PH 102, Electromagnetism,

Post Mid Semester

Lecture 1

Magnetostatics:

The Lorentz Force Law,
The Bio-Savart law
and
Div & Curl of Magnetostatic fields.

D. J. Griffiths: 5.1 - 5.3.2

Sovan Chakraborty, Department of Physics, IITG

- Classes will be held mostly on Wednesday-Thursday, check the tentative lecture plan.

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Division
Lec 1	05-03-2020	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 2	11-03-2020	Application of Ampere's Law, Magnetic Vector Potential	5.3, 5.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 1	12-03-2020	Lec 1		
Tut 2	17-03-2020	Lec 2		
Lec 3	18-03-2020	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic materials, magnetization	5.4, 6.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 4	19-03-2020	Field of a magnetized object, Boundary conditions	6.2, 6.3, 6.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 3	24-03-2020	Lec 3, 4		
Lec 5	25-03-2020	Ohm's law, motional emf, electromotive force	7.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 6	26-03-2020	Faraday's law, Lenz's law, Self & Mutual inductance, Energy stored in magnetic field	7.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 4	31-03-2020	Lec 5, 6		
Lec 7	01-04-2020	Maxwell's equations	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 8	02-04-2020	Discussions, Problem solving	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
	07-04-2020	Quiz II		
Lec 9	08-04-2020	Continuity equation, Poynting theorem	8.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 10	16-04-2020	Wave solution of Maxwell's equations, polarisation	9.1, 9.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 5	21-04-2020	Lec 9, 10		
Lec 11	22-04-2020	Electromagnetic waves in matter, reflection & transmission: normal incidence	9.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 12	23-04-2020	Reflection & transmission: oblique incidence	9.3, 9.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 6	28-4-2020	Lec 11, 12		
Lec 13	29-04-2020	Relativity and electromagnetism: Galilean & special relativity	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 14	30-04-2020	Discussions, problem solving	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

- Classes will be held mostly on Wednesday-Thursday, check the lecture plan.
- All relevant course materials will be uploaded in the Moodle page,
  https://intranet.iitg.ernet.in/moodle/
  Course name: PH 102 (2020)
  Enrollment key: em2020
- The details of the tutorial groups and the pre-midsem part, you may still access via,
  http://www.iitg.ac.in/phy/ph102.php.

You may reach me by email, sovan@iitg.ac.in,
My office is at the physics department.
ANEX building, room number 303A.

Strict attendance (75% with tutorial + lectures) for the Final Semester Exam

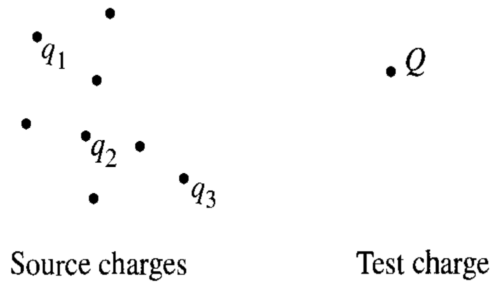
The Lorentz Force Law :

Magnetostatics

Let's go back to electrostatics:

For a collection of charges (Source) force on 'test' charge.

The source charge is at rest



$$\vec{F} = \sum_i \vec{F}_i$$

How are magnetic fields generated ??

The Lorentz Force Law :

How are magnetic fields generated and calculated ??

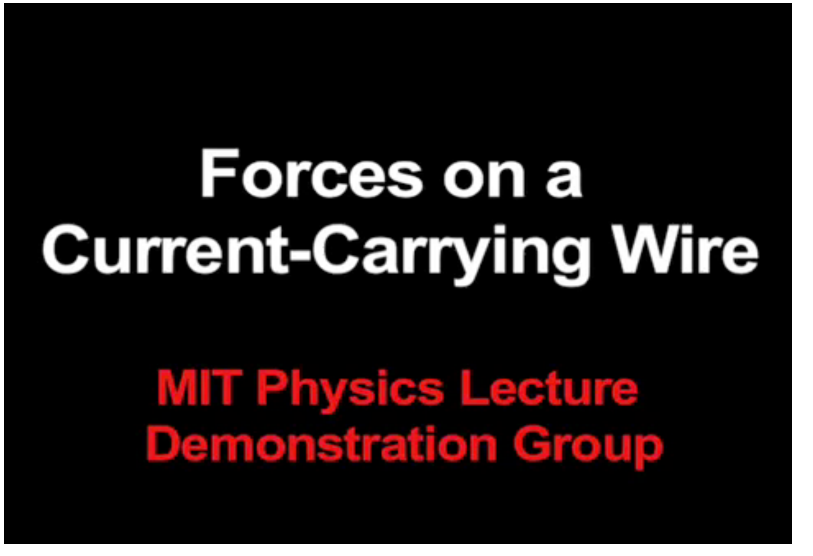
What is the force between the charges in motion??

Two current carrying wires can repel / attract each other :

Accumulation of charges can not be the reason as that would only repel,
why attract !!

Not Electrostatic in nature!

Magnetic Force



http://techtv.mit.edu/videos/813

(Bring a small compass near the current carrying wires)

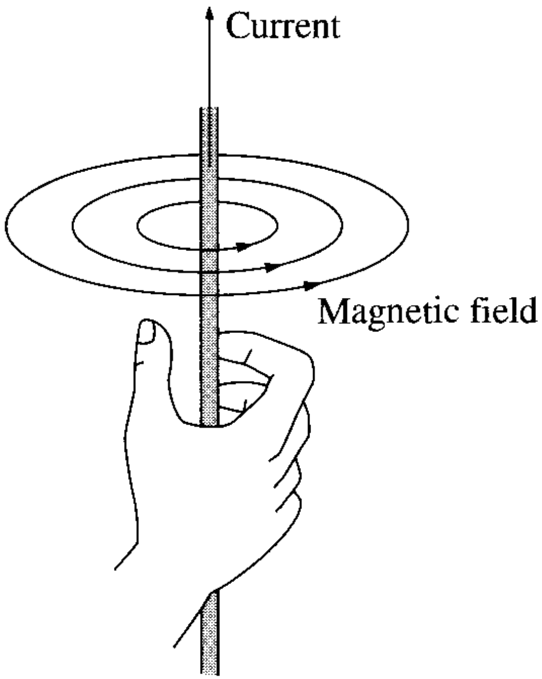
The Lorentz Force Law :

Magnetic Force due to moving charges :

Stationary charge: only electric field (***E***),

Moving charge: Also magnetic field (*B*)

However, ***B*** circles around the wire !! Thus the force is into the 'page'



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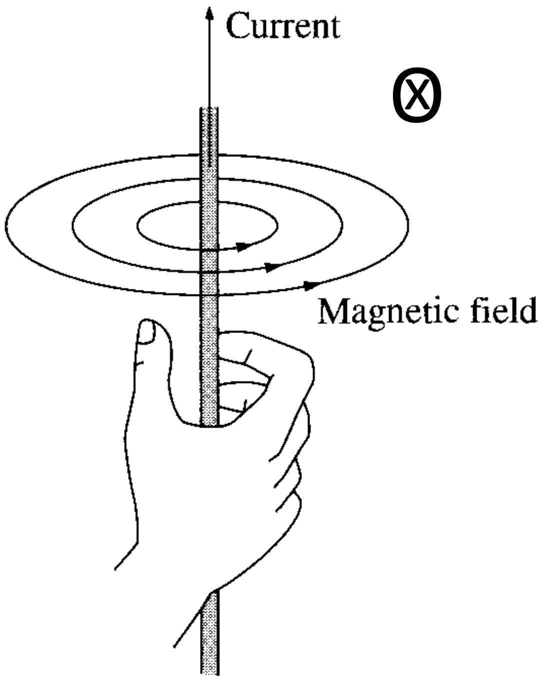
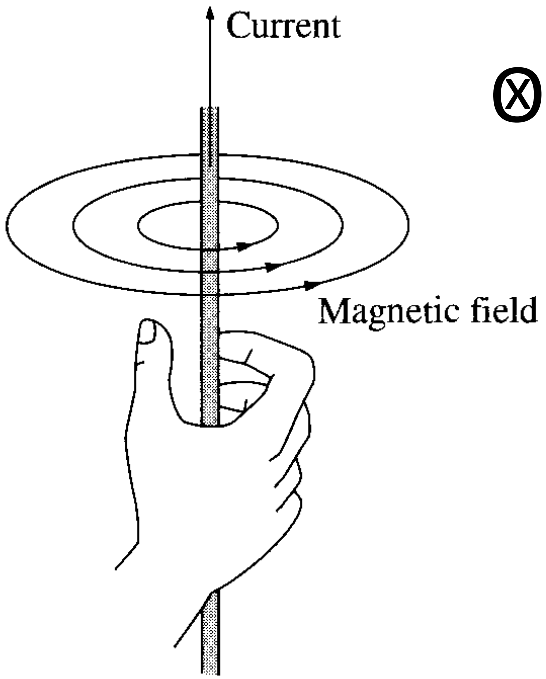


