PH 102, Electromagnetism,

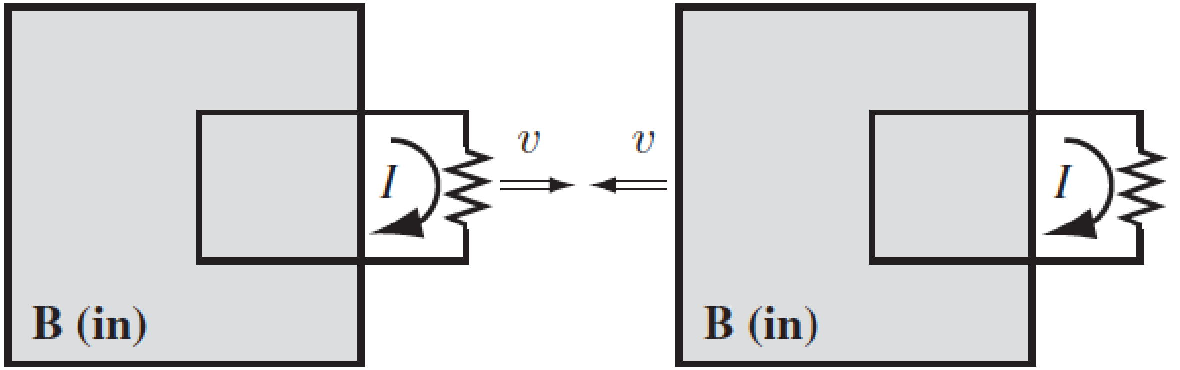
Post Mid Semester

Lecture 13

Electrodynamics and Relativity

Sovan Chakraborty, Department of Physics, IITG

Faraday's Law



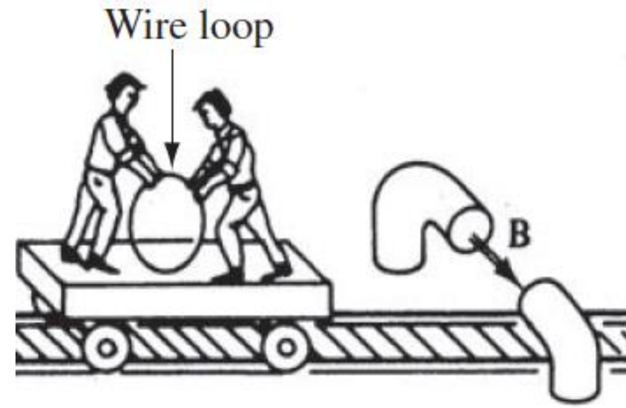
(a) (b)

(a) Case of motional emf; $\mathbf{v} \times \mathbf{B}$ drives the current ***I*** not ***E***

(b) 2nd Experiment has the same emf, relative motion of loop and the magnet. Induced ***E*** drives the current ***I***

- A charge in motion produces magnetic field (B) , but no B in the rest frame of the charge!
- Do we need one unique stationary reference frame?

Origin of the Postulates: STR



1) Motional EMF for observer on station

2) Change in Magnetic field for observer on train,
No magnetic force!

- Before Einstien:

- It is a Coincidence that the emf is same!
- Can both the interpretation be true? NO
- The velocity of the ultimate inertial frame **Ether** has to be taken into account!

- Electric and magnetic fields are stains in a invisible jelly like medium called **Ether**.

- The speed of the charge was to be measured wrt Ether, as the motion relative to Ether shd be considered.

Origin of the Postulates: STR

- Electric and magnetic fields are stains in a invisible jelly like medium called **Ether**.
- The speed of the charge was to be measured wrt Ether, as the motion relative to Ether shd be considered.

This also predicts that the electromagnetic waves travel through the vacuum

at a speed $\frac{1}{\sqrt{\epsilon_0 \mu_0}} = 3.00 \times 10^8 \mathrm{m/s},$ relative (presumably) to the ether .

