DEPARTMENT OF MATHEMATICS, IIT - GUWAHATI
Odd Semester of the Academic year 2019 - 2020
MA 101 Mathematics I
Problem Sheet 1: Revision of vectors, equations of lines and planes, vector differentiation, limits and continuities of functions of several variables.
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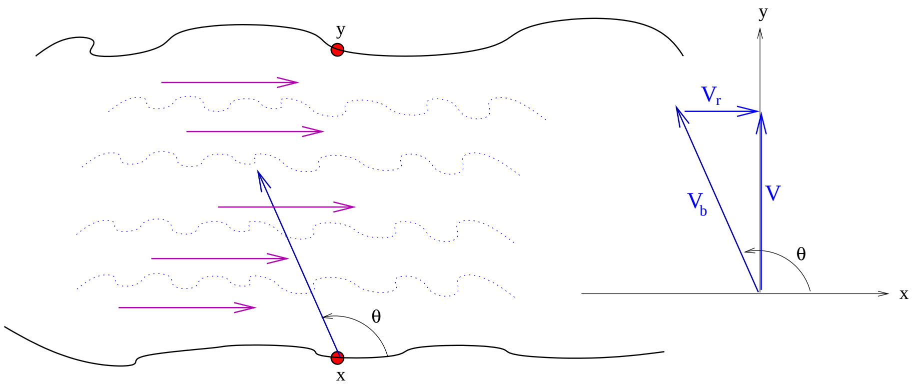
1. 

A man wants to paddle his boat across a river from point X to the point Y on the opposite shore directly across from X. If he can paddle the boat at the rate of 5 kilometers per hour and the current in the river is 3 kilometers per hour, in what direction θ should he steer his boat in order to go straight across the river? Also what is his resultant speed across the river?

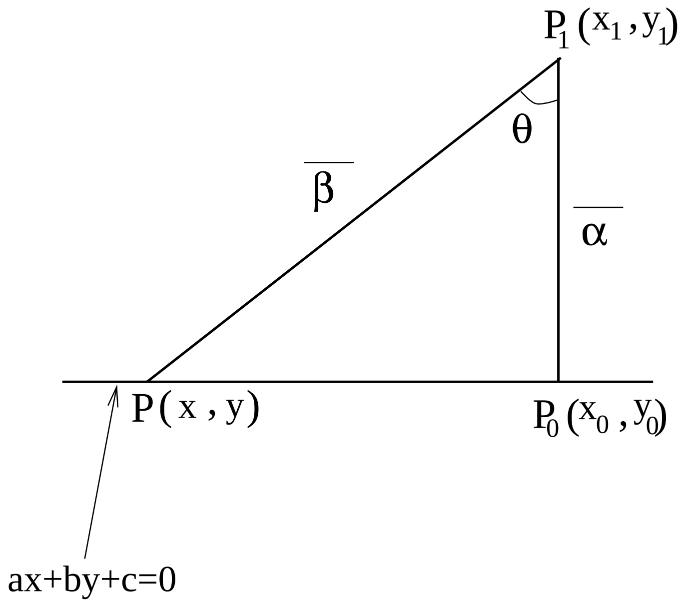
Solution: Let $\overline{V}_b$ be the velocity of the boat and $\overline{V}_r$ be the velocity of the river and let $\overline{V}$ be the resultant velocity of the boat across the river.
Clearly if $\overline{V}_r = 3\mathbf{i}$ then $\overline{V}_b = 5\cos(\theta)\mathbf{i} + 5\sin(\theta)\mathbf{j}$ and $\overline{V} = \alpha\mathbf{j}$ for some $\alpha > 0$, where $\alpha = |\overline{V}|$ is the relative speed of the boat across the river.
Also $\overline{V}_b + \overline{V}_r = \overline{V} \Rightarrow (5\cos(\theta) + 3)\mathbf{i} + 5\sin(\theta)\mathbf{j} = \alpha\mathbf{j}$ **(1)**
Equating the components of the vectors on both sides of equation **(1)** we get
$5\cos(\theta) + 3 = 0 \ \Rightarrow \cos(\theta) = -\frac{3}{5} \ \Rightarrow \theta = 2.214$ radians, or $\theta = 126.87^o$.

