Department of Mathematics
Indian Institute of Technology Guwahati

MA 101: Mathematics I

Tutorial Sheet-6

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1. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$

   (a) Show that f is Riemann integrable on $[-1, 1]$ and that $\int\limits_{-1}^{1} f(x)\, dx = 0$.

   (b) If $F(x) = \int\limits_{-1}^{x} f(t)\, dt$ for all $x \in [-1, 1]$, then show that $F : [-1, 1] \to \mathbb{R}$ is differentiable, and in particular, $F'(0) = f(0)$, although f is not continuous at 0.

2. Let $f : [a, b] \to \mathbb{R}$ be continuous such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int\limits_{a}^{b} f(x)\, dx = 0$. Show that $f(x) = 0$ for all $x \in [a, b]$.
   Equivalently, if $f : [a, b] \to \mathbb{R}$ is continuous such that $f(x) \geq 0$ for all $x \in [a, b]$ and $f(c) \neq 0$ for some $c \in [a, b]$, then $\int\limits_{a}^{b} f(x)\, dx > 0$.
   (The above result need not be true if f is assumed to be only Riemann integrable on $[a, b]$.)

3. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = \begin{cases} x & \text{if } x \text{ is rational,} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$
   Examine whether f is Riemann integrable on $[0, 1]$.

4. If $f : [0, 1] \to \mathbb{R}$ is Riemann integrable, then find $\lim\limits_{n\to\infty} \int\limits_{0}^{1} x^n f(x)\, dx$.

5. If $f : [0, 2\pi] \to \mathbb{R}$ is continuous such that $\int\limits_{0}^{\frac{\pi}{2}} f(x)\, dx = 0$, then show that there exists $c \in (0, \frac{\pi}{2})$ such that $f(c) = 2\cos 2c$.

6. Evaluate the limit: $\lim\limits_{n\to\infty} \left(\frac{1^8 + 3^8 + \cdots + (2n-1)^8}{n^9} \right)$.

7. Let $f : [0, \infty) \to \mathbb{R}$ be such that $f \in \mathcal{R}[0, x]$ for all $x > 0$. If $x \sin(\pi x) = \displaystyle\int_0^{x^2} f(t) dt$, find the value of $f(4)$.

8. Examine whether the integral $\int\limits_{0}^{\infty} \sin(x^2)\, dx$ is convergent.

9. Determine all real values of p for which the integral $\int\limits_{0}^{\infty} \frac{x^{p-1}}{1+x}\, dx$ is convergent.

10. Determine all real values of p for which the integral $\int\limits_{0}^{\infty} \frac{e^{-x}-1}{x^p}\, dx$ is convergent.

11. Examine whether the improper integral $\int\limits_{-\infty}^{\infty} te^{-t^2}\, dt$ is convergent.