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MA 101: Mathematics I
Series of real numbers
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Let (a_n) be a sequence of real numbers. An expression of the form

$$a_1 + a_2 + \cdots + a_n + \cdots$$

is called an infinite series. We use the notation: $\sum\limits_{n=1}^{\infty} a_n$.

(1) The number a_n is called the n-th term of the series.

(2) $s_n = \sum\limits_{k=1}^{n} a_k$ is called the n-th partial sum of the series.

(3) If the sequence of partial sums (s_n) converges to a limit ℓ, we say that the series converges and its sum is ℓ.

(4) If (s_n) diverges, we say that the series diverges.

Example 1. *We have*

(1) *The geometric series* $\sum_{n=1}^{\infty} ar^{n-1}$ *(where* $a \neq 0$*) converges if and only if* $|r| < 1$.

(2) *The series* $\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$ *is convergent with sum* 1.

(3) *The series* $1 - 1 + 1 - 1 + \cdots$ *is not convergent.*

(4) *The series* $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ *diverges.*

Theorem 1 (Algebraic operations on series)**.** *Let* $\sum_{n=1}^{\infty} x_n$ *and* $\sum_{n=1}^{\infty} y_n$ *be convergent with sums* x *and* y *respectively. Then*

(a) $\sum_{n=1}^{\infty} (x_n + y_n)$ *is convergent with sum* $x + y$

(b) $\sum_{n=1}^{\infty} \alpha x_n$ *is convergent with sum* αx, *where* $\alpha \in \mathbb{R}$

Theorem 2 (Necessary condition for convergence)**.** *If* $\sum\limits_{n=1}^{\infty} a_n$ *converges then* $\lim\limits_{n\to\infty} a_n = 0$.

Proof. Let $s_n = \sum_{k=1}^{n} a_k$. Then $a_n = s_n - s_{n-1}$. Since $\sum_{n=1}^{\infty} a_n$ converges, so s_n and s_{n-1} will converge to the same limit, and hence $a_n \to 0$. $\square$

Hence if $x_n \not\to 0$, then $\sum_{n=1}^{\infty} x_n$ cannot be convergent.

Remark 1. *The condition $\lim_{n\to\infty} a_n = 0$ is not sufficient for the convergence of $\sum_{n=1}^{\infty} a_n$. For example, $\frac{1}{n} \to 0$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.*

Example 2. *The following series are not convergent.*
(a) $\sum_{n=1}^{\infty} \frac{n^2+1}{(n+3)(n+4)}$ *(b)* $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+2}$.

Theorem 3 (Monotone criterion)**.** *A series $\sum_{n=1}^{\infty} x_n$ of non-negative terms is convergent if and only if the sequence (s_n) is bounded above.*

Proof. Since $x_n \geq 0$, so the sequence (s_n) of partial sums is increasing. By Monotone Convergence Theorem, (s_n) is convergent if and only if it is bounded above. Equivalently, $\sum_{n=1}^{\infty} x_n$ is convergent if and only if (s_n) is bounded above. $\square$

Example 3. (a) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is convergent.*

Solution. We have

$$\begin{aligned} s_n &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \\ &\leq 1 + \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{(n-1)\cdot n} \\ &= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 2 - \frac{1}{n} \\ &< 2. \end{aligned}$$

Thus, (s_n) is bounded above. Hence the sereis $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. $\square$

(b) $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent.*

Solution. We know that $s_n = \sum_{k=1}^{n} \frac{1}{n}$ is not bounded above. Hence $\sum_{n=1}^{\infty} \frac{1}{n}$ is divergent. $\square$

Theorem 4 (Cauchy criterion)**.** *A series* $\sum_{n=1}^{\infty} x_n$ *is convergent if and only if for each* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that*

$$|x_{n+1} + \cdots + x_m| < \varepsilon \quad \textit{for all} \quad m > n \geq n_0.$$

