UNIVERSITY OF MARYLAND

Control of Robotic Systems

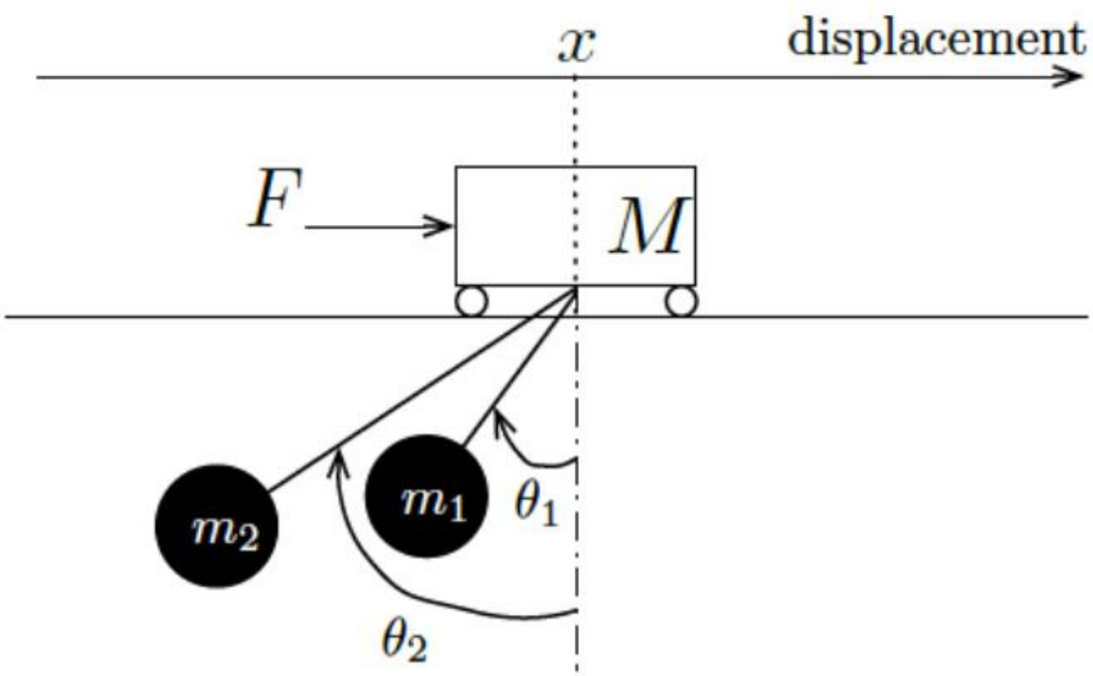
ENPM - 667

Project 2

Pradeep Gopal (116885027)
Sahana Anbazhagan (116702720)
Section 0201

Problem statement 1:

Consider a crane that moves along a one-dimensional track. It behaves as a frictionless cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m_1 and m_2, and the lengths of the cables are l_1 and l_2, respectively. The following figure depicts the crane and associated variables used throughout this project.



1.A Equations of motion for the system and the corresponding nonlinear state-space representation.

Considering P_{m1} as the position vector of mass m_1 and velocity is given by $V_{m1} = \dot{P}_{m1}$, then,

$$P_{m1} = (x - l_1 \sin\theta_1)\hat{\imath} + (-l_1 \cos\theta_1)\hat{\jmath}$$

$$V_{m1} = (\dot{x} - l_1 \cos\theta_1 \dot{\theta}_1)\hat{\imath} + (l_1 \sin\theta_1 \dot{\theta}_1)\hat{\jmath}$$

$$V_{m1}{}^2 = (\dot{x} - l_1 \cos\theta_1 \dot{\theta}_1)^2 + (l_1 \sin\theta_1 \dot{\theta}_1)^2 = \dot{x}^2 - 2\dot{x}\, l_1 \cos\theta_1 \dot{\theta}_1 + l_1{}^2 \dot{\theta}_1{}^2$$

Considering P_{m2} as the position vector of mass $m_2$and velocity is given by $v_{m2} = \dot{P}_{m2}$, then

$$p_{m2} = (x - l_2 \sin\theta_2)\hat{\imath} + (-l_2 \cos\theta_2)\hat{\jmath}$$

$$V_{m2} = \left(\dot{x} - l_2 \cos\theta_2\, \dot{\theta_2}\right)\hat{\imath} + \left(l_2 \sin\theta_2\, \dot{\theta_2}\right)\hat{\jmath}$$

$$V_{m2}{}^2 = \left(\dot{x} - l_2 \cos\theta_2\, \dot{\theta_1}\right)^2 + \left(l_2 \sin\theta_2\, \dot{\theta_2}\right)^2 = \dot{x}^2 - 2\dot{x}\, l_2 \cos\theta_2\, \dot{\theta_2} + l_2{}^2\dot{\theta_2}{}^2$$

Kinetic energy is given by, $K.E = \frac{1}{2}Mv^2$

$$K.E = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m_1 V_{m1}{}^2 + \frac{1}{2}m_2 V_{m2}{}^2$$

$$K.E = \frac{1}{2}\, M\, x(\dot{t})^2 + \frac{1}{2}\, m_1(x(\dot{t}) - l_1\theta_1(\dot{t})\cos\,(\theta_1(t))^2 + \frac{1}{2}\, m_1\, l_1(\theta_1(\dot{t})\sin\,(\theta_1(t))^2 + \frac{1}{2}\, m1(x(\dot{t}) - l_1\theta_2(\dot{t})\cos\,(\theta_2(t))^2 + \frac{1}{2}\, m_1(l_2\theta_2(\dot{t})\sin\,(\theta_2(t))^2$$

The Potential Energy is given by $P.E = mgh$

$$P.E = -\, m_2\, l_2 g \cos\left(\theta_1(t)\right) - m_2\, l_2 g \cos\left(\theta_2(t)\right)$$

We know that the Total Lagrange Energy can be calculated,

L = K.E – P.E

$$\begin{aligned} L = \frac{1}{2}\, M\, x(\dot{t})^2 + \frac{1}{2}\, m_1\, x(\dot{t})^2 + \frac{1}{2}\, m_1\, l_1{}^2\theta_1(\dot{t})^2\cos\,(\theta_1(t)^2 \\ - m_1 x(\dot{t})\, l_1\dot{\theta}_1 \cos\left(\theta_1(t)\right) + m_2\, l_2{}^2\theta_1(\dot{t})^2\sin\,(\theta_1(t)^2 + \frac{1}{2}\, m_2\, x(\dot{t})^2 \\ + \frac{1}{2}\, m_2\, l_2{}^2\theta_2(\dot{t})^2\cos\,(\theta_2(t)^2 - m_2 x(\dot{t})\, l_2\dot{\theta}_2 \cos\left(\theta_2(t)\right) \\ + \frac{1}{2}\, m_2\, l_2{}^2\theta_2(\dot{t})^2\cos\,(\theta_2(t)^2 + m_1 g\, l_1\cos\left(\theta_1(t)\right) \\ + m_2 g\, l_2\cos\left(\theta_2(t)\right) \end{aligned}$$

Simplifying this equation, we get

$$L = \frac{1}{2} M\, x\dot{(t)}^2 + \frac{1}{2} x\dot{(t)}^2(\mathrm{m}_1 + \mathrm{m}_2) + \frac{1}{2} \mathrm{m}_1\, \mathrm{l}_1{}^2 \theta_1\dot{(t)}^2 + \frac{1}{2} \mathrm{m}_2\, \mathrm{l}_2{}^2 \theta_2\dot{(t)}^2 - \mathrm{m}_1 x\dot{(t)}\, \mathrm{l}_1 \dot{\theta}_1 \cos(\theta_1(t)) - \mathrm{m}_2 x\dot{(t)}\, \mathrm{l}_2 \dot{\theta}_2 \cos(\theta_2(t)) + g[\mathrm{m}_1\, \mathrm{l}_1 \cos(\theta_1(t)) + \mathrm{m}_2\, \mathrm{l}_2 \cos(\theta_2(t))]$$

We know the Lagrange Equations are as follows:

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F$ ------------------------------------- (i)

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0$ ------------------------------------- (ii)

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0$ ------------------------------------- (iii)

Considering the first equation we have,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \left(\frac{\partial L}{\partial x}\right) = F$$

$$\frac{\partial L}{\partial \dot{x}} = M\dot{x} + x\dot{(t)}(\mathrm{m}_1 + \mathrm{m}_2) - \mathrm{m}_1\, \mathrm{l}_1 \dot{\theta}_1 \cos(\theta_1(t)) - \mathrm{m}_2\, \mathrm{l}_2 \dot{\theta}_2 \cos(\theta_2(t))$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = M\ddot{x} + \ddot{x}(\mathrm{m}_1 + \mathrm{m}_2) + \mathrm{m}_1\, \mathrm{l}_1 \dot{\theta}_1{}^2 \sin(\theta_1(t)) - \mathrm{m}_1\, \mathrm{l}_1 \ddot{\theta}_1 \cos(\theta_1(t)) + \mathrm{m}_2\, \mathrm{l}_2 \dot{\theta}_2{}^2 \cos(\theta_1(t)) - \mathrm{m}_2\, \mathrm{l}_2 \ddot{\theta}_2 \cos(\theta_2(t))$$

Since there is no x component in L,

$$\frac{\partial L}{\partial x} = 0$$

Substituting in (i),

$$F = \ddot{x}[M + (\mathrm{m}_1 + \mathrm{m}_2)] + \mathrm{m}_1\, \mathrm{l}_1 \dot{\theta}_1{}^2 \sin(\theta_1(t)) - \mathrm{m}_1\, \mathrm{l}_1 \ddot{\theta}_1 \cos(\theta_1(t)) + \mathrm{m}_2\, \mathrm{l}_2 \dot{\theta}_2{}^2 \cos(\theta_1(t)) - \mathrm{m}_2\, \mathrm{l}_2 \ddot{\theta}_2 \cos(\theta_2(t)) \text{ ----- (1)}$$

Now considering the second equation,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \left(\frac{\partial L}{\partial \theta_1}\right) = 0$$

$$\frac{\partial L}{\partial \dot{\theta_1}} = \ \mathrm{m_1\ l_1}\dot{\theta}_1 - \mathrm{m_1\ l_1}\dot{x}\cos(\theta_1)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_1}}\right) = \ \mathrm{m_1\ l_1}^2\ddot{\theta_1} + \mathrm{m_1\ l_1}\dot{x}\sin(\theta_1)\dot{\theta}_1 - \mathrm{m_1\ l_1}\ddot{x}\cos(\theta_1)$$

$$\frac{\partial L}{\partial \theta_1} = \ \mathrm{m_1\ l_1}\dot{x}\sin(\theta_1)\dot{\theta}_1 - \ g\mathrm{m_1\ l_1}\sin(\theta_1)$$

Substituting in (ii),

$$\mathrm{m_1\ l_1}^2\ddot{\theta_1} + \mathrm{m_1\ l_1}\dot{x}\sin(\theta_1)\dot{\theta}_1 - \mathrm{m_1\ l_1}\ddot{x}\cos(\theta_1) - \mathrm{m_1\ l_1}\dot{x}\sin(\theta_1)\dot{\theta}_1 + \ g\mathrm{m_1\ l_1}\sin(\theta_1) = 0$$

$$\mathrm{m_1\ l_1}^2\ddot{\theta_1} - \mathrm{m_1\ l_1}\ddot{x}\cos(\theta_1) + \ g\mathrm{m_1\ l_1}\sin(\theta_1) = 0$$

$$\mathrm{m_1\ l_1}(\ \mathrm{l_1}\ddot{\theta_1} - \ddot{x}\cos(\theta_1) + g\,\mathrm{l_1}\sin(\theta_1)) = 0$$

Since $\mathrm{m_1\ l_1} \neq 0$,

$$\mathrm{l_1}\ddot{\theta_1} - \ddot{x}\cos(\theta_1) + g\,\mathrm{l_1}\sin(\theta_1) = 0 \qquad \text{---------------- (2)}$$

The third equation is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_2}}\right) - \left(\frac{\partial L}{\partial \theta_2}\right) = 0$$

$$\frac{\partial L}{\partial \dot{\theta_2}} = \ \mathrm{m_2\ l_2}\dot{\theta}_2 - \mathrm{m_2\ l_2}\dot{x}\cos(\theta_2)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta_2}}\right) = \ \mathrm{m_2\ l_2}^2\ddot{\theta_2} + \mathrm{m_2\ l_2}\dot{x}\sin(\theta_2)\dot{\theta}_2 - \mathrm{m_2\ l_2}\ddot{x}\cos(\theta_2)$$

$$\frac{\partial L}{\partial \theta_2} = \ \mathrm{m_2\ l_2}\dot{x}\sin(\theta_2)\dot{\theta}_2 - \ g\mathrm{m_2\ l_2}\sin(\theta_2)$$

Substituting above equations in (iii),

$$\mathrm{m_2\ l_2}^2\ddot{\theta_2} + \mathrm{m_2\ l_2}\dot{x}\sin(\theta_2)\dot{\theta}_2 - \mathrm{m_2\ l_2}\ddot{x}\cos(\theta_2) - \mathrm{m_2\ l_2}\dot{x}\sin(\theta_2)\dot{\theta}_2 + \ g\mathrm{m_2\ l_2}\sin(\theta_2) = 0$$

$$\mathrm{m_2\ l_2}^2\ddot{\theta_2} - \mathrm{m_2\ l_2}\ddot{x}\cos(\theta_2) + \ g\mathrm{m_2\ l_2}\sin(\theta_2) = 0$$

$$\mathrm{m_2\ l_2}(\ \mathrm{l_2}\ddot{\theta_2} - \ddot{x}\cos(\theta_2) + g\sin(\theta_2)) = 0$$

Since $m_2\, l_2 \neq 0$,

$$l_2 \ddot{\theta}_2 - \ddot{x}\cos(\theta_2) + g\sin(\theta_2) = 0 \qquad \text{--------------------- (3)}$$

We have,

$f_1 = \dot{x}$

$f_2 = \dot{\theta}_1$

$f_3 = \dot{\theta}_2$

$f_4 = \ddot{x} = \frac{1}{(M + m_1 + m_2)}(F - m_1 l_1 \sin(\theta_1)\dot{\theta}_1^2 - m_2 l_2 \sin(\theta_2)\dot{\theta}_2^2 + m_1 l_1 \cos(\theta_1)\ddot{\theta}_1 + m_2 l_2 \cos(\theta_2)\ddot{\theta}_2)$