How come the force is towards Left or Right BUT not into the page



The Lorentz Force Law :

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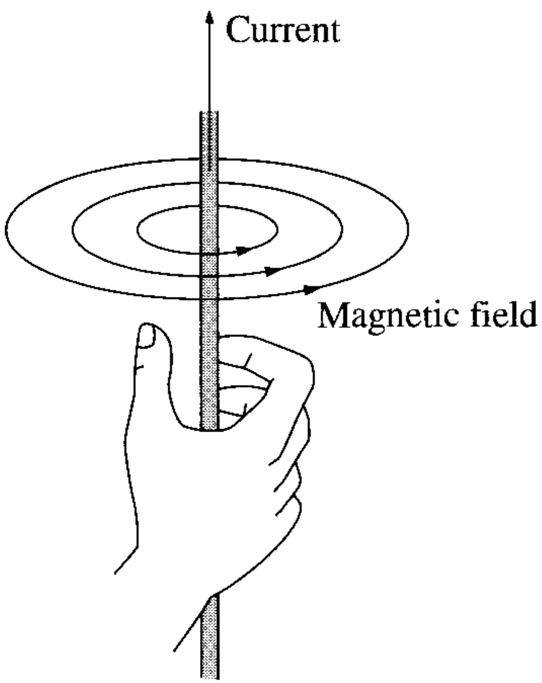
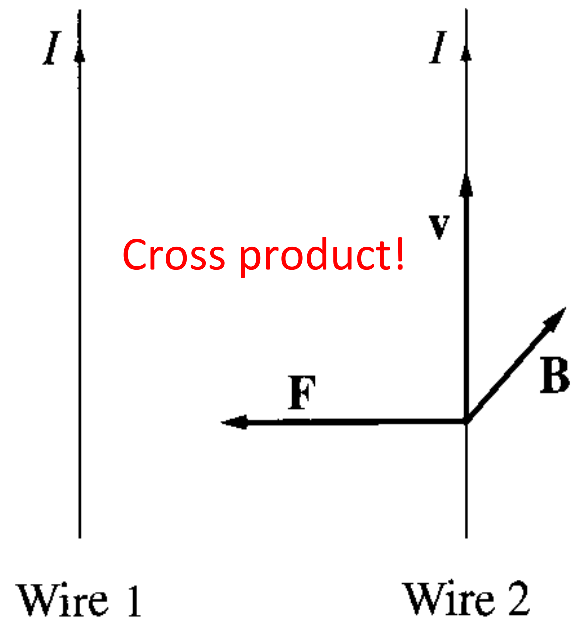


Charge moving Upwards

Magnetic field into the page

Resulting force On left! How?



Cross product!

The Lorentz Force Law :

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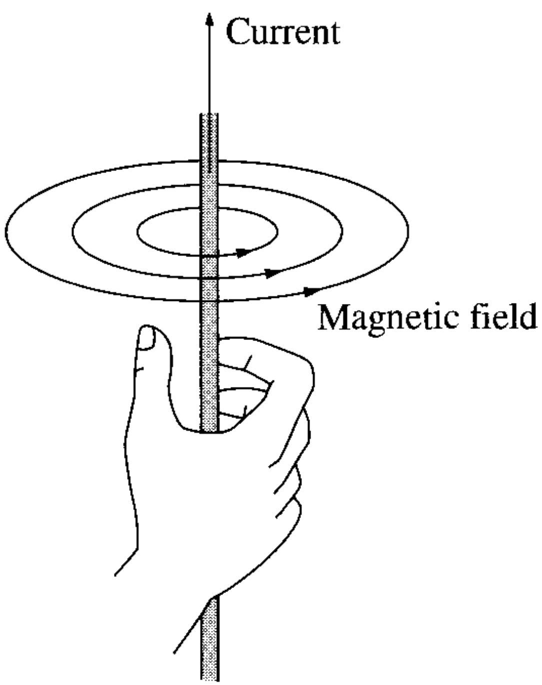
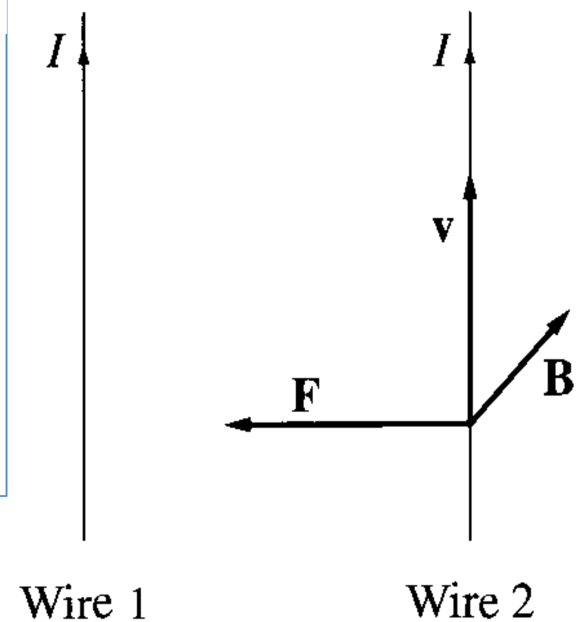


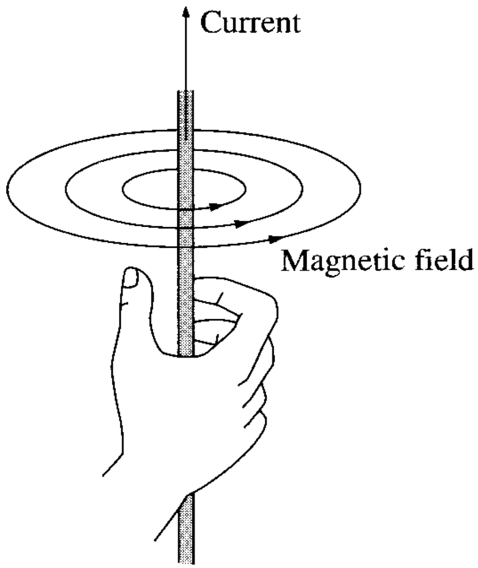
$$\mathbf{F}_{\text{mag}} = Q(\mathbf{v} \times \mathbf{B}).$$

Lorentz Force Law

Cross Product!!



The Lorentz Force Law :



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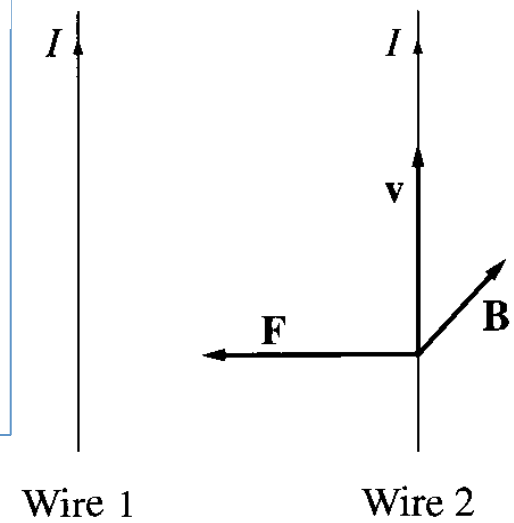
Lorentz Force Law

Cross Product!!



