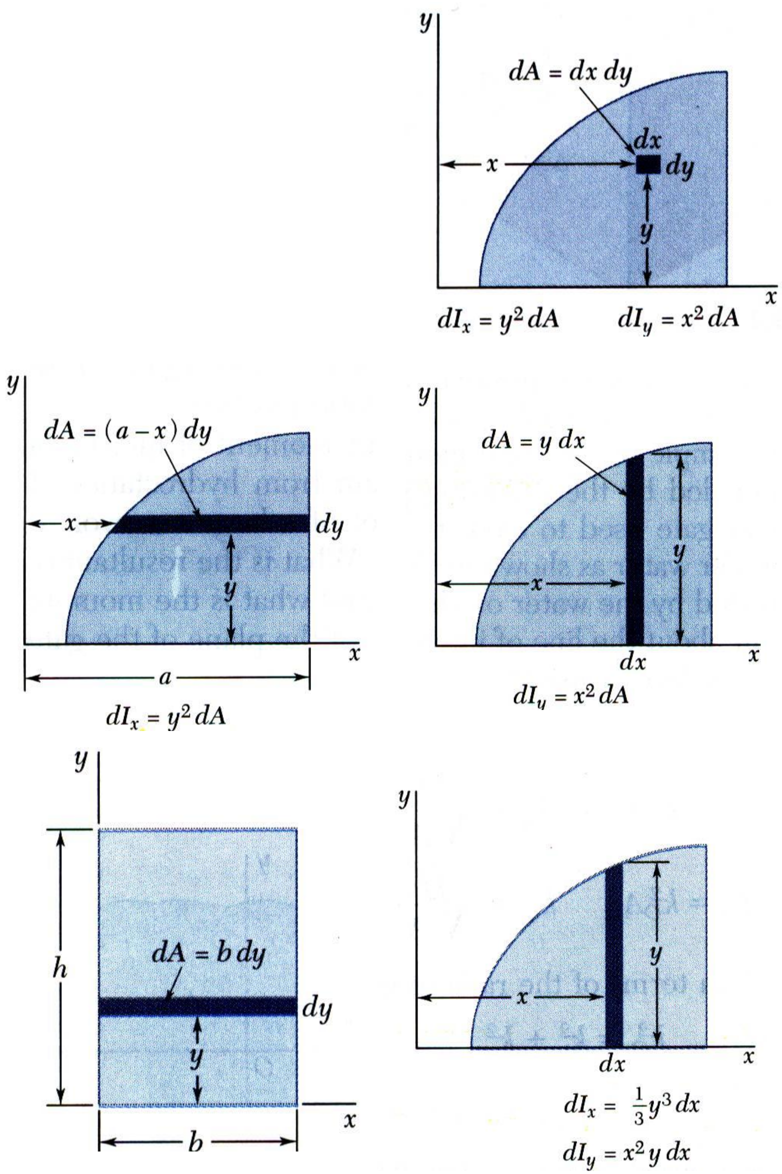
ME101: Engineering Mechanics (3 1 0 8)

2019-20 (II Semester)

LECTURE: 16

Area Moments of Inertia by Integration



- *Second moments* or *moments of inertia* of an area with respect to the x and y axes,

$$I_x = \int y^2 dA \qquad I_y = \int x^2 dA$$

- Evaluation of the integrals is simplified by choosing dA to be a thin strip parallel to one of the coordinate axes.

- For a rectangular area,

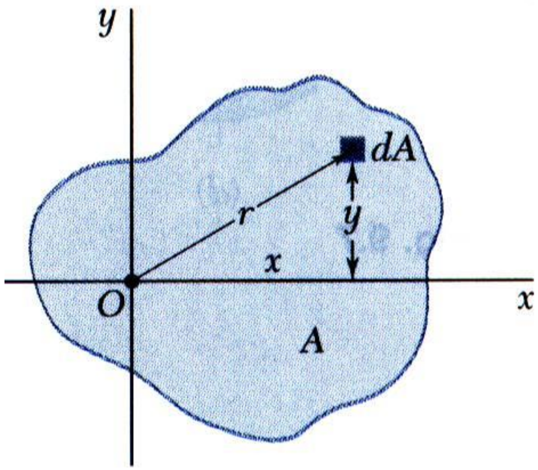
$$I_x = \int y^2 dA = \int_0^h y^2 b\,dy = \tfrac{1}{3} b h^3$$

- The formula for rectangular areas may also be applied to strips parallel to the axes,

$$dI_x = \tfrac{1}{3} y^3 dx \qquad dI_y = x^2 dA = x^2 y\,dx$$

Area Moments of Inertia

Polar Moment of Inertia



- The *polar moment of inertia* is an important parameter in problems involving torsion of cylindrical shafts and rotations of slabs.