$f_5 = \ddot{\theta}_1 = \frac{1}{l1}(\ddot{x}\cos\theta_1 - g\sin\theta_1)$

$f_6 = \ddot{\theta}_2 = \frac{1}{l2}(\ddot{x}\cos\theta_2 - g\sin\theta_2)$

Substituting the values of $\ddot{\theta}_1$ and $\ddot{\theta}_2$ in $\ddot{x}$,

$$\ddot{x} = \frac{1}{M + m_1 + m_2}\Big(F - m_1\, l_1 \dot{\theta}_1^{\,2} \sin\big(\theta_1(t)\big) - m_2\, l_2 \dot{\theta}_2^{\,2} \sin\big(\theta_2(t)\big) + \frac{m_1\, l_1}{l1}(\ddot{x}\cos(\theta_1) - g\sin(\theta_1))\cos(\theta_1) + \frac{m_2\, l_2}{l_2}(\ddot{x}\cos(\theta_2) - g\sin(\theta_2))\cos(\theta_2)$$

$$\ddot{x}(M + m_1 + m_2 - m_1\cos(\theta_1)^2 - m_2\cos(\theta_2)^2) = F - m_1\, l_1 \dot{\theta}_1^{\,2} \sin\big(\theta_1(t)\big) - m_2\, l_2 \dot{\theta}_2^{\,2} \sin\big(\theta_2(t)\big) - m_1\, g\sin(\theta_1)\cos(\theta_1) - m_2\, g\sin(\theta_2)\cos(\theta_2)$$

$$\ddot{x} = \frac{(F - m_1\, l_1 \dot{\theta}_1^{\,2} \sin\big(\theta_1(t)\big) - m_2\, l_2 \dot{\theta}_2^{\,2} \sin\big(\theta_2(t)\big) - \frac{m_1}{2} g\sin(2\theta_1) - \frac{m_2}{2} g\sin(2\theta_2)}{(M + m_1 + m_2 - m_1\cos(\theta_1)^2 - m_2\cos(\theta_2)^2)}$$

$\ddot{\theta}_1 = \frac{1}{l_1}(\ddot{x}\,\cos\theta_1 - g\sin\theta_1)$

$\ddot{\theta}_2 = \frac{1}{l_2}(\ddot{x}\,\cos\theta_2 - g\sin\theta_2)$

Where,

$$\ddot{x} = \frac{(F - m_1\, l_1 \dot{\theta}_1^{\,2} \sin\big(\theta_1(t)\big) - m_2\, l_2 \dot{\theta}_2^{\,2} \sin\big(\theta_2(t)\big) - \frac{m_1}{2} g\sin(2\theta_1) - \frac{m_2}{2} g\sin(2\theta_2)}{(M + m_1 + m_2 - m_1\cos(\theta_1)^2 - m_2\cos(\theta_2)^2)}$$

Now our simplified equations are,

$f_1 = \dot{x}$

$f_2 = \dot{\theta}_1$

$f_3 = \dot{\theta}_2$

$$f4 = \quad \ddot{x} = \frac{(F - \mathrm{m}_1\,\mathrm{l}_1\dot{\theta}_1{}^2\sin\big(\theta_1(t)\big) - \mathrm{m}_2\,\mathrm{l}_2\dot{\theta}_2{}^2\sin\big(\theta_2(t)\big) - \frac{\mathrm{m}_1}{2}g\sin(2\theta_1) - \frac{\mathrm{m}_2}{2}g\sin(2\theta_2)}{(M + \mathrm{m}_1 + \mathrm{m}_2 - \mathrm{m}_1\cos(\theta_1)^2 - \mathrm{m}_2\cos(\theta_2)^2)}$$

$$f_5 = \ddot{\theta}_1 = \frac{1}{l1}(\ddot{x}\cos\theta_1 - g\sin\theta_1)$$

$$f_6 = \ddot{\theta}_2 = \frac{1}{l2}(\ddot{x}\cos\theta_2 - g\sin\theta_2)$$

Where,

$$\ddot{x} = \frac{(F - \mathrm{m}_1\,\mathrm{l}_1\dot{\theta}_1{}^2\sin\big(\theta_1(t)\big) - \mathrm{m}_2\,\mathrm{l}_2\dot{\theta}_2{}^2\sin\big(\theta_2(t)\big) - \frac{\mathrm{m}_1}{2}g\sin(2\theta_1) - \frac{\mathrm{m}_2}{2}g\sin(2\theta_2)}{(M + \mathrm{m}_1 + \mathrm{m}_2 - \mathrm{m}_1\cos(\theta_1)^2 - \mathrm{m}_2\cos(\theta_2)^2)}$$

1.B State Space Representation of the linearized system around the equilibrium point $x = \theta_1 = \theta_2 = 0$

In order to linearize we need to calculate the following Jacobian matrix,

$$\begin{bmatrix}
\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} & \frac{\partial f_1}{\partial \dot{x}} & \frac{\partial f_1}{\partial \dot{\theta}_1} & \frac{\partial f_1}{\partial \dot{\theta}_2} \\
\frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial \theta_1} & \frac{\partial f_2}{\partial \theta_2} & \frac{\partial f_2}{\partial \dot{x}} & \frac{\partial f_2}{\partial \dot{\theta}_1} & \frac{\partial f_2}{\partial \dot{\theta}_2} \\
\frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial \theta_1} & \frac{\partial f_3}{\partial \theta_2} & \frac{\partial f_3}{\partial \dot{x}} & \frac{\partial f_3}{\partial \dot{\theta}_1} & \frac{\partial f_3}{\partial \dot{\theta}_2} \\
\frac{\partial f_4}{\partial x} & \frac{\partial f_4}{\partial \theta_1} & \frac{\partial f_4}{\partial \theta_2} & \frac{\partial f_4}{\partial \dot{x}} & \frac{\partial f_4}{\partial \dot{\theta}_1} & \frac{\partial f_4}{\partial \dot{\theta}_2} \\
\frac{\partial f_5}{\partial x} & \frac{\partial f_5}{\partial \theta_1} & \frac{\partial f_5}{\partial \theta_2} & \frac{\partial f_5}{\partial \dot{x}} & \frac{\partial f_5}{\partial \dot{\theta}_1} & \frac{\partial f_5}{\partial \dot{\theta}_2} \\
\frac{\partial f_6}{\partial x} & \frac{\partial f_6}{\partial \theta_1} & \frac{\partial f_6}{\partial \theta_2} & \frac{\partial f_6}{\partial \dot{x}} & \frac{\partial f_6}{\partial \dot{\theta}_1} & \frac{\partial f_6}{\partial \dot{\theta}_2}
\end{bmatrix}$$

From the above matrix we find that the terms,

$$\frac{\partial f_1}{\partial x} = \frac{\partial f_1}{\partial \theta_1} = \frac{\partial f_1}{\partial \theta_2} = \frac{\partial f_1}{\partial \dot{\theta}_1} = \frac{\partial f_1}{\partial \dot{\theta}_2} = \frac{\partial f_2}{\partial x} = \frac{\partial f_2}{\partial \theta_1} = \frac{\partial f_2}{\partial \theta_2} = \frac{\partial f_2}{\partial \dot{x}} = \frac{\partial f_2}{\partial \dot{\theta}_2} = \frac{\partial f_3}{\partial x} = \frac{\partial f_3}{\partial \theta_1} = \frac{\partial f_3}{\partial \theta_2} = \frac{\partial f_3}{\partial \dot{x}} = \frac{\partial f_3}{\partial \dot{\theta}_1} = \frac{\partial f_4}{\partial x} = \frac{\partial f_4}{\partial \dot{x}} = \frac{\partial f_4}{\partial \dot{\theta}_1} = \frac{\partial f_4}{\partial \dot{\theta}_2} = \frac{\partial f_5}{\partial x} = \frac{\partial f_5}{\partial \dot{x}} = \frac{\partial f_5}{\partial \dot{\theta}_1} = \frac{\partial f_5}{\partial \dot{\theta}_2} = \frac{\partial f_6}{\partial x} = \frac{\partial f_6}{\partial \dot{x}} = \frac{\partial f_6}{\partial \dot{\theta}_1} = \frac{\partial f_6}{\partial \dot{\theta}_2} = 0$$

We get $\frac{\partial f_1}{\partial \dot{x}}, \frac{\partial f_2}{\partial \dot{\theta}_1}, \frac{\partial f_3}{\partial \dot{\theta}_2} = 1$, since differentiation variable is same as the function variable to be differentiated.

The remaining values to be differentiated and calculated are:

$$a_{42} = \frac{\partial f_4}{\partial \theta_1} = -\frac{\mathrm{m}_1 g}{M}$$

$$a_{43} = \frac{\partial f_4}{\partial \theta_2} = -\frac{\mathrm{m}_2 g}{M}$$

$$a_{52} = \frac{\partial f_5}{\partial \theta_1} = -\frac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_1}$$

$$a_{53} = \frac{\partial f_5}{\partial \theta_2} = -\frac{\mathrm{m}_2 g}{M\,\mathrm{l}_1}$$

$$a_{62} = \frac{\partial f_6}{\partial \theta_1} = -\frac{\mathrm{m}_1 g}{M\,\mathrm{l}_2}$$

$$a_{63} = \frac{\partial f_6}{\partial \theta_2} = -\frac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_2}$$

Substituting the above values in the 6x6 matrix we have, we obtain

$$A_F = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\frac{\mathrm{m}_1 g}{M} & -\frac{\mathrm{m}_2 g}{M} & 0 & 0 & 0 \\ 0 & -\frac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_1} & -\frac{\mathrm{m}_2 g}{M\,\mathrm{l}_1} & 0 & 0 & 0 \\ 0 & -\frac{\mathrm{m}_1 g}{M\,\mathrm{l}_2} & -\frac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_2} & 0 & 0 & 0 \end{bmatrix}$$

Since, $A_F = \nabla_x F(x,u)$

Similarly, B_F matrix can be calculated from $B_F = \nabla_u F(x,u)$

$$B_F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{\mathrm{M}} \\ \dfrac{1}{M\,\mathrm{l}_1} \\ \dfrac{1}{M\,\mathrm{l}_2} \end{bmatrix}$$

So, representing the above derived equations and matrices by substituting them in the state space form, we obtain,

$$X(\dot{t}) = AX(t) + BU(t)$$

$$\begin{bmatrix} \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{x} \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\dfrac{\mathrm{m}_1 g}{M} & -\dfrac{\mathrm{m}_2 g}{M} & 0 & 0 & 0 \\ 0 & -\dfrac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_1} & -\dfrac{\mathrm{m}_2 g}{M\,\mathrm{l}_1} & 0 & 0 & 0 \\ 0 & -\dfrac{\mathrm{m}_1 g}{M\,\mathrm{l}_2} & -\dfrac{g(M+\mathrm{m}_1)}{M\,\mathrm{l}_2} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ \theta_1 \\ \theta_2 \\ \dot{x} \\ \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \dfrac{1}{\mathrm{M}} \\ \dfrac{1}{M\,\mathrm{l}_1} \\ \dfrac{1}{M\,\mathrm{l}_2} \end{bmatrix} F$$

This is the linearized system around the equilibrium points $x = \theta_1 = \theta_2 = 0$

1.C Controllability constraints on M, m1, m2, l1 and l2

Theorem: For the Linear Time Invariant system state equation, the grammian of controllability is invertible if and only if the n x nm controllability matrix satisfies,

$$rank[B_k \; AB_k \ldots\ldots \; A^{n-1}B_k] = n$$

Where n is the dimension of the A matrix.

Therefore, for our system we compute the controllability matrix and check for the rank as follows,

$$rank[B_k \; AB_k \; A^2B_k \; A^3B_k \; A^4B_k \; A^5B_k]$$

If the rank of this controllability matrix equals to n = 6, the system is controllable.

Following is the determinant of this controllability matrix,

$$\frac{g^6(\mathrm{l1}-\mathrm{l2})^2}{\mathrm{l1}^6\mathrm{l2}^6M^6}$$

It can be noted that if $l1 = l2$, the determinant equals to zero and hence the determinant is not invertible and the system in not controllable.

We test for the controllability of this matrix by considering different constraints on M, m1, m2, l1 and l2.

```
%-------------------------------------------------------------%

%First Component - Part  C - Controllability constraints on
 m1,m2,M,l1,l2%
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%-------------------------------------------------------------%

clc;
clear all;
syms m1 m2 M g l1 l2;

%Matrices in state space format
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)];
```

```
%Computing the individual components of the controllability matrix
AB = A*B;
A2B = A*AB;
A3B = A*A2B;
A4B = A*A3B;
A5B = A*A4B;

%Controllability matrix
C = [B AB A2B A3B A4B A5B];

%Determinant of the controllability matrix
DET = det(C);
DET = simplify(DET);

%Different constraints

%case 1: l1 = l2
disp('case 1: l1 = l2');
C1=subs(C,l1,l2); %substitung l1=l2 in the controllabillity matrix
Rank=rank(C1)
if(Rank<6);
    disp('system is not controllable');
else
    disp('system is controllable');
end

disp(' ');

%case 2: m1 = m2
disp('case 2: m1 = m2');
C1=subs(C,m1,m2); %substitung m1=m2 in the controllabillity matrix
Rank=rank(C1)

if(Rank<6)
    disp('system is not controllable');
else
    disp('system is controllable');
end

disp(' ');

%case 3: m1 = M
disp('case 3: m1 = M');
C1=subs(C,m1,M); %substitung m1=M in the controllabillity matrix
Rank=rank(C1)
```

```
if(Rank<6)
    disp('system is not controllable');
else
    disp('system is controllable');
end

disp(' ');

%case 4: m2 = M
disp('case 4: m2 = M');
C1=subs(C,m2,M); %substitung m1=M in the controllabillity matrix
Rank=rank(C1)
if(Rank<6)
    disp('system is not controllable');
else
    disp('system is controllable');
end
```

Matlab command window

```
case 1: l1 = l2

Rank =

     4

system is not controllable

case 2: m1 = m2

Rank =

     6

system is controllable

case 3: m1 = M

Rank =

      6

 system is controllable
```

```
case 4: m2 = M

Rank =

     6

system is controllable
```

Inference

It can be inferred that the controllability of the system is not affected by m1, m2 and M as the rank remains 6 in all the cases. But when l1 = l2, the rank reduces to 4 and the system becomes uncontrollable.

1.D Linear Quadratic Regulator Design

Checking for Controllability

We check for controllability by substituting the values of m1, m2, M, l1 and l2 by checking the rank of the controllability matrix.

```
%------------------------------------------------------------%

%First Component - Part  C - Controllability check%
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%------------------------------------------------------------%

clc;
clear all;
syms m1 m2 M g l1 l2;

%Entering the values of the system
m1=100;
m2=100;
l1=20;
l2=10;
M=1000;
```

```
%Matrices in state space format
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)];

%Computing the individual components of the controllability matrix
AB = A*B;
A2B = A*AB;
A3B = A*A2B;
A4B = A*A3B;
A5B = A*A4B;

%Controllability matrix
C = [B AB A2B A3B A4B A5B];
```

Matlab command window

```
rank_of_the_controllability_matrix =

     6
```

Inference

It can be inferred that the system is controllable as the rank of the controllability matrix is n = 6.

Optimal K Value design – LQR Controller

```
%-------------------------------------------------------------%

%First Component - Part  C - Controllability constraints on
 m1,m2,M,l1,l2%
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%-------------------------------------------------------------%

clc;
clear all;
syms m1 m2 l1 l2 g M
```

```
%Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.8;

%Initial Conditions
x0=[10,10,20,10,10,10];

%Matrices
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
Q = [1 0 0 0 0 0; 0 1 0 0 0 0; 0 0 1 0 0 0; 0 0 0 70 0 0; 0 0 0 0 160
 0; 0 0 0 0 0 160];
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)];
C = eye(6);
D = 0;
R = 0.00001;
[K,P,poles] = lqr(A,B,Q,R)

disp(K);
```

<u>Matlab command window</u>

```
K =

   1.0e+03 *

    0.3162    1.9288    3.0822    2.8429    0.9467    0.4941
```

Linearized system with LQR Controller – Initial condition response

The LQR controller is designed for the linearized system and the response is simulated for the initial conditions of the state variables as follows.