Proof. We know that a sequence in $\mathbb{R}$ is convergent if and only if it is a Cauchy sequence. Hence, the series $\sum_{n=1}^{\infty} x_n$ is convergent if and only if the sequence (s_n) of partial sums is Cauchy. Now, $s_n = \sum_{k=1}^{n} x_n$ is Cauchy if for given $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ such that for all $m > n \geq n_0$,

$$|s_m - s_n| = |x_{n+1} + \cdots + x_m| < \varepsilon.$$

This completes the proof of the theorem. □

Tests for convergence:

Theorem 5 (Comparison test)**.** *Let* (x_n) *and* (y_n) *be sequences in* $\mathbb{R}$ *such that for some* $n_0 \in \mathbb{N}$, $0 \leq x_n \leq y_n$ *for all* $n \geq n_0$*. Then*

(a) $\sum_{n=1}^{\infty} y_n$ *is convergent* $\Rightarrow \sum_{n=1}^{\infty} x_n$ *is convergent.*

(b) $\sum_{n=1}^{\infty} x_n$ *is divergent* $\Rightarrow \sum_{n=1}^{\infty} y_n$ *is divergent.*

Proof. Clearly, $(a) \Leftrightarrow (b)$. So, we prove (a).

(a) Suppose that $\sum_{n=1}^{\infty} y_n$ is convergent. Then for given $\varepsilon > 0$, there is some $n_1 \in \mathbb{N}$ such that $|y_{n+1} + y_{n+2} + \cdots + y_m| < \varepsilon$ for all $m > n \geq n_1$. We also have $0 \leq x_n \leq y_n$ for all $n \geq n_0$. Let $n_2 = \max\{n_0, n_1\}$. Then for $m > n \geq n_2$, we have

$$\begin{aligned}|x_{n+1} + x_{n+2} + \cdots + x_m| &= x_{n+1} + x_{n+2} + \cdots + x_m \\ &\leq y_{n+1} + y_{n+2} + \cdots + y_m \\ &= |y_{n+1} + y_{n+2} + \cdots + y_m| \\ &< \varepsilon.\end{aligned}$$

By Cauchy's criterion, the sereis $\sum_{n=1}^{\infty} x_n$ is convergent.

Alternative proof: Without loss of generality, we may assume that $0 \leq x_n \leq y_n$ for all $n \geq 1$ as we can always add or remove finitely many terms without affecting the nature (convergence/divergence) of the series. Let $s_n = \sum_{k=1}^{n} x_n$ and $t_n = \sum_{k=1}^{n} y_n$. Since $y_n \geq x_n \geq 0$ for all n, so (s_n) and (t_n) are increasing sequences and $s_n \leq t_n$. If $\sum_{n=1}^{\infty} y_n$ is convergent, then (t_n) is convergent and hence bounded above. This implies that (s_n) is bounded above, and since (s_n) is increasing so it is convergent. This proves that $\sum_{n=1}^{\infty} x_n$ is convergent. This completes the proof of (a). □

Example 4. (a) $\sum_{n=1}^{\infty} \frac{1+\sin n}{1+n^2}$ *is convergent.*

Solution. We have $0 \leq \frac{1+\sin n}{1+n^2} \leq \frac{2}{n^2}$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. $\square$

(b) $\sum\limits_{n=2}^{\infty} \frac{1}{\sqrt{n(n-1)}}$ *is not convergent.*

Solution. We have $\frac{1}{\sqrt{n(n-1)}} > \frac{1}{n} > 0$ for all $n \geq 2$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges. $\square$

Theorem 6 (Limit comparison test)**.** *Let (x_n) and (y_n) be sequences of positive real numbers such that $\frac{x_n}{y_n} \to \ell \in \mathbb{R}$.*

(a) *If $\ell \neq 0$, then $\sum\limits_{n=1}^{\infty} x_n$ is convergent if and only if $\sum\limits_{n=1}^{\infty} y_n$ is convergent.*

(b) *If $\ell = 0$, then $\sum\limits_{n=1}^{\infty} y_n$ is convergent $\Rightarrow \sum\limits_{n=1}^{\infty} x_n$ is convergent.*

(c) *If $\ell = \infty$ and $\sum\limits_{n=1}^{\infty} y_n$ is divergent $\Rightarrow \sum\limits_{n=1}^{\infty} x_n$ diverges.*

Proof. Note that $\ell \geq 0$.

