PH 102 : Electromagnetism
Tutorial 4
31/03/20

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1. A rectangular loop of wire is situated so that one end (height h) is between the plates of a parallel plate capacitor (shown in figure 1), oriented parallel to the field $\vec{E}$. The other end is way outside, where the field is essentially zero. What is the emf in this loop? If the total resistance is R, what current flows? Explain.

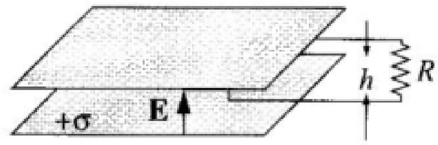

Figure 1: Figure for take home problem 1.

Solution:
For all electrostatic fields
$$\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = 0.$$
However, the electric field in between the two plates of a parallel plate capacitor is $E = \sigma/\epsilon_0$ where σ is the surface charge density of the plates. Taking the field outside to be exactly zero, the line integral would look like $\mathcal{E} = \oint \vec{E} \cdot d\vec{l} = (\sigma/\epsilon_0)h$. However, the field outside the plates is never exactly zero, there always exists some fringing field a the edges (figure 4.31, Introduction to Electrodynamics, D J Griffiths) that makes sure that the non-zero contribution from the left edge of the loop to the line integral gets cancelled out. This is consistent with the fact $\oint \vec{E} \cdot d\vec{l} = 0$ for electrostatics. Therefore, the emf in the loop and hence the current through the resistor R are zero.

2. A square loop is cut out of a thick sheet of aluminium. It is then placed so that the top portion is in a uniform magnetic field $\vec{B}$, and allowed to fall under gravity (shown in figure 2 where the shading indicates the field region and $\vec{B}$ points into the page). If the magnetic field is 1 T, find the terminal velocity of the loop (in m/s). Find the velocity of the loop as a function of time. How long does it take (in seconds) to reach, say, 90% of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit? Write your final answer in numbers by using acceleration due to gravity $g = 9.8$ m/s^2, mass density of aluminium $\eta = 2.7 \times 10^3$ kg/m^3, resistivity of aluminium $\rho = 2.8 \times 10^{-8}$ Ωm.

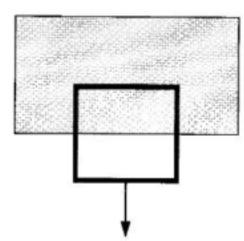
Figure 2: Figure for take home problem 2.

Solution:

The magnitude of the induced emf is

$$\mathcal{E} = Bl\frac{dx}{dt} = Blv = IR.$$

The induced current is, therefore, given by $I = Blv/R$. The current flows in such a direction that opposes the motion of the loop downward (Lenz's law). The upward force on the loop is therefore $F = IlB = B^2l^2v/R$. Since this force opposes the gravitational force downward, applying Newton's law for the loop we get

$$mg - \frac{B^2l^2}{R}v = m\frac{dv}{dt} \implies \frac{dv}{dt} = g - \alpha v, \quad \alpha = \frac{B^2l^2}{mR}.$$

The terminal velocity (v_t) can be found by equating $dv/dt = 0$:

$$g - \alpha v_t = 0 \implies v_t = \frac{g}{\alpha} = \frac{mgR}{B^2l^2}.$$

The velocity as a function of time can be found by solving the above differential equation:

$$\frac{dv}{dt} = g - \alpha v \implies \frac{dv}{g - \alpha v} = dt \implies -\frac{1}{\alpha}\ln(g - \alpha v) = t + C, \quad C \equiv \text{Constant}$$

$$\implies g - \alpha v = Ae^{-\alpha t}$$

where the constant C is absorbed in A. Since the loop started from rest that is at $t = 0$, $v = 0$, we have $A = g$. Thus the velocity as a function of time is

$$v = \frac{g}{\alpha}(1 - e^{-\alpha t}) = v_t(1 - e^{-\alpha t}).$$

For 90% of terminal velocity

$$v/v_t = 0.9 = 1 - e^{-\alpha t} \implies e^{-\alpha t} = 1 - 0.9 = 0.1$$

$$\implies \ln(0.1) = -\alpha t \implies \alpha t = \ln 10 \implies t = \frac{1}{\alpha}\ln 10$$

$$\implies t_{90\%} = \frac{v_t}{g}\ln 10.$$

If A is the cross sectional area of the aluminium sheet, l is the length of each side of the square loop, the mass of the loop is $m = 4\eta Al$. The resistance of the loop is $R = 4l\rho/A$. Using these in the expression for terminal velocity

$$v_t = \frac{g}{B^2l^2}4\eta Al\frac{4l\rho}{A} = \frac{16g\eta\rho}{B^2}.$$

Using the given numerical values, we get $v_t = 1.2$ cm/s. Also $t_{90\%} = 2.8$ ms.

