Department of Mathematics
Indian Institute of Technology Guwahati
MA 101: Mathematics I
Continuity
July-December 2019

Definition 1. *Let $D \subseteq \mathbb{R}$ and let $x_0 \in \mathbb{R}$ such that for some $h > 0$, $(x_0-h, x_0+h)\setminus\{x_0\} \subseteq D$. If $f : D \to \mathbb{R}$, then $\ell \in \mathbb{R}$ is said to be the limit of f at x_0 if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in D$ satisfying $0 < |x - x_0| < \delta$.*

We write: $\lim\limits_{x\to x_0} f(x) = \ell$. In the following theorem, we prove a sequential criterion for limit.

Theorem 1 (Sequential criterion)**.** *Let $D \subseteq \mathbb{R}$ and let $x_0 \in \mathbb{R}$ such that for some $h > 0$, $(x_0 - h, x_0 + h) \setminus \{x_0\} \subseteq D$. Let $f : D \to \mathbb{R}$. Then the following are equivalent.*

(a) $\lim\limits_{x\to x_0} f(x) = \ell$.

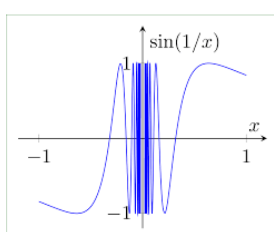
(b) *For any sequence (x_n) in D with $x_n \neq x_0$ for all $n \geq 1$ and $x_n \to x_0$, the sequence $(f(x_n))$ converges to ℓ.*

Proof. (a) $\Rightarrow$ (b): Suppose that $\lim\limits_{x\to x_0} f(x) = \ell$. Then for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - \ell| < \varepsilon$ for all $x \in D$ satisfying $0 < |x - x_0| < \delta$. Suppose that (x_n) is a sequence in D converging to x_0. Then there exists $n_0 \in \mathbb{N}$ such that $|x_n - x_0| < \delta$ for all $n \geq n_0$. Then we have $|f(x_n) - \ell| < \varepsilon$ for all $n \geq n_0$, and hence $f(x_n) \to \ell$.

(b) $\Rightarrow$ (a): Suppose that $x_n \to x_0$ implies $f(x_n) \to \ell$. We claim that $\lim\limits_{x\to x_0} f(x) = \ell$. To see why the cliam must be true, assume otherwise. Then there exists some $\varepsilon_0 > 0$ such that for any $\delta > 0$, there is $x \in D$ such that $0 < |x - x_0| < \delta$ and $|f(x) - \ell| \geq \varepsilon_0$. We now apply this to each $\delta = 1, \frac{1}{2}, \ldots, \frac{1}{n}, \ldots$. For $\delta = \frac{1}{n}$, we pick $x_n \in D$ satisfying $0 < |x_n - x_0| < \frac{1}{n}$. Then $x_n \to x_0$ but $|f(x_n) - \ell| \geq \varepsilon_0$. Thus, we have found a sequence (x_n) in D with $x_n \to x_0$, but the sequence $(f(x_n))$ does not converge to ℓ. Therefore, (a) must be true. $\square$

Example 1. $\lim\limits_{x\to 0} \sin\frac{1}{x}$ *does not exist.*

Solution. We have



Let $x_n = \frac{2}{(4n+1)\pi}$ and $y_n = \frac{1}{n\pi}$ for all $n \in \mathbb{N}$. Then $x_n \to 0$ and $y_n \to 0$. Since $\sin\frac{1}{x_n} = 1$ and $\sin\frac{1}{y_n} = 0$ for all $n \in \mathbb{N}$, we get $\sin\frac{1}{x_n} \to 1$ and $\sin\frac{1}{y_n} \to 0$. Therefore by the sequential criterion for limit, $\lim\limits_{x\to 0} \sin\frac{1}{x}$ does not exist. $\square$

Definition 2. *A function $f : D \to \mathbb{R}$ is called bounded if there exists $M > 0$ such that $|f(x)| < M$ for all $x \in D$.*