(a) If $\ell \neq 0$, then $\varepsilon = \frac{\ell}{2} > 0$. Since $\frac{x_n}{y_n} \to \ell$, so there is some $n_0 \in \mathbb{N}$ such that $\frac{\ell}{2} < \frac{x_n}{y_n} < \frac{3\ell}{2}$ for all $n \geq n_0$. This implies $\frac{\ell}{2} y_n < x_n < \frac{3\ell}{2} y_n$ for all $n \geq n_0$. Applying the Comparison Test twice, we complete the proof.

(b) Let $\varepsilon > 0$. Since $\frac{x_n}{y_n} \to 0$, so there is some $n_0 \in \mathbb{N}$ such that $0 < \frac{x_n}{y_n} < \varepsilon$ for all $n \geq n_0$. This implies $0 < x_n < \varepsilon \cdot y_n$ for all $n \geq n_0$. We now complete the proof by applying the Comparison Test.

(c) If $\frac{x_n}{y_n}$ diverges to ∞, then for given $M > 0$ there is some n_0 such that $\frac{x_n}{y_n} > M$ for all $n \geq n_0$. This is, $x_n > M \cdot y_n$ for all $n \geq n_0$. Now applying the Comparison Test, we complete the proof.

$\square$

Example 5. $\sum\limits_{n=1}^{\infty} \frac{n}{4n^3-2}$ *is convergent.*

Solution. Let $x_n = \frac{n}{4n^3-2}$ and $y_n = \frac{1}{n^2}$ for all $n \in \mathbb{N}$. Then $\lim\limits_{n\to\infty} \frac{x_n}{y_n} = \lim\limits_{n\to\infty} \frac{n^3}{4n^3-2} = \lim\limits_{n\to\infty} \frac{1}{4-\frac{2}{n^3}} = \frac{1}{4} \neq 0$. By Limit Comparison Test, the given series is convergent. $\square$

Theorem 7 (Cauchy's condensation test)**.** *Let (x_n) be a decreasing sequence of nonnegative real numbers. Then $\sum\limits_{n=1}^{\infty} x_n$ is convergent if and only if $\sum\limits_{n=1}^{\infty} 2^n x_{2^n}$ is convergent.*

Proof. Let $s_n = \sum_{k=1}^{n} x_k$ and $t_n = \sum_{k=1}^{n} 2^k x_{2^k}$. Since (x_n) is decreasing, we have

$$\begin{aligned}
s_{2^n} &= \sum_{k=1}^{2^n} x_k \\
&= x_1 + x_2 + (x_3 + x_4) + (x_5 + x_6 + x_7 + x_8) + \cdots + (x_{2^{n-1}+1} + x_{2^{n-1}+2} + \cdots + x_{2^n}) \\
&\geq x_1 + x_2 + 2x_4 + 4x_8 + \cdots + 2^{n-1} x_{2^n} \\
&= x_1 + \frac{1}{2}(2x_2 + 2^2 x_4 + 2^3 x_8 + \cdots + 2^n x_{2^n}) = x_1 + \frac{t_n}{2}.
\end{aligned}$$

