Strassen's Matrix Multiplication

Strassen's Matrix Multiplication is a **divide and conquer algorithm** used to multiply two square matrices more efficiently than the classical method. It was proposed by **Volker Strassen in 1969** and reduces the time complexity from

$$O(n^3) \text{ to } O(n^{\log_2 7}) \approx O(n^{2.81})$$

This algorithm is particularly useful for multiplying **large matrices**.

Classical Matrix Multiplication

For two matrices A and B of size $n \times n$:

$$C[i][j] = \sum_{k=1}^{n} A[i][k] \cdot B[k][j]$$

Time Complexity:

$$O(n^3)$$

Idea of Strassen's Algorithm

Instead of performing 8 multiplications for two 2×2 matrices, Strassen's algorithm performs only **7 multiplications** by using additional additions and subtractions.

Since multiplication is more expensive than addition, this reduces overall time complexity.

Matrix Division

Given two matrices:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Each submatrix is of size $\frac{n}{2} \times \frac{n}{2}$.

Strassen's 7 Multiplications

$$\begin{aligned}
M_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) \\
M_2 &= (A_{21} + A_{22})B_{11} \\
M_3 &= A_{11}(B_{12} - B_{22}) \\
M_4 &= A_{22}(B_{21} - B_{11}) \\
M_5 &= (A_{11} + A_{12})B_{22} \\
M_6 &= (A_{21} - A_{11})(B_{11} + B_{12}) \\
M_7 &= (A_{12} - A_{22})(B_{21} + B_{22})
\end{aligned}$$

Result Matrix Computation

The resulting matrix $C = A \times B$ is obtained as:

$$\begin{aligned}
C_{11} &= M_1 + M_4 - M_5 + M_7 \\
C_{12} &= M_3 + M_5 \\
C_{21} &= M_2 + M_4 \\
C_{22} &= M_1 - M_2 + M_3 + M_6
\end{aligned}$$

Example (2 × 2 Matrix)

Let:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Using Strassen's formulas, the product matrix obtained is:

$$C = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

Time Complexity Analysis

Strassen's recurrence relation:

$$T(n) = 7T\left(\frac{n}{2}\right) + O(n^2)$$

Using Master's Theorem:

$$T(n) = O(n^{\log_2 7}) \approx O(n^{2.81})$$

Advantages

- Faster than classical multiplication for large matrices
- Reduces number of costly multiplication operations
- Basis for advanced matrix multiplication algorithms

Limitations

- Not efficient for small matrices
- Increased memory usage
- Numerical stability issues due to many additions/subtractions
- Matrix size must be a power of 2 (padding required otherwise)

Applications

- Scientific computing
- Computer graphics
- Machine learning and AI
- Large-scale numerical simulations

Conclusion

Strassen's Matrix Multiplication algorithm significantly improves performance for large matrix multiplication problems by reducing time complexity using a divide-and-conquer approach. However, practical implementations often use it only beyond a certain matrix size due to overhead.