Theorem 2. *Let $f : D \to \mathbb{R}$. Suppose that $\lim_{x \to x_0} f(x) = \ell$. Then there exists some $\delta > 0$ such that f is bounded on $(x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$. That is, there exists $M > 0$ such that $|f(x)| < M$ for all $x \in (x_0 - \delta, x_0 + \delta)$ with $x \neq x_0$.*

Proof. Let $\varepsilon > 0$. Since $\lim_{x \to x_0} f(x) = \ell$, so there is some $\delta > 0$ such that

$$|f(x) - \ell| < \varepsilon \quad \text{whenever} \quad 0 < |x - x_0| < \delta.$$

Take $M = \varepsilon + |\ell|$. Then we have $|f(x)| < M$ for all $x \in (x_0 - \delta, x_0 + \delta)$ with $x \neq x_0$. $\square$

Remark 1. *If $x_0 \in D$, then by taking $M = \max\{\varepsilon + |\ell|, |f(x_0)|\}$ we have $|f(x)| < M$ for all $x \in (x_0 - \delta, x_0 + \delta)$.*

Theorem 3 (Limit Theorems)**.** *Let $D \subseteq \mathbb{R}$ and let $x_0 \in \mathbb{R}$ such that for some $h > 0$, $(x_0 - h, x_0 + h) \setminus \{x_0\} \subseteq D$. Let $f, g, j : D \to \mathbb{R}$. Suppose that $\lim_{x \to x_0} f(x) = \ell$ and $\lim_{x \to x_0} g(x) = m$. Then*

(1) $\lim_{x \to x_0} (f(x) \pm g(x)) = \ell \pm m$.

(2) *If $f(x) \leq g(x)$ for all $x \in (x_0 - h, x_0 + h) \setminus \{x_0\}$, then $\ell \leq m$.*

(3) $\lim_{x \to x_0} (fg)(x) = \ell m$ *and if $m \neq 0$ and $g(x) \neq 0$ for all $x \in D$, then* $\lim_{x \to x_0} \dfrac{1}{g(x)} = \dfrac{1}{m}$.

(4) *If $f(x) \leq j(x) \leq g(x)$ for all $x \in (x_0 - h, x_0 + h) \setminus \{x_0\}$ and $\ell = m$, then $\lim_{x \to x_0} j(x) = \ell$.*

Proof. (2) We will use sequential criterion to prove the result. Let (x_n) be a sequence in D with $x_n \neq x_0$ for all $n \geq 1$ and $x_n \to x_0$. Since $\lim_{x \to x_0} f(x) = \ell$ and $\lim_{x \to x_0} g(x) = m$, so $\lim_{n \to \infty} f(x_n) = \ell$ and $\lim_{n \to \infty} g(x_n) = m$. Also $x_n \to x_0$, and therefore there is some $n_0 \in \mathbb{N}$ such that $x_n \in (x_0 - h, x_0 + h)$ for all $n \geq n_0$. Since $f(x) \leq g(x)$ for all $x \in (x_0 - h, x_0 + h) \setminus \{x_0\}$, so $f(x_n) \leq g(x_n)$ for all $n \geq n_0$. This yields $\lim_{n \to \infty} f(x_n) \leq \lim_{n \to \infty} g(x_n)$, and hence $\ell \leq m$.

(3) We will use the definition to prove this result. Since $\lim_{x \to x_0} f(x) = \ell$, so there exist $\delta_1 > 0$ and $M > 0$ such that

$$|f(x)| < M \quad \text{for all} \quad x \in (x_0 - \delta_1, x_0 + \delta_1) \quad \text{with} \quad x \neq x_0.$$