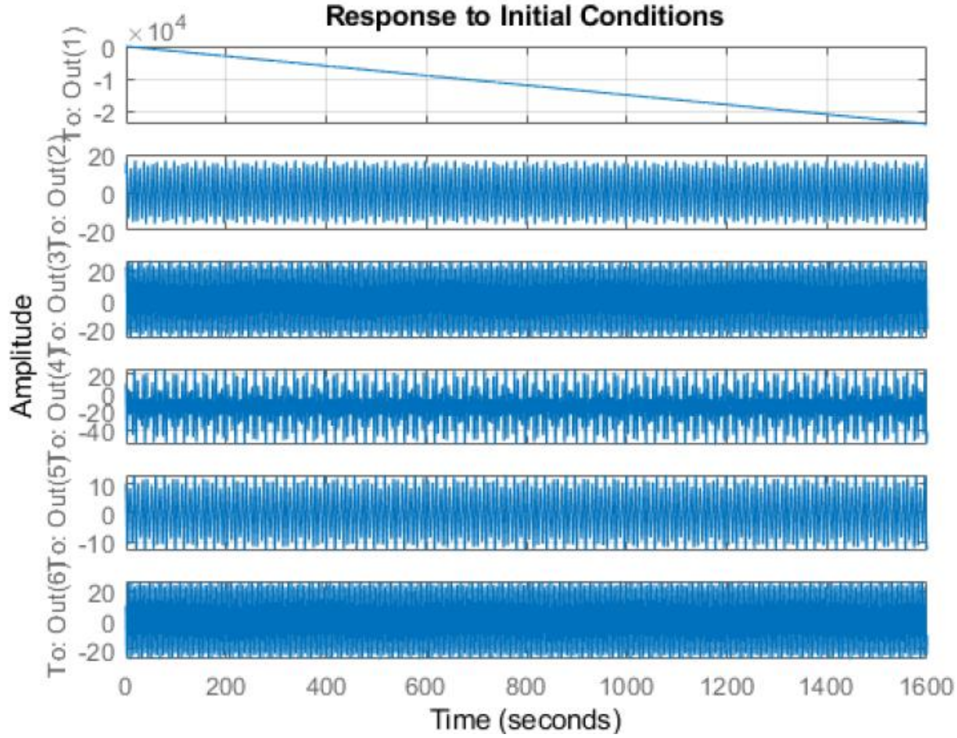
The initial conditions of the system are chosen to be,

X0 = [10, 10, 20, 10, 10, 10]

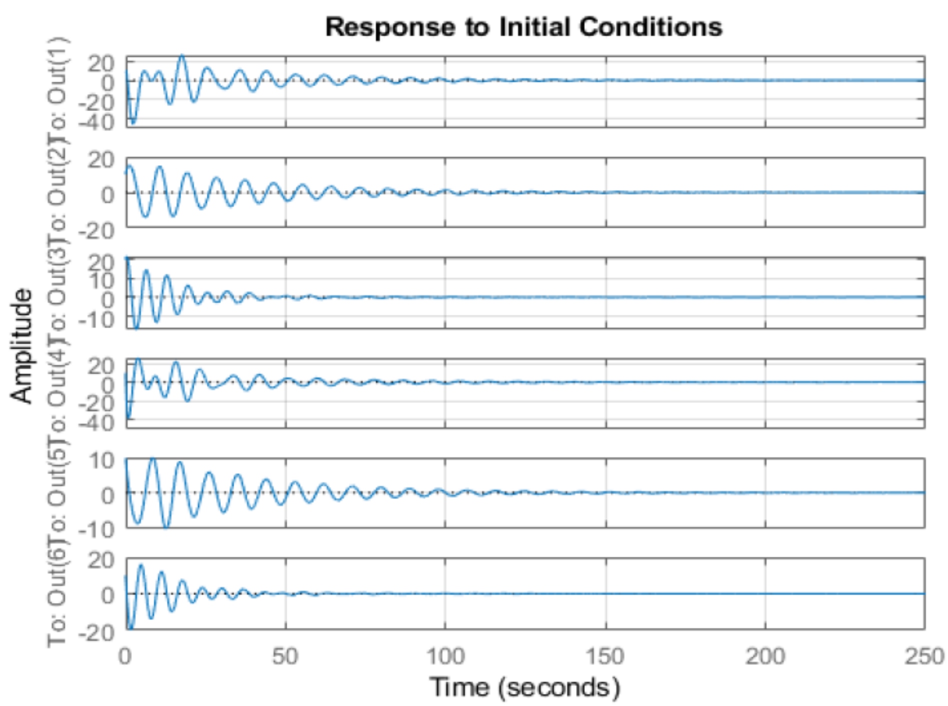
```
%-------------------------------------------------------------%

%First Component - Part  D - LQR controller_Initial conditions
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%-------------------------------------------------------------%

clc;
clear all;
syms m1 m2 l1 l2 g M

%Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.8;

option = 1
%1 for response of the system with Initial conditions & without LQR
%2 for response of the system with Initial conditions & with LQR
%3 for The step response without the LQR controller
%4 for The step response with the LQR controller

%Initial Conditions
x0=[10,10,20,10,10,10];

%Matrices
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
Q = [1 0 0 0 0 0; 0 1 0 0 0 0; 0 0 1 0 0 0; 0 0 0 70 0 0; 0 0 0 0 160
 0; 0 0 0 0 0 160];
```

```
Q = [1 0 0 0 0 0; 0 1 0 0 0 0; 0 0 1 0 0 0; 0 0 0 70 0 0; 0 0 0 0 160
 0; 0 0 0 0 0 160];
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)];
C = eye(6);
D = 0;
R = 0.00001;
[K,P,poles] = lqr(A,B,Q,R)



disp(K);

%Response with the Initial conditions & without LQR
if option == 1
    sys = ss(A,B,C,D);
    initial(sys,x0)
end

%Response with the Initial conditions & with LQR
if option == 2
    [K,P,poles] = lqr(A,B,Q,R)
    sys = ss(A-(B*K),B,C,D);
    initial(sys,x0)
end

%The step response without the LQR controller
if option == 3
    sys = ss(A,B,C,D);
    t=0:0.01:250;
    step(sys,t)
    hold on;
end

%The step response with the LQR controller
if option == 4
    [K,P] = lqr(A,B,Q,R);
    sys = ss(A-(B*K),B,C,D);
    t=0:0.01:250;
    step(sys,t)

end
```

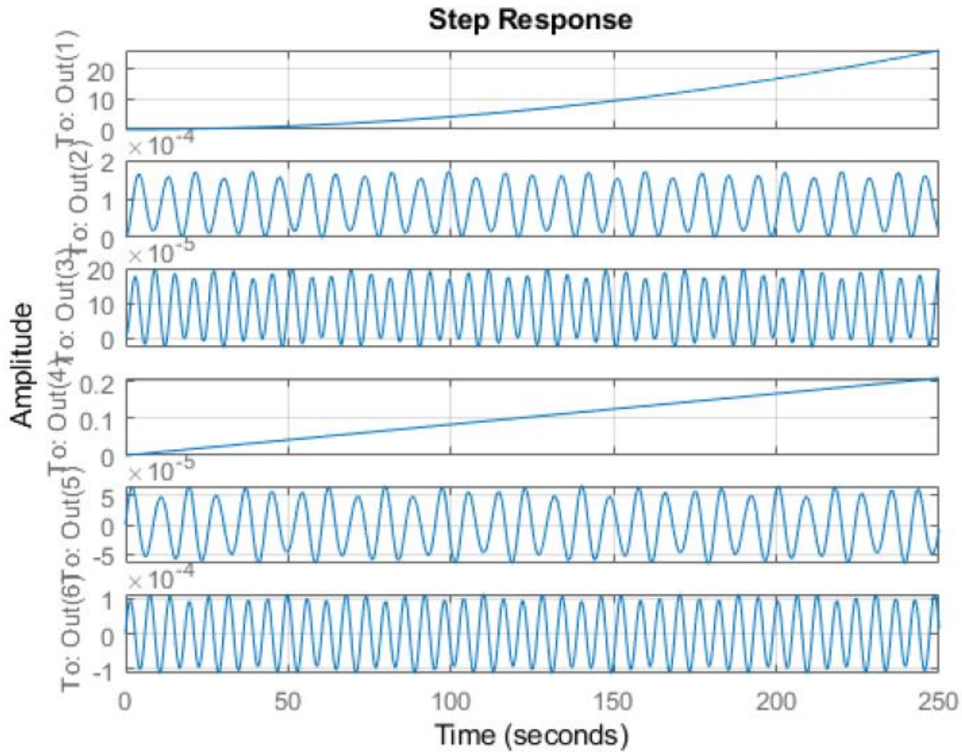
Response to Initial conditions without an LQR Controller to the Linearized system



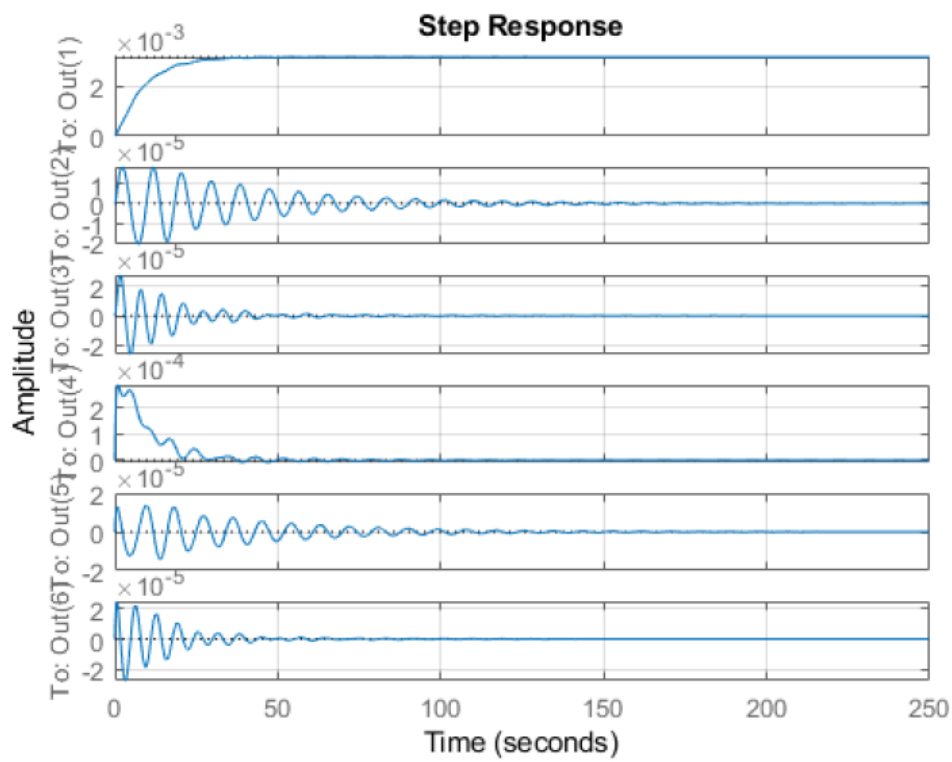
Response to Initial conditions without an LQR Controller to the Linearized system



Response to unit step input without an LQR Controller to the Linearized system



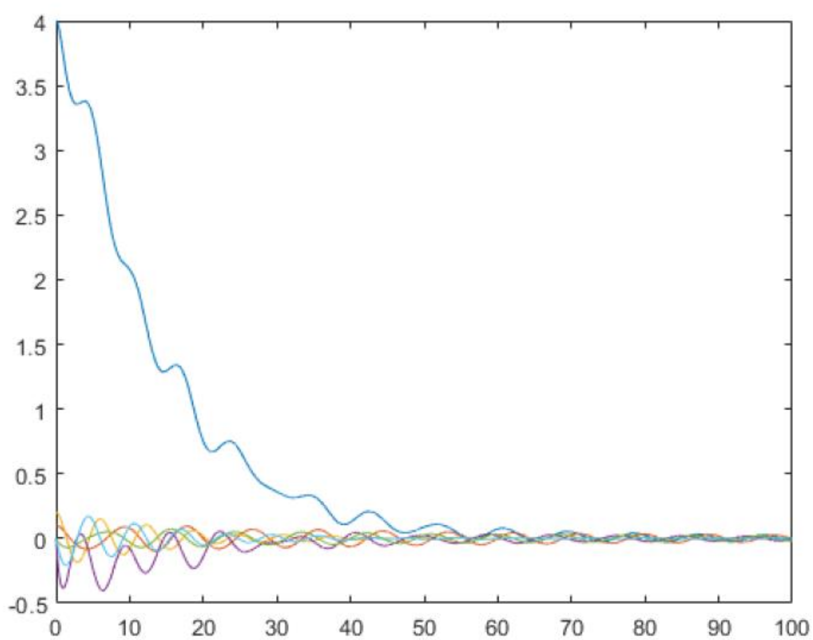
Response to unit step input with an LQR Controller to the Linearized system



Non-Linear system with LQR Controller – Initial condition response

```
%-------------------------------------------------------------%

%First Component
%Part  D - Non linear system with LQR controller
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%-------------------------------------------------------------%

clc;
clear all;

x_0=[4;0.10;0.2;0;0;0];
t=0:0.01:100;

 [t,y] = ode45(@non_linear,t,x_0);
plot(t,y);


function diff=non_linear(t,x)

%Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.8;


%Matrices
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
Q = [1 0 0 0 0 0; 0 10 0 0 0 0; 0 0 150 0 0 0; 0 0 0 150 0 0; 0 0 0 0
 10 0; 0 0 0 0 0 300];
```

```
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)];
C = [0 0 0.01 0 0.001 0];
D = 0;
R = 0.0001;

K =lqr(A,B,Q,R);
U=-K*x;

diff=zeros(6,1);
diff(1)= x(4);

diff(2)= x(5);

diff(3)= x(6);

diff(4)=((1/(M+m1+m2-m1*(cos(x(2))^2)-m2*(cos(x(3))^2)))*(U-
(m1*l1*sin(x(2))*(x(5)^2))-(m2*l2*sin(x(3))*(x(6)^2))-
m1*cos(x(2))*g*sin(x(2))-m2*cos(x(3))*g*sin(x(3))));

diff(5)=(1/l1)*((cos(x(2))*(((1/(M+m1+m2-m1*(cos(x(2))^2)-
m2*(cos(x(3))^2)))*(U-(m1*l1*sin(x(2))*(x(5)^2))-
(m2*l2*sin(x(3))*(x(6)^2))-m1*cos(x(2))*g*sin(x(2))-
m2*cos(x(3))*g*sin(x(3))))))-g*sin(x(2)));

diff(6)=(1/l2)*((cos(x(3))*(((1/(M+m1+m2-m1*(cos(x(2))^2)-
m2*(cos(x(3))^2)))*(U-(m1*l1*sin(x(2))*(x(5)^2))-
(m2*l2*sin(x(3))*(x(6)^2))-m1*cos(x(2))*g*sin(x(2))-
m2*cos(x(3))*g*sin(x(3))))))-g*sin(x(3)));
```