Now, if $\sum\limits_{n=1}^{\infty} x_n$ is convergent, then (s_n) is convergent. This implies that (s_{2^n}) is bounded and hence (t_n) is bounded above. By Monotone criterion, the series $\sum\limits_{n=1}^{\infty} 2^n x_{2^n}$ is convergent. On the other hand,

$$
\begin{aligned}
s_{2^n} &= \sum_{k=1}^{2^n} x_k \\
&= x_1 + (x_2 + x_3) + (x_4 + x_5 + x_6 + x_7) + \cdots + (x_{2^{n-1}} + x_{2^{n-1}+1} + \cdots + x_{2^n-1}) + x_{2^n} \\
&\leq x_1 + 2x_2 + 4x_4 + 8x_8 + \cdots + 2^{n-1} x_{2^{n-1}} + x_{2^n} \\
&\leq x_1 + 2x_2 + 4x_4 + 8x_8 + \cdots + 2^{n-1} x_{2^{n-1}} + 2^n x_{2^n} \\
&= x_1 + t_n.
\end{aligned}
$$

Now, if $\sum\limits_{n=1}^{\infty} 2^n x_{2^n}$ is convergent then (t_n) is bounded and hence (s_{2^n}) is bounded above. Since (s_n) is increasing, so $s_n \leq s_{2^n}$ for all $n \geq 1$. This proves that (s_n) is bounded above and by Monotone criterion, the series $\sum\limits_{n=1}^{\infty} x_n$ is convergent. □

Example 6. *We have*

(a) *p-series:* $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ *is convergent if and only if* $p > 1$.

Solution. If $p \leq 0$, then clearly the series diverges as $\frac{1}{n^p} \not\to 0$. Let $p > 0$. Then the sequence $(1/n^p)$ of non negative terms is decreasing. By Cauchy's condensation test, $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if and only if $\sum\limits_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p}$ is convergent. Now, the geometric series $\sum\limits_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum\limits_{n=1}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n$ is convergent if and only if $\frac{1}{2^{p-1}} < 1$, that is, if and only if $p > 1$. □

(b) $\sum\limits_{n=2}^{\infty} \frac{1}{n(\log n)^p}$ *is convergent if and only if* $p > 1$.

Solution. Let $f(x) = \frac{1}{x(\log x)^p}$ for all $x > 1$. Then $f : (1, \infty) \to \mathbb{R}$ is differentiable and $f'(x) = -\frac{(\log x)^{p-1}(\log x + p)}{x^2 (\log x)^{2p}} \leq 0$ for all $x > \max\{1, e^{-p}\} = a$ (say). Hence f is decreasing on (a, ∞) and so $f(n+1) \leq f(n)$ for all $n \geq n_0$, where $n_0 \in \mathbb{N}$ is chosen to satisfy $n_0 > a$. Thus the sequence $\left(\frac{1}{n(\log n)^p}\right)_{n=n_0}^{\infty}$ of non-negative real numbers is decreasing. Since the series $\sum\limits_{n=n_0}^{\infty} 2^n \cdot \frac{1}{2^n (\log 2^n)^p} = \sum\limits_{n=n_0}^{\infty} \frac{1}{(\log 2)^p n^p}$ is convergent if and only if $p > 1$, by Cauchy's condensation test, $\sum\limits_{n=n_0}^{\infty} \frac{1}{n(\log n)^p}$ is convergent if and only if $p > 1$. Consequently the given series is convergent if and only if $p > 1$. □

Alternating series: An alternating series is an infinite series whose terms alternate in sign.

Example 7. *(a)* $\sum_{n=1}^{\infty}(-1)^n$ *(b)* $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$.

Theorem 8 (Leibniz's test)**.** *Let (x_n) be a decreasing sequence of positive real numbers such that $x_n \to 0$. Then the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ is convergent.*

Proof. Since (x_n) is decreasing, we have

$$\begin{aligned} s_{2n+1} = \sum_{k=1}^{2n+1}(-1)^{k+1}x_k &= x_1 - x_2 + x_3 - \cdots + x_{2n-1} - x_{2n} + x_{2n+1} \\ &= s_{2n-1} - x_{2n} + x_{2n+1} = s_{2n-1} - (x_{2n} - x_{2n+1}) \le s_{2n-1}. \end{aligned}$$

Hence, the subsequence (s_{2n+1}) is decreasing. Since (x_n) is decreasing and x_n's are positive, we have

$$\begin{aligned} s_{2n+1} &= x_1 - x_2 + x_3 - \cdots + x_{2n-1} - x_{2n} + x_{2n+1} \\ &= (x_1 - x_2) + (x_3 - x_4) + \cdots + (x_{2n-1} - x_{2n}) + x_{2n+1} \ge 0. \end{aligned}$$

Thus, the subsequence (s_{2n+1}) is bounded below. Hence, it is convergent.

