Physics II (PH 102)

Electromagnetism (Lecture 2)

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Gradient Operator (∇ or grad)

Suppose $\phi(x, y, z)$ is a scalar field in $\mathbb{R}^3$ with continuous partial derivatives, the GRADIENT of $\phi(x, y, z)$ is given by

$$\operatorname{grad} \phi(x, y, z) \equiv \nabla \phi(x, y, z) = \left(\hat{\mathbf{i}} \frac{\partial}{\partial x} + \hat{\mathbf{j}} \frac{\partial}{\partial y} + \hat{\mathbf{k}} \frac{\partial}{\partial z} \right) \phi(x, y, z)$$

- The gradiant is defined *only* for scalar fields which yields *vector fields.*
- The relation $\cancel{\nabla \mathbf{V}}$ is *meaningless* for a vector field $\mathbf{V}$.
- *The magnitude $|\nabla \phi|$ gives the maximum value of the directional derivative of ϕ at any given point , while the direction of the gradient points along the fastest rate of change of ϕ at that point.*
- *$\nabla \phi$ points in the direction normal to the level surface $\phi = const.$ ie.,*

$$\hat{\mathbf{N}} = \pm \frac{\nabla \phi}{|\nabla \phi|}$$

Example of Gradient in 2D

Examples

1. $f(x,y) = xy$ then $\frac{\partial f}{\partial x} = y$ and $\frac{\partial f}{\partial y} = x$ and
$$\vec{\nabla} f = y\hat{\mathbf{i}} + x\hat{\mathbf{j}}$$

2. $f(x,y) = e^x y$ then $\frac{\partial f}{\partial x} = e^x y$ and $\frac{\partial f}{\partial y} = e^x$ and
$$\vec{\nabla} f = \left(y\hat{\mathbf{i}} + \hat{\mathbf{j}}\right) e^x$$

3. $f(x,y) = \sin\left(5x^2 + 3y\right)$ then $\frac{\partial f}{\partial x} = 10x\cos\left(5x^2 + 3y\right)$ and
$\frac{\partial f}{\partial y} = 3\cos\left(5x^2 + 3y\right)$ and
$$\vec{\nabla} f = \left(10x\hat{\mathbf{i}} + 3\hat{\mathbf{j}}\right)\cos\left(5x^2 + 3y\right)$$

Example of Gradients in 3D

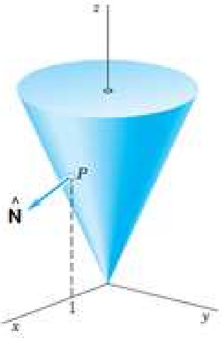
Find a unit normal vector $\hat{\mathbf{N}}$ of the cone of revolution $z^2 = 4(x^2 + y^2)$ at the point P: $(1, 0, 2)$.

Solution. The cone is the level surface $f = 0$ of $f(x, y, z) = 4(x^2 + y^2) - z^2$.

$$\operatorname{grad} f = [8x, \quad 8y, \quad -2z], \qquad \operatorname{grad} f(P) = [8, \quad 0, \quad -4]$$

$$\hat{\mathbf{N}} = \frac{\operatorname{grad} f(P)}{|\operatorname{grad} f(P)|} = \left[\frac{2}{\sqrt{5}}, \quad 0, \quad -\frac{1}{\sqrt{5}}\right].$$

$\hat{\mathbf{N}}$ points downward since it has a negative z-component. The other unit normal vector of the cone at P is $-\hat{\mathbf{N}}$



Divergence Operator ($\nabla\cdot$ or "div")

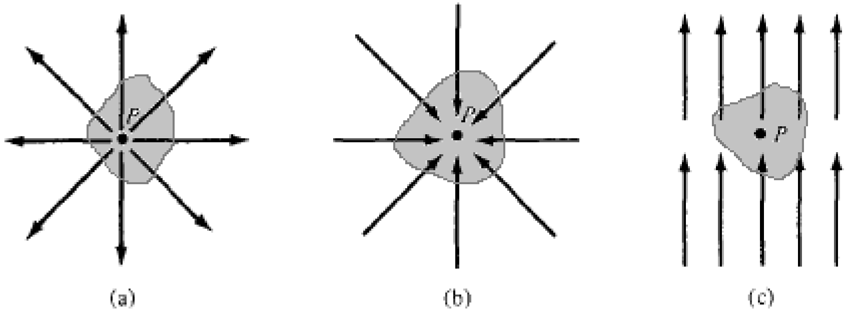
Let $\mathbf{V}(x,y,z)$ is a differentiable vector field in $\mathbb{R}^3$with real Cartesian components V_x, V_y, and V_z. The DIVERGENCE of $\mathbf{V}(x,y,z)$ is obtained by taking the scalar "dot-product" operation with ∇:

$$\nabla\cdot\mathbf{V}(x,y,z)=\left(\hat{\mathbf{i}}\frac{\partial}{\partial x}+\hat{\mathbf{j}}\frac{\partial}{\partial y}+\hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\mathbf{i}}V_x+\hat{\mathbf{j}}V_x+\hat{\mathbf{k}}V_x\right)$$

$$\boxed{\operatorname{div}\mathbf{V}(x,y,z)\equiv\nabla\cdot\mathbf{V}(x,y,z)=\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+\frac{\partial V_z}{\partial z}}$$

