PH 102: Physics II

Lecture 12 (Post midsem, Spring 2020)

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IIT Guwahati

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

SN	Date	Topic	Griffith's section	Division
Lec 1	05-03-2020	Lorentz Force, Biot-Savart law, Divergence & Curl of Magnetostatic Fields	5.1, 5.2, 5.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 2	11-03-2020	Application of Ampere's Law, Magnetic Vector Potential	5.3, 5.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 1	12-03-2020	Lec 1		
Tut 2	17-03-2020	Lec 2		
Lec 3	18-03-2020	Magnetic dipole, Force & torque on a magnetic dipole, Magnetic materials, magnetization	5.4, 6.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 4	19-03-2020	Field of a magnetized object, Boundary conditions	6.2, 6.3, 6.4	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 3	24-03-2020	Lec 3, 4		
Lec 5	25-03-2020	Ohm's law, motional emf, electromotive force	7.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 6	26-03-2020	Faraday's law, Lenz's law, Self & Mutual inductance, Energy stored in magnetic field	7.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 4	31-03-2020	Lec 5, 6		
Lec 7	01-04-2020	Maxwell's equations	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 8	02-04-2020	Discussions, Problem solving	7.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
	07-04-2020	Quiz II		
Lec 9	08-04-2020	Continuity equation, Poynting theorem	8.1	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 10	16-04-2020	Wave solution of Maxwell's equations, polarisation	9.1, 9.2	I, II (4-4:55 pm) III, IV (11-11:55 am)
Tut 5	21-04-2020	Lec 9, 10		
Lec 11	22-04-2020	Electromagnetic waves in matter, reflection & transmission: normal incidence	9.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 12	23-04-2020	Reflection & transmission: oblique incidence	9.3, 9.4	I, II (4-4:55 pm) III, IV (11-11:55

LECTURE PLAN (TENTATIVE) OF PH 102 (POST MID-SEM)

				am)
Tut 6	28-4-2020	Lec 11, 12		
Lec 13	29-04-2020	Relativity and electromagnetism: Galilean & special relativity	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)
Lec 14	30-04-2020	Discussions, problem solving	12.1, 12.2, 12.3	I, II (4-4:55 pm) III, IV (11-11:55 am)

Reflection & Transmission: Oblique Incidence

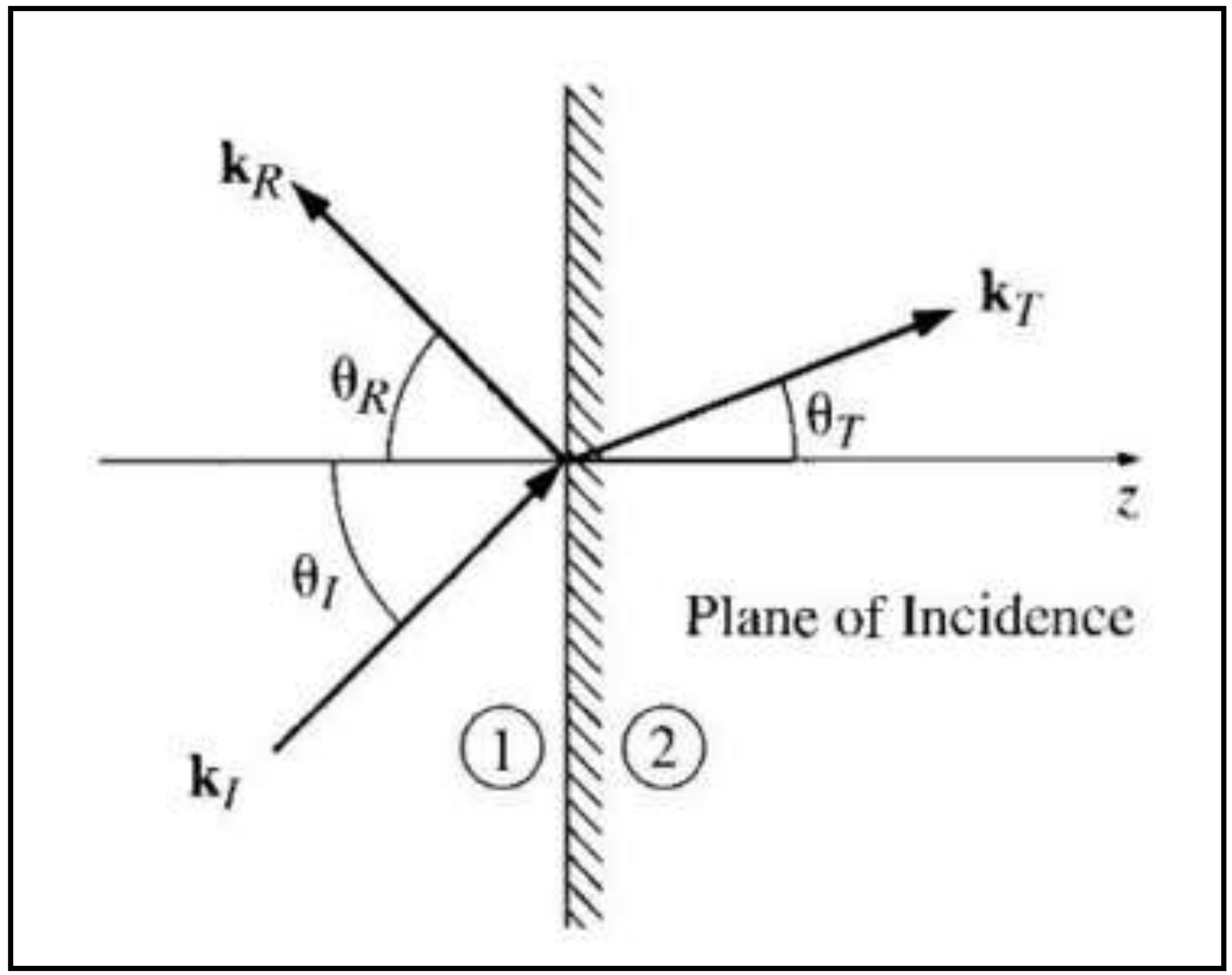


Figure 9.14, Introduction to Electrodynamics, D J Griffiths

Reflection & Transmission: Oblique Incidence

Let the incident monochromatic plane wave be

$$\tilde{\vec{E}}_I(\vec{r},t) = \tilde{\vec{E}}_{0I} e^{i(\vec{k}_I\cdot\vec{r}-\omega t)},\ \tilde{\vec{B}}_I(\vec{r},t) = \frac{1}{v_1}(\hat{k}_I \times \tilde{\vec{E}}_I)$$

that gives rise to a reflected wave

$$\tilde{\vec{E}}_R(\vec{r},t) = \tilde{\vec{E}}_{0R} e^{i(\vec{k}_R\cdot\vec{r}-\omega t)},\ \tilde{\vec{B}}_R(\vec{r},t) = \frac{1}{v_1}(\hat{k}_R \times \tilde{\vec{E}}_R)$$

and a transmitted wave

$$\tilde{\vec{E}}_T(\vec{r},t) = \tilde{\vec{E}}_{0T} e^{i(\vec{k}_T\cdot\vec{r}-\omega t)},\ \tilde{\vec{B}}_T(\vec{r},t) = \frac{1}{v_2}(\hat{k}_T \times \tilde{\vec{E}}_T).$$