With Both Electric and Magnetic Field

$$\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

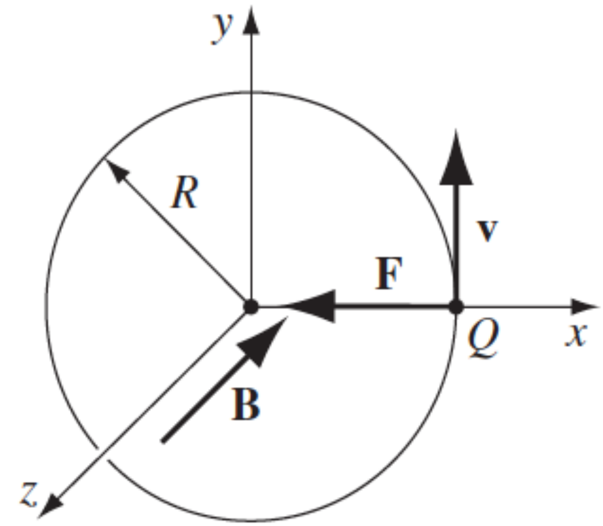
Note:

- It is not **derived** but rather an axiom
- Bizarre particle trajectory, e.g. Cyclotron, Cycloid motion.

How are magnetic fields (B) generated and calculated ??

The Lorentz Force Law :

Cyclotron motion:



Cyclotron Formula: $$QvB = m\frac{v^2}{R}, \quad \text{or} \quad p = QBR,$$

To find particle momentum: Measure B and R

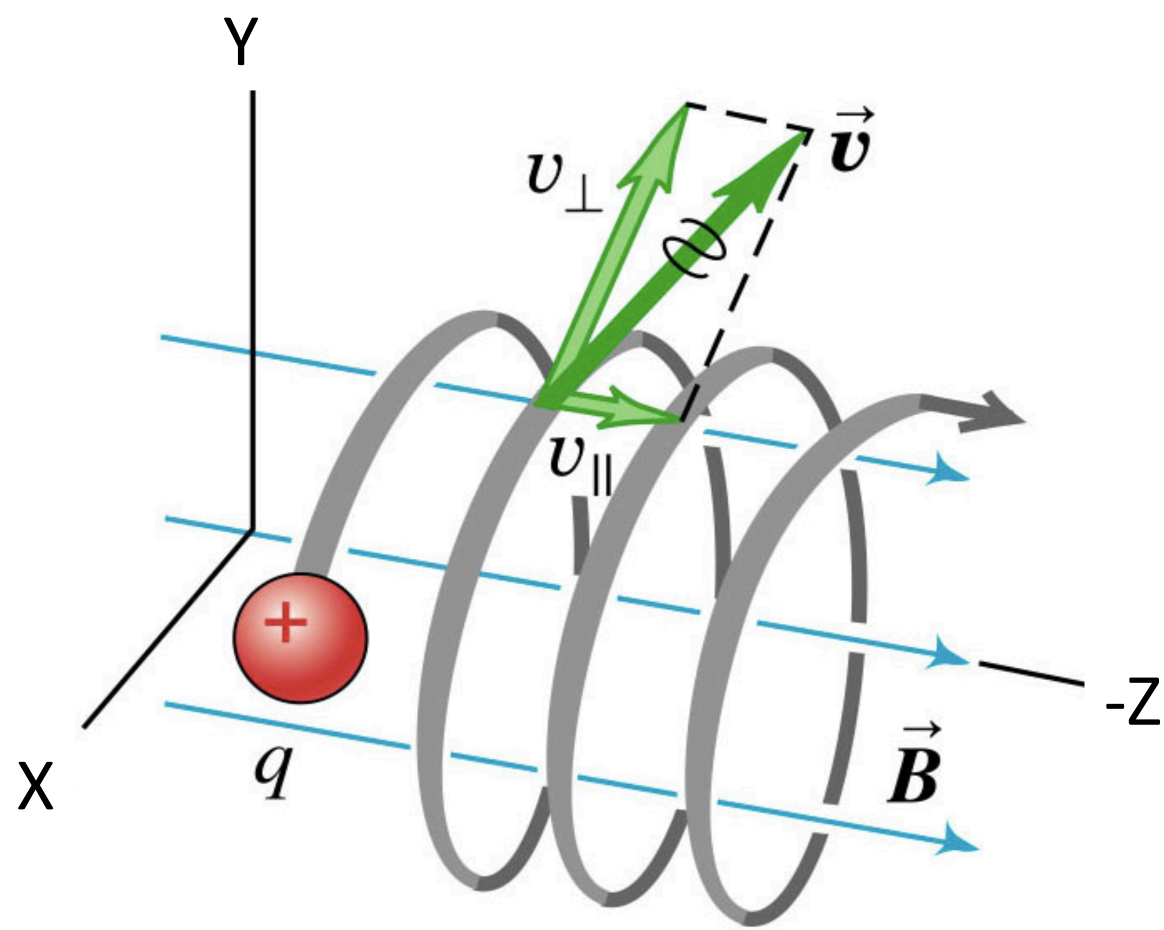
Note:
velocity is in a plane perpendicular to B

If v has component parallel to B, then $\mathbf{v}_{||} \times \mathbf{B} = 0$ ➔ motion in parallel direction is unaffected

➔ Results in Helical Motion

The Lorentz Force Law :

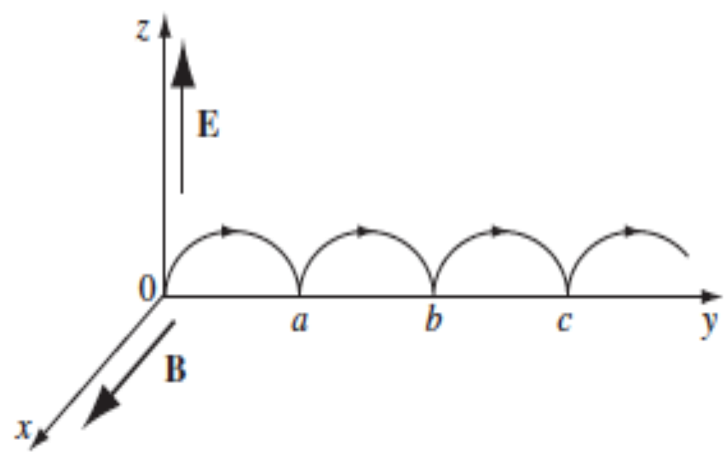
Helical Motion :



The Lorentz Force Law :

Cycloid Motion : Example 5.2, D J. G

Example 5.2. Cycloid Motion. A more exotic trajectory occurs if we include a uniform electric field, at right angles to the magnetic one. Suppose, for instance, that **B** points in the x-direction, and **E** in the z-direction, as shown in Fig. 5.7. A positive charge is released from the origin; what path will it follow?



The Lorentz Force Law :

Cycloid Motion : Example 5.2, D J. G

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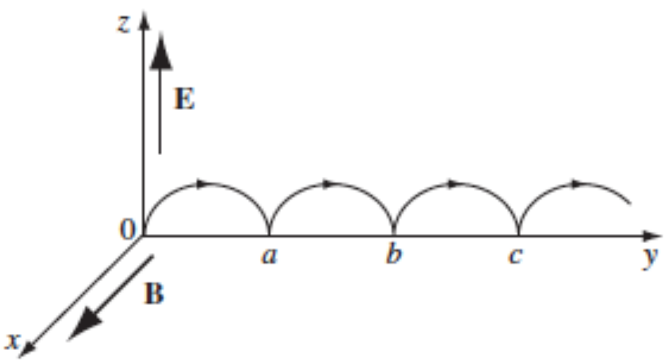

$$v(t=0) = 0;$$

$$v(t=\Delta t) = v_z = qE\Delta t$$

$$F_{mag} = qv_zB\,\hat{y},$$

$$\mathbf{v} = (0, \dot{y}, \dot{z})$$

No Force in x direction!