However, Michelson – Morley experiment did not discover any 'Ether wind'

Einstien proposed,

1. The principle of relativity: The laws of physics apply in all inertial reference systems.

2. The universal speed of light: speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

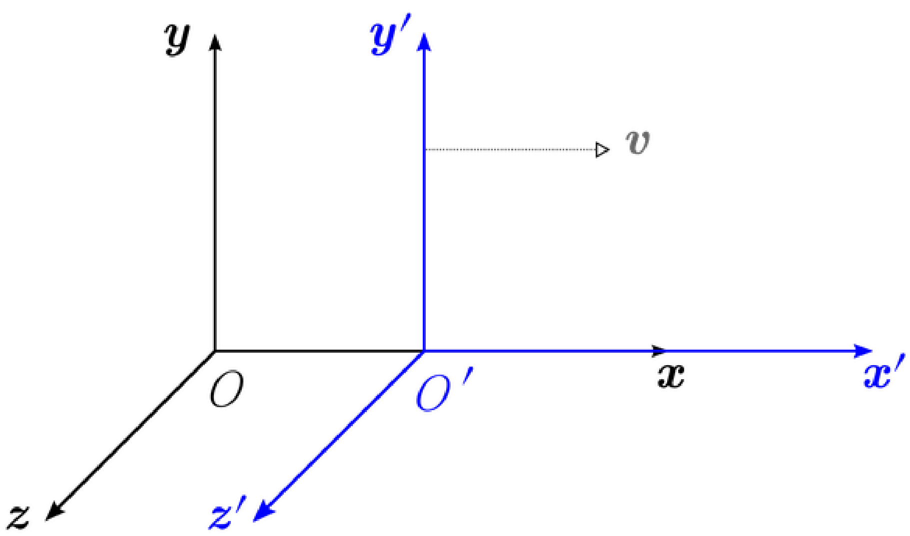
Special Theory of Relativity: Recap

Einstien proposed,

1. The principle of relativity: The laws of physics apply in all inertial reference systems.

2. The universal speed of light: speed of light in vacuum is the same for all inertial observers, regardless of the motion of the source.

Lorentz Transformation:



$$x' = \gamma(x - Vt)$$

$$y' = y$$

$$z' = z$$

$$t' = \gamma\left(t - \frac{V}{c^2}x\right),$$

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}}.$$

Special Theory of Relativity: Recap

Distance between two spacetime events is Lorentz invariant,

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

- Events simultaneous in one frame may not remain so for an observer moving with uniform speed.
- Clocks run slower in a moving frame:

$$\Delta\bar{t} = \sqrt{1 - v^2/c^2}\,\Delta t.$$

- Objects shorten in a moving frame:

$$\Delta\bar{x} = \frac{1}{\sqrt{1 - v^2/c^2}}\Delta x.$$

- Velocity transformations: Particle moving with speed u in frame S, an observer in the frame S’ will measure its velocity as

$$\bar{u}_x = \frac{d\bar{x}}{d\bar{t}} = \frac{u_x - v}{(1 - vu_x/c^2)}, \quad \bar{u}_y = \frac{d\bar{y}}{d\bar{t}} = \frac{u_y}{\gamma(1 - vu_x/c^2)}, \quad \bar{u}_z = \frac{d\bar{z}}{d\bar{t}} = \frac{u_z}{\gamma(1 - vu_x/c^2)}.$$

- Relativistic momentum is also Lorentz invariant:

$$\mathbf{p} \equiv m\eta = \frac{m\mathbf{u}}{\sqrt{1 - u^2/c^2}};$$

Special Theory of Relativity: Recap

Relativistic Energy:
$$E \equiv \frac{mc^2}{\sqrt{1-u^2/c^2}}; \;\; E^2 - p^2c^2 = m^2c^4.$$

