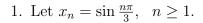
Department of Mathematics Indian Institute of Technology Guwahati MA 101: Mathematics I

Model solutions of Quiz-I



(a) Find
$$\limsup x_n$$
 and $\limsup x_n$.

Solution. We have
$$x_1 = \sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}, x_2 = \sin \frac{2\pi}{3} = \frac{\sqrt{3}}{2}, x_3 = \sin \frac{3\pi}{3} = 0,$$

$$x_4 = \sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}, x_5 = \sin \frac{5\pi}{3} = -\frac{\sqrt{3}}{2}, x_6 = \sin \frac{6\pi}{3} = 0, \dots$$
Hence, $x_n \in \{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\}$ for all $n \ge 1$. [1]

Since x_n takes the value $-\frac{\sqrt{3}}{2}$ for infinitely many values of n, so $z_n = \inf\{x_k : k \ge n\} = -\frac{\sqrt{3}}{2}$ for all n, and hence $\liminf x_n = -\frac{\sqrt{3}}{2}$. Also x_n takes the value $\frac{\sqrt{3}}{2}$ for infinitely many values of n, so $y_n = \sup\{x_k : k \ge n\} = \frac{\sqrt{3}}{2}$ for all n. Hence, $\limsup x_n = \frac{\sqrt{3}}{2}$. [1]

(b) Let (x_{n_k}) be a subsequence of (x_n) such that $x_{n_k} \to \ell$. Find all possible values of ℓ . To each possible values of ℓ , find a subsequence of (x_n) converging to ℓ .

Solution. Let $x_{n_k} \to \ell$. If $\ell \notin \{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\}$, then either $\ell < -\frac{\sqrt{3}}{2}$ or $-\frac{\sqrt{3}}{2} < \ell < 0$ or $0 < \ell < \frac{\sqrt{3}}{2}$ or $\ell > \frac{\sqrt{3}}{2}$. Let $\ell < -\frac{\sqrt{3}}{2}$. Take $\varepsilon = \frac{|-\frac{\sqrt{3}}{2} - \ell|}{2}$. Then $x_n \notin (\ell - \varepsilon, \ell + \varepsilon)$ for any n. Hence, (x_{n_k}) does not converge to ℓ . Using similar argument, it follows that (x_{n_k}) does not converge to ℓ if $\ell \notin \{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\}$. Hence, $\ell \in \{-\frac{\sqrt{3}}{2}, 0, \frac{\sqrt{3}}{2}\}$. [1]

We have
$$x_{3n} = \sin \frac{3n\pi}{3} = \sin n\pi \to 0$$
.

$$x_{6n+1} = \sin \frac{(6n+1)\pi}{3} = \sin(2n\pi + \frac{\pi}{3}) \to \frac{\sqrt{3}}{2}.$$

$$x_{6n-2} = \sin \frac{(6n-2)\pi}{3} = \sin(2n\pi - \frac{2\pi}{3}) \to -\frac{\sqrt{3}}{2}.$$
[1]

2. Let
$$x_1 = 1$$
 and $x_{n+1} = \frac{3n}{3n+1}x_n^2$, $n \ge 1$.

(a) Show that
$$(x_n)$$
 is bounded.

Solution. We have $x_n^2 \ge 0$ for all n, and so $x_{n+1} = \frac{3n}{3n+1}x_n^2 \ge 0$ for all $n \ge 1$. Also, $x_1 = 1$. Hence $x_n \ge 0$ for all n. Suppose that $x_n \le 1$. Then $x_{n+1} \le 1$. Since $x_1 = 1$, by the principle of mathematical induction $x_n \le 1$ for all n. Thus, $0 \le x_n \le 1$ for all n.

(b) Show that
$$(x_n)$$
 is decreasing.

Solution. We have $x_n - x_{n+1} = x_n - \frac{3n}{3n+1}x_n^2 = x_n(1 - \frac{3n}{3n+1}x_n) \ge 0$ for all n. Hence (x_n) is decreasing.

(c) Find
$$\lim x_n$$
.

Solution. By monotone convergence theorem, (x_n) is convergent. Suppose that $x_n \to \ell$. Then $\ell = \ell^2$. Hence, $\ell = 0$ or $\ell = 1$. We have $x_2 = \frac{3}{4}$. Since (x_n) is decreasing, so $x_n \leq \frac{3}{4}$ for all $n \geq 2$. This yields $\ell \leq \frac{3}{4}$. Therefore, $\ell = 0$.

3. Test the convergence/divergence of the infinite series
$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{2019}}$$
.

Solution. Here $x_n = \frac{1}{(\log n)^{2019}}$ is decreasing and positive for all $n \geq 2$. By Cauchy condensation test,

$$\sum_{n=2}^{\infty} \frac{1}{(\log n)^{2019}} \text{ is convergent} \Leftrightarrow \sum_{n=2}^{\infty} 2^n \frac{1}{n^{2019} (\log 2)^{2019}} \text{ is convergent}$$

$$\Leftrightarrow \frac{1}{(\log 2)^{2019}} \sum_{n=2}^{\infty} \frac{2^n}{n^{2019}} \text{ is convergent}$$

$$[1\frac{1}{2}]$$

Let
$$y_n = \frac{2^n}{n^{2019}}$$
. Then $\frac{y_{n+1}}{y_n} = \frac{2^{n+1}}{(n+1)^{2019}} \times \frac{n^{2019}}{2^n} = \frac{2}{(1+\frac{1}{n})^{2019}} \to 2$.
By Ratio test, $\sum_{n=1}^{\infty} y_n$ is divergent and hence $\sum_{n=2}^{\infty} \frac{1}{(\log n)^{2019}}$ diverges. $[1\frac{1}{2}]$