Historical overview:

1666 - Newton invents ODEs, Solves planetary orbits, 2-body problem

Late 1800 - Poncaire introduces the concept of phase space and shows the problem of solving chaotic systems.

> **Chaos:** Aperiodic, seemingly unpredictable behaviour in deterministic systems that display "sensitive dependence on initial conditions."

1920 - 1950 : Nonlinear oscillators in physics, Engineering - radio, radar, phase-locked loops, lasers.

1950 : Computer power rapidly improved.

1960's : Lorenz @ MIT - chaotic system of convection rolls in the atmosphere (Famous paper published in meteorological journal "Deterministic aperiodic flow.")

* - Work by Smale, KAM (Pure mathematical work) on Chaos.

1975 : May, a biologist, notices chaos in iterated maps $x_{n+1} = f(x_n)$
- Wrote a paper, "Complicated dynamics in simple dynamic systems."

Mandlebrot - developed fractals...
Winfree - non-linear oscillators in biology.
Ruelle and Tackens - Link between chaos and turbulence.

1978 : Figenbaum discovered universal route to chaos for completely unrelated system (connections to phase transitions in statistical physics.)
↓
Renormalization.

1980s : Chaos, non-linear dynamics, fractals get HOT!

1990s : Engineering applications of chaos, subject starts to peak and drift to systems with many variables called complex systems.

2000s : Networks...

Logical map of dynamics:

- Differential eq^{n}s:

$$\vec{\dot{x}} = \vec{f}(x)$$
$$x \in \mathbb{R}^n, \quad \vec{x} = [x_1, x_2, \dots x_n]^T$$

$$\dot{x}_1 = f_1(x_1, x_2, \dots x_n)$$
$$\dot{x}_2 = f_2(x_1, x_2, \dots x_n)$$
$$\vdots$$
$$\dot{x}_n = f_n(x_1, x_2, \dots x_n)$$

A system is linear if all x_i on RHS apper only to the first power. (no products, x_2^2, $x_1 x_2$ ...) } Other systems are called non-linear.

- Only considering autonomous system. } Better to visualising phase space geometry.

Example: Basic harmonic oscillator

$$m\ddot{x} + kx = 0, \quad \text{Let } z_1 = x, \quad z_2 = \dot{x}$$

$$\dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} \dot{z}_0 \\ -kx \end{bmatrix} \quad \text{(linear)}.$$

Suppose the solution to a system is known: We have $x_1(t), x_2(t)$. There could be a point that moves along a (x_2, x_1) space, tracing out ~~the~~ a trajectory.

Without solving the differential eq^n analytically we can qualitatively plot all the different trajectories, called a phase portrait.

	n = 1	n = 2	n = 3	n = a lot	n = ∞
Linear	RC	SHM			wave eq^n, EM, Schrodinger eq^n
Non-linear	Logistic	Pendulum	fractals, iterated maps, Lorenz (chaos).	Networks, Complex systems... [Not alot known about them...]	General relatively, turbulence fibrilliation..

* Plan for the course are underlined.

1-D systems:

Example: $\dot{x} = \sin(x)$

Traditional way,

$$\int \frac{dx}{\sin x} = \int dt = t + c$$

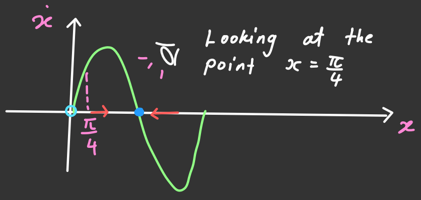
$$-\ln|\csc(x) + \cot(x)| = t + C$$

At $t = 0$, $x = x_0$

$$t = \ln\left|\frac{\csc(x_0) - \cot(x_0)}{\csc(x) - \cot(x)}\right|$$

This result is not that helpful. Suppose $x_0 = \frac{\pi}{4}$, what is the limit of $x(t)$ as $t \to \infty$? Its' not obvious from the sol^n.

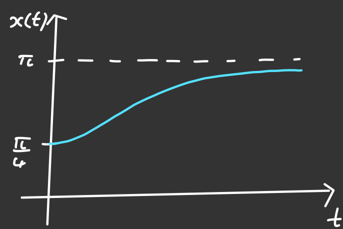
Picture method:



$\dot{x} = \sin(x)$

x at $\frac{\pi}{4}$ has a $\dot{x} > 0$, so x moves to the right. It's changing at the fastest rate when $x = \frac{\pi}{2}$, after which it's rate starts to drop (but still positive.) Eventually the velocity of x, drops to π at π.

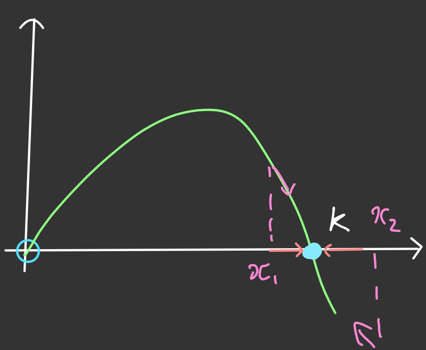
We can sketch this description.