Similarly,

$$\begin{aligned} s_{2n+2} &= x_1 - x_2 + x_3 - \cdots + x_{2n-1} - x_{2n} + x_{2n+1} - x_{2n+2} \\ &= s_{2n} + (x_{2n+1} - x_{2n+2}) \ge s_{2n} \end{aligned}$$

and

$$\begin{aligned} s_{2n} &= x_1 - x_2 + x_3 - \cdots - x_{2n-2} + x_{2n-1} - x_{2n} \\ &= x_1 - (x_2 - x_3) - (x_4 - x_5) - \cdots - (x_{2n-2} - x_{2n-1}) - x_{2n} \le x_1. \end{aligned}$$

Thus, the subsequence (s_{2n}) is increasing and bounded above.
Let $s_{2n-1} \to \ell_1$ and $s_{2n} \to \ell_2$. We have $x_{2n} = s_{2n-1} - s_{2n}$. Since $x_n \to 0$, so $\ell_1 = \ell_2$. This proves that the subsequences (s_{2n-1}) and (s_{2n}) converge to the same limit, and hence (s_n) is convergent. That is, the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}x_n$ is convergent. □

Example 8. *By Leibniz's test, the alternating harmonic series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$ converges.*

Definition 1. *$\sum_{n=1}^{\infty} x_n$ is called absolutely convergent if $\sum_{n=1}^{\infty}|x_n|$ is convergent. $\sum_{n=1}^{\infty} x_n$ is called conditionally convergent if $\sum_{n=1}^{\infty} x_n$ is convergent but $\sum_{n=1}^{\infty}|x_n|$ is divergent.*

Example 9. *The series $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ converges. But $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges. Hence, $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n}$ converges conditionally.*

Theorem 9. *Every absolutely convergent series is convergent.*

Proof. Let $\sum_{n=1}^{\infty} x_n$ be absolutely convergent. Let $t_n = \sum_{k=1}^{n} |x_k|$. Then (t_n) is convergent and hence Cauchy. Let $\varepsilon > 0$. Then there is some $n_0 \in \mathbb{N}$ such that $|t_m - t_n| < \varepsilon$ for all $m, n \geq n_0$. Let $s_n = \sum_{k=1}^{n} x_k$. Now, for $m > n \geq n_0$,

$$\begin{aligned}
|s_m - s_n| &= |x_m + x_{m-1} + \cdots + x_{n+1}| \\
&\leq |x_m| + |x_{m-1}| + \cdots + |x_{n+1}| \\
&= t_m - t_n \\
&= |t_m - t_n| \\
&< \varepsilon.
\end{aligned}$$

Hence, (s_n) is Cauchy and so it is convergent. Equivalently, $\sum_{n=1}^{\infty} x_n$ is convergent. $\square$

Theorem 10 (Comparison test-II)**.** *Let (x_n) be a series of real numbers. Then $\sum_{n=1}^{\infty} x_n$ converges absolutely if there is an absolutely convergent series $\sum_{n=1}^{\infty} y_n$ and some $n_0 \in \mathbb{N}$ satisfying $|x_n| \leq |y_n|$ for all $n \geq n_0$.*

Theorem 11 (Limit comparison test-II)**.** *Let (x_n) and (y_n) be sequences of nonzero real numbers such that $\left|\frac{x_n}{y_n}\right| \to \ell \in \mathbb{R}$.*

