Department of Mathematics
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MA 101: Mathematics I
Differentiation
July-December 2019

The derivative of a function is usually tied up with the notion of the slope of the tangent to the curve which represents the function, or with the rate of change of the function. These interpretations are, in fact, the reason differentiation was introduced.

Definition 1. *Let $D \subseteq \mathbb{R}$ and let $x_0 \in D$ be an interior point, that is, there exists some $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq D$. A function $f : D \to \mathbb{R}$ is said to be differentiable at x_0 if*

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \quad \textit{or, equivalently} \quad \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists in $\mathbb{R}$. If f is differentiable at x_0, then the derivative of f at x_0 is

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Further, $f : D \to \mathbb{R}$ is said to be differentiable if f is differentiable at each $x_0 \in D$.

In the following theorem, we prove that differentiability implies continuity.

Theorem 1. *If $f : D \to \mathbb{R}$ is differentiable at $x_0 \in D$, then f is continuous at x_0.*

Proof. For $x \neq x_0$, we write

$$f(x) = (x - x_0) \times \frac{f(x) - f(x_0)}{x - x_0} + f(x_0).$$

Since $\lim\limits_{x \to x_0} (x - x_0) = 0$ and $\lim\limits_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)$, we find that

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Therefore, f is continuous at x_0. $\square$

Example 1. *$f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is not differentiable at 0.*

Solution. Let $x \neq 0$. Then

$$\lim_{x \to 0+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0+} \frac{|x|}{x} = 1 \text{ and } \lim_{x \to 0-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0-} \frac{|x|}{x} = -1.$$

Therefore, $\lim\limits_{x \to 0} \frac{f(x) - f(0)}{x - 0}$ does not exist, and hence f is not differentiable at 0. $\square$

Example 2. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x \sin \frac{1}{x} & \textit{if } x \neq 0, \\ 0 & \textit{if } x = 0. \end{cases}$*
Then f is not differentiable at 0.

Solution. Since $\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=\lim\limits_{x\to 0}\sin\frac{1}{x}$ does not exist, f is not differentiable at 0. $\square$

Example 3. $f(x)=\begin{cases} x^2 & \text{if } x\in\mathbb{Q}, \\ 0 & \text{if } x\in\mathbb{R}\setminus\mathbb{Q}. \end{cases}$ *Then $f:\mathbb{R}\to\mathbb{R}$ is differentiable only at* 0 *and* $f'(0)=0$.

Solution. If $x_0(\neq 0)\in\mathbb{Q}$, then there exists a sequence (t_n) in $\mathbb{R}\setminus\mathbb{Q}$ such that $t_n\to x_0$. Since $f(t_n)=0$ for all $n\in\mathbb{N}$, $f(t_n)\to 0\neq x_0^2=f(x_0)$. Hence f is not continuous at x_0. Also, if $u_0\in\mathbb{R}\setminus\mathbb{Q}$, then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n\to u_0$. Since $f(r_n)=r_n^2\to u_0^2\neq 0=f(u_0)$, f is not continuous at u_0. Thus f is not continuous at any $x(\neq 0)\in\mathbb{R}$ and therefore f cannot be differentiable at any $x(\neq 0)\in\mathbb{R}$.
Again, for each $\varepsilon>0$, choosing $\delta=\varepsilon>0$, we find that $|\frac{f(x)-f(0)}{x-0}|\le|x|<\varepsilon$ for all $x\in\mathbb{R}$ satisfying $0<|x|<\delta$. Hence $\lim\limits_{x\to 0}\frac{f(x)-f(0)}{x-0}=0$ and consequently f is differentiable at 0 with $f'(0)=0$. $\square$

Theorem 2 (Rules for finding derivatives)**.** *Let $D\subseteq\mathbb{R}$ and let $x_0\in D$ be an interior point. Suppose $f,g:D\to\mathbb{R}$ are differentiable at x_0. Then*

(a) *If $\alpha\in\mathbb{R}$, then the function αf is differentiable at x_0 and $(\alpha f)'(x_0)=\alpha f'(x_0)$.*

(b) *The function $f+g$ is differentiable at x_0 and $(f+g)'(x_0)=f'(x_0)+g'(x_0)$.*

(c) (*Product rule*) *The function fg is differentiable at x_0 and $(fg)'(x_0)=f'(x_0)g(x_0)+f(x_0)g'(x_0)$.*

(d) (*Quotient rule*) *If $g(x_0)\neq 0$, then the function f/g is differentiable at x_0 and*

$$(f/g)'(x_0)=\frac{f'(x_0)g(x_0)-f(x_0)g'(x_0)}{(g(x_0))^2}.$$

Theorem 3 (Chain rule for derivative)**.** *Let $f:D\to\mathbb{R}$ and $g:E\to\mathbb{R}$. Let $f(D)\subseteq E$. Suppose that f is differentiable at x_0 and g is differentiable at $f(x_0)$. Then $g\circ f$ is differentiable at x_0 and $(g\circ f)'(x_0)=g'(f(x_0))f'(x_0)$.*