Let $\varepsilon > 0$. Then there exist $\delta_2, \delta_3 > 0$ such that $|f(x) - \ell| < \frac{\varepsilon}{2(|m|+1)}$ whenever $0 < |x - x_0| < \delta_2$ and $|g(x) - m| < \varepsilon/2M$ whenever $0 < |x - x_0| < \delta_3$. Let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then

$$\begin{aligned} |f(x)g(x) - \ell m| &= |f(x)g(x) - f(x)m + f(x)m - \ell m| \\ &\leq |f(x)||g(x) - m| + |m||f(x) - \ell| \\ &< \varepsilon \quad \text{whenever} \quad 0 < |x - x_0| < \delta. \end{aligned}$$

This proves that $\lim_{x \to x_0} (fg)(x) = \ell m$. $\square$

Theorem 4. *Suppose that $f(x)$ is bounded in $(x_0-h, x_0+h)\setminus\{x_0\}$ for some $h>0$ and $\lim_{x\to x_0} g(x)=0$. Then $\lim_{x\to x_0} f(x)g(x)=0$.*

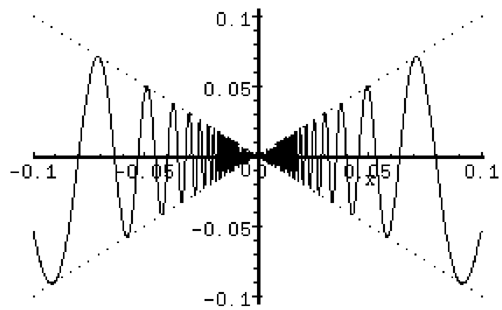
Proof. Let $M>0$ be such that $|f(x)|<M$ for all $x\in(x_0-h, x_0+h)\setminus\{x_0\}$. Let $\varepsilon>0$ be given. Since $\lim_{x\to x_0} g(x)=0$, so there exists $\delta_1>0$ such that $|g(x)-0|<\varepsilon/M$ whenever $0<|x-x_0|<\delta_1$. Let $\delta=\min\{\delta_1, h\}$. Then,

$$|f(x)g(x)-0|<\varepsilon \quad \text{whenever} \quad 0<|x-x_0|<\delta.$$

Hence, $\lim_{x\to x_0} f(x)g(x)=0$. □

Example 2. $\lim_{x\to 0} x\sin\frac{1}{x}=0$.

Solution. We have $\sin\frac{1}{x}$ is bounded on $\mathbb{R}\setminus\{0\}$. Hence the result follows by the previous theorem.



□

Definition 3. *Let $D\subseteq\mathbb{R}$ and let $x_0\in\mathbb{R}$ such that for some $h>0$, $(x_0, x_0+h)\subseteq D$. If $f: D\to\mathbb{R}$, then $\ell\in\mathbb{R}$ is said to be the right hand limit of f at x_0 if for each $\varepsilon>0$, there exists $\delta>0$ such that*

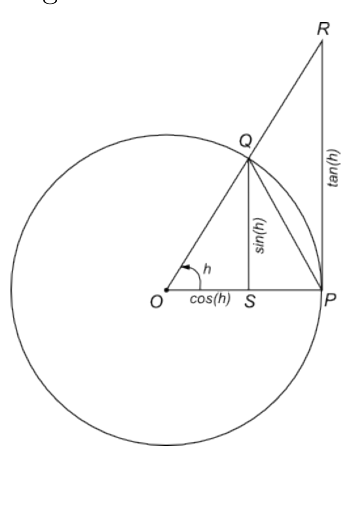
$$|f(x)-\ell|<\varepsilon \quad \textit{whenever} \quad x\in D \quad \textit{and} \quad 0<x-x_0<\delta.$$

Notation for right hand limit: $\lim_{x\to x_0+} f(x)=\ell$.
Similarly one defines left hand limit of f at x_0 and is denoted by $\lim_{x\to x_0-} f(x)$.

Theorem 5. $\lim_{x\to x_0} f(x)=\ell \Leftrightarrow \lim_{x\to x_0+} f(x)=\lim_{x\to x_0-} f(x)=\ell$.

Example 3. *Show that* $\lim_{h\to 0}\frac{\sin h}{h}=1$.