(a) *If $\ell \neq 0$, then $\sum\limits_{n=1}^{\infty} x_n$ is absolutely convergent iff $\sum\limits_{n=1}^{\infty} y_n$ is absolutely convergent.*

(b) *If $\ell = 0$, then $\sum\limits_{n=1}^{\infty} y_n$ is absolutely convergent $\Rightarrow$ $\sum\limits_{n=1}^{\infty} x_n$ is absolutely convergent.*

Theorem 12 (Ratio Test)**.** *Let $\sum\limits_{n=1}^{\infty} x_n$ be a series of nonzero real numbers. Let*

$$a = \liminf \left|\frac{x_{n+1}}{x_n}\right| \quad \textit{and} \quad A = \limsup \left|\frac{x_{n+1}}{x_n}\right|.$$

Then

(1) *If $A < 1$, then $\sum\limits_{n=1}^{\infty} x_n$ is absolutely convergent.*

(2) *If $a > 1$, then $\sum\limits_{n=1}^{\infty} x_n$ is divergent.*

Proof. (1) Let $A < 1$. Let $B \in \mathbb{R}$ be such that $A < B < 1$. Put $\varepsilon = B - A$. We have

$$A = \limsup \left|\frac{x_{n+1}}{x_n}\right| = \lim_{n\to\infty} y_n,$$

where $y_n = \sup\left\{\frac{|x_{k+1}|}{|x_k|} : k \geq n\right\}$. Since $\varepsilon = B - A > 0$, so there is some $n_0 \in \mathbb{N}$ such that

$$0 < y_n < A + \varepsilon = B \ \text{ for all } \ n \geq n_0, \ \text{ that is, } \ \frac{|x_{n+1}|}{|x_n|} < B \ \text{ for all } \ n \geq n_0.$$

This yields $|x_{n_0+k}| < |x_{n_0}|B^k$ for all $k \geq 1$. Since $0 < B < 1$, so $\sum_{k=1}^{\infty} |x_{n_0}|B^k = |x_{n_0}| \sum_{k=1}^{\infty} B^k$ is convergent. Therefore, by Comparison Test, $\sum_{k=1}^{\infty} |x_{n_0+k}|$ is also convergent. This proves that $\sum_{n=1}^{\infty} x_n$ converges absolutely.

(2) Suppose that $a = \liminf |\frac{x_{n+1}}{x_n}| > 1$. Let $b \in \mathbb{R}$ such that $1 < b < a$. Let $\varepsilon = a - b$. We have

$$a = \liminf |\frac{x_{n+1}}{x_n}| = \lim_{n\to\infty} z_n,$$

where $z_n = \inf \left\{ \frac{|x_{k+1}|}{|x_k|} : k \geq n \right\}$. Since $\varepsilon = a - b > 0$, so there is some $n_0 \in \mathbb{N}$ such that

$$a - \varepsilon < z_n \quad \text{for all} \quad n \geq n_0, \quad \text{that is,} \quad \frac{|x_{n+1}|}{|x_n|} > b \quad \text{for all} \quad n \geq n_0.$$

This yields $|x_{n_0+k}| > |x_{n_0}|b^k > |x_{n_0}|$ for all $k \geq 1$. Thus, $\lim_{n\to\infty} |x_n| \geq |x_{n_0}|$. This proves that $x_n \not\to 0$, and therefore $\sum_{n=1}^{\infty} x_n$ is divergent. $\square$

Remark 2. *If* $|\frac{x_{n+1}}{x_n}| \to \ell$, *then* $a = A = \ell$.

Example 10. *We have*

(a) *The series* $\sum_{n=1}^{\infty} \frac{n}{2^n}$ *is convergent.*

(b) *The series* $\sum_{n=1}^{\infty} \frac{(2n)!}{(n!)^2}$ *is not convergent.*

(c) *The series* $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ *is convergent for any* $x \in \mathbb{R}$.