Proof. We define a function $h:E\to\mathbb{R}$ by

$$h(y)=\begin{cases}\frac{g(y)-g(f(x_0))}{y-f(x_0)}, & y\neq f(x_0);\\ g'(f(x_0)), & y=f(x_0).\end{cases}$$

We have $\lim\limits_{y\to f(x_0)}h(y)=g'(f(x_0))=h(f(x_0))$. Also, $g(y)-g(f(x_0))=h(y)\times(y-f(x_0))$ for all $y\in E$ including $y=f(x_0)$. So, for $x\neq x_0$ we have

$$\begin{aligned}\frac{(g\circ f)(x)-(g\circ f)(x_0)}{x-x_0}&=\frac{g(f(x))-g(f(x_0))}{x-x_0}\\ &=h(f(x))\frac{f(x)-f(x_0)}{x-x_0}\end{aligned}$$

Hence, $(g\circ f)'(x_0)=\lim\limits_{x\to x_0}h(f(x))\times\lim\limits_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}=h(f(x_0))f'(x_0)=g'(f(x_0))f'(x_0)$. $\square$

Example 4. *Let $f : \mathbb{R} \to \mathbb{R}$ defined by* $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
Then f is differentiable at 0. But $f' : \mathbb{R} \to \mathbb{R}$ is not continuous at 0.

Solution. Clearly f is differentiable at all $x\ (\neq 0) \in \mathbb{R}$ and $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ for all $x\ (\neq 0) \in \mathbb{R}$. Also, for each $\varepsilon > 0$, choosing $\delta = \varepsilon > 0$, we find that $|\frac{f(x)-f(0)}{x-0}| = |x \sin \frac{1}{x}| \leq |x| < \varepsilon$ for all $x \in \mathbb{R}$ satisfying $0 < |x| < \delta$. Hence $\lim\limits_{x \to 0} \frac{f(x)-f(0)}{x-0} = 0$ and consequently f is differentiable at 0 with $f'(0) = 0$. Thus $f : \mathbb{R} \to \mathbb{R}$ is differentiable.
Again, since $\frac{1}{2n\pi} \to 0$ but $f'(\frac{1}{2n\pi}) \to -1 \neq f'(0)$, $f' : \mathbb{R} \to \mathbb{R}$ is not continuous at 0. $\square$

Definition 2. *Let $f : D \to \mathbb{R}$. f has a local maximum at $x_0 \in D$ if there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap D$. Similarly, f has a local minimum at $x_0 \in D$ if there exists $\delta > 0$ such that $f(x) \geq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap D$.*

Theorem 4. *If $f : D \to \mathbb{R}$ has a local maximum or local minimum at an interior point x_0 of D and if f is differentiable at x_0, then $f'(x_0) = 0$.*

Proof. Suppose that f has a local maximum at x_0, where x_0 is an interior point of D. Then there exists $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \subseteq D$. Hence,

$$\frac{f(x) - f(x_0)}{x - x_0} \leq 0 \ \text{ for all } \ x \in (x_0, x_0 + \delta) \Rightarrow \lim_{x \to x_0+} \frac{f(x) - f(x_0)}{x - x_0} \leq 0;$$
$$\frac{f(x) - f(x_0)}{x - x_0} \geq 0 \ \text{ for all } \ x \in (x_0 - \delta, x_0) \Rightarrow \lim_{x \to x_0-} \frac{f(x) - f(x_0)}{x - x_0} \geq 0.$$

Since f is differentiable at x_0, so $f'(x_0) = 0$. If f has a local minimum at x_0 then following as above we readily obtain that $f'(x_0) = 0$. $\square$

Theorem 5 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$. Suppose that*

(a) *f is continuous on $[a, b]$.*

(b) *f is differentiable on (a, b).*

(c) *$f(a) = f(b)$.*

Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Proof. Since f is continuous on $[a, b]$, so f has maximum and minimum values on $[a, b]$. If both these extreme values are equal, then f is the constant function, and we can choose c to be any point in (a, b). If f is not constant, then one of the extreme values is not equal to $f(a)$. Since $f(a) = f(b)$, so f has a local maximum/minimum at a point c other than a and b. In other words, $c \in (a, b)$. Since f is differentiable at c, so $f'(c) = 0$. $\square$

Example 5. *The equation $x^2 = x \sin x + \cos x$ has exactly two (distinct) real roots.*

