Digital Signal Processing

I Sai Pradeep

Contents

Abstract—**This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution:

```
import soundfile as sf
from scipy import signal

#read .wav file
input_signal,fs = sf.read('Sound_Noise.wav'
    )

#sampling frequency of Input signal
sampl_freq=fs

#order of the filter
order=4

#cutoff frquency 4kHz
cutoff_freq=4000.0

#digital frequency
Wn=2*cutoff_freq/sampl_freq

# b and a are numerator and denominator
    polynomials respectively
b, a = signal.butter(order,Wn, 'low')

#filter the input signal with butterworth filter
output_signal = signal.filtfilt(b, a,
    input_signal)
#output_signal = signal.lfilter(b, a,
    input_signal)

#write the output signal into .wav file
sf.write('Sound_With_ReducedNoise.wav',
    output_signal, fs)
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?

Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/Pradeep8802/
    EE3900−Digital−Signal−
    Processing/blob/main/
    Assignment1/codes/3.1.py
```

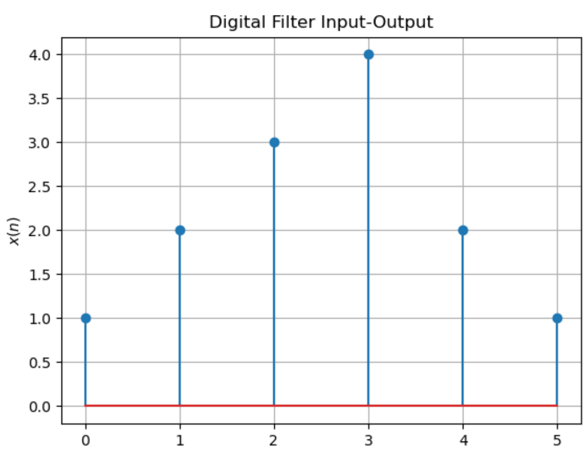

Fig. 3.1

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.
Solution: The following code yields Fig. 3.2.

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/3.2.py
```

3.3 Repeat the above exercise using a C code.
Solution: The following c code is used to find $y(n)$

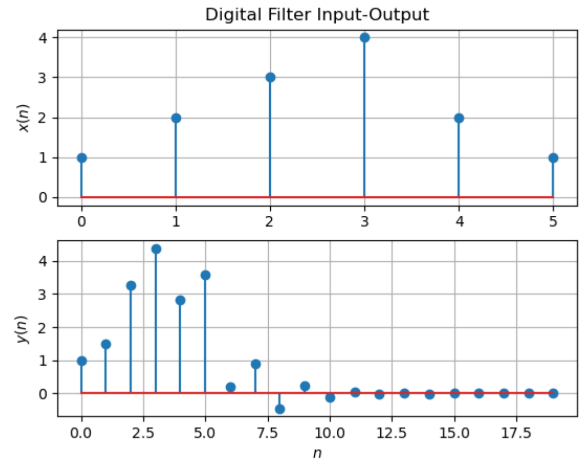

Fig. 3.2

```
wget https://github.com/Pradeep8802/
    EE3900−Digital−Signal−
    Processing/blob/main/
    Assignment1/codes/3.3.c
```

The following code yields Fig. 3.3.