Solution. We consider the following unit circle.



We have

$$\begin{aligned}
&\text{Area of } \ \Delta OPQ < \ \text{Area of the circular section } \ OPQO < \ \text{Area of } \ \Delta OPR \\
&\Rightarrow 1 < \frac{\text{Area of the circular section } \ OPQO}{\text{Area of } \ \Delta OPQ} < \frac{\text{Area of } \ \Delta OPR}{\text{Area of } \ \Delta OPQ} \\
&\Rightarrow 1 < \frac{h}{\sin h} < \frac{\text{Area of } \ \Delta OPR}{\text{Area of } \ \Delta OPQ}
\end{aligned}$$

Now $\dfrac{\text{Area of } \ \Delta OPR}{\text{Area of } \ \Delta OPQ} \to 1$ as $h \to 0+$. Therefore, we have $\lim\limits_{h\to 0+} \frac{\sin h}{h} = 1$. Since $\frac{\sin h}{h}$ is an even function, so $\lim\limits_{h\to 0-} \frac{\sin h}{h} = 1$. This proves that $\lim\limits_{h\to 0} \frac{\sin h}{h} = 1$. □

Definition 4. *$f(x)$ has limit ℓ as x approaches $+\infty$, if for any given $\varepsilon > 0$, there exists $M > 0$ such that*

$$x > M \implies |f(x) - \ell| < \varepsilon.$$

Similarly, one can define limit of $f(x)$ as x approaches $-\infty$.

Example 4. (i) $\lim\limits_{x\to\infty} \dfrac{1}{x} = 0$, (ii) $\lim_{x\to-\infty} \frac{1}{x} = 0$, (iii) $\lim\limits_{x\to\infty} \sin x$ *does not exist.*

Definition 5. *A function $f(x)$ approaches ∞ as $x \to x_0$ if for every real $M > 0$, there exists $\delta > 0$ such that*

$$0 < |x - x_0| < \delta \implies f(x) > M.$$

Similarly, one can define limit of $f(x)$ approaching $-\infty$.

Example 5. (i) $\lim\limits_{x\to 0} \dfrac{1}{x^2} = \infty$, (ii) $\lim\limits_{x\to 0} \dfrac{1}{x^2} \sin(1/x)$ *does not exist.*

Solution. For (ii), let $x_n = \frac{1}{\frac{\pi}{2} + 2n\pi}$ and $y_n = \frac{1}{n\pi}$. Then $x_n, y_n \to 0$ as $n \to \infty$.
But $\lim\limits_{n\to\infty} f(x_n) = \dfrac{1}{x_n^2} \to \infty$ and $\lim\limits_{n\to\infty} f(y_n) = 0$. □

Theorem 6. *Suppose that $\lim\limits_{x\to x_0} f(x) = \ell$. If $\ell \neq 0$, then there exists some δ such that $f(x) \neq 0$ for all $x \in (x_0 - \delta, x_0 + \delta) \setminus \{x_0\}$.*

Proof. **Case-I:** Let $\ell > 0$. Take $\varepsilon = \frac{\ell}{2} > 0$. Then there exists $\delta > 0$ such that

$$f(x) > \ell - \varepsilon = \ell - \frac{\ell}{2} = \frac{\ell}{2} > 0 \ \text{ whenever } \ 0 < |x - x_0| < \delta.$$

Case-II: Let $\ell < 0$. Take $\varepsilon = -\frac{\ell}{2} > 0$. Then there exists $\delta > 0$ such that

$$f(x) < \ell + \varepsilon = \ell - \frac{\ell}{2} = \frac{\ell}{2} < 0 \ \text{ whenever } \ 0 < |x - x_0| < \delta.$$

□

Definition 6. *Let D be a nonempty subset of $\mathbb{R}$ and let $f : D \to \mathbb{R}$. We say that f is continuous at $x_0 \in D$ if for each $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in D$ satisfying $|x - x_0| < \delta$.*
We say that $f : D \to \mathbb{R}$ is continuous if f is continuous at each $x_0 \in D$.

