PH 102: Electromagnetism
Tutorial 1
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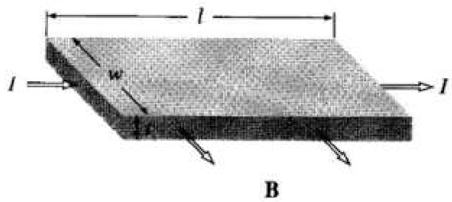

Figure 1: Figure for problem 1.

1. A current I flows to the right through a rectangular bar of conducting material, in the presence of a uniform magnetic field $\vec{B}$ pointing out of the page, as shown in figure 1.

   (a) If the moving charges are positive, in which direction are they deflected by the magnetic field? This deflection results in an accumulation of charge on the upper and lower surfaces of the bar, which in turn produces an electric force to counteract the magnetic one. Equilibrium occurs when the two exactly cancel each other. (A phenomenon known as the Hall effect, to be studied as a part of an experiment in PH 110).

   (b) Find the resulting potential difference (the Hall voltage) between the top and bottom of the bar, in terms of B, v (the speed of the charges), and the relevant dimensions of the bar.

   (c) How would your analysis change if the moving charges were negative? (The Hall effect is the classic way of determining the sign of the mobile charge carriers in a material, which is also one of the objectives of the Hall effect related experiment in PH 110).

   Solution: (a) Positive charges are deflected downwards as evident from the direction of $\vec{v} \times \vec{B}$ in figure 1. Therefore, the bottom edge of the bar acquires a net positive charge.

   (b) At equilibrium magnetic force equals electric force that is, $qvB = qE \implies E = vB$. The potential difference between the top and bottom ends of the bar is $V = Et = vBt$ with the bottom at higher potential.

   (c) If the moving charges are negative, by convention, they move to the left (as current is towards right). The direction of magnetic force $-q(\vec{v} \times \vec{B})$ is again downward like before. But this time, the bottom edge of the bar will acquire a net negative charge and hence at a lower potential compared to the top edge that is at higher potential.

2. (a) A rotating disk (angular velocity ω) carries a uniform density of "static electricity" σ. Find the surface current density K at a distance r from the center.

   (b) Consider a uniformly charged solid sphere of radius R and total charge Q, centered at the origin and spinning at a constant angular velocity ω about the z axis. Find the current density $\vec{J}$ at any point (r, θ, ϕ)

   Solution:

   (a) The surface current density, $\vec{K}$, i.e, the current per unit width - perpendicular to the flow can be written as
$$\vec{K} = \sigma \vec{v},$$
   here $\vec{v}$ is the velocity of the surface charge and σ is the surface charge density. For the rotating disk $v = \omega r$, thus $K = \sigma \omega r$ with direction along $\hat{\phi}$ in spherical polar coordinates.

   (b) For the sphere rotating about the z axis, the velocity at any point on the sphere $\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\,\dot{\theta}\,\hat{\boldsymbol{\theta}} + r\,\dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$, with $\dot{r} = \dot{\theta} = 0$. Therefore, current density $\mathbf{J} = \rho\mathbf{v} = \rho r\,\dot{\varphi}\sin\theta\,\hat{\boldsymbol{\varphi}}$, where $\rho = Q/((4/3)\pi R^3)$.

3. Find the magnetic field at a point $z > R$ on the axis of (a) the rotating disk and (b) the rotating sphere, in problem 2.

   Solution:

   (a) Consider the shaded region of the disk with a width of dr (see figure 2). This ring of charge $dq = 2\pi r\sigma dr$ will have the current $I = dq/dt = \sigma\omega r dr$, where $dt = 2\pi/\omega$. As discussed in the class, magnetic field at the axis (z-axis) of a circular coil of radius R (center at z=0) is
$$B(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}$$
   Similarly, the magnetic field on the z axis for this ring is $d\vec{B} = \frac{\mu_0}{2}\sigma\omega r\ \frac{r^2}{(r^2+z^2)^{3/2}}\ dr\hat{z}$. Therefore the total field of the disk is,

$$\vec{B} = \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^3 dr}{(r^2+z^2)^{3/2})}\hat{z} = \frac{\mu_0\sigma\omega}{4}[2(\frac{u+2z^2}{\sqrt{u+z^2}})]\Big|_0^{R^2} = \frac{\mu_0\sigma\omega}{2}[(\frac{R^2+2z^2}{\sqrt{R^2+z^2}}) - 2z]\hat{z}. \quad (\text{with } u = r^2)$$

