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Code Documentation for The UWB Beam summation scheme for RCS calculations in the high frequency regime

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Chapter 1

Code Documentation

This documentation describes the sub-routines/configurations of the UWB-PS-BS algorithm for calculation of the RCS $\sigma(\hat{\mathbf{r}}_0, \hat{\mathbf{r}}; \omega)$ of a given smooth target over a multi-octave band $\omega \in \Omega = (\omega_{\min}, \omega_{\max})$. All equations in this document refer to the final paper.

Input Parameters:

- Illumination direction : $\hat{\mathbf{r}}_0 = (\theta_0, \phi_0)$
  we shall also use $\boldsymbol{\xi}_0 = (\sin\theta_0, \cos\phi_0, \sin\theta_0 \sin\phi_0)$
- Observation direction : $\hat{\mathbf{r}} = (\theta, \phi)$
- Frequency band : $\Omega = (\omega_{\min}, 2^J \omega_{\min})$. $J > 1$ is the number of one-octave sub-bands.

Target geometry: Here it is a sphere with radius a and center c_0

The “wave modeler’s input are:

- The location of the expansion plane: z_0
- The beam collimation distance: $\bar{b}$, typically taken as $\bar{b} \gtrsim a$.
- The maximum overcompleteness parameter in each band: $\bar{\nu}$
- The Upsampling scheme: $Sch = \pm 1$ (see below),

$$Sch = \begin{cases} -1 & x\text{-}\xi - \text{upsampling} \\ 1 & \xi\text{-}x - \text{upsampling} \end{cases}$$

1.1 List of subroutine

The list of all the subroutine are stated below:

1. Expplane: Construction of the expansion plane
2. ExpansionCoeff: Calculating the expansion coefficients
3. Coefftrunc: PS truncation by including the beams that are excited.
4. Raytrack: Tracking the rays corresponding the PS-lattice
5. Raytrunc: Truncation via the geometrical observation
6. Beaminit: Defining the beam-characteristics (i.e amplitude, complex curvature matrix and the beam coordinate system)
7. Beamtranspec: Transformation of the amplitude and the complex curvature matrix of the beams from the waist to the specular point
8. POIsys: Construction of the plane of incidences coordinate system at specular point
9. Beamirot: Rotation of the beam coordinate sytem and the complex curvature matrix in the plane of incidences
10. GBReflection: Computation of the reflected beam coordinate system and the complex curvature matrix
11. N2FFGammarrot: Rotation of the exit beam coordinate system in the angle coordinate system.
12. FFbeamchar: Calculating the far-field beam-characteristics (i.e the differential solid angle, The far-zone phase projection, the amplitudes divergence until the exit point)
13. FFcont: Calculating the far-field beam contribution
14. RCStarg: Calculating the RCS of the target at the frequency

1.2 Algorithm

1.2.1 Preamble stage

The preamble stage is used for a fast and rough estimate of the phase space configuration by using a sparse lattice of rays.

1. Choose a frequency $\omega' < \omega_{\min}$ such that ray asymptotic is still valid.

2. Define the initial phase-space regime $\mathcal{X}_0' = (\mathbf{x}', \boldsymbol{\xi}')$ where we assume that the data resides. Here we choose: $|\mathbf{x}'| \leq 2a$ and $|\boldsymbol{\xi}'| \leq 1$.

3. Construct the phase space lattice at z_0 to span the target domain at the frequency ω' using (10).

$$\text{Expplane}(\mathcal{X}_0', \bar{b}, , \bar{\nu}, \omega_{\max} = \omega') \to \mathcal{X}'$$

   where here we choose $\omega_{\max}$ in (10) as ω'.

   The output: initial phase space unit-cell $(\bar{x}', \bar{\xi}')$ and the discretized phase-space $\mathcal{X}' = (\mathbf{x_m}, \boldsymbol{\xi_n})$ covering $\mathcal{X}_0'$.

4. Computation of the expansion coefficients $a_{\boldsymbol{\mu}}(\omega')$ for $\mathcal{X}'$ using equation (15):

$$\text{ExpansionCoeff}(\mathcal{X}', \boldsymbol{\xi}_0, \bar{b}, \bar{\nu}, \omega_{\max} = \omega', \omega = \omega') \to a_{\boldsymbol{\mu}}(\omega')$$

   where ω and $\omega_{\max}$ are the operating frequency and the max frequency in (15). Output: The expansion coefficient: $a_{\boldsymbol{\mu}}(\omega')$ at $\mathcal{X}'$

5. Phase-space truncation by including only the beams whose excitation amplitudes are larger than ϵ.

$$\text{Coefftrunc}(\mathcal{X}', a_{\boldsymbol{\mu}}(\omega'), \epsilon = 10^{-6}) \to [\mathcal{X}_a', a_{\boldsymbol{\mu}}(\omega')]$$

   Input: The input phase-space zone $\mathcal{X}'$ and the expansion coefficient $a_{\boldsymbol{\mu}}(\omega')$. The truncation parameter: $\epsilon = 10^{-6}$

   Output:

   - $\mathcal{X}_a'$ = the truncated phase-space zone. Note: $\mathcal{X}_a' \ll \mathcal{X}'$.
   - $a_{\boldsymbol{\mu}}(\omega')$ = the corresponding set of expansion coefficient.

6. Tracking the rays corresponding to $\mathcal{X}_a'$:

$$\text{Raytrack}(\mathcal{X}_a', z_0, c_0, a) \to [\mathcal{X}_{ar}', \mathcal{R}_r']$$
$$\mathcal{R}_r' = (\mathbf{Q}_{\boldsymbol{\mu}}', \hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}}', \bar{\Theta}_{D_{\boldsymbol{\mu}}}'(\omega'), \sigma'(\mathbf{Q}_{\boldsymbol{\mu}}'), \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}', \theta_{i_{\boldsymbol{\mu}}}')$$

   Input:

   - The phase-space $\mathcal{X'}_a$ and the expansion plane z_0.
   - The "target geometry: The center c_0 and the radius a of the sphere. (in general: a file describing the "geometry").

Output:

- $\mathcal{X}'_{ar}$ = a contraction of $\mathcal{X}'_a$ consisting of rays that hit the target. It is a “block” phase space (i.e. $X_{1a} < x_{1_m} < X_{1b}$, $\Xi_{1a} < \xi_{1_n} < \Xi_{1b}$ etc). In that block, there are some rays $\boldsymbol{\mu}$ that do not hit the target. In $\mathcal{X}'_{ar}$ they are denoted by the symbol #. The symbol # is used throughout.
- A data-set $\mathcal{R}'_r$ describing the reflected ray data corresponding to $\mathcal{X}'_{ar}$. It describes:
  - The exit points and directions of the rays: $\mathbf{Q}'_{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\sigma}}'_{\boldsymbol{\mu}}$.
  - $\Theta'_{D_{\boldsymbol{\mu}}}(\omega')$ = the diffraction angle of the beams (used in the following routine “Raytrunc”)
  - The local outward normal and the respective angle of incidence at the exit point: $\hat{\boldsymbol{\nu}}'_{\boldsymbol{\mu}}$ and $\theta'_{i_{\boldsymbol{\mu}}}$ (not used in the preamble)
  - The phase accumulation of the ray field along its propagation path up to $\mathbf{Q}'_{\boldsymbol{\mu}}$: $\sigma'(\mathbf{Q}'_{\boldsymbol{\mu}})$ (not used in the preamble)
  - Comments: This is a generic program that is used throughout. Here we do not use all its output parameters.

7. Ray truncation via “geometrical observation”.

$$\text{Raytrunc}(\mathcal{X}'_{ar}, \mathcal{R}'_r, \hat{\mathbf{r}}, K = 5) \rightarrow [\mathcal{X}'_{aB}, \mathcal{R}'_{rB}]$$

Input:

- the phase space data-set $\mathcal{X}'_{ar}$ and $\mathcal{R}'_r$ calculated before.
- The observation direction: $\hat{\mathbf{r}}$
- The truncation parameter: K

Output: This routine selects out of $\mathcal{X}'_{ar}$ only the rays whose directions $\hat{\boldsymbol{\sigma}}'_{\boldsymbol{\mu}}$ fall within a cone of angle $K\bar{\Theta}'_{D_{\boldsymbol{\mu}}}(\omega')$ about the observation direction $\hat{\mathbf{r}}$. It returns the contracted phase-space data set $\mathcal{X}'_{aB}$ and $\mathcal{R}'_{rB} = (\mathbf{Q}'_{\boldsymbol{\mu}}, \hat{\boldsymbol{\sigma}}'_{\boldsymbol{\mu}}, \bar{\Theta}'_{D_{\boldsymbol{\mu}}}(\omega'), \sigma'(\mathbf{Q}'_{\boldsymbol{\mu}}), \hat{\boldsymbol{\nu}}'_{\boldsymbol{\mu}}, \theta'_{i_{\boldsymbol{\mu}}})$.