The Lorentz Force Law :

Cycloid Motion : Example 5.2, D J. G

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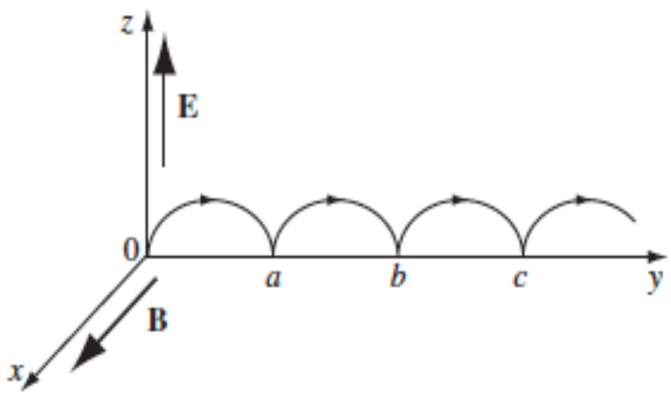


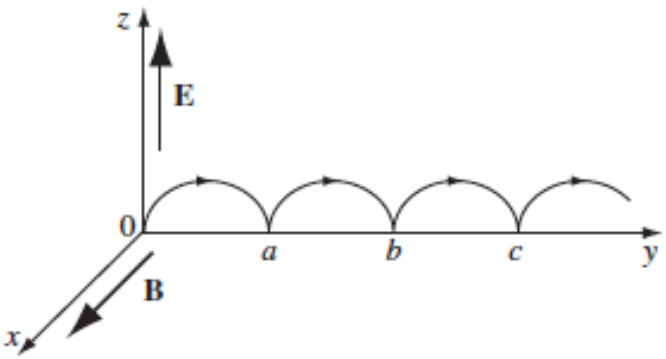
v (t = 0) = 0, v (t= Δt) = $v_z = qE\Delta t$

$\boldsymbol{F}_{\text{mag}} = qv_zB\,\hat{\boldsymbol{y}}$, $\mathbf{v} = (0, \dot{y}, \dot{z})$

Total Force,
$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = Q(E\,\hat{\mathbf{z}} + B\dot{z}\,\hat{\mathbf{y}} - B\,\dot{y}\,\hat{\mathbf{z}})$$
$$= m\mathbf{a} = m(\ddot{y}\,\hat{\mathbf{y}} + \ddot{z}\,\hat{\mathbf{z}}).$$

No Force in x direction!

The Lorentz Force Law :

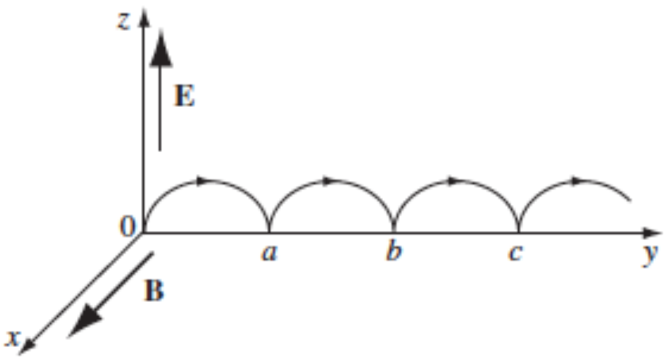


$$\mathbf{F} = Q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) = Q(E\,\hat{\mathbf{z}} + B\dot{z}\,\hat{\mathbf{y}} - B\dot{y}\,\hat{\mathbf{z}})$$
$$= m\mathbf{a} = m(\ddot{y}\,\hat{\mathbf{y}} + \ddot{z}\,\hat{\mathbf{z}}).$$

$$QB\dot{z} = m\ddot{y}, \qquad QE - QB\dot{y} = m\ddot{z}$$

$$\ddot{y} = \omega\dot{z}, \quad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right) \qquad \omega \equiv \frac{QB}{m}$$

The Lorentz Force Law :



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$$QB\dot{z} = m\ddot{y}, \qquad QE - QB\dot{y} = m\ddot{z}$$

$$\ddot{y} = \omega\dot{z}, \quad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right) \qquad \boxed{\omega \equiv \frac{QB}{m}}$$

Coupled differential equation!

[$\dddot{y} = \omega^2 (E/B - \dot{y})$ $\rightarrow$ $y(t) = C_1 \cos \omega t - C_2 \sin \omega t + (E/B)t + C_3$

$\ddot{z} = \omega^2 (C_1 \sin \omega t - C_2 \cos \omega t\,)$ $\rightarrow$ $z(t) = C_2 \cos \omega t - C_1 \sin \omega t + C_4$]

The Lorentz Force Law :

$$y(t) = C_1 \cos \omega t + C_2 \sin \omega t + (E/B)t + C_3,$$
$$z(t) = C_2 \cos \omega t - C_1 \sin \omega t + C_4.$$

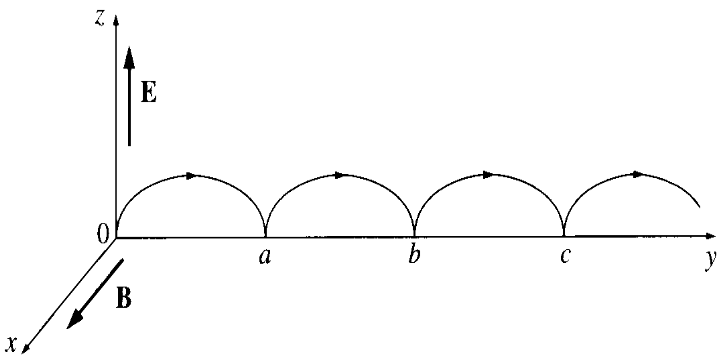


Figure 5.7

Particle started at origin at rest

$$y(0) = z(0) = 0 \quad \dot{y}(0) = \dot{z}(0) = 0$$

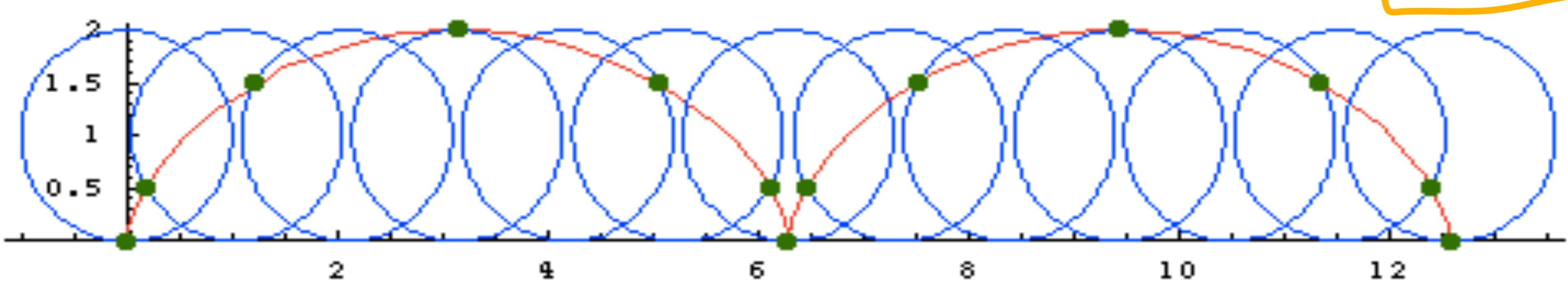
$C_1 = -C_3 = 0$, $C_2 = -C_4 = -E/(\omega B)$,

$$\boxed{y(t) = \frac{E}{\omega B}(\omega t - \sin \omega t), \quad z(t) = \frac{E}{\omega B}(1 - \cos \omega t).}$$

$$(y - R\omega t)^2 + (z - R)^2 = R^2$$

$$\boxed{R \equiv \frac{E}{\omega B}}$$

Spot on the rim of a wheel



The Lorentz Force Law :

Magnetic Forces do not work

$$d\mathbf{l} = \mathbf{v}\,dt,$$

$$dW_{\text{mag}} = \mathbf{F}_{\text{mag}} \cdot d\mathbf{l} = Q(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\,dt = 0.$$

- Magnetic forces can not speed up or slow down a particle.
- Then, who is doing the work, say in the EM cranes?

