Homework 2

- Text classification exercise in Python
- Due Weds September 27 at 11:59pm
- Allow time for debugging, analysis, and review
- HW2 Blackboard discussion group available

Generalized Linear Models

CS-585

Natural Language Processing

Sonjia Waxmonsky

Statistical learning for classification

- Text categorization is a classification task
  - We want to associate every document with a class (label) using some **statistical** model
- We have seen one way to do this (naïve Bayes), but let’s be more explicit about the problem setting

Statistical learning for classification

- A document is represented as a vector $\vec{x}$ of features (typically, words or other lexical representations)
  - We use X to represent a random variable ranging over values of $\vec{x}$
- The label for each document is a categorical value $y \in Y$
  - We also use Y to represent a random variable ranging over values of y
- We are given a training set of labeled documents $(\vec{x}_1, y_1), (\vec{x}_2, y_2), \cdots, (\vec{x}_N, y_N)$
- The task is to identify some scoring function $\Psi(\vec{x}, y; \Theta)$
  - Ψ tells us how compatible $\vec{x}$ is with the label y
  - Θ is the set of parameters we can adaptively modify to change

What we've see already: Naïve Bayes

(from session 8, slide 23:)

$$score(y, w_1, \cdots, w_n) = \log \frac{d\_count[y]}{d\_count} + \sum_{i=1}^{n} \log \frac{w\_count[w_i][y] + \alpha}{w\_count[y] + \alpha|V|}$$

Parameters

- Estimates of priors and likelihood probabilities

When and how are these parameters set?

Linear models

- Classification rule is

$$\hat{y} = \underset{y \in Y}{\operatorname{argmax}} \Psi(\vec{x}, y; \Theta)$$

- In the case of a linear model,

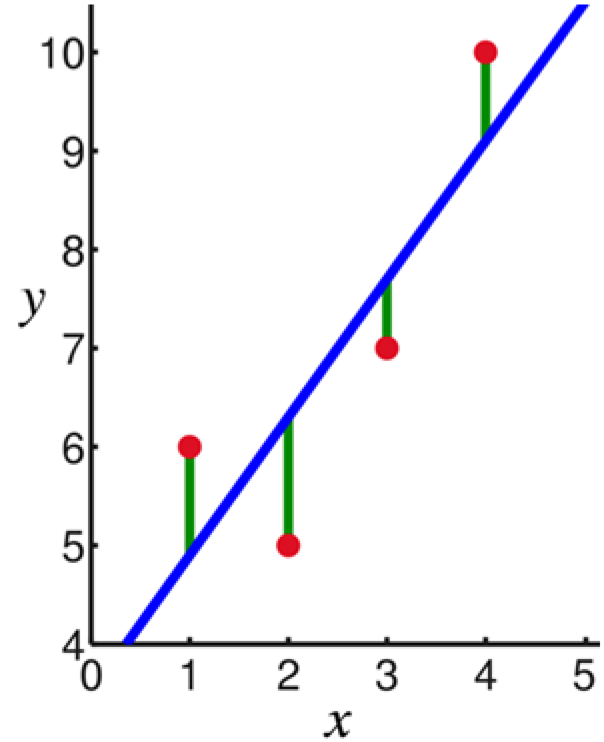
$$\Psi(\vec{x}, y; \Theta) = \Theta^T f(\vec{x}, y)$$

Θ: Model Parameters

- Typically, the feature vector $f(\vec{x}, y)$ represents each value of $\vec{x}$ in combination with each class y.

