Department of Mathematics
Indian Institute of Technology Guwahati

MA 101: Mathematics I

Model solutions of mid-semester examination

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1. Let $a > 0$. Let $x_1 = 1$ and $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$ for all $n \in \mathbb{N}$. Prove that (x_n) is convergent and find $\lim\limits_{n\to\infty} x_n$. $\boxed{\mathbf{3}}$

Solution: Since $a > 0$, so $x_n > 0$ for all $n \in \mathbb{N}$. By AM-GM inequality, we have

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \geq \sqrt{a} \quad \Rightarrow x_{n+1}^2 \geq a \text{ for all } n \in \mathbb{N}. \qquad [1]$$

Now,

$$\begin{aligned} x_{n+1} - x_n &= \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - x_n \\ &= \frac{1}{2}\left(\frac{a}{x_n} - x_n\right) \\ &= \frac{a - x_n^2}{2x_n} \\ &\leq 0 \ \text{ for all } \ n \geq 2. \end{aligned}$$

Hence, $(x_n)_{n=2}^{\infty}$ is decreasing. [1]

Since (x_n) is bounded below, so (x_n) is convergent. Let $x_n \to \ell$. Then, $\ell = \pm\sqrt{a}$. Since $x_n > 0$ for all n, so $\ell \geq 0$. Hence $\ell = \sqrt{a}$. [1]

2. Determine all the values of $x \in \mathbb{R}$ for which the series $\displaystyle\sum_{n=1}^{\infty}\left(\sin\frac{1}{n}\right)x^n$ converges. $\boxed{\mathbf{4}}$

Solution: Let $a_n = \sin\frac{1}{n}$. We have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\sin\frac{1}{n+1}}{\sin\frac{1}{n}}\right| = \lim_{n\to\infty}\left|\frac{(\sin\frac{1}{n+1})/(1/(n+1))}{(\sin\frac{1}{n})/(1/n)}\right| \times \lim_{n\to\infty}\frac{n}{n+1} = 1.$$

Hence, the series converges absolutely for all $x \in (-1, 1)$; and diverges if $x < -1$ or $x > 1$. [$1\frac{1}{2}$]

When $x = -1$, the series $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\sin\frac{1}{n}\right)$ is an alternating series. Since $(\sin\frac{1}{n})$ is decreasing and $\sin\frac{1}{n} \to 0$, so by Leibniz test, the series converges. [1]

When $x = 1$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\sin\frac{1}{n}$. The equation of the line passing through the points $(0, 0)$ and $(\pi/2, 1)$ is given by $y = \frac{2}{\pi}x$. Hence

$$\sin x \geq \frac{2}{\pi}x \ \text{ for all } \ x \in [0, \pi/2].$$

This implies that $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n} \geq \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}$, and hence the series diverges when $x = 1$.

Another proof: We have $\displaystyle\lim_{x\to 0}\frac{\sin x}{x} = 1$, and hence $\displaystyle\lim_{n\to\infty}\frac{\sin\frac{1}{n}}{\frac{1}{n}} = 1$. Since $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, by limit comparison test, the series $\displaystyle\sum_{n=1}^{\infty}\sin\frac{1}{n}$ also diverges.

Thus, the given series converges if and only if $x \in [-1, 1)$. $[1\frac{1}{2}]$

3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable and $|f'(x)| < \frac{1}{2}$ for all $x \in \mathbb{R}$. Let $a_1 \in \mathbb{R}$ and $a_{n+1} = f(a_n)$ for all $n \in \mathbb{N}$. Show that (a_n) is a Cauchy sequence. Also, prove that the equation $f(x) = x$ has at least one real root. $\boxed{\mathbf{3}}$

Solution: We have $|a_{n+2} - a_{n+1}| = |f(a_{n+1}) - f(a_n)|$. By Mean Value Theorem, there exists c_n between a_n and a_{n+1} such that

$$|f(a_{n+1}) - f(a_n)| = |f'(c_n)||a_{n+1} - a_n|.$$

[1]

This implies that $|a_{n+2} - a_{n+1}| < \frac{1}{2}|a_{n+1} - a_n|$ for all $n \in \mathbb{N}$. Hence, (a_n) is a Cauchy sequence. [1]