Rest Energy:
$$E_{\text{rest}} \equiv mc^2.$$

Kinetic Energy
$$E_{\text{kin}} \equiv E - mc^2 \;\; = mc^2\left(\frac{1}{\sqrt{1-u^2/c^2}} - 1\right).$$

Transformation of Momentum-Energy

$$p'_x = \gamma\left(p_x - v\frac{E}{c^2}\right) \qquad \frac{E'}{c^2} = \gamma\left(\frac{E}{c^2} - \frac{vp_x}{c^2}\right) \qquad \& \qquad \begin{aligned} p'_y &= p_y \\ p'_z &= p_z \end{aligned}$$

Special Theory of Relativity: Recap

Transformation of Forces

$$F'_x = \frac{dp'_x}{dt'} = \frac{\gamma(dp_x - vdE/c^2)}{\gamma(dt - vdx/c^2)} = \frac{\frac{dp_x}{dt} - \frac{v}{c^2}\frac{dE}{dt}}{1 - \frac{v}{c^2}\frac{dx}{dt}} = \frac{F_x - \frac{v}{c^2}\frac{dE}{dt}}{1 - \frac{v}{c^2}u_x}$$

and using,

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{mc^2}{\sqrt{1-u^2/c^2}}\right) = \frac{m\vec{u}}{(1-u^2/c^2)^{3/2}} \cdot \frac{d\vec{u}}{dt} = \frac{d}{dt}\left(\frac{m\vec{u}}{\sqrt{1-u^2/c^2}}\right) \cdot \vec{u} = \frac{d\vec{p}}{dt} \cdot \vec{u}$$

$$\boxed{F'_x = \frac{F_x - \frac{v}{c^2}(\vec{u} \cdot \vec{F})}{1 - \frac{v}{c^2}u_x}}$$

and

$$\boxed{F'_{y,z} = \frac{dp'_y}{dt'} = \frac{F_{y,z}}{\gamma\left(1 - \frac{v}{c^2}u_x\right)}}$$

For particle at rest, **u** = 0, $\quad F'_\perp = F_\perp/\gamma, F'_\parallel = F_\parallel$

Special Theory of Relativity: Recap

Transformation of Electromagnetic Fields

$$\bar{E}_x = E_x, \qquad \bar{E}_y = \gamma(E_y - vB_z), \qquad \bar{E}_z = \gamma(E_z + vB_y),$$

$$\bar{B}_x = B_x, \qquad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \qquad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$

Case I: $\mathbf{B} = \mathbf{0}$ in $\mathcal{S}$

$$\bar{\mathbf{B}} = \gamma\frac{v}{c^2}(E_z\,\hat{\mathbf{y}} - E_y\,\hat{\mathbf{z}}) = \frac{v}{c^2}(\bar{E}_z\,\hat{\mathbf{y}} - \bar{E}_y\,\hat{\mathbf{z}}) \quad \text{and} \quad \mathbf{v} = v\,\hat{\mathbf{x}},$$

Thus, $$\bar{\mathbf{B}} = -\frac{1}{c^2}(\mathbf{v} \times \bar{\mathbf{E}}).$$

Case II: $\mathbf{E} = \mathbf{0}$ in $\mathcal{S}$

$$\bar{\mathbf{E}} = \mathbf{v} \times \bar{\mathbf{B}}.$$

Special Theory of Relativity: Recap

Transformation of Electromagnetic Fields

$$\bar{E}_x = E_x, \qquad \bar{E}_y = \gamma(E_y - vB_z), \qquad \bar{E}_z = \gamma(E_z + vB_y),$$

$$\bar{B}_x = B_x, \qquad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \qquad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$

Case I: $\mathbf{B} = \mathbf{0} \text{ in } \mathcal{S}$

$$\bar{\mathbf{B}} = \gamma\frac{v}{c^2}(E_z\,\hat{\mathbf{y}} - E_y\,\hat{\mathbf{z}}) = \frac{v}{c^2}(\bar{E}_z\,\hat{\mathbf{y}} - \bar{E}_y\,\hat{\mathbf{z}}) \quad \text{and} \quad \mathbf{v} = v\,\hat{\mathbf{x}},$$