Since $\alpha^2 + 3^2 = 5^2$, $\alpha = 4$, hence the resultant speed is 4 kilometers per hour.

2. Use a scalar projection to show that the distance from a point $P_1(x_1, y_1)$ to the line $ax + by + c = 0$ is
$$\frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$$
Use this formula to find the distance from the point $(-2, 3)$ to the line $3x - 4y + 5 = 0$.

Solution:

Let $P_0(x_0, y_0)$ be the foot of the perpendicular vector from $P_1(x_1, y_1)$ to the line $ax + by + c = 0$,
then distance of $P_1(x_1, y_1)$ from the line $ax + by + c = 0$ is given by $\left|\overrightarrow{P_1P_0}\right|$.



Let $P(x, y)$ be a point on the line $ax + by + c = 0$,
then $P_0(x_0, y_0)$ and $P(x, y)$ satisfy the equations $ax_0+by_0+c = 0$ and $ax+by+c = 0$, respectively
$\Rightarrow \quad a(x - x_0) + b(y - y_0) = 0,$
$\Rightarrow \langle a, b\rangle \cdot \langle x_0 - x, y_0 - y\rangle = 0.$
Hence $\langle a, b\rangle \perp \langle x_0 - x, y_0 - y\rangle$ (or $\langle a, b\rangle$ is orthogonal to $\langle x_0 - x, y_0 - y\rangle$)
$\Rightarrow \langle a, b\rangle \perp \overrightarrow{PP_0}$.

But $\overrightarrow{P_1P_0} \perp \overrightarrow{PP_0}$ therefore any nonzero vector $\overline{\alpha}$ along $\overrightarrow{P_1P_0}$ must be parallel to $\langle a, b\rangle$, and $\overline{\alpha} = \lambda \langle a, b\rangle$ for some $\lambda \in \mathbf{R}$, $\lambda \neq 0$.
If we denote $\overrightarrow{P_1P}$ by $\overline{\beta}$, then
$\left|\overrightarrow{P_1P_0}\right| = \left|proj_{\overline{\alpha}}(\overline{\beta})\right| = \left|\dfrac{\overline{\alpha}.\overline{\beta}}{|\overline{\alpha}|}\right|$
$= \left|\dfrac{\lambda(a(x_1 - x) + b(y_1 - y))}{|\lambda| \sqrt{a^2 + b^2}}\right| = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}.$
Using the above formula the distance from $(-2, 3)$ to the line $3x - 4y + 5 = 0$ is given by $\dfrac{|3(-2) + (-4)3 + 5|}{\sqrt{3^2 + (-4)^2}} = \dfrac{13}{5}$.

3. (a) Find a point at which the given lines intersect:

$$\mathbf{r}_1(t) =< 1, 1, 0 > +t < -1, 1, 2 >, \quad t \in \mathbf{R}$$
$$\mathbf{r}_2(s) =< 2, 0, 2 > +s < -1, 1, 0 >, \quad s \in \mathbf{R}$$

(b) Find the equation of the plane that contains these lines.

Solution: (a) We can denote the lines as

$$\mathbf{r}_1(t) = \langle 1 - t, 1 + t, 2t\rangle\,, \quad t \in \mathbf{R}.$$

$$\mathbf{r}_2(s) = \langle 2-s, s, 2\rangle, \quad s \in \mathbf{R}.$$

If the lines intersect then there exists $t, s \in \mathbf{R}$ such that $2-s = 1-t$, $s = 1+t$ and $2t = 2$, which implies $t = 1$ and $s = 2$ and the lines intersect at $P_0 = (0,2,2)$.
(b) The direction vectors of the two lines are given by $\langle -1,1,2\rangle$ and $\langle -1,1,0\rangle$.
A normal to the plane containing these two lines is given by

$$\langle -1,1,2\rangle \times \langle -1,1,0\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 1 & 2 \\ -1 & 1 & 0 \end{vmatrix} = -2\mathbf{i} - 2\mathbf{j} = \langle -2,-2,0\rangle.$$

