)(4\pi a^2) = \left(\frac{q^2}{8\pi\epsilon_0 r}\right)_{r=a}$$

- Extend V_a to include **ALL SPACE**:

$$\mathrm{V}_a \to \mathcal{V}_\infty \equiv \mathbb{R}^3 \quad \& \quad S_a \to S_\infty \Longrightarrow a \to \infty$$

- Configuration energy (self-energy) of the charged sphere V_R:

$$\begin{aligned}
U_E &= \frac{\epsilon_0}{2} \underset{V_a \to \mathcal{V}_\infty}{\iiint} |\mathbf{E}(\mathbf{r})|^2 \, d\tau + \frac{\epsilon_0}{2} \underset{S_a \to S_\infty}{\oiint} V(\mathbf{r})\, \mathbf{E}(\mathbf{r}) \cdot d\mathbf{a} \\
&= \lim_{a \to \infty} \left[\frac{q^2}{8\pi\epsilon_0} \left\{ \frac{1}{5R} + \left(\frac{1}{R} - \cancelto{0}{\frac{1}{a}} \right) \right\} + \cancelto{0}{\frac{q^2}{8\pi\epsilon_0 a}} \right] \\
&= \frac{1}{4\pi\epsilon_0} \left(\frac{3q^2}{5R} \right) \\
U_E &\equiv \frac{\epsilon_0}{2} \underset{\text{All Space}}{\iiint} |\mathbf{E}(\mathbf{r})|^2 \, d\tau = \frac{1}{4\pi\epsilon_0} \left(\frac{3q^2}{5R} \right).
\end{aligned}$$

- Check by <u>direct</u> integration over the **original** charged sphere V_R:

$$U_E = \frac{1}{2} \underset{\mathrm{V}_R}{\iiint} \rho(\mathbf{r}) V(\mathbf{r}) \, d\tau = \frac{\epsilon_0}{2} \left(\frac{q}{4\pi\epsilon_0} \right)^2 \int_0^R \left(\frac{r^2}{R^6} \right) r^2 dr \int_0^{4\pi} d\Omega = \frac{1}{4\pi\epsilon_0} \left(\frac{3q^2}{5R} \right).$$

Self-energy of a Point Charge

- Electric field of a point charge placed at origin:

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$$

- Self-Energy:

$$\begin{aligned} W = \frac{\epsilon_0}{2}\iiint_{\text{All Space}} |\mathbf{E}(\mathbf{r})|^2 d\tau &= \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \left\{\int_0^\infty \left(\frac{1}{r^4}\right) r^2 dr \int_0^\pi \sin\theta d\theta \int_0^{2\pi} d\phi\right\} \\ &= \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \left\{\int_0^\infty \left(\frac{1}{r^4}\right) 4\pi r^2 dr\right\} \\ &= \frac{q^2}{8\pi\epsilon_0}\int_0^\infty \frac{1}{r^2} dr \\ &= \frac{q^2}{8\pi\epsilon_0}\lim_{\delta R\to 0}\int_{\delta R}^\infty \frac{1}{r^2} dr = \frac{q^2}{8\pi\epsilon_0}\lim_{\delta R\to 0}\frac{1}{\delta R} \Rightarrow \infty \end{aligned}$$

- Since the radius δR of the point charge vanishes, the self-energy blow up!
- Point charges are idealized concepts. In reality $\delta R \neq 0$ (say, for electrons), so **Self-Energies** (classically) of spherical objects of finite radius δR is:

$$W_{\text{sphere}} = \frac{1}{4\pi\epsilon_0}\left(\frac{3q^2}{5\,\delta R}\right) \rightarrow \text{finite}$$

Interaction Energy of two Point Charges

Example

Find the Interaction energy of two charges, q_1 and q_2 located at $\mathbf{r}_1$ and $\mathbf{r}_2$, respectively.

Net Electric field at any point $\mathbf{r}$ (Superposition Principle)

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r}),$$

Total Configuration Energy:

$$\begin{aligned} W &= \frac{\epsilon_0}{2} \iiint_{\text{All Space}} |\mathbf{E}(\mathbf{r})|^2 \, d\tau = \frac{\epsilon_0}{2} \iiint_{\text{All Space}} |\mathbf{E}_1(\mathbf{r}) + \mathbf{E}_2(\mathbf{r})|^2 \, d\tau \\ &= \frac{\epsilon_0}{2} \iiint_{\text{All Space}} |\mathbf{E}_1(\mathbf{r})|^2 \, d\tau + \frac{\epsilon_0}{2} \iiint_{\text{All Space}} |\mathbf{E}_2(\mathbf{r})|^2 \, d\tau + \frac{\epsilon_0}{2} \iiint_{\text{All Space}} 2\mathbf{E}_1(\mathbf{r}) \cdot \mathbf{E}_2(\mathbf{r}) \, d\tau \end{aligned}$$

Interaction Energy:

$$W^{\text{int}} = \epsilon_0 \iiint_{\text{All Space}} \mathbf{E}_1(\mathbf{r}) \cdot \mathbf{E}_2(\mathbf{r}) \, d\tau = \frac{q_1 q_2}{16\pi^2\epsilon_0} \iiint_{\text{All Space}} \frac{(\mathbf{r} - \mathbf{r}_1) \cdot (\mathbf{r} - \mathbf{r}_2)}{|\mathbf{r} - \mathbf{r}_1|^3 |\mathbf{r} - \mathbf{r}_2|^3} d\tau = \frac{q_1 q_2}{4\pi\epsilon_0 r_{12}}$$

$\Rightarrow$ This is exactly the work done by an external agent in bringing q_2 from ∞ to $\mathbf{r}_2$ with q_1 already present at $\mathbf{r}_1$.

Total Configuration Energy & Density

- *Interesting Question: Where is the total energy stored?*
- *...in the charges?...in the fields?* *No unique answer to that question!*
- The equation involves integration over all charge distributions:

$$U_E = \frac{1}{2} \iiint_{\mathcal{V}} \rho(\mathbf{r}) V(\mathbf{r}) \, d^3 r$$

$\Longmapsto$ suggests that the energy may be stored in the charges.

- The equation involves integration over all field configurations:

$$U_E = \frac{\epsilon_0}{2} \iiint_{\text{All Space}} E^2(\mathbf{r}) \, d^3 r$$

$\Longmapsto$ suggests that the energy may be stored in the fields.

- It is conventional to define an ENERGY DENSITY:

$$u(\mathbf{r}) = \frac{\epsilon_0}{2} |\mathbf{E}(\mathbf{r})|^2$$

$\Longmapsto$ a volume dv will contain Electrostatic Potential Energy equal to $u(\mathbf{r})dv$.