```
wget https://github.com/Pradeep8802/
    EE3900−Digital−Signal−
    Processing/blob/main/
    Assignment1/codes/3.3.py
```

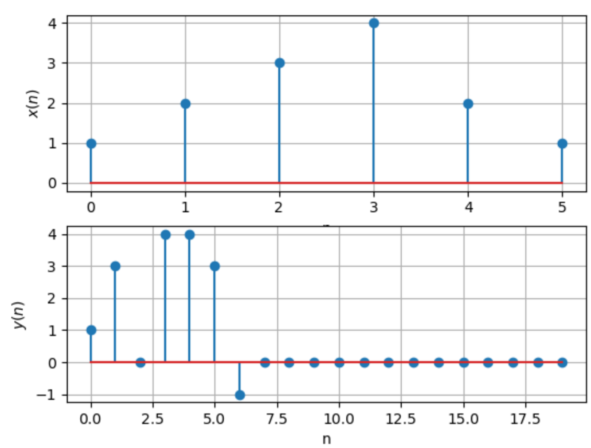

Fig. 3.3

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1. **Solution:** Z-transform of x(n),$X(z)$ is given by

$$\mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

$$= \sum_{n=0}^{5} x(n)z^{-n} \tag{4.8}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.9}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.10}$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.11}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.12}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.15}$$

Solution: The Z-transform of $\delta(n)$ is defined as

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n)z^{-n} \tag{4.16}$$

$$= \delta(0)z^{-0} \tag{4.17}$$

$$= 1 \tag{4.18}$$

Hence we can say that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.19}$$

and from (4.14),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.20}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.21}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.22}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n)z^{-n} \tag{4.23}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.24}$$

$$= \sum_{n=0}^{\infty} (z^{-1}a)^n \tag{4.25}$$

$$= \frac{1}{1 - az^{-1}}, \quad \left|z^{-1}a\right| < 1 \tag{4.26}$$

$$= \frac{1}{1 - az^{-1}}, \quad |z| > |a| \tag{4.27}$$

using the fomula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.28}$$

Plot $|H(e^{\jmath\omega})|$. Is it periodic? If so, find the period. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $h(n)$.
Solution:

```
wget https://github.com/Pradeep8802/EE3900
```

−Digital−Signal−Processing/blob/main/
Assignment1/codes/4.5.py

$H(e^{jw})$ is given by

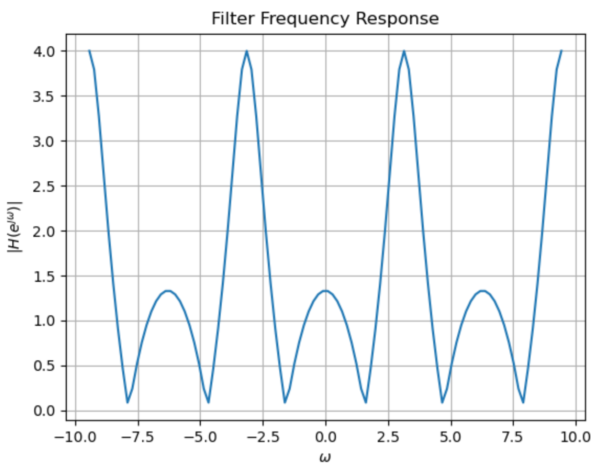

Fig. 4.6: $|H(e^{j\omega})|$

$$H(e^{jw}) = \frac{1 + (e^{jw})^{-2}}{1 + \frac{1}{2}(e^{jw})^{-1}} \tag{4.29}$$

$$= 2\frac{1 + \cos(-2\omega) + j\sin(-2\omega)}{2 + \cos(-\omega) + j\sin(-\omega)} \tag{4.30}$$

$$= 2\frac{1 + \cos(2\omega) - j\sin(2\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.31}$$

$$= 2\frac{2\cos^2(\omega) - 2j\sin(\omega)\cos(\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.32}$$

$$= 4\cos(\omega)\frac{\cos(\omega) - j\sin(\omega)}{2 + \cos(\omega) - j\sin(\omega)} \tag{4.33}$$

$$= 4|\cos(\omega)|\frac{e^{jw}}{2 + e^{jw}} \tag{4.34}$$

So,

$$|H(e^{jw})| = 4|\cos(\omega)|\frac{|e^{jw}|}{|2 + e^{jw}|} \tag{4.35}$$

$$= \frac{|4\cos(\omega)|}{\sqrt{5 + 4\cos(\omega)}} \tag{4.36}$$

The period of $|\cos(\omega)|$ is π. The period of $5 + 4\cos(\omega)$ is 2π. Hence $|H(e^{j\omega})|$ is periodic with period 2π.(The LCM of the period of $|\cos(\omega)|$ and $5 + 4\cos(\omega)$ is 2π) The graph of $|H(e^{j\omega})|$ is symmetric with respect to y-axis. It is continuous over ω. The following code plots Fig. 4.6.

4.7 Express $h(n)$ in terms of $H(e^{j\omega})$.
Solution: We know that

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k} \tag{4.37}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.38}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n}d\omega \tag{4.39}$$

$$= \frac{1}{2\pi}\sum_{k=-\infty}^{\infty} h(k)\int_{-\pi}^{\pi} e^{j\omega(n-k)}d\omega \tag{4.40}$$

$$\tag{4.41}$$

case-1:If$(n \neq k)$

$$= \frac{1}{2\pi}\sum_{k\neq n} h(k)\left.\frac{e^{j\omega(n-k)}}{j(n-k)}\right]_{-\pi}^{\pi} \tag{4.42}$$

$$= \frac{1}{2\pi}\sum_{k\neq n} h(k)\frac{2\sin(\pi(n-k))}{(n-k)} \tag{4.43}$$

$$= 0 \tag{4.44}$$

case-2:If$(n = k)$

$$= \frac{1}{2\pi}h(n)\int_{-\pi}^{\pi} d\omega \tag{4.45}$$

$$= h(n) \tag{4.46}$$

Hence, we can say,

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{4.47}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), \quad n < 5 \tag{5.1}$$

for H(z) in (4.12).
Solution: $H(z)$ is given by

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} = \frac{2 + 2z^{-2}}{2 + z^{-1}} \tag{5.2}$$

$$2z^{-1} - 4 \tag{5.3}$$
$$z^{-1} + 2 \;\big)\; 2z^{-2} + 2 \tag{5.4}$$
$$2z^{-2} + 4z^{-1} \tag{5.5}$$
$$-4z^{-1} + 2 \tag{5.6}$$
$$-\,4z^{-1} - 8 \tag{5.7}$$
$$10 \tag{5.8}$$

So,

$$H(z) = 2z^{-1} - 4 + \frac{10}{z^{-1} + 2} \tag{5.9}$$
$$= 2z^{-1} - 4 + \frac{5}{\frac{1}{2}z^{-1} + 1} \tag{5.10}$$
$$= 2z^{-1} - 4 + 5\sum_{n=0}^{\infty}\left(-\frac{z^{-1}}{2}\right)^{n} \tag{5.11}$$
$$= 1 - \frac{1}{2}z^{-1} + 5 \times \sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^{n} z^{-n} \tag{5.12}$$

So,h(n) will be given by

$$h(n) = \begin{cases} 5 \times \left(-\frac{1}{2}\right)^{n} & n \geq 2 \\ \left(-\frac{1}{2}\right)^{n} & 2 > n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.13}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.14}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.12),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.15}$$
$$\implies h(n) = \left(-\frac{1}{2}\right)^{n} u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.16}$$

using (4.22) and (4.6).

5.3 Sketch $h(n)$. Is it bounded? Justify theoretically.
Solution: The following code plots Fig. 5.3.

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/5.3.py
```

on simplfying we get h(n) as

$$\begin{cases} 5 \times \left(-\frac{1}{2}\right)^{n} & n \geq 2 \\ \left(-\frac{1}{2}\right)^{n} & 2 > n \geq 0 \\ 0 & n < 0 \end{cases} \tag{5.17}$$

$$\because 5 \times \left(-\frac{1}{2}\right)^{n} \to 0 \quad \text{for} \quad n \to \infty \tag{5.18}$$

So, we can conclude that h(n) is bounded.

