Physics II (PH 102)

Electromagnetism (Lecture 13)

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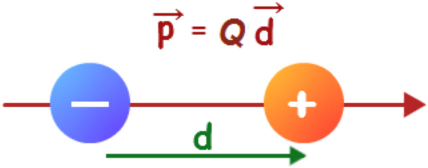
Dielectric Materials

- *Dielectrics*, like parafin, wax, glass, mica, etc. are insulators that have no free mobile charge carriers. The charges are rather bound to their respective atoms & molecules.
- In absence of external fields, the positive and negative charge centers overlap, so thay they remain UNPOLARIZED.
- In applied external electric fields, the positive and negative charge centers suffer minute displacements in opposite directions about their equilibrium positions which are of the order of *atomic dimensions* . The dielectric material is then said to become ELECTRICALLY POLARIZED.
- Such atoms/molecules with *displaced equal and opposite charge centers* behave as ELECTRIC DIPOLES .



Dielectrics: 2 Properties of Atomic & Molecular Dipoles

Definition

ELECTRIC DIPOLE MOMENT: It refers to a vector quantity, usually denoted by $\mathbf{p}$, and given by the product of the magnitude of one of either charges Q and the separation/displacement vector $\mathbf{d}$ between them. The sense of the vector is conventionally given by the direction from $-Q$ to $+Q$. (SI Unit: $\mathrm{C{\cdot}m}$)

$$\mathbf{p} = Q\mathbf{d}.$$

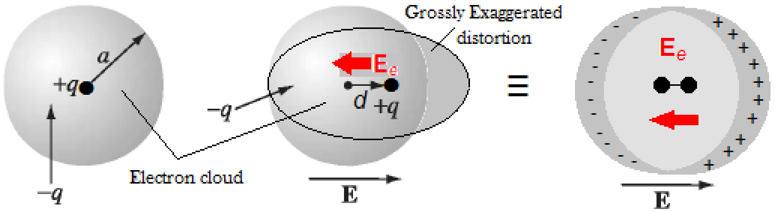
Definition

ELECTRIC POLARIZABILITY: The electric polarizability α of an atomic or a molecular dipole is a measure of its degree of polarization. It is naively defined as the ratio of the induced dipole moment $\mathbf{p}$ to the external electric field $\mathbf{E}_{\mathrm{ext}}$ that produces its dipole moment. (SI Unit: $\mathrm{C^2{\cdot}m/N}$)

$$\mathbf{p} = \alpha\mathbf{E}_{\mathrm{ext}}$$

Atomic Polarization

Crude Atomic Model: A neutral atom consisting of a nucleus of charge $+q$ surrounded by a uniformly charged spherical cloud of charge $-q$ and radius a, in the absence of external fields.



The displacement $\mathbf{d}$ will be such that the internal induced electric field $\mathbf{E}_e$ of the electron cloud will balance the external electric field $\mathbf{E}$ under equilibrium.

Recall Problem 4.03 (Tutorial 4): The disorted atom is equivalent to the intersection of two equal and opposite uniformly charged spheres. Assuming negligible distortion from spherical shape under equilibrium $\rho \approx q/\left(\frac{4}{3}\pi a^3\right)$

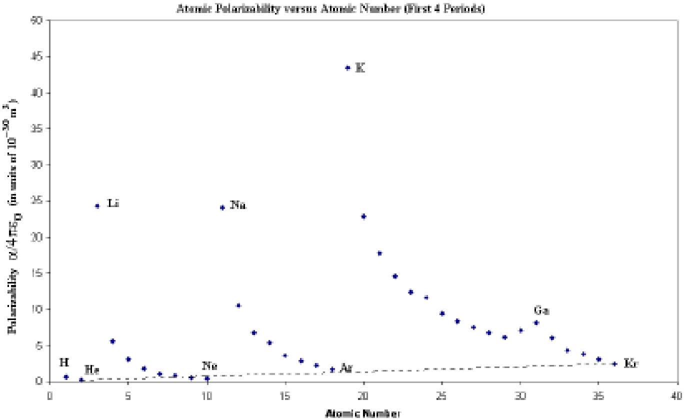
$$\mathbf{E}_e = \mathbf{E}_- + \mathbf{E}_+ \;=\; -\frac{\rho}{3\epsilon_0}\mathbf{r}_- + \frac{\rho}{3\epsilon_0}\mathbf{r}_+ = -\frac{q}{4\pi a^3\epsilon_0}(\mathbf{r}_- - \mathbf{r}_+) = -\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}}{a^3} \equiv -\mathbf{E}$$