Inference

Response of the linear system and non - linear system with initial conditions and unit step input is plotted with and without the controller. It can be inferred that the response of the linear system becomes stable with the application of an LQR controller in both the cases. Thus, the LQR controlled designed with the optimal gain K stabilizes the closed loop system (A+$B_k K$)

Lyapunov Indirect Method – Stability test

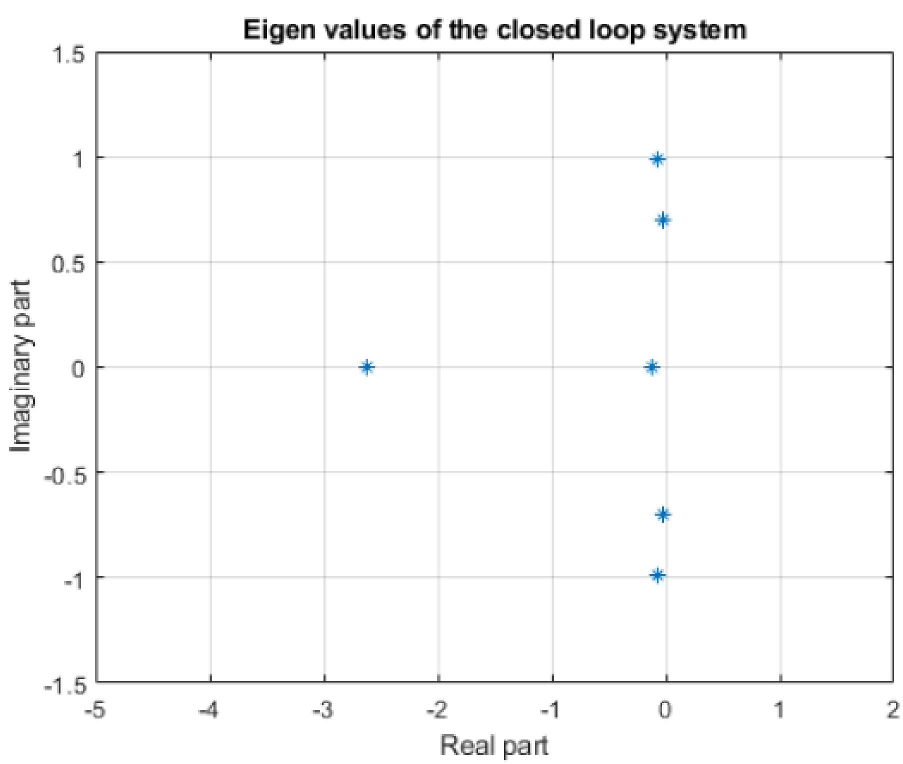
The linearized system is stabilized around the equilibrium point of interest and it is found that the Eigen values of the closed loop system (A+BK) have negative real part. It can be inferred that the system is atleast locally stable around this equilibrium point.

```
%--------------------------------------------------------------%

%First Component - Part  D - Lyapunov Indirect stability test
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%--------------------------------------------------------------%

clc;
clear all;
syms m1 m2 l1 l2 g M

%Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.8;

%Matrices
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
Q = [1 0 0 0 0 0; 0 1 0 0 0 0; 0 0 1 0 0 0; 0 0 0 70 0 0; 0 0 0 0 160
 0; 0 0 0 0 0 160];
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)];
C = eye(6);
D = 0;
```

```
R = 0.00001;
[K,P,poles] = lqr(A,B,Q,R)

eigen=eig(A-B*K)
plot(real(eigen),imag(eigen),'*')
xlim([-5,2])
ylim([-1.5 1.5])
xlabel('Real part')
ylabel('Imaginary part');
title('Eigen values of the closed loop system')
grid on
```

<u>Matlab Command Window</u>

```
eigen =

  -2.6224 + 0.0000i
  -0.1197 + 0.0000i
  -0.0724 + 0.9883i
  -0.0724 - 0.9883i
  -0.0264 + 0.7014i
  -0.0264 - 0.7014i
```



2. E Observability Test for different output vectors on the linearized system

The different output vectors are,

$$C(1) = [1\ 0\ 0\ 0\ 0\ 0]\ corresponding\ to\ x(t)$$

$$C(2) = [0\ 1\ 0\ 0\ 0\ 0; 0\ 0\ 1\ 0\ 0\ 0]\ corresponding\ to\ (\theta_1(t), \theta_2(t))$$

$$C(3) = [1\ 0\ 0\ 0\ 0\ 0; 0\ 0\ 1\ 0\ 0\ 0]\ corresponding\ to\ (x(t), \theta_2(t))$$

$$C(4) = [1\ 0\ 0\ 0\ 0\ 0; 0\ 1\ 0\ 0\ 0\ 0; 0\ 0\ 1\ 0\ 0\ 0] corresponding\ to\ x(t), \theta_1(t))\theta_2(t)$$

We know that, if the pair (A^T, C^T) is controllable, then we say that (A, C) is observable.

For the time – invariant case the state equation is observable if and only if the $(npxn)$ observability matrix satisfies,

$$\text{Rank } [C^T\ A^T C^T \ldots\ldots (A^T)^{n-1} C^T] == n$$

In our case, the rank of the observability matrix should be equal to 6, for the system to be observable.

```
%-------------------------------------------------------------%

%First Component - Part  E - Observability check%
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/13/2019%

%-------------------------------------------------------------%

clc;
clear all;

%Value of Parameters
m1=100;
m2=100;
l1=20;
l2=10;
g=9.8;%m/s2
M=1000;
```

```
%System State Matrix
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 (-m1*g)/M (-m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) (-m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) (-g*(M
+m2))/(M*l2) 0 0 0];
B=[0;0;0;1/M;1/(M*l1);1/(M*l2)];
C1 = [1 0 0 0 0 0];
C2 = [0 0 0 0 0 0;0 1 0 0 0 0;0 0 1 0 0 0];
C3 = [1 0 0 0 0 0;0 0 0 0 0 0;0 0 1 0 0 0];
C4 = [1 0 0 0 0 0;0 1 0 0 0 0;0 0 1 0 0 0];

%Observability Matrices for different output vectors
O1 = [C1; (C1*A);(C1*A*A);(C1*A*A*A);(C1*A*A*A*A);(C1*A*A*A*A*A)];
O2 = [C2; (C2*A);(C2*A*A);(C2*A*A*A);(C2*A*A*A*A);(C2*A*A*A*A*A)];
O3 = [C3; (C3*A);(C3*A*A);(C3*A*A*A);(C3*A*A*A*A);(C3*A*A*A*A*A)];
O4 = [C4; (C4*A);(C4*A*A);(C4*A*A*A);(C4*A*A*A*A);(C4*A*A*A*A*A)];

%Rank of the Observability Matrices
if(rank(O1))==6
    disp('The system is observable for the output vector x(t)');
else
    disp('The system is not observable for the output vector x(t)');
end
disp(' ');
if(rank(O2))==6
    disp('The system is observable for the output vector
 theta1(t),theta2(t)');
else
    disp('The system is not observable for the output vector
 theta1(t),theta2(t)');
end
disp(' ');
if(rank(O3))==6

    disp('The system is observable for the output vector
 x(t),theta2(t)');
else
    disp('The system is not observable for the output vector
 x(t),theta2(t)');
end
disp(' ');
if(rank(O3))==6
    disp('The system is observable for the output vector
 x(t),theta1(t),theta2(t)');
else
    disp('The system is not observable for the output vector
 x(t),theta1(t),theta2(t)');
end
```

Matlab Command Window

```
The system is observable for the output vector x(t)

The system is not observable for the output vector theta1(t),theta2(t)

The system is observable for the output vector x(t),theta2(t)

The system is observable for the output vector
 x(t),theta1(t),theta2(t)
```

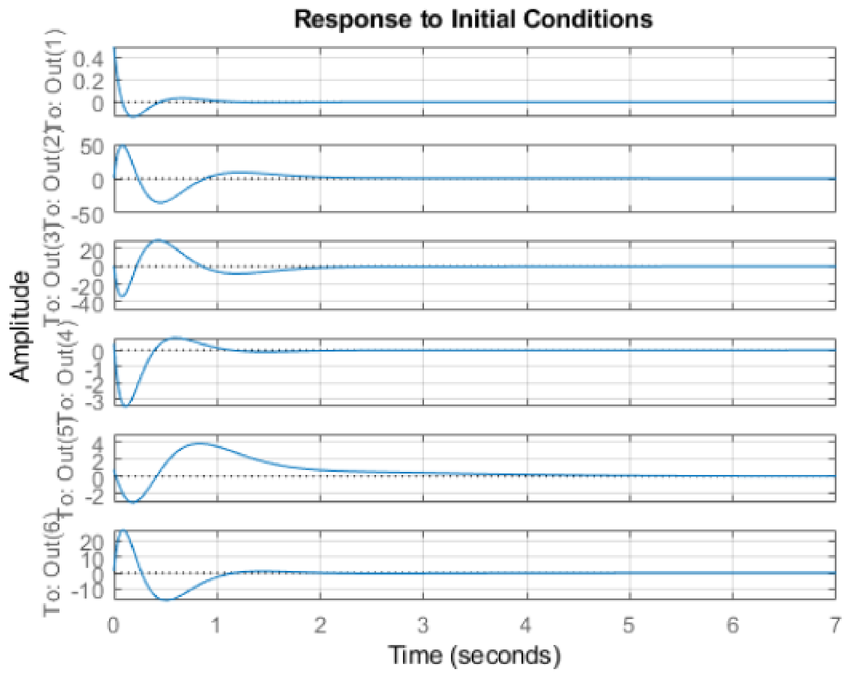
2.F Luenberger Observer

1. Linear system with x(t) as output vector (C(t) = [1 0 0 0 0 0])

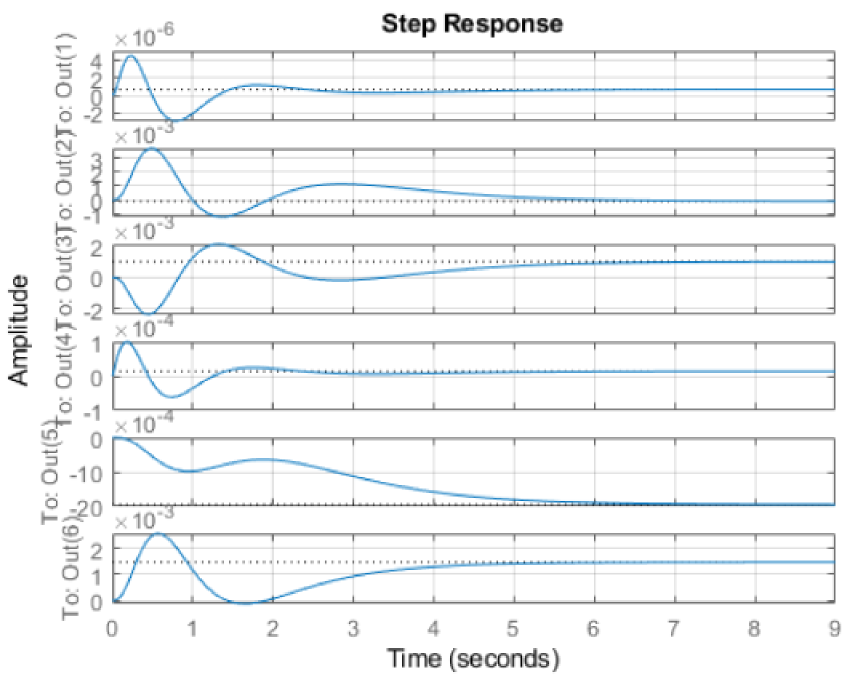
```
%--------------------------------------------------------------%

%Second Component - Part  F - luenberger Observer_Linear system
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%--------------------------------------------------------------%
syms m1 m2 l1 l2 g M

%Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.8;
poles=[-2,-1,-4,-5,-6,-3]; %Defining Eigen values with neg real part
poles=transpose(poles);

%Initial Conditions
x01=[0,10,20,0,0,0,0,0,0,0,0,0];

option = 1;
%1 - Response of the observer taking the error to zero with initial
 conditions
%2 - Response of the observer taking the error to zero with unit step
 input
%3 - Response with the Initial conditions of the entire system
%4 - Response with the input step signal of the entire system
```

```
%Matrices
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
Q = [1 0 0 0 0 0; 0 10 0 0 0 0; 0 0 150 0 0 0; 0 0 0 150 0 0; 0 0 0 0
 10 0; 0 0 0 0 0 300];
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)];
C = [1 0 0 0 0 0]; %First Output vector
D = 0;
R = 0.0001;
C1=eye(6);
K=lqr(A,B,Q,R);
L = place(A',C',poles)'


%Response of the observer taking the error to zero with initial
 conditions
if option == 1
    sys =ss((A-L*C),B,C1,D);
    initial(sys,x0);


end

%Response of the observer taking the error to zero with unit step
 input
if option == 2
    sys =ss((A-L*C),B,C1,D);
    step(sys);
end

%Response with the Initial conditions of the entire system
if option == 3
    sys = ss([(A-B*K) B*K; zeros(size(A)) (A-L*C)],
 [B;zeros(size(B))],[C1 zeros(size(C1))], [0]);
    initial(sys,x01)
end

%Response with the input step signal of the entire system
if option == 4
    sys = ss([(A-B*K) B*K; zeros(size(A)) (A-L*C)],
 [B;zeros(size(B))],[C1 zeros(size(C1))], [0]);
    step(sys)
end

grid on
```

a) Response of the observer system with Initial conditions driving the error to zero with x(t) as the output vector

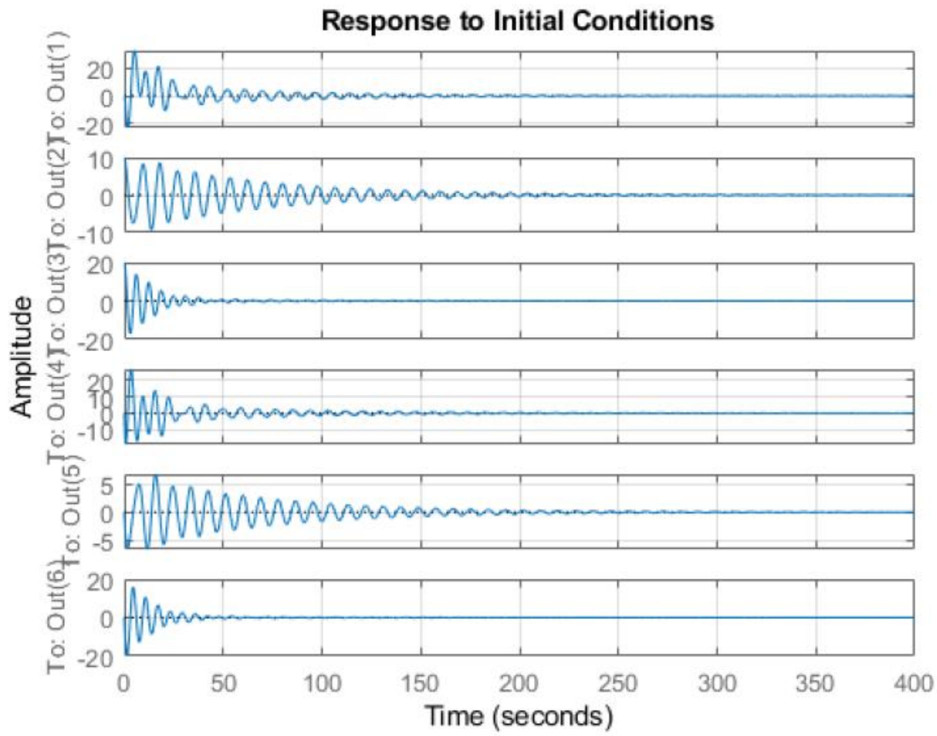


b) Response of the observer system with unit step input, driving the error to zero with x(t) as the output vector