$$J_0 = I_z = \int r^2 dA$$

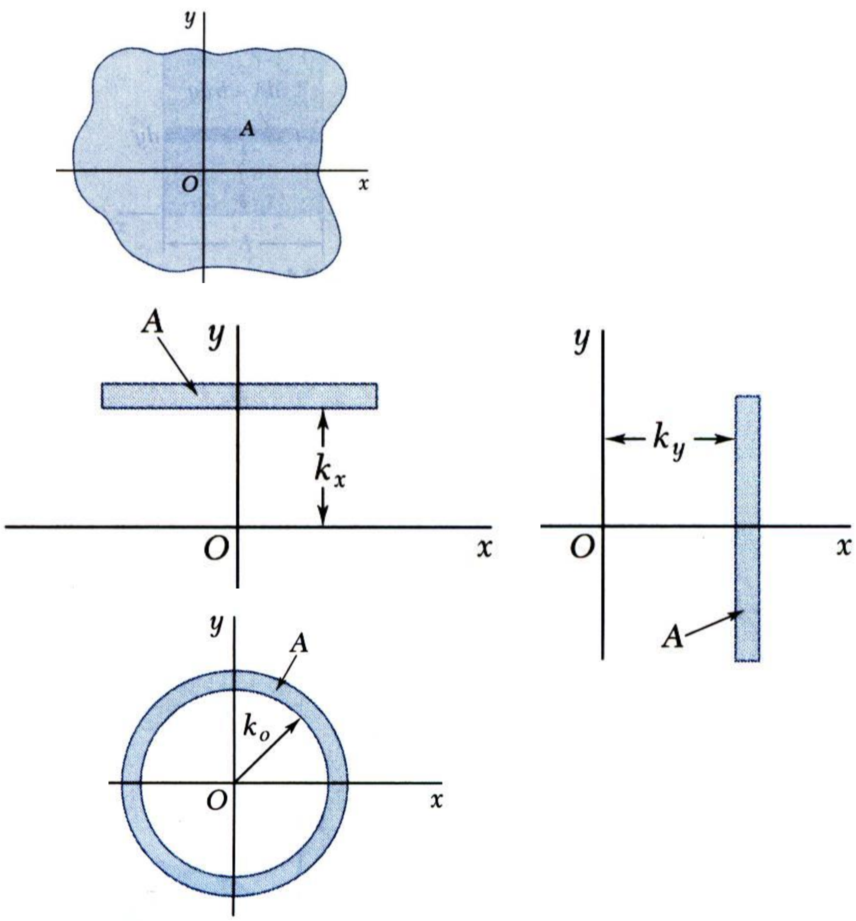
- The polar moment of inertia is related to the rectangular moments of inertia,

$$J_0 = I_z = \int r^2 dA = \int \left(x^2 + y^2\right) dA = \int x^2 dA + \int y^2 dA$$
$$= I_y + I_x$$

Moment of Inertia of an area is purely a mathematical property of the area and in itself has no physical significance.

Area Moments of Inertia

Radius of Gyration of an Area



- Consider area A with moment of inertia I_x. Imagine that the area is concentrated in a thin strip parallel to the x axis with equivalent I_x.

$$I_x = k_x^2 A \qquad k_x = \sqrt{\frac{I_x}{A}}$$

$k_x =$ *radius of gyration* with respect to the x axis

- Similarly,

$$I_y = k_y^2 A \qquad k_y = \sqrt{\frac{I_y}{A}}$$

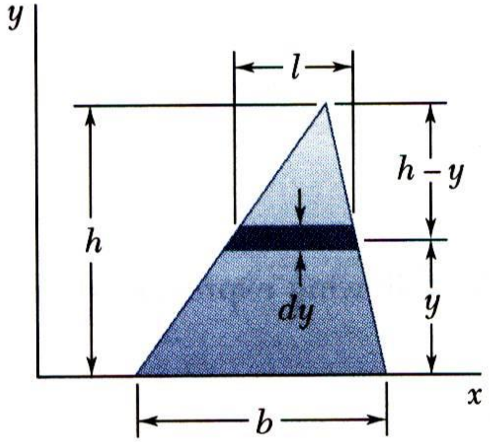
$$J_O = I_z = k_O^2 A = k_z^2 A \qquad k_O = k_z = \sqrt{\frac{J_O}{A}}$$

$$k_O^2 = k_z^2 = k_x^2 + k_y^2$$

Radius of Gyration, *k* is a measure of distribution of area from a reference axis
Radius of Gyration is different from centroidal distances

Area Moments of Inertia

Example: Determine the moment of inertia of a triangle with respect to its base.



<u>SOLUTION</u>:

- A differential strip parallel to the x axis is chosen for dA.

$$dI_x = y^2 dA \qquad dA = l\,dy$$

- For similar triangles,

$$\frac{l}{b} = \frac{h-y}{h} \qquad l = b\frac{h-y}{h} \qquad dA = b\frac{h-y}{h}dy$$

- Integrating dI_x from $y = 0$ to $y = h$,

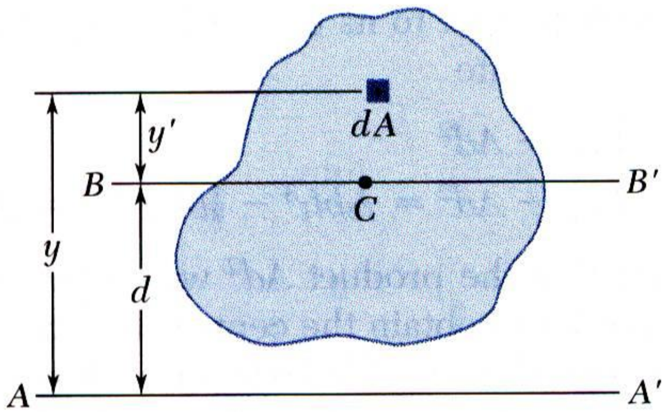
$$I_x = \int y^2 dA = \int_0^h y^2 b\frac{h-y}{h}dy = \frac{b}{h}\int_0^h \left(hy^2 - y^3\right)dy$$

$$= \frac{b}{h}\left[h\frac{y^3}{3} - \frac{y^4}{4}\right]_0^h$$

$$\boxed{I_x = \frac{bh^3}{12}}$$

Area Moments of Inertia

Parallel Axis Theorem



- Consider moment of inertia I of an area A with respect to the axis AA'

$$I = \int y^2 dA$$

- The axis BB' passes through the area centroid and is called a *centroidal axis.*

$$I = \int y^2 dA = \int (y' + d)^2 dA$$
$$= \int y'^2 dA + 2d \int y' dA + d^2 \int dA$$

- Second term = 0 since centroid lies on BB' ($\int y' dA = y_c A$, and $y_c = 0$

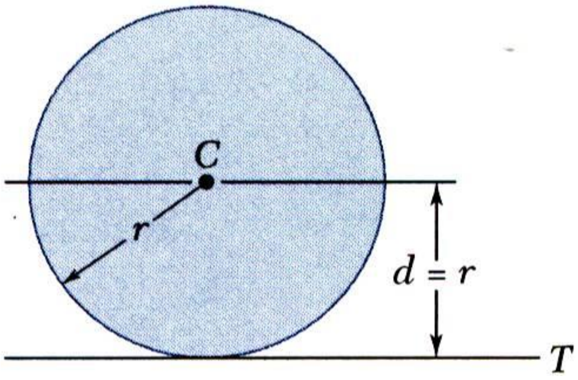
$$I = \bar{I} + Ad^2$$ *Parallel Axis theorem*

Parallel Axis theorem:
MI @ any axis =
MI @ centroidal axis + Ad^2

The two axes should be parallel to each other.