Remark 3. *If* $\ell = \lim_{n\to\infty} |\frac{x_{n+1}}{x_n}| = 1$, *then the Ratio test is inconclusive. For example, for both the series* $\sum_{n=1}^{\infty} \frac{1}{n}$ *and* $\sum_{n=1}^{\infty} \frac{1}{n^2}$, $\ell = 1$. *However,* $\sum_{n=1}^{\infty} \frac{1}{n}$ *is divergent and* $\sum_{n=1}^{\infty} \frac{1}{n^2}$ *is convergent.*

Theorem 13 (Root Test)**.** *Let* $\sum_{n=1}^{\infty} x_n$ *be a series of real numbers. Let* $A = \limsup \sqrt[n]{|x_n|}$. *Then*

(1) *If* $A < 1$, *then* $\sum_{n=1}^{\infty} x_n$ *is absolutely convergent.*

(2) *If* $A > 1$, *then* $\sum_{n=1}^{\infty} x_n$ *is divergent.*

(3) *The test is inconclusive if* $A = 1$.

Proof. (1) Let $A < 1$. Let $B \in \mathbb{R}$ be such that $A < B < 1$. Put $\varepsilon = B - A$. We have

$$A = \limsup \sqrt[n]{|x_n|} = \lim_{n\to\infty} y_n,$$

where $y_n = \sup\left\{\sqrt[k]{|x_k|} : k \geq n\right\}$. Since $\varepsilon = B - A > 0$, so there is some $n_0 \in \mathbb{N}$ such that

$$0 < y_n < A + \varepsilon = B \quad \text{for all} \quad n \geq n_0, \quad \text{that is,} \quad |x_n| < B^n \quad \text{for all} \quad n \geq n_0.$$

Since $0 < B < 1$, so $\displaystyle\sum_{n=1}^{\infty} B^n$ is convergent. Therefore, by Comparison Test, $\displaystyle\sum_{n=1}^{\infty} |x_n|$ is also convergent. This proves that $\displaystyle\sum_{n=1}^{\infty} x_n$ converges absolutely.

(2) Let $A > 1$. Let $B \in \mathbb{R}$ be such that $A > B > 1$. Put $\varepsilon = A - B$. Then there exists some $n_0 \in \mathbb{N}$ such that

$$B = A - \varepsilon < y_n \quad \text{for all} \quad n \geq n_0.$$

Thus, $y_n > 1$ for all $n \geq n_0$. From this, we find that $|x_n| > 1$ for infinitely many values of n. Hence x_n does not converse to 0. This proves that $\sum\limits_{n=1}^{\infty} x_n$ is divergent.

(3) Let $x_n = \frac{1}{n}$. Then $A = 1$ and we know that $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ is divergent. Again, if $x_n = \frac{1}{n^2}$ then also $A = 1$. However, $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ is convergent. □

Remark 4. *If* $\sqrt[n]{|x_n|} \to \ell$, *then* $A = \limsup \sqrt[n]{|x_n|} = \ell$.

Example 11. (a) *The series* $\sum\limits_{n=1}^{\infty} \frac{(n!)^n}{n^{n^2}}$ *is convergent.*

Solution. Taking $x_n = \frac{(n!)^n}{n^{n^2}}$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} |x_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{n!}{n^n} = 0$ (since $\lim\limits_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim\limits_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1$). Hence by the root test, the given series is convergent. □

(b) *The series* $\sum\limits_{n=1}^{\infty} \frac{5^n}{3^n+4^n}$ *is not convergent.*

Solution. Taking $x_n = \frac{5^n}{3^n+4^n}$ for all $n \in \mathbb{N}$, we have $\lim\limits_{n\to\infty} |x_n|^{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{5}{(3^n+4^n)^{\frac{1}{n}}} = \frac{5}{4}$ (since $\lim\limits_{n\to\infty} (3^n + 4^n)^{\frac{1}{n}} = 4$, as shown earlier by using Sandwich theorem). Hence by the root test, the given series is not convergent. □

Given a series $\sum_{n=1}^{\infty} x_n$, we can construct many other series $\sum_{n=1}^{\infty} y_n$ by leaving the order of the terms x_n fixed, but inserting parentheses that group together finite number of terms. For example, the following series

$$1 - \frac{1}{2} + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{5} - \frac{1}{6} + \frac{1}{7}\right) - \frac{1}{8} + \left(\frac{1}{9} - \cdots + \frac{1}{13}\right) - \cdots$$

is obtained by grouping the terms in the alternating harmonic series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$.

