DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
Odd Semester of the Academic year 2019 - 2020
MA 101 Mathematics I
Problem Sheet 1: Revision of vectors, equations of lines and planes, vector differentiation, limits and continuities of functions of several variables.
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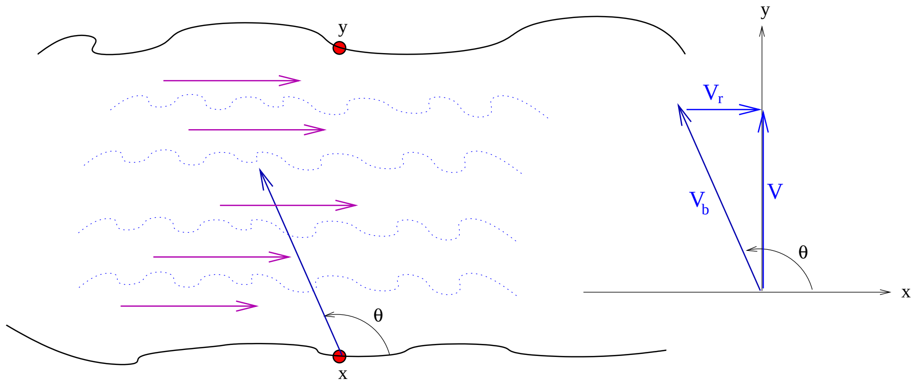
1. A man wants to paddle his boat across a river from point X to the point Y on the



opposite shore directly across from X. If he can paddle the boat at the rate of 5 kilometers per hour and the current in the river is 3 kilometers per hour, in what direction θ should he steer his boat in order to go straight across the river? Also what is his resultant speed across the river?

2. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is

$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

3. (a) Find a point at which the given lines intersect:

$$\mathbf{r} =< 1, 1, 0 > +t < -1, 1, 2 >$$
$$\mathbf{r} =< 2, 0, 2 > +s < -1, 1, 0 >$$

(b) Find an equation of the plane that contains these lines.

4. Consider the three planes $x - y + z = 1$, $x + \lambda y - 2z + 10 = 0$ and $2x - 3y + z + \mu = 0$ where λ and μ are parameters. Determine the values of λ and μ such that the planes
(a) intersect at a single point.
(b) intersect in a line.
(c) intersect (taken two at a time) in three distinct parallel lines.

5. Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

6. Find a point on the curve $\mathbf{r} = 4\cos(t)\mathbf{i} + 4\sin(t)\mathbf{j} + 3t\mathbf{k}$ at a distance 10π units along the curve from the origin in the direction of increasing arc length.

7. Reparametrize the curve

$$\mathbf{r}(t) = \left(\frac{2}{t^2+1} - 1\right)\mathbf{i} + \frac{2t}{t^2+1}\mathbf{j}$$

with respect to the arc length measured from the point $(1,0)$ in the direction of increasing t. Express the parametrization in its simplest form. What can you conclude about the curve?

8. Show that the curvature of a plane parametric curve $x = f(t),\ y = g(t)$ is

$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{\frac{3}{2}}}$$

where the dots indicate the derivatives with respect to t.

9. At what point does the curve $y = e^x$ have maximum curvature? What happens to the curvature as $x \to \infty$?

10. Find the unit tangent vector, unit normal vector and the binormal at the given point (a) $\mathbf{r}(t) = \left\langle t^2, \frac{2}{3}t^3, t \right\rangle,\ \left(1, \frac{2}{3}, 1\right)$ and (b) $\mathbf{r}(t) =< \cos(t), \sin(t), \ln(\cos(t)) >,\ (1,0,0)$.

11. The helix $\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1+t)\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at the point $(1,0,0)$. Find the angle of intersection of these curves.