- *Defined for vector fields which yields scalar fields.*
- *Gives a measure of how much a vector field tends to diverge from or converge to a given point.*
- *A SOURCE is a point of +ve divergence and a SINK is a point of -ve divergence.*
- *A non-trivial (i.e.,* $\mathbf{V}\neq 0$*) vector field with zero divergence identically is said to be SOLENOIDAL (e.m. theory) or INCOMPRESSIBLE (fluid mechanics).*

Divergence: Physical significance

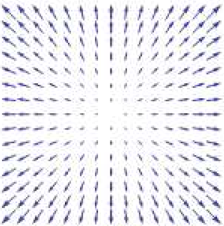


div $\mathbf{V}$ at P: Illustration of the divergence of a vector field at P; **(a)** positive divergence, **(b)** negative divergence, **(c)** zero divergence.

Examples of Divergence

Example

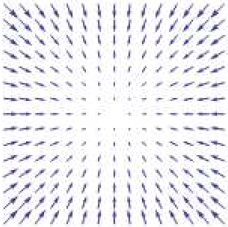
$$\nabla \cdot \vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$



$$\vec{V} = \hat{e}_x x + \hat{e}_y y$$

$$\nabla \cdot \vec{V} = 1 + 1 = 2$$

Positive divergence: source



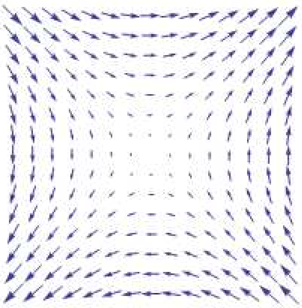
$$\vec{V} = -\hat{e}_x x - \hat{e}_y y$$

$$\nabla \cdot \vec{V} = -1 - 1 = -2$$

Negative divergence: sink

Examples of Divergence

Example



$$\vec{V} = \hat{e}_x y + \hat{e}_y x$$

$$\nabla \cdot \vec{V} = 0 + 0 = 0$$

Divergence Free

Curl Operator ($\nabla \times$ or "rot")

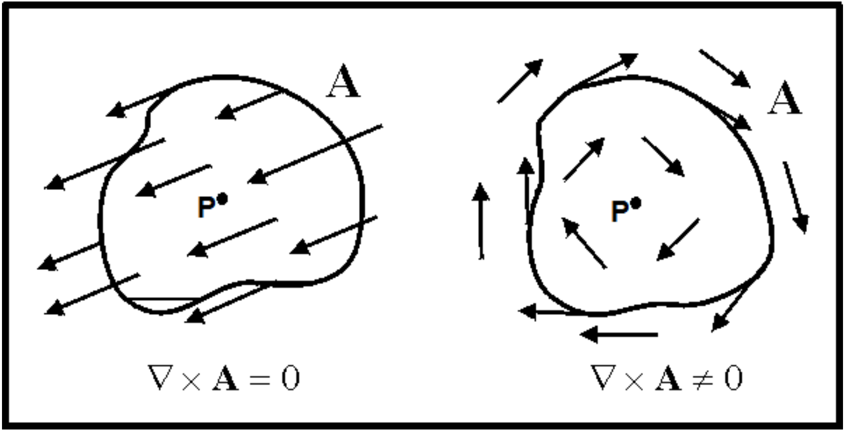
Suppose $\mathbf{A}(x, y, z)$ is a differentiable vector field in $\mathbb{R}^3$ with real Cartesian components A_x, A_y, A_z, the CURL or ROTATION of $\mathbf{A}(x, y, z)$ is obtained by taking the vector "cross-product" operation with ∇:

$$\nabla \times \mathbf{A}(x, y, z) = \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \times \left(\hat{\mathbf{i}}A_x + \hat{\mathbf{j}}A_y + \hat{\mathbf{k}}A_z\right)$$

$$\boxed{\operatorname{curl} \mathbf{A}(x, y, z) = \hat{\mathbf{i}}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{\mathbf{j}}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{\mathbf{k}}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)}$$

- *Defined for vector fields which also yields vector fields.*
- *Unlike the gradient or divergence oparator, the curl oparator is defined only in 3D, like a vector cross-product.*
- *Its magnitude gives the tendency of the vector field to rotate/circulate about a given point, while its direction lies along the axis of rotation as determined by the right-hand rule.*
- *An IRROTATIONAL vector field is one for which the curl vanishes identically.*

Curl: Physical significance



Non-vanishing curl implies the vector field to be rotational about the point P

Product Identities for Gradient, Divergence and Curl (Prove them!)

