Quantum Mechanics (PH101)
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Tutorial-8
due on Wednesday, 6th of November, 2019 (8:00Hrs IST)

1. The lifetime of a given atom in an excited state is $10^{-8}\ s$. It comes to the ground state by emitting a photon of wavelength 5800 $\mathring{A}$. Find the energy uncertainty and wavelength uncertainty of the photon. Use the minimum time-Energy uncertainty principle $\Delta E \Delta t = \hbar/2$.

Solution:

Here we will use the time-Energy uncertainty principle $\Delta E \Delta t = \hbar/2$.

For the given problem the photon can be emitted at any instant during the time interval $\Delta t = 10^{-8}\ s$.

$\therefore$ The energy uncertainty of the photon is

$$\Delta E = \frac{\hbar}{2\Delta t} = \frac{0.527 \times 10^{-34}\ J.s}{10^{-8}\ s} = 0.527 \times 10^{-26}\ J$$

If λ is the wavelength of the photon then,

$$E = \frac{hc}{\lambda} \Rightarrow \lambda = \frac{hc}{E} \Rightarrow \Delta\lambda = hc\left(\frac{-\Delta E}{E^2}\right)$$

So the uncertainty in wavelength is

$$\Delta\lambda = \frac{hc}{E^2}\Delta E = \frac{hc\lambda^2}{h^2c^2}\Delta E = \frac{\lambda^2}{hc}\Delta E$$

$$= \frac{(5.8 \times 10^{-7} m)^2 \times (0.527 \times 10^{-26} J)}{(6.62 \times 10^{-34} Js) \times (3 \times 10^8 m/s)} = 0.89 \times 10^{-14} m \sim 9 \times 10^{-8} \mathring{A}$$

2. Find the uncertainty in the velocity of a particle if the uncertainty in its position is equal to its (a) de Broglie wavelength (b) Compton wavelength. Use the minimum position and momentum uncertainty relation.

Solution:

From the uncertainty principle we know that

$$\Delta x \Delta p = \hbar/2$$

$$\Delta v = \frac{\hbar}{2m\Delta x}$$

(a) If the uncertainty in position is equal to the de Broglie wavelength

$$\Delta x = \lambda = h/mv$$

So the uncertainty in velocity is

$$\Delta v = \frac{\hbar}{2m\Delta x} = v/4\pi$$

(b)If the uncertainty in position is equal to the Compton wavelength

$$\Delta x = \lambda_C = \frac{h}{mc}$$

So,

$$\Delta v = \frac{\hbar}{2m\Delta x} = c/4\pi$$

Therefore, the uncertainty in velocity is of the order to the speed of light in vacuum.

3. Check if $\Psi = Ae^{i(kx-\omega t)}$ and $\Psi = Asin(kx - \omega t)$ are acceptable solutions of the time-dependent Schroedinger's equation. The time-dependent Schroedinger'e equation is given by

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial^2 x} + U\Psi$$

Solution:
The time-dependent Schroedinger'e equation is given by

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial^2 x} + U\Psi$$

For $\Psi = Ae^{i(kx-\omega t)}$**:**

$$\frac{\partial\Psi}{\partial t} = -i\omega Ae^{i(kx-\omega t)} = -i\omega\Psi$$

$$\frac{\partial \Psi}{\partial x} = ik\Psi$$

$$\frac{\partial^2 \Psi}{\partial x^2} = i^2k^2\Psi = -k^2\Psi$$

Inserting these in the Schroedinger's equation yields

$$i\hbar(-i\omega\Psi) = -\frac{\hbar^2}{2m}(-k^2\Psi) + U\Psi$$

$$\Rightarrow \left(\hbar\omega - \frac{\hbar^2 k^2}{2m} - U\right)\Psi = 0$$

Using $E = h\nu = \hbar\omega$ and $p = \hbar k$, we obtain

$$\left(E - \frac{p^2}{2m} - U\right)\Psi = 0$$

Note that the quantity on the left is equal to zero for the non-relativistic case as $E = K.E + U = \frac{p^2}{2m} + U$.

Thus $\Psi = Ae^{i(kx-\omega t)}$ is a solution of the Schroedinger's equation.

For $\Psi = Asin(kx - \omega t)$**:**

Following the similar process as above we will reach at

$$-i\hbar\omega cos(kx - \omega t) = \left(\frac{\hbar^2 k^2}{2m} + U\right)sin(kx - \omega t)$$

This equation is generally not satisfied for all x and t. Hence $\Psi = Asin(kx - \omega t)$ is not an acceptable solution of the time-dependent Schroedinger equation. This function however, is a solution of the classical wave equation.

4. The normalized wave function of the ground state of the Quantum harmonic oscillator is given by $\psi(x) = C_0 e^{-\alpha x^2}$, where $C_0 = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4}$ and $\alpha = \frac{m\omega}{2\hbar}$. m is the mass and ω is the angular frequency of the oscillator.
Compute the $\Delta x \Delta p$ for this state, where Δx and Δp are the uncertainties in the position x and momentum p, respectively. Please comment over the result whether it is consistent with the uncertainty principle. **Use the Gaussian integral** $\int_{-\infty}^{\infty} e^{\alpha x^2 + \beta x} dx = \sqrt{\frac{\pi}{\alpha}} e^{\beta^2/4\alpha}$.

Solution: We have $< x >= \int_{-\infty}^{\infty} x|\psi(x)|^2 = 0$ as inegrand is odd function.

$< x^2 >= \int_{-\infty}^{\infty} x^2|\psi(x)|^2$. Solving the integral by integration by parts and using the Gaussian integral formula we have $< x^2 >= \dfrac{\hbar}{2m\omega}$.