Since they all have the same frequency, their wave-numbers are related by

$$k_I v_1 = k_R v_1 = k_T v_2 = \omega,\ k_I = k_R = \frac{v_2}{v_1} k_T = \frac{n_1}{n_2} k_T.$$

Reflection & Transmission: Oblique Incidence

The boundary conditions at the interface (z=0) give rise to generic equations like:

$$(\ )e^{i(\vec{k}_I\cdot\vec{r}-\omega t)} + (\ )e^{i(\vec{k}_R\cdot\vec{r}-\omega t)} = (\ )e^{i(\vec{k}_T\cdot\vec{r}-\omega t)}$$

where the contents inside parentheses depend upon the parallel or perpendicular components of E or B. Since these boundary conditions should be valid for all points (x,y) on the interface (z=0), the exponentials depending upon (x,y) must be equal. Therefore,

$$\vec{k}_I\cdot\vec{r} = \vec{k}_R\cdot\vec{r} = \vec{k}_T\cdot\vec{r},\ \text{at } z=0$$

Or, more explicitly,

$$x(k_I)_x + y(k_I)_y = x(k_R)_x + y(k_R)_y = x(k_T)_x + y(k_T)_y$$

For x=0, this gives $$(k_I)_y = (k_R)_y = (k_T)_y$$

For y=0, this gives $$(k_I)_x = (k_R)_x = (k_T)_x$$

Parallel components of wave vector are separately equal

Since the parallel components of incident, reflected and transmitted wave vectors are separately equal it means if we choose the incident wave vector to be in x-z plane say (so that $(k_I)_y = 0$), it would mean that the reflected and transmitted wave vectors will also lie in the x-z plane.

First Law:

The incident, reflected and transmitted wave vectors form a plane (called the plane of incidence), which also includes the normal to the interface (z-axis).

Thus, we have

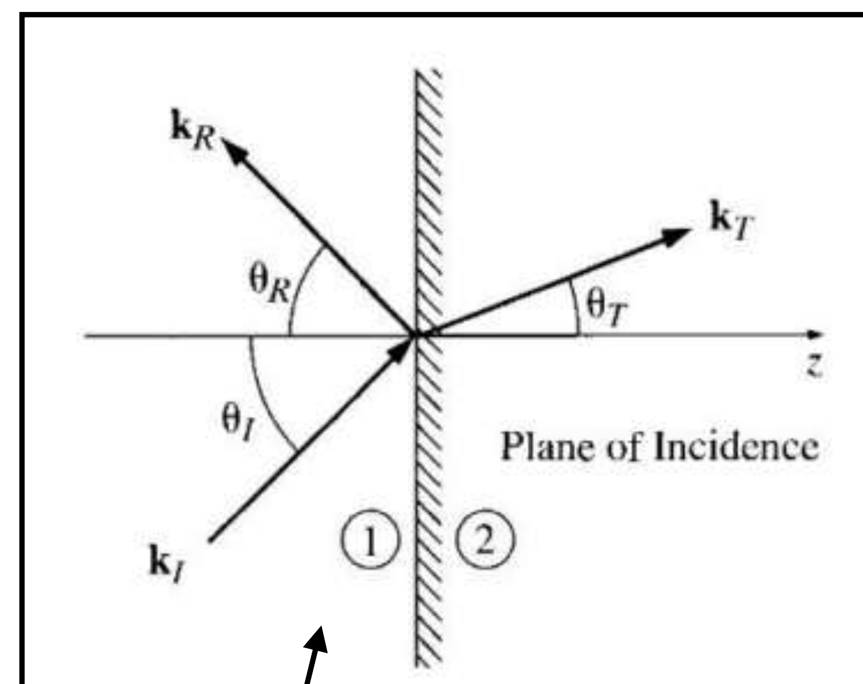
$$(k_I)_x = (k_R)_x = (k_T)_x$$

$$k_I \sin\theta_I = k_R \sin\theta_R = k_T \sin\theta_T$$

θ_I : angle of incidence

θ_R : angle of reflection

θ_T : angle of transmission/refraction.



x-z plane

Second Law:

The angle of incidence is equal to the angle of reflection $\theta_I = \theta_R$

This is obvious by using $k_I = k_R$ in $k_I \sin\theta_I = k_R \sin\theta_R$

This is also known as the **law of reflection**.

Similarly, using $k_I = \dfrac{n_1}{n_2} k_T$ in $k_I \sin\theta_I = k_T \sin\theta_T$ we get

Third Law:

$$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

This is known as the **law of refraction** or **Snell's law**.

These are the three fundamental laws of geometrical optics.

Since the exponential terms cancel out from both sides of the generic boundary conditions, we can write the exact boundary conditions in terms of the amplitudes:

$$(i)\ \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp} \implies \epsilon_1(\vec{\tilde{E}}_{0I} + \vec{\tilde{E}}_{0R})_z = \epsilon_2(\vec{\tilde{E}}_{0T})_z$$

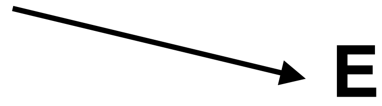
$$(ii)\ B_1^{\perp} = B_2^{\perp} \implies (\vec{\tilde{B}}_{0I} + \vec{\tilde{B}}_{0R})_z = (\vec{\tilde{B}}_{0T})_z$$

$$(iii)\ E_1^{\parallel} = E_2^{\parallel} \implies (\vec{\tilde{E}}_{0I} + \vec{\tilde{E}}_{0R})_{x,y} = (\vec{\tilde{E}}_{0T})_{x,y}$$

$$(iv)\ \frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel} \implies \frac{1}{\mu_1}(\vec{\tilde{B}}_{0I} + \vec{\tilde{B}}_{0R})_{x,y} = \frac{1}{\mu_2}(\vec{\tilde{B}}_{0T})_{x,y}$$

Here $\vec{\tilde{B}}_0 = \frac{1}{v}(\hat{k} \times \vec{\tilde{E}}_0)$ and the last two boundary conditions contain two equations each, for x and y components respectively.

Incident polarisation parallel to the plane of incidence



In this case, the reflected and transmitted waves are also polarised in this plane.

The boundary condition (i) becomes, in this case:

$$\epsilon_1(-\tilde{E}_{0I}\sin\theta_I + \tilde{E}_{0R}\sin\theta_R) = \epsilon_2(-\tilde{E}_{0T}\sin\theta_T)$$

The boundary condition (ii) is trivial as magnetic field has no component perpendicular to the interface (z=0).