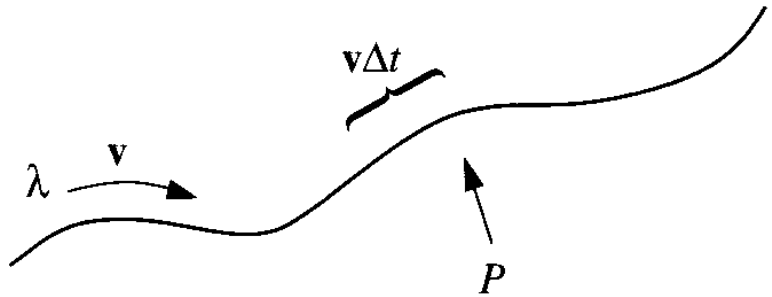
See Example. 5.3

Example 5.3. A rectangular loop of wire, supporting a mass m, hangs vertically with one end in a uniform magnetic field $\mathbf{B}$, which points into the page in the shaded region of Fig. 5.10. For what current I, in the loop, would the magnetic force upward exactly balance the gravitational force downward?

The Lorentz Force Law :

We are yet to figure out **B!**
Let us understand the moving charges bit more.

Current : Charge per unit time passing a given point



$$I = \frac{(\lambda \, \mathrm{v}\Delta t)}{\Delta t} = \lambda \, \mathrm{v}$$

$$\vec{I} = \lambda \, \vec{\mathrm{v}}$$

Amperes (A) = Coloumbs (C) / sec (s)

$$\vec{F}_{mag} = \int (\lambda \, dl)(\vec{\mathrm{v}} \times \vec{B}) = \int (\vec{I} \times \vec{B}) \, dl = \int I (d\vec{l} \times \vec{B})$$

The Lorentz Force Law :

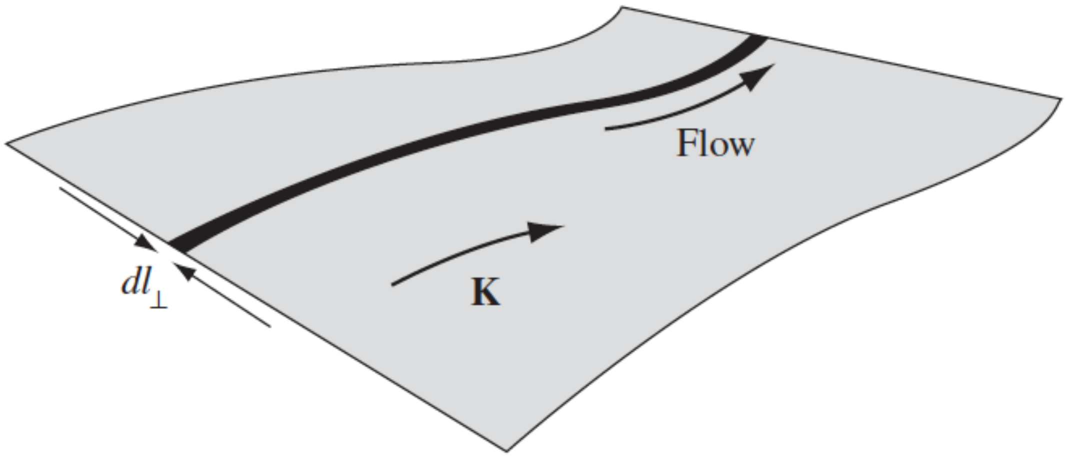
The Surface current density :

The current per unit length perpendicular to flow, $\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}}$

The magnetic force on a surface current ($\mathbf{K} = \sigma \mathbf{v}$) is,

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B}) \sigma \, da = \int (\mathbf{K} \times \mathbf{B}) \, da.$$

$$\mathbf{K} \equiv \frac{d\mathbf{I}}{dl_{\perp}}$$



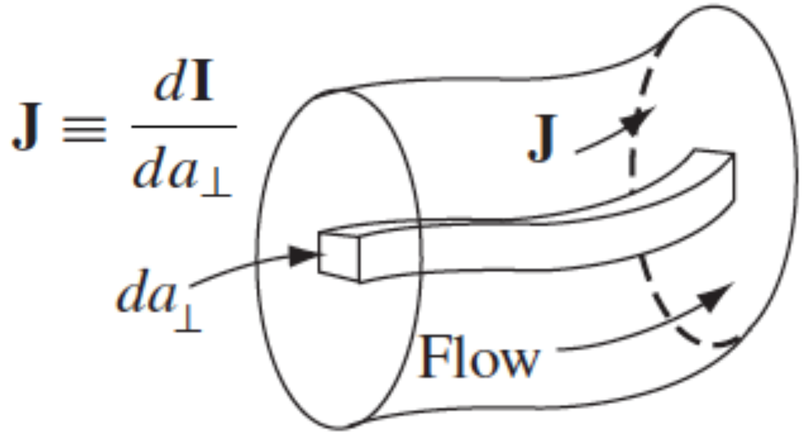
The Lorentz Force Law :

The Volume current density:

The current per unit area perpendicular to flow, $\mathbf{J} \equiv \frac{d\mathbf{I}}{da_{\perp}}$

The magnetic force on a volume current ($\mathbf{J} = \rho\mathbf{v}$) is,

$$\mathbf{F}_{\text{mag}} = \int (\mathbf{v} \times \mathbf{B})\rho \, d\tau = \int (\mathbf{J} \times \mathbf{B}) \, d\tau.$$



Continuity Equation:

Continuity Equation :

The current through the closed surface ***S*** containing volume ***V***

$$\vec{J} = \frac{d\vec{I}}{da_{\perp}} \Rightarrow \quad I = \oint_{s} \vec{J} \cdot d\vec{a}_{\perp} = \int_{V} \nabla \cdot \vec{J}\, d\tau$$

(Using Divergence theorem)

The total charge per unit time leaving the volume ***V*** is ***I***

&

Total charge being conserved, outgoing flow is at the expense of the charge remaining inside

$$\int_{V} \nabla . \mathbf{J}\, d\tau = -\frac{d}{dt}\int_{V} \rho\, d\tau = -\int_{V} \frac{d\rho}{dt} d\tau$$

$$\nabla \cdot \mathbf{J} + \frac{d\rho}{dt} = 0$$

Continuity Equation Or Local Charge Conservation

Time Independent Case

steady current

$$\frac{d\rho}{dt} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{J} = 0$$

What does it imply?

Steady Current:

How are magnetic fields generated and calculated ??

Moving charge generates magnetic field!

What are the characteristics of that magnetic field?

Why ‘magneto<u>statics</u>’??

‘Electro<u>statics</u>’ ?? Source charges at rest ! Electric fields are ……..?

‘Magneto<u>statics</u>’ ?? Steady Current (what is that?) Magnetic fields are ……..?

Steady Current:

How are magnetic fields generated and calculated ??

Moving charge generates magnetic field!

What are the characteristics of that magnetic field?

Why 'magneto<u>statics</u>'??

'Electro<u>statics</u>' ?? Source charges at rest ! Electric fields are constant in time

Magneto<u>statics</u>' ?? Steady Current! Magnetic fields are constant in time.

Steady current: Continuous flow for ever,
no change and no piling !!