1.2.2 RCS over $\Omega^{(j)}$ for $j = 1$

Short description: We consider only the phase-space subset $\mathcal{X}'_{aB}$ calculated in the preamble. This *a priori localization* has an important effect on the overall run-time. Next we calculate the corresponding expansion coefficients $a^{(1)}_{\boldsymbol{\mu}}$ at the lowest frequency of the band (where the amplitude distribution is less localized) via (15) and use it to truncate the phase space further. Then we trace the resulting beams via interaction with the target and consider only the phase space rays that are scattered by the target. We use the N2FF transformation to calculate the far zone contributions in the $\hat{\mathbf{r}}$ direction, and define the contracted phases-space subset $\mathcal{X}'^{(1)}_{aB}$ by retaining only the terms whose contribution $a_{\boldsymbol{\mu}} B_{\boldsymbol{\mu}}(\mathbf{r})$ is larger than some threshold. The subset $\mathcal{X}'^{(1)}_{aB}$ will then be used for all $\omega \in \Omega^{(j=1)}$.

1. Construction of the expansion plane at the frequency band $\omega_{\max}$ via (10).

$$\text{Expplane}(\mathcal{X}'_{aB}, \bar{b}, \bar{\nu}, \omega_{\max} = 2\omega_{\min}) \to \mathcal{X}^{(1)}$$

   Note: $\omega_{\max}$, the max frequency in (10), is taken here to be the largest frequency in the $j = 1$ band, i.e., twice the lowest frequency in the band, i.e., $2\omega_{\min}$.

   The input is the phase-space zone $\mathcal{X}'_{aB}$ calculated in the preamble and the output in the discretized phase space zone $(\mathbf{x}^{(j)}_{\mathbf{m}}, \boldsymbol{\xi}^{(j)}_{\boldsymbol{n}})$ covering $\mathcal{X}'_{aB}$.

2. Computation of the expansion coefficients (15) corresponding to $\mathcal{X}^{(1)}$ at the lowest frequency of the band $\omega = \omega_{\min}$.

$$\text{ExpansionCoeff}(\mathcal{X}^{(1)}, \boldsymbol{\xi}_0, \bar{b}, \bar{\nu}, \omega_{\max} = 2\omega_{\min}, \omega = \omega_{\min}) \to a^{(1)}_{\boldsymbol{\mu}}(\omega_{\min})$$

3. Truncation of weakly excited beams

$$\text{Coefftrunc}(\mathcal{X}^{(1)}, a^{(1)}_{\boldsymbol{\mu}}(\omega_{\min}), \epsilon = 10^{-6}) \to [\mathcal{X}^{(1)}_a, a^{(1)}_{\boldsymbol{\mu}}(\omega_{\min})]$$

4. Tracking of the beam axes in $\mathcal{X}^{(1)}_a$:

$$\text{Raytrack}(\mathcal{X}^{(1)}_a, z_0, c_0, a) \to [\mathcal{X}^{(1)}_{ar}, \mathcal{R}^{(1)}_r]$$
$$\mathcal{R}^{(1)}_r = (\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}, \hat{\boldsymbol{\sigma}}^{(1)}_{\boldsymbol{\mu}}, \bar{\Theta}^{(1)}_{D_{\boldsymbol{\mu}}}(\omega_{\min}), \sigma^{(1)}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}), \hat{\boldsymbol{\nu}}^{(1)}_{\boldsymbol{\mu}}, \theta^{(1)}_{i_{\boldsymbol{\mu}}})$$

5. Ray truncation via “geometrical observation”.

$$\text{Raytrunc}(\mathcal{X}^{(1)}_{ar}, \mathcal{R}^{(1)}_r, \hat{\mathbf{r}}, K = 5) \to [\mathcal{X}^{(1)}_{aB}, \mathcal{R}^{(1)}_{rB}]$$

6. Calculating the initial values for the “beam-characteristics database” $\mathcal{B}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{r}^{(1)}_{0_{\boldsymbol{\mu}}})$ corresponding to $\mathcal{X}^{(1)}_{aB}$ [see (2.16)–(2.18) in the Thesis with $\sigma_{\boldsymbol{\mu}} = 0$]

$$\text{Beaminit}(\mathcal{X}^{(1)}_{aB}, \bar{b}, z_0) \to \mathcal{B}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{r}^{(1)}_{0_{\boldsymbol{\mu}}}) = (D^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{r}^{(1)}_{0_{\boldsymbol{\mu}}}), \boldsymbol{\Gamma}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{r}^{(1)}_{0_{\boldsymbol{\mu}}}), (\hat{\boldsymbol{\eta}}^{(1)}_{i_{1_{\boldsymbol{\mu}}}}, \hat{\boldsymbol{\eta}}^{(1)}_{i_{2_{\boldsymbol{\mu}}}}, \hat{\boldsymbol{\sigma}}^{(1)}_{i_{\boldsymbol{\mu}}}))$$

   Output: $\mathcal{B}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{r}^{(1)}_{0_{\boldsymbol{\mu}}})$ = the “beam-characteristics database” at the initial points $\mathbf{r}^{(1)}_{0_{\boldsymbol{\mu}}}$ corresponding to $\mathcal{X}^{(1)}_{aB}$.

This database consists of the amplitudes, the complex curvature matrixes at $\mathbf{\Gamma}_{i_{\boldsymbol{\mu}}}(\mathbf{r}^{(1)}_{0_{\boldsymbol{\mu}}})$, and the beam coordinate system.

Note: The amplitudes are described in $\mathcal{B}$ without the propagation phases $e^{-jk\sigma}$. Therefore we denote them as $D_{\boldsymbol{\mu}}$ instead of $A_{\boldsymbol{\mu}}$. The propagation of $D_{\boldsymbol{\mu}}$ includes the beam divergence (see "Beamtranspec" below). The propagation distances are included in the ray-databases $\mathcal{R}$. Here, the initial values are $\sigma_{\boldsymbol{\mu}} = 0$.

7. Propagation of the beam-characteristics from the initial points to the reflection points: [see (24)–(25) in the paper and (2.17) in the Thesis]

$$\begin{aligned}&\text{Beamtranspec}(\mathcal{B}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{r}^{(1)}_{0_{\boldsymbol{\mu}}}), \mathcal{R}^{(1)}_{rB})\\ &\to \mathcal{B}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}) = (D^{(1)}_{\boldsymbol{\mu}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}), \mathbf{\Gamma}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}), (\hat{\boldsymbol{\eta}}^{(1)}_{i_{1_{\boldsymbol{\mu}}}}, \hat{\boldsymbol{\eta}}^{(1)}_{i_{2_{\boldsymbol{\mu}}}}, \hat{\boldsymbol{\sigma}}^{(1)}_{i_{\boldsymbol{\mu}}}))\end{aligned}$$

Output:

- $\mathcal{B}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}})-$ the beam-characteristics database at the reflection point
- $\mathbf{\Gamma}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}) = \big([\mathbf{\Gamma}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{r}^{(1)}_{0_{\boldsymbol{\mu}}})]^{-1} + \sigma^{(1)}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}})\mathbf{I}\big)^{-1}$ is the complex curvature matrix at the specular point [see 25 in paper].
- The amplitudes divergence at $\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}$ (without the propagation phase; see (2.17) in the Thesis): $D^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}) = D^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{r}^{(1)}_{0_{\boldsymbol{\mu}}})\sqrt{\frac{\det \mathbf{\Gamma}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}})}{\det \mathbf{\Gamma}^{(1)}_{i_{\boldsymbol{\mu}}}(0)}}$
- The propagation distances $\sigma^{(1)}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}})$ are included in the ray-databases $\mathcal{R}^{(1)}_{rB}$

8. Construction of the plane of incidences coordinate system at the specular point: [see Eqs,(B1)–(B4) and Fig.4 in the paper]

$$\text{POIsys}(\mathcal{B}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}), \mathcal{R}^{(1)}_{rB}) \to (\hat{\mathbf{t}}^{(1)}_{1_{\boldsymbol{\mu}}}, \hat{\mathbf{t}}^{(1)}_{2_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\nu}}^{(1)}_{\boldsymbol{\mu}})$$

Input: $\hat{\boldsymbol{\sigma}}^{(1)}_{i_{\boldsymbol{\mu}}}$ and $(\hat{\boldsymbol{\nu}}^{(1)}_{\boldsymbol{\mu}}, \theta^{(1)}_{i_{\boldsymbol{\mu}}})$ are included in $\mathcal{R}^{(1)}_{rB}$ and $B^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}))$ respectively and are defined above.