Since every Cauchy sequence in $\mathbb{R}$ is convergence, so let $a_n \to \ell$. Since f is continuous so

$$\ell = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} f(a_n) = f(\lim_{n\to\infty} a_n) = f(\ell).$$

Hence, the equation $f(x) = x$ has at least one real root. [1]

4. Determine the radius of convergence of the Taylor series of $\sqrt{x}$ about $x = 1$; and prove that the series converges to $\sqrt{x}$ for each $x \in (1, 2)$. $\boxed{\mathbf{4}}$

Solution: The Taylor series of $f(x) = \sqrt{x}$ about $x = 1$ is given by $\displaystyle\sum_{n=0}^{\infty} a_n(x-1)^n$, where $a_n = \frac{f^{(n)}(1)}{n!}$. We have $f(1) = 1$, $f'(x) = \frac{1}{2}x^{-1/2}$, $f''(x) = -\frac{1}{4}x^{-3/2}$, $f'''(x) = \frac{3}{8}x^{-5/2}$, and by induction

$$f^{(n)}(x) = (-1)^{n-1}\frac{1\cdot 3\cdots(2n-3)}{2^n} \times x^{-(2n-1)/2}.$$

Hence, $a_n = (-1)^{n-1} \times \dfrac{1\cdot 3\cdots(2n-3)}{2^n\cdot n!}$. $[1\frac{1}{2}]$

Now,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{1\cdot 3\cdots(2n-3)(2n-1)\cdot 2^n\cdot n!}{1\cdot 3\cdots(2n-3)\cdot 2^{n+1}\cdot(n+1)!} = \lim_{n\to\infty}\frac{2n-1}{2n+2} = 1.$$

Hence, the radius of convergence of the Taylor series of $\sqrt{x}$ is 1. [1]

The remainder term is

$$R_{n-1}(x) = \frac{f^{(n)}(c)}{n!}(x-1)^n,$$

where $|x-1|<1$ (that is, $x \in (0,2)$) and c is a point between x and 1. Now,

$$|R_{n-1}(x)| \leq \frac{|f^{(n)}(c)|}{n!} = \frac{1\cdot 3\cdots(2n-3)}{2^n \cdot n!} \times \left|\frac{1}{c}\right|^{(2n-1)/2}.$$

Note that

$$\frac{1\cdot 3\cdots(2n-3)}{2^n \cdot n!} = \frac{1\cdot 3\cdots(2n-3)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot 2n} < \frac{1}{2n}.$$

Now, if $x \in (1,2)$, then $c > 1$ and so $\frac{1}{c} < 1$. Hence, $|R_{n-1}(x)| < \frac{1}{2n}(1/c)^{\frac{2n-1}{2}} \to 0$ as $n \to \infty$. Hence, the Taylor series of $\sqrt{x}$ about $x=1$ converges to $\sqrt{x}$ for each $x \in (1,2)$. $[1\frac{1}{2}]$.

5. Let $f : (-1,2) \to \mathbb{R}$ be twice differentiable. Suppose that $f(1-\frac{1}{n}) = 1$ for all $n \in \mathbb{N}$.

   (a) Find $f'(1)$. $\boxed{2}$

   (b) Find $f''(1)$. $\boxed{2}$

Solution: (a) f is continuous at 1 and $1-\frac{1}{n} \to 1$. Hence, $f(1-1/n) \to f(1)$. Since $f(1-1/n) = 1$ for all n, so $f(1) = 1$. [1]

We have $f'(1) = \lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$ exists. Hence,

$$f'(1) = \lim_{n\to\infty} \frac{f(1-1/n)-f(1)}{1/n} = 0. \qquad [1]$$

(b) We have $f(1-1/n) = 1 = f(1)$ for all $n \in \mathbb{N}$. By Rolle's theorem, there exists $c_n \in (1-1/n, 1)$ such that $f'(c_n) = 0$ for all n. [1]

Again, $f''(1) = \lim\limits_{h\to 0} \dfrac{f'(1+h)-f'(1)}{h}$ exists and $c_n \to 1$. Hence,

$$f''(1) = \lim_{n\to\infty} \frac{f'(c_n)-f'(1)}{1-c_n} = 0. \qquad [1]$$

6. Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1, \text{if } x \neq \frac{1}{2}; \\ 0, \text{if } x = \frac{1}{2}. \end{cases}$

   Using *Riemann's criterion*, prove that f is Riemann integrable over $[0,1]$. $\boxed{3}$

Solution: Let $\varepsilon > 0$. We consider the partition $P_\varepsilon = \{0, \frac{1}{2}-\frac{\varepsilon}{3}, \frac{1}{2}+\frac{\varepsilon}{3}, 1\}$.