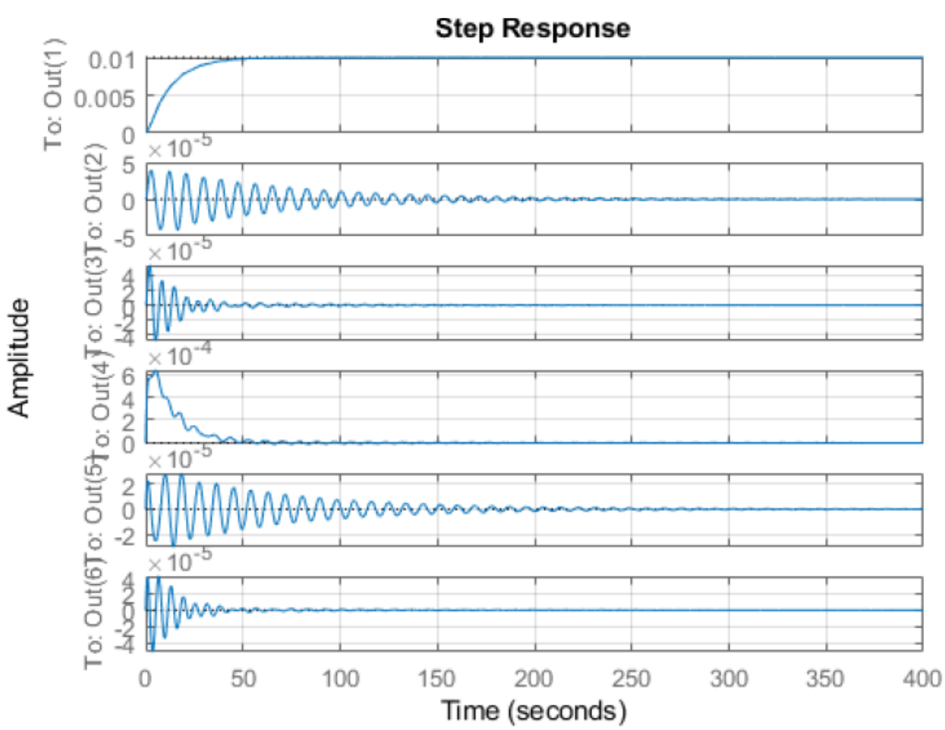


—

c) Response of the entire system with the LQR controller and the Luenberger Observer under initial conditions with x(t) as the output vector

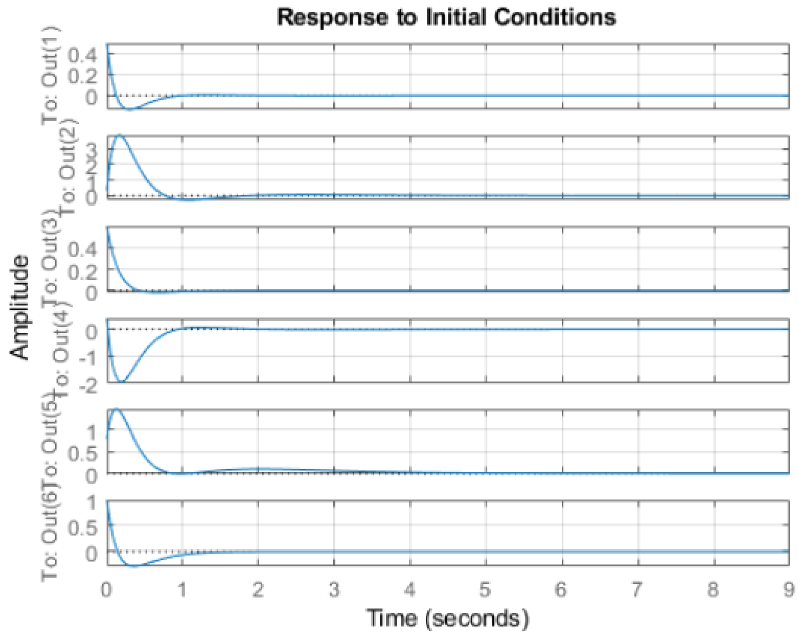


d) Response of the entire system with the LQR controller and the Luenberger Observer under unit step input with x(t) as the output vector

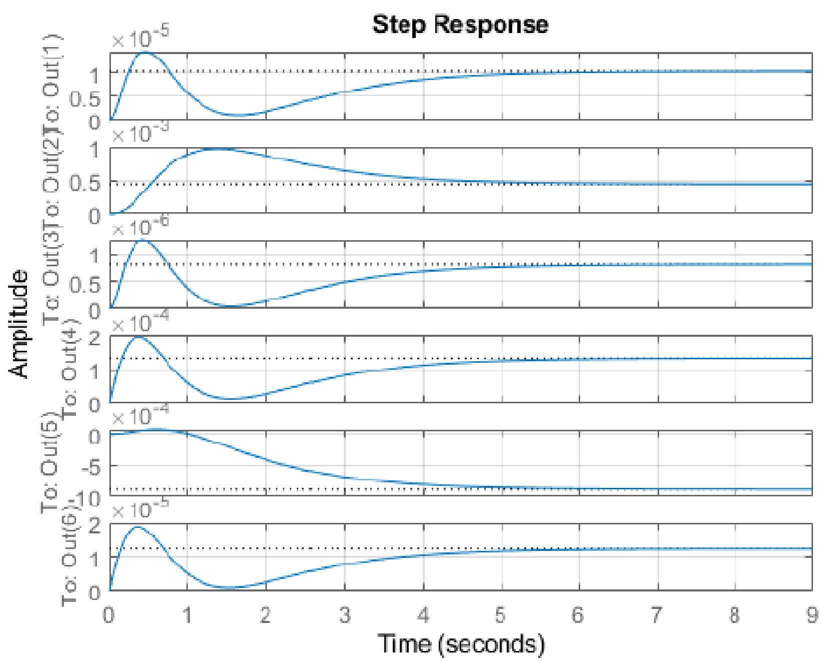


2. Linear system with $(x(t), \theta_2(t))$ as output vector (C(t) = [1 0 0 0 0 0; 0 0 1 0 0 0])

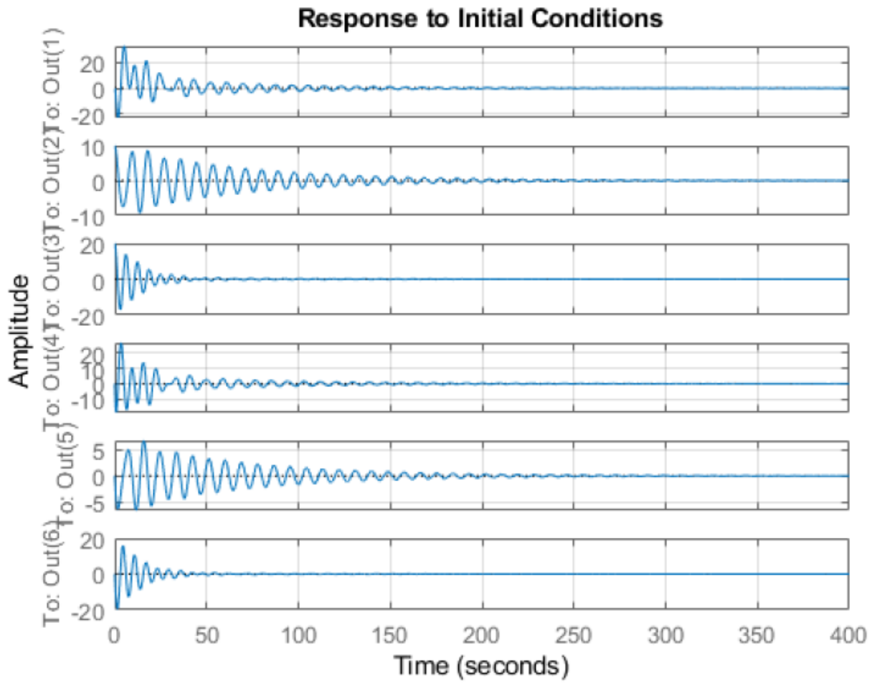
a) Response of the observer system with Initial conditions driving the error to zero with $(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{\theta}_2(\boldsymbol{t}))$ as the output vector



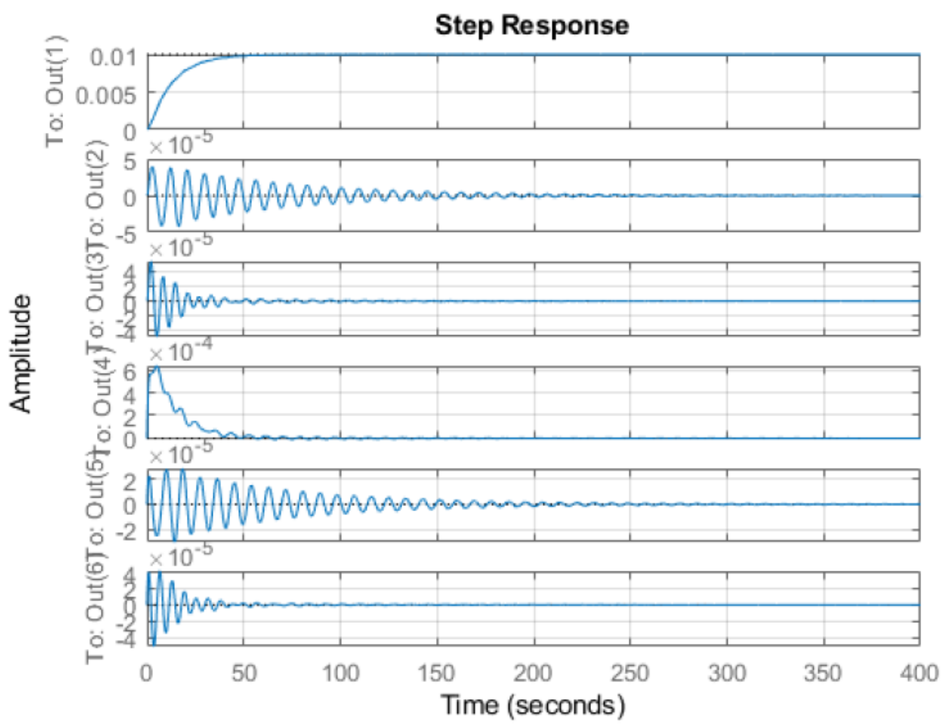
b) Response of the observer system with unit step input, driving the error to zero with $(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{\theta}_2(\boldsymbol{t}))$ as the output vector



c) Response of the entire system with the LQR controller and the Luenberger Observer under initial conditions with $(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{\theta}_2(\boldsymbol{t}))$ as the output vector

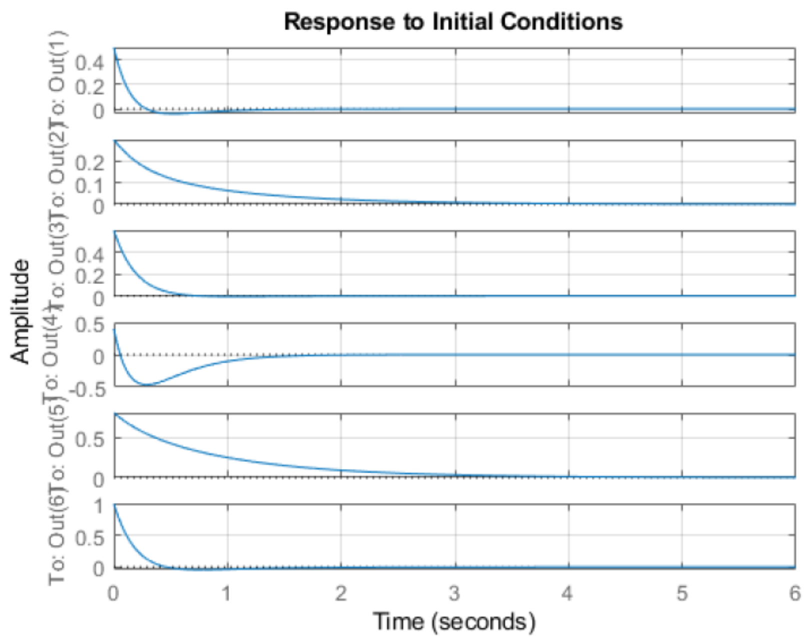


d) Response of the entire system with the LQR controller and the Luenberger Observer under unit step input with $(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{\theta}_2(\boldsymbol{t}))$ as the output vector

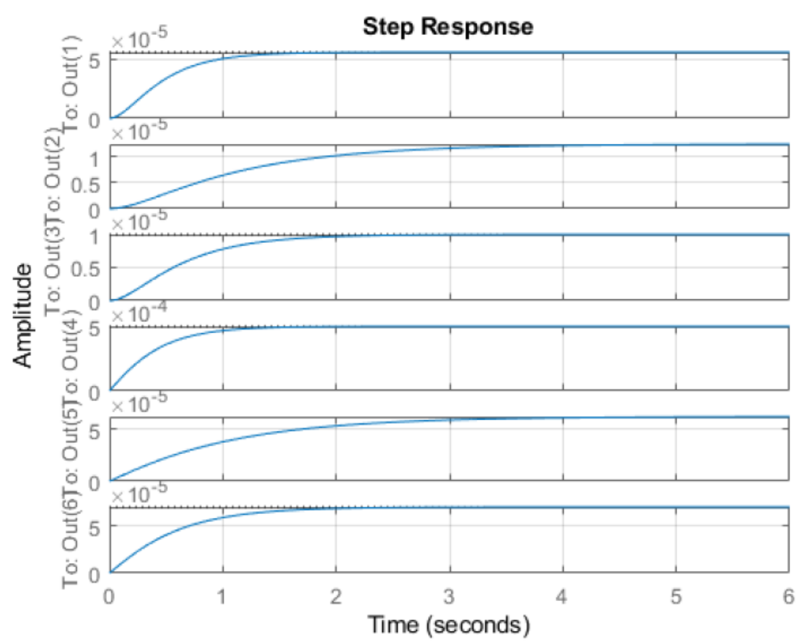


3. Linear system with $(x(t), \theta_1(t), \theta_2(t))$ as output vector (C(t) = [1 0 0 0 0 0; 0 1 0 0 0 0; 0 0 1 0 0 0])