Output: The plane of incident coordinates: $(\hat{\mathbf{t}}^{(1)}_{1_{\boldsymbol{\mu}}}, \hat{\mathbf{t}}^{(1)}_{2_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\nu}}^{(1)}_{\boldsymbol{\mu}})$

Calculated via Eq.(B4): $\hat{\mathbf{t}}^{(1)}_{1_{\boldsymbol{\mu}}} = -\frac{\hat{\boldsymbol{\nu}}^{(1)}_{\boldsymbol{\mu}} \times \hat{\boldsymbol{\sigma}}^{(1)}_{i_{\boldsymbol{\mu}}}}{\sin\theta^{(1)}_{i_{\boldsymbol{\mu}}}}, \quad \hat{\mathbf{t}}^{(1)}_{2_{\boldsymbol{\mu}}} = \hat{\boldsymbol{\nu}}^{(1)}_{\boldsymbol{\mu}} \times \hat{\mathbf{t}}^{(1)}_{1_{\boldsymbol{\mu}}}$

9. Rotation of the beam coordinate system and the complex curvature matrix in the plane of incidences:

$$\begin{aligned}&\text{Beamirot}(\mathcal{B}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}), (\hat{\mathbf{t}}^{(1)}_{1_{\boldsymbol{\mu}}}, \hat{\mathbf{t}}^{(1)}_{2_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\nu}}^{(1)}_{\boldsymbol{\mu}}))\\ &\to \tilde{\mathcal{B}}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}) = (D^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}), \tilde{\mathbf{\Gamma}}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}), (\hat{\boldsymbol{\eta}}^{(1)}_{Qi_{1_{\boldsymbol{\mu}}}}, \hat{\boldsymbol{\eta}}^{(1)}_{Qi_{2_{\boldsymbol{\mu}}}}, \hat{\boldsymbol{\sigma}}^{(1)}_{i_{\boldsymbol{\mu}}}))\end{aligned}$$

Input: $\mathcal{B}_{i_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)})$ and $(\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}}^{(1)}, \hat{\mathbf{t}}_{2_{\boldsymbol{\mu}}}^{(1)}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}^{(1)})$ are def above.

Output: The beam-characteristics database at the specular point: $\tilde{\mathcal{B}}_{i_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)})$ where $\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}} = \hat{\boldsymbol{\eta}}_{Qi_{1_{\boldsymbol{\mu}}}}$ and $\hat{\boldsymbol{\eta}}_{Qi_{2_{\boldsymbol{\mu}}}} = \hat{\boldsymbol{\sigma}}_{i_{\boldsymbol{\mu}}} \times \hat{\boldsymbol{\eta}}_{Qi_{1_{\boldsymbol{\mu}}}}$ and $\tilde{\boldsymbol{\Gamma}}_{i_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)})$ is the complex curvature matrix in the plane of incidences

10. Computation of the reflected beam coordinate system and the complex curvature matrix via (34) and (35)

$$\begin{aligned}&\text{GBReflection}(\theta_{i_{\boldsymbol{\mu}}}^{(1)}, (\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}}^{(1)}, \hat{\mathbf{t}}_{2_{\boldsymbol{\mu}}}^{(1)}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}^{(1)}), \tilde{B}_{i_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)}))\\ &\to \mathcal{B}_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)}) = [D_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)}), \tilde{\boldsymbol{\Gamma}}_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)}), (\hat{\boldsymbol{\eta}}_{Qr_{1_{\boldsymbol{\mu}}}}^{(1)}, \hat{\boldsymbol{\eta}}_{Qr_{2_{\boldsymbol{\mu}}}}^{(1)}, \hat{\boldsymbol{\sigma}}_{r_{\boldsymbol{\mu}}}^{(1)})]\end{aligned}$$

Input: $\theta_{i_{\boldsymbol{\mu}}}^{(1)}, (\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}}^{(1)}, \hat{\mathbf{t}}_{2_{\boldsymbol{\mu}}}^{(1)}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}^{(1)}), \tilde{B}_{i_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)})$ are def. above

Output: The reflected beam-characteristics database at the specular point: $\mathcal{B}_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)})$ where $\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}} = \hat{\boldsymbol{\eta}}_{Qr_{1_{\boldsymbol{\mu}}}}$ and $\hat{\boldsymbol{\eta}}_{Qr_{2_{\boldsymbol{\mu}}}} = \hat{\boldsymbol{\sigma}}_{r_{\boldsymbol{\mu}}} \times \hat{\boldsymbol{\eta}}_{Qr_{1_{\boldsymbol{\mu}}}}$ and $\tilde{\boldsymbol{\Gamma}}_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)})$ is the complex curvature matrix of the reflected field at the specular point in the plane of incidences

11. Rotation of the exit beam's into the global spherical coordinate system via (43)

$$\begin{aligned}&\text{N2FFGammarrot}(\mathcal{B}_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)}))\\ &\to \tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)}) = [D_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)}), \tilde{\Gamma}_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)}), \hat{\tilde{\boldsymbol{\eta}}}_{Qr_{1_{\boldsymbol{\mu}}}}^{(1)}, \hat{\tilde{\boldsymbol{\eta}}}_{Qr_{2_{\boldsymbol{\mu}}}}^{(1)}, \hat{\boldsymbol{\sigma}}_{r_{\boldsymbol{\mu}}}^{(1)})]\end{aligned}$$

Input: The output of the subroutine "GBReflection"

Output: The reflected field beam-characteristics: $\tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)})$ expressed in the rotated transversal coordinate system corresponding to the global spherical coordinate system [see (43)]

12. Calculating the far-field beam-characteristics $\mathcal{B}_{f_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)})$ via (45)

$$\begin{aligned}&\text{FFbeamchar}(\tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)}), \mathcal{R}_{rB}^{(1)}, \hat{\mathbf{r}})\\ &\to \mathcal{B}_{f_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)}) = [A_{\mathbf{Q}}^{(1)}, \mathbf{P}_{r_{\boldsymbol{\mu}}}^{(1)}, \boldsymbol{\Omega}_{\boldsymbol{\mu}}^{(1)}]\end{aligned}$$

Input: $\tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)}), \mathcal{R}_{rB}^{(1)}, \hat{\mathbf{r}}$ are def. above

Output: The far-field beam-characteristics database at the exit point $\mathcal{B}_{f_{\boldsymbol{\mu}}}^{(1)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(1)})$: where

- The differential solid angle $\boldsymbol{\Omega}_{\boldsymbol{\mu}}^{(1)}$ for a beam exit direction $\hat{\boldsymbol{\sigma}}_{r_{\boldsymbol{\mu}}}^{(1)} = (\theta_{r_{\boldsymbol{\mu}}}^{(1)}, \phi_{r_{\boldsymbol{\mu}}}^{(1)})$ and the observation direction $\hat{\mathbf{r}}$:

$$\boldsymbol{\Omega}_{\boldsymbol{\mu}}^{(1)} = \begin{pmatrix} \sin\theta\cos\theta_{r_{\boldsymbol{\mu}}}^{(1)}\cos(\phi-\phi_{r_{\boldsymbol{\mu}}}^{(1)}) - \cos\theta\sin\theta_{r_{\boldsymbol{\mu}}}^{(1)} \\ \sin\theta\sin(\phi-\phi_{r_{\boldsymbol{\mu}}}^{(1)}) \end{pmatrix} \approx \begin{pmatrix} \delta_\theta \\ \delta_\phi \sin\theta_{r_{\boldsymbol{\mu}}}^{(1)} \end{pmatrix}$$

- The far-zone phase projection: $\mathbf{P}^{(1)}_{r_{\boldsymbol{\mu}}} = \hat{\mathbf{r}} \cdot \mathbf{Q}^{(1)}_{\boldsymbol{\mu}}$
- The amplitudes divergence until the exit point: $A^{(1)}_{\mathbf{Q}} = D^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{r}^{(1)}_{0_{\boldsymbol{\mu}}})\sqrt{\frac{\det \boldsymbol{\Gamma}^{(1)}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}})}{\det \boldsymbol{\Gamma}^{(1)}_{i_{\boldsymbol{\mu}}}(0)}}$

13. Calculating the RCS at $\omega \in \Omega^{(1)}$ by calculating the beam field and the expansion coefficient above in the far zone via [(42) and (45)].