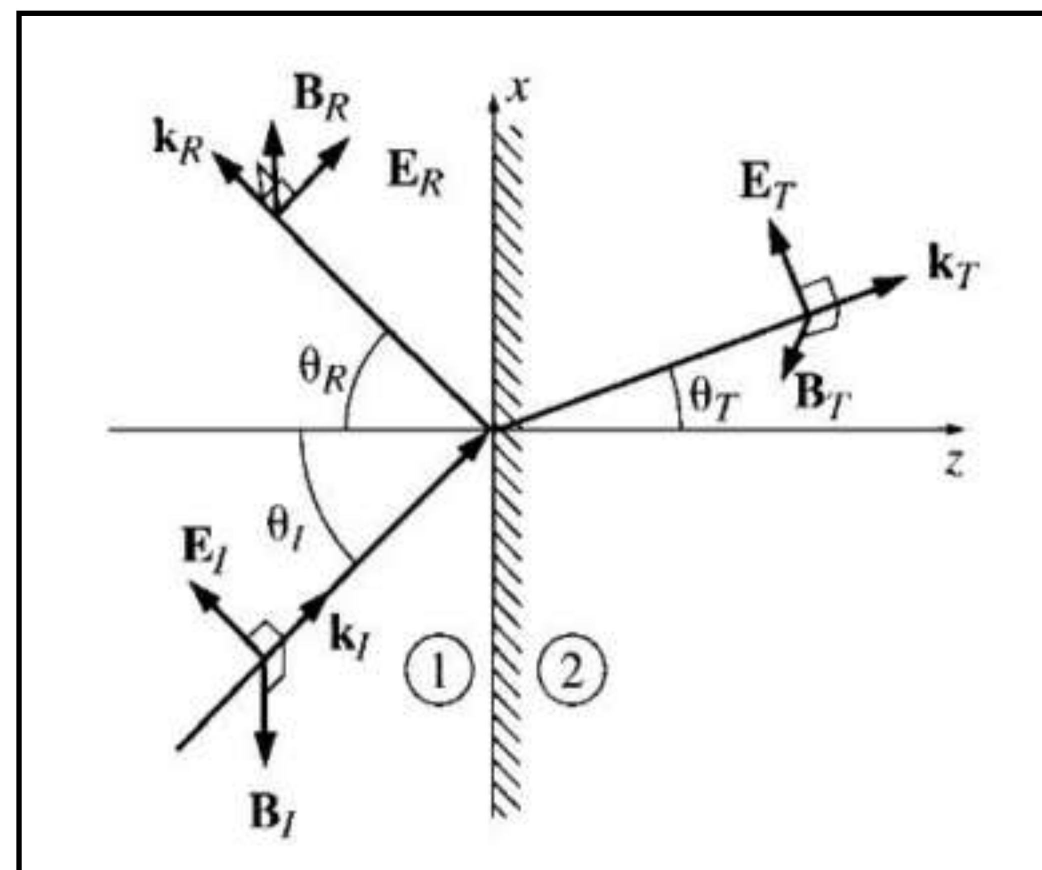


Figure 9.14, Introduction to Electrodynamics, D J Griffiths

The boundary conditions (iii), (iv) give:

$$\tilde{E}_{0I}\cos\theta_I + \tilde{E}_{0R}\cos\theta_R = \tilde{E}_{0T}\cos\theta_T \implies \tilde{E}_{0I} + \tilde{E}_{0R} = \alpha\tilde{E}_{0T},\ \alpha = \frac{\cos\theta_T}{\cos\theta_I},$$

$$\frac{1}{\mu_1 v_1}(\tilde{E}_{0I} - \tilde{E}_{0R}) = \frac{1}{\mu_2 v_2}\tilde{E}_{0T} \implies \tilde{E}_{0I} - \tilde{E}_{0R} = \beta\tilde{E}_{0T},\ \beta = \frac{\mu_1 v_1}{\mu_2 v_2} = \frac{\mu_1 n_2}{\mu_2 n_1}.$$

Here we have used the law of reflection $\theta_I = \theta_R$. Using the above two equations we can find

$$\tilde{E}_{0R} = \left(\frac{\alpha - \beta}{\alpha + \beta}\right)\tilde{E}_{0I},\ \tilde{E}_{0T} = \left(\frac{2}{\alpha + \beta}\right)\tilde{E}_{0I}$$

These are called **Fresnel's Equations**, for the case of polarisation in the plane of incidence. They imply:

1. Transmitted wave is always in phase with the incident one.

2. The reflected wave is either in phase (*right side up*) or 180° out of phase (*upside down*) depending upon $\alpha > \beta (\alpha < \beta)$.

Using the law of refraction

$$\alpha = \frac{\cos\theta_T}{\cos\theta_I} = \frac{\sqrt{1-\sin^2\theta_T}}{\cos\theta_I} = \frac{\sqrt{1-[(n_1/n_2)\sin\theta_I]^2}}{\cos\theta_I}$$

Using $\theta_I = 0 \implies \alpha = 1$, we recover the results for normal incidence discussed in Lecture 25.

For grazing incidence $\theta_I = 90^\circ, \alpha \to \infty$ resulting in a totally reflected wave (which as Griffiths says, "*is a fact that is painfully familiar to anyone who has driven at night on a wet road*").

For an intermediate angle of incidence $\theta_I = \theta_B$ that satisfies

$$\alpha = \beta \implies \sin^2\theta_B = \frac{1-\beta^2}{(n_1/n_2)^2 - \beta^2}$$

gives rise to a completely extinguished reflected wave. It is known as the **Brewster's angle**.