$$\mathbf{p} = q(\mathbf{r}_- - \mathbf{r}_+) \;=\; q\mathbf{d} = \left(4\pi a^3\epsilon_0\right)\mathbf{E} \equiv \alpha\mathbf{E} \quad (\alpha \rightarrow \text{atomic polarizability})$$

$$\alpha \;=\; 4\pi\epsilon_0 a^3 = 3\epsilon_0 \times \text{atomic volume.}$$

Atomic Polarizabilities

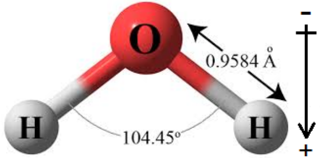
- Atomic polarizabilty indicates a measure of the ease with which an atomic electron cloud can be distorted.

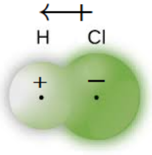


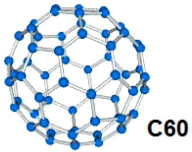
- Alkali atoms with very loosely bound electron clouds have the largest polarizabilities.
- Noble atoms have very tightly bound compact cores of electron clouds with smallest atomic polarizabilities.

Molecular Polarization: Polar and Non-Polar Molecules

Symmetrical and Asymmetrical types of molecules:







- NON-POLAR MOLECULES: *Spherically-Symmetrical* (like Fullerenes), or *Linear-Symmetrical* (like $\mathrm{CO_2}$) Molecules, have no *intrinsic dipole moment* since the positive and negative charge centers coincide. Dipole moments are *induced* only by external electric fields.

  $\Longrightarrow$ Dielectrics made of *non-polar molecues* exhibits no polarization without external applied fields. In the presence of applied fields $\mathbf{E} \neq 0$, they attain *weak polarizations*.

- POLAR MOLECULES: *Linear-Asymmetric* Molecules (like HCl), or *Triangular* Molecules (like $\mathrm{H_2O}$), with distinct centers of positive and negative charges have *permanent* or *intrinsic dipole moments*.

  $\Longrightarrow$ Dielectrics composed of *polar molecules* still do not exhibit polarization without external fields as the dipoles remain randomly oriented yielding a net zero polarization. For $\mathbf{E} \neq 0$, these materials get *strongly polarized*.

Molecular Polarizibility

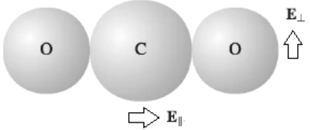
In general for molecules the situation is more complicated due to the presence of different axes of symmetry or even completely asymmetrical. They polarize more readily in some directions than others. In such *anisotropic* cases, the polarizibilty is NOT a simple scalar. Instead the dipole moment $\mathbf{p} = (p_x, p_y, p_z)$ is related to $\mathbf{E} = (E_x, E_y, E_z)$ through the *tensorial* relationship $\mathbf{p} \equiv \hat{\alpha} \cdot \mathbf{E}$, i.e.,

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

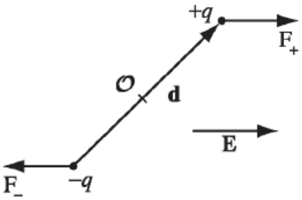
where $\hat{\alpha}$ is the **Polarizability Tensor** of the molecule.