Let $f : [a,b] \to \mathbb{R}$. Then f is continuous at $c \in (a,b)$ if $\lim_{x\to c} f(x) = f(c)$. Also, f is continuous at a if $\lim_{x\to a+} f(x) = f(a)$. Similarly, f is continuous at b if $\lim_{x\to b-} f(x) = f(b)$.

Theorem 7 (Sequential criterion of continuity)**.** *Let $f : D \to \mathbb{R}$. Then f is continuous at $x_0 \in D$ if and only if for every sequence (x_n) in D such that $x_n \to x_0$, we have $f(x_n) \to f(x_0)$.*

Example 6. *We have*

(a) $f(x) = \begin{cases} 3x+2 & \text{if } x < 1, \\ 4x^2 & \text{if } x \geq 1 \end{cases}$ *is not continuous at $x = 1$.*

Solution. We have $\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (3x+2) = 5$ and $\lim_{x\to 1+} f(x) = \lim_{x\to 1+} 4x^2 = 4$. Hence $\lim_{x\to 1} f(x)$ does not exist and so f is not continuous at 1. □

(b) $f(x) = \begin{cases} x \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$ *is continuous at* 0.

Solution. For all $x\ (\neq 0) \in \mathbb{R}$, $|f(x) - f(0)| = |x \sin\frac{1}{x}| \leq |x|$ and hence given any $\varepsilon > 0$, choosing $\delta = \varepsilon > 0$, we get $|f(x) - f(0)| < \varepsilon$ for all $x \in \mathbb{R}$ satisfying $|x - 0| < \delta$. Therefore f is continuous at 0. □

(c) $f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$ *is not continuous at* 0.

Solution. Since $\lim_{x\to 0} \sin\frac{1}{x}$ does not exists. Therefore f is not continuous at 0. □

(d) $f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$ *is not continuous at any point of* $\mathbb{R}$.

Solution. If $x_0 \in \mathbb{Q}$, then there exists a sequence (t_n) in $\mathbb{R}\setminus\mathbb{Q}$ such that $t_n \to x_0$. Since $f(t_n) = 0$ for all $n \in \mathbb{N}$, $f(t_n) \to 0 \neq 1 = f(x_0)$. Hence f is not continuous at x_0. Again, if $x_0 \in \mathbb{R}\setminus\mathbb{Q}$, then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to x_0$. Since $f(r_n) = 1$ for all $n \in \mathbb{N}$, $f(r_n) \to 1 \neq 0 = f(x_0)$. Hence f is not continuous at x_0. □

(e) $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ -x & \text{if } x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$ *is continuous only at* 0.

Solution. Given any $\varepsilon > 0$, choosing $\delta = \varepsilon > 0$, we have $|f(x) - f(0)| = |x| < \varepsilon$ for all $x \in \mathbb{R}$ satisfying $|x - 0| < \delta$. Therefore f is continuous at 0. If $x_0\ (\neq 0) \in \mathbb{Q}$, then there exists a sequence (t_n) in $\mathbb{R}\setminus\mathbb{Q}$ such that $t_n \to x_0$. Since $f(t_n) = -t_n$ for all $n \in \mathbb{N}$, $f(t_n) \to -x_0 \neq x_0 = f(x_0)$. Hence f is not continuous at x_0. Again, if $x_0 \in \mathbb{R}\setminus\mathbb{Q}$, then there exists a sequence (r_n) in $\mathbb{Q}$ such that $r_n \to x_0$. Since $f(r_n) = x_0$ for all $n \in \mathbb{N}$, $f(r_n) \to x_0 \neq -x_0 = f(x_0)$. Hence f is not continuous at x_0. □