    (a) Calculating the far-field $B^{(1)}_{\mu}(\hat{\mathbf{r}}, \omega)$ lies in $X^{(1)}_{aB}$:

    $$\text{FFcont}(\mathcal{B}^{(1)}_{f_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}), \tilde{\mathcal{B}}^{(1)}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}), \omega) \to B^{(1)}_{\mu}(\hat{\mathbf{r}}, \omega)$$

    Input: $\mathcal{B}^{(1)}_{f_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}}), \tilde{\mathcal{B}}^{(1)}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}})$ are def. above
    Output: The far-field beam contribution that lies in $X^{(1)}_{aB}$:

    $$B^{(1)}_{\mu}(\hat{\mathbf{r}}, \omega) \simeq \frac{A^{(1)}_{\mathbf{Q}}}{\sqrt{\det \tilde{\Gamma}^{(1)}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}})}} e^{-jk\frac{1}{2}\boldsymbol{\Omega}^T[\tilde{\tilde{\Gamma}}^{(1)}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}^{(1)}_{\boldsymbol{\mu}})]^{-1}\boldsymbol{\Omega}} e^{jk\mathbf{P}^{(1)}_{r_{\boldsymbol{\mu}}}}$$

    (b) Calculating the expansion coefficient $a^{(1)}_{\boldsymbol{\mu}}(\omega)$ at ω

    $$\text{ExpansionCoeff}(\mathcal{X}^{(1)}_{aB}, \boldsymbol{\xi}_0, \omega_{\max} = 2\omega_{\min}, \omega, \bar{b}) \to a^{(1)}_{\boldsymbol{\mu}}(\omega)$$

    (c) Calculating the RCS of the target at the frequency ω : $\sigma^{(1)}(\hat{\mathbf{r}}_0, \hat{\mathbf{r}}; \omega)$

    $$\text{RCStarg}(a^{(1)}_{\boldsymbol{\mu}}(\omega), B^{(1)}_{\mu}(\hat{\mathbf{r}}, \omega)) \to \sigma^{(1)}(\hat{\mathbf{r}}_0, \hat{\mathbf{r}}; \omega)$$

    Input: $a^{(1)}_{\boldsymbol{\mu}}(\omega)$ and $B^{(1)}_{\mu}(\hat{\mathbf{r}}, \omega)$
    Output: RCS of the target at the frequency ω : $\sigma^{(1)}(\hat{\mathbf{r}}_0, \hat{\mathbf{r}}; \omega) = 4\pi|\sum_{\boldsymbol{\mu}} a^{(1)}_{\boldsymbol{\mu}}(\omega) B^{(1)}_{\mu}(\hat{\mathbf{r}}, \omega)|^2$

1.2.3 RCS for $\omega \in \Omega^{(j)}$ and $j = 2$

For $j = 2$ and the interlaced ξ-x upsampling, we repeat the same steps as in 1.2.2 to obtain $\sigma^{(2)}(\hat{\mathbf{r}}_0, \hat{\mathbf{r}}; \omega)$, but for $\omega_{\max} = 2^2\omega_{\min}$ and $b^{(2)} = 2\bar{b}$, and $\mathcal{X}^{(1)}_{aB}$ as a priori localization (instead of the preamble phase).

Similarly for the interlaced x-ξ upsampling, we use $b^{(2)} = \bar{b}/2$

1.2.4 RCS for $j \geq 2$

The process above can be systematized as follows

- We introduce the upsampling switch with the initial value

$$Upsamp^{(1)} = \begin{cases} 1 & \text{for } \xi\text{-}x \text{ upsampling} \\ -1 & \text{for } x\text{-}\xi \text{ upsampling} \end{cases}$$

  and the recursive relation

$$Upsamp^{(j)} = -Upsamp^{(j-1)}$$

- $b^{(j)} = b^{(j-1)} 2^{Upsamp^{(j-1)}}$
- Expplane$(\mathcal{X}_{aB}^{(j-1)}, b^{(j)}, \bar{\nu}, \omega_{\max} = 2^j \omega_{\min}) \to \mathcal{X}^{(j)}$
- ExpansionCoeff$(\mathcal{X}^{(j)}, \boldsymbol{\xi}_0, b^{(j)}, \bar{\nu}, \omega_{\max} = 2^j \omega_{\min}, \omega = 2^{j-1}\omega_{\min}) \to a_{\boldsymbol{\mu}}^{(j)}(2^{j-1}\omega_{\min})$
- Coefftrunc$(\mathcal{X}^{(j)}, a_{\boldsymbol{\mu}}^{(j)}(2^{j-1}\omega_{\min}), \epsilon = 10^{-6}) \to [\mathcal{X}_a^{(j)}, a_{\boldsymbol{\mu}}^{(j)}(2^{j-1}\omega_{\min})]$
- Raytrack$(\mathcal{X}_a^{(j)}, z_0, c_0, a) \to [\mathcal{X}_{ar}^{(j)}, \mathcal{R}_r^{(j)}]$
  $\mathcal{R}_r^{(j)} = (\mathbf{Q}_{\boldsymbol{\mu}}^{(j)}, \hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}}^{(j)}, \bar{\Theta}_{D_{\boldsymbol{\mu}}}^{(j)}(2^{j-1}\omega_{\min}), \sigma^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)}), \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}^{(j)}, \theta_{i_{\boldsymbol{\mu}}}^{(j)})$
- Raytrunc$(\mathcal{X}_{ar}^{(j)}, \mathcal{R}_r^{(j)}, \hat{\mathbf{r}}, K = 5, \epsilon = 10^{-6}) \to [\mathcal{X}_{aB}^{(j)}, \mathcal{R}_{rB}^{(j)}]$
- Beaminit$(\mathcal{X}_{aB}^{(j)}, b^{(j)}, z_0) \to \mathcal{B}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{r}_{0_{\boldsymbol{\mu}}}^{(j)})$
  $\mathcal{B}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{r}_{0_{\boldsymbol{\mu}}}^{(j)}) = (D_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{r}_{0_{\boldsymbol{\mu}}}^{(j)}), \boldsymbol{\Gamma}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{r}_{0_{\boldsymbol{\mu}}}^{(j)}), (\hat{\boldsymbol{\eta}}_{i_{1_{\boldsymbol{\mu}}}}^{(j)}, \hat{\boldsymbol{\eta}}_{i_{2_{\boldsymbol{\mu}}}}^{(j)}, \hat{\boldsymbol{\sigma}}_{i_{\boldsymbol{\mu}}}^{(j)}))$
- Beamtranspec$(\mathcal{B}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{r}_{0_{\boldsymbol{\mu}}}^{(j)}), \mathcal{R}_{rB}^{(j)}) \to \mathcal{B}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)})$
  $\mathcal{B}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)}) = (D_{\boldsymbol{\mu}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)}), \boldsymbol{\Gamma}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)}), (\hat{\boldsymbol{\eta}}_{i_{1_{\boldsymbol{\mu}}}}^{(j)}, \hat{\boldsymbol{\eta}}_{i_{2_{\boldsymbol{\mu}}}}^{(j)}, \hat{\boldsymbol{\sigma}}_{i_{\boldsymbol{\mu}}}^{(j)}))$
- POIsys$(\mathcal{B}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{r}_{0_{\boldsymbol{\mu}}}^{(j)}), \mathcal{R}_{rB}^{(j)}) \to (\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}}^{(j)}, \hat{\mathbf{t}}_{2_{\boldsymbol{\mu}}}^{(j)}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}^{(j)})$
- Beamirot$(\mathcal{B}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{r}_{0_{\boldsymbol{\mu}}}^{(j)}), (\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}}^{(j)}, \hat{\mathbf{t}}_{2_{\boldsymbol{\mu}}}^{(j)}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}^{(j)}), \mathcal{R}_{rB}^{(j)}) \to \tilde{B}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)})$
  $\tilde{B}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)}) = (D_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)}), \tilde{\boldsymbol{\Gamma}}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)}), (\hat{\boldsymbol{\eta}}_{Qi_{1_{\boldsymbol{\mu}}}}^{(j)}, \hat{\boldsymbol{\eta}}_{Qi_{2_{\boldsymbol{\mu}}}}^{(j)}, \hat{\boldsymbol{\sigma}}_{i_{\boldsymbol{\mu}}}^{(j)}))$
- GBReflection$(\theta_{i_{\boldsymbol{\mu}}}^{(j)}, (\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}}^{(j)}, \hat{\mathbf{t}}_{2_{\boldsymbol{\mu}}}^{(j)}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}^{(j)}), \tilde{B}_{i_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)})) \to \mathcal{B}_{r_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)})$
- N2FFGammarrot$(\mathcal{B}_{r_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)})) \to B_{r_{\boldsymbol{\mu}}}^{\prime(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)})$
- FFbeamchar$(\tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)}), \mathcal{R}_{rB}^{(j)}, \hat{\mathbf{r}}, \omega) \to \mathcal{B}_{f_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)})$
- Calculating the RCS at $\omega \in \Omega^{(j)}$
  FFcont$(\mathcal{B}_{f_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)}), \tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}^{(j)}(\mathbf{Q}_{\boldsymbol{\mu}}^{(j)}), \omega) \to B_{\mu}^{(j)}(\hat{\mathbf{r}}, \omega)$
  ExpansionCoeff$(\mathcal{X}_{aB}^{(j)}, \boldsymbol{\xi}_0, b^{(j)}, \bar{\nu}, \omega_{\max} = 2^j \omega_{\min}, \omega = \omega) \to a_{\boldsymbol{\mu}}^{(j)}(\omega)$
  RCStarg$(a_{\boldsymbol{\mu}}^{(j)}(\omega), B_{\mu}^{(j)}(\hat{\mathbf{r}}, \omega)) \to \sigma^{(j)}(\hat{\mathbf{r}}_0, \hat{\mathbf{r}}; \omega)$