12. If $\mathbf{u}(t) = \hat{\imath} - 2t^2\hat{\jmath} + 3t^3\hat{k}$ and $\mathbf{v}(t) = t\hat{\imath} + \cos t\hat{\jmath} + \sin t\hat{k}$, find (a) $D_t[\mathbf{u}(t)\cdot\mathbf{v}(t)]$ and (b) $D_t[\mathbf{u}(t)\times\mathbf{v}(t)]$. Also verify that (c) $\lim\limits_{t\to\pi}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \lim\limits_{t\to\pi}[\mathbf{u}(t)]\cdot\lim\limits_{t\to\pi}[\mathbf{v}(t)]$ and $\lim\limits_{t\to\pi}[\mathbf{u}(t)\times\mathbf{v}(t)] = \lim\limits_{t\to\pi}[\mathbf{u}(t)]\times\lim\limits_{t\to\pi}[\mathbf{v}(t)]$

13(a) A particle moves with constant speed along a curve in space. Show that its velocity and acceleration vectors are always perpendicular.
(b) Let $\mathbf{r}(t) = (2t^3+3)\hat{\imath} + (\ln t)\hat{\jmath} + 3\hat{k}$ be the position vector of a moving particle at time $t > 0$. Find the time(s) at which velocity and acceleration vectors are perpendicular.

14. Find the limit if it exists, or show that the limit does not exist.
(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$ (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$ (d) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^3}{x^2+y^6}$
(e) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x\sin(x^2+y^2)}{x^2+y^2}$ (f) $\displaystyle\lim_{(x,y)\to(4,\pi)} x^2\sin\left(\frac{y}{x}\right)$ (g) $\displaystyle\lim_{(x,y)\to(0,1)} \frac{x+y-1}{\sqrt{x}-\sqrt{1-y}}$

15. Examine whether

$$f(x,y) = \begin{cases} \frac{xy(y^2-x^2)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

is a continuous function.

16. Let $f(x,y) = \begin{cases} \frac{x^2-xy}{x+y} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0). \end{cases}$
Find (a) $f_x(0,0)$, $f_y(0,0)$ and (b) also find $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$, and check whether it is equal to $f_x(0,0)$.

17. Let $f(x,y) = \sqrt{x^2+y^2}$.
(a) Find $f_x(x,y)$ and $f_y(x,y)$ for $(x,y) \neq (0,0)$. (b) Show that $f_x(0,0)$ and $f_y(0,0)$ do not exist.

18. Let

$$f(x,y) = \begin{cases} \frac{xy^3}{x^2+y^6} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

(a) Calculate $f_x(x,y)$ and $f_y(x,y)$ at all points where $(x,y) \neq (0,0)$.
(b) Compute all first and second order partial derivatives at $(0,0)$ if they exist.
(c) Show that f is discontinuous at $(0,0)$.

Extra Questions

1. Show that the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ represents the volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

2. If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are noncoplanar vectors, let

$$\mathbf{k_1} = \frac{\mathbf{v_2} \times \mathbf{v_3}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})}, \quad \mathbf{k_2} = \frac{\mathbf{v_3} \times \mathbf{v_1}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})} \quad \text{and} \quad \mathbf{k_3} = \frac{\mathbf{v_1} \times \mathbf{v_2}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})}.$$

Show that
(a) $\mathbf{k}_i$ is perpendicular to to $\mathbf{v}_j$ if $i \neq j$.
(b) $\mathbf{k}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$.
(c) $\mathbf{k_1} \cdot (\mathbf{k_2} \times \mathbf{k_3}) = \dfrac{\mathbf{1}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})}$.

3. Given the vectors $\mathbf{a} = (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$ and $\mathbf{b} = (\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3})$, verify that

$$|\mathbf{a} \times \mathbf{b}|^{\mathbf{2}} = |\mathbf{a}|^{\mathbf{2}} |\mathbf{b}|^{\mathbf{2}} - (\mathbf{a} \cdot \mathbf{b})^{\mathbf{2}},$$

by computing each side in terms of the components of $\mathbf{a}$ and $\mathbf{b}$.