Now $\Delta x = \sqrt{< x^2 > - < x >^2} = \sqrt{\dfrac{\hbar}{2m\omega}}$.

$< p >= -i\hbar \int_{-\infty}^{\infty} \psi^*(x)\dfrac{d\psi(x)}{dx}dx = 0$. Again inetgrand here is odd function.

$< p^2 >= -\hbar^2 \int_{-\infty}^{\infty} \psi^*(x)\dfrac{d^2\psi(x)}{dx^2}dx$. Again performing the integral by parts and using the Gaussian integral we have,

$< p^2 >= \dfrac{m\hbar\omega}{2}$.

Therefore, $\Delta p = \sqrt{(< p^2 > - < p >^2)} = \sqrt{\dfrac{m\hbar\omega}{2}}$.

Now $\Delta x \Delta p = \hbar/2$. This is, for the optimum state (Gaussian state) the product of the uncertainties of position and momentum is the smallest value allowed by Heisenberg's Uncertainty relation:

$\Delta p \Delta x \geq \hbar/2$.

5. An electron is described by the wave function

$$\psi(x) = \begin{cases} 0, & \text{for } x \leq 0 \\ Ce^{-x}(1-e^{-x}), & \text{for } x > 0, \end{cases}$$

where x is in nm and C is a constant.

(a) Determine the value of C that normalizes $\psi(x)$.

(b) Where is the electron most likely to be found?

(c) Calculate the average position or expectation value of the position $< x >$ for the electron. Compare this with the most likely position, and comment on the difference.

Solution:

(a) We have the normalization condition:

$\int_{-\infty}^{\infty} |\psi^2| = 1$. We get $|C| = 2\sqrt{3}\ \ nm^{-1/2}$.

(b)The most likely place x_m for the electron to be is where $|\psi(x)|^2$ is maximum, or, in this case where $\psi(x)$ is maximum. We have

$$\frac{d\psi(x)}{dx} = 0 \Rightarrow C[e^{-2x} - e^{-x}(1-e^{-x}) = 0 \Rightarrow x = ln2 \quad nm = 0.693 \quad nm.$$

(c) Since electron state is in the stationary state. Its average position is given by

$$< x >= \int_{-\infty}^{\infty} \psi(x) x \psi^*(x) = C^2 \int_0^{\infty} xe^{-2x}[1-e^{-2x}]^2 = 12 \int_0^{\infty} xe^{-2x}[1-2e^{-x}+e^{-2x}]$$

Using integration by parts we have,

$$< x >= C^2 \left[\frac{1}{4} - \frac{2}{9} + \frac{1}{16}\right] = \frac{13}{12} \quad nm \simeq 1.083 \quad nm$$

6. A particle is represented by the wavefunction at time $t = 0$ by

$\Psi(x) = A(a^2 - x^2)$ if $-a \leq x \leq a$ and zero at all other places. Here A and a are constant.

(a) Determine the normalization constant A.
(b) What is the expectation value of x at $t = 0$?
(c) What is the expectation value of p at $t = 0$?
(d) Evaluate $\langle x^2 \rangle$ and $\langle p^2 \rangle$ at $t = 0$.
(e) Obtain the uncertainty relation $(\Delta x \Delta p)$ and comment over your result whether you are getting minimum uncertainty relation or not.

Solution:

(a) The normalization condition is

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} |\Psi(x)|^2 dx \\ &= \int_{-a}^{a} |\Psi(x)|^2 dx \\ &= \int_{-a}^{a} A^2(a^2-x^2)^2 dx \ = 2A^2\left[a^5 - \frac{2}{3}a^5 + \frac{a^5}{5}\right] = \frac{16}{15}A^2 a^5 \end{aligned}$$

$$\Rightarrow A = \sqrt{\frac{15}{16a^5}}$$

(b) The expectation value of x at $t = 0$ is

$$\langle x \rangle = \int_{-\infty}^{\infty} x|\Psi(x)|^2 dx$$

$$\langle x \rangle = \int_{-a}^{a} xA^2(a^2 - x^2)^2 dx$$

This integral is zero since the integrand is an odd function of x.

(c) $\langle p \rangle$ will also be zero due to the above reason given in (b).

(d) Expectation value of x^2 at $t = 0$ is

$$\langle x^2 \rangle = \int_{-\infty}^{\infty} \Psi^*(a) x^2 \Psi(x) dx = \int_{-\infty}^{\infty} x^2 |\Psi(x)|^2 dx$$

$$= \int_{-a}^{a} x^2 A^2 (a^2 - x^2)^2 dx = \frac{a^2}{7}$$

(e) Expectation value of p^2 at $t = 0$ is

$$\langle p^2 \rangle = \int_{-\infty}^{\infty} \Psi^*(a) p^2 \Psi(x) dx = \int_{-a}^{a} \Psi^*(a) \left(- \hbar^2 \frac{d^2}{dx^2} \right) \Psi(x) dx$$

$$= \frac{5\hbar^2}{2a^2}$$

(f) $(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2 = \langle x^2 \rangle = \dfrac{a^2}{7}$

$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2 = \langle p^2 \rangle = \dfrac{5\hbar^2}{2a^2}$

$(\Delta x)^2 (\Delta p)^2 = \dfrac{5}{14} \hbar^2$

$\Delta x \Delta p = \sqrt{\dfrac{5}{14}} \hbar$

Note that here we get $\Delta x \Delta p > \hbar/2$ because the given wave function is non-Gaussian.