1.3 Code functions

The following presented are the several subroutines consisting of their corresponding inputs, functionality, and outputs. Those are as follows:

1.3.1 Construction of an expansion plane

This subroutine in the MATLAB code is referred as *Expplane*, whose input parameters are:

- The spatial and the spectral regime : $\bar{\mathcal{X}} = (\mathbf{x}, \boldsymbol{\xi})$, discretization of the spatial and spectral regime in PS-lattice
- The expansion frequency ω
- The beam collimation distance $\bar{b}$
- The maximum overcompleteness parameter in the band $\bar{\nu}$

The functionality of the subroutine is based on the construction of the beam lattice at the expansion plane z_0 at the operating frequency ω. The expansion plane that spans the entire target domain and even beyond. The unit-cell of the beam lattice is determined via (10) or (47), described as

$$\left(\bar{x},\, \bar{\xi}\right) = \sqrt{2\pi\bar{\nu}/\omega}\left(\bar{b}^{1/2},\, \bar{b}^{-1/2}\right) \tag{1.1}$$

,which indeed is dependent on the collimation distance $\bar{b} \sim a$. Thus, the wave modeler must punctiliously decide the algorithm's input parameter.

The output of this subroutine is the database $\mathcal{X}$ containing phase-space lattice construction at the expansion plane.

$$\mathcal{X} = \mathrm{Expplane}(\mathbf{x}, \boldsymbol{\xi}, \bar{b}, \omega, \bar{\nu})$$
$$\mathcal{X} = (\mathbf{x}_m, \boldsymbol{\xi}_n)$$

The output of the subroutine:

- The discretized phase-space $\mathcal{X} = (\mathbf{x}_m, \boldsymbol{\xi}_n) = (\mathbf{x}, \boldsymbol{\xi})$ covering $\bar{\mathcal{X}}$

1.3.2 Expansion coefficients

This subroutine in the MATLAB code is referred as *ExpansionCoeff* whose input parameters are

- The input phase-space: $\mathcal{X}$
- The illumination direction: $\boldsymbol{\xi}_0$
- The max operating frequency in the single octave sub-band: $\omega_{\max}$
- The operating frequency in single octave sub-band: ω
- The beam collimation distance: $\bar{b}$
- The maximum overcompleteness parameter in the band: $\bar{\nu}$

The expansion coefficient $a_{\boldsymbol{\mu}}(\omega)$ are computed at the operating frequency ω via (15)

$$a_{\boldsymbol{\mu}}(\omega) = 2\bar{\nu}^2 \frac{\omega^2}{\omega_j^2} e^{-kb|\boldsymbol{\xi}_{\mathbf{n}}-\boldsymbol{\xi}_i|^2/2} e^{-jk\boldsymbol{\xi}_0\cdot\mathbf{x}_{\mathbf{m}}}. \tag{1.2}$$

$$a_{\boldsymbol{\mu}}(\omega) = \text{ExpansionCoeff}(\mathcal{X}, \boldsymbol{\xi}_0, \omega_{\max}, \omega, \bar{b}, \bar{\nu})$$

where $\boldsymbol{\xi}_i = (\sin\theta_i \cos\phi_i, \sin\theta_i \sin\phi_i)$ and $\boldsymbol{\xi}_n = (\sin\theta_{\mathrm{n}} \cos\phi_{\mathrm{n}}, \sin\theta_{\mathrm{n}} \sin\phi_{\mathrm{n}})$.
Where the output of the subroutine is:

- The expansion coefficients: $a_{\boldsymbol{\mu}}(\omega)$

1.3.3 Coefficient Truncation

This subroutine in the MATLAB code is referred as *Coefftrunc* whose input parameters are

- The input phase-space: $\mathcal{X}$
- The expansion coefficient: $a_{\boldsymbol{\mu}}(\omega)$
- The truncation parameter: ϵ

Imposing the "geometrical radiation line" condition, $|a_{\boldsymbol{\mu}}(\omega)| < \epsilon$ (where $\epsilon = 10^{-6}$ is a threshold parameter) are eliminated from the database $\mathcal{X}$ to obtain a modified database $\mathcal{X}_a$.

$$[\mathcal{X}_a, a_{\boldsymbol{\mu}}(\omega)] = \text{Coefftrunc}(\mathcal{X}, a_{\boldsymbol{\mu}}(\omega), \epsilon) \tag{1.3}$$

Thus, the output of the subroutine is

- The truncated phase-space: $\mathcal{X}_a$
- The truncated expansion coefficient: $a_{\boldsymbol{\mu}}(\omega)$

1.3.4 Ray-tracing

This subroutine in the MATLAB code is referred as *Raytrack* whose input parameters:

- The phase-space: $\mathcal{X}_a$
- The location of the expansion plane: z_0
- The center and the radius of the sphere c_0 and a respectively.

The phase-space lattice expressed in the database $\mathcal{X}_a$ consists of the beam origin $\mathbf{r}_0 = (m_1\bar{x}, m_2\bar{x}, z_0)$ and the beam direction $\hat{\boldsymbol{\sigma}}_n$. These data points are retrieved from the database via simple query command. The "rays" are then locally tracked to compute the intersection of the sphere via *Line-sphere intersection*. These line-sphere intersection are further sub-divided into two case, the line intersects the sphere or the line is tangent and doesn't hit the sphere. The former provides with the specular point $\mathbf{Q}_{\boldsymbol{\mu}}$, the distance traversed by the ray from the origin $\mathbf{r}_0$ to specular point on the sphere $\sigma_{\min_{\boldsymbol{\mu}}}$, the local outward normal $\hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}$ to the surface at the specular point $\mathbf{Q}$ and the angle of incidences $\theta_{i_{\boldsymbol{\mu}}}$ relative to the local outward normal. The later discarded from the database $\mathcal{X}_a$ altogether.

By the law of reflection, the direction of reflected ray $\hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}} = \hat{\boldsymbol{\sigma}}_n - (\hat{\boldsymbol{\sigma}}_n \cdot \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}})\hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}$ is computed. Until these beam emerges out of the target domain at the exit direction $\hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}} = (\theta'_{\boldsymbol{\mu}}, \phi'_{\boldsymbol{\mu}})$ and the specular point $\mathbf{Q}_{\boldsymbol{\mu}}$. Therefore the subroutine is described as:

$$\begin{aligned}
&[\mathcal{X}_{ar}, \mathcal{R}_r] = \text{Raytrack}(\mathcal{X}_a, z_0, c_0, a) \\
&\mathcal{R}_r = (\mathbf{Q}_{\boldsymbol{\mu}}, \hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}}, \bar{\Theta}_{D_{\boldsymbol{\mu}}}(\omega), \sigma_{\min_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}, \theta_{i_{\boldsymbol{\mu}}})
\end{aligned}$$

Where $\mathcal{R}_r$ is the database containing the reflected field-characteristics: exit direction $\hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}}$, the diffraction angle of the beam $\bar{\Theta}_{D_{\boldsymbol{\mu}}}$, the exit point $\mathbf{Q}_{\boldsymbol{\mu}}$, the local outward normal at the exit point $\hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}$, the phase accumulation of the ray field along its propagation path $\sigma_{\min_{\boldsymbol{\mu}}}$ and the angle of inidence from the local outward normal $\theta_{i_{\boldsymbol{\mu}}}$ for every element in $\mathcal{X}_{ar}$.