Generalized linear models

- A linear model $\Psi(\vec{x}, y; \Theta) = \Theta^T f(\vec{x}, y)$ gives us a score for a given document, which is a real number. But in order to learn a good function, we need to turn this into a meaningful number that we can use for optimization
- In linear regression, the output of our linear function is itself our $\hat{y}$, and we can optimize based on the reconstruction error $||\hat{y} - y||_2$



https://en.wikipedia.org/wiki/Linear_regression

Generalized linear models (GLMs)

- A framework for associating linear functions with distributions for use in optimization/estimation
- Idea: define a $\boxed{\textit{link function } g}$ that determines the relationship between the score function Ψ (linear model) and the expected value of the distribution we are interested in

$$g(E[Y = y|X = \vec{x}]) = \Theta^T f(\vec{x}, y)$$

- For linear regression, g is the identity function, so the linear model provides our estimate directly

Generalized linear models

GLM Type	Link function	Uses
Linear regression	$g(E[\cdot]) = E[\cdot]$	Estimate real-valued quantity
Poisson regression	$g(E[\cdot]) = \ln(E[\cdot])$	Estimate number of events in fixed time window
Logistic regression	$g(E[\cdot]) = \ln\left(\frac{E[\cdot]}{1 - E[\cdot]}\right)$	Estimate the probability of a binary outcome variable (Bernoulli distribution)

Text Classification
(one application)

↑
Goal: Estimate
expected value

Logistic regression

For logistic regression,

$$g(E[\cdot]) = \ln\left(\frac{E[\cdot]}{1 - E[\cdot]}\right) = \Theta^T f(\vec{x}, y)$$

…and the expectation is the estimate of a probability distribution over two outcomes, so we can write

$$\ln\left(\frac{P(Y = 1|X = \vec{x})}{1 - P(Y = 1|X = \vec{x})}\right) = \Theta^T f(\vec{x}, y)$$

$$\boxed{1 - P(Y = 1|X = \vec{x}) = P(Y = 0|X = \vec{x})}$$

Logistic regression

$$\ln\left(\frac{P(Y=1|X=\vec{x})}{1-P(Y=1|X=\vec{x})}\right) = \Theta^T f(\vec{x},y)$$

So the $\boxed{\text{logit function}}$ is the link that transforms our probabilities into the linear output of our model.

What if we want to get from the linear outputs to the probability space?