Let f be a differentiable scalar field, $\mathbf{A}$ and $\mathbf{B}$ be differentiable vector fields, and $k = const.$, then the following product identities hold:

- $\nabla \cdot (k\mathbf{A}) = k\,(\nabla \cdot \mathbf{A})$
- $\nabla \times (k\mathbf{A}) = k\,(\nabla \times \mathbf{A})$
- $\nabla \cdot (f\mathbf{A}) = f\,(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f$
- $\nabla \times (f\mathbf{A}) = f\,(\nabla \times \mathbf{A}) - \mathbf{A} \times \nabla f$
- $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
- $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A}(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) - (\mathbf{A} \cdot \nabla)\,\mathbf{B}$
- curl of a grad is zero identically: $\nabla \times (\nabla f) = 0$
- div of a curl is zero identically: $\nabla \cdot (\nabla \times \mathbf{A}) = 0$

Fact

The last two identities are very important and we shall use them very often.

Laplacian Operator $\nabla^2 \equiv \nabla \cdot \nabla$

Operation on Scalar Fields yields other Scalar Fields: Suppose the scalar function, $\phi(x,y,z)$ defines a differentiable scalar field in $\mathbb{R}^3$ with continuous higher order partial derivatives, then a **second order scalar operator** is obtained by first taking the "dot-product" of two ∇'s and then operating on $\phi(x,y,z)$, or equivalently, by taking the gradient first and then evaluating the divergence:

$$(\nabla \cdot \nabla)\,\phi(x,y,z) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\phi(x,y,z) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\begin{aligned}
\nabla \cdot (\nabla \phi(x,y,z)) &= \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\mathbf{i}}\frac{\partial \phi}{\partial x} + \hat{\mathbf{j}}\frac{\partial \phi}{\partial y} + \hat{\mathbf{k}}\frac{\partial \phi}{\partial z}\right) \\
&= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \equiv \nabla^2 \phi(x,y,z)
\end{aligned}$$

Operation on Vector Fields yields other Vector Fields: Suppose the vector function, $\mathbf{V}(x,y,z) = \hat{\mathbf{i}}V_x + \hat{\mathbf{j}}V_y + \hat{\mathbf{k}}V_z$ defines a differentiable vector field in $\mathbb{R}^3$ with continuous higher order partial derivatives, then instead the following sequence of operation makes sense:

$$\begin{aligned}
(\nabla \cdot \nabla)\mathbf{V}(x,y,z) &= \left[\left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right) \cdot \left(\hat{\mathbf{i}}\frac{\partial}{\partial x} + \hat{\mathbf{j}}\frac{\partial}{\partial y} + \hat{\mathbf{k}}\frac{\partial}{\partial z}\right)\right]\mathbf{V}(x,y,z) \\
\nabla^2\mathbf{V}(x,y,z) &= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\mathbf{V} \equiv \hat{\mathbf{i}}(\nabla^2 V_x) + \hat{\mathbf{j}}(\nabla^2 V_y) + \hat{\mathbf{k}}(\nabla^2 V_z)
\end{aligned}$$

Laplacian Operator $\nabla^2 \equiv \nabla \cdot \nabla$

$$\boxed{\nabla^2 \equiv \nabla \cdot \nabla = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)}$$

- *Defined for both scalar and vector fields which also yield other scalar and vector fields, respectively.*
- The relation $\nabla^2 \mathbf{V} \neq \nabla \cdot \cancel{(\nabla \mathbf{V})}$ is *meaningless* for a vector field $\mathbf{V}$. However, a direct operation on its components only make sense, namely, $\nabla^2 \mathbf{V} \equiv \hat{\mathbf{i}}(\nabla^2 V_x) + \hat{\mathbf{j}}(\nabla^2 V_y) + \hat{\mathbf{k}}(\nabla^2 V_z)$
- *A HARMONIC field is one whose Laplacian vanishes identically.*

Ordinary Integrals of Vector Functions (in 1D or single variable)

Let $\mathbf{A}(u) = A_1(u)\hat{\mathbf{i}} + A_2(u)\hat{\mathbf{j}} + A_3(u)\hat{\mathbf{k}}$ be a vector valued function of a parameter $u \in \mathbb{R}$, where the components $A_{1,2,3} \in \mathbb{R}$ are assumed to be continuous in 1D domain $[a, b] \in \mathbb{R}$. If $\exists$ a vector function $\mathbf{S}(u)$ such that

$$\mathbf{A}(u) = \frac{d\mathbf{S}(u)}{du}$$

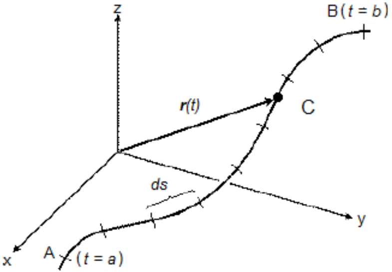
then,

$$\int_a^b \mathbf{A}(u)\, du = \int_a^b \left(\frac{d\mathbf{S}(u)}{du}\right) du = \mathbf{S}(b) - \mathbf{S}(a)$$

is defined as the DEFINITE INTEGRAL of $\mathbf{A}(u)$ over the domain $[a, b]$ and yields a constant vector.