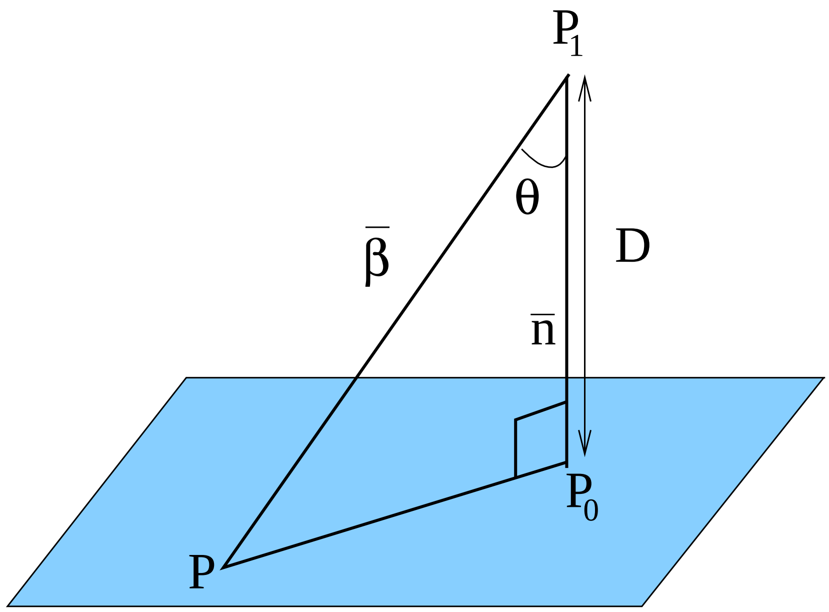
Hence equation of the required plane (which passes through the point (0,2,2)) is given by

$$-2(x-0) - 2(y-2) + 0(z-2) = 0$$

$$\text{or} \quad x + y = 2.$$

4. Find a formula for the distance D from a point $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$.

Solution:



Let $P_0(x_0, y_0, z_0)$ be the foot of the perpendicular vector from $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$, then $\overrightarrow{P_1P_0}$ is a normal vector $\overline{n}$ to the plane and the distance of $P_1(x_1, y_1, z_1)$ to the plane $ax + by + cz + d = 0$ is given by $\left|\overrightarrow{P_1P_0}\right|$.
Let $P(x, y, z)$ be a point on the plane $ax + by + cz + d = 0$, since $P_0(x_0, y_0, z_0)$ also lies on the plane $ax + by + cz + d = 0$,
$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$.
Therefore any nonzero normal vector $\overline{n}$ of the plane given by $\overline{\alpha}$ is of the form
$\overline{\alpha} = \lambda \langle a, b, c\rangle$ for some $\lambda \in \mathbf{R}$, $\lambda \neq 0$.
If we denote $\overrightarrow{P_1P}$ by $\overline{\beta}$, then

$$\left|\overrightarrow{P_1P_0}\right| = \left|proj_{\overline{\alpha}}(\overline{\beta})\right| = \left|\frac{\overline{\alpha}.\overline{\beta}}{|\overline{\alpha}|}\right|$$
$$= \left|\frac{\lambda(a(x_1-x)+b(y_1-y)+c(z_1-z))}{|\lambda|\sqrt{a^2+b^2+c^2}}\right| = \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}.$$

5. Find a point on the curve $\mathbf{r}(t) = 4\cos(t)\mathbf{i} + 4\sin(t)\mathbf{j} + 3t\mathbf{k}$ at a distance 10π units from the origin along the curve in the direction of increasing arc length.

Solution: Since $\mathbf{r}(t) = 4\cos(t)\mathbf{i} + 4\sin(t)\mathbf{j} + 3t\mathbf{k}$ is of the form $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where $f(t), g(t), h(t)$ are continuously differentiable functions in t,
$\mathbf{r}'(t) = -4\sin(t)\mathbf{i} + 4\cos(t)\mathbf{j} + 3\mathbf{k}$
and the arc length $L(t_1)$ of $\mathbf{r}(t)$ from $t = 0$ to $t = t_1$ is given by:
$L(t_1) = \int_0^{t_1} |\mathbf{r}'(t)|\, dt = \int_0^{t_1} \sqrt{16(\sin^2(t)+\cos^2(t))+9}\, dt = 5t_1.$
If $L(t_1) = 5t_1 = 10\pi$, then $t_1 = 2\pi$, and the corresponding point on the curve is given by:
$\mathbf{r}(2\pi) = 4\cos(2\pi)\mathbf{i} + 4\sin(2\pi)\mathbf{j} + 3(2\pi)\mathbf{k} = 4\mathbf{i} + 6\pi\mathbf{k}.$

6. Reparametrize the curve

$$\mathbf{r}(t) = \left(\frac{2}{t^2+1} - 1\right)\mathbf{i} + \frac{2t}{t^2+1}\mathbf{j}$$

with respect to the arc length measured from the point $(1,0)$ in the direction of increasing t. Express the parametrization in its simplest form. What can you conclude about the curve?