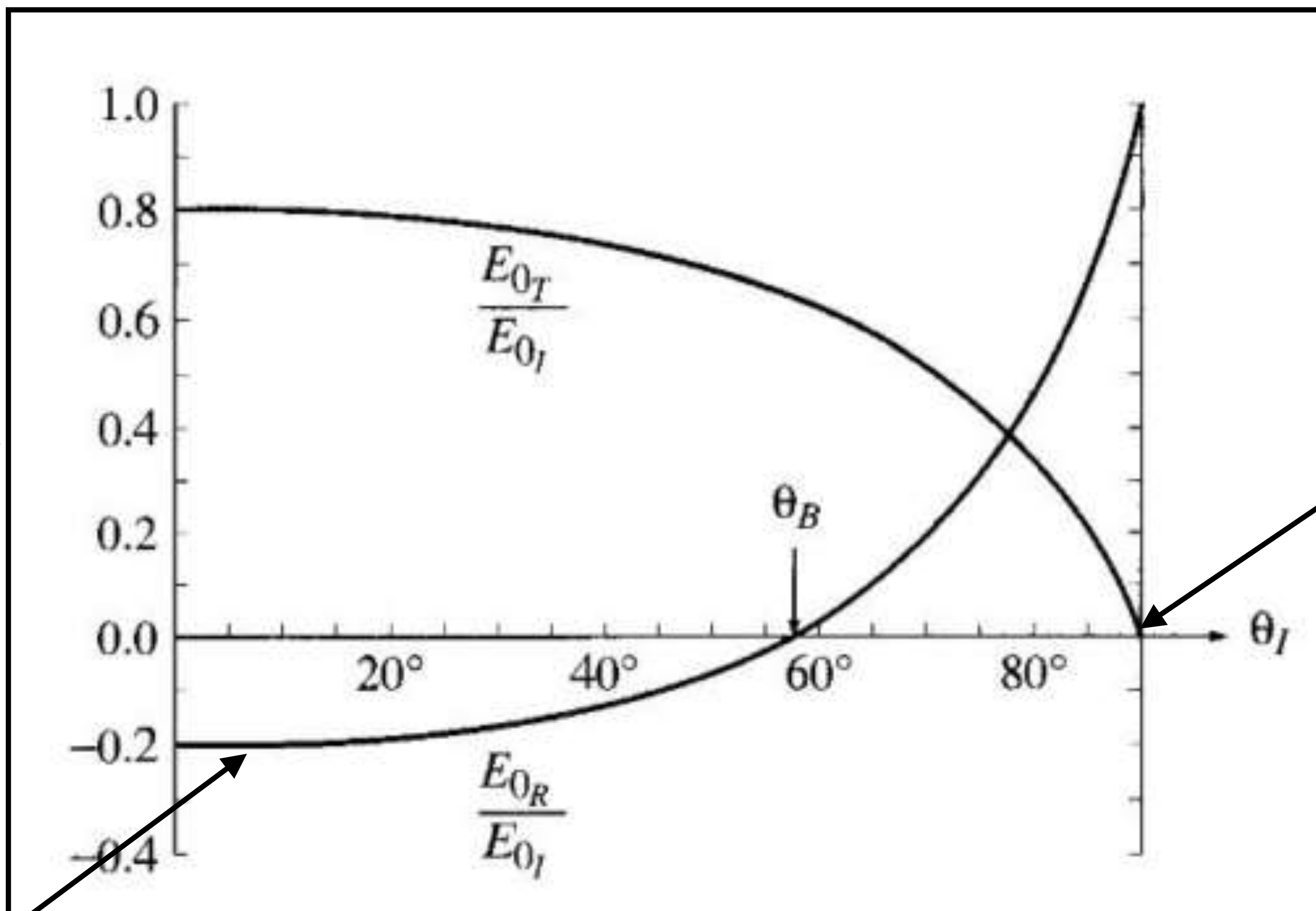
Sir David Brewster (1781–1868)

For $\mu_1 \approx \mu_2 \implies \beta \approx \dfrac{n_2}{n_1}$, we can write

$$\sin^2\theta_B \approx \frac{\beta^2(1-\beta^2)}{1-\beta^4} = \frac{\beta^2}{1+\beta^2} \implies \tan\theta_B \approx \frac{n_2}{n_1}$$

$n_2 = 1.5$ (Glass)
$n_1 = 1.0$ (Air)

Grazing incidence



Reflected & Incident waves out of phase

Figure 9.16, Introduction to Electrodynamics, D J Griffiths

The incident intensity is $I_I = \frac{1}{2}\epsilon_1 v_1 E_{0I}^2 \cos\theta_I$

Here the appearance of $\cos\theta_I$ is due to the mismatch in the direction of power flow (Poynting vector **S**) and the normal to the interface. In the present case, the power per unit area striking the interface is $\vec{S}\cdot\hat{z}$.

Similarly, the reflected and transmitted intensities are

$$I_R = \frac{1}{2}\epsilon_1 v_1 E_{0R}^2 \cos\theta_R, \; I_T = \frac{1}{2}\epsilon_2 v_2 E_{0T}^2 \cos\theta_T$$

The corresponding reflection and transmission coefficients are:

$$R = \frac{I_R}{I_I} = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{\alpha-\beta}{\alpha+\beta}\right)^2,$$

$$T = \frac{I_T}{I_I} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\left(\frac{E_{0T}}{E_{0I}}\right)^2 \frac{\cos\theta_T}{\cos\theta_I} = \alpha\beta\left(\frac{2}{\alpha+\beta}\right)^2$$

In the last equation, we have used $\frac{\epsilon_2 v_2}{\epsilon_1 v_1} = \frac{\mu_1}{\mu_2}\frac{\epsilon_2 \mu_2 v_2}{\epsilon_1 \mu_1 v_1} = \frac{\mu_1 v_1}{\mu_2 v_2} = \beta$

It is obvious to check that in this case also R+T=1.

$n_2 = 1.5$ (Glass)

$n_1 = 1.0$ (Air)

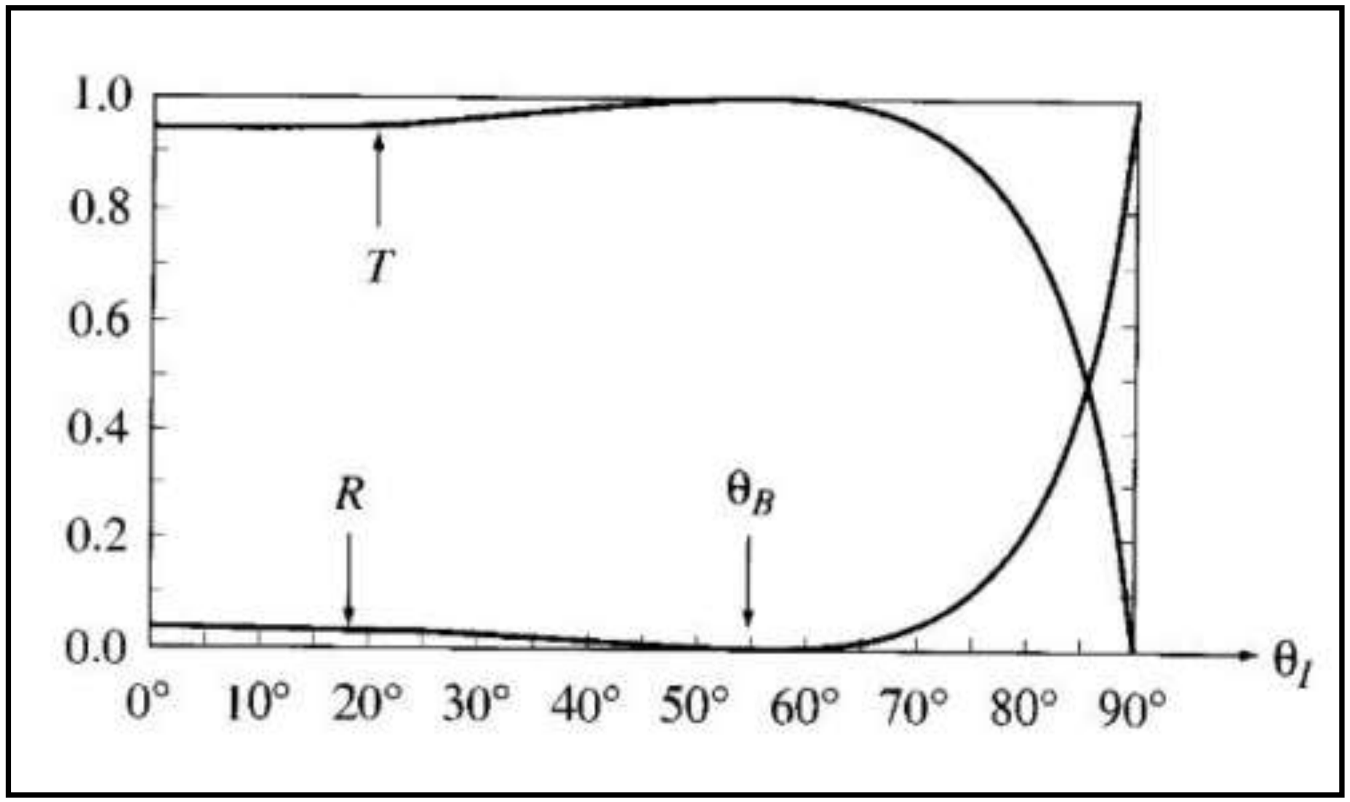


Figure 9.17, Introduction to Electrodynamics, D J Griffiths

Incident polarisation perpendicular to the plane of incidence

E

Let the incident polarisation be in y direction, perpendicular to the plane of incidence (x-z). The incident, reflected and transmitted waves can be written as:

$$\vec{\tilde{E}}_I = \tilde{E}_{0I} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)} \hat{y},$$