Thus, the output of the subroutine is

- The exit direction of the rays: $\hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}}$
- The diffraction angle of the beam: $\bar{\Theta}_{D_{\boldsymbol{\mu}}}$
- The exit point: $\mathbf{Q}_{\boldsymbol{\mu}}$
- The local outward normal at the exit point: $\hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}$,
- The phase accumulation of the ray field along its propagation path: $\sigma_{\min_{\boldsymbol{\mu}}}$
- The angle of inidence from the local outward normal: $\theta_{i_{\boldsymbol{\mu}}}$

1.3.5 Ray-tracing-truncation

This subroutine in the MATLAB code is referred as *Raytrunc* whose input parameters:

- The PS whose rays hits the target surface: $\mathcal{X}_{ar}$
- The reflected field database: $\mathcal{R}_r = (\mathbf{Q}_{\boldsymbol{\mu}}, \hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}}, \bar{\Theta}_{D_{\boldsymbol{\mu}}}(\omega), \sigma_{min_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}, \theta_{i_{\boldsymbol{\mu}}})$ corresponding to $\mathcal{X}_{ar}$
- Observation direction: $\hat{\mathbf{r}}$
- The constant: K
- The truncation parameter: ϵ

We consider only the subset of rays whose exit directions $\hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}}$ are within a cone about the observation direction $\hat{\mathbf{r}}$, whose apex angle is K times the diffraction beam angle $\bar{\Theta}_{D_{\boldsymbol{\mu}}}$ (33), where K is typically taken to be between 3 to 6. Another effective way is to calculate $\hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}} \cdot \hat{\mathbf{r}} \geq \cos(K\bar{\Theta}_{D_{\boldsymbol{\mu}}})$. Further accounting for subsets of rays in the vicinity by discarding all the beams from the database $\mathcal{X}_a$ and $\mathcal{R}_r$ to obtain a new database $\mathcal{X}_{aB}$ and $\mathcal{R}_{rB}$ respectively.

$$[\mathcal{X}_{aB}, \mathcal{R}_{rB}] = \text{Raytrunc}(\mathcal{R}_r, \hat{\mathbf{r}}, K = 5, \epsilon = 10^{-6})$$

Thus, the output of the subroutine is

- The truncated PS zone whose exit direction falls within the vicinity of the observation direction: $\mathcal{X}_{aB}$
- The truncated reflected ray database containing the reflected field characteristics lying within the vicinity of the observation direction $\mathcal{R}_{rB} = (\mathbf{Q}_{\boldsymbol{\mu}}, \hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}}, \bar{\Theta}_{D_{\boldsymbol{\mu}}}(\omega'), \sigma_{\min_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}, \theta_{i_{\boldsymbol{\mu}}})$ corresponding to $\mathcal{X}_{aB}$.

1.3.6 Beam charachteristics at the expansion plane

The sub-routine in the MATLAB code is referred as *Beaminit* whose input parameters are:

- The PS database: $\mathcal{X}_{aB} = (\mathbf{x}_m, \boldsymbol{\xi}_n)_{aB}$
- The collimation distance: $\bar{b}$
- The location of the expansion plane: z_0

The input parameters from the database $\mathcal{X}_{aB}$ are extracted in order to compute the direction of the propagation $\hat{\boldsymbol{\sigma}}_n = (\theta_{_n}, \phi_{_n})$ and the location of the waist on the expansion plane $\mathbf{x}_m = (m_1\bar{x}, m_2\bar{x})$, therefore beam coordinate system

$$\begin{pmatrix} \eta_{i_{1_\mu}} \\ \eta_{i_{2_\mu}} \\ \sigma_{i_\mu} \end{pmatrix} = \begin{pmatrix} \cos\theta_{\mathbf{n}}\cos\phi_{\mathbf{n}} & \cos\theta_{\mathbf{n}}\sin\phi_{\mathbf{n}} & -\sin\theta_{\mathbf{n}} \\ -\sin\phi_{\mathbf{n}} & \cos\phi_{\mathbf{n}} & 0 \\ \sin\theta_{\mathbf{n}}\cos\phi_{\mathbf{n}} & \sin\theta_{\mathbf{n}}\sin\phi_{\mathbf{n}} & \cos\theta_{\mathbf{n}} \end{pmatrix} \begin{pmatrix} x_1 - m_1\bar{x} \\ x_2 - m_2\bar{x} \\ z \end{pmatrix} \tag{1.4}$$

where $\sin\theta_{\mathbf{n}} = \boldsymbol{\xi}_{\mathbf{n}} = \mathbf{n}\bar{\xi}$, $\cos\theta_{\mathbf{n}} = \sqrt{1 - |\boldsymbol{\xi}_{\mathbf{n}}|^2}$, $\cos\phi_{\mathbf{n}} = n_1/\sqrt{n_1^2 + n_2^2}$ and $\sin\phi_{\mathbf{n}} = n_1/\sqrt{n_1^2 + n_2^2}$.

where, the complex curvature matrix of the each beams are

$$\mathbf{\Gamma}_{i_\mu}(\mathbf{r}_{0_\mu} = (\mathcal{X}_{aB}, z_0), \sigma = 0) = \begin{bmatrix} \frac{1}{j\bar{b}\cos^2\theta_n} & 0 \\ 0 & \frac{1}{j\bar{b}} \end{bmatrix} \tag{1.5}$$

Hence, the output of the subroutine is:

$$\begin{aligned} B_{i_\mu}(\mathbf{r}_{0_\mu}) &= \text{Beaminit}(\mathcal{X}_{aB}, \bar{b}, z_0) \\ B_{i_\mu}(\mathbf{r}_{0_\mu}) &= (A_{i_\mu}(\mathbf{r}_{0_\mu}), \mathbf{\Gamma}_{i_\mu}(\mathbf{r}_{0_\mu}), (\hat{\boldsymbol{\eta}}_{i_{1_\mu}}, \hat{\boldsymbol{\eta}}_{i_{2_\mu}}, \hat{\boldsymbol{\sigma}}_{i_\mu})) \end{aligned}$$

- The beam characteristics database at the waist for PS $\mathcal{X}_{aB}$: $B_{i_\mu}(\mathbf{r}_{0_\mu}) = (A_{i_\mu}(\mathbf{r}_{0_\mu}), \mathbf{\Gamma}_{i_\mu}(\mathbf{r}_{0_\mu}), (\hat{\boldsymbol{\eta}}_{i_{1_\mu}}, \hat{\boldsymbol{\eta}}_{i_{2_\mu}}, \hat{\boldsymbol{\sigma}}_{i_\mu}))$ [see 2.16 in thesis]
- The complex curvature matrix at the beam waist: $\mathbf{\Gamma}_{i_\mu}(\mathbf{r}_{0_\mu})$ [see 2.17 in thesis]

1.3.7 Beam characteristics transformation at the specular point

The sub-routine in the MATLAB code is referred as *Beamtransspec* whose input parameters are:

Input:

- $\mathbf{\Gamma}_{i_\mu}(\mathbf{r}_{0_\mu})$ and $\sigma_{\min_\mu}$ are extracted from beam database $B_{i_\mu}(\mathbf{r}_{0_\mu})$ and reflected ray database $\mathcal{R}_{rB}$ respectively and are def. above

where the beam characteristics are translated from the waist location of the beam to the specular point. Thereby accounting for the complex curvatre matrix behaviour along its beam axis as well as the amplitude decay along the beam axis[see 25 in paper], which is described as

$$\begin{aligned} \mathbf{\Gamma}_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}) &= \left(\left[\mathbf{\Gamma}_{i_\mu}(\mathbf{r}_{0_\mu})\right]^{-1} + \sigma_{\min_\mu}\mathbf{I}\right)^{-1} \\ A_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}) &= \sqrt{\frac{\det \mathbf{\Gamma}_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}})}{\det \mathbf{\Gamma}_{i_\mu}(0)}} \end{aligned} \tag{1.6}$$

Thus the subroutine in the MATLAB code is expressed as

$$\begin{aligned} B_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}) &= \mathrm{Beamtranspec}(B_{i_\mu}(\mathbf{r}_{0_\mu}), \mathcal{R}_{rB}) \\ B_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}) &= (A_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}), \mathbf{\Gamma}_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}), (\hat{\boldsymbol{\eta}}_{i_{1_\mu}}, \hat{\boldsymbol{\eta}}_{i_{2_\mu}}, \hat{\boldsymbol{\sigma}}_{i_\mu})) \end{aligned}$$