Thus, $$\bar{\mathbf{B}} = -\frac{1}{c^2}(\mathbf{v} \times \bar{\mathbf{E}}).$$

Case II: $\mathbf{E} = \mathbf{0} \text{ in } \mathcal{S}$

$$\bar{\mathbf{E}} = \mathbf{v} \times \bar{\mathbf{B}}.$$

Special Theory of Relativity: Recap

Transformation of Electromagnetic Fields

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \quad \bar{E}_z = \gamma(E_z + vB_y),$$

$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$

Fields written in components parallel & perpendicular to the velocity of the moving frame,

$$\mathbf{E} = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp}, \qquad \mathbf{E}' = \mathbf{E}'_{\parallel} + \mathbf{E}'_{\perp},$$

$$\mathbf{B} = \mathbf{B}_{\parallel} + \mathbf{B}_{\perp}, \qquad \mathbf{B}' = \mathbf{B}'_{\parallel} + \mathbf{B}'_{\perp}.$$

The transformation of the parallel & perpendicular components,

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \qquad \mathbf{E}'_{\perp} = \gamma\left(\mathbf{E}_{\perp} + \mathbf{v} \times \mathbf{B}_{\perp}\right)$$

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \qquad \mathbf{B}'_{\perp} = \gamma\left(\mathbf{B}_{\perp} - (\mathbf{v}/c^2) \times \mathbf{E}_{\perp}\right)$$

Special Theory of Relativity: Recap

Transformation of Electromagnetic Fields

$$\bar{E}_x = E_x, \quad \bar{E}_y = \gamma(E_y - vB_z), \quad \bar{E}_z = \gamma(E_z + vB_y),$$

$$\bar{B}_x = B_x, \quad \bar{B}_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right), \quad \bar{B}_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right).$$

Case I: $\mathbf{B} = \mathbf{0}$ in $\mathcal{S}$

$$\bar{\mathbf{B}} = \gamma\frac{v}{c^2}(E_z\,\hat{\mathbf{y}} - E_y\,\hat{\mathbf{z}}) = \frac{v}{c^2}(\bar{E}_z\,\hat{\mathbf{y}} - \bar{E}_y\,\hat{\mathbf{z}}) \quad \text{and} \quad \mathbf{v} = v\,\hat{\mathbf{x}},$$

Thus, $$\bar{\mathbf{B}} = -\frac{1}{c^2}(\mathbf{v} \times \bar{\mathbf{E}}).$$

Case II: $\mathbf{E} = \mathbf{0}$ in $\mathcal{S}$

$$\bar{\mathbf{E}} = \mathbf{v} \times \bar{\mathbf{B}}.$$

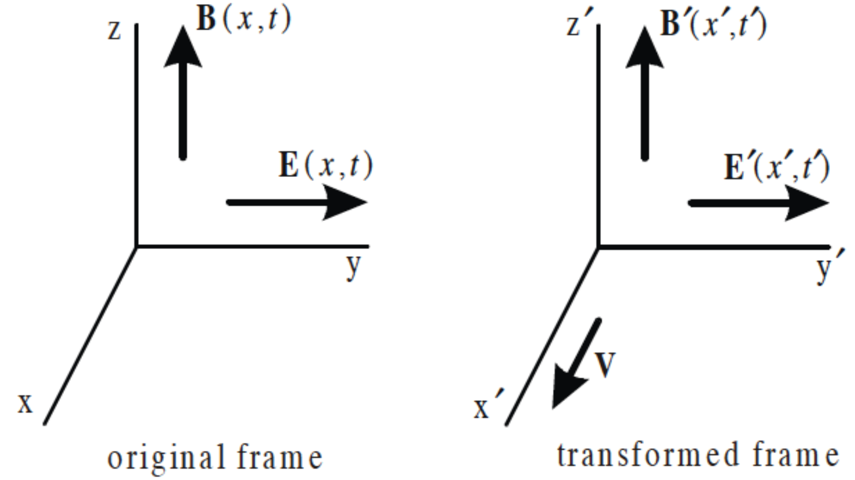
Maxwell's Equations

Two important consequences of Maxwell's equations:

- Possibility of electromagnetic waves
- Breakdown of Newtonian mechanics, leading to Theory of relativity.