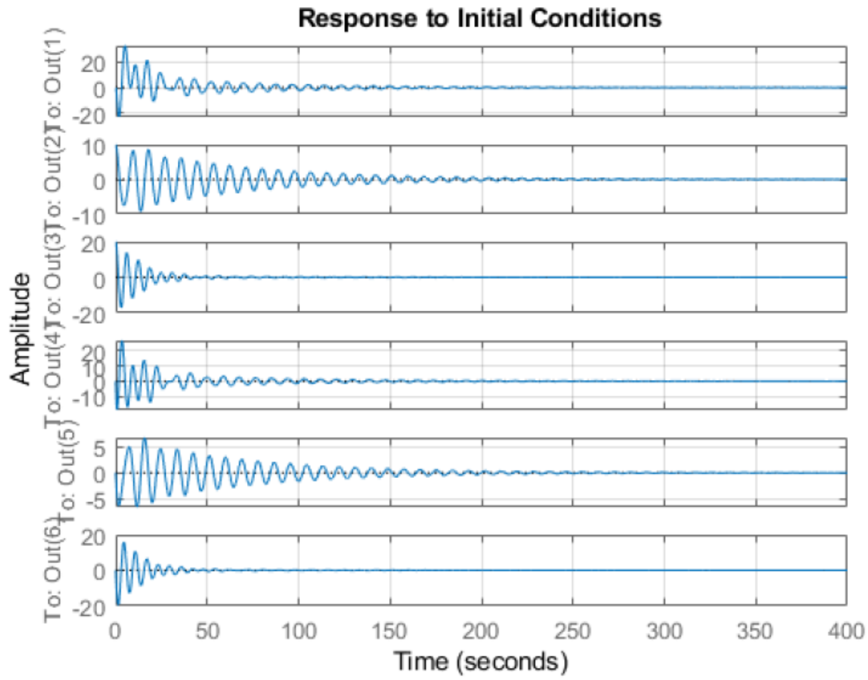
a) Response of the observer system with Initial conditions driving the error to zero with $(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{\theta_1}(\boldsymbol{t}), \boldsymbol{\theta_2}(\boldsymbol{t}))$ as the output vector



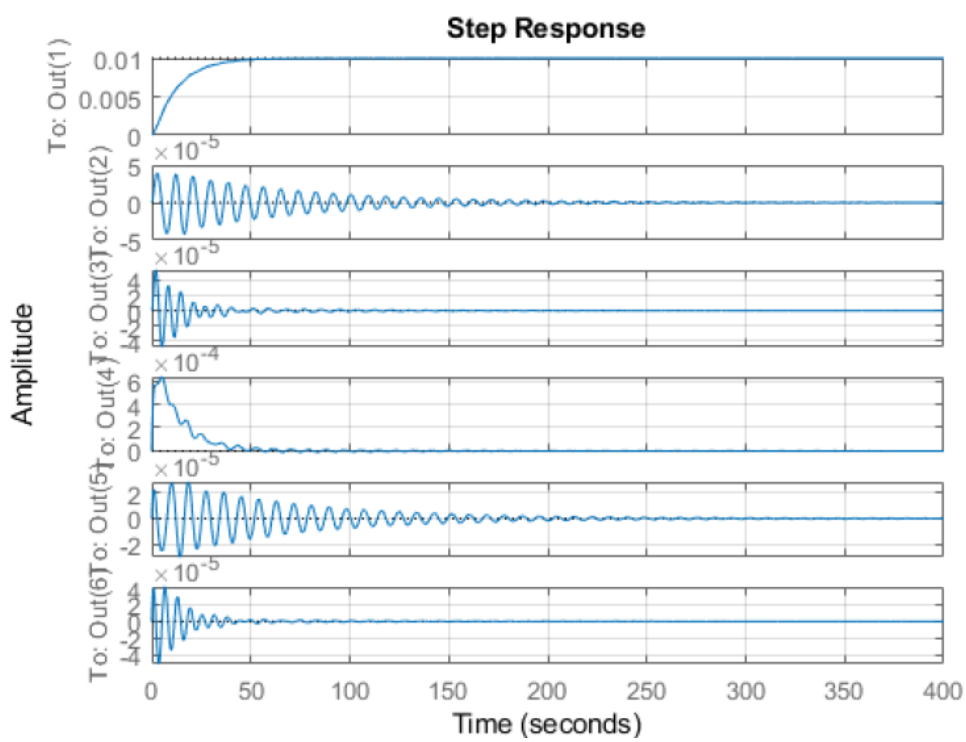
b) Response of the observer system with unit step input, driving the error to zero with $(\boldsymbol{x}(\boldsymbol{t}), \boldsymbol{\theta_1}(\boldsymbol{t}), \boldsymbol{\theta_2}(\boldsymbol{t}))$ as the output vector



c) Response of the entire system with the LQR controller and the Luenberger Observer under initial conditions with $(\boldsymbol{x(t)}, \boldsymbol{\theta_1(t)}, \boldsymbol{\theta_2(t)})$ as the output vector



d) Response of the entire system with the LQR controller and the Luenberger Observer under unit step input with $(\boldsymbol{x(t)}, \boldsymbol{\theta_1(t)}, \boldsymbol{\theta_2(t)})$ as the output vector

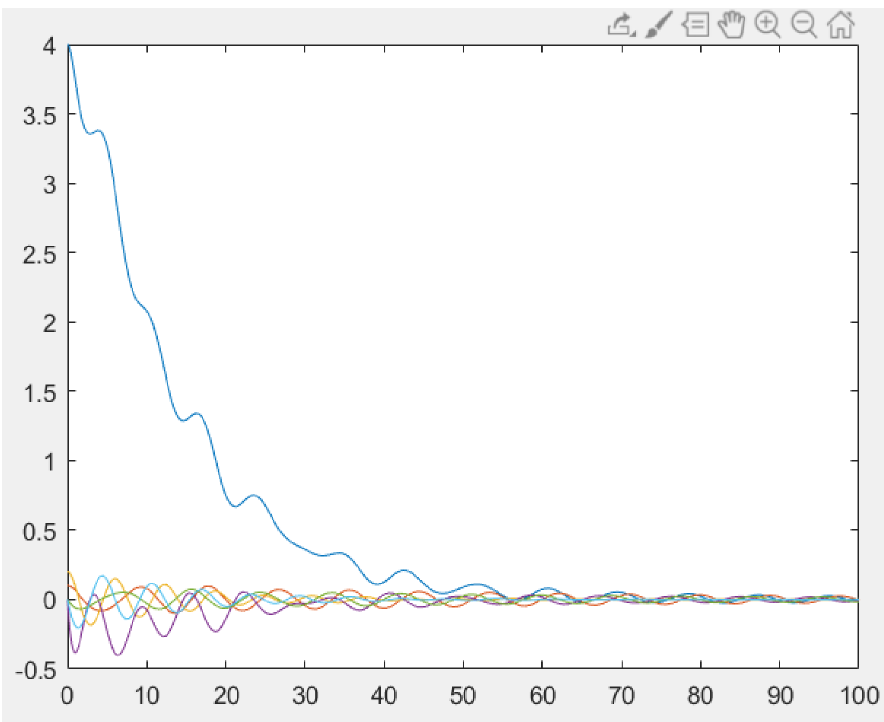


4. Non-Linear system Response with x(t) as output vector (C(t) = [1 0 0 0 0 0])

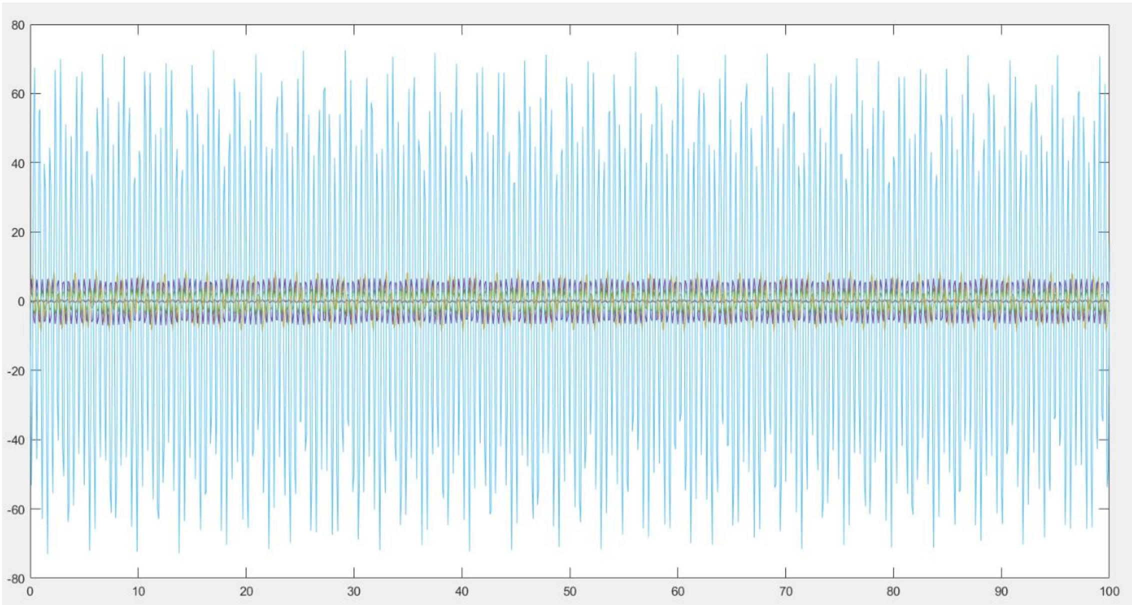
```
%-------------------------------------------------------------%

%First Component - Part  C - Controllability constraints on m1,m2,M,l1,l2%
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%-------------------------------------------------------------%
clc;
clear all;

x0 = [4;0.10;0.2;0;0;0;0.5;0;0;0;0;0];
t=0:0.1:100;

[t,x] = ode45(@nonlinear_lqg,t,x0);
plot(t,x(:,1:6));



function dx=nonlinear_lqg(t,x)
% syms m1 m2 l1 l2 g M F U

%Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.8;
poles=[-2,-1,-4,-5,-6,-3];
poles=transpose(poles);

%Matrices
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
Q = [1 0 0 0 0 0; 0 10 0 0 0 0; 0 0 150 0 0 0; 0 0 0 150 0 0; 0 0 0 0
 10 0; 0 0 0 0 0 300];
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)];
C = [1 0 0 0 0 0];
D = 0;
R = 0.0001;
K =lqr(A,B,Q,R);
U=-K*x(1:6);
```

```
L = place(A',C',poles)'

errordot=(A-L*C)*x(7:12);
dx=zeros(12,1);
dx(1)= x(4);
dx(2)= x(5);
dx(3)= x(6);
dx(4)=((1/(M+m1+m2-m1*(cos(x(2))^2)-m2*(cos(x(3))^2)))*(U-
(m1*l1*sin(x(2))*(x(5)^2))-(m2*l2*sin(x(3))*(x(6)^2))-
m1*cos(x(2))*g*sin(x(2))-m2*cos(x(3))*g*sin(x(3))));
dx(5)=(1/l1)*((cos(x(2))*(((1/(M+m1+m2-m1*(cos(x(2))^2)-
m2*(cos(x(3))^2)))*(U-(m1*l1*sin(x(2))*(x(5)^2))-
(m2*l2*sin(x(3))*(x(6)^2))-m1*cos(x(2))*g*sin(x(2))-
m2*cos(x(3))*g*sin(x(3))))))-g*sin(x(2)));
dx(6)=(1/l2)*((cos(x(3))*(((1/(M+m1+m2-m1*(cos(x(2))^2)-
m2*(cos(x(3))^2)))*(U-(m1*l1*sin(x(2))*(x(5)^2))-
(m2*l2*sin(x(3))*(x(6)^2))-m1*cos(x(2))*g*sin(x(2))-
m2*cos(x(3))*g*sin(x(3))))))-g*sin(x(3)));
dx(7)= x(4)-x(10);
dx(8)= x(5)-x(11);
dx(9)= x(6)-x(12);
dx(10)= dx(4)-errordot(4);
dx(11)= dx(5)-errordot(5);
dx(12)= dx(6)-errordot(6);
```

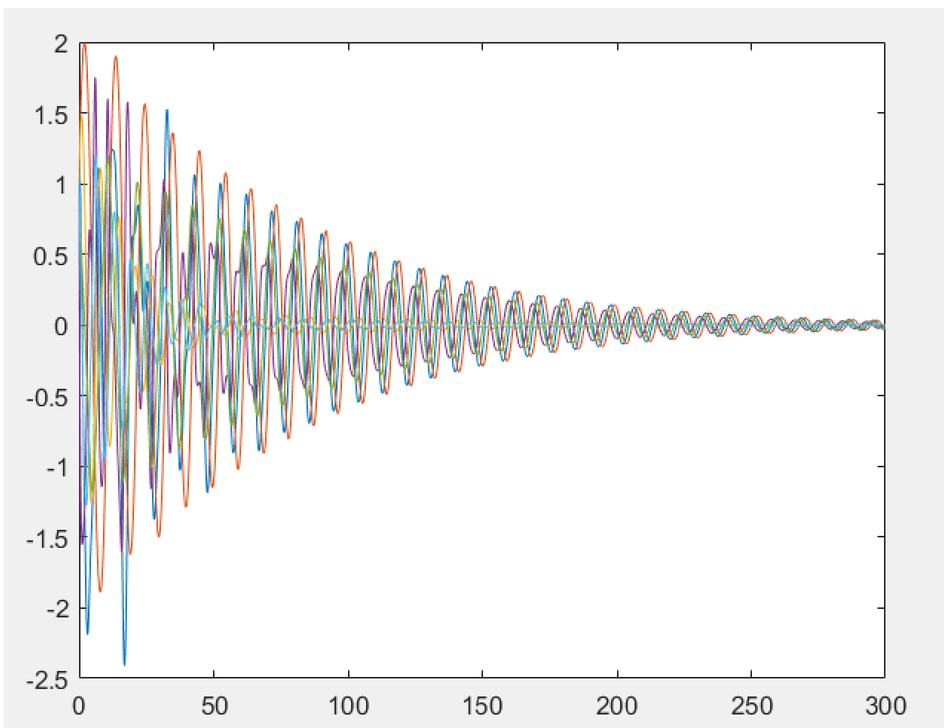
a) Response of the nonlinear system with the LQR controller and the Luenberger Observer under initial conditions with x(t) as the output vector



b) Response of the error calculated under initial conditions with x(t) as output state vector. Error = State – Estimate