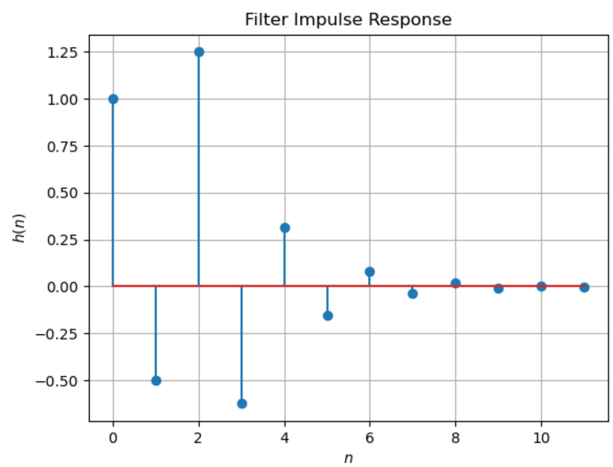

Fig. 5.3: $h(n)$ wrt n

5.4 Convergent? Justify using the ratio test.
Solution: We can say a given real sequence $\{x_n\}$ is convergent if

$$\lim_{n\to\infty}\left|\frac{x_{n+1}}{x_n}\right| < 1 \tag{5.19}$$

This is known as Ratio test.
In this case the limit will become,

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{5\left(\frac{-1}{2}\right)^{n+1}}{5\left(\frac{-1}{2}\right)^{n}}\right| \tag{5.20}$$
$$= \lim_{n\to\infty}\left|\frac{-1}{2}\right| \tag{5.21}$$
$$= \frac{1}{2} \tag{5.22}$$

As $\frac{1}{2} < 1$, from root test we can say that $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.23}$$

Is the system defined by (3.2) stable for the

impulse response in (5.14)?
Solution: For system of 3.2 ,$h(n)$ is defined in (5.13) So,

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=2}^{\infty} 5 \times \left(-\frac{1}{2}\right)^n + \sum_{n=0}^{1} \left(-\frac{1}{2}\right)^n + \sum_{n=-\infty}^{-1} 0 \tag{5.24}$$

$$= 5 \times \frac{1}{6} + \frac{1}{2} \tag{5.25}$$

$$= \frac{4}{3} \tag{5.26}$$

Since the sum is finite so the system is stable for impulsive response

5.6 Verify the above result using a python code.
Solution: The above result is verified using the below python code

```
wget https://github.com/Pradeep8802/EE3900
    -Digital-Signal-Processing/tree/main/
    Assignment1/codes/5.6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.27}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 3.1.

```
wget https://github.com/Pradeep8802/EE3900
    -Digital-Signal-Processing/tree/main/
    Assignment1/codes/5.7.py
```

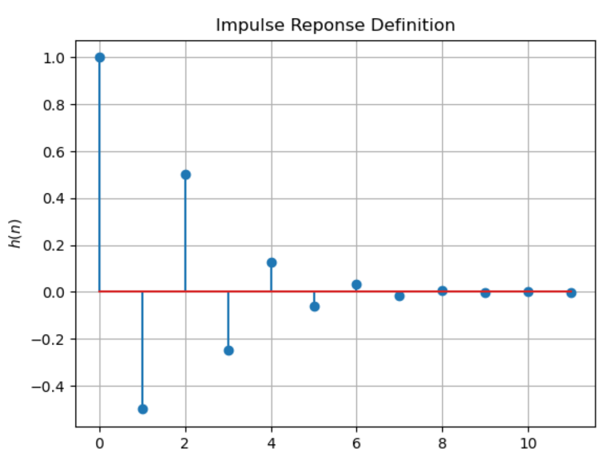

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.28}$$

Comment. The operation in (5.28) is known as *convolution*.
Solution: The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.1.