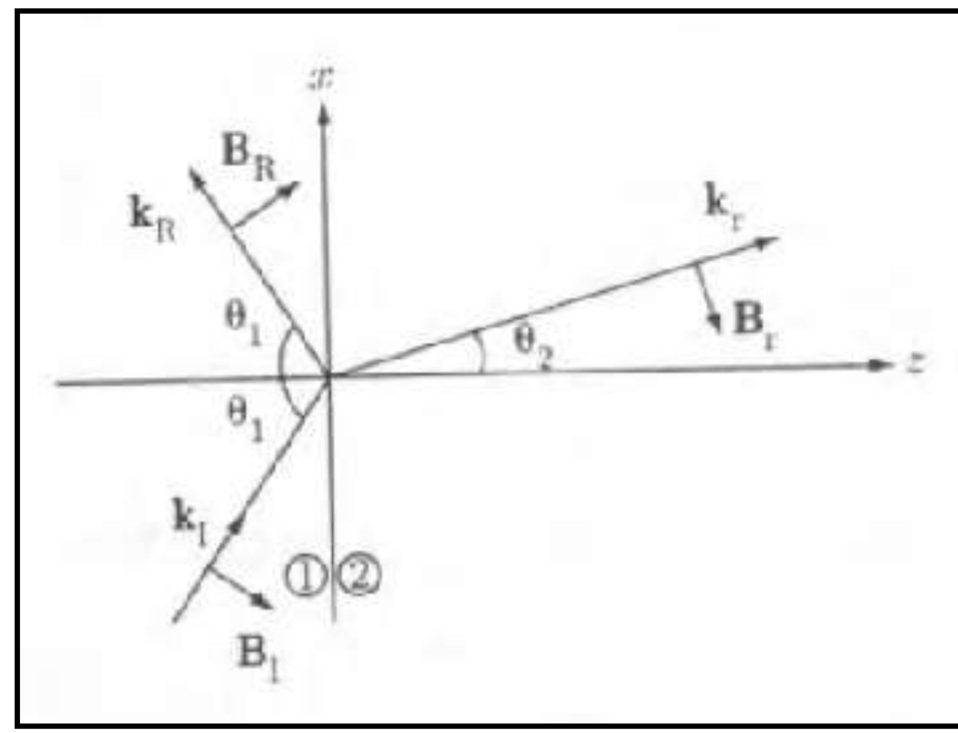
$$\vec{\tilde{B}}_I = \frac{1}{v_1} \tilde{E}_{0I} e^{i(\vec{k}_I \cdot \vec{r} - \omega t)} (-\cos\theta_I \hat{x} + \sin\theta_I \hat{z}),$$

$$\vec{\tilde{E}}_R = \tilde{E}_{0R} e^{i(\vec{k}_R \cdot \vec{r} - \omega t)} \hat{y},$$

$$\vec{\tilde{B}}_R = \frac{1}{v_1} \tilde{E}_{0R} e^{i(\vec{k}_R \cdot \vec{r} - \omega t)} (\cos\theta_I \hat{x} + \sin\theta_I \hat{z}),$$

$$\vec{\tilde{E}}_T = \tilde{E}_{0T} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)} \hat{y},$$

$$\vec{\tilde{B}}_T = \frac{1}{v_2} \tilde{E}_{0T} e^{i(\vec{k}_T \cdot \vec{r} - \omega t)} (-\cos\theta_T \hat{x} + \sin\theta_T \hat{z}).$$



Law of reflection $\theta_I = \theta_R$ is used

The electromagnetic boundary conditions:

$$(i)\ \epsilon_1 E_1^{\perp} = \epsilon_2 E_2^{\perp},\ (iii)\ E_1^{\parallel} = E_2^{\parallel},$$

$$(ii)\ B_1^{\perp} = B_2^{\perp},\ (iv)\ \frac{1}{\mu_1} B_1^{\parallel} = \frac{1}{\mu_2} B_2^{\parallel}.$$

Law of refraction: $$\frac{\sin\theta_T}{\sin\theta_I} = \frac{v_2}{v_1}$$

The boundary condition (i) is trivial. The boundary condition (iii) gives:

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \tilde{E}_{0T}$$

(ii) gives:

$$\frac{1}{v_1}\tilde{E}_{0I}\sin\theta_I + \frac{1}{v_1}\tilde{E}_{0R}\sin\theta_I = \frac{1}{v_2}\tilde{E}_{0T}\sin\theta_T$$

$$\tilde{E}_{0I} + \tilde{E}_{0R} = \left(\frac{v_1\sin\theta_T}{v_2\sin\theta_I}\right)\tilde{E}_{0T} = \tilde{E}_{0T}.$$

The boundary condition (iv) gives:

$$\frac{1}{\mu_1}\left[\frac{1}{v_1}\tilde{E}_{0I}(-\cos\theta_I)+\frac{1}{v_1}\tilde{E}_{0R}\cos\theta_I\right]=\frac{1}{\mu_2 v_2}\tilde{E}_{0T}(-\cos\theta_T)$$

$$\implies \tilde{E}_{0I}-\tilde{E}_{0R}=\alpha\beta\tilde{E}_{0T};\ \alpha=\frac{\cos\theta_T}{\cos\theta_I},\ \beta=\frac{\mu_1 v_1}{\mu_2 v_2}.$$

Solving these for reflected, transmitted amplitudes:

$$2\tilde{E}_{0I}=(1+\alpha\beta)\tilde{E}_{0T}\implies \tilde{E}_{0T}=\left(\frac{2}{1+\alpha\beta}\right)\tilde{E}_{0I},$$

$$\tilde{E}_{0R}=\tilde{E}_{0T}-\tilde{E}_{0I}=\left(\frac{2}{1+\alpha\beta}-1\right)\tilde{E}_{0I}\implies \tilde{E}_{0R}=\left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)\tilde{E}_{0I}.$$

Transmitted wave is **in phase** with the incident one. Reflected wave is **in phase** if $\alpha\beta<1$ and **out of phase** by 180 degree if $\alpha\beta>1$

The real amplitudes are related as

$$E_{0T} = \left(\frac{2}{1+\alpha\beta}\right) E_{0I}, \; E_{0R} = \left|\frac{1-\alpha\beta}{1+\alpha\beta}\right| E_{0I}.$$

These are the **Fresnel Equations** for polarisation perpendicular to the plane of incidence.