Example: Logistic eq^n

$$\dot{x} = r\left(1 - \frac{x}{k}\right)$$

The phase plot would then be,



Starting on the left of k, the population would increase until it reaches k.
Starting on the right it would decrease...

Linearization: Suppose we want to write the dynamics around a fixed point x^*, $f(x^*) = 0$.

$$x(t) = x^* + \eta(t)$$

$$\dot{x} = \frac{d}{dt}(x^* + \eta(t)) = \dot{\eta}(t)$$

$$f(x) = \underbrace{f(x^* + \eta(t))}_{\text{use taylor expansion}} = f(x^*) + \eta f'(x^*) + \underbrace{\frac{\eta^2}{2} f''(x^*)}_{}$$

Can be neglected IF: ① η is small ② $f'(x) \neq 0$.

If $f'(x^*) = 0$, the we cannot get any insight on the stability of the system, the fixed point could be stable, unstabe, ..., half-stable

We then get,

$$\dot{x}(t) = \dot{\eta}(t) = \eta f'(x^*) = \eta r$$

If $r > 0$, exponential growth since

$$\dot{\eta} = r\eta$$

$$\int \frac{1}{\eta} d\eta = \int r \cdot dt \Rightarrow \ln \eta = rt + c$$

$$\eta = Be^{rt}, \ \underline{r > 0}$$

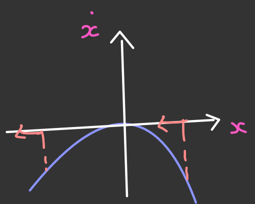
If $r < 0$, we have the opposite of exponential decay.

$$\eta = Be^{-rt}, \ \underline{r < 0}.$$

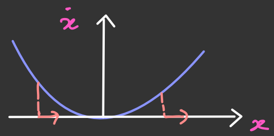
* Example of how we can't conclude anything from $f'(x^*) = 0$

Consider, $\dot{x} = x^2$

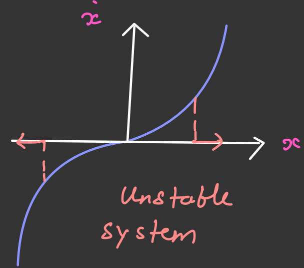
$\Rightarrow x^* = 0$, since $f(x^*) = 0$



Opposite to the case when $f(x) = x^2$, stable from the right, unstable from the left.



Stable from the left, but unstable from the right.



Unstable system

Example: Logistic eq^n:

$$\dot{x} = rx\left(1 - \frac{x}{k}\right)$$

$\dot{x} = 0$ is satisfied when

$$\underbrace{x = 0}_{x^*_1}, \quad \underbrace{x = k}_{x^*_2}.$$

$$f'(x^*_1) = r - \frac{2rx}{k} = r > 0 \ \text{[Unstable]}$$

$$f'(x^*_2) = r - \frac{2rx}{k} = -r. < 0 \ \text{[Stable]}$$

Consistent with what was seen earlier, with the point at the origin being unstable, and the point at $x = k$ was stable.

Existence of uniqueness:

Solutions to $\dot{x} = f(x)$ <u>exist</u> and are <u>unique</u>

IF: <u>$f(x)$</u> and <u>$f'(x)$</u> <u>are continuous</u>.

* There a milder conditions for existence and uniqueness, but for this course this would suffice.

Impossibility of oscillations:

Possible behaviour of $x(t)$ as $t \to \infty$ for $\dot{x}(t) = f(x)$?

Only two possibilities exist:

(i) $x(t) \to \pm\infty$
(ii) $x(t) \to x^*$

} This is because $x(t)$ always increases monotically or stays fixed } Any function you draw in the $\dot{x}, x$ space for 1D systems would have this.

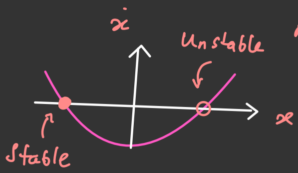
Bifurcations: (In the context of 1-D systems).

- Think of it as changing a parameter of a system, the qualitative structure of the vector field may change dramatically (fixed points changed or destroyed, stability changed).
  - ↳ Such a change is called bifurcation...
    - The value of the parameter at which the change occurs are called <u>bifurcation points</u>.
    - Examples: Reynold's number in fluid dynamics, Arythmia, ...

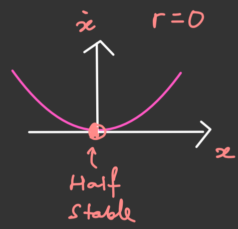
Saddle-node bifurcation: Basic mechanism for creation and destruction of fixed points..