Solution. Let $f(x) = x^2 - x \sin x - \cos x$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is differentiable (and hence continuous) with $f'(x) = x(2 - \cos x)$ for all $x \in \mathbb{R}$. Since $\cos x \neq 2$ for any $x \in \mathbb{R}$, the equation $f'(x) = 0$ has exactly one real root, namely $x = 0$. As a consequence of Rolle's theorem, it follows that the equation $f(x) = 0$ has at most two real roots. Also, since $f(-\pi) = \pi^2 + 1 > 0$, $f(0) = -1 < 0$ and $f(\pi) = \pi^2 + 1 > 0$, by the intermediate value property of continuous functions, the equation $f(x) = 0$ has at least one root in $(-\pi, 0)$ and at least one root in $(0, \pi)$. Thus the equation $f(x) = 0$ has exactly two (distinct) real roots and so the given equation has exactly two (distinct) real roots. $\square$

Example 6. *Find the number of (distinct) real roots of the equation* $x^4+2x^2-6x+2=0$.

Solution. Taking $f(x) = x^4 + 2x^2 - 6x + 2$ for all $x \in \mathbb{R}$, we find that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable with $f'(x) = 4x^3 + 4x - 6$ and $f''(x) = 12x^2 + 4$ for all $x \in \mathbb{R}$. Since $f''(x) \neq 0$ for all $x \in \mathbb{R}$, as a consequence of Rolle's theorem, it follows that the equation $f'(x) = 0$ has at most one real root and hence the equation $f(x) = 0$ has at most two real roots. Again, since $f(0) = 2 > 0$, $f(1) = -2 < 0$ and $f(2) = 14 > 0$, by the intermediate value property of continuous functions, the equation $f(x) = 0$ has at least one real root in $(0, 1)$ and at least one real root in $(1, 2)$. Therefore the given equation has exactly two (distinct) real roots. □

Theorem 6 (Mean Value Theorem (MVT)). *If* $f : [a, b] \to \mathbb{R}$ *is continuous and if* f *is differentiable on* (a, b), *then there exists* $c \in (a, b)$ *such that* $f(b) - f(a) = f'(c)(b - a)$.

Proof. Define the function $g : [a, b] \to \mathbb{R}$ by

$$g(x) = f(x) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(x - a)\right].$$

Clearly g satisfies the conditions of Rolle's theorem. Therefore, there exists $c \in (a, b)$ such that $g'(c) = 0$. This gives $f(b) - f(a) = f'(c)(b - a)$. □

Theorem 7. *Let* I *be an interval and let* $f : I \to \mathbb{R}$ *be differentiable. Then*

(a) $f'(x) = 0$ *for all* $x \in I \Leftrightarrow f$ *is constant on* I.

(b) $f'(x) \geq 0$ *for all* $x \in I \Leftrightarrow f$ *is increasing on* I.

(c) $f'(x) \leq 0$ *for all* $x \in I \Leftrightarrow f$ *is decreasing on* I.

(d) $f'(x) > 0$ *for all* $x \in I \Rightarrow f$ *is strictly increasing on* I.

(e) $f'(x) < 0$ *for all* $x \in I \Rightarrow f$ *is strictly decreasing on* I.

(f) $f'(x) \neq 0$ *for all* $x \in I \Rightarrow f$ *is one-one on* I.

Proof of (a). If f is constant on I then it easily follows that $f'(x) = 0$ for all $x \in I$. Conversely, suppose that $f'(x) = 0$ for all $x \in I$. Let $x_1, x_2 \in I$ be such that $x_1 < x_2$. By MVT, there exists $c \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$. Since $c \in (x_1, x_2)$, $f'(c) = 0$ and therefore $f(x_1) = f(x_2)$. This proves that f is a constant function. □

Proof of (b) and (c). Suppose that $f'(x) \geq 0$ for all $x \in I$. Let $x_1, x_2 \in I$ be such that $x_1 < x_2$. Then applying MVT to $[x_1, x_2]$, there exists some $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) \geq 0.$$

This gives $f(x_2) \geq f(x_1)$, and hence f is increasing. Conversely, suppose that f is differentiable and increasing on I. Let $c \in I$. Then for any $x \neq c$, we have

$$\frac{f(x) - f(c)}{x - c} \geq 0, \quad \text{and hence} \quad f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} \geq 0.$$

The proof of (c) follows similarly. □

Proof of (d) and (e). Suppose that $f'(x) > 0$ for all $x \in I$. Let $x_1, x_2 \in I$ be such that $x_1 < x_2$. Then applying MVT to $[x_1, x_2]$, there exists some $c \in (x_1, x_2)$ such that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0.$$

This gives $f(x_2) > f(x_1)$, and hence f is strictly increasing. The proof of (e) follows similarly. ☐