Then $U(f, P_\varepsilon) = 1$. [1]

$L(f, P_\varepsilon) = (1/2-\varepsilon/3) + 0 + (1-1/2-\varepsilon/3) = 1-2\varepsilon/3$. [1]

Hence, $U(f,P_\varepsilon) - L(f,\varepsilon) = 2\varepsilon/3 < \varepsilon$. By Riemann's criterion, f is Riemann integrable over $[0,1]$. [1]

7. For $n \in \mathbb{N}$, let $a_n = \dfrac{1}{n^{\frac{5}{2}}} \displaystyle\sum_{k=1}^{n} k^{\frac{3}{2}}$. Determine $\lim\limits_{n\to\infty} a_n$ by expressing a_n as a Riemann sum of a suitable function. $\boxed{3}$

Solution: We have $a_n = \dfrac{1}{n^{5/2}} \sum_{k=1}^{n} k^{3/2} = \dfrac{1}{n} \sum_{k=1}^{n} (k/n)^{3/2}$.
Let $P_n = \{0, 1/n, 2/n, \ldots, (n-1)/n, 1\}$. [1]

Consider the function $f(x) = x^{3/2}$. Then $a_n = U(f, P_n)$. [1]

Since f is continuous on $[0,1]$, so it is Riemann integrable on $[0,1]$. Using 1st fundamental thm of calculus, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} U(f, P_n) = \int_0^1 f(x)dx = \int_0^1 x^{3/2}dx = 2/5. \qquad [1]$$

8. Using Gamma function, evaluate the integral $\displaystyle\int_0^1 x(\log x)^4 dx$. $\boxed{\mathbf{3}}$

   Solution: Taking $y = -\log x$, that is, $x = e^{-y}$, we have $dx = -e^{-y}dy$. Hence,

$$\int_0^1 x(\log x)^4 dx = \int_\infty^0 e^{-y} y^4 (-e^{-y}) dy = \int_0^\infty e^{-2y} y^4 dy. \qquad [1\tfrac{1}{2}]$$

   Let $u = 2y$. Then $dy = \frac{1}{2}du$. Now,

$$\int_0^1 x(\log x)^4 dx = \frac{1}{2^5}\int_0^\infty e^{-u} u^4 du = \frac{1}{2^5}\Gamma(5) = \frac{4!}{2^5} = \frac{3}{4}. \qquad [1\tfrac{1}{2}]$$

9. Find all the values of $p \in \mathbb{R}$ for which the following improper integral converges: $\boxed{\mathbf{3}}$

$$\int_0^1 \frac{x^p e^{-x}}{\log(1+x)} dx.$$

   Solution: Let $f(x) = \dfrac{x^p e^{-x}}{\log(1+x)}$. Since $\log(1+x) \to 0$ as $x \to 0+$, the improper integral is of Type-II.
   Let $g(x) = x^{p-1} = \dfrac{1}{x^{1-p}}$. Then,

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} \frac{xe^{-x}}{\log(1+x)} = \lim_{x\to 0+} \frac{e^{-x} - x^2 e^{-x}}{1/(1+x)} = 1. \qquad [1\tfrac{1}{2}]$$

   By limit comparison test, $\displaystyle\int_0^1 f(x)dx$ converges if and only if $\displaystyle\int_0^1 g(x)dx$ converges.
   We know that $\displaystyle\int_0^1 g(x)dx = \int_0^1 \frac{dx}{x^{1-p}}$ converges if and only if $1 - p < 1$, that is, if and only if $p > 0$.
   Hence, the given improper integral converges if and only if $p > 0$. [$1\frac{1}{2}$]