Theorem 8. *Let $f, g : D \to \mathbb{R}$ be continuous at $x_0 \in D$. Then*

(a) *$f + g$, fg and $|f|$ are continuous at x_0,*

(b) *f/g is continuous at x_0 if $g(x) \neq 0$ for all $x \in D$.*

Proof. Here we prove that if f is continuous at x_0, then $|f|$ is also continuous at x_0. The remaining results can be easily proved using the sequential criterion for continuity. We note that the function $|f|$ is defined as $|f|(x) := |f(x)|$ for all $x \in D$. Let $\varepsilon > 0$ be given. Since f is continuous at x_0, so there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$. Now,

$$||f|(x) - |f|(x_0)| = ||f(x)| - |f(x_0)|| \leq |f(x) - f(x_0)| < \varepsilon \quad \text{whenever} \quad |x - x_0| < \delta.$$

This proves that $|f|$ is continuous at x_0. $\square$

Theorem 9. *Composition of two continuous functions is continuous.*

Proof. Let A and B be nonempty subsets of $\mathbb{R}$. Let $f : A \to B$ be continuous at $x_0 \in A$ and g be continuous at $f(x_0)$. We now prove that $g \circ f$ is continuous at x_0. Let (x_n) be a sequence in A such that $x_n \to x_0$. Since f is continuous at x_0, so $f(x_n) \to f(x_0)$. Now, $(f(x_n))$ is a sequence in B and g is continuous at $f(x_0)$. Therefore, $g(f(x_n)) \to g(f(x_0))$, that is, $(g \circ f)(x_n) \to (g \circ f)(x_0)$. Hence, by using sequential criterion, $g \circ f$ is continuous at x_0. $\square$

Example 7 (Further examples of continuous functions)**.** *Polynomial function, Rational function, sine function, cosine function, exponential function, etc.*

Theorem 10. *If $f : D \to \mathbb{R}$ is continuous at x_0 and $f(x_0) \neq 0$, then there exists $\delta > 0$ such that $f(x) \neq 0$ for all $x \in D$ satisfying $|x - x_0| < \delta$.*

Proof. **Case-I:** Let $f(x_0) > 0$. Take $\varepsilon = \frac{f(x_0)}{2} > 0$. Then there exists $\delta > 0$ such that

$$f(x) > f(x_0) - \varepsilon = f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2} > 0 \text{ whenever } x \in D \text{ and } |x - x_0| < \delta.$$

Case-II: Let $f(x_0) < 0$. Take $\varepsilon = -\frac{f(x_0)}{2} > 0$. Then there exists $\delta > 0$ such that

$$f(x) < f(x_0) + \varepsilon = f(x_0) - \frac{f(x_0)}{2} = \frac{f(x_0)}{2} < 0 \text{ whenever } x \in D \text{ and } |x - x_0| < \delta.$$

$\square$

Theorem 11. *If $f : [a, b] \to \mathbb{R}$ is continuous and if $f(a) \cdot f(b) < 0$, then there exists $c \in (a, b)$ such that $f(c) = 0$.*

Proof. Assume that $f(a) < 0 < f(b)$. Let $S = \{x \in [a, b] : f(x) < 0\}$. Let $c = \sup S$. We claim that $f(c) = 0$. Since $f(b) > 0$, so there exists $\delta > 0$ such that $f(x) > 0$ for all $x \in (b - \delta, b]$. Hence $c \leq b - \delta < b$. Choose $n_0 \in \mathbb{N}$ such that $c + \frac{1}{n_0} \in (c, b)$. Then $x_n = c + \frac{1}{n} \notin S$ for all $n \geq n_0$. Clearly, $x_n \to c$. Therefore, $f(c) = \lim f(x_n) \geq 0$. On the otherhand, note that $c - \frac{1}{n}$ is not an upper bound of S for each n. Therefore, there exists a point $y_n \in (c - \frac{1}{n}, c) \cap S$. Note that $y_n \to c$ and $f(c) = \lim f(y_n) \leq 0$. Hence $f(c) = 0$. $\square$