Time Independent Case

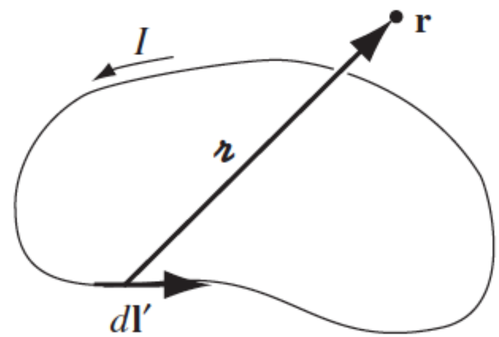
$$\frac{d\rho}{dt} = 0 \quad \Rightarrow \quad \nabla \cdot \mathbf{J} = 0$$

The Biot-Savart Law :

For a steady line current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{I} \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, dl' = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.$$

$$\mu_0 = 4\pi \times 10^{-7}\ \mathrm{N/A^2}$$



For surface currents:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{K}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, da'$$

For volume currents:

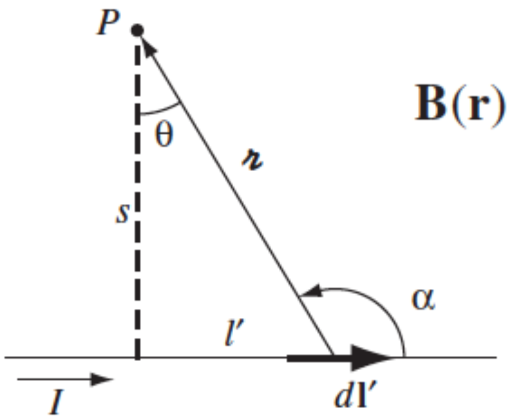
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'$$

Analogous to Coulomb's law of electrostatics

Does it apply to moving point charges? NO

The Biot-Savart Law :

Magnetic field at distance from a steady current long straight wire:



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\imath}}}{\imath^2}$$

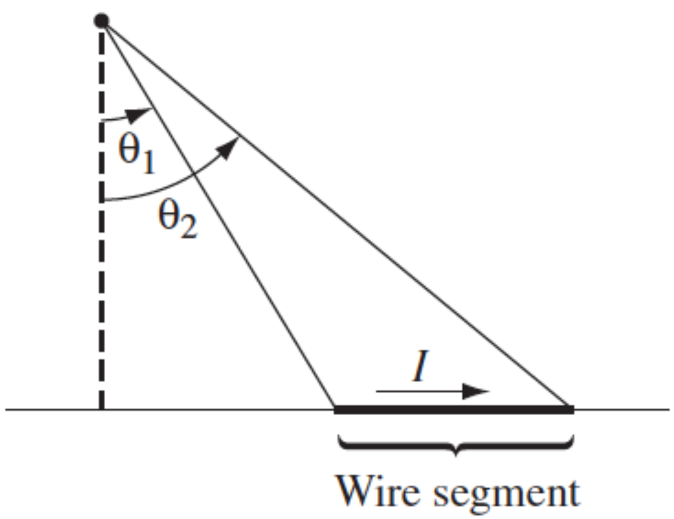
$(d\mathbf{l}' \times \hat{\boldsymbol{\imath}})$ points out of the page with magnitude, $dl' \sin\alpha = dl' \cos\theta$

Also, $l' = s\tan\theta$, hence, $dl' = \dfrac{s}{\cos^2\theta}\, d\theta$,

and $s = \imath\cos\theta$, hence, $\dfrac{1}{\imath^2} = \dfrac{\cos^2\theta}{s^2}$

$$B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\, d\theta$$

$$= \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta\, d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1).$$



The Biot-Savart Law :

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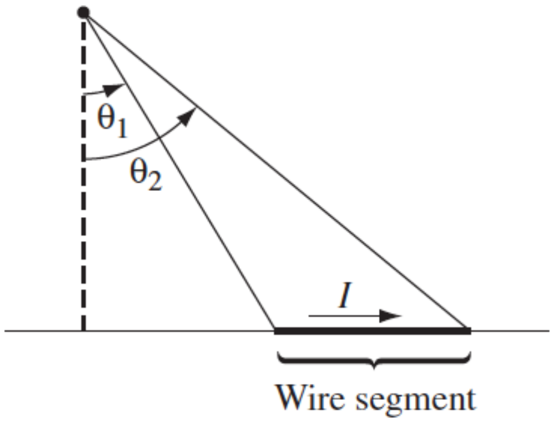


$$B = \frac{\mu_0 I}{4\pi} \int_{\theta_1}^{\theta_2} \left(\frac{\cos^2\theta}{s^2}\right)\left(\frac{s}{\cos^2\theta}\right)\cos\theta\, d\theta$$

$$= \frac{\mu_0 I}{4\pi s} \int_{\theta_1}^{\theta_2} \cos\theta\, d\theta = \frac{\mu_0 I}{4\pi s}(\sin\theta_2 - \sin\theta_1).$$

For an infinite wire,

$\theta_1 = -\pi/2$ and $\theta_2 = \pi/2$

$$B = \frac{\mu_0 I}{2\pi s}$$

As B circles around the wire,

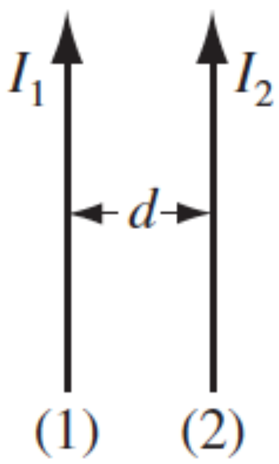
$$\boxed{\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}}.}$$

Finite segment can not support **steady** current!

Can be part of a closed circuit

The Biot-Savart Law :

Force of attraction between two long, parallel wires :



Field at (2) due to (1) $\quad B = \frac{\mu_0 I_1}{2\pi d},\quad$ into the page

Lorentz force law : force towards (1)

$$F = I_2\left(\frac{\mu_0 I_1}{2\pi d}\right)\int dl.$$

$$\boxed{\mathbf{F}_{\mathrm{mag}} = I\int (d\mathbf{l}\times\mathbf{B}).}$$

Force per unit length,

$$f = \frac{\mu_0}{2\pi}\frac{I_1 I_2}{d}.$$

The Biot-Savart Law :

Example 5.6, D J. G

Example 5.6. Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I (Fig. 5.21).

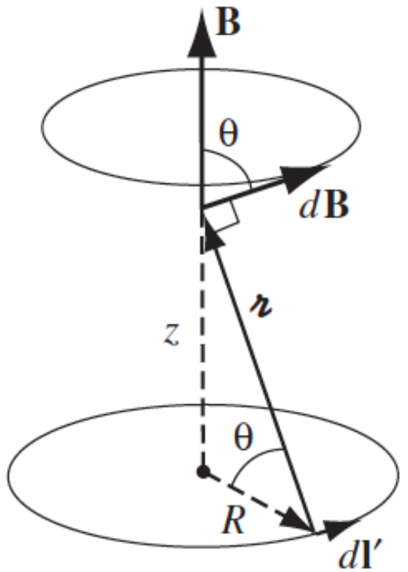


FIGURE 5.21

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

The Biot-Savart Law :

Example 5.6, D J. G

Example 5.6. Find the magnetic field a distance z above the center of a circular loop of radius R, which carries a steady current I (Fig. 5.21).

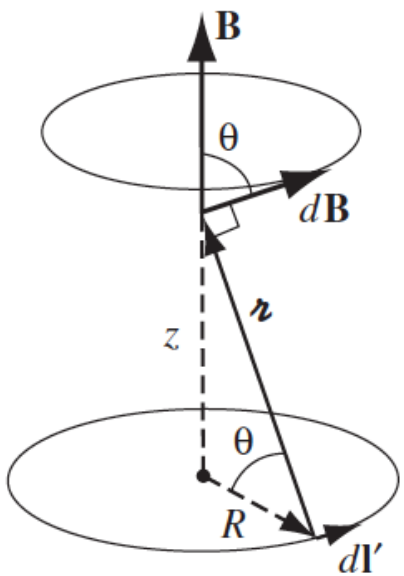


FIGURE 5.21

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\mathbf{l}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}$$

The Horizontal components from different segments cancel each other

The vertical components combine to,

$$B(z) = \frac{\mu_0}{4\pi} I \int \frac{dl'}{\mathscr{r}^2} \cos\theta.$$

$$= \frac{\mu_0 I}{4\pi} \left(\frac{\cos\theta}{\mathscr{r}^2} \right) \int dl'$$

$$= \frac{\mu_0 I}{4\pi} \left(\frac{\cos\theta}{\mathscr{r}^2} \right) 2\pi R$$

$$= \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}}.$$

So Far ....