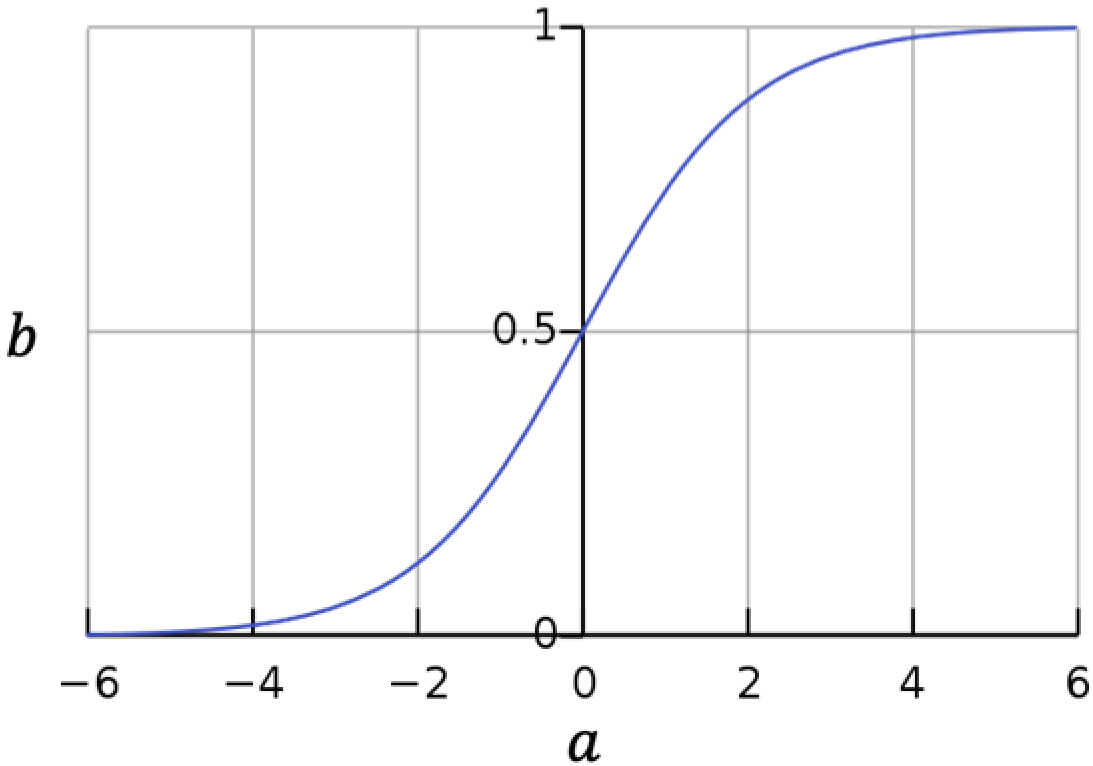
$$a = \ln\left(\frac{b}{1-b}\right)$$

$\boxed{\text{Inverse logit}}$ (or *sigmoid squashing function*)

$$b = \frac{e^a}{1+e^a}$$

$$P(Y=1|X=\vec{x}) = \frac{e^{\Theta^T f(\vec{x},y)}}{1+e^{\Theta^T f(\vec{x},y)}}$$

Inverse logit / sigmoid



$$b = \frac{e^a}{1 + e^a}$$

↑

Inverse Logit

$$P(Y = 1|X = \vec{x}) = \frac{e^{\Theta^T f(\vec{x},y)}}{1 + e^{\Theta^T f(\vec{x},y)}}$$

Inverse logit and softmax

Inverse logit maps from linear outputs to normalized probabilities for binary classification

$$P(Y = 1|X = \vec{x}) = \frac{e^{\Theta^T f(\vec{x},y)}}{1 + e^{\Theta^T f(\vec{x},y)}}$$

What if we have more than 2 classes? (Categorical LR)

Softmax:

$$P(Y = y|X = \vec{x}) = \frac{e^{\Theta^T f(\vec{x},y)}}{\sum_{y' \in Y} e^{\Theta^T f(\vec{x},y')}}, Y = \{y_1, y_2, \cdots, y_N\}$$

… remember this from `word2vec`?

Maximum Likelihood Estimate

- So for categorical logistic regression, the probability of an observed label y given the features/words x is

$$P(Y = y|X = \vec{x}) = \frac{e^{\Theta^T f(\vec{x},y)}}{\sum_{y' \in Y} e^{\Theta^T f(\vec{x},y')}}$$

- If we aggregate this over all of our training examples, we get the *likelihood*:

$$L(\Theta) = \prod_{i=1}^{N} P(y_i|\vec{x}_i; \Theta)$$

- One way of choosing parameters Θ is trying to find the value of Θ that maximizes $L(\Theta)$: the *maximum likelihood estimate*