Solution: Since

$$\mathbf{r}(t) = \left(\frac{2}{t^2+1} - 1\right)\mathbf{i} + \frac{2t}{t^2+1}\mathbf{j},$$

the point $(1,0)$ corresponds to $t = 0$.
Also $f(t) = \dfrac{2}{t^2+1} - 1$ and $h(t) = \dfrac{2t}{t^2+1}$ are continuously differentiable at all $t \in \mathbf{R}$.

$$\mathbf{r}'(t) = \left(\frac{-4t}{(t^2+1)^2}\right)\mathbf{i} + \left(\frac{-2t^2+2}{(t^2+1)^2}\right)\mathbf{j}, \text{ and}$$

$$|\mathbf{r}'(t)| = \sqrt{\left(\frac{-4t}{(t^2+1)^2}\right)^2 + \left(\frac{-2t^2+2}{(t^2+1)^2}\right)^2} = \frac{2}{t^2+1}.$$

Hence $s(t) = \int_0^t |\mathbf{r}'(\xi)|\, d\xi = \int_0^t \dfrac{2}{\xi^2+1} d\xi = 2\tan^{-1}(t)$
$\Rightarrow t(s) = \tan(\dfrac{s}{2})$, for $0 \le s < \pi$.

$$\mathbf{r}(t(s)) = \left(\frac{2}{(\tan(\frac{s}{2}))^2+1} - 1\right)\mathbf{i} + \frac{2\tan(\frac{s}{2})}{(\tan(\frac{s}{2}))^2+1}\mathbf{j} = \cos(s)\mathbf{i} + \sin(s)\mathbf{j} \quad 0 \le s < \pi.$$

Hence with this reparametrization, the points on the curve represent points on the upper half part of the unit circle centered at the origin, excluding the point $(-1,0)$.

7. At what point does the curve $y = e^x$ have maximum curvature? What happens to the curvature as $x \to \infty$?

Solution: $\kappa(x) = \dfrac{|y''(x)|}{(1 + (y'(x))^2)^{\frac{3}{2}}} = \dfrac{e^x}{(1 + e^{2x})^{\frac{3}{2}}}$.

$\Rightarrow \kappa'(x) = \dfrac{e^x(1 - 2e^{2x})}{(1 + e^{2x})^{\frac{5}{2}}}$.

$\Rightarrow \kappa'(x) = 0$ if and only if $1 - 2e^{2x} = 0$ or $x = -\dfrac{1}{2}\ln(2)$.

Note that for $x < -\dfrac{1}{2}\ln(2)$, $\kappa'(x) > 0$ which implies

$\kappa(x)$ is a strictly increasing function in $(-\infty, -\dfrac{1}{2}\ln(2))$.

For $x > -\dfrac{1}{2}\ln(2)$, $\kappa'(x) < 0$ which implies

$\kappa(x)$ is a strictly decreasing function in $(-\dfrac{1}{2}\ln(2), \infty)$.

Hence the point of the curve at which curvature is maximum is given by

$(-\dfrac{1}{2}\ln(2), e^{-\frac{1}{2}\ln(2)}) = (-\dfrac{1}{2}\ln(2), \dfrac{1}{\sqrt{2}})$.