If the loop is cut to break the circuit, there will not be any induced current and hence no upward force opposing the fall. The loop will then fall freely with acceleration g.

3. Find the energy stored in a section of length l of a long solenoid (radius R, current I, n turns per unit length) using the following formulas discussed in the class:
(a) $W = \frac{1}{2}LI^2$ where L is the inductance.
(b) $W = \frac{1}{2}\oint(\vec{A}\cdot\vec{I})dl$ where $\vec{A}$ is the magnetic vector potential.
(c) $W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau$.
(d) $W = \frac{1}{2\mu_0}\left[\int_V B^2 d\tau - \oint_S(\vec{A}\times\vec{B})\cdot d\vec{a}\right]$, where S is the surface bounding the volume V. Take as your volume the cylindrical tube from radius $a < R$ out to radius $b > R$.

Solution:
(a) The magnetic field inside a solenoid is $B = \mu_0 nI$. The magnetic flux through a single turn is $\Phi_1 = \mu_0 nI\pi R^2$. For length l, there are nl number of turns and hence the total flux is $\Phi = \mu_0 n^2\pi R^2 Il$. Since the flux is related to the self-inductance as $\Phi = LI$, therefore, $L = \mu_0 n^2\pi R^2 l$. The energy stored is

$$W = \frac{1}{2}LI^2 = \frac{1}{2}\mu_0 n^2\pi R^2 lI^2.$$

(b) The vector potential inside a solenoid can be calculated as

$$\oint \vec{A}\cdot d\vec{l} = A(2\pi r) = \int \vec{B}\cdot d\vec{a} = \mu_0 nI(\pi r^2)$$

$$\implies \vec{A} = \frac{\mu_0 nI}{2}r\hat{\phi}$$

At the surface of the solenoid, $\vec{A} = \frac{\mu_0 nI}{2}R\hat{\phi}$. The energy stored for one turn is

$$W_1 = \frac{1}{2}\oint(\vec{A}\cdot\vec{I})dl = \frac{1}{2}\frac{\mu_0 nI}{2}RI(2\pi R).$$

For length l, there are nl number of turns and hence the stored energy is

$$W = \frac{1}{2}\frac{\mu_0 nI}{2}RI(2\pi R)(nl) = \frac{1}{2}\mu_0 n^2\pi R^2 lI^2.$$

(c) The magnetic field inside is $B = \mu_0 nI$ and zero outside. The stored energy is

$$W = \frac{1}{2\mu_0}\int_{\text{all space}} B^2 d\tau = \frac{1}{2\mu_0}(\mu_0 nI)^2\int d\tau = \frac{1}{2\mu_0}(\mu_0 nI)^2(\pi R^2 l) = \frac{1}{2}\mu_0 n^2\pi R^2 lI^2.$$

(d) $W = \frac{1}{2\mu_0}\left[\int_V B^2 d\tau - \oint_S(\vec{A}\times\vec{B})\cdot d\vec{a}\right]$ where S is the surface bounding the volume V. Taking the volume as the cylindrical tube from radius $a < R$ out to radius $b > R$

and noting that $B = 0$ outside the solenoid, we have

$$\int_V B^2 d\tau = \mu_0^2 n^2 I^2 \pi (R^2 - a^2) l.$$

For the surface enclosing this volume $\vec{A} \times \vec{B} = 0$ at $s = b$ surface lying outside the solenoid. On the inside surface at $s = a$, the vector potential and magnetic field are $\vec{A} = \frac{\mu_0 n I}{2} a\hat{\phi}, \vec{B} = \mu_0 n I \hat{z}$. Therefore, $\vec{A} \times \vec{B} = \frac{1}{2}\mu_0^2 n^2 I^2 a(\hat{\phi} \times \hat{z}) = \frac{1}{2}\mu_0^2 n^2 I^2 a\hat{s}$. The surface integral of this will get contribution only from the inner surface at $s = a$ with area vector pointing in the $-\hat{s}$ direction. Therefore,

$$\oint (\vec{A} \times \vec{B}) \cdot d\vec{a} = \int \left(\frac{1}{2}\mu_0^2 n^2 I^2 a\hat{s}\right) [ad\phi dz(-\hat{s})] = -\frac{1}{2}\mu_0^2 n^2 I^2 a^2 (2\pi l).$$

The stored energy is therefore,

$$W = \frac{1}{2\mu_0}\left[\mu_0^2 n^2 I^2 \pi (R^2 - a^2) l + \mu_0^2 n^2 I^2 a^2 \pi l\right] = \frac{1}{2}\mu_0 n^2 I^2 R^2 \pi l.$$

4. A circular loop of radius a is at a distance D above a tiny magnetic dipole of infinitesimal area dS carrying a current I_1, as shown in figure 3. Assume current through the circular loop $I_2 = 0$, for the time being. Also, the distance D and loop radius a are related as $D = \sqrt{3}a$. Write your final answers only in terms of I_1, dS, a and fundamental constants.