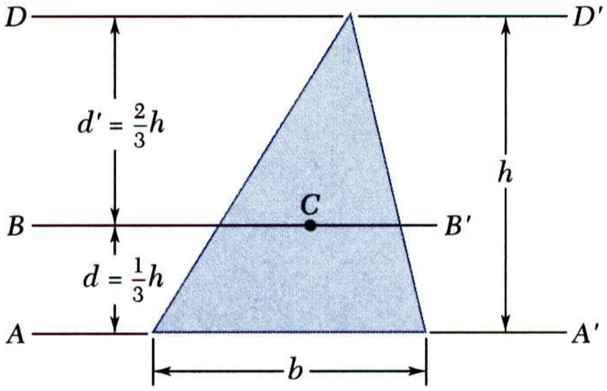
Area Moments of Inertia

Parallel Axis Theorem



- Moment of inertia I_T of a circular area with respect to a tangent to the circle,

$$I_T = \bar{I} + Ad^2 = \tfrac{1}{4}\pi r^4 + \left(\pi r^2\right)r^2$$
$$= \tfrac{5}{4}\pi r^4$$

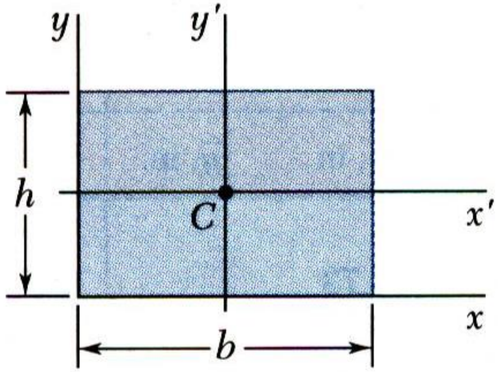


- Moment of inertia of a triangle with respect to a centroidal axis,

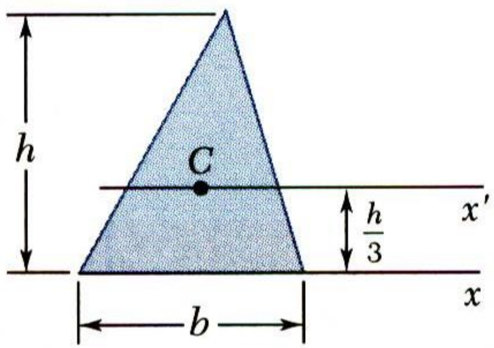
$$I_{AA'} = \bar{I}_{BB'} + Ad^2$$
$$I_{BB'} = I_{AA'} - Ad^2 = \tfrac{1}{12}bh^3 - \tfrac{1}{2}bh\left(\tfrac{1}{3}h\right)^2$$
$$= \tfrac{1}{36}bh^3$$

- The moment of inertia of a composite area A about a given axis is obtained by adding the moments of inertia of the component areas A_1, A_2, A_3, ... , with respect to the same axis.

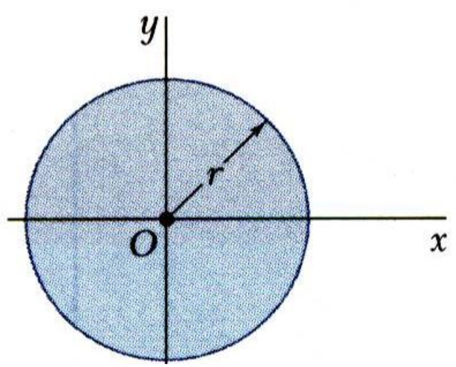
Area Moments of Inertia: Standard MIs



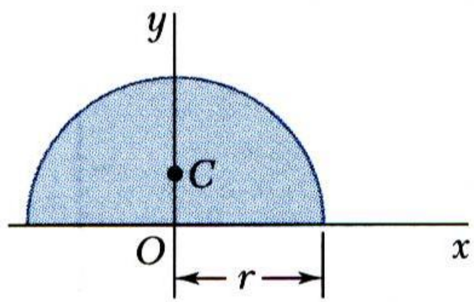
$$\bar{I}_{x'} = \frac{1}{12}bh^3$$
$$\bar{I}_{y'} = \frac{1}{12}b^3h$$
$$I_x = \frac{1}{3}bh^3$$
$$I_y = \frac{1}{3}b^3h$$
$$J_C = \frac{1}{12}bh(b^2 + h^2)$$



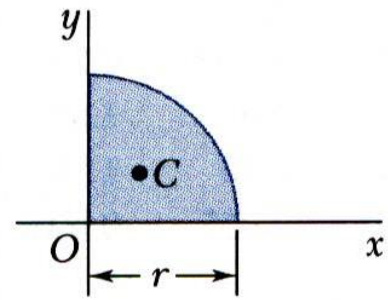
$$\bar{I}_{x'} = \frac{1}{36}bh^3$$
$$I_x = \frac{1}{12}bh^3$$