Theorem 12 (Intermediate value theorem)**.** *Let I be an interval of $\mathbb{R}$ and let $f : I \to \mathbb{R}$ be continuous. If $a, b \in I$ with $a < b$ and if $f(a) < k < f(b)$, then there exists $c \in (a, b)$ such that $f(c) = k$.*

Proof. We define a new function $g : [a, b] \to \mathbb{R}$ by $g(x) = f(x) - k$. Since f is continuous so g is also continuous. Now, $g(a) = f(a) - k < 0$ and $g(b) = f(b) - k > 0$. By the previous theorem, there exists $c \in (a, b)$ such that $g(c) = 0$. This yields $f(c) = k$. $\square$

Example 8. *The following are some consequences of intermediate value theorem.*

(a) *The equation $x^2 = x \sin x + \cos x$ has at least two real roots.*

Solution. Let $f(x) = x^2 - x \sin x - \cos x$ for all $x \in \mathbb{R}$. Then $f : \mathbb{R} \to \mathbb{R}$ is continuous and $f(-\pi) = \pi^2 + 1 > 0$, $f(0) = -1 < 0$ and $f(\pi) = \pi^2 + 1 > 0$. Hence by the intermediate value theorem, the equation $f(x) = 0$ has at least one root in $(-\pi, 0)$ and at least one root in $(0, \pi)$. Thus the equation $f(x) = 0$ has at least two real roots. $\square$

(b) (Fixed point theorem). *If $f : [a, b] \to [a, b]$ is continuous, then there exists $c \in [a, b]$ such that $f(c) = c$.*

Solution. Let $g(x) = f(x) - x$ for all $x \in [a, b]$. Since f is continuous, $g : [a, b] \to \mathbb{R}$ is continuous. If $f(a) = a$ or $f(b) = b$, then we get the result by taking $c = a$ or $c = b$ respectively. Otherwise $g(a) = f(a) - a > 0$ and $g(b) = f(b) - b < 0$ (since it is given that $a \le f(x) \le b$ for all $x \in [a, b]$). Hence by the intermediate value theorem, there exists $c \in (a, b)$ such that $g(c) = 0$, that is, $f(c) = c$. $\square$

(c) *Let $f : [0, 2] \to \mathbb{R}$ be continuous such that $f(0) = f(2)$. Then there exist $x_1, x_2 \in [0, 2]$ such that $x_1 - x_2 = 1$ and $f(x_1) = f(x_2)$.*

Solution. Let $g(x) = f(x + 1) - f(x)$ for all $x \in [0, 1]$. Since f is continuous, $g : [0, 1] \to \mathbb{R}$ is continuous. Also, $g(0) = f(1) - f(0)$ and $g(1) = f(2) - f(1) = -g(0)$, since $f(0) = f(2)$. If $g(0) = 0$, then $f(1) = f(0)$ and we get the result by taking $x_1 = 1$ and $x_2 = 0$. If $g(0) \neq 0$, then $g(0)$ and $g(1)$ are of opposite signs and hence by the intermediate value theorem, there exists $c \in (0, 1)$ such that $g(c) = 0$, that is, $f(c + 1) = f(c)$. We get the result by taking $x_1 = c + 1$ and $x_2 = c$. $\square$

Theorem 13. *If $f : [a, b] \to \mathbb{R}$ is continuous, then $f : [a, b] \to \mathbb{R}$ is bounded.*

Proof. We need to prove that the set $f([a, b])$ is bounded. Suppose that the set $f([a, b])$ is unbounded. Firstly, suppose that it is unbounded above. Then for each $n \in \mathbb{N}$, there exists $x_n \in [a, b]$ such that $f(x_n) > n$. Since (x_n) is a sequence in $[a, b]$, so it is bounded. By Bolzano Weierstrass theorem, (x_n) has a convergent subsequence, say (x_{n_k}). Let $x_{n_k} \to x_0$. Since $a \le x_n \le b$, so $x_0 \in [a, b]$. (Note that this is the place where we need a closed interval as the domain of f). We are given that f is continuous on $[a, b]$. Therefore, $f(x_{n_k}) \to f(x_0)$. By the definition of subsequence, we have $n_k \ge k$ and hence $f(x_{n_k}) > n_k \ge k$ for all $k \ge 1$. This is a contradiction to the fact the sequence $f(x_{n_k})$ is bounded. We get a similar contradiction if we assume that $f([a, b])$ is unbounded below. Therefore, f must be a bounded function. $\square$

Example 9. *There does not exist any continuous function from* $[0,1]$ *onto* $(0,\infty)$.

