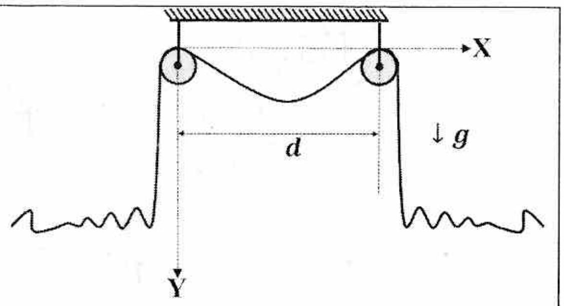
4

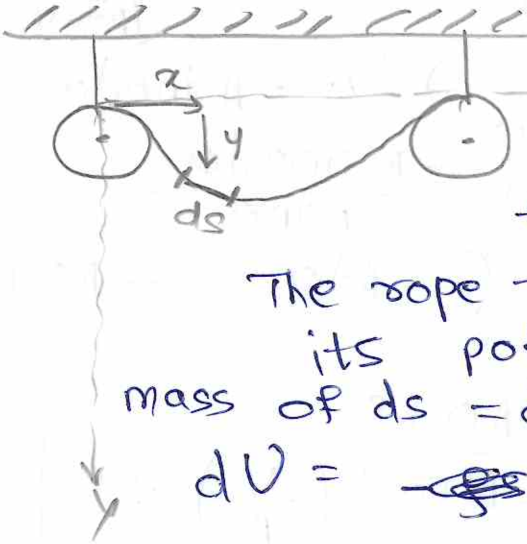
Name	Pradnesh P. Kalkar	Roll No	190123046

Q6 **[5 marks]** A very long rope has mass per unit length ρ and it passes through two frictionless pulleys separated by a distance d. Acceleration due to gravity (g) acts downwards.

Due to its weight, the rope hangs between the pulleys making a curve as shown in the figure. Employing variational method, calculate the equation of the rope in between the pulleys. Determination of integration constants are not required.



Soln



As, every system tries to achieve stability,

The rope tries to minimize its potential energy U.

mass of $ds = dm = \rho ds$

$dU = -dmgy$

$\therefore dU = -\rho ds g y \quad ; ds = \sqrt{dx^2+dy^2}$

$= -\rho g \sqrt{dx^2+dy^2}\, y$

1 1/2

$\therefore U = \int dU = -\int \rho g y \sqrt{1+x'^2}\, dy$

We have to make this integral stationary. i.e. effectively to make

$\int y\sqrt{1+x'^2}\, dy$ stationary

$\therefore F(x, x', y) = y\sqrt{1+x'^2}$

$\therefore$ By using variational principle,

$\therefore$ From the E–L eq^n, we have

$$\frac{d}{dy}\left(\frac{\partial F}{\partial x'}\right) - \frac{\partial F}{\partial x} = 0$$

Now, $\frac{\partial F}{\partial x} = 0 \Rightarrow \frac{\partial F}{\partial x'} = \text{const}$

$$\therefore \frac{y x'}{\sqrt{1+x'^2}} = c = A$$

$$\therefore \frac{y^2 x'^2}{1+x'^2} = A^2$$

$$\therefore \cancel{x^2} \quad \frac{y^2}{A^2} = 1 + \frac{1}{x'^2}$$

$$\Rightarrow x'^2 = \frac{A^2}{A^2-y^2}$$

$\frac{A^2}{y^2-A^2}$

11½

$$x' = \frac{A}{\sqrt{A^2-y^2}} \Rightarrow dx = \frac{A\,dy}{\sqrt{A^2-y^2}}$$

Put, $y = A\cos\theta \Rightarrow dy = -A\sin\theta\, d\theta$

$$\therefore \int dx = \int \frac{-A^2 \sin\theta\, d\theta}{A\sin\theta}$$

$$= \int -A\, d\theta = -A\theta + C$$

$$\therefore x = -A\theta + C$$

$$\therefore \boxed{x = -A\cos^{-1}\left(\frac{y}{A}\right) + C}$$

This is the required equation of rope

~~$x = 0, y = 0$~~