

Fake Coin Problem

Introduction

The Fake Coin Problem is a classic problem in the Design and Analysis of Algorithms. In this problem, we are given a set of coins that appear identical, but one coin is fake. The fake coin may be either heavier or lighter than the genuine coins. A balance scale is used to compare coins, which can show three outcomes: left side heavier, right side heavier, or both sides equal. The objective is to identify the fake coin and determine whether it is heavier or lighter using the minimum number of weighings.

Problem Statement

Given 12 coins, where one coin is fake and can be either heavier or lighter, determine:

- Which coin is fake
 - Whether the fake coin is heavier or lighter
 - Using the minimum number of weighings
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Key Observation

A balance scale has three possible outcomes:

1. Left side is heavier
2. Right side is heavier
3. Both sides balance

With 3 weighings, we get $3^3 = 27$ outcomes, which are sufficient to identify the fake coin among 12 coins (24 possibilities).

Step 1: Label the Coins

Let the coins be labeled as:

$$C_1, C_2, C_3, C_4, C_5, C_6, C_7, C_8, C_9, C_{10}, C_{11}, C_{12}$$

Weighing 1

Compare:

$$(C_1, C_2, C_3, C_4) \quad \text{vs} \quad (C_5, C_6, C_7, C_8)$$

Case 1: Both Sides Balance

This means all 8 coins are genuine. The fake coin must be among:

$$C_9, C_{10}, C_{11}, C_{12}$$

Weighing 2 (Case 1)

Compare:

$$(C_9, C_{10}) \quad \text{vs} \quad (C_{11}, C_{12})$$

- If they balance, all coins are genuine (impossible case)
 - If not, the fake coin is in the heavier or lighter pair
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Weighing 3 (Case 1)

Compare one suspected coin with a known genuine coin.

- If heavier, the coin is fake and heavier
- If lighter, the coin is fake and lighter

Thus, the fake coin is identified.

Case 2: Weighing 1 Does Not Balance

Assume:

$$(C_1, C_2, C_3, C_4) \text{ is heavier than } (C_5, C_6, C_7, C_8)$$

This indicates that:

- One of C_1 to C_4 is heavier, or
 - One of C_5 to C_8 is lighter
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Weighing 2 (Case 2)

Compare:

$$(C_1, C_2, C_5) \text{ vs } (C_3, C_4, C_9)$$

where C_9 is a known genuine coin.

- If balanced: fake coin is C_5 or C_6 (lighter)
 - If left is heavier: fake coin is C_1 or C_2 (heavier)
 - If right is heavier: fake coin is C_3 or C_4 (lighter)
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Weighing 3 (Case 2)

Compare the two suspected coins directly:

$$C_i \text{ vs } C_j$$

- The heavier or lighter coin is identified as fake
 - If they balance, the remaining coin is fake
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Why This Strategy Works

- Coins are divided into equal groups to maximize information
 - Balance results eliminate large sets of coins
 - Direction of imbalance indicates heavy or light
 - Three weighings are sufficient to identify the fake coin
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Final Answer

- Minimum number of weighings required: **3**
- Technique used: Divide and compare using balance scale
- Result: Fake coin and its nature are identified efficiently