Proof of (f). If f is not injective, there would be two points $x_1, x_2 \in I$ such that $x_1 < x_2$ and $f(x_1) = f(x_2)$. Applying MVT to $[x_1, x_2]$, we would then conclude that there is a $c \in (x_1, x_2) \subseteq I$ such that $f'(c) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = 0$. This contradicts the hypothesis that $f'(x) \neq 0$ for all $x \in I$. ☐

Remark 1. *The converse of (d) and (e) are not true in general. For example, $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3$ is strictly increasing but $f'(0) = 0$.*

Example 7. *We have $\sin x \leq x$ for all $x \geq 0$.*

Proof. Let $f(x) = \sin x - x$ for all $x \in \mathbb{R}$. Then $f'(x) = \cos x - 1$ for all $x \in \mathbb{R}$. Since $f'(x) \leq 0$ for all $x \in \mathbb{R}$, so f is decreasing on $\mathbb{R}$. Hence $f(x) \leq f(0) = 0$ for all $x \geq 0$. Therefore $\sin x \leq x$ for all $x \geq 0$. ☐

Example 8. $\sin x \geq x - \frac{x^3}{6}$ *for all* $x \in [0, \frac{\pi}{2}]$.

Solution. Let $f(x) = \sin x - x + \frac{x^3}{6}$ for all $x \in [0, \frac{\pi}{2}]$. Then $f : [0, \frac{\pi}{2}] \to \mathbb{R}$ is infinitely differentiable and $f'(x) = \cos x - 1 + \frac{x^2}{2}$, $f''(x) = \sin x + x$ and $f'''(x) = 1 - \cos x$ for all $x \in [0, \frac{\pi}{2}]$. Since $f'''(x) \geq 0$ for all $x \in [0, \frac{\pi}{2}]$, f'' is increasing on $[0, \frac{\pi}{2}]$. Hence $f''(x) \geq f''(0) = 0$ for all $x \in [0, \frac{\pi}{2}]$. This shows that f' is increasing on $[0, \frac{\pi}{2}]$ and so $f'(x) \geq f'(0) = 0$ for all $x \in [0, \frac{\pi}{2}]$. Thus f is increasing on $[0, \frac{\pi}{2}]$ and so $f(x) \geq f(0) = 0$ for all $x \in [0, \frac{\pi}{2}]$. Therefore $\sin x \geq x - \frac{x^3}{6}$ for all $x \in [0, \frac{\pi}{2}]$. ☐

Example 9. *If $f(x) = x^3 + x^2 - 5x + 3$ for all $x \in \mathbb{R}$, then f is one-one on $[1, 5]$ but not one-one on $\mathbb{R}$.*

Proof. $f : \mathbb{R} \to \mathbb{R}$ is differentiable with $f'(x) = 3x^2 + 2x - 5$ for all $x \in \mathbb{R}$. Clearly $f'(x) \neq 0$ for all $x \in (1, 5)$ and hence f is one-one on $[1, 5]$. Again, since $f(0) = 3$, $f(1) = 0$ and $f(2) = 5$, by the intermediate value property of continuous functions, there exist $x_1 \in (0, 1)$ and $x_2 \in (1, 2)$ such that $f(x_1) = 1 = f(x_2)$. Therefore f is not one-one on $\mathbb{R}$. ☐

Theorem 8 (L'Hôpital's rules)**.** *Let $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$.*

(1) *Suppose that f and g are differentiable at $x_0 \in (a, b)$, $f(x_0) = g(x_0) = 0$ and $g'(x_0) \neq 0$. Then $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \frac{f'(x_0)}{g'(x_0)}$.*

(2) *Suppose that $f : (a, b) \to \mathbb{R}$ and $g : (a, b) \to \mathbb{R}$ are differentiable such that $\lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a+} g(x) = 0$ and $g'(x) \neq 0$ for all $x \in (a, b)$. If $\lim\limits_{x \to a+} \frac{f'(x)}{g'(x)} = \ell$, then $\lim\limits_{x \to a+} \frac{f(x)}{g(x)} = \ell$.*

Example 10. $\lim\limits_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{1}{2}$.

Solution. Applying (first version of) L'Hôpital's rule, we obtain

$$\lim_{x\to 0}\frac{\sqrt{1+x}-1}{x}=\frac{\frac{d}{dx}(\sqrt{1+x}-1)|_{x=0}}{\frac{d}{dx}(x)|_{x=0}}=\frac{1}{2}.$$

□

Example 11. $\lim\limits_{x\to\frac{\pi}{2}}\frac{1-\sin x}{1+\cos 2x}=\frac{1}{4}$.