Example

A linear molecule like, $\mathrm{CO_2}$ has a polarizability $\alpha_{||} = 4.5 \times 10^{-40}\mathbf{C^2{\cdot}m/N}$ when field is applied along the axis of symmetry of the molecule, and $\alpha_{\perp} = 2 \times 10^{-40}\mathbf{C^2{\cdot}m/N}$ when applied in the perpendicular direction. For such linear molecules, an applied field in a general direction <u>induces</u> a dipole moment given by the linear relation: $\mathbf{p} = \alpha_{||}\mathbf{E}_{||} + \alpha_{\perp}\mathbf{E}_{\perp}$.



Forces & Torques on Dipoles: Uniform Fields



- In an UNIFORM external field $\mathbf{E}$, the total force on a dipole is zero:

$$\mathbf{F}_{Total} = \mathbf{F}_{-} + \mathbf{F}_{+} = -q\mathbf{E} + q\mathbf{E} = 0.$$

- In an UNIFORM external field $\mathbf{E}$ there is a net non-zero torque that tends to allign the dipole $\mathbf{p} = q\mathbf{d}$ along the $\mathbf{E}$ direction:

$$\begin{aligned}\boldsymbol{\tau}_{Total} &= \boldsymbol{\tau}_{-} + \boldsymbol{\tau}_{+} = (\mathbf{r}_{-} \times \mathbf{F}_{-}) + (\mathbf{r}_{+} \times \mathbf{F}_{+}) \\ &= [(-\mathbf{d}/2) \times (-q\mathbf{E})] + [(\mathbf{d}/2) \times (q\mathbf{E})] = (q\mathbf{d}) \times \mathbf{E} = \mathbf{p} \times \mathbf{E} \rightarrow \text{const.}\end{aligned}$$

- What happens in a NON-UNIFORM Electric Field $\mathbf{E} = \mathbf{E}(\mathbf{r})$?

Forces & Torques on Dipoles: Non-uniform Fields

- In NON-UNIFORM external field $\mathbf{E} = \mathbf{E}(\mathbf{r})$, the net force is non-zero. If the center of the dipole be

$$\mathbf{r} = (\mathbf{r}_+ + \mathbf{r}_-)/2,$$

and the displacement vector is

$$\mathbf{d} \equiv \Delta\mathbf{r} = \mathbf{r}_+ - \mathbf{r}_- .$$

The total force on the dipole is given by

$$\begin{aligned}
\mathbf{F}_{Total}(\mathbf{r}) &= \mathbf{F}_+(\mathbf{r}_+) + \mathbf{F}_-(\mathbf{r}_-) = q\mathbf{E}_+(\mathbf{r}_+) - q\mathbf{E}_-(\mathbf{r}_-) \equiv q\Delta\mathbf{E}(\mathbf{r}) \neq 0 \\
&= q\left(\Delta E_x \hat{\mathbf{i}} + \Delta E_y \hat{\mathbf{j}} + \Delta E_z \hat{\mathbf{k}}\right) \\
&= q\left(\nabla E_x \cdot \Delta\mathbf{r}\right)\hat{\mathbf{i}} + q\left(\nabla E_y \cdot \Delta\mathbf{r}\right)\hat{\mathbf{j}} + q\left(\nabla E_z \cdot \Delta\mathbf{r}\right)\hat{\mathbf{k}} \\
&\equiv q\left(\Delta\mathbf{r} \cdot \nabla\right)\mathbf{E}(\mathbf{r}) = q\left(\mathbf{d} \cdot \nabla\right)\mathbf{E}(\mathbf{r}) \\
\mathbf{F}_{Total}(\mathbf{r}) &= \left(\mathbf{p} \cdot \nabla\right)\mathbf{E}(\mathbf{r}) \neq 0.
\end{aligned}$$