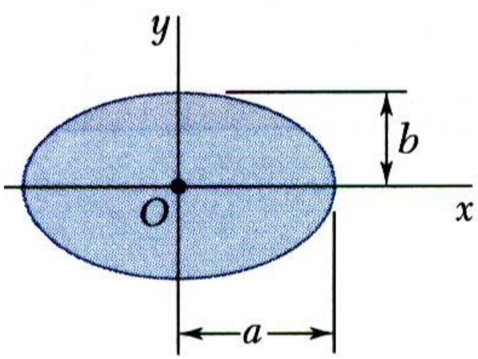
$$\bar{I}_x = \bar{I}_y = \frac{1}{4}\pi r^4$$
$$J_O = \frac{1}{2}\pi r^4$$



$$I_x = I_y = \frac{1}{8}\pi r^4$$
$$J_O = \frac{1}{4}\pi r^4$$



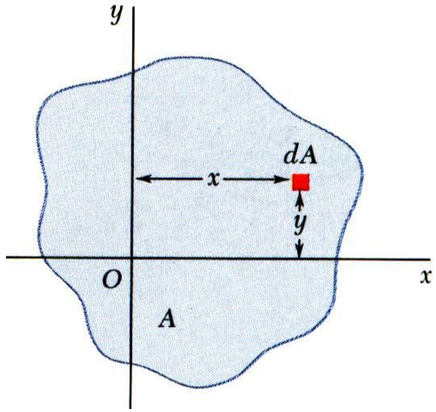
$$I_x = I_y = \frac{1}{16}\pi r^4$$
$$J_O = \frac{1}{8}\pi r^4$$



$$\bar{I}_x = \frac{1}{4}\pi ab^3$$
$$\bar{I}_y = \frac{1}{4}\pi a^3b$$
$$J_O = \frac{1}{4}\pi ab(a^2 + b^2)$$

Product of Inertia

Products of Inertia: for problems involving unsymmetrical cross-sections and in calculation of MI about rotated axes, it may be +ve, -ve, or zero

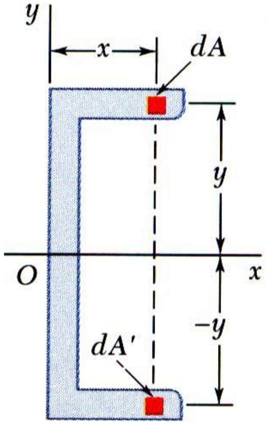


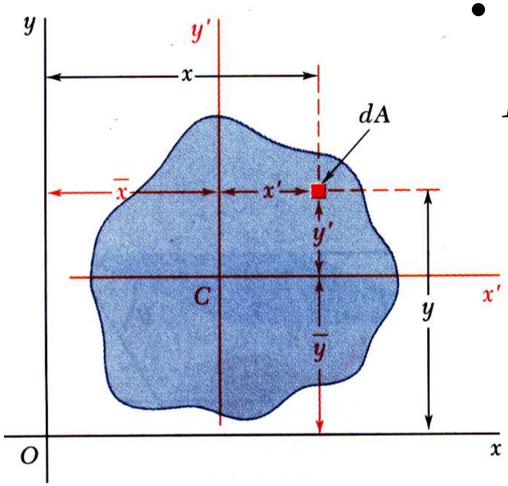
- Product of Inertia of area A w.r.t. x-y axes:

$$I_{xy} = \int xy\, dA$$

x and y are the coordinates of the element of area dA=xy

- When the x axis, the y axis, or both are an axis of symmetry, the product of inertia is zero.

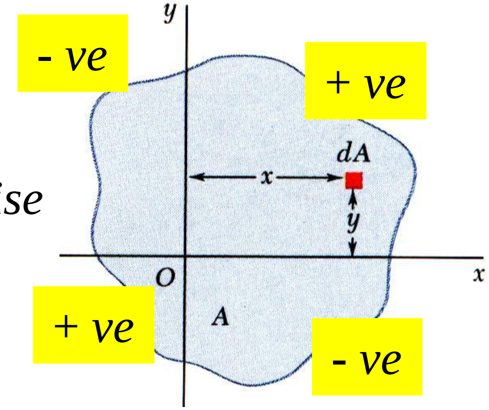




- Parallel axis theorem for products of inertia:

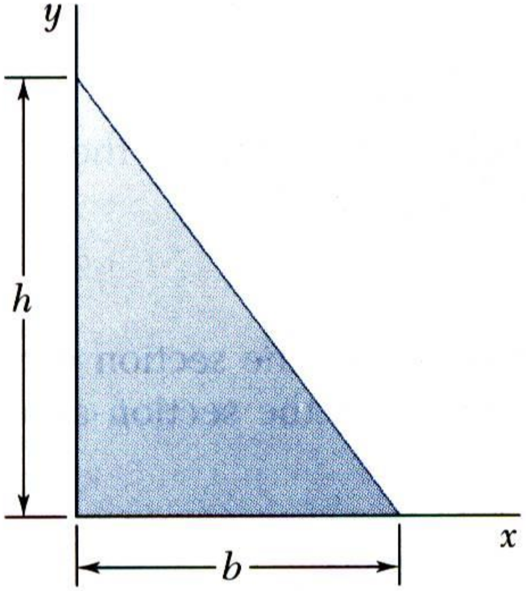
$$I_{xy} = \bar{I}_{xy} + \bar{x}\bar{y}A$$

Quadrant-wise contribution



Product of Inertia

Example: Product of Inertia



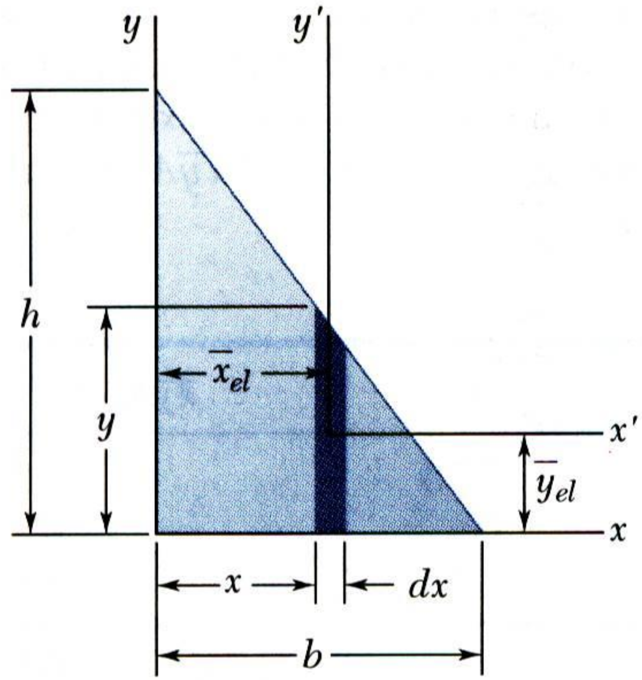
Determine the product of inertia of the right triangle *(a)* with respect to the *x* and *y* axes and
(b) with respect to centroidal axes parallel to the *x* and *y* axes.