Solution. Applying L'Hôpital's rule twice, we obtain

$$\lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{1+\cos 2x}=\lim_{x\to\frac{\pi}{2}}\frac{-\cos x}{-2\sin 2x}=\lim_{x\to\frac{\pi}{2}}\frac{\sin x}{-4\cos 2x}=\frac{1}{4}.$$

□

Example 12. $\lim\limits_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}=0$.

Solution. For all $x(\neq 0)\in\mathbb{R}$, we have $0\le|x\sin\frac{1}{x}|\le|x|$. Since $\lim\limits_{x\to 0}|x|=0$, by sandwich theorem (for limit of functions), we get $\lim\limits_{x\to 0}|x\sin\frac{1}{x}|=0$ and hence $\lim\limits_{x\to 0}x\sin\frac{1}{x}=0$. It follows that $\lim\limits_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}=\lim\limits_{x\to 0}\frac{x\sin\frac{1}{x}}{\frac{\sin x}{x}}=\frac{\lim\limits_{x\to 0}x\sin\frac{1}{x}}{\lim\limits_{x\to 0}\frac{\sin x}{x}}=\frac{0}{1}=0$. □

Example 13. $\lim\limits_{x\to\infty}\frac{x-\sin x}{2x+\sin x}=\frac{1}{2}$.

Solution. Since $|\frac{\sin x}{x}|\le\frac{1}{x}$ for all $x>0$ and since $\lim\limits_{x\to\infty}\frac{1}{x}=0$, we get $\lim\limits_{x\to\infty}\frac{\sin x}{x}=0$. Consequently

$$\lim_{x\to\infty}\frac{x-\sin x}{2x+\sin x}=\lim_{x\to\infty}\frac{1-\frac{\sin x}{x}}{2+\frac{\sin x}{x}}=\frac{1}{2}.$$

□

A very useful technique in the analysis of real functions is the approximation of functions by polynomials.

Theorem 9 (Taylor's theorem)**.** *Let $f:[a,b]\to\mathbb{R}$ be such that f and its derivatives $f',f'',\ldots,f^{(n)}$ are continuous on $[a,b]$ and that $f^{(n+1)}$ exists on (a,b). If $x_0\in[a,b]$, then for any $x\in[a,b]$ there exists a point c between x and x_0 such that*

$$\begin{aligned}f(x)&=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n\\&\quad+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}\\&=P_n(x)+R_n(x),\end{aligned}$$

where $P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$ and $R_n(x)=\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$. The polynomial $P_n(x)$ is called the nth Taylor polynomial of f and $R_n(x)$ is called the remainder.

Proof. Let x and x_0 be given and let J denote the closed interval with endpoints x_0 and x. We define the function F on J by

$$F(t) := f(x) - f(t) - f'(t) \cdot (x-t) - \cdots - \frac{f^{(n)}(t)}{n!}(x-t)^n, \tag{1}$$

where $t \in J$. We have

$$F'(t) = -\frac{(x-t)^n}{n!} f^{(n+1)}(t).$$

If we define G on J by

$$G(t) := F(t) - \left(\frac{x-t}{x-x_0}\right)^{n+1} F(x_0)$$

for $t \in J$, then $G(x_0) = G(x) = 0$. Applying Rolle's theorem, there exists c between x and x_0 such that

$$0 = G'(c) = F'(c) + (n+1)\frac{(x-c)^n}{(x-x_0)^{n+1}} F(x_0).$$

Hence, we obtain

$$\begin{aligned} F(x_0) &= -\frac{1}{n+1}\frac{(x-x_0)^{n+1}}{(x-c)^n} F'(c) \\ &= \frac{1}{n+1}\frac{(x-x_0)^{n+1}}{(x-c)^n}\frac{(x-c)^n}{n!} f^{n+1}(c) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}. \end{aligned} \tag{2}$$

From (1) and (2) we readily obtain the required result. □

Example 14. *We have* $1 + \frac{x}{2} - \frac{x^2}{8} \le \sqrt{1+x} \le 1 + \frac{x}{2}$ *for all* $x > 0$.

Solution. Let $x > 0$ and let $f(t) = \sqrt{1+t}$ for all $x \in [0, x]$. Then $f : [0, x] \to \mathbb{R}$ is twice differentiable and $f'(t) = \frac{1}{2\sqrt{1+t}}$, $f''(t) = -\frac{1}{4(1+t)^{3/2}}$ for all $t \in [0, x]$. By Taylor's theorem, there exists $c \in (0, x)$ such that $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(c) = 1 + \frac{x}{2} - \frac{x^2}{8} \cdot \frac{1}{(1+c)^{3/2}}$. Since $0 < \frac{1}{(1+c)^{3/2}} < 1$, we get $1 + \frac{x}{2} - \frac{x^2}{8} \le \sqrt{1+x} \le 1 + \frac{x}{2}$. □

Theorem 10 (Results on local maxima and local minima)**.** *Let* $x_0 \in (a, b)$ *and let* $n \ge 2$. *Also, let* $f, f', ..., f^{(n)}$ *be continuous on* (a, b) *and* $f'(x_0) = f''(x_0) = \cdots = f^{(n-1)}(x_0) = 0$ *but* $f^{(n)}(x_0) \ne 0$.