   (b) We use the above disk solution to solve the sphere one. Slice the sphere in infinitesimal slabs (see figure 2) and summing over all such slabs will give the solution. The thickness of such slabs $|d(Rcos\theta)| = Rsin\theta d\theta$; $\sigma \to \rho|d(Rcos\theta)| = \rho Rsin\theta d\theta$; $R \to Rsin\theta$; $z \to z - Rcos\theta$ and $R^2 + z^2 \to R^2 + z^2 - 2Rz\ cos\theta$.

   From 'part a' solution, the field for such a slab would consist the term $[(\frac{R^2+2z^2}{\sqrt{R^2+z^2}}) - 2z]$ $= 2[\sqrt{R^2+z^2} - \frac{R^2/2}{\sqrt{R^2+z^2}} - z]$. Rewriting this term with the above substitutions,
$$2[\sqrt{R^2 + z^2 - 2Rz\ cos\theta} - \frac{(R^2/2)sin^2\theta}{\sqrt{R^2 + z^2 - 2Rz\ cos\theta}} - (z - R\ cos\theta)].$$

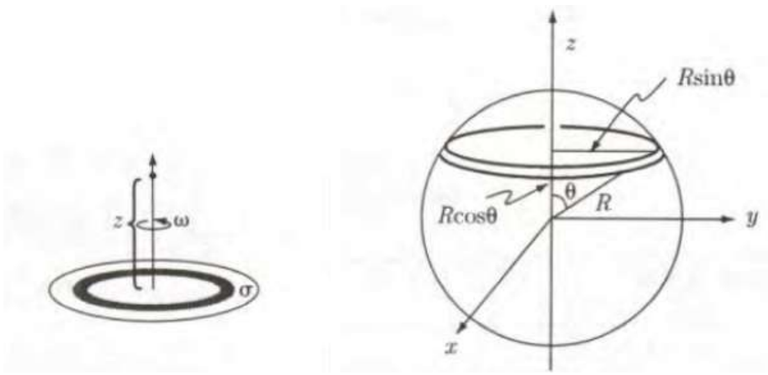

Figure 2: Figure for problem 3.

Therefore, total B_z is given by,

$$= \frac{\mu_0 \rho \omega R}{2} 2 \int_0^{\pi} sin\theta \, d\theta \big[\sqrt{R^2 + z^2 - 2Rz \, cos\theta} - \frac{(R^2/2) sin^2\theta}{\sqrt{R^2 + z^2 - 2Rz \, cos\theta}} - (z - R \, cos\theta) \big]$$

$$= \mu_0 \rho \omega R \int_{-1}^{1} \big[\sqrt{R^2 + z^2 - 2Rzu} - \frac{(R^2/2) sin^2\theta}{\sqrt{R^2 + z^2 - 2Rzu}} - z + Ru) \big] du \qquad (u = cos\theta)$$

$$= \mu_0 \rho \omega R \big[I_1 - \frac{R^2}{2}(I_2 - I_3) - I_4 + I_5 \big].$$

$$I_1 = \int_{-1}^{1} \sqrt{R^2 + z^2 - 2Rzu} \, du = -\frac{1}{3Rz}(R^2 + z^2 - 2Rzu)^{3/2} \Big|_{-1}^{1} = \frac{2}{3z}(3z^2 + R^2).$$

$$I_2 = \int_{-1}^{1} \frac{1}{\sqrt{R^2 + z^2 - 2Rzu}} \, du = -\frac{1}{Rz}(R^2 + z^2 - 2Rzu)^{1/2} \Big|_{-1}^{1} = \frac{2}{z}.$$

$$I_3 = \int_{-1}^{1} \frac{u^2}{\sqrt{R^2 + z^2 - 2Rzu}} \, du$$

$$= -\frac{1}{60R^3z^3}(8(R^2 + z^2)^2 + 4(R^2 + z^2)2Rzu + 3(2Rz)^2u^2)(R^2 + z^2 - 2Rzu)^{1/2} \Big|_{-1}^{1}$$

$$= \frac{4}{15z^3}(R^2 + (5/2)z^2).$$

$$I_4 = z \int_{-1}^{1} du = 2z; I_5 = R \int_{-1}^{1} u du = 0.$$

$$B_z = \frac{2\mu_0 \rho \omega R^5}{15 z^3} \implies \vec{B} = \frac{2\mu_0 \rho \omega R^5}{15 z^3} \hat{z}.$$

4. A semicircular wire carries a steady current $\vec{I}$. Find the magnetic field at a point P on the other semicircle (see figure 3). The semicircular wire must be connected to some other wire to complete the circuit. Neglect this wire needed to complete the circuit.