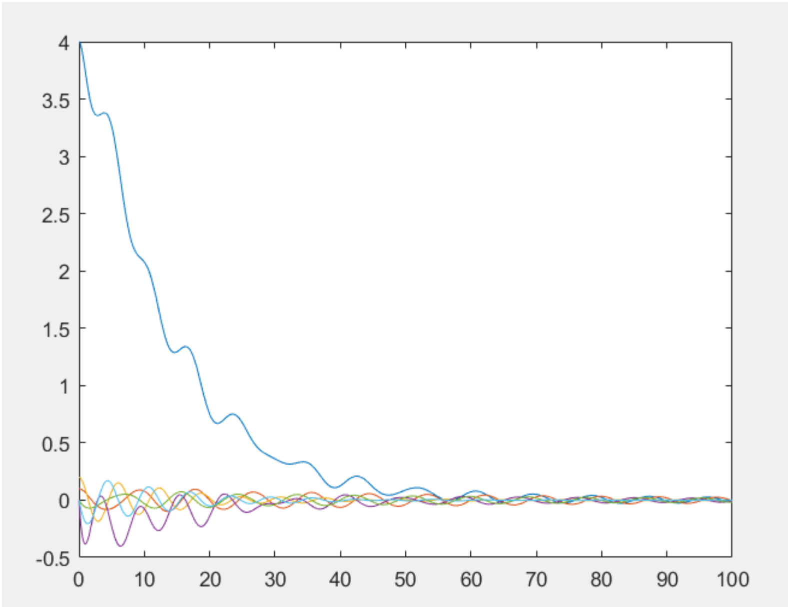


c) Response of the nonlinear system with the LQR controller and the Luenberger Observer under unit step input with x(t) as the output vector

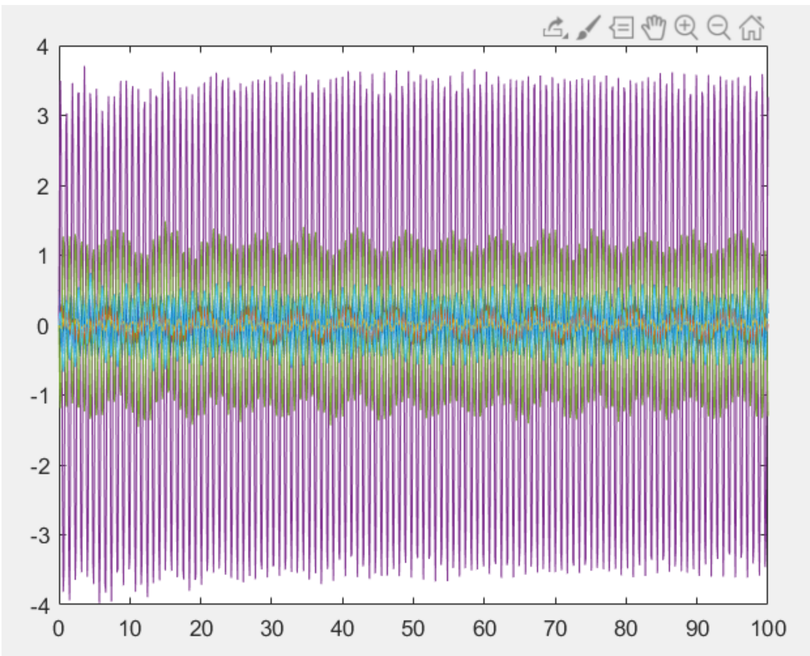


5. Non-Linear system Response with (x(t), θ_2) as output vector (C(t) = [1 0 0 0 0 0; 0 0 1 0 0 0])

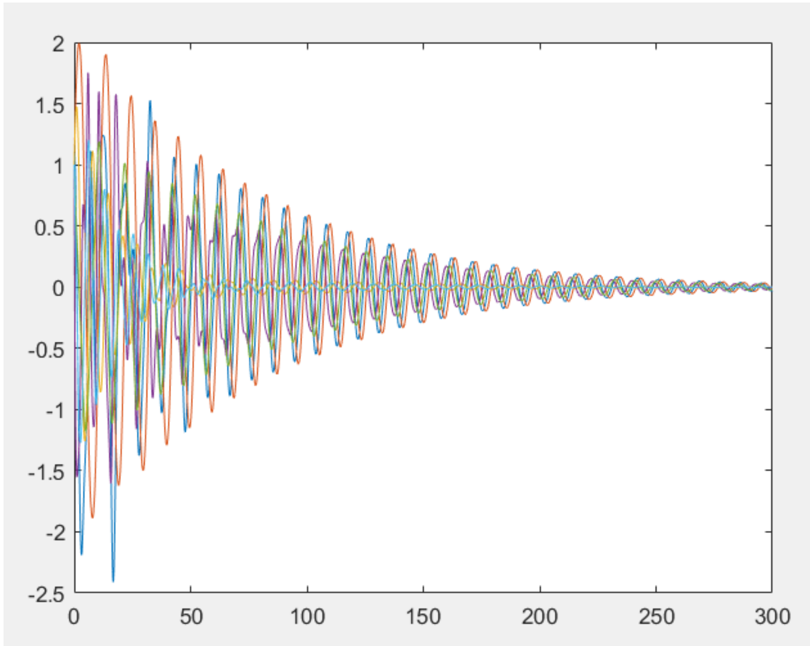
a) Response of the nonlinear system with the LQR controller and the Luenberger Observer under initial conditions with **(x(t), θ_2)** as the output vector



b) Response of the error calculated under initial conditions with **(x(t), θ_2)** as output state vector. Error = State – Estimate

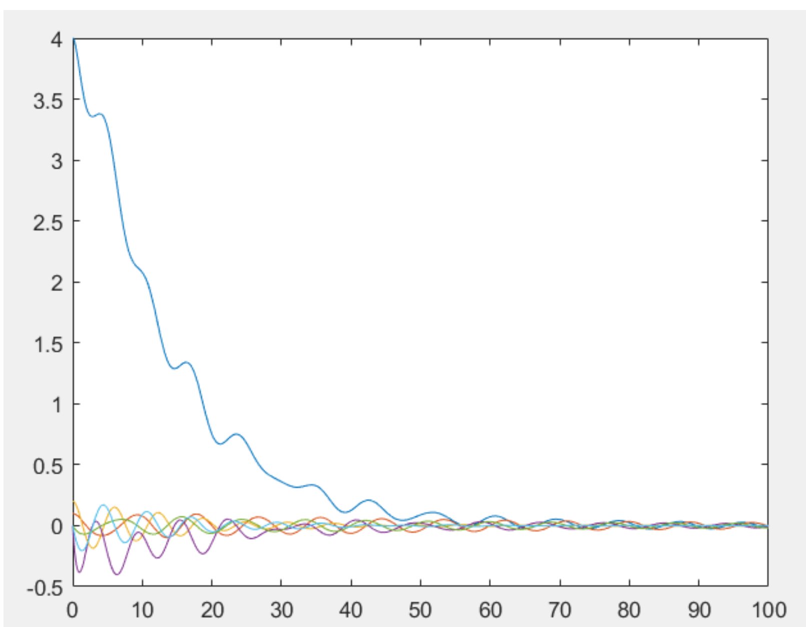


c) Response of the nonlinear system with the LQR controller and the Luenberger Observer under unit step input with **(x(t), $\boldsymbol{\theta_2}$)** as the output vector

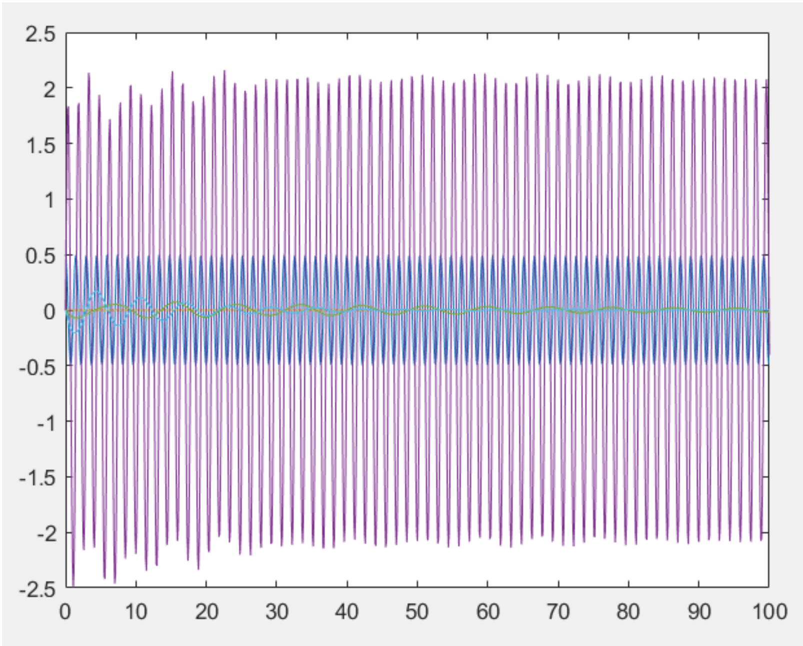


6. Non-Linear system Response with (x(t), $\boldsymbol{\theta_1}$, $\boldsymbol{\theta_2}$) as output vector (C(t) = [1 0 0 0 0 0; 0 1 0 0 0 0;0 0 1 0 0 0])

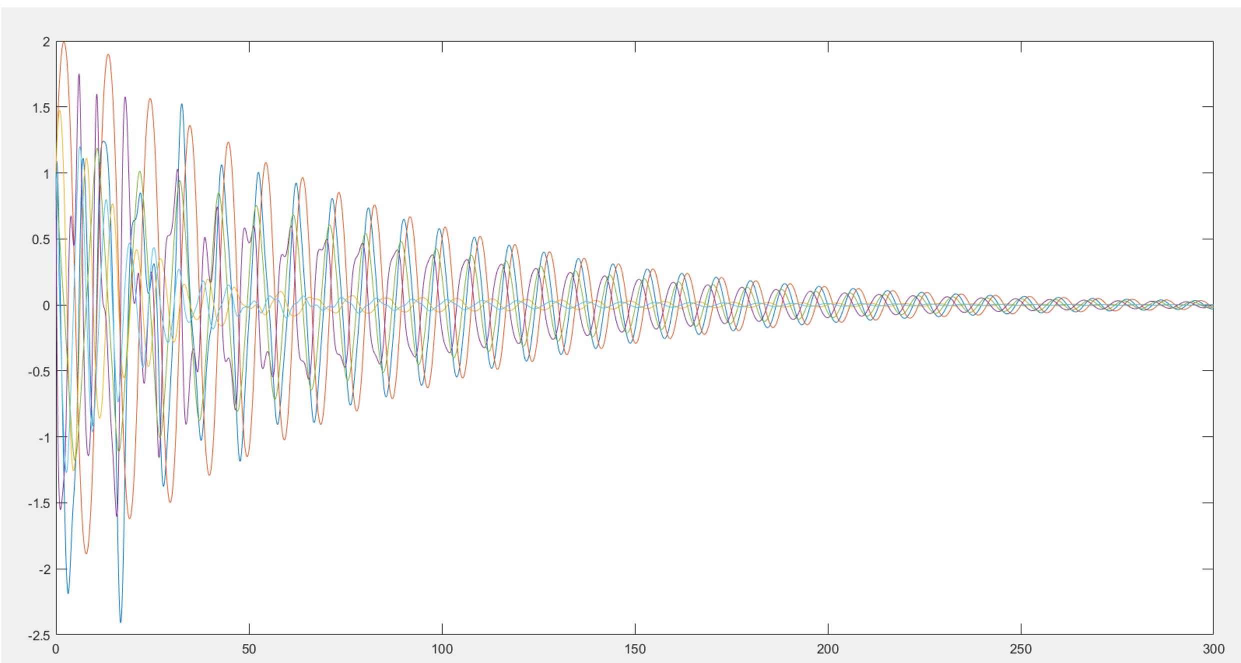
a) Response of the nonlinear system with the LQR controller and the Luenberger Observer under initial conditions with **(x(t), $\boldsymbol{\theta_1}$, $\boldsymbol{\theta_2}$)** as the output vector



b) Response of the error calculated under initial conditions with **(x(t), $\boldsymbol{\theta_1}$, $\boldsymbol{\theta_2}$)** as output state vector. Error = State – Estimate



c) Response of the nonlinear system with the LQR controller and the Luenberger Observer under unit step input with **(x(t), $\boldsymbol{\theta_1}$, $\boldsymbol{\theta_2}$)** as the output vector



2.E Linear Quadratic Gaussian Method

The smallest output vector is chosen to be x(t) and its corresponding vector is

C1(t) = [1 0 0 0 0 0]

The Kalman Bucy Filter is designed with this output vector and the optimal gain K is calculated as follows,

```
%------------------------------------------------------------%

%First Component - Part  C - LQG Controller
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%------------------------------------------------------------%

function dx=nonlinear_lqg(t,x)
% syms m1 m2 l1 l2 g M F U

%Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.8;


%Matrices
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
Q = [1 0 0 0 0 0; 0 10 0 0 0 0; 0 0 150 0 0 0; 0 0 0 150 0 0; 0 0 0 0
 10 0; 0 0 0 0 0 300];
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)]
C = [1 0 0 0 0 0];
D = 0;
R = 0.0001;

K =lqr(A,B,Q,R);
vd=0.3*eye(6);
vn=1;
KFG=lqr(A',C',vd,vn)
```

Matlab Command Window

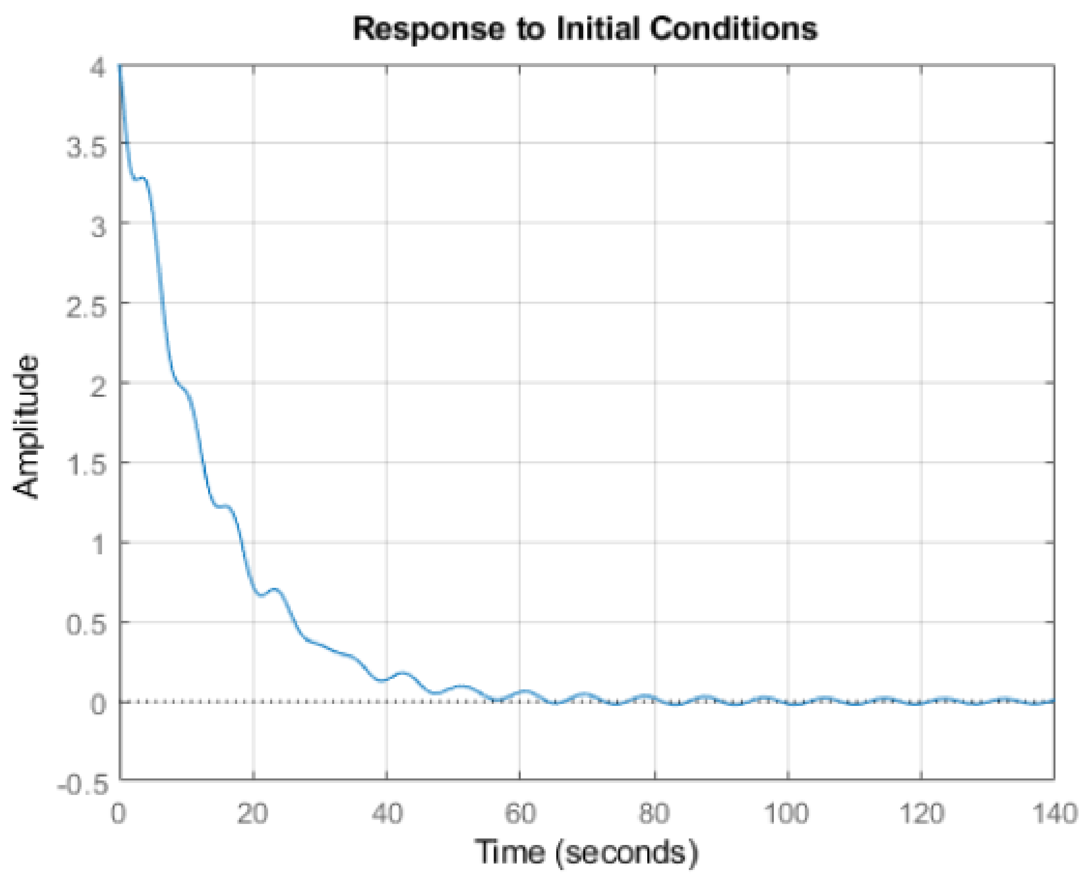
```
KFG =

    2.3383   -0.4225   -0.5185    2.5838    0.6580    0.3946
```