Reflection and Transmission Coefficients:

$$R = \left(\frac{E_{0R}}{E_{0I}}\right)^2 = \left(\frac{1-\alpha\beta}{1+\alpha\beta}\right)^2, \; T = \frac{\epsilon_2 v_2}{\epsilon_1 v_1}\frac{\cos\theta_T}{\cos\theta_I}\left(\frac{E_{0T}}{E_{0I}}\right)^2 = \alpha\beta\left(\frac{2}{1+\alpha\beta}\right)^2$$

$$\implies R + T = \frac{(1-\alpha\beta)^2 + 4\alpha\beta}{(1+\alpha\beta)^2} = 1.$$

For vanishing reflection coefficient in this case

$$E_{0R} = 0 \implies \alpha\beta = 1$$

$$\alpha = \frac{\sqrt{1 - \left(\frac{v_2}{v_1}\right)^2 \sin^2\theta}}{\cos\theta} = \frac{1}{\beta} = \frac{\mu_2 v_2}{\mu_1 v_1}$$

$$1 - \left(\frac{v_2}{v_1}\right)^2 \sin^2\theta = \left(\frac{\mu_2 v_2}{\mu_1 v_1}\right)^2 \cos^2\theta$$

$$1 = \left(\frac{v_2}{v_1}\right)^2 \left[\sin^2\theta + \left(\frac{\mu_2}{\mu_1}\right)^2 \cos^2\theta\right]$$

For $\mu_1 \approx \mu_2$ this leads to the trivial condition $v_1 = v_2$ which can be true only if the two media are indistinguishable.

There exists no **Brewster's angle** in this case, if the two media are distinguishable but have $\mu_1 \approx \mu_2$.

For $\mu_1 \neq \mu_2$, one can find the Brewster's angle as:

$$\left(\frac{v_1}{v_2}\right)^2 = 1 - \cos^2\theta_B + \left(\frac{\mu_2}{\mu_1}\right)^2 \cos^2\theta_B$$

$$\implies \cos^2\theta_B = \frac{\left(\frac{v_1}{v_2}\right)^2 - 1}{\left(\frac{\mu_2}{\mu_1}\right)^2 - 1} = \frac{\frac{\epsilon_2}{\epsilon_1} - \frac{\mu_1}{\mu_2}}{\frac{\mu_2}{\mu_1} - \frac{\mu_1}{\mu_2}}$$

In this case, the phase difference between reflected and incident wave does not change from 0 to 180 degrees depending upon the angle of incidence. It is either always 0 or always 180 degrees out of phase.

In general,

$$\beta > 1, \alpha\beta > 1 \implies \delta_R = \pi$$
$$\beta < 1, \alpha\beta < 1 \implies \delta_R = 0.$$

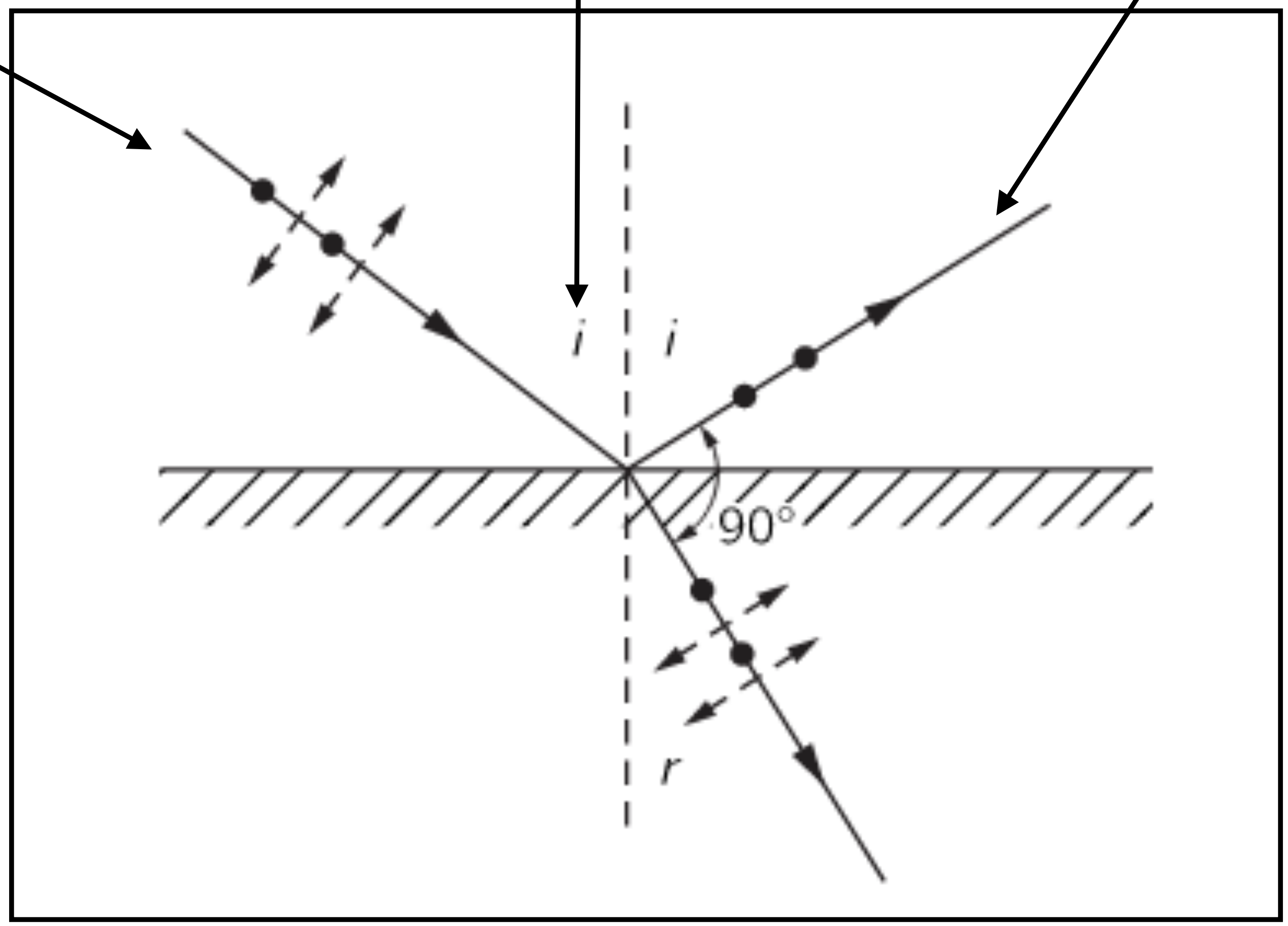
Verify!