Note: The vector function may be a 3D vector, but the integral is a *one-dimensional* or a single variable definite integral.

Line Integrals over Parametric 3D Space Curves



The *domain of integration* can be generalized to an *arbitrary 3D path* in $\mathbb{R}^3$ having a *1D parametric representation*

$$\mathbf{r}(t) = g(t)\hat{\mathbf{i}} + h(t)\hat{\mathbf{j}} + k(t)\hat{\mathbf{k}}$$

where
$$x = g(t),\ y = h(t),\ z = k(t),$$
are smooth functions of the variable $t \in [a, b] \in \mathbb{R}$.

Example

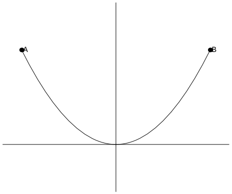
Time parameter t describes point $\mathbf{r}(t)$ on the space curve C of a moving particle in 3D. If $f[\mathbf{r}(t)]$ be any smooth scaler function defined on C, then $\int_C f[\mathbf{r}]ds$ defines a scaler **LINE INTEGRAL** of $f(\mathbf{r})$ over C.

Examples of I-dim Parametrization Space Curves

Examples

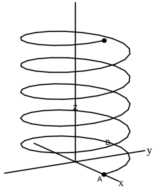
Parabolic path in 2D

$$\left.\begin{aligned} x(s) &= s \\ y(s) &= s^2 \end{aligned}\right\} s \in [-1, 1]$$



Helical path in 3D

$$\left.\begin{aligned} x(\theta) &= \cos\theta \\ y(\theta) &= \sin\theta \\ z(\theta) &= \theta/2\pi \end{aligned}\right\} \theta \in [0, 10\pi]$$



Line Integral of Scalar Fields

QUESTION: How to evaluate the LINE INTEGRAL of the scalar function $f(x, y, z)$ over a given space curve $C : \mathbf{r}(t), \ \ a \leq t \leq b$?

- Split the given path C into differential segments ds beween the end-points.
- Find a smooth (continuous derivatives) 1D parametrization for C :

$$x = g(t), \ y = h(t), \ z = k(t)$$

$$\mathbf{r}(t) = g(t)\hat{\mathbf{i}} + h(t)\hat{\mathbf{j}} + k(t)\hat{\mathbf{k}}, \quad a \leq t \leq b$$

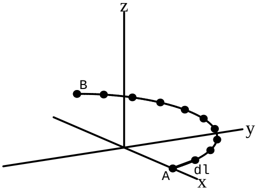
- Line Integral over path C is converted to a definite integral over $t \in [a, b]$:

$$\begin{aligned}
\int_C f(x,y,z)ds &= \int_a^b f(x,y,z)\left(\frac{ds(t)}{dt}\right) dt = \boxed{\int_a^b f\left[\mathbf{r}(t)\right] \left|\frac{d\mathbf{r}(t)}{dt}\right| dt} \\
&= \int_a^b f(x,y,z)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt \\
&\equiv \int_a^b f[g(t),h(t),k(t)]\sqrt{\left(\frac{dg}{dt}\right)^2 + \left(\frac{dh}{dt}\right)^2 + \left(\frac{dk}{dt}\right)^2}\ dt
\end{aligned}$$

Example of Line Integral of Scalar Fields

Example

Find the length of a circular arc AB of radius R for $\theta \in [0, \alpha]$.



- Plane-polar Parametric form: $\mathbf{r}(\theta) = (R\cos\theta, R\sin\theta)$.
- $\mathbf{r}'(\theta) = (-R\sin\theta, R\cos\theta)$ and $|\mathbf{r}'(\theta)| = R$.
- The function we need to integrate here is $f[\mathbf{r}(\theta)] = 1$.
- Length of arc is given by the line integral:

$$\begin{aligned} L &= \int_C f(\mathbf{r})\, dl = \int_C \left(\frac{dl}{d\theta}\right) d\theta = \int_0^\alpha |\mathbf{r}'(\theta)|\, d\theta \\ &= \int_0^\alpha R d\theta = R\alpha \end{aligned}$$