Output:

- The beam characteristics database at the specular point:
  $B_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}) = (A_{\boldsymbol{\mu}}(\mathbf{Q}_{\boldsymbol{\mu}}), \mathbf{\Gamma}_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}), (\hat{\boldsymbol{\eta}}_{i_{1_\mu}}, \hat{\boldsymbol{\eta}}_{i_{2_\mu}}, \hat{\boldsymbol{\sigma}}_{i_\mu}))$
- The divergence factor: $A_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}})$

1.3.8 Construction of the plane of incidence system

The sub-routine in the MATLAB code is referred as *POIsys* whose input parameters are:

Input:

- $\hat{\boldsymbol{\sigma}}_{i_\mu}$ and $(\hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}, \theta_{i_\mu})$ are extracted $B_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}})$ and $\mathcal{R}_{rB}$ respectively and are def. above

Consequently, the plane of incident coordinate system $(\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}}, \hat{\mathbf{t}}_{2_{\boldsymbol{\mu}}})$ are constructed via the local outward normal $\hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}$, the angle of incidence $\theta_{i_{\boldsymbol{\mu}}}$ and the direction of propagation $\hat{\boldsymbol{\sigma}}_n = (\theta_n, \phi_n)$ via

$$\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}} = -\frac{\hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}} \times \hat{\boldsymbol{\sigma}}_n}{\sin\theta_{i_{\boldsymbol{\mu}}}}, \quad \hat{\mathbf{t}}_{2_{\boldsymbol{\mu}}} = \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}} \times \hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}} \tag{1.7}$$

Thus the subroutine in the MATLAB code is expressed as

$$(\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}}, \hat{\mathbf{t}}_{2_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}) = \mathrm{POIsys}(B_{i_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), \mathcal{R}_{rB})$$

Output:

- The plane of incident coordinate of the plane of incidence : $(\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}}, \hat{\mathbf{t}}_{2_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}})$

1.3.9 Rotation of the beam coordinate system and the complex coordinate system in the plane of incidence

The sub-routine in the MATLAB code is referred as *Beamirot* whose input parameters are:

Input:

- $B_{i_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}})$ and $(\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}}, \hat{\mathbf{t}}_{2_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}})$ are def above.

Then to obtain the transverse coordinate system and the complex curvature matrix in the plane of incidence, the translated incidence beam coordinate system in the plane of incidences and normal to the plane of incidences are obtained via substituting $\hat{\mathbf{t}}_{1_{\boldsymbol{\mu}}} = \hat{\boldsymbol{\eta}}_{Qi_{1_{\boldsymbol{\mu}}}}$ and $\hat{\boldsymbol{\eta}}_{Qi_{2_{\boldsymbol{\mu}}}} = \hat{\boldsymbol{\sigma}}_{i_{\boldsymbol{\mu}}} \times \hat{\boldsymbol{\eta}}_{Qi_{1_{\boldsymbol{\mu}}}}$

$$\begin{pmatrix} \hat{\boldsymbol{\eta}}_{Qi_{1_{\boldsymbol{\mu}}}} \\ \hat{\boldsymbol{\eta}}_{Qi_{2_{\boldsymbol{\mu}}}} \end{pmatrix} = \boldsymbol{\Theta}_{\boldsymbol{\eta}} \begin{pmatrix} \hat{\boldsymbol{\eta}}_{i_{1_{\boldsymbol{\mu}}}} \\ \hat{\boldsymbol{\eta}}_{i_{2_{\boldsymbol{\mu}}}} \end{pmatrix} \qquad \boldsymbol{\Theta}_{\boldsymbol{\eta}} = \begin{pmatrix} \hat{\boldsymbol{\eta}}_{Qi_{1_{\boldsymbol{\mu}}}} \cdot \hat{\boldsymbol{\eta}}_{i_{1_{\boldsymbol{\mu}}}} & \hat{\boldsymbol{\eta}}_{Qi_{1_{\boldsymbol{\mu}}}} \cdot \hat{\boldsymbol{\eta}}_{i_{2_{\boldsymbol{\mu}}}} \\ \hat{\boldsymbol{\eta}}_{Qi_{2_{\boldsymbol{\mu}}}} \cdot \hat{\boldsymbol{\eta}}_{i_{1_{\boldsymbol{\mu}}}} & \hat{\boldsymbol{\eta}}_{Qi_{2_{\boldsymbol{\mu}}}} \cdot \hat{\boldsymbol{\eta}}_{i_{2_{\boldsymbol{\mu}}}} \end{pmatrix} \tag{1.8}$$

where $\hat{\boldsymbol{\eta}}_{i_{1_{\boldsymbol{\mu}}}} = (\cos\theta_n \cos\phi_n, \cos\theta_n \sin\phi_n, -\sin\theta_n)$ and $\hat{\boldsymbol{\eta}}_{i_{2_{\boldsymbol{\mu}}}} = (-\sin\phi_n, \cos\phi_n, 0)$.

Hence the $\tilde{\boldsymbol{\Gamma}}_{Qi_{\boldsymbol{\mu}}}$ matrix in the plane of incidence coordinate system is readily identified as

$$\tilde{\boldsymbol{\Gamma}}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}) = \boldsymbol{\Theta}_{\boldsymbol{\eta}} \boldsymbol{\Gamma}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}) \boldsymbol{\Theta}_{\boldsymbol{\eta}}^{T} \tag{1.9}$$

Thus the subroutine in the MATLAB code is expressed as

$$\tilde{B}_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}) = \text{Beamirot}(B_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}), (\hat{\mathbf{t}}_{1_\mu}, \mathbf{t}_{2_\mu}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}))$$

Output:

- The beam characteristics database in the plane of incidence: $\tilde{B}_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}) = (A_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}), \tilde{\boldsymbol{\Gamma}}_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}), (\hat{\boldsymbol{\eta}}_{Qi_{1_\mu}}, \hat{\boldsymbol{\eta}}_{Qi_{2_\mu}}, \hat{\boldsymbol{\sigma}}_{i_\mu}))$

1.3.10 Reflected field calculations

The sub-routine in the MATLAB code is referred as *GBReflection* whose input parameters are:

- The beam characteristics database in the plane of incidence: $\tilde{B}_{i_\mu} = (A_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}), \tilde{\boldsymbol{\Gamma}}_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}), (\hat{\boldsymbol{\eta}}_{Qi_{1_\mu}}, \hat{\boldsymbol{\eta}}_{Qi_{2_\mu}}, \hat{\boldsymbol{\sigma}}_{i_\mu}))$
- The angle of incidence: θ_{i_μ}

The reflected beam coordinate system in the plane of incidences and normal to the plane of incidences are obtained via substituting $\hat{\mathbf{t}}_{1_\mu} = \hat{\boldsymbol{\eta}}_{Qi_{1_\mu}} = \hat{\boldsymbol{\eta}}_{Qr_{1_\mu}}$ and are normal to the plane of incidence $(\hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}, \hat{\boldsymbol{\sigma}}_{i_\mu})$, so that $\mathbf{t}_{2_\mu} = \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}} \times \hat{\mathbf{t}}_{1_\mu}$, $\hat{\boldsymbol{\eta}}_{Qi_{2_\mu}} = \hat{\boldsymbol{\sigma}}_{i_\mu} \times \hat{\boldsymbol{\eta}}_{Qi_{1_\mu}}$ and $\hat{\boldsymbol{\eta}}_{Qr_{2_\mu}} = \hat{\boldsymbol{\sigma}}_{r_\mu} \times \hat{\boldsymbol{\eta}}_{Qr_{1_\mu}}$ lie in the plane of incidence.

Hence the reflected field's complex curvature matrix is obtained via (34) and (35), described by

$$\boldsymbol{\Gamma}_{r_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}) = \mathbf{J}\tilde{\boldsymbol{\Gamma}}_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}})\mathbf{J} + 2\cos\theta_{i_\mu}\mathbf{J}\boldsymbol{\Theta}_i^{-1}\mathbf{C}_{Q_\mu}\boldsymbol{\Theta}_i^{-1}\mathbf{J} \tag{1.10}$$

where

$$\mathbf{J}\boldsymbol{\Theta}_i^{-1} = \boldsymbol{\Theta}_i^{-1}\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\cos\theta_{i_\mu}} \end{pmatrix} \tag{1.11}$$

and $\mathbf{C}_{Q_\mu}$ is a 2×2 real symmetrical matrix which defines the Gaussian curvature of $\mathcal{S}$ at $Q_{\boldsymbol{\mu}}$.