1. Linear system Response with LQG Controller

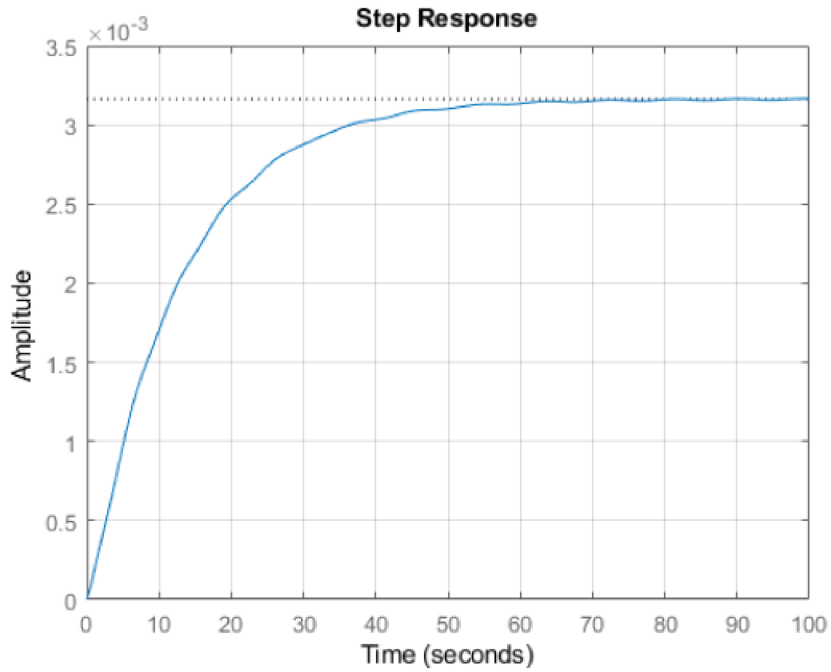
```
%-------------------------------------------------------------%

%First Component - Part  C - LQG Controller_Linear
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%-------------------------------------------------------------%
syms m1 m2 l1 l2 g M

%Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.8;

option= 1;
%Option = 1 Response with the Initial conditions & with LQR
%Option = 2 Response with the unit step input & with LQR

%Initial Conditions
x0 = [4;0.10;0.2;0;0;0;0;0;0;0;0;0];


%Matrices
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
Q = [1 0 0 0 0 0; 0 10 0 0 0 0; 0 0 150 0 0 0; 0 0 0 150 0 0; 0 0 0 0
 10 0; 0 0 0 0 0 300];
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)];
C = [1,0,0,0,0,0];
D = 0;
R = 0.00001;

K =lqr(A,B,Q,R);
vd=0.3*eye(6);
vn=1;
KF=lqr(A',C',vd,vn);
KF=KF';
```

```
%Response with the Initial conditions & with LQR
if option == 1
    sys = ss([(A-B*K) B*K; zeros(size(A)) (A-KF*C)],
 [B;zeros(size(B))],[C zeros(size(C))], [0]);
    initial(sys,x0)
end

%Response with the unit step input & with LQR
if option == 2

   sys = ss([(A-B*K) B*K; zeros(size(A)) (A-KF*C)],
 [B;zeros(size(B))],[C zeros(size(C))], [0]);
   step(sys)
end

grid on
```

a) Linear system Response with LQG Controller under initial conditions



b) Linear system Response with LQG Controller under unit step input

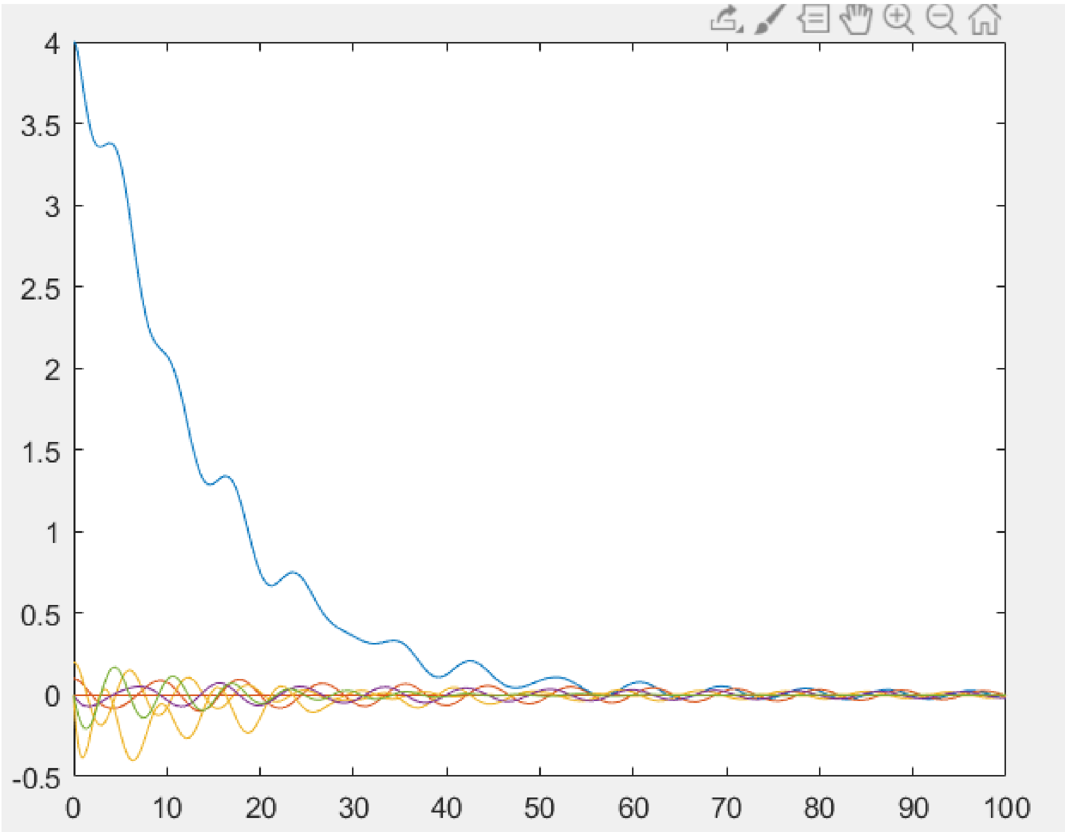


2. Non - Linear system Response with LQG Controller

```
%------------------------------------------------------------%

%First Component - Part  C - LQG Controller_NonLinearsystem
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%------------------------------------------------------------%
clc;
clear all;

init = [4;0.10;0.2;0;0;0;0;0;0;0;0;0];
t=0:0.1:100;

[t,y] = ode45(@nonlinear_lqg,t,init);
plot(t,y(:,:))
```

```
function dx=nonlinear_lqg(t,x)
% syms m1 m2 l1 l2 g M F U

%Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.8;


%Matrices
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
Q = [1 0 0 0 0 0; 0 10 0 0 0 0; 0 0 150 0 0 0; 0 0 0 150 0 0; 0 0 0 0
 10 0; 0 0 0 0 0 300];
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)]
C = [1 0 0 0 0 0];
D = 0;
R = 0.0001;

K =lqr(A,B,Q,R);
vd=0.3*eye(6);
vn=1;
KFG=lqr(A',C',vd,vn)
KFG=KFG';
U=-K*x(1:6);


errordot=(A-KFG*C)*x(7:12);
dx=zeros(12,1);
dx(1)= x(4);
dx(2)= x(5);
dx(3)= x(6);
dx(4)=((1/(M+m1+m2-m1*(cos(x(2))^2)-m2*(cos(x(3))^2)))*(U-
(m1*l1*sin(x(2))*(x(5)^2))-(m2*l2*sin(x(3))*(x(6)^2))-
m1*cos(x(2))*g*sin(x(2))-m2*cos(x(3))*g*sin(x(3))));
dx(5)=(1/l1)*((cos(x(2))*(((1/(M+m1+m2-m1*(cos(x(2))^2)-
m2*(cos(x(3))^2)))*(U-(m1*l1*sin(x(2))*(x(5)^2))-
(m2*l2*sin(x(3))*(x(6)^2))-m1*cos(x(2))*g*sin(x(2))-
m2*cos(x(3))*g*sin(x(3))))))-g*sin(x(2)));

dx(6)=(1/l2)*((cos(x(3))*(((1/(M+m1+m2-m1*(cos(x(2))^2)-
m2*(cos(x(3))^2)))*(U-(m1*l1*sin(x(2))*(x(5)^2))-
(m2*l2*sin(x(3))*(x(6)^2))-m1*cos(x(2))*g*sin(x(2))-
m2*cos(x(3))*g*sin(x(3))))))-g*sin(x(3)));
dx(7)= x(4)-x(10);
dx(8)= x(5)-x(11);
dx(9)= x(6)-x(12);
dx(10)= dx(4)-errordot(4);
dx(11)= dx(5)-errordot(5);
dx(12)= dx(6)-errordot(6);
```

a) Non - Linear system Response with LQG Controller under initial conditions



3. Reconfiguring Controller to track a constant reference on x

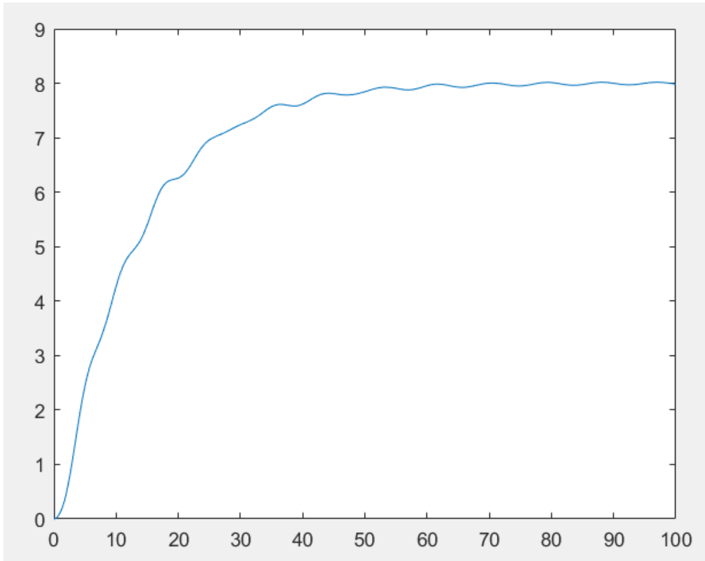
The reference on x is chosen to be 8. Therefore, the input to the system U = -Kx is now altered as follows,

$$U = -K(x - xdesired)$$

```
%-------------------------------------------------------------%

%First Component - Part  C - LQG Controller_NonLinearsystem
%Course - Control of Robotic systems%
%Authors - Pradeep Gopal and Sahana Anbazhagan%
%Date - 12/10/2019%

%-------------------------------------------------------------%
clc;
clear all;

init = [0;0.10;0.2;0;0;0;0;0;0;0;0;0];
t=0:0.1:100;
xdes=[8 0 0 0 0 0]';
[t,y] = ode45(nonlinear_lqg,init,t);
plot(t,y(:,1))


function dx=nonlinear_lqg(t,x)
% syms m1 m2 l1 l2 g M F U

%Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.8;


%Matrices
A = [0 0 0 1 0 0; 0 0 0 0 1 0; 0 0 0 0 0 1; 0 -(m1*g)/M -(m2*g)/M 0 0
 0; 0 (-g*(M+m1))/(M*l1) -(m2*g)/(M*l1) 0 0 0; 0 (-m1*g)/(M*l2) -(g*(M
+m2))/(M*l2) 0 0 0];
Q = [1 0 0 0 0 0; 0 10 0 0 0 0; 0 0 150 0 0 0; 0 0 0 150 0 0; 0 0 0 0
 10 0; 0 0 0 0 0 300];
B = [0; 0; 0; 1/M; 1/(M*l1); 1/(M*l2)];
C = [1 0 0 0 0 0];
D = 0;
R = 0.0001;
xdes=[8 0 0 0 0 0]';
```

```
K =lqr(A,B,Q,R);
vd=0.3*eye(6);
vn=1;
KFG=lqr(A',C',vd,vn);
KFG=KFG';
U= -(K*(x(1:6)-xdes));

errordot=(A-KFG*C)*x(7:12);
dx=zeros(12,1);
dx(1)= x(4);
dx(2)= x(5);
dx(3)= x(6);
dx(4)=((1/(M+m1+m2-m1*(cos(x(2))^2)-m2*(cos(x(3))^2)))*(U-
(m1*l1*sin(x(2))*(x(5)^2))-(m2*l2*sin(x(3))*(x(6)^2))-
m1*cos(x(2))*g*sin(x(2))-m2*cos(x(3))*g*sin(x(3))));
dx(5)=(1/l1)*((cos(x(2))*(((1/(M+m1+m2-m1*(cos(x(2))^2)-
m2*(cos(x(3))^2)))*(U-(m1*l1*sin(x(2))*(x(5)^2))-
(m2*l2*sin(x(3))*(x(6)^2))-m1*cos(x(2))*g*sin(x(2))-
m2*cos(x(3))*g*sin(x(3))))))-g*sin(x(2)));

dx(6)=(1/l2)*((cos(x(3))*(((1/(M+m1+m2-m1*(cos(x(2))^2)-
m2*(cos(x(3))^2)))*(U-(m1*l1*sin(x(2))*(x(5)^2))-
(m2*l2*sin(x(3))*(x(6)^2))-m1*cos(x(2))*g*sin(x(2))-
m2*cos(x(3))*g*sin(x(3))))))-g*sin(x(3)));
dx(7)= x(4)-x(10);
dx(8)= x(5)-x(11);
dx(9)= x(6)-x(12);
dx(10)= dx(4)-errordot(4);
dx(11)= dx(5)-errordot(5);
dx(12)= dx(6)-errordot(6);
end
```

It can be seen that the resultant output shows the x vector successfully tracks the given desired input 8.



4. Constant Force Disturbance

No, the design will not reject constant force disturbance applied on the cart as there is no Integration term as one of the states which will integrate the error and account for the steady state disturbance.

$$X_I(t) = \int_0^t \tilde{X}(\tau)d\tau$$