(a) *If* n *is even and* $f^{(n)}(x_0) < 0$, *then* f *has a local maximum at* x_0.

(b) *If* n *is even and* $f^{(n)}(x_0) > 0$, *then* f *has a local minimum at* x_0.

(c) *If* n *is odd, then* f *has neither a local maximum nor a local minimum at* x_0.

Proof. Applying Taylor's theorem at x_0, we find that for $x \in (a, b)$ we have

$$f(x) = P_{n-1}(x) + R_{n-1}(x) = f(x_0) + \frac{f^{(n)}(c)}{n!}(x-x_0)^n,$$

where c is some point between x_0 and x. Since $f^{(n)}$ is continuous and $f^{(n)}(x_0) \ne 0$, so there exists $\delta > 0$ such that $f^{(n)}(x)$ will have the same sign as $f^{(n)}(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta)$. If $x \in (x_0 - \delta, x_0 + \delta)$, then the point $c \in (x - x_0, x + x_0)$ and consequently $f^{(n)}(c)$ and $f^{(n)}(x_0)$ will have the same sign.

(a) If n is even and $f^{(n)}(x_0) < 0$, then we have $f^{(n)}(c) < 0$ and $(x - x_0)^n \geq 0$ so that $R_{n-1}(x) \leq 0$ for all $x \in (x_0-\delta, x_0+\delta)$. Hence $f(x) \leq f(x_0)$ for all $x \in (x_0-\delta, x_0+\delta)$, and therefore f has a local maximum at x_0.

(b) If n is even and $f^{(n)}(x_0) > 0$, then we have $f^{(n)}(c) > 0$ and $(x - x_0)^n \geq 0$ so that $R_{n-1}(x) \geq 0$ for all $x \in (x_0-\delta, x_0+\delta)$. Hence $f(x) \geq f(x_0)$ for all $x \in (x_0-\delta, x_0+\delta)$, and therefore f has a local minimum at x_0.

(c) If n is odd then $(x - x_0)^n$ is positive if $x > x_0$ and negative if $x < x_0$. Consequently, if $x \in (x_0 - \delta, x_0 + \delta)$ then $R_{n-1}(x)$ will have opposite signs to the left and to the right of x_0. Therefore f has neither a local maximum nor a local minimum at x_0.

$\square$

Theorem 11. *If $f(x) = x^5 - 5x^4 + 5x^3 + 12$ for all $x \in \mathbb{R}$, then f has a local maximum only at 1 and a local minimum only at 3.*

Proof. $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and $f'(x) = 5x^2(x - 1)(x - 3)$, $f''(x) = 10x(2x^2 - 6x + 3)$, $f'''(x) = 30(2x^2 - 4x + 1)$ for all $x \in \mathbb{R}$. Since $f'(x) = 0$ iff $x = 0$, 1, or 3, f has neither a local maximum nor a local minimum at any point of $\mathbb{R} \setminus \{0, 1, 3\}$. Again, since $f''(1) = -10 < 0$, $f''(3) = 90 > 0$, $f''(0) = 0$ and $f'''(0) = 30 \neq 0$, f has a local maximum at 1 (with local maximum value $f(1) = 13$), f has a local minimum at 3 (with local minimum value $f(3) = -15$) and f has neither a local maximum nor a local minimum at 0. $\square$

Definition 3. *A power series is a series of the form $\sum_{n=0}^{\infty} a_n(x - x_0)^n$, where x_0, $a_n \in \mathbb{R}$ for $n = 0, 1, 2, \ldots$ and $x \in \mathbb{R}$.*

Remark 2. *Since the transformation $X = x - c$ reduces a power series around c to a power series around 0, it is sufficient to consider the series $\sum_{n=0}^{\infty} a_n x^n$.*

Example 15. *(a)* $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ *(b)* $\sum_{n=0}^{\infty} n!x^n$ *(c)* $\sum_{n=0}^{\infty} x^n$

Theorem 12. *We have*

(a) *If $\sum_{n=0}^{\infty} a_n x^n$ converges for $x = x_1 \neq 0$, then it converges absolutely for all $x \in \mathbb{R}$ satisfying $|x| < |x_1|$.*

(b) *If $\sum_{n=0}^{\infty} a_n x^n$ diverges for $x = x_2$, then it diverges for all $x \in \mathbb{R}$ satisfying $|x| > |x_2|$.*

Proof. Observe that $(a) \Leftrightarrow (b)$. So we prove part (a) only.