Therefore the subroutine is expressed as

$$B_{r_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}) = \text{GBReflection}(\theta_{i_\mu}, (\hat{\mathbf{t}}_{1_\mu}, \hat{\mathbf{t}}_{2_\mu}, \hat{\boldsymbol{\nu}}_{\boldsymbol{\mu}}), \tilde{B}_{i_\mu}(\mathbf{Q}_{\boldsymbol{\mu}}))$$

The output of the subroutine is:

Output:

- The beam characteristics database at the specular point:
  $B_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}) = [A_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), \boldsymbol{\Gamma}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), (\hat{\boldsymbol{\eta}}_{Qr1_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\eta}}_{Qr2_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\sigma}}_{r_{\boldsymbol{\mu}}})]$

1.3.11 N2FF Transformation

The sub-routine in the MATLAB code is referred as *N2FFGammarrot* whose input parameters are:

- The reflected field beam characteristics: $B_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}) = [D_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), \boldsymbol{\Gamma}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), (\hat{\boldsymbol{\eta}}_{Qr1_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\eta}}_{Qr2_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\sigma}}_{r_{\boldsymbol{\mu}}})]$

Functionality: Transformation of the near field beam coordinate into the angle system coordinate system by rotation transverse beam coordinate of the last reflected GB in the target domain into the angle coordinate system.

In order to compute the curvature matrix and the beam coordinate system in the far zone, the transversal coordinate system of a beam is rotated in the angle coordinate system as described in (43).

$$\begin{pmatrix} \tilde{\hat{\boldsymbol{\eta}}}_{Qr1_{\boldsymbol{\mu}}} \\ \tilde{\hat{\boldsymbol{\eta}}}_{Qr2_{\boldsymbol{\mu}}} \end{pmatrix} = \boldsymbol{\Phi}_{\boldsymbol{\mu}} \begin{pmatrix} \hat{\boldsymbol{\eta}}_{Qr1_{\boldsymbol{\mu}}} \\ \hat{\boldsymbol{\eta}}_{Qr2_{\boldsymbol{\mu}}} \end{pmatrix} \qquad \boldsymbol{\Phi}_{\boldsymbol{\mu}} = \begin{pmatrix} \cos\alpha_{\boldsymbol{\mu}} & \sin\alpha_{\boldsymbol{\mu}} \\ -\sin\alpha_{\boldsymbol{\mu}} & \cos\alpha_{\boldsymbol{\mu}} \end{pmatrix} \tag{1.12}$$

where one readily find that $\alpha_{\boldsymbol{\mu}} = \arctan\left(\eta_{Qr2z_{\boldsymbol{\mu}}}/\eta_{Qr1z_{\boldsymbol{\mu}}}\right)$. The π ambiguity is resolved by requiring $\eta'_{Qr1_{\boldsymbol{\mu}}} < 0$.

Hence the $\tilde{\boldsymbol{\Gamma}}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}})$ matrix in the angle coordinate system is readily identified as

$$\tilde{\boldsymbol{\Gamma}}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}) = \boldsymbol{\Phi}_{\boldsymbol{\mu}} \boldsymbol{\Gamma}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}) \boldsymbol{\Phi}_{\boldsymbol{\mu}}^{T} \tag{1.13}$$

Therefore the subroutine is expressed as

$$\begin{aligned} \tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}) &= \text{N2FFGammarrot}(\mathcal{B}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}})) \\ \tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}) &= [D_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), \tilde{\Gamma}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), \tilde{\hat{\boldsymbol{\eta}}}_{Qr1_{\boldsymbol{\mu}}}, \tilde{\hat{\boldsymbol{\eta}}}_{Qr2_{\boldsymbol{\mu}}}, \hat{\boldsymbol{\sigma}}_{r_{\boldsymbol{\mu}}})] \end{aligned} \tag{1.14}$$

1.3.12 Far-field beam-characteristics

The sub-routine in the MATLAB code is referred as *FFbeamchar* whose input parameters are: Calculating the far-field beam-characteristics $\mathcal{B}_{f_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}})$ via (45)

$$\begin{aligned} \mathcal{B}_{f_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}) &= \text{FFbeamchar}(\tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), \mathcal{R}_{rB}, \hat{\mathbf{r}}) \\ \mathcal{B}_{f_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}) &= [A_{\mathbf{Q}}, \mathbf{P}_{r_{\boldsymbol{\mu}}}, \boldsymbol{\Omega}_{\boldsymbol{\mu}}] \end{aligned} \tag{1.15}$$

Input: $\tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), \mathcal{R}_{rB}, \hat{\mathbf{r}}$ are def. above

Output: The far-field beam-characteristics database at the exit point $\mathcal{B}_{f_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}})$: where

- The differential solid angle $\boldsymbol{\Omega}_{\boldsymbol{\mu}}$ for a beam exit direction $\hat{\boldsymbol{\sigma}}_{r_{\boldsymbol{\mu}}} = (\theta_{r_{\boldsymbol{\mu}}}, \phi_{r_{\boldsymbol{\mu}}})$ and the observation direction $\hat{\mathbf{r}}$:
$$\boldsymbol{\Omega}_{\boldsymbol{\mu}} = \begin{pmatrix} \sin\theta\cos\theta_{r_{\boldsymbol{\mu}}}\cos(\phi-\phi_{r_{\boldsymbol{\mu}}}) - \cos\theta\sin\theta_{r_{\boldsymbol{\mu}}} \\ \sin\theta\sin(\phi-\phi_{r_{\boldsymbol{\mu}}}) \end{pmatrix} \approx \begin{pmatrix} \delta_\theta \\ \delta_\phi \sin\theta_{r_{\boldsymbol{\mu}}} \end{pmatrix}$$
- The far-zone phase projection: $\mathbf{P}_{r_{\boldsymbol{\mu}}} = \hat{\mathbf{r}} \cdot \mathbf{Q}_{\boldsymbol{\mu}}$
- The amplitudes divergence until the exit point: $A_{\mathbf{Q}} = D_{i_{\boldsymbol{\mu}}}(\mathbf{r}_{0_{\boldsymbol{\mu}}})\sqrt{\frac{\det \boldsymbol{\Gamma}_{i_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}})}{\det \boldsymbol{\Gamma}_{i_{\boldsymbol{\mu}}}(0)}}$

1.3.13 Farzone contributions

The sub-routine in the MATLAB code is referred as *N2FFcontr* whose input parameters are:

Input: $\mathcal{B}_{f_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), \tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}})$ are def. above

Calculating the RCS at $\omega \in \Omega$ by calculating the beam field in the far zone via [(42) and (45)].

1. Calculating the far-field $B_\mu(\hat{\mathbf{r}}, \omega)$ lies in X_{aB}:

   For a given observation direction $\hat{\mathbf{r}}$ and the beam propagation direction $\hat{\boldsymbol{\sigma}}_{\boldsymbol{\mu}} = (\theta'_{\boldsymbol{\mu}}, \phi'_{\boldsymbol{\mu}})$, the beam field in the far zone is represented as (46). The beams contribution $|a_{\boldsymbol{\mu}}(\omega)\Psi_{\boldsymbol{\mu}}(\hat{\mathbf{r}}, \omega)|$ in the far-field are coherently summed (accounting the phase shift terms, divergence factor etc.)

   Hence the beam field at ω is computed via
$$B_\mu(\hat{\mathbf{r}}, \omega) \simeq \frac{A_{\boldsymbol{\mu}} e^{-jk\sigma_{\min_{\boldsymbol{\mu}}}}}{\sqrt{\det \tilde{\Gamma}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}})}} e^{-jk\frac{1}{2}\boldsymbol{\Omega}_\mu^T \tilde{\Gamma}_{r_\mu}^{-1}(\mathbf{Q}_\mu)\boldsymbol{\Omega}_\mu} e^{jk\hat{\mathbf{r}}\cdot\mathbf{Q}_\mu} \frac{e^{-jkr}}{r} \tag{1.16}$$
   In practice, using vectorization, the operating frequency ω is extended into a frequency band Ω in the MATLAB code.

   Therefore the subroutine is expressed as
$$B_\mu(\hat{\mathbf{r}}, \omega) = \mathrm{FFcont}(\mathcal{B}_{f_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), \tilde{\mathcal{B}}_{r_{\boldsymbol{\mu}}}(\mathbf{Q}_{\boldsymbol{\mu}}), \omega) \tag{1.17}$$
   The output of the subroutine is:

- The beam field at the far-zone: $B_\mu(\hat{\mathbf{r}}, \omega)$