<u>SOLUTION</u>:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips
- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

Product of Inertia

Examples



SOLUTION:

- Determine the product of inertia using direct integration with the parallel axis theorem on vertical differential area strips

$$y = h\left(1 - \frac{x}{b}\right) \qquad dA = y\,dx = h\left(1 - \frac{x}{b}\right)dx$$

$$\bar{x}_{el} = x \qquad \bar{y}_{el} = \tfrac{1}{2}y = \tfrac{1}{2}h\left(1 - \frac{x}{b}\right)$$

Integrating dI_x from $x = 0$ to $x = b$,

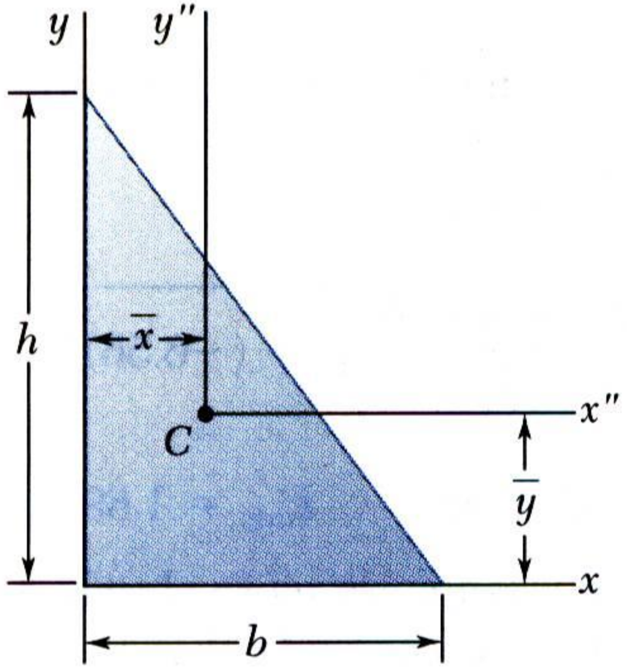
$$I_{xy} = \int dI_{xy} = \int \bar{x}_{el}\bar{y}_{el}\,dA = \int_0^b x\left(\tfrac{1}{2}\right)h^2\left(1 - \frac{x}{b}\right)^2 dx$$

$$= h^2\int_0^b\left(\frac{x}{2} - \frac{x^2}{b} + \frac{x^3}{2b^2}\right)dx = h^2\left[\frac{x^2}{4} - \frac{x^3}{3b} + \frac{x^4}{8b^2}\right]_0^b$$

$$\boxed{I_{xy} = \tfrac{1}{24}b^2h^2}$$

Product of Inertia

Examples



SOLUTION

- Apply the parallel axis theorem to evaluate the product of inertia with respect to the centroidal axes.

$$\bar{x} = \tfrac{1}{3}b \qquad \bar{y} = \tfrac{1}{3}h$$

With the results from part *a*,

$$I_{xy} = \bar{I}_{x''y''} + \bar{x}\bar{y}A$$

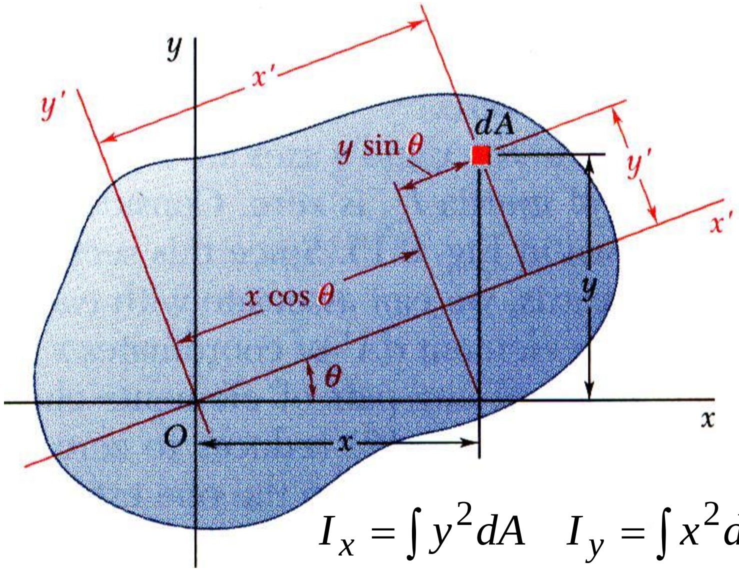
$$\bar{I}_{x''y''} = \tfrac{1}{24}b^2h^2 - \left(\tfrac{1}{3}b\right)\left(\tfrac{1}{3}h\right)\left(\tfrac{1}{2}bh\right)$$

$$\boxed{\bar{I}_{x''y''} = -\tfrac{1}{72}b^2h^2}$$

Area Moments of Inertia

Rotation of Axes

→ Determination of axes about which the MI is a maximum and a minimum



$$I_x = \int y^2 dA \quad I_y = \int x^2 dA$$

$$I_{xy} = \int xy\, dA$$

Moments and product of inertia w.r.t. new axes *x'* and *y'*?