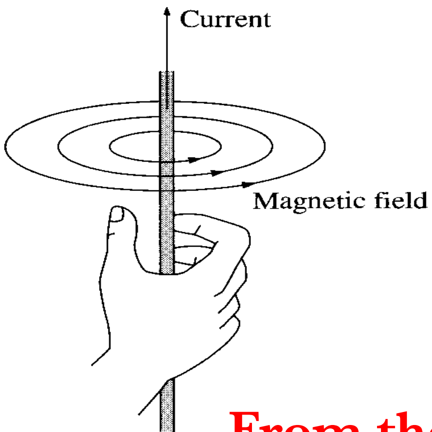
$\boxed{\text{The Lorentz Force Law}}$

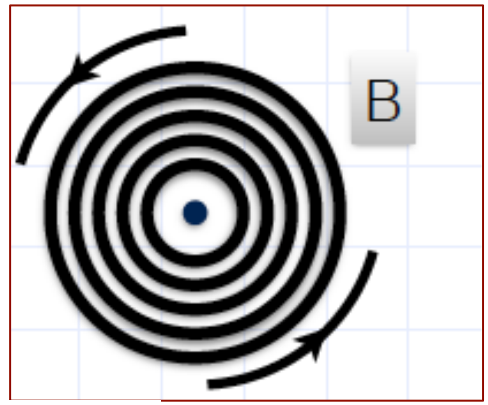
$\boxed{\text{The Biot-Savart Law}}$

For different observers, i.e. different velocities:
Do we have different magnetic fields ???

Divergence and Curl of B

For a steady line current of an infinite straight wire,





From the figure itself
Curl of *B* is non Zero

Line integral around a circular path,

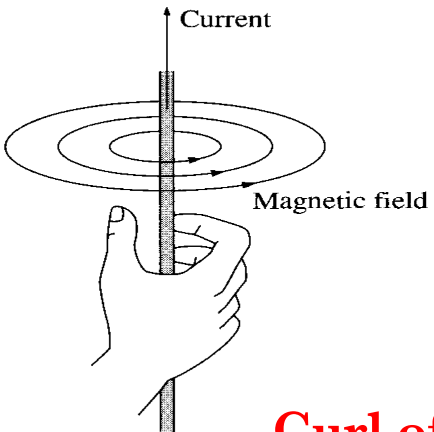
$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} \, dl$$

$$= \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I.$$

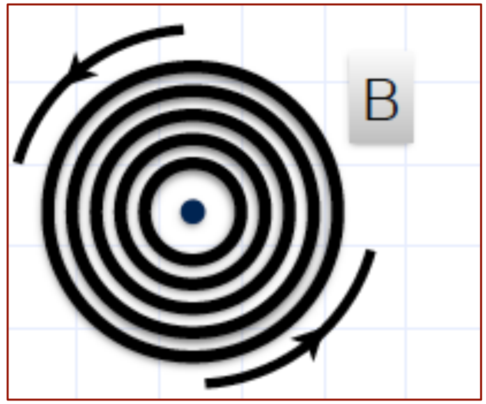
- Independent of s!
- Is it true for non-circular path as well? → yes

Divergence and Curl of B

For a steady line current



Curl of *B* is non Zero



$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s} \, dl$$

$$= \frac{\mu_0 I}{2\pi s} \oint dl = \mu_0 I.$$

Cylindrical coordinate system (s, ϕ, z)

$$\boxed{d\mathbf{l} = ds\,\hat{\mathbf{s}} + s\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}}}$$

$$\mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\boldsymbol{\phi}},$$

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 I}{2\pi} \oint \frac{1}{s} \, s \, d\phi$$

$$= \frac{\mu_0 I}{2\pi} \int_0^{2\pi} d\phi = \mu_0 I.$$

Divergence and Curl of B

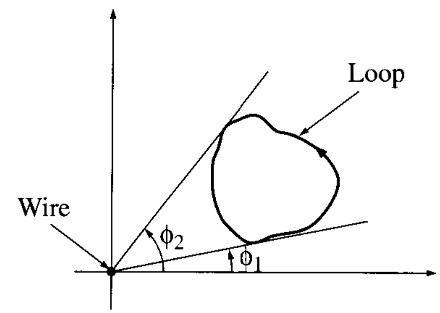
$$\oint \mathbf{B} \cdot d\mathbf{l} = \oint \frac{\mu_0 I}{2\pi s}\, dl$$

What is this closed path?

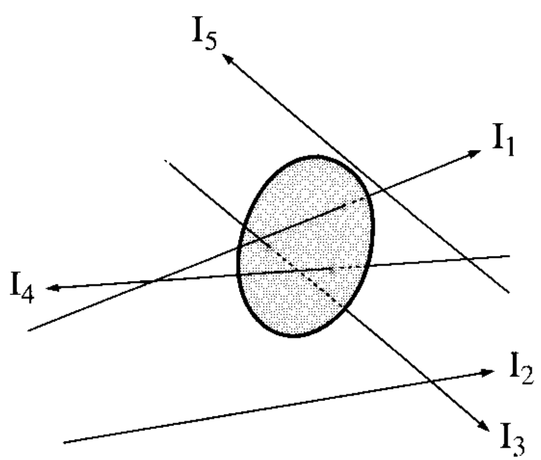
Loop enclosing wire

What if it didn't enclose the wire

$$\int_{\varphi_1}^{\varphi_2} d\varphi + \int_{\varphi_2}^{\varphi_1} d\varphi = 0$$



Bundle of straight wires : ***J*** volume current density



$$\boxed{\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}},}$$

$$\boxed{I_{\text{enc}} = \int \mathbf{J} \cdot d\mathbf{a},}$$

Using Stokes' Theorem

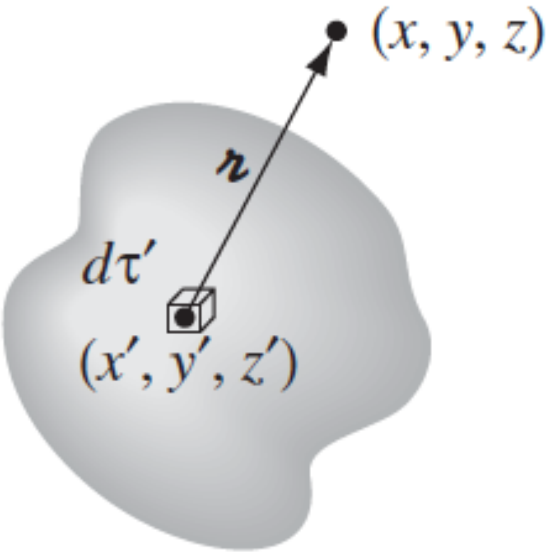
$$\int (\nabla \times \mathbf{B}) \cdot d\mathbf{a} = \mu_0 \int \mathbf{J} \cdot d\mathbf{a},$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J}.}$$

Excellent !!

Derivation, true for infinite straight wires, **ONLY**

Divergence and Curl of B



B (*x, y, z*) and ***J*** (*x′, y′, z′*)

$$\boldsymbol{\mathscr{r}} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$
$$d\tau' = dx'\,dy'\,dz'.$$

Biot-Savart Law for volume current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\, d\tau'.$$

Divergence & curl of ***B*** (*x, y, z*) should be w.r.t unprimed co-ord

Integration volume is ***dτ′***, integration over primed co-ord

$$\nabla \cdot \mathbf{B} = \frac{\mu_0}{4\pi}\int \nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) d\tau'.$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}),$$

$$\nabla \cdot \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \cdot (\nabla \times \mathbf{J}) - \mathbf{J} \cdot \left(\nabla \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right).$$

J is only primed co-ord dependent

$$\nabla \times \mathbf{J} = 0,$$

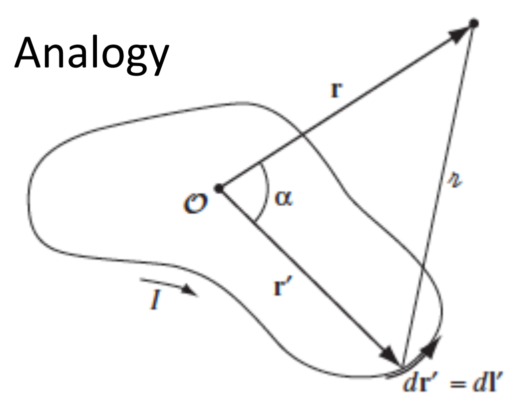
$$\boxed{\nabla \times (\hat{\boldsymbol{\mathscr{r}}}/\mathscr{r}^2) = 0}$$

Check yourself $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$

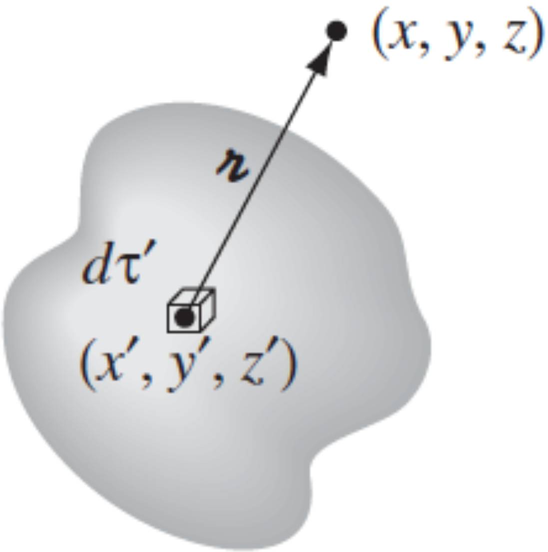
$$\boxed{\nabla \cdot \mathbf{B} = 0.}$$

Divergence of the magnetic field is zero !