```
wget https://github.com/Pradeep8802/EE3900
    -Digital-Signal-Processing/tree/main/
    Assignment1/codes/5.4.py
```

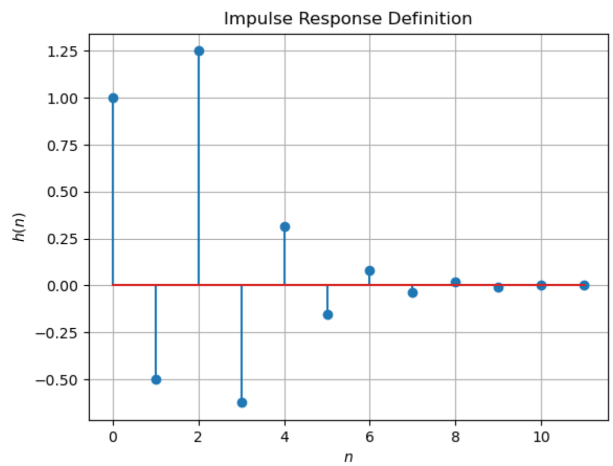

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Toeplitz matrix.
Solution:

```
wget https://github.com/Pradeep8802/EE3900
    -Digital-Signal-Processing/tree/main/
    Assignment1/codes/5.9.py
```

From (5.28),we express $y(n)$ as

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k) \tag{5.29}$$

To understand how we can use a Toeplitz matrix, we will see what we are doing in (5.28)

$$y(0) = x(0)\,h(0) \tag{5.30}$$

$$y(1) = x(0)\,h(1) + x(1)\,h(0) \tag{5.31}$$

$$y(2) = x(0)\,h(2) + x(1)\,h(1) + x(2)\,h(0) \tag{5.32}$$

.

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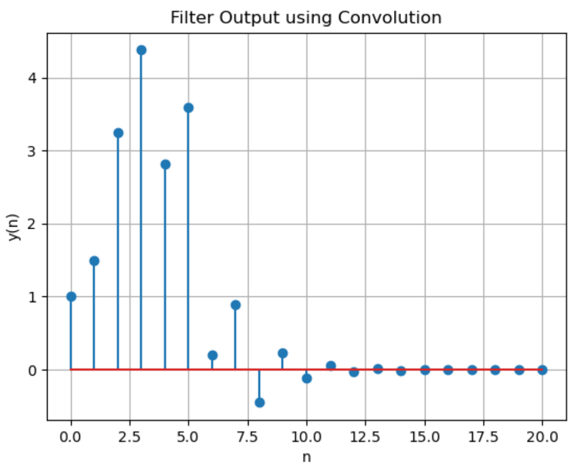

Fig. 5.9: Convolution of $x(n)$ and $h(n)$ using toeplitz matrix

The same thing can be written as,

$$y(0) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.33}$$

$$y(1) = \begin{pmatrix} h(1) & h(0) & 0 & 0 & . & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.34}$$

$$y(2) = \begin{pmatrix} h(2) & h(1) & h(0) & 0 & . & .0 \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.35}$$

.

.

Using Toeplitz matrix of $h(n)$ we can simplify it as,

$$y(n) = \begin{pmatrix} h(0) & 0 & 0 & . & . & .0 \\ h(1) & h(0) & 0 & . & . & .0 \\ h(2) & h(1) & h(0) & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & .h(m-1) \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.36}$$

Now from (3.1) we will take n

$$x(n) = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{5.37}$$

And from (5.13) we will take some values of n,

$$h(n) = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ . \\ . \end{pmatrix} \tag{5.38}$$

Now using (5.36),

$$y(n) = x(n) * h(n) \tag{5.39}$$

$$= \begin{pmatrix} 1 & 0 & 0 & . & . & .0 \\ -0.5 & 1 & 0 & . & . & .0 \\ 1.25 & -0.5 & 1 & . & . & .0 \\ & & .. & & & \\ & & .. & & & \\ 0 & 0 & 0 & . & . & . \end{pmatrix} \begin{pmatrix} x(0) \\ x(1) \\ x(2) \\ . \\ . \\ x(5) \end{pmatrix} \tag{5.40}$$

$$= \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ . \\ . \\ . \end{pmatrix} \tag{5.41}$$

The above equation (5.41) is the convolution of $x(n)$ and $h(n)$

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.42}$$

Solution: From (5.28),

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.43}$$

Replacing $n-k$ with a,we get

$$y(n) = \sum_{n-a=-\infty}^{\infty} x(n-a)h(a) \tag{5.44}$$

$$= \sum_{-a=-\infty}^{\infty} x(n-a)h(a) \tag{5.45}$$

$$= \sum_{a=-\infty}^{\infty} x(n-a)h(a) \tag{5.46}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution: Run the following codes to compute $X(k)$ which is plotted in 6.1.

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/6.1.py
```