Note: $x' = x\cos\theta + y\sin\theta$

$y' = y\cos\theta - x\sin\theta$

$$I_{x'} = \int y'^2 dA = \int (y\cos\theta - x\sin\theta)^2 dA$$

$$I_{y'} = \int x'^2 dA = \int (x\cos\theta + y\sin\theta)^2 dA$$

$$I_{x'y'} = \int x' y' dA = \int (x\cos\theta + y\sin\theta)(y\cos\theta - x\sin\theta) dA$$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\sin\theta\cos\theta = 1/2\sin 2\theta \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta$$

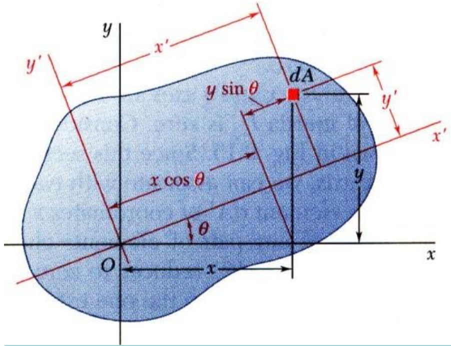
$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

Area Moments of Inertia

Rotation of Axes



$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

Adding first two eqns:

$I_{x'} + I_{y'} = I_x + I_y = I_z$ → The Polar MI @ O

Angle which makes $I_{x'}$ and $I_{y'}$ either max or min can be found by setting the derivative of either $I_{x'}$ or $I_{y'}$ w.r.t. θ equal to zero:

$$\frac{dI_{x'}}{d\theta} = \left(I_y - I_x\right)\sin 2\theta - 2I_{xy}\cos 2\theta = 0$$

Denoting this critical angle by α

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}$$

→ two values of 2α which differ by π since $\tan 2\alpha = \tan(2\alpha + \pi)$

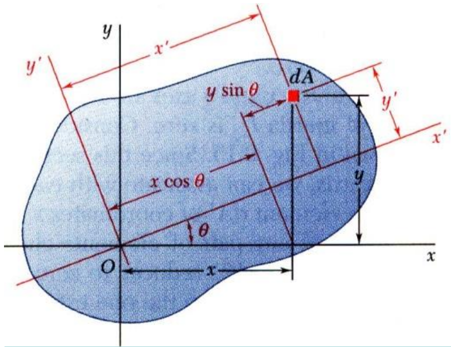
→ two solutions for α will differ by $\pi/2$

→ one value of α will define the axis of maximum MI and the other defines the axis of minimum MI

→ These two rectangular axes are called the principal axes of inertia

Area Moments of Inertia

Rotation of Axes



$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x} \Rightarrow \sin 2\alpha = \cos 2\alpha \frac{2I_{xy}}{I_y - I_x}$$

Substituting in the third eqn for critical value of 2θ: $I_{x'y'} = 0$

→ Product of Inertia $I_{x'y'}$ is zero for the Principal Axes of inertia

Substituting sin2α and cos2α in first two eqns for **Principal Moments of Inertia**:

$$I_{\max} = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4I_{xy}^2}$$

$$I_{\min} = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{\left(I_x - I_y\right)^2 + 4I_{xy}^2}$$

$$I_{xy@\alpha} = 0$$

Area Moments of Inertia

Mohr's Circle of Inertia :: Graphical representation of the MI equations

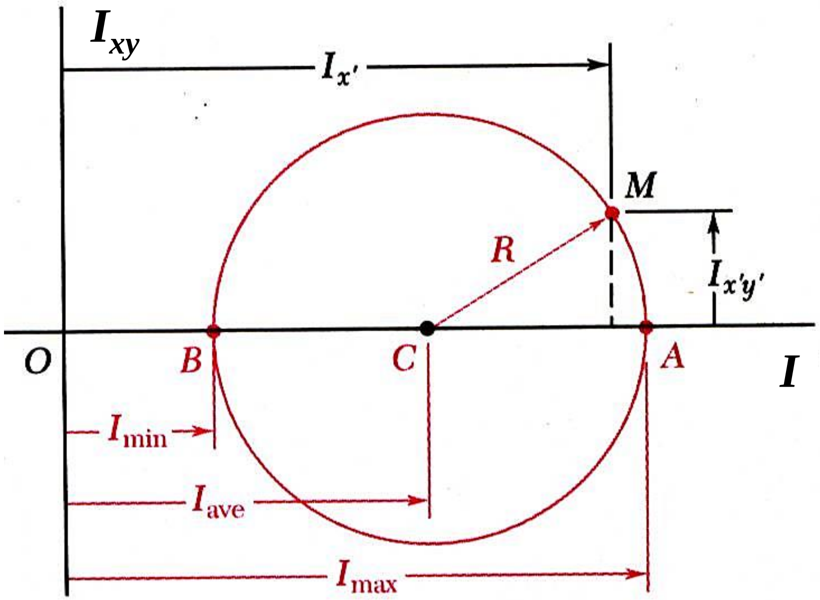
- For given values of I_x, I_y, & I_{xy}, corresponding values of $I_{x'}$, $I_{y'}$, & $I_{x'y'}$ may be determined from the diagram for any desired angle θ.

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}$$



$$(I_{x'} - I_{ave})^2 + I_{x'y'}^2 = R^2$$

$$I_{ave} = \frac{I_x + I_y}{2} \qquad R = \sqrt{\left(\frac{I_x - I_y}{2}\right)^2 + I_{xy}^2}$$

$$I_{\max} = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$

$$I_{\min} = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$

$$I_{xy@\alpha} = 0$$

- At the points A and B, $I_{x'y'} = 0$ and $I_{x'}$ takes the maximum and minimum values