- In NON-UNIFORM field $\mathbf{E} = \mathbf{E}(\mathbf{r})$, the net torque is also non-zero:

$$\begin{aligned}
\boldsymbol{\tau}_{Total}(\mathbf{r}) &= \left(q\frac{\mathbf{d}}{2}\right) \times \left[\mathbf{E}_+(\mathbf{r}_+) + \mathbf{E}_-(\mathbf{r}_-)\right] \\
&= \mathbf{p} \times \frac{\left[\mathbf{E}_+(\mathbf{r}_+) + \mathbf{E}_-(\mathbf{r}_-)\right]}{2} = \mathbf{p} \times \mathbf{E}(\mathbf{r}) \neq 0.
\end{aligned}$$

Electric Polarization

If N be the total number of "elementary" (atomic or molecular) dipoles in a small volume Δv of a dielectric material, then the net electric dipole moment of that sample is given by

$$\Delta \mathbf{p}_{tot} = Q_1\mathbf{d}_1 + Q_2\mathbf{d}_2 + \cdots + Q_N\mathbf{d}_N \overset{N\to\infty}{\Longrightarrow} \sum_{k=1}^{k=N} Q_k\mathbf{d}_k \begin{cases} \approx 0 & \text{if} \quad \mathbf{E} = 0 \\ \neq 0 & \text{if} \quad \mathbf{E} \neq 0 \end{cases} .$$

Definition

ELECTRIC POLARIZATION: *This is the total electric dipole moment per unit volume of the dielectric material, i.e.,*

$$\mathbf{P} = \lim_{\Delta v \to 0} \frac{\Delta \mathbf{p}_{tot}}{\Delta v} = \frac{d\mathbf{p}}{dv}.$$

Fact

Potential due to dipole moment $d\mathbf{p}(\mathbf{r}') = \mathbf{P}(\mathbf{r}')dv'$ *at a target point* $T(\mathbf{r})$*:*

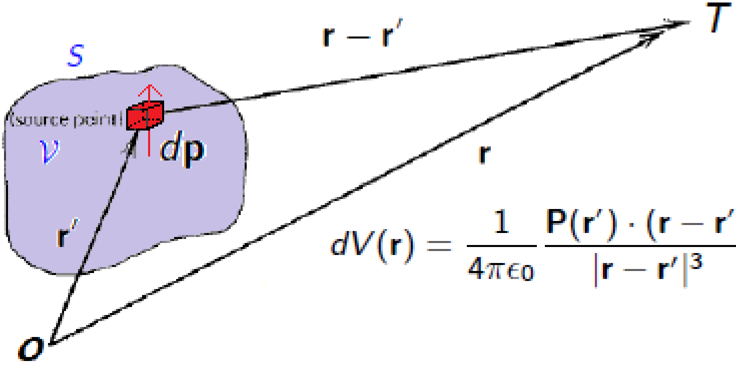
$$dV(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{d\mathbf{p}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dv'$$

Polarized Dielectric

Consider a polarized dielectric sample $\mathcal{V}$, bounded by smooth closed surface S. If $\mathbf{P}(\mathbf{r}')$ be the polarization vector at the source point $\mathbf{r}'$, then the total electric dipole moment in the elemental volume dv' is

$$d\mathbf{p}(\mathbf{r}') = \mathbf{P}(\mathbf{r}')dv'.$$

Thus, the Electrostatic Potential $dV(\mathbf{r}')$ at the target point $T(\mathbf{r})$ outside the dielectric due to all the dipoles within elemental volume dv' is given by the follwing expression:



$$dV(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{P}(\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dv'$$

Polarized Dielectric (contd.)