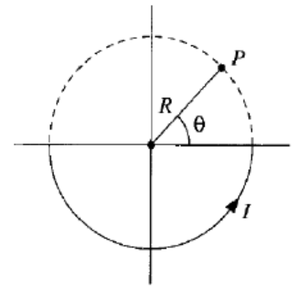

Figure 3: The path

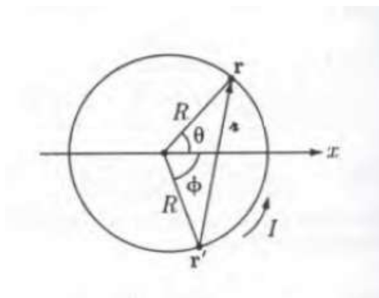

Figure 4: Solution to problem 4.

Solution:

According to the diagram, the vector connecting dl and the center is $\vec{r'} = R\ \cos\phi\ \hat{x} - R\ \sin\phi\ \hat{y}$, and the coordinate of point P is $\vec{r} = R\ \cos\theta\ \hat{x} + R\ \sin\theta\ \hat{y}$. Thus vector connecting $\vec{dl}$ and the point P is $\vec{\mathscr{r}} = \vec{r} - \vec{r'} = R\ (\cos\theta - \cos\phi)\ \hat{x} + R\ (\sin\theta + \sin\phi)\ \hat{y}$ and $\vec{dl} = \hat{z} \times \vec{r'} d\phi = R\ \sin\phi\ d\phi\ \hat{x} + R\ \cos\phi\ d\phi\ \hat{y} = R\ (\sin\phi\ \hat{x} + \cos\phi\ \hat{y}) d\phi$.

$$\vec{dl} \times \vec{\mathscr{r}} = R^2 d\phi \big[1 - \cos(\theta + \phi)\big] \hat{z}.$$

Therefore,

$$\vec{B} = \frac{\mu_0 I}{4\pi} \int \frac{\vec{dl} \times \vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0 I}{4\pi} R^2 \hat{z} \int_0^{\pi} \frac{\big[1 - \cos(\theta + \phi)\big]}{\big[2R^2 - 2R^2 \cos(\theta + \phi)\big]^{3/2}} d\phi = \frac{\mu_0 I}{8\sqrt{2}\pi R} \hat{z} \int_0^{\pi} \frac{d\phi}{\sqrt{2} \sin\frac{\theta + \phi}{2}}$$

$$= \frac{\mu_0 I}{16\pi R} \hat{z} \left(2\ \ln\big[\tan\frac{\theta + \phi}{4}\big]\right)\Bigg|_0^{\pi} = \frac{\mu_0 I}{8\pi R} \ln\left[\frac{\tan\frac{\theta + \pi}{4}}{\tan\frac{\theta}{4}}\right] \hat{z}.$$

Note: Instead of using ϕ one can use the corresponding polar angle $\theta' = 2\pi - \phi$ and take limit on θ' as $\pi - 2\pi$. Using $\phi = 2\pi - \theta'$, we get

$$\vec{B} = \frac{\mu_0 I}{8\sqrt{2}\pi R}\hat{z}\int_{2\pi}^{\pi} \frac{d(2\pi - \theta')}{\sqrt{2}\sin\frac{\theta+2\pi-\theta'}{2}} = \frac{\mu_0 I}{16\pi R}\hat{z}\int_{\pi}^{2\pi} \frac{d\theta'}{\sin\frac{\theta+2\pi-\theta'}{2}}$$

$$\vec{B} = \frac{\mu_0 I}{16\pi R}\hat{z}\left(-2\ \ln\left[\tan\frac{\theta+2\pi-\theta'}{4}\right]\right)\Bigg|_{\pi}^{2\pi} = \frac{\mu_0 I}{8\pi R}\ln\left[\frac{\tan\frac{\theta+\pi}{4}}{\tan\frac{\theta}{4}}\right]\hat{z}$$

which is the same result as before.