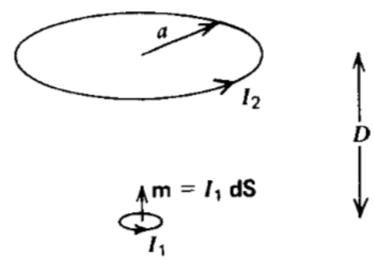

Figure 3: Figure for problem 4

(i) How much magnetic flux of the dipole passes through the circular loop?
(ii) What is the mutual inductance between the dipole and the loop?
(iii) Now, consider the loop to be carrying a current $I_2 \neq 0$. The relation between D and a remains same as before $D = \sqrt{3}a$. How much magnetic flux due to I_2 passes through the magnetic dipole? What is the mutual inductance between the loop and the dipole in this case?

Solution:
(i) Magnetic vector potential due to the dipole is $\vec{A} = \frac{\mu_0}{32\pi}\frac{I_1 dS}{a^2}\hat{\phi}$ (See tutorial 3). Now, magnetic flux can be calculated using

$$\oint \vec{A} \cdot \vec{dl} = \int \vec{B} \cdot \vec{da} = \Phi$$

$$\implies A(2\pi a) = \Phi$$

$$\implies \Phi = \frac{\mu_0}{32\pi}\frac{I_1 dS}{a^2}(2\pi a) = \frac{\mu_0 I_1 dS}{16a}$$

(ii) Mutual inductance $M = \frac{\Phi}{I_1} = \frac{\mu_0 dS}{16a}$.

(iii) If the loop carries a current $I_2 \neq 0$, the magnetic field at its axis and at the location of the tiny loop can be found to be (see tutorial 3) $\vec{B} = \frac{\mu_0 I_2}{16a}\hat{z}$. Assuming the field to remain same throughout the area of the tiny dipole, the magnetic flux can be calculated to be

$$\Phi = \int \vec{B} \cdot \vec{da} = \frac{\mu_0 I_2 dS}{16a}$$

Mutual inductance is $M = \frac{\Phi}{I_2} = \frac{\mu_0 dS}{16a}$.

5. Replace the tiny magnetic dipole in above problem by a loop L_1 with no current. It is now placed on top (at the axis which coincides with z-axis) of a loop L_2 of radius a, as shown in figure 4. If the loop L_1 is made up from a paramagnetic material, which direction it will move once a current I_2 is sent through loop L_2 in anticlockwise direction?

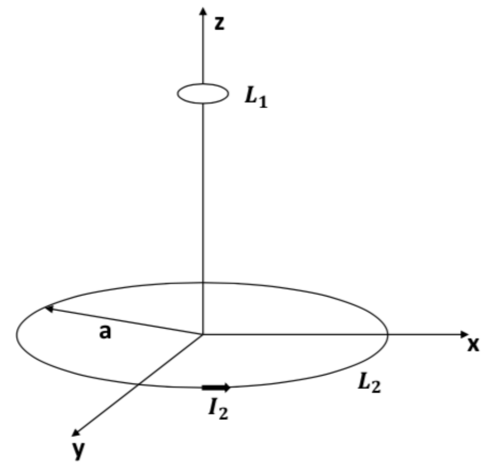

Figure 4: Figure for problem 5

Solution:
$-z$ direction.

6. Consider a rectangular loop of wire of width a, length b, rotating with an angular velocity ω about the axis PQ (dashed line in figure 5) and lying in a uniform, time dependent magnetic field $B = B_0 \sin \omega t$ perpendicular to the plane of the loop at $t = 0$. Find the angular frequency at which the induced emf of the loop alternates.

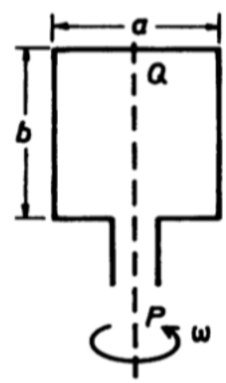

Figure 5: Figure for problem 6

Solution:

Flux through the loop is

$$\Phi = \int \vec{B} \cdot \vec{da} = B_0 \sin \omega t (ab) \cos \omega t = \frac{1}{2} B_0 ab \sin 2\omega t$$

Induced emf is

$$\epsilon = -\frac{d\Phi}{dt} = -B_0 ab\omega \cos 2\omega t$$

Thus, emf alternates at angular frequency 2ω.