$$
\begin{aligned}
V(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \iiint\limits_{\mathcal{V}} \mathbf{P}(\mathbf{r}') \cdot \left(\frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \right) dv' \\
&= \frac{1}{4\pi\epsilon_0} \iiint\limits_{\mathcal{V}} \mathbf{P}(\mathbf{r}') \cdot \nabla' \left(\frac{1}{|\mathbf{r}-\mathbf{r}'|} \right) dv' \qquad \Longrightarrow \text{grad w.r.t "source" variable } \mathbf{r}' \\
&= \frac{1}{4\pi\epsilon_0} \iiint\limits_{\mathcal{V}} \left[\nabla' \cdot \left(\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right) - \frac{1}{|\mathbf{r}-\mathbf{r}'|} \left(\nabla' \cdot \mathbf{P}(\mathbf{r}') \right) \right] dv' \\
&= \frac{1}{4\pi\epsilon_0} \oiint\limits_{S} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \left(\mathbf{P}(\mathbf{r}') \cdot \hat{\mathbf{n}} \right)_S da' + \frac{1}{4\pi\epsilon_0} \iiint\limits_{\mathcal{V}} \frac{1}{|\mathbf{r}-\mathbf{r}'|} \left(-\nabla' \cdot \mathbf{P}(\mathbf{r}') \right) dv' \\
V(\mathbf{r}) &\equiv \frac{1}{4\pi\epsilon_0} \oiint\limits_{S} \frac{\sigma_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} da' + \frac{1}{4\pi\epsilon_0} \iiint\limits_{\mathcal{V}} \frac{\rho_b(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} dv' \qquad \Longrightarrow \text{Coulomb's Law for } V
\end{aligned}
$$

$$
\sigma_b(\mathbf{r}') = \left(\mathbf{P}(\mathbf{r}') \cdot \hat{\mathbf{n}} \right)_S \qquad ; \qquad \rho_b(\mathbf{r}') = -\nabla' \cdot \mathbf{P}(\mathbf{r}')
$$

Here we used the Gauss's Divergence Theorem.

Vector Calculus Identity: $\nabla \cdot (f\,\mathbf{A}) = f\,(\nabla \cdot \mathbf{A}) + (\nabla f) \cdot \mathbf{A}$

Bound or Polarization Charges

Thus, net effect of the polarized dielectric sample is to replace the dielectric material by equivalent "fictitious" *bound surface charge distribution* with surface density σ_b, and a *bound volume charge distribution* with volume density ρ_b:

$$\boxed{\sigma_b(\mathbf{r}') = \left(\mathbf{P}(\mathbf{r}')\cdot\hat{\mathbf{n}}\right)_S \qquad ; \qquad \rho_b(\mathbf{r}') = -\nabla'\cdot\mathbf{P}(\mathbf{r}')}$$

- BOUND CHARGES: These are fictitious charges that result from tiny displacements of positive and negative charge centers over atomic or molecular scales ($\sim 0.1 - 10\ \mathrm{Am}$) during dielectric polarization. They are incapable of moving freely over macroscopic distances.
- FREE CHARGES: These are true charges which are capable of moving freely over macroscopic distances, e.g., electrons and ions in conductors and electrolytes.
- Total bound charge (surface + volume) is always zero:
  Using Gauss's Divergence Theorem

$$Q_b = \oint\!\!\!\oint_S (\mathbf{P}\cdot\hat{\mathbf{n}})_S\, da' + \iiint_{\mathcal{V}} (-\nabla'\cdot\mathbf{P})dv' = \cancel{\iiint_{\mathcal{V}} \nabla'\cdot\mathbf{P}dv'} - \cancel{\iiint_{\mathcal{V}} \nabla'\cdot\mathbf{P}dv'} = 0.$$

Polarized Sphere

Example

Consider an electrically neutral (i.e., $Q_{\text{free}} = 0$) dielectric sphere of radius R with polarization $\mathbf{P} = k\mathbf{r}$, k being a constant. Determine all the bound charges and determine the Electric fields inside and outside the sphere.