Note that $e^x < 1 + e^{2x}$, for all $x \in R$,

$$\Rightarrow 0 \leq \kappa(x) = \frac{e^x}{(1 + e^{2x})^{\frac{3}{2}}} \leq \frac{1}{\sqrt{(1 + e^{2x})}} \qquad (\mathbf{1})$$

Since $\displaystyle\lim_{x\to\infty} \frac{1}{\sqrt{(1 + e^{2x})}} = 0, \quad (\mathbf{1}) \quad \Rightarrow \quad \lim_{x\to\infty} \kappa(x) = 0.$

8. Find the unit tangent vector, unit normal vector and the binormal vector of the curves at the corresponding points given below:

   (a) $\mathbf{r}(t) = \left\langle t^2, \dfrac{2}{3}t^3, t \right\rangle$, $\left(1, \dfrac{2}{3}, 1\right)$

   (b) $\mathbf{r}(t) = \langle \cos(t), \sin(t), \ln(\cos(t)) \rangle$, $(1, 0, 0)$.

Solution: (a) $\mathbf{r}(t) = \left\langle t^2, \dfrac{2}{3}t^3, t \right\rangle$,

$\Rightarrow \mathbf{r}'(t) = \langle 2t, 2t^2, 1 \rangle$, hence the unit tangent vector at $\mathbf{r}(t)$ of the curve is given by:

$\Rightarrow \mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \left\langle \dfrac{2t}{2t^2+1}, \dfrac{2t^2}{2t^2+1}, \dfrac{1}{2t^2+1} \right\rangle$.

Since the point $\left(1, \dfrac{2}{3}, 1\right)$ of the curve $\mathbf{r}(t)$ corresponds to $t = 1$, the the unit tangent vector at that point is given by:

$\mathbf{T}(1) = \left\langle \dfrac{2}{3}, \dfrac{2}{3}, \dfrac{1}{3} \right\rangle$.

Also $\mathbf{T}'(t) = \left\langle \dfrac{2 - 4t^2}{(2t^2+1)^2}, \dfrac{4t}{(2t^2+1)^2}, \dfrac{-4t}{(2t^2+1)^2} \right\rangle$.

Since $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$, $\mathbf{N}(1) = \dfrac{\mathbf{T}'(1)}{|\mathbf{T}'(1)|} = \left\langle -\dfrac{1}{3}, \dfrac{2}{3}, -\dfrac{2}{3} \right\rangle$, and

$\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \left\langle -\dfrac{2}{3}, \dfrac{1}{3}, \dfrac{2}{3} \right\rangle$.

(b) Since the point $(1,0,0)$ of the curve $\mathbf{r}(t)$ corresponds to $t=0$, by following the same procedure as in part(a) we get:
$\mathbf{T}(0) = \langle 0,1,0\rangle$
$\mathbf{N}(0) = \left\langle -\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right\rangle$
$\mathbf{B}(0) = \left\langle -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right\rangle$.

9. The helix $\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ intersects the curve $\mathbf{r}_2(t) = (1+t)\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$ at the point $(1,0,0)$. Find the angle of intersection of these curves.

Solution: The angle of intersection of these curves at $(1,0,0)$ is the angle between the tangents of these two curves at the point $(1,0,0)$.
$\mathbf{r}_1(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k} \Rightarrow \mathbf{r}_1'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}$.
Also $\mathbf{r}_2(t) = (1+t)\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k} \Rightarrow \mathbf{r}_2'(t) = \mathbf{i} + 2t\mathbf{j} + 3t^2\mathbf{k}$.
Since the point $(1,0,0)$ corresponds to $t=0$ for both $\mathbf{r}_1(t)$ and $\mathbf{r}_2(t)$, if θ is the required angle then
$\cos(\theta) = \dfrac{\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(0)}{|\mathbf{r}_1'(0)|\,|\mathbf{r}_2'(0)|} = 0$, or $\theta = \frac{\pi}{2}$.

10. (a) A particle moves with constant speed along a curve in space. Show that its velocity and acceleration vectors are always perpendicular.
    (b) Let $\mathbf{r}(t) = (2t^3+3)\mathbf{i} + (\ln t)\mathbf{j} + 3\mathbf{k}$ be the position vector of a moving particle at time $t>0$. Find the time(s) at which velocity and acceleration vectors are perpendicular.

Solution: (a) If $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ denotes the above curve, then its velocity, speed, acceleration at any time $t>0$ is given by
$\mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$, $\quad |\mathbf{r}'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$ and
$\mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$, respectively.