Compatibility of Maxwell's equations and Newton's laws:

Consider, ***E*** and ***B*** in two different reference frames, one moving with a constant velocity V with respect to the other.



E & ***B*** satisfy the Maxwell's equations in original frame.

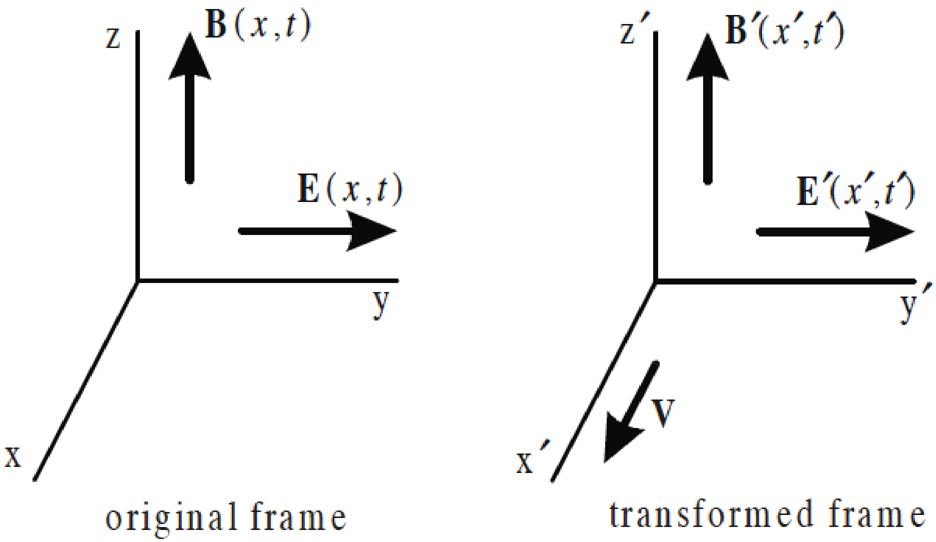
Consider, ***E*** in the y & ***B*** in the z direction and ***E*** & ***B*** depend on x and t, only.

Moving frame quantities signified with primes.

Original frame fields E(x,t) and B(x,t) obey Maxwell's equations

Lets us see, whether the transformed fields E′(x′,t′) and B′(x′, t′) satisfy Maxwell's equations or not!

Maxwell's Equations



$$x' = x - Vt$$
$$y' = y$$
$$z' = z$$
$$t' = t\,.$$

Newtonian principle: Force same in frames moving with constant velocity with respect to each other.

Invariance of the Lorentz force: Force on a moving charge is independent of ref frame.

Charge q moving with velocity v in the x direction

$$q\{E - vB\} = q\{E' - v'B'\}$$

[-from directions of v and B]

However, the velocity transforms as $v' = v - V$,

$$\text{hence, } E' - v'B' = (E' + VB') - vB' = E - vB.$$

Equating the coefficients of v,

[Sum of a v-independent term and one proportional to v.]

$$E = (E' + VB') \text{ and } B = B'.$$

This is the rule by which the fields transform.