Maximum likelihood estimation

Find

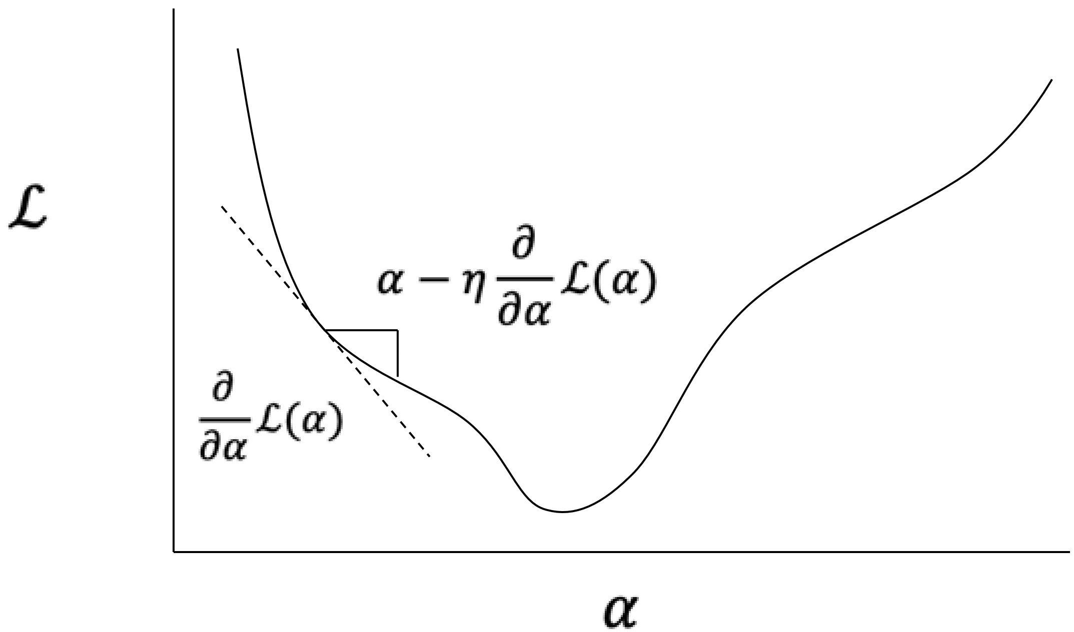
$$
\begin{aligned}
\operatorname*{argmax}_{\Theta} L(\Theta) &= \operatorname*{argmax}_{\Theta} \prod_{i=1}^{N} P(y_i|\vec{x}_i;\Theta) \\
&= \operatorname*{argmax}_{\Theta} \prod_{i=1}^{N} \frac{e^{\Theta^T f(\vec{x}_i, y_i)}}{\sum_{y' \in Y} e^{\Theta^T f(\vec{x}_i, y')}} \\
&= \operatorname*{argmax}_{\Theta} \sum_{i=1}^{N} \left(\Theta^T f(\vec{x}_i, y_i) - \log \sum_{y' \in Y} e^{\Theta^T f(\vec{x}_i, y')} \right)
\end{aligned}
$$

”log likelihood”

Optimization

- How do we actually find the parameter values Θ that maximize

$$\sum_{i=1}^{N}\left(\Theta^T f(\vec{x}_i, y_i) - \log \sum_{y' \in Y} e^{\Theta^T f(\vec{x}_i, y')}\right)?$$