Polarisation by Reflection

Unpolarised Light

Brewster's angle

Polarised perpendicular to the plane of incidence



Credit: Feynman Lectures in Physics

Total Internal Reflection

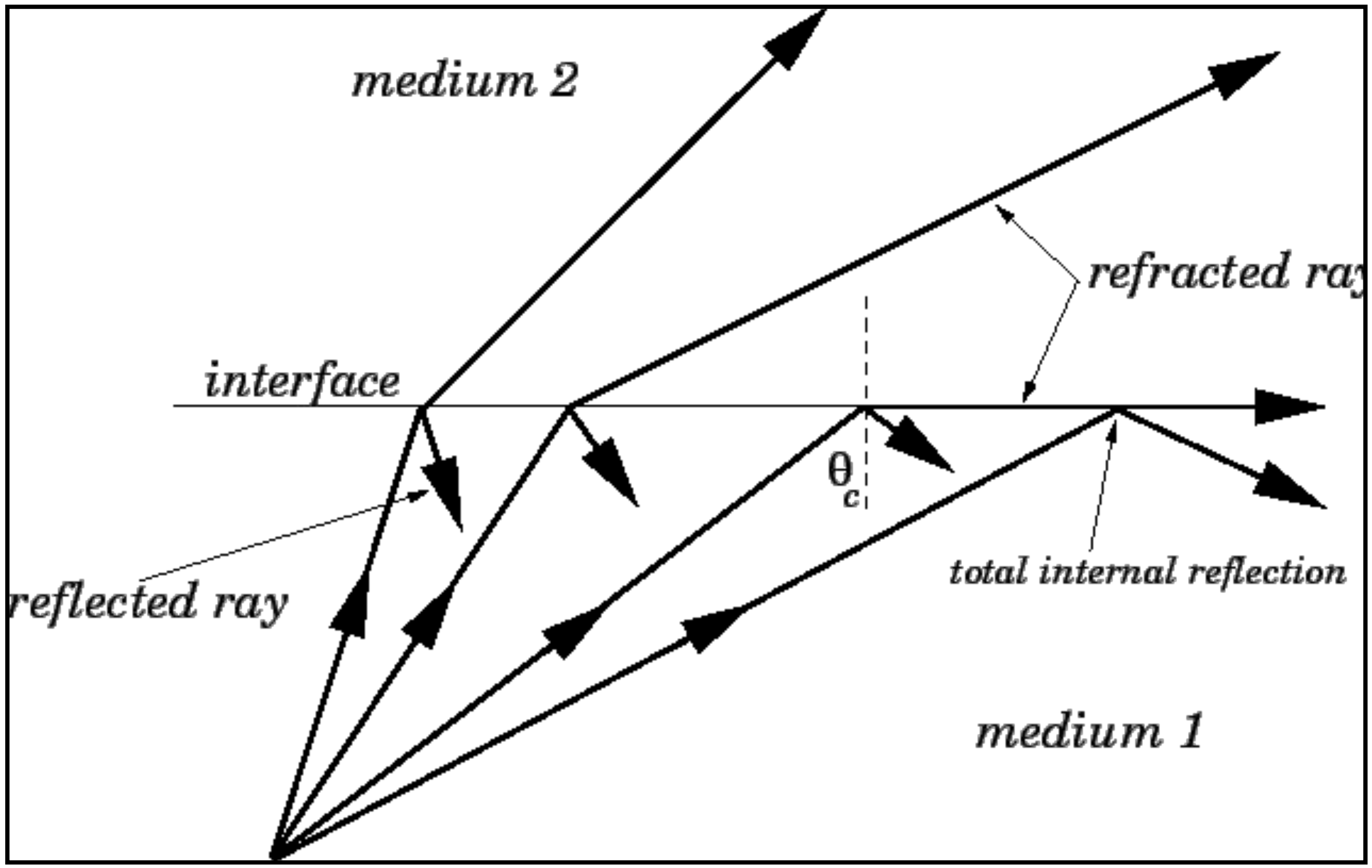
According to Snell's law: $$\frac{\sin\theta_T}{\sin\theta_I} = \frac{n_1}{n_2}$$

For incident angle having the critical value $\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$ the angle of refraction is $\theta_T = \pi/2$ and the transmitted wave just grazes the interface ($n_1 > n_2$).

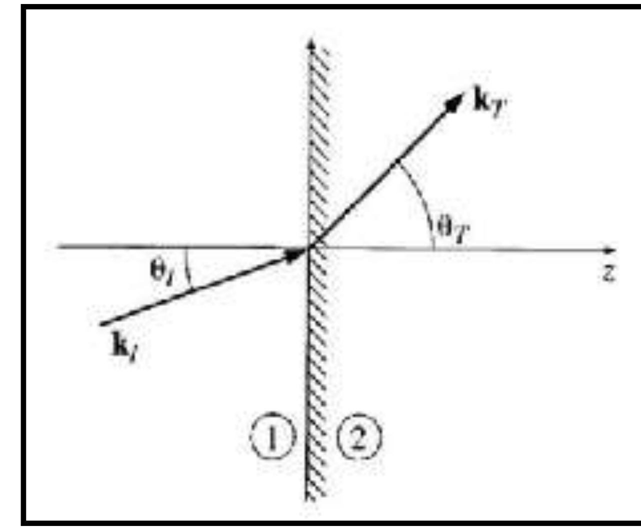
For angle of incidence greater than this critical value, there is no transmitted/refracted ray at all, but only a reflected one. This phenomenon is called the **total internal reflection (TIR)**.

What are the electric and magnetic fields in the second medium? Are they zero for such angle of incidence?

Total Internal Reflection



Credit: Univ of Texas



For the transmitted wave: $\vec{k}_T = k_T(\sin\theta_T \hat{x} + \cos\theta_T \hat{z})$

For TIR, $\sin\theta_T = \frac{n_1}{n_2}\sin\theta_I > \frac{n_1}{n_2}\sin\theta_c > 1$

Therefore, $\cos\theta_T = \sqrt{1-\sin^2\theta} = i\sqrt{\sin^2\theta_T - 1}$ is imaginary. We can write

$$\vec{k}_T \cdot \vec{r} = k_T(x\sin\theta_T + z\cos\theta_T) = xk_T\sin\theta_T + izk_T\sqrt{\sin^2\theta_T - 1} = kx + i\kappa z$$

$$k \equiv k_T \sin\theta_T = \left(\frac{\omega n_2}{c}\right)\frac{n_1}{n_2}\sin\theta_I = \frac{\omega n_1}{c}\sin\theta_I$$

$$\kappa \equiv k_T\sqrt{\sin^2\theta_T - 1} = \frac{\omega n_2}{c}\sqrt{\left(\frac{n_1}{n_2}\right)^2\sin^2\theta_I - 1} = \frac{\omega}{c}\sqrt{n_1^1\sin^2\theta_I - n_2^2}.$$