(a) Let $x_1 \neq 0$ and put $y_n = a_n x_1^n$ for all $n \geq 0$. It is given that $\displaystyle\sum_{n=0}^{\infty} y_n$ is convergent. Hence, there exists $M > 0$ such that $|y_n| = |a_n x_1^n| < M$. Let $x \in \mathbb{R}$ be such that $|x| < |x_1|$. Let $r = \frac{|x|}{|x_1|}$. Now,

$$|a_n x^n| = |a_n x_1^n| r^n < M r^n \text{ for all } n.$$

Since $0 \le r < 1$, so $\sum_{n=0}^{\infty} r^n$ converges, and by comparison test $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

$\square$

Definition 4 (Radius of convergence)**.** *For every power series $\sum_{n=0}^{\infty} a_n x^n$, there exists a unique R satisfying $0 \le R \le \infty$ such that the series converges absolutely if $|x| < R$ and diverges if $|x| > R$.*

Theorem 13. *Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. Let $\beta = \limsup \sqrt[n]{|a_n|}$ and $R = \frac{1}{\beta}$ (we define $R = 0$ if $\beta = \infty$ and $R = \infty$ if $\beta = 0$). Then*

(a) *$\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$*

(b) *$\sum_{n=0}^{\infty} a_n x^n$ diverges for $|x| > R$.*

(c) *No conclusion if $|x| = R$.*

Proof. Let $x_n = a_n x^n$. Then $\sqrt[n]{|x_n|} = |x| \sqrt[n]{|a_n|}$. By Root test, $\sum_{n=0}^{\infty} x_n = \sum_{n=0}^{\infty} a_n x^n$ converges absolutely if $\limsup \sqrt[n]{|x_n|} = |x| \limsup \sqrt[n]{|a_n|} = |x|\beta < 1$. Equivalently, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < \frac{1}{\beta} = R$. Again, by Root test $\sum_{n=0}^{\infty} a_n x^n$ diverges for $|x| > R$. $\square$

Theorem 14. *Consider the power series $\sum_{n=0}^{\infty} a_n x^n$. Suppose $\beta = \lim \left|\frac{a_{n+1}}{a_n}\right|$ and $R = \frac{1}{\beta}$ (We define $R = 0$ if $\beta = \infty$ and $R = \infty$ if $\beta = 0$). Then*

(a) *$\sum_{n=0}^{\infty} a_n x^n$ converges for $|x| < R$*

(b) *$\sum_{n=0}^{\infty} a_n x^n$ diverges for $|x| > R$.*

(c) *No conclusion if $|x| = R$.*

Proof. The proof readily follows using the Ratio test. $\square$

Example 16. *The power series $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ converges if and only if $x \in [-1, 1]$.*

Solution. If $x = 0$, then the given series becomes $0 + 0 + \cdots$, which is clearly convergent. Let $x\ (\neq 0) \in \mathbb{R}$ and let $a_n = \frac{x^n}{n^2}$ for all $n \in \mathbb{N}$. Then $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = |x|$. Hence by ratio test, $\sum_{n=1}^{\infty} a_n$ is convergent (absolutely) if $|x| < 1$, *i.e.* if $x \in (-1, 1)$ and is not convergent if $|x| > 1$, *i.e.* if $x \in (-\infty, -1) \cup (1, \infty)$. Therefore the radius of convergence of the given power series is 1. Again, if $|x| = 1$, then $\sum\limits_{n=1}^{\infty} |a_n| = \sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent and hence $\sum\limits_{n=1}^{\infty} a_n$ is also convergent. Therefore the interval of convergence of the given power series is $[-1, 1]$. $\square$

Example 17. *For the power series* $\sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 4^n}(x-1)^n$*, the radius of convergence is* 4 *and the interval of convergence is* $(-3, 5]$.

Solution. **Method-I**: If $x = 1$, then the given series becomes $0 + 0 + \cdots$, which is clearly convergent. Let $x (\neq 1) \in \mathbb{R}$ and let $a_n = \frac{(-1)^n}{n.4^n}(x-1)^n$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \frac{1}{4}|x-1|$. Hence by ratio test, $\sum\limits_{n=1}^{\infty} a_n$ is convergent (absolutely) if $\frac{1}{4}|x-1| < 1$, *i.e.* if $x \in (-3, 5)$ and is not convergent if $\frac{1}{4}|x-1| > 1$, *i.e.* if $x \in (-\infty, -3) \cup (5, \infty)$. Therefore the radius of convergence of the given power series is 4. Again, if $x = -3$, then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{1}{n}$ is not convergent. If $x = 5$, then $\sum\limits_{n=1}^{\infty} a_n = \sum\limits_{n=1}^{\infty} \frac{(-1)^n}{n}$ is convergent by Leibniz test. Therefore the interval of convergence of the given power series is $(-3, 5]$.