Since speed is constant for the moving particle, $(x'(t))^2 + (y'(t))^2 + (z'(t))^2 = c$ for all t in the domain of the curve,
$\Rightarrow 2(x'(t)x''(t) + y'(t)y''(t) + z'(t)z''(t)) = 0$.
$\Rightarrow \langle x'(t), y'(t), z'(t)\rangle \cdot \langle x''(t), y''(t), z''(t)\rangle = \mathbf{r}'(t)\cdot\mathbf{r}''(t) = 0$, or $\mathbf{r}'(t) \perp \mathbf{r}''(t)$.

(b) $\mathbf{r}'(t) = (6t^2)\mathbf{i} + (\frac{1}{t})\mathbf{j}$ and $\mathbf{r}''(t) = (12t)\mathbf{i} + (\frac{-1}{t^2})\mathbf{j}$, for $t>0$.
$\mathbf{r}'(t)\cdot\mathbf{r}''(t) = 0 \Rightarrow 72t^3 - \frac{1}{t^3} = 0, \Rightarrow t^6 = \frac{1}{72}$ or $t = \sqrt[6]{\frac{1}{72}}$.

11. Find the limit if it exists, or show that the limit does not exist.
    (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$
    (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$
    (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3-y^3}{x^2+y^2}$

(d) $\lim_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6}$

(e) $\lim_{(x,y)\to(0,0)} \dfrac{x\sin(x^2+y^2)}{x^2+y^2}$

(f) $\lim_{(x,y)\to(4,\pi)} x^2 \sin\left(\dfrac{y}{x}\right)$

(g) $\lim_{(x,y)\to(0,1)} f(x,y)$,

where $f(x,y) = \frac{x+y-1}{\sqrt{x}-\sqrt{1-y}}$ if $x+y\neq 1$
$= 0$ if $x+y=1$.

Solution: (a) $f(x,y) = \dfrac{xy}{x^2+y^2} = \dfrac{m}{1+m^2}$, for all (x,y) such that $y=mx$, $x\neq 0$,

$\Rightarrow \lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2} = \dfrac{m}{1+m^2}$ along $y=mx$,

which is different for different straight lines passing through the origin.

hence $\lim_{(x,y)\to(0,0)} \dfrac{xy}{x^2+y^2}$ does not exist.

(For a more detailed discussion refer to (**) below).

((**) For all (x,y) on the line $y=0$, $f(x,y)=f(x,0)=0$.

Similarly for all (x,y) on the line $y=x$, $f(x,y)=f(x,x)=\dfrac{1}{2}$,

$\Rightarrow |f(x,0)-f(x,x)| = \dfrac{1}{2}$ for all $x\in \mathbf{R}$. (1)

Take any $\epsilon \leq \frac{1}{4}$, say $\epsilon = \frac{1}{8}$.

Since for all $\delta > 0$, however small, there exists points (x,y) of the straight line $y=mx$

such that $0<\sqrt{x^2+y^2} = \sqrt{x^2+m^2x^2} < \delta$, (2)

(1) and (2) implies that given $\epsilon = \frac{1}{8}$, there exists no $c\in R$ and no $\delta>0$,

such that $|f(x,y)-c| < \epsilon = \dfrac{1}{8}$ if $0<\sqrt{x^2+y^2}<\delta$.)

(b) Clearly $|x|\leq\sqrt{x^2+y^2}$ and $|y|\leq\sqrt{x^2+y^2}$

$\Rightarrow |x|\,|y| \leq (x^2+y^2)$

$\Rightarrow |f(x,y)| = \dfrac{|x|\,|y|}{\sqrt{x^2+y^2}} \leq \sqrt{x^2+y^2}.$ (1)

Given $\epsilon>0$, take $\delta=\epsilon$, then from (1) it follows

$|f(x,y)-0|<\epsilon$ if $0<\sqrt{x^2+y^2}<\delta$.

$\Rightarrow \lim_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+y^2}} = 0.$

(c) Clearly $\left|x^3\right| = |x|\,x^2 \leq \sqrt{x^2+y^2}(x^2+y^2)$ and $\left|y^3\right| = |y|\,y^2 \leq \sqrt{x^2+y^2}(x^2+y^2)$,

$|f(x,y)| = \left|\dfrac{x^3-y^3}{x^2+y^2}\right| \leq \left|\dfrac{x^3}{x^2+y^2}\right| + \left|\dfrac{y^3}{x^2+y^2}\right|$

$\leq 2\dfrac{\sqrt{x^2+y^2}(x^2+y^2)}{x^2+y^2} = 2\sqrt{x^2+y^2}.$ (1)

Given $\epsilon>0$ take $\delta = \frac{\epsilon}{2}$ then from (1) it follows

$|f(x,y)-0|<\epsilon$ if $0<\sqrt{x^2+y^2}<\delta$.