Theorem 14. *Grouping of terms of a convergent series does not change the convergence and the sum. However, a divergent series can become convergent after grouping of terms.*

Proof. Let $\sum_{n=1}^{\infty} x_n$ be convergent. Suppose that the series $\sum_{n=1}^{\infty} y_n$ is obtained from $\sum_{n=1}^{\infty} x_n$ by grouping the terms. Then we have

$$y_1 = x_1 + x_2 + \cdots + x_{n_1}, \ y_2 = x_{n_1+1} + x_{n_1+2} + \cdots + x_{n_2}, \ \cdots$$

Let (s_n) and (t_n) be the sequences of partial sums of $\sum_{n=1}^{\infty} x_n$ and $\sum_{n=1}^{\infty} y_n$, respectively. Then

$$t_1 = y_1 = s_{n_1}, \ t_2 = y_1 + y_2 = s_{n_2}, \cdots .$$

Thus, (t_n) is a subsequence of (s_n). Since $\sum_{n=1}^{\infty} x_n$ is convergent, so (s_n) is convergent. Therefore, (t_n) is also convergent and converges to the limit of (s_n). This proves that the grouped series $\sum_{n=1}^{\infty} y_n$ is convergent and its sum is same as $\sum_{n=1}^{\infty} x_n$.

It is clear that the converse to this theorem is not true. We know that the series $\sum_{n=1}^{\infty} (-1)^n$ diverges. However, the grouping

$$(-1+1) + (-1+1) + \cdots + (-1+1) + \cdots$$

converges to 0. □

Definition 2 (Rearrangement of series)**.** *A series $\sum_{n=1}^{\infty} y_n$ is called a rearrangement of a series $\sum_{n=1}^{\infty} x_n$ if there is a bijection f of $\mathbb{N}$ onto $\mathbb{N}$ such that $y_n = x_{f(n)}$ for all $n \in \mathbb{N}$.*

Theorem 15. *Rearrangement of terms does not change the convergence and the sum of an absolutely convergent series.*

Example 12. *Let $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = s$. Then,*

$$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots = \frac{3}{2}s.$$

Solution. We first note that by Leibniz's test, the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ converges. Let

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = s. \tag{1}$$

Then the series $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \cdots = \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \cdots)$ converges to $\frac{1}{2}s$. It follows that the series

$$0 + \frac{1}{2} - 0 - \frac{1}{4} + 0 + \frac{1}{6} - 0 - \frac{1}{8} + \cdots = \frac{1}{2}s \tag{2}$$

Hence the series $(1+0) + (-\frac{1}{2} + \frac{1}{2}) + (\frac{1}{3} - 0) + (-\frac{1}{4} - \frac{1}{4}) + (\frac{1}{5} + 0) + \cdots$, which is the sum of the series (1) and (2), converges to $s + \frac{1}{2}s = \frac{3}{2}s$. Therefore it follows that $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \cdots = \frac{3}{2}s$. □

Theorem 16 (Riemann's rearrangement theorem)**.** *Let* $\sum_{n=1}^{\infty} x_n$ *be a conditionally convergent series.*

(1) *If* $s \in \mathbb{R}$*, then there exists a rearrangement of terms of* $\sum_{n=1}^{\infty} x_n$ *such that the rearranged series has the sum* s*.*

(2) *There exists a rearrangement of terms of* $\sum_{n=1}^{\infty} x_n$ *such that the rearranged series diverges.*