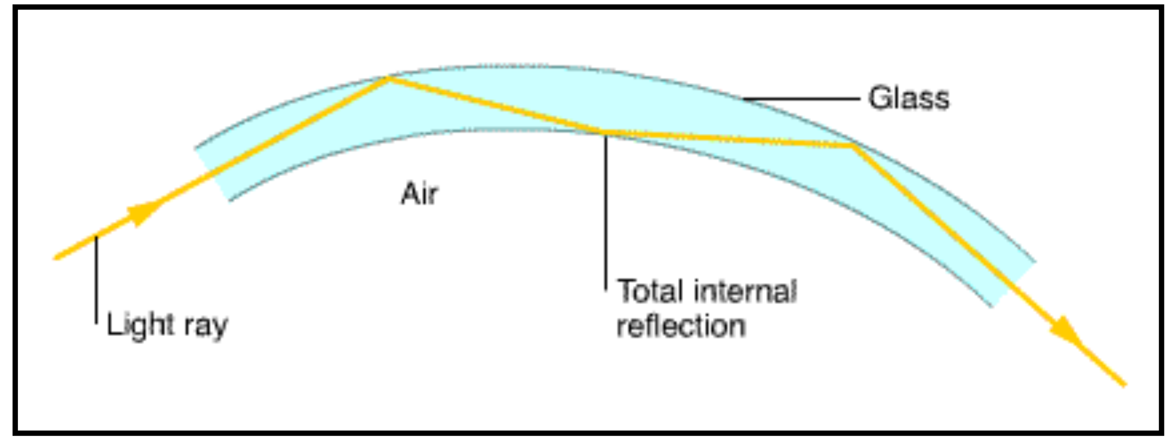
n_1^2

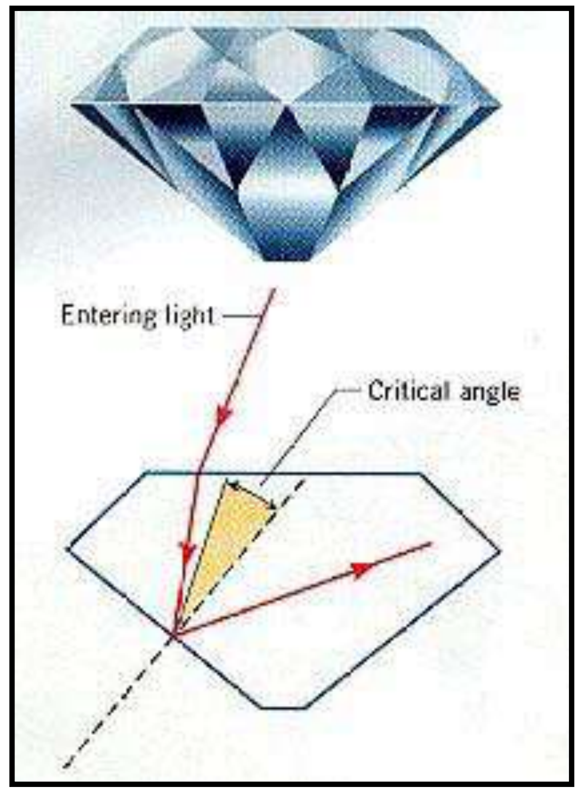
The transmitted field is:

$$\tilde{\vec{E}}(\vec{r},t) = \tilde{\vec{E}}_{0T}e^{i(\vec{k}_T\cdot\vec{r}-\omega t)} = \tilde{\vec{E}}_{0T}e^{-\kappa z}e^{i(kx-\omega t)}.$$

Almost vanishing amplitude away from z=0 interface!

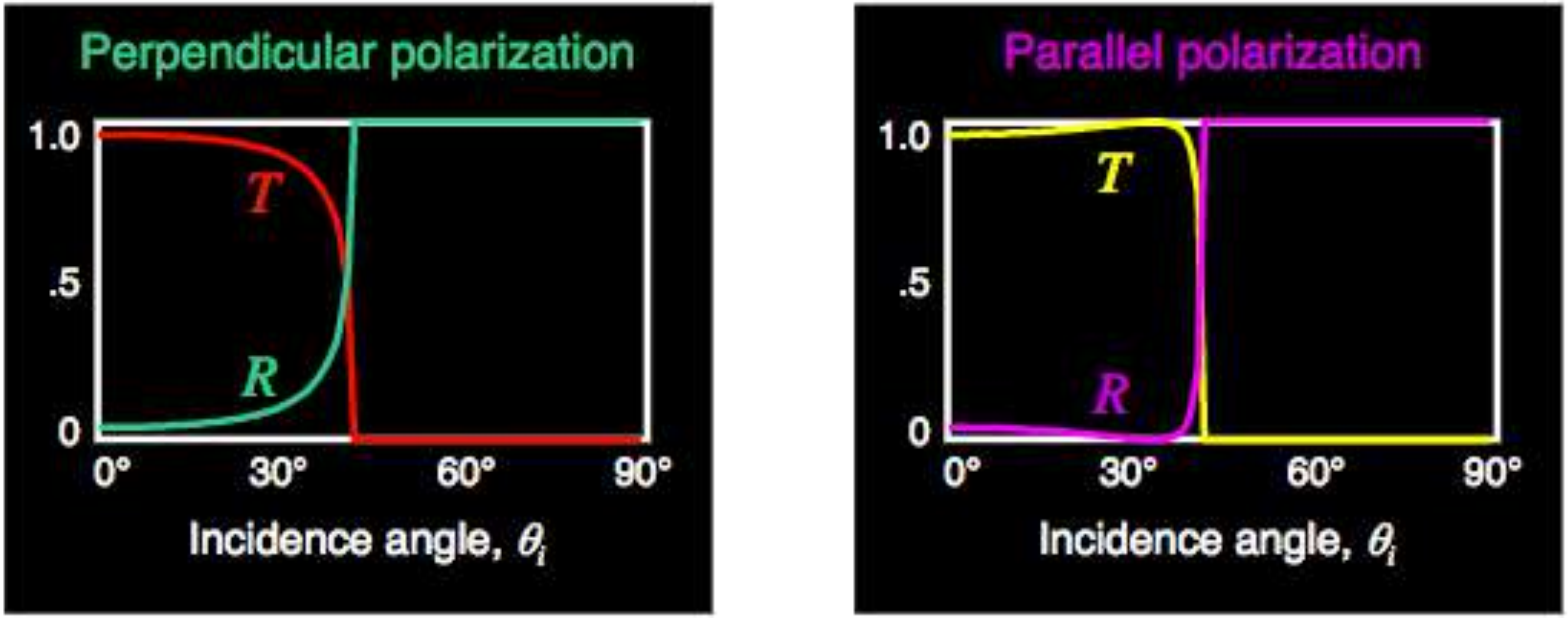
Applications of TIR:





Exercise: Show that for both types of polarisation (parallel as well as perpendicular to the plane of incidence), the reflection coefficient in case of TIR is 1, as expected.

Glass to Air (TIR)



$$n_1 = 3/2, n_2 = 1, \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \approx 41.8^\circ$$

$$\theta_B = \tan^{-1}\left(\frac{n_2}{n_1}\right) \approx 33.7^\circ.$$ For parallel polarisation

Credit: Brown Univ

Exercise: The water surface in the picture is behaving like a very good reflector. Is it due to TIR?