5. Find the magnetic field due to a current I in an infinitely long coaxial cable whose inner conductor has radius a and the outer conductor has the radii $b, c(b < c)$ (currents in outer and inner conductors are in opposite directions). Also, express the magnetic field as a vector in terms of the current density.

Solution:
The current is uniformly distributed in both the conductors, see figure 5. The current

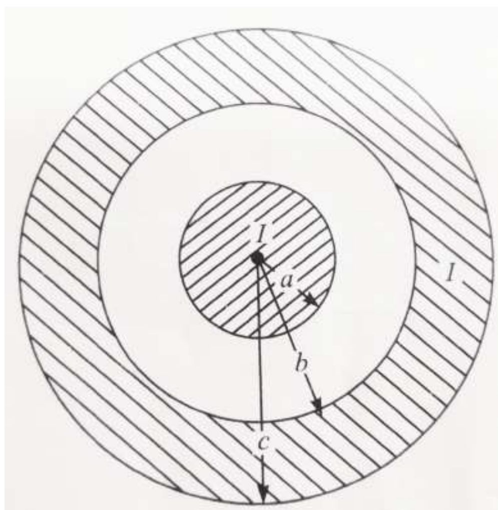

Figure 5

densities in the inner and outer conductors are $J_{\text{in}} = \frac{I}{\pi a^2}$ and $J_{\text{out}} = \frac{I}{\pi(c^2-b^2)}$, respectively with the densities are in the $\hat{z}, -\hat{z}$ directions respectively. Due to the symmetry of the problem the magnetic field ($\vec{B}$) is in the $\hat{\phi}$ direction.

Inside the inner conductor ($r < a$) by Ampere's law,

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{\text{enc}} = \mu_0 J_{\text{in}}\pi r^2$$

$$2\pi r B_\phi = \mu_0 J_{\text{in}}\pi r^2 \implies B_\phi = \mu_0\frac{J_{\text{in}}}{2}r \implies \vec{B} = \mu_0\frac{\vec{J}_{\text{in}}\times\vec{r}}{2}$$

.

In the region between the inner and outer conductors, $a < r < b$,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{\text{enc}} = \mu_0 J_{\text{in}} \pi a^2$$

$$B_\phi = \mu_0 \frac{J_{\text{in}}}{2} \frac{a^2}{r} \implies \vec{B} = \mu_0 \frac{\vec{J_{\text{in}}} \times \vec{r}}{2} \left(\frac{a}{r}\right)^2$$

In the outer conductor, $b \leq r \leq c$,

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 \left(I - J_{\text{out}} \pi \left(r^2 - b^2\right)\right) = \mu_0 \left(I - \frac{I\pi \left(r^2 - b^2\right)}{\pi \left(c^2 - b^2\right)}\right) = \frac{\mu_0 I}{\pi \left(c^2 - b^2\right)} \pi \left(c^2 - r^2\right)$$

$$B_\phi = \frac{\mu_0 J_{\text{out}}}{2} \frac{c^2 - r^2}{r} \implies \vec{B} = \mu_0 \frac{\vec{J_{\text{out}}} \times \vec{r}}{2} \left(\frac{c^2 - r^2}{r^2}\right)$$

Note that outside the conductor $(r > c)$ $\vec{B} = 0$. The figure 6 shows the magnetic field variation with radius.

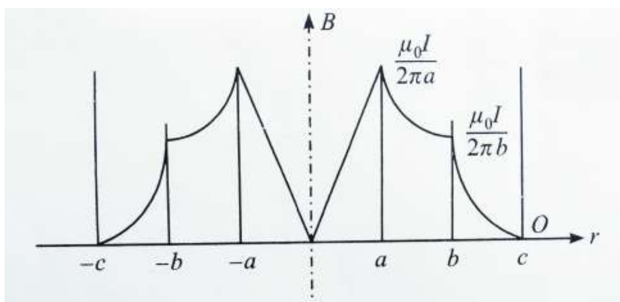

Figure 6

For Students: Solve the problem by taking the currents in outer and inner conductors to be in the same direction.