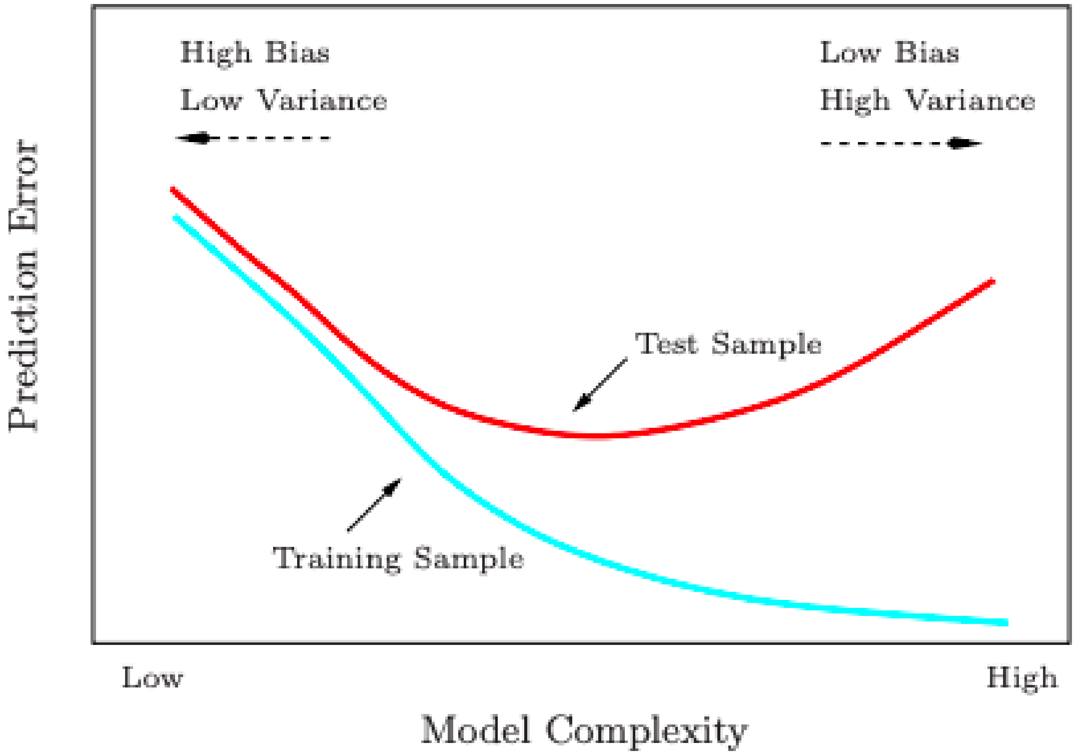


Gradient descent → Minimize negative log likelihood

Overfitting

- In NLP problems, we typically have a very large feature set – if every word is a feature, we will have $|V|$ features (or $|V|\times|Y|$ for multiclass classification)
- In many applications, $|V| \gg |D|$, the number of documents, so the model may be able to learn the characteristics of the training documents very exactly ("memorize" it)
- When this happens, we see that performance on the training set increases to near-perfection, while performance on test ("unseen") data degrades
- The model fails to "generalize"

Overfitting



What have we seen that can increase complexity in BOW text categorization models ?

Hastie, Tibshirani & Friedman. *Elements of Statistical Learning*

Overfitting

- Ways to deal with overfitting
  1. Get more data (increase $|D|$)
  2. Simplify the model (decrease $|V|$)
  3. Use *regularization* (constrain Θ)

Regularization

- Bias-Variance tradeoff
  - Bias: parameters should stay within "reasonable" bounds – not too many of them should be too large
  - Variance: parameters should be allowed to **vary** to capture the observed structure of the training data
- Regularization is a mechanism for increasing **bias** at the expense of **variance**

L_2 regularization for logistic regression

- Discourage model weights from getting too large by adding a penalty on the norm of the parameter vector Θ
- Instead of maximizing the log likelihood, minimize the penalized negative log likelihood

$$\log L(\Theta) = -\sum_{i=1}^{N} \log P(y_i|\vec{x}_i;\Theta) + \frac{\lambda}{2}\|\Theta\|_2^2$$

L_2 / ridge penalty

L_1 regularization for logistic regression

L_1 penalty: Sum of absolute values of coefficients

$$\log L(\Theta) = -\sum_{i=1}^{N} \log P(y_i | \vec{x}_i; \Theta) + \lambda \|\Theta\|_1$$

Encourages sparsity by pushing coefficients toward zero

→ "feature selection"

Generative vs. discriminative models

- Generative models
  - Model the joint probability distribution $P(Y, X)$
  - If we want to get the conditional probability $P(Y|X)$, we can normalize across classes: $P(Y = y|X) = \frac{P(Y=y,X)}{\sum_{y' \in Y} P(Y=y',X)}$
  - Can be understood in terms of a "generative story" – e.g., in a naïve Bayes framework, documents are the result of a process according to which
    - First, we choose a category according to distribution $P(Y)$
    - Then, we choose a bunch of words to fill the document up, according to $P(W|Y)$
- Discriminative models
  - Model $P(Y|X)$ directly
  - Most supervised classification methods are discriminative