- Bound volume and surface charge densities are constants:

$$\rho_b = -\nabla \cdot \mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr\right) = -3k \quad ; \quad \sigma_b = \left(\mathbf{P}\cdot\hat{\mathbf{r}}\right)_{r=R} = kR$$

- Total bound charge:

$$Q_b = Q_b^{(\text{volume})} + Q_b^{(\text{surface})} = \left[(-3k)\left(\frac{4}{3}\pi R^3\right)\right]_{\text{volume}} + \left[(kR)\left(4\pi R^2\right)\right]_{\text{surface}} = 0$$

- Electric field outside the dielectric sphere with no free charges, $Q_{\text{free}} = 0$:

$$\mathbf{E}^b_{\text{out}}(\mathbf{r}) = \left[\frac{1}{4\pi\epsilon_0}\frac{Q_b^{(\text{surface})}}{r^2}\hat{\mathbf{r}}\right] + \left[\frac{1}{4\pi\epsilon_0}\frac{Q_b^{(\text{volume})}}{r^2}\hat{\mathbf{r}}\right] = 0$$

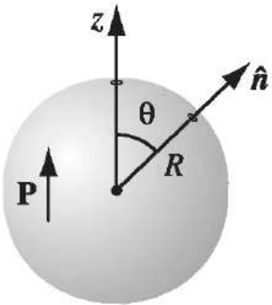
- Electric field inside will have contributions ONLY from enclosed bound volume charges within a given Gaussian surface S:

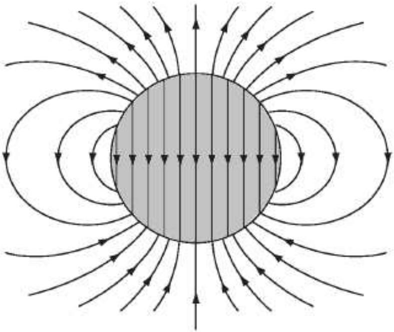
$$\oiint_S \mathbf{E}^b_{\text{in}}(\mathbf{r}) \cdot d\mathbf{S} = \frac{Q_b^{\text{encl}}(r)}{\epsilon_0} = \frac{\left(\frac{4}{3}\pi r^3\right)\rho_b}{\epsilon_0} \quad \Longrightarrow \quad \mathbf{E}^b_{\text{in}}(\mathbf{r}) = -\frac{kr}{\epsilon_0}\hat{\mathbf{r}}.$$

Uniformly Polarized Sphere

Example

Consider an electrically neutral but uniformly polarized sphere of radius R carrying polarization $\mathbf{P} = P_0\hat{\mathbf{k}}$, where P_0 is a constant. Determine all the bound charges, as well as the Electric fields inside and outside the sphere.





Note: This arrangement is called a *Spherical "Electret"* (in analogy to a "magnet") if the polarization is permanent or "frozen-in", i.e., the dielectric material retains its polarization even after the external field is removed.

Uniformly Polarized Sphere (contd.)

- Bound volume and surface charge densities:

$$\rho_b = -\nabla \cdot (P_0 \hat{\mathbf{k}}) = 0 \quad ; \quad \sigma_b(\theta) = \mathbf{P} \cdot \hat{\mathbf{n}} = P_0 \hat{\mathbf{k}} \cdot \hat{\mathbf{r}} = P_0 \cos\theta$$

- Potential and Electric Field <u>outside</u> the dielectric sphere is identical to that of an electric dipole $\mathbf{p}$ where,

$$\mathbf{p} = \left(\frac{4}{3}\pi R^3\right)\mathbf{P} = \frac{4}{3}\pi R^3 P_0 \hat{\mathbf{k}}$$

$$V_{\rm dip}^{\rm (out)}(r > R, \theta) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} = \frac{P_0}{3\epsilon_0}\frac{R^3}{r^2}\cos\theta$$

$$\Longrightarrow \mathbf{E}_{\rm dip}^{\rm (out)}(r > R, \theta) = -\nabla V_{\rm dip}^{\rm (out)}(r, \theta) = \frac{P_0}{3\epsilon_0}\frac{R^3}{r^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)$$

- "De-polarizing" Electric Field <u>inside</u> can be obtained using <u>2 Methods</u>:

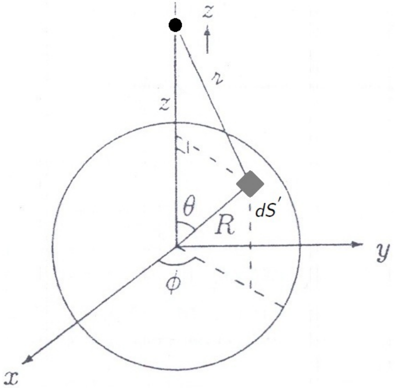
1. Using the Crude Atomic Model.
2. Rigorous surface integration using Coulomb's Law.