Theorem 14. *If* $f:[a,b]\to\mathbb{R}$ *is continuous, then there exist* $x_0, y_0 \in [a,b]$ *such that* $f(x_0)\le f(x)\le f(y_0)$ *for all* $x\in[a,b]$.

Proof. Since $f:[a,b]\to\mathbb{R}$ is continuous, so f is bounded. That is, the set $A=f([a,b])$ is bounded. Let $M=\sup(A)$ and $m=\inf(A)$. We claim that there exist $x_0, y_0\in[a,b]$ such that $m=f(x_0)$ and $M=f(y_0)$. Since $M=\sup(A)$, so for each $n\in\mathbb{N}$ there exists $y_n\in[a,b]$ such that $M-\frac{1}{n}<f(y_n)\le M$. This implies that $f(y_n)\to M$. Again, (y_n) is a bounded sequence and hence by Bolzano Weierstrass theorem it has a convergent subsequence, say (y_{n_k}). Let $y_{n_k}\to y_0$. Clearly, $y_0\in[a,b]$. Since f is continuous at y_0, so $f(y_{n_k})\to f(y_0)$. Hence $M=f(y_0)\in f([a,b])$. Equivalently, the maximum of $f(x)$ is attained at y_0.
The proof of $m=f(x_0)$ for some $x_0\in[a,b]$ follows along similar lines. □

Definition 7 (Limit point of a set)**.** *Let* $A\subseteq\mathbb{R}$. *A real number* x *is called a limit point of* A *if there exists a sequence* (x_n) *in* A *converging to* x.

Definition 8 (Closed set)**.** *Let* $A\subseteq\mathbb{R}$. *Then* A *is called a closed set if* A *contains all its limit points. That is, if* (x_n) *is a sequence in* A *converging to* x*, then* $x\in A$.

Example 10. $\mathbb{R}$, $[a,b]$, $\{x_1,x_2,\ldots,x_n\}$, $\mathbb{N}$ *are closed sets. But,* (a,b), $\mathbb{Q}$ *are not closed sets.*

Theorem 15. *Let* A *be a closed and bounded subset of* $\mathbb{R}$. *If* $f:A\to\mathbb{R}$ *is continuous, then* f *is bounded.*

Proof. We need to prove that the set $f(A)$ is bounded. Suppose that the set $f(A)$ is unbounded. Firstly, suppose that $f(A)$ is unbounded above. Then for each $n\in\mathbb{N}$, there exists $x_n\in A$ such that $f(x_n)>n$. Since (x_n) is a sequence in A, so it is bounded. By Bolzano Weierstrass theorem, (x_n) has a convergent subsequence, say (x_{n_k}). Let $x_{n_k}\to x_0$. Since (x_{n_k}) is a sequence in A converging to x_0 and A is closed, so $x_0\in A$. We are given that f is continuous on A. Therefore, $f(x_{n_k})\to f(x_0)$. By the definition of subsequence, we have $n_k\ge k$ and hence $f(x_{n_k})>n_k\ge k$ for all $k\ge 1$. This is a contradiction to the fact the sequence $f(x_{n_k})$ is bounded. We get a similar contradiction if we assume that $f([a,b])$ is unbounded below. Therefore, f must be a bounded function. □

Remark 2. *The above result is not true if* A *is bounded but not closed. For example* $f(x)=1/x$ *on* $(0,1)$. *Also, the result is not true if* A *is closed but not bounded. For example,* $f(x)=x$ *on* $\mathbb{R}$.