Hence $\lim_{(x,y)\to(0,0)} \dfrac{x^3-y^3}{x^2+y^2} = 0.$

(d) $f(x,y) = \dfrac{m}{1+m^2}$, for all (x,y) such that $x = my^3$, $y \neq 0$.
Hence by a similar argument as in part(a),
$\lim\limits_{(x,y)\to(0,0)} \dfrac{xy^3}{x^2+y^6}$ does not exist.

(e) Note that $\lim\limits_{(x,y)\to(0,0)} x = 0$, and $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{r^2\to 0} \dfrac{\sin r^2}{r^2} = 1$,
where $x = r\cos(\theta)$ and $y = r\sin(\theta)$.
Hence $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\sin(x^2+y^2)}{x^2+y^2} = \left(\lim\limits_{(x,y)\to(0,0)} x\right) \times \left(\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}\right) = 0 \times 1 = 0.$

Aliter: Since $\left|\sin(x^2+y^2)\right| \leq (x^2+y^2)$,
$|f(x,y)| = \left|\dfrac{x\sin(x^2+y^2)}{x^2+y^2}\right| \leq |x| \leq \sqrt{x^2+y^2}.$
Given $\epsilon > 0$, take $\delta = \epsilon$, then
$|f(x,y) - 0| < \epsilon$ if $0 < \sqrt{x^2+y^2} < \delta$.

(f) $\lim\limits_{(x,y)\to(4,\pi)} x^2 = \lim\limits_{x\to 4} x^2 = 16$ and $\lim\limits_{(x,y)\to(4,\pi)} \sin(\dfrac{y}{x}) = \lim\limits_{u\to\frac{\pi}{4}} \sin(u) = \sin(\dfrac{\pi}{4})$
$\Rightarrow \lim\limits_{(x,y)\to(4,\pi)} x^2 \sin\left(\dfrac{y}{x}\right) = \left(\lim\limits_{(x,y)\to(4,\pi)} x^2\right) \times \left(\lim\limits_{(x,y)\to(4,\pi)} \sin(\dfrac{y}{x})\right) = 16\sin(\dfrac{\pi}{4}) = 8\sqrt{2}.$

(g) $f(x,y) = \dfrac{x+y-1}{\sqrt{x}-\sqrt{1-y}}$ if $x+y-1 \neq 0$,
$= \dfrac{(x+y-1)(\sqrt{x}+\sqrt{1-y})}{(\sqrt{x}-\sqrt{1-y})(\sqrt{x+\sqrt{1-y}})} = \sqrt{x} + \sqrt{1-y}.$
Since $\lim\limits_{(x,y)\to(0,1)} \sqrt{x} = \lim\limits_{x\to 0} \sqrt{x} = 0$ and $\lim\limits_{(x,y)\to(0,1)} \sqrt{1-y} = \lim\limits_{y\to 1} \sqrt{1-y} = 0$,
$\lim\limits_{(x,y)\to(0,1)} f(x,y) = 0$, if $x+y-1 \neq 0$. **(1)**
Also $f(x,y) = 0$ if $x+y-1 = 0$. **(2)**
From **(1)** and **(2)** it follows, $\lim\limits_{(x,y)\to(0,1)} f(x,y) = 0$.

12. Examine whether

$$f(x,y) = \begin{cases} \frac{xy(y^2-x^2)}{x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

is a continuous function.

Solution: Since $f(x,y)$ is of the form $f(x,y) = \dfrac{g(x,y)}{h(x,y)}$
where g,h are continuous functions and $h(x,y) \neq 0$ for all $(x,y) \neq (0,0)$,
$f(x,y)$ is continuous at all $(x,y) \neq (0,0)$.
To check the continuity of f at $(x,y) = (0,0)$.
$|f(x,y)| = \dfrac{|xy|\,|(y^2-x^2)|}{x^2+y^2} \leq \dfrac{|xy|\,(y^2+x^2)}{x^2+y^2} \leq |xy| \leq (x^2+y^2)$ for $(x,y) \neq (0,0)$.