$$I_{\max,\min} = I_{ave} \pm R$$

Area Moments of Inertia

Mohr's Circle of Inertia: **Construction**

$$\tan 2\alpha = \frac{2I_{xy}}{I_y - I_x}$$

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta$$

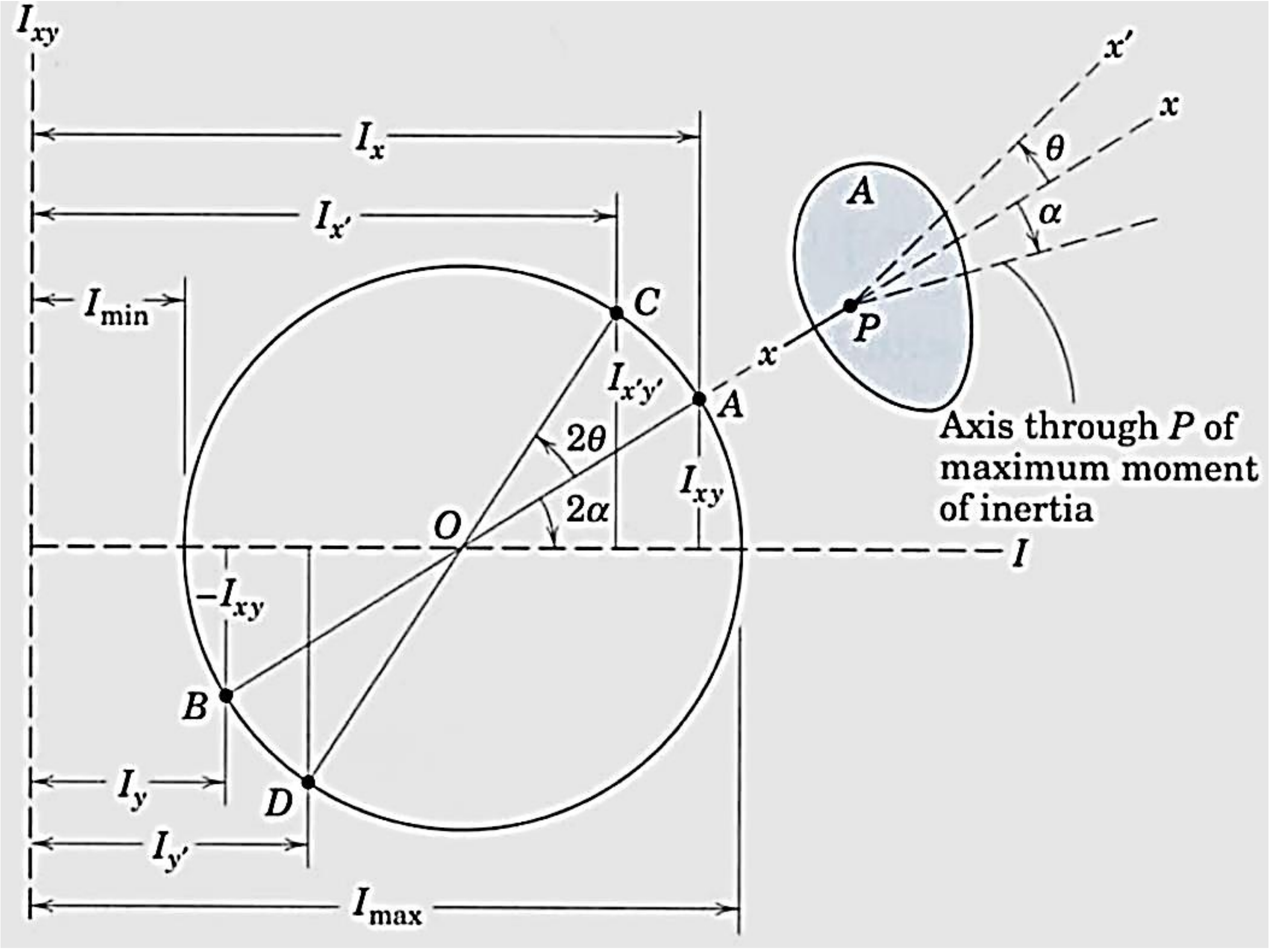
$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2}\cos 2\theta + I_{xy}\sin 2\theta$$

$$I_{x'y'} = \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta$$

$$I_{max} = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$

$$I_{min} = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$

$$I_{xy@\alpha} = 0$$



Choose horz axis → MI

Choose vert axis → PI

Point A – known {I_x, I_{xy}}

Point B – known {I_y, $-I_{xy}$}

Circle with dia AB

Angle α for Area

→ Angle 2α to horz (same sense) → I_{max}, I_{min}

Angle x to x' = θ

→ Angle OA to OC = 2θ

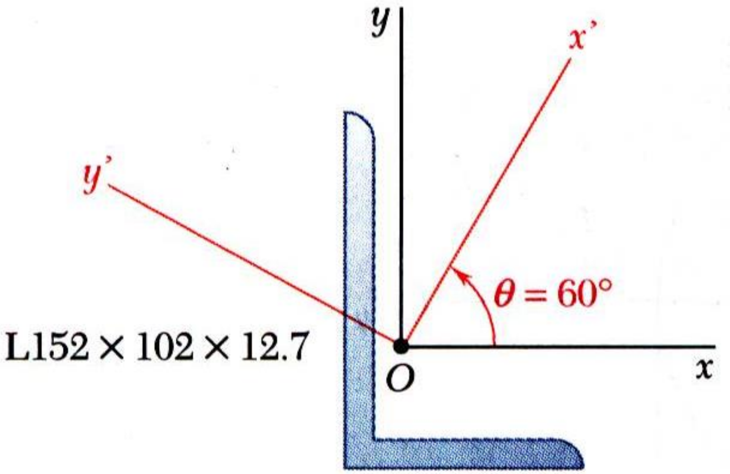
→ Same sense

Point C → $I_{x'}$, $I_{x'y'}$

Point D → $I_{y'}$

Area Moments of Inertia

Example: Mohr's Circle of Inertia



The moments and product of inertia with respect to the *x* and *y* axes are I_x = 7.24x106 mm^4, I_y = 2.61x106 mm^4, and I_{xy} = -2.54x10^6 mm^4.

