Special Theory of Relativity (PH101)
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Quiz-II Solution
Wednesday, 30th of October, 2019 (08:00 Hrs.-08:50 Hrs. IST), Total Marks-10

1. A train with proper length L_0 moves at speed $c/2$ with respect to the ground. An observer in the back of the train throws a ball at $t = 0$ towards the front wall of the train with a speed $c/3$. How long the ball will take to hit the front wall of the train, and what distance will it cover in (a) the train frame, (b)the ground frame, and (c)the ball frame? Compute the event separation interval ($\mathbf{ds^2}$) in all three frames.[**0.5+2.0+2.0+1.5**]

Solution: (a)In the train frame, the distance is simply $d = L_0$. And the time is $t = L_0/(c/3) = 3L_0/c$.

(b) The velocity of the ball with respect to the ground is (with $u' = c/3$ and $v = c/2$)

$$u_g = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{\frac{c}{2} + \frac{c}{3}}{1 + \frac{1}{3} \cdot \frac{1}{2}} = \frac{5c}{7}.$$

The length of the train from the ground frame is $L_0/\gamma_v = \sqrt{3}L_0/2$. Therefore, at time t the position of the front of train is $\sqrt{3}L_0/2 + vt$. And the position of the ball is $u_g t$. These two positions are equal when

$$(u_g - v)t = \frac{\sqrt{3}L}{2} \Longrightarrow t = \frac{\frac{\sqrt{3}L_0}{2}}{\frac{5c}{7} - \frac{c}{2}} = \frac{7L_0}{\sqrt{3}c}$$

.

Alternate solution

In the train frame, the space and time intervals are $x' = L_0$ and $t' = 3L_0/c$, from part (a). Also $\gamma_v = 2/\sqrt{3}$, so the Lorentz transformations give the coordinates in the ground frame as

$$x = \gamma_v(x' + vt) = \frac{2}{\sqrt{3}}\left(L_0 + \frac{c}{2}\left(\frac{3L_0}{c}\right)\right) = \frac{5L_0}{\sqrt{3}}.$$

$$t = \gamma(t' + vx'/c^2) = \frac{2}{\sqrt{3}}\left(\frac{3L_0}{c} + \frac{\frac{c}{2}(L_0)}{c^2}\right) = \frac{7L_0}{\sqrt{3}c}$$

.

(c) In the ball frame, the train length is $L_0/\gamma_{1/3} = \sqrt{8}L_0/3$. Therefore, the time it takes the train to fly past the ball at speed $c/3$ is $t = (\sqrt{8}L_0/3)/(c/3) = 2\sqrt{2}L_0/c$ and $d = 0$ (because ball does not move in its frame.)

The value of event separation interval are:

Train frame: $c^2t^2 - x^2 = c^2(3L_0/c)^2 - L_0^2 = 8L_0^2$.

Ground frame: $c^2t^2 - x^2 = c^2(7L_0/\sqrt{3}c)^2 - (5L_0/\sqrt{3})^2 = 8L_0^2$

Ball frame: $c^2t^2 - x^2 = c^2(2\sqrt{2}L_0/c)^2 - (0)^2 = 8L_0^2$

2. Two balls of rest mass m_0 and M_0 moving towards each other along the x-axis collide and after collision they stick to each other. Assume that there is no energy loss during the entire process. Considering that the speed of each balls is $c/2$ w.r.t. an observer at rest, compute the velocity and the kinetic energy of the combined mass. **[2+2]**
Solution:
Consider that the two objects have rest masses m_0 and M_0 moving along +x and -x directions respectively.

The rest mass of the combined object formed after the collision is M_0' .
The goal is to find the velocity and kinetic energy of M_0'.
Given that the speed of the balls before collision $u = c/2$.
Let's denote the velocity of the combined object is v.
We will use the principle of conservation of energy and momentum for this event as seen from the observer at rest.
Energy conservation leads to:

$$E_{m_0} + E_{M_0} = E_{M_0'}$$

$$\Rightarrow \Gamma_u m_0 c^2 + \Gamma_u M_0 c^2 = \Gamma_v M_0' c^2$$

$$\Rightarrow \Gamma_u m_0 + \Gamma_u m_0 = \Gamma_v M_0', \tag{1}$$

Where $\Gamma_u = \dfrac{1}{\sqrt{1 - u^2/c^2}} = 2/\sqrt{3}$.
Using the value of Γ_u in Eq. 1 we have,

$$\frac{2}{\sqrt{3}}(m_0 + M_0) = \Gamma_v M_0' \tag{2}$$

Similarly using the momentum conservation we have ,

$$\Gamma_u m_0 c/2 - \Gamma_u M_0 c/2 = \Gamma_v M_0' v$$

$$\Rightarrow \frac{2}{\sqrt{3}} m_0 c/2 - \frac{2}{\sqrt{3}} M_0 c/2 = \Gamma_v M_0' v$$

$$\Rightarrow \frac{c}{\sqrt{3}}(m_0 - M_0) = \Gamma_v M_0' v \tag{3}$$

Dividing Eq. 3 by Eq. 2 we will get the expression for v as,

$$v = \frac{c}{2}\left(\frac{m_0 - M_0}{m_0 + M_0}\right) \tag{4}$$

With this v we can obtain the expression of Γ_v for the combined mass M_0' as :

$$\Gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \dfrac{\frac{c^2}{4}\left(\frac{m_0 - M_0}{m_0 + M_0}\right)^2}{c^2}}} = \frac{1}{\sqrt{1 - \frac{1}{4}\left(\frac{m_0 - M_0}{m_0 + M_0}\right)^2}} \tag{5}$$

Substituting the expression for Γ_v in Eq. 2 we can obtain the value of M_0' as

$$M_0' = \frac{\frac{2}{\sqrt{3}}(m_0 + M_0)}{\dfrac{1}{\sqrt{1 - \frac{1}{4}\left(\frac{m_0 - M_0}{m_0 + M_0}\right)^2}}} = \frac{2}{\sqrt{3}}(m_0 + M_0)\sqrt{1 - \frac{1}{4}\left(\frac{m_0 - M_0}{m_0 + M_0}\right)^2}$$

$\therefore$ the K.E. of the combined mass M_0' after the collision can be given by

$$K.E = (\Gamma_v - 1) M_0' c^2$$

$$= \left(\frac{1}{\sqrt{1 - \frac{1}{4}\left(\frac{m_0 - M_0}{m_0 + M_0}\right)^2}} - 1\right)\frac{2}{\sqrt{3}}(m_0 + M_0)\sqrt{1 - \frac{1}{4}\left(\frac{m_0 - M_0}{m_0 + M_0}\right)^2}\, c^2$$

$$= \frac{2}{\sqrt{3}}(m_0 + M_0)\left[1 - \sqrt{1 - \frac{1}{4}\left(\frac{m_0 - M_0}{m_0 + M_0}\right)^2}\,\right] c^2$$

Note that the above problem can be solved by considering that m_0 is moving in the -ve x-direction and M_0 is moving in the +ve direction. In that case $v = -\frac{c}{2}\left(\frac{m_0 - M_0}{m_0 + M_0}\right)$. Other quantities will remain unaltered.