Transformation of Maxwell's Equations

Let us investigate the transformation of the curl B equation (in the absence of any currents).

$$\operatorname{curl}\mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\,.$$

we assumed, $\mathbf{E} = E(x,t)\,\mathbf{j}$ and $\mathbf{B} = B(x,t)\,\mathbf{k}$

Hence, $\operatorname{curl}\mathbf{B} = -\frac{\partial B}{\partial x}\mathbf{j}$ and $\frac{\partial \mathbf{E}}{\partial t} = \frac{\partial E}{\partial t}\mathbf{j}$

The Maxwell's equation for these fields, $\frac{\partial B}{\partial x} = -\frac{1}{c^2}\frac{\partial E}{\partial t}\,.$

From the co-ordinate transformations.

$$\frac{\partial x'}{\partial x} = 1 \qquad \frac{\partial x'}{\partial t} = -V$$

$$\frac{\partial t'}{\partial t} = 1 \qquad \frac{\partial t'}{\partial x} = 0\,.$$

Transformation of Maxwell's Equations

$$\frac{\partial B}{\partial x} = \frac{\partial B}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial B}{\partial t'}\frac{\partial t'}{\partial x}$$

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial E}{\partial x'}\frac{\partial x'}{\partial t}.$$

$$\frac{\partial x'}{\partial x} = 1 \qquad \frac{\partial x'}{\partial t} = -V$$

$$\frac{\partial t'}{\partial t} = 1 \qquad \frac{\partial t'}{\partial x} = 0\,.$$

$$\frac{\partial B}{\partial x} = \frac{\partial B}{\partial x'}$$

$$\frac{\partial E}{\partial t} = \frac{\partial E}{\partial t'} - V\frac{\partial E}{\partial x'}.$$

E = (E′ +VB′) & B = B′, using these relations in $\frac{\partial B}{\partial x} = \frac{\partial B}{\partial x'}$ $\frac{\partial E}{\partial t} = \frac{\partial E}{\partial t'} - V\frac{\partial E}{\partial x'}.$ would give us the equation in transformed frame,

$$\frac{\partial B'}{\partial x'} = -\frac{1}{c^2}\frac{\partial E'}{\partial t'} + \frac{1}{c^2}\left\{V\frac{\partial E'}{\partial x'} - V\frac{\partial B'}{\partial t'} + V^2\frac{\partial B'}{\partial x'}\right\}$$

[Extra term ????]

Maxwell's equation are not invariant under Galilean transformation

Transformation of Maxwell's Equations

The Maxwell's equation for these fields, $\frac{\partial B}{\partial x} = -\frac{1}{c^2}\frac{\partial E}{\partial t}$.

$$\frac{\partial B'}{\partial x'} = -\frac{1}{c^2}\frac{\partial E'}{\partial t'} + \frac{1}{c^2}\left\{V\frac{\partial E'}{\partial x'} - V\frac{\partial B'}{\partial t'} + V^2\frac{\partial B'}{\partial x'}\right\}$$

[Extra term ????]

Maxwell's equation are not invariant under Galilean transformation

What about Lorentz transformation?

$$x' = \gamma\left(x - Vt\right)$$
$$y' = y$$
$$z' = z$$
$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} .$$

What about E and B?

$$t' = \gamma\left(t - \frac{V}{c^2}x\right),$$

Transformation of Maxwell's Equations

The Maxwell's equation for these fields, $\dfrac{\partial B}{\partial x} = -\dfrac{1}{c^2}\dfrac{\partial E}{\partial t}$.

$$\frac{\partial B'}{\partial x'} = -\frac{1}{c^2}\frac{\partial E'}{\partial t'} + \frac{1}{c^2}\left\{V\frac{\partial E'}{\partial x'} - V\frac{\partial B'}{\partial t'} + V^2\frac{\partial B'}{\partial x'}\right\}$$

[Extra term ????]

Maxwell's equation are not invariant under Galilean transformation

What about Lorentz transformation?

$$x' = \gamma(x - Vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - \frac{V}{c^2}x\right),$$

$$\gamma = \frac{1}{\sqrt{1 - V^2/c^2}} .$$

$$E = \gamma\{E' + VB'\}$$

$$B = \gamma\left\{B' + \frac{V}{c^2}E'\right\};$$

Check for yourself, (Homework)