Method 1: $\Longrightarrow \quad \mathbf{E}_{\rm in}^{\rm (depol)}(r \leq R) = -\frac{\rho \mathbf{d}}{3\epsilon_0} = -\frac{\mathbf{P}}{3\epsilon_0} = -\frac{P_0}{3\epsilon_0}\hat{\mathbf{k}}$

Note: $\mathbf{E}_{\rm in}^{\rm (depol)}$ is UNIFORM and directed opposite to $\mathbf{P}$.

Method 2: Electric field at a target point **r** on the z-axis.

- $\sigma_b(\theta) = P_0 \cos\theta$: Surface integration is only needed.
- $\rho_b = 0$: We do not need to consider volume integration.



$$\mathbf{E}^{(\mathrm{pol})}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \oiint_S \frac{\sigma_b(\mathbf{r}')\,(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\, dS'$$

- Target/Field Point $\mathbf{r} = (0, 0, z)$, Source Point $\mathbf{r}' = (R, \theta, \phi)$
- Parametric form $\mathbf{r}'(\theta, \phi) = R(\sin\theta\cos\phi\hat{\mathbf{i}} + \sin\theta\sin\phi\hat{\mathbf{j}} + \cos\theta\hat{\mathbf{k}})$
- $\mathbf{r} - \mathbf{r}'(\theta, \phi) = -R(\sin\theta\cos\phi)\hat{\mathbf{i}} - R(\sin\theta\sin\phi)\hat{\mathbf{j}} + (z - R\cos\theta)\,\hat{\mathbf{k}}$
- $\left|\mathbf{r} - \mathbf{r}'(\theta, \phi)\right| = \sqrt{R^2 + z^2 - 2R\,z\cos\theta}$
- Elemental surface area at $\mathbf{r}'$: $dS' = R^2\sin\theta d\theta d\phi$

$$
\begin{aligned}
\mathbf{E}^{(\mathrm{pol})}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\oiint_S \frac{\sigma(\mathbf{r}')\left(\mathbf{r} - \mathbf{r}'(\theta, \phi)\right)}{|\mathbf{r} - \mathbf{r}'(\theta, \phi)|^3}\,dS' \\
&= \frac{P_0}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\frac{R^2\sin\theta d\theta d\phi}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}} \\
&\quad\times\left[\cancelto{0}{-R\sin\theta\cos\theta\cos\phi\hat{\mathbf{i}}} - \cancelto{0}{R\sin\theta\cos\theta\sin\phi\hat{\mathbf{j}}} + \cos\theta\,(z - R\cos\theta)\,\hat{\mathbf{k}}\right] \\
&= \frac{P_0R^2}{2\epsilon_0}\int_0^{\pi}\frac{\cos\theta\,(z - R\cos\theta)\sin\theta d\theta}{(R^2 + z^2 - 2Rz\cos\theta)^{3/2}}\,\hat{\mathbf{k}} = \frac{P_0R^2}{2\epsilon_0}\int_{-1}^{1}\frac{(z - Ru)\,du}{(R^2 + z^2 - 2Rzu)^{3/2}}\,\hat{\mathbf{k}}
\end{aligned}
$$

$$
\mathbf{E}^{(\mathrm{pol})}(z\hat{\mathbf{k}}) = \begin{cases} \frac{2P_0R^3}{3\epsilon_0z^3}\hat{\mathbf{k}} \equiv \mathbf{E}^{(\mathrm{out})}_{\mathrm{dip}}(z, \theta = 0) & \text{if} \quad z > R \\ -\frac{P_0}{3\epsilon_0}\hat{\mathbf{k}} & \text{if} \quad z \le R \end{cases}.
$$