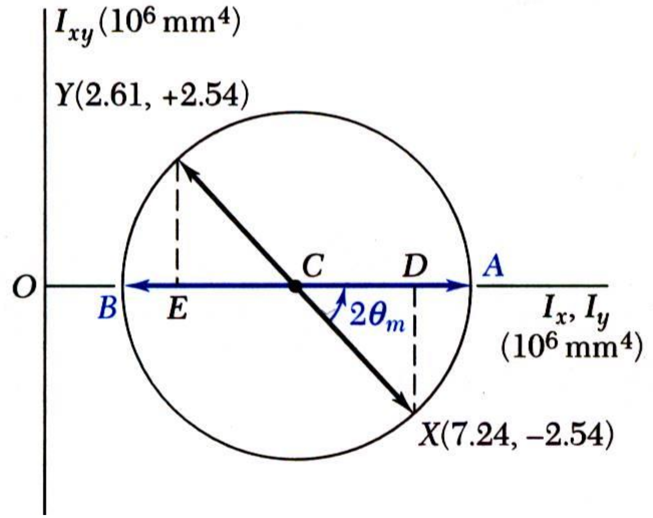
Using Mohr's circle, determine *(a)* the principal axes about *O*, *(b)* the values of the principal moments about *O*, and *(c)* the values of the moments and product of inertia about the *x'* and *y'* axes

SOLUTION:

- Plot the points (I_x , I_{xy}) and (I_y ,-I_{xy}). Construct Mohr's circle based on the circle diameter between the points.
- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.
- Based on the circle, evaluate the moments and product of inertia with respect to the *x'y'* axes.

Area Moments of Inertia

Example: Mohr's Circle of Inertia



$$I_x = 7.24 \times 10^6 \text{mm}^4$$

$$I_y = 2.61 \times 10^6 \text{mm}^4$$

$$I_{xy} = -2.54 \times 10^6 \text{mm}^4$$

SOLUTION:

- Plot the points (I_x , I_{xy}) and (I_y ,-I_{xy}). Construct Mohr's circle based on the circle diameter between the points.

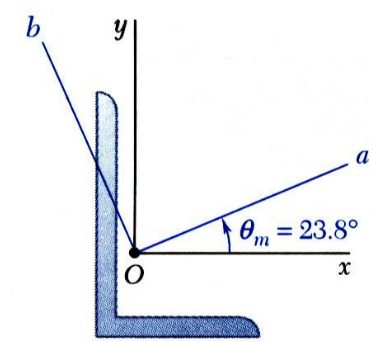
$$OC = I_{ave} = \tfrac{1}{2}\left(I_x + I_y\right) = 4.925 \times 10^6 \text{mm}^4$$

$$CD = \tfrac{1}{2}\left(I_x - I_y\right) = 2.315 \times 10^6 \text{mm}^4$$

$$R = \sqrt{(CD)^2 + (DX)^2} = 3.437 \times 10^6 \text{mm}^4$$

- Based on the circle, determine the orientation of the principal axes and the principal moments of inertia.

$$\tan 2\theta_m = \frac{DX}{CD} = 1.097 \quad 2\theta_m = 47.6° \quad \boxed{\theta_m = 23.8°}$$

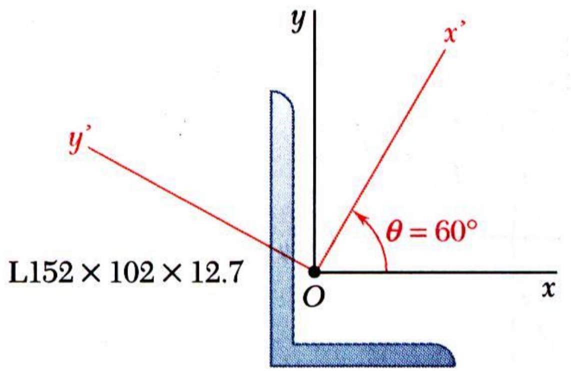


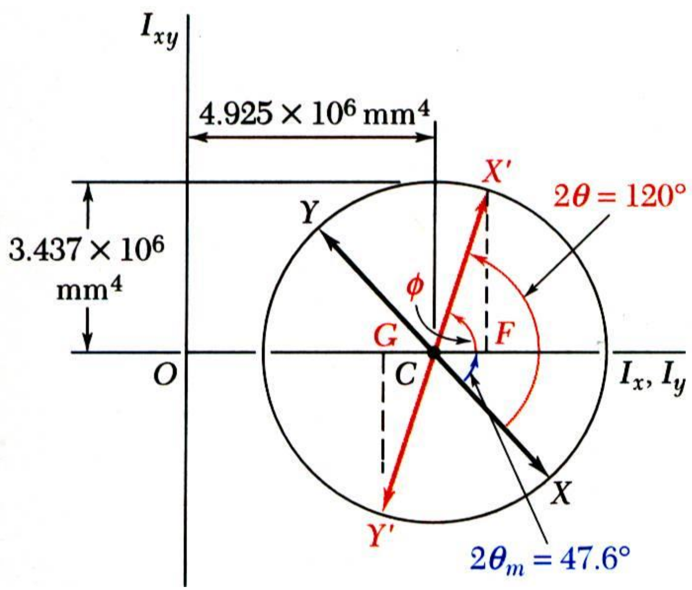
Area Moments of Inertia

Example: Mohr's Circle of Inertia

$$OC = I_{ave} = 4.925 \times 10^6 \text{mm}^4$$

$$R = 3.437 \times 10^6 \text{mm}^4$$





- Based on the circle, evaluate the moments and product of inertia with respect to the *x'y'* axes.

The points *X'* and *Y'* corresponding to the *x'* and *y'* axes are obtained by rotating *CX* and *CY* counterclockwise through an angle $\theta = 2(60^o) = 120^o$. The angle that *CX'* forms with the horz is $\phi = 120^o - 47.6^o = 72.4^o$.

$$I_{x'} = OF = OC + CX'\cos\varphi = I_{ave} + R\cos 72.4^o$$

$$\boxed{I_{x'} = 5.96 \times 10^6 \text{mm}^4}$$

$$I_{y'} = OG = OC - CY'\cos\varphi = I_{ave} - R\cos 72.4^o$$

$$\boxed{I_{y'} = 3.89 \times 10^6 \text{mm}^4}$$

$$I_{x'y'} = FX' = CY'\sin\varphi = R\sin 72.4^o$$

$$\boxed{I_{x'y'} = 3.28 \times 10^6 \text{mm}^4}$$