Method-II: Let $a_n = \frac{(-1)^n}{n \cdot 4^n}$ for all $n \in \mathbb{N}$. Clearly, $\beta = \lim |\frac{a_{n+1}}{a_n}| = \frac{1}{4}$. Therefore the radius of convergence of the given power series is 4. Thus the given series is convergent (absolutely) if $|x-1| < 4$, that is, if $x \in (-3, 5)$ and is not convergent if $|x-1| > 4$, that is, if $x \in (-\infty, -3) \cup (5, \infty)$. Rest of the proof is similar. $\square$

Theorem 15 (Term by term differentiation of power series)**.** *A power series can be differentiated term by term within the interval of convergence. In fact, if* $\sum\limits_{n=0}^{\infty} a_n x^n$ *has radius of convergence* R*, then* $f'(x) = \sum\limits_{n=1}^{\infty} n a_n x^{n-1}$ *for* $|x| < R$.

Proof. It is clear that $\limsup \sqrt[n]{|na_n|} = \limsup\{n^{\frac{1}{n}} \sqrt[n]{|a_n|}\} = \limsup \sqrt[n]{|a_n|} = \beta$. Thus, both the series $\sum\limits_{n=0}^{\infty} a_n x^n$ and $\sum\limits_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence. However, to prove that the series $\sum_{n=1}^{\infty} n a_n x^{n-1}$ converges to $f'(x)$ requires another concept called *uniform convergence* which is beyond the scope of this course. $\square$

Definition 5. *If a function* f *has derivatives of all orders at a point* $c \in \mathbb{R}$*, then we can calculate the Taylor coefficients by* $a_0 = f(c), a_n = \frac{f^{(n)}(c)}{n!}$ *for all* $n \in \mathbb{N}$*. In this way, we obtain a power series* $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$*. The power series* $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ *converges to* $f(x)$ *for* $|x-c| < R$ *if and only if the sequence* $(R_n(x))$ *of remainders converges to* 0 *for each* x *in* $|x-c| < R$*. The power series* $\sum\limits_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$ *is called the* Taylor series *of* f *at* c*. The Taylor series of a function* f *at* $c = 0$ *is known as* Maclaurin's series.

Example 18. *The Maclaurin series* $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$ *of* $\sin x$ *converges to* $\sin x$ *for all* $x \in \mathbb{R}$.

Solution. If $f(x) = \sin x$ for all $x \in \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is infinitely differentiable and $f^{(2n-1)}(x) = (-1)^{n+1} \cos x$, $f^{(2n)}(x) = (-1)^n \sin x$ for all $x \in \mathbb{R}$ and for all $n \in \mathbb{N}$. Hence the Maclaurin series for $\sin x$ is the series $\sum\limits_{n=1}^{\infty} (-1)^{n+1} \frac{x^{2n-1}}{(2n-1)!}$, where $x \in \mathbb{R}$. For $x = 0$, the Maclaurin series of $\sin x$ becomes $0 - 0 + 0 - \cdots$, which clearly converges to $\sin 0 = 0$. Let $x (\neq 0) \in \mathbb{R}$. The remainder term in the Taylor expansion of $\sin x$ about the point 0 is given by $R_n(x) = \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(c_n)$, where c_n lies between 0 and x. Since $|\sin c_n| \leq 1$ and $|\cos c_n| \leq 1$, we get $|R_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!}$. Also, since $\lim\limits_{n\to\infty} \frac{|x|^{n+2}}{(n+2)!} \cdot \frac{(n+1)!}{|x|^{n+1}} = \lim\limits_{n\to\infty} \frac{|x|}{n+2} = 0 < 1$, we get $\lim\limits_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ and hence it follows that $\lim\limits_{n\to\infty} R_n(x) = 0$. Therefore the Maclaurin series of $\sin x$ converges to $\sin x$. $\square$

Example 19. *Let $f : \mathbb{R} \to \mathbb{R}$ be defined by*

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Then f is differentiable any number of times, and that $f^{(n)}(0) = 0$ for every $n \in \mathbb{N}$. Hence, in a neighborhood of 0, the Maclaurin series of f is identically zero. But the function takes nonzero value at $x \neq 0$. Thus, the Maclaurin series of f does not converge to f. The reason is that the remainder term $R_n(c)$ does not converge to 0 for any $c \neq 0$. Therefore, an infinitely differentiable function may not have Taylor series representation.