Given $\epsilon > 0$ take $\delta = \sqrt{\epsilon}$, then
$|f(x, y) - 0| \leq (x^2 + y^2) < \epsilon \quad \text{if} \quad 0 < \sqrt{x^2 + y^2} < \delta$,
hence $\lim\limits_{(x,y)\to(0,0)} f(x, y) = 0$.
Since $f(0, 0) = 0$, f is continuous at $(0, 0)$.

Extra Questions

1. Show that the scalar triple product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ represents the volume of the parallelepiped determined by the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

2. If $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are noncoplanar vectors, let
$$\mathbf{k_1} = \frac{\mathbf{v_2} \times \mathbf{v_3}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})}, \quad \mathbf{k_2} = \frac{\mathbf{v_3} \times \mathbf{v_1}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})} \quad \text{and} \quad \mathbf{k_3} = \frac{\mathbf{v_1} \times \mathbf{v_2}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})}.$$
Show that
   (a) $\mathbf{k}_i$ is perpendicular to to $\mathbf{v}_j$ if $i \neq j$
   (b) $\mathbf{k}_i \cdot \mathbf{v}_i = 1$ for $i = 1, 2, 3$
   (c) $\mathbf{k_1} \cdot (\mathbf{k_2} \times \mathbf{k_3}) = \dfrac{\mathbf{1}}{\mathbf{v_1} \cdot (\mathbf{v_2} \times \mathbf{v_3})}$.

3. Given the vectors $\mathbf{a} = (\mathbf{a_1}, \mathbf{a_2}, \mathbf{a_3})$ and $\mathbf{b} = (\mathbf{b_1}, \mathbf{b_2}, \mathbf{b_3})$, verify that
$$|\mathbf{a} \times \mathbf{b}|^{\mathbf{2}} = |\mathbf{a}|^{\mathbf{2}}|\mathbf{b}|^{\mathbf{2}} - (\mathbf{a} \cdot \mathbf{b})^{\mathbf{2}},$$
by computing each side in terms of the components of $\mathbf{a}$ and $\mathbf{b}$.

4. Show that the curvature of a plane parametric curve $x = f(t),\ y = g(t)$ is
$$\kappa = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{[\dot{x}^2 + \dot{y}^2]^{\frac{3}{2}}}$$
where the dots indicate the derivatives with respect to t.

Solution: $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$,
$\Rightarrow \mathbf{r}'(t) = \dot{x(t)}\mathbf{i} + \dot{y(t)}\mathbf{j}$, $\mathbf{r}''(t) = \ddot{x(t)}\mathbf{i} + \ddot{y(t)}\mathbf{j}$.
$\mathbf{r}'(t) \times \mathbf{r}''(t) = (\dot{x(t)}\ddot{y(t)} - \dot{y(t)}\ddot{x(t)})\mathbf{k}$.
$\kappa(t) = \left|\dfrac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{|\mathbf{r}'(t)|^3}\right| = \dfrac{|\dot{x(t)}\ddot{y(t)} - \dot{y(t)}\ddot{x(t)}|}{((\dot{x(t)})^2 + (\dot{y(t)})^2)^{\frac{3}{2}}}$.

5. If $\mathbf{u}(t) = \mathbf{i} - 2t^2\mathbf{j} + 3t^3\mathbf{k}$ and $\mathbf{v}(t) = t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$, find
   (a) $D_t[\mathbf{u}(t) \cdot \mathbf{v}(t)]$
   (b) $D_t[\mathbf{u}(t) \times \mathbf{v}(t)]$.
   (c) $\lim\limits_{t\to\pi}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \lim\limits_{t\to\pi}[\mathbf{u}(t)] \cdot \lim\limits_{t\to\pi}[\mathbf{v}(t)]$ and $\lim\limits_{t\to\pi}[\mathbf{u}(t) \times \mathbf{v}(t)] = \lim\limits_{t\to\pi}[\mathbf{u}(t)] \times \lim\limits_{t\to\pi}[\mathbf{v}(t)]$