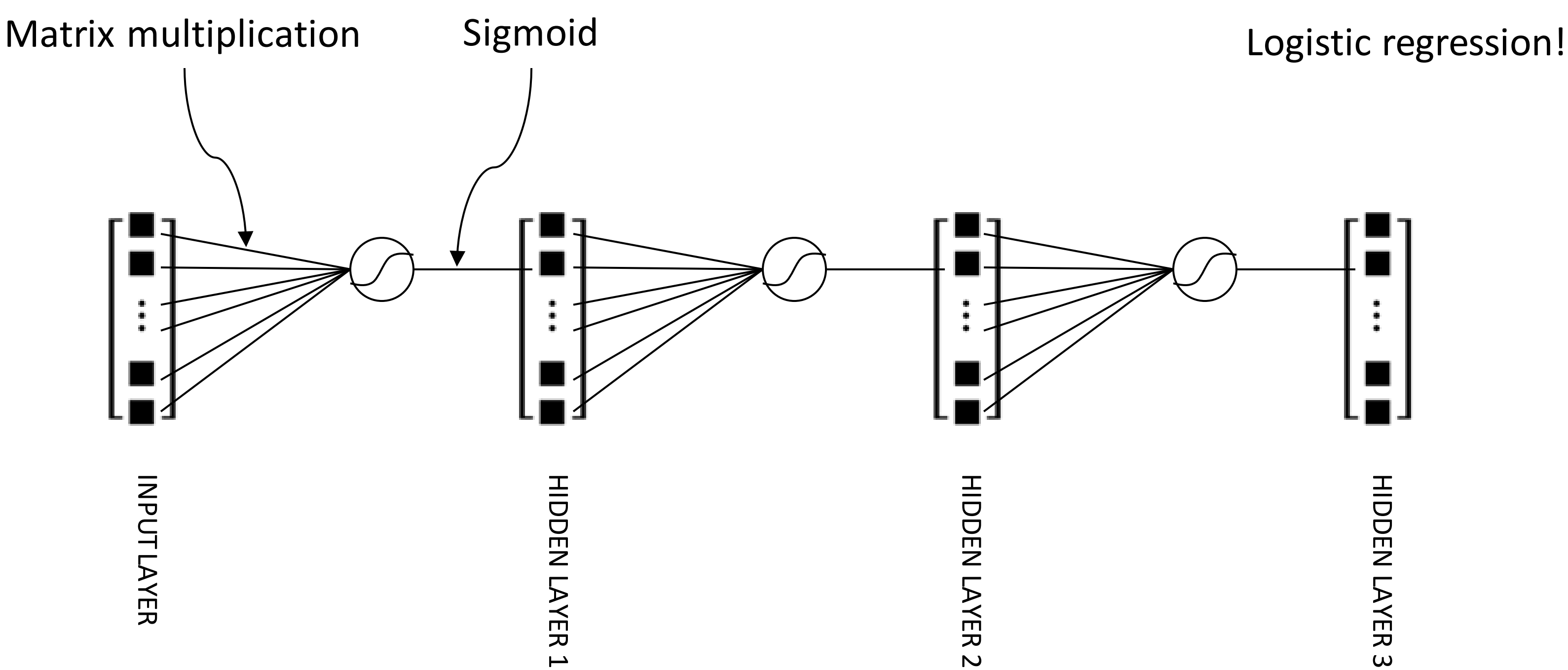
Connection to neural networks

- Generally, neural networks are sequences of matrix multiplications. Each layer of activations (a vector) is produced by multiplying the previous layer by a matrix

- But as we learned, a sequence of affine (linear) transformations, is itself an affine transform. So in order to allow neural networks to learn more complex functions, we have to introduce **nonlinearities**

- Softmax is one nonlinear transform. The **sigmoid squashing** function is another.

Connection to neural networks



$$\vec{h}_1 = \frac{e^{W\vec{x}}}{1 + e^{W\vec{x}}} \quad \vec{h}_2 = \frac{e^{W\vec{h_1}}}{1 + e^{W\vec{h_1}}} \quad \vec{h}_3 = \frac{e^{W\vec{h_2}}}{1 + e^{W\vec{h_2}}}$$

Example: naïve Bayes vs. logistic regression

[Notebook]