Analogy



Divergence and Curl of B



B (*x, y, z*) and ***J*** (*x′, y′, z′*)

$$\boldsymbol{\mathscr{r}} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

$$d\tau' = dx'\,dy'\,dz'.$$

Biot-Savart Law for volume current

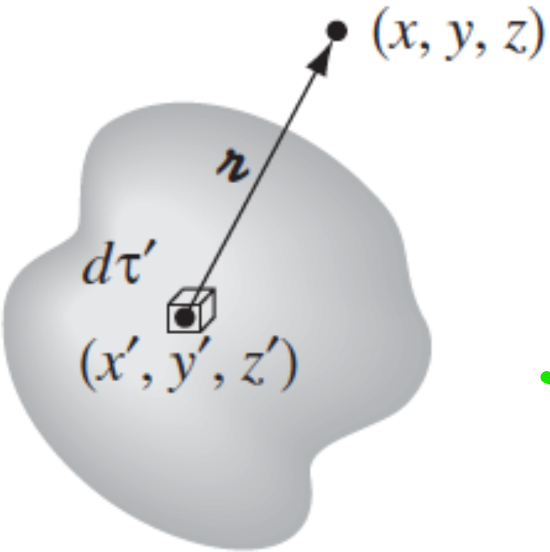
$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'.$$

Divergence & curl of ***B*** (*x, y, z*) should be w.r.t unprimed co-ord

Integration volume is ***dτ′***, integration over primed co-ord

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi}\int \nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right)d\tau'.$$

Divergence and Curl of B



B (*x, y, z*) and ***J*** (*x′, y′, z′*)

$$\boldsymbol{\mathscr{r}} = (x - x')\,\hat{\mathbf{x}} + (y - y')\,\hat{\mathbf{y}} + (z - z')\,\hat{\mathbf{z}},$$

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Biot-Savart Law for volume current

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\,d\tau'.$$

Divergence & curl of ***B*** (*x, y, z*) should be w.r.t unprimed co-ord

Integration volume is ***dτ′***, integration over primed co-ord

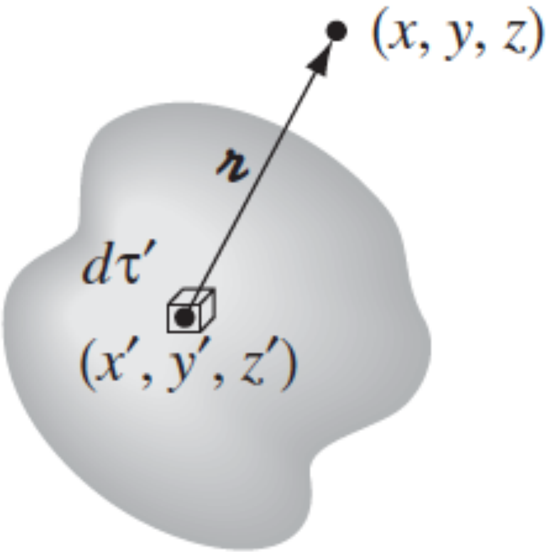
$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi}\int \nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) d\tau'.$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}).$$

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) = \mathbf{J}\left(\nabla \cdot \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}\right) - (\mathbf{J} \cdot \nabla)\frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.$$

J derivative terms are dropped ***J*** = ***J*** (*x′, y′, z′*)

Divergence and Curl of B



$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'.$$

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}.$$

The 2nd term integrates to zero!

$$-(\mathbf{J} \cdot \nabla) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} = (\mathbf{J} \cdot \nabla') \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}.$$

As $\mathscr{r}$ depends only on the co-ordinate difference

$[(\partial/\partial x) f(x - x') = -(\partial/\partial x') f(x - x').]$

The 2nd term, *x* comp:

$$(\mathbf{J} \cdot \nabla') \left(\frac{x - x'}{\mathscr{r}^3} \right) = \nabla' \cdot \left[\frac{(x - x')}{\mathscr{r}^3} \mathbf{J} \right] - \left(\frac{x - x'}{\mathscr{r}^3} \right) (\nabla' \cdot \mathbf{J})$$

$$\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot (\nabla f),$$

$$\left[-(\mathbf{J} \cdot \nabla) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right]_x = \nabla' \cdot \left[\frac{(x - x')}{\mathscr{r}^3} \mathbf{J} \right]$$

For Steady currents $\nabla' \cdot \mathbf{J} = \boldsymbol{0}$

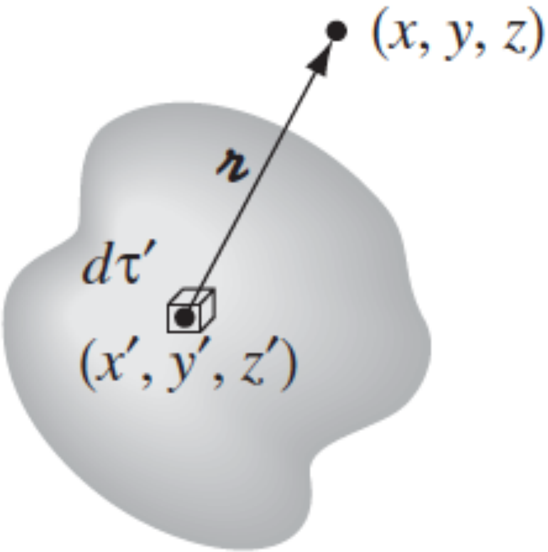
$$\int_{\mathcal{V}} \nabla' \cdot \left[\frac{(x - x')}{\mathscr{r}^3} \mathbf{J} \right] d\tau' = \oint_{\mathcal{S}} \frac{(x - x')}{\mathscr{r}^3} \mathbf{J} \cdot d\mathbf{a}' = \boldsymbol{0}$$

The integration is over $\boldsymbol{d\tau'}$

$J = 0$, on the boundary

Similarly, the other components can be also shown to be zero

Divergence and Curl of B



$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}') \times \hat{\mathscr{r}}}{\mathscr{r}^2}\, d\tau'.$$

$$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) d\tau'.$$

$$\nabla \times \left(\mathbf{J} \times \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = \mathbf{J} \left(\nabla \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) - (\mathbf{J} \cdot \nabla) \frac{\hat{\mathscr{r}}}{\mathscr{r}^2}.$$

The first term: $\nabla \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) = 4\pi \delta^3(\mathscr{r}).$

Thus in general: $$\nabla \times \mathbf{B} = \frac{\mu_0}{4\pi} \int \mathbf{J}(\mathbf{r}') 4\pi \delta^3(\mathbf{r} - \mathbf{r}')\, d\tau' = \mu_0 \mathbf{J}(\mathbf{r}),$$

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J},}$$

is the Ampere's law.

Post Mid Semester

Lecture 15

The Lorentz Force Law

The Biot-Savart Law

Divergence and Curl of B

$$\boxed{\nabla \times \mathbf{B} = \mu_0 \mathbf{J},} \qquad \boxed{\nabla \cdot \mathbf{B} = 0.}$$