Let, $\dot{x} = f(x) = r + x^2$

We can then plot the phase plot,



$r < 0$, As r is increased... →



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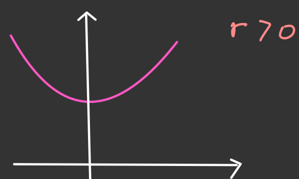
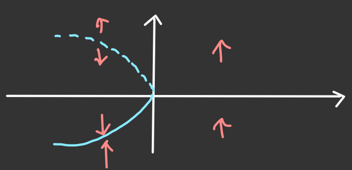


No fixed points, they are destroyed.

The above series of pictures can be represented by a <u>bifurcation</u> <u>diagram</u>:



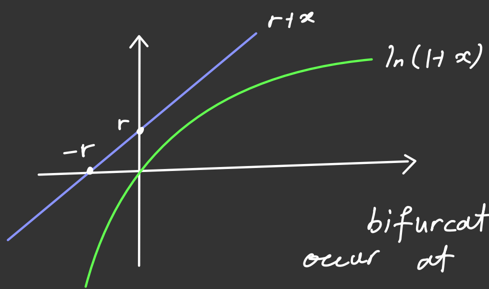
Example:

$$\dot{x} = r + x - \ln(1+x)$$

To find fixed points,

$$\underbrace{r + x = \ln(1+x)}_{\text{Difficult to solve analytically}}$$

} Just use a graphical approach



As r is reduced, the blue curve would eventually intersect. The bifurcation would the occur at

$$r + x = \ln(1+x)$$

$$\frac{d}{dx}(r+x) = \frac{d}{dx}(\ln(1+x))$$

Solving the gradient eq^n:

$$\frac{d}{dx}(r+x) = 1 = \frac{d}{dx}(\ln(1+x)) = \frac{1}{1+x^*} \Rightarrow x^* = 1.$$

From the eq^n then

$$r + x^* = 0$$

$$r = 0.$$

Bifurcation at $(x, r) = (0, 0)$

$$x = x^* + \eta$$

$$\dot{x} = \underbrace{\dot{x}^*}_{\text{constant}} + \dot{\eta}$$

$$\dot{x} = \dot{\eta} \underbrace{=}_{0} r + x - \ln(1+x)$$

$$= r + x - \underbrace{\left(x - \frac{x^2}{2} + \frac{x^3}{3} + \dots\right)}_{\text{Mclaurin's expansion}} = \underbrace{r + \frac{x^2}{2} + \underbrace{O(x^3)}_{\text{small.}}}_{\text{Normal form of saddle node bifurcation.}}$$

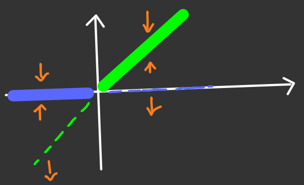
Transcritical bifurcations:

Let,

$$\dot{x} = rx - x^2$$
$$= x(r - x)$$
$$x_1^* = 0, \quad x_2^* = r$$

$$f'(x_1^*) = r - 2x$$
$$= r, \quad r < 0 \text{ (stable)}$$
$$f'(x_2^*) = r - 2x$$
$$= -r, \quad r > 0 \text{ (stable)}$$

The bifurcation diagram then is,

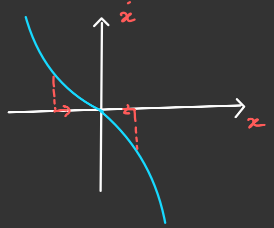


Pitchfork bifurcation:

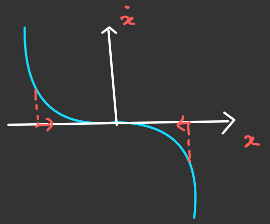
Let $\dot{x} = rx - x^3$

Drawing the phase plot,

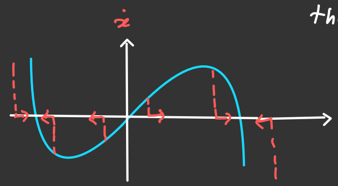
If $r < 0$,



If $r = 0$



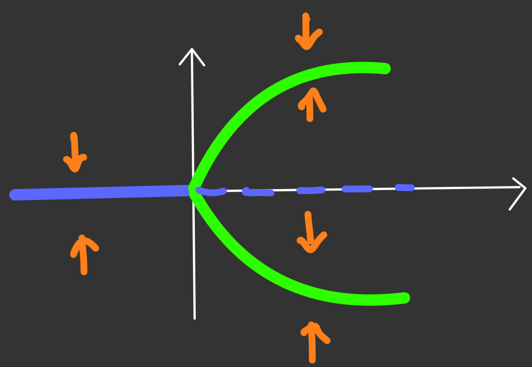
If $r > 0$



- We find that there is a symmetric pair of fixed points. And the origin is unstable...

The above is called a super-critical pitchfork...

The bifurcation diagram is then :



At $x = 0$, $r < 0$, stable

$x = \sqrt{r}$, $x = -\sqrt{r}$ are not fixed points

At $r > 0$, $x = 0$ is unstable

and $x = \sqrt{r}$, $x = -\sqrt{r}$ are stable .