Run the following codes to compute $H(k)$.

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/6.1_2.py
```

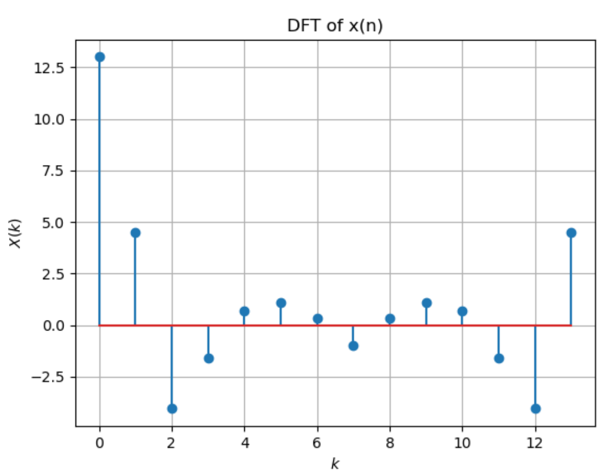

Fig. 6.1: DFT of $x(k)$

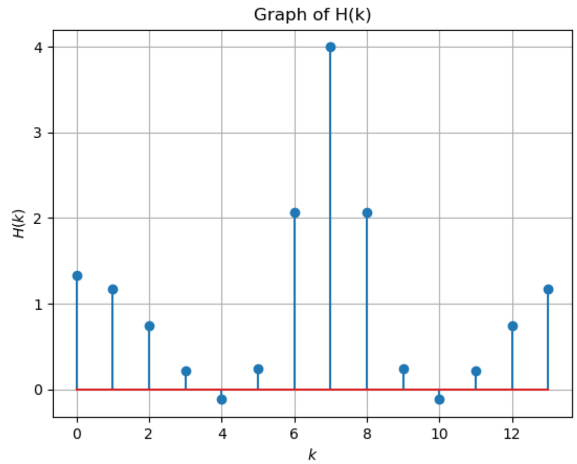

Fig. 6.1: $H(n)$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: Run the following codes to compute $Y(k)$ respectively.

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/6.2.py
```

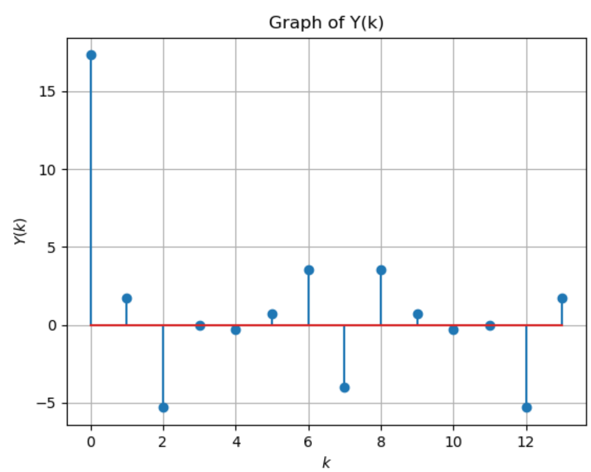

Fig. 6.2: DFT of $xyn)$

6.3 Compute

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) \cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 6.2. Note that this is the same as

$y(n)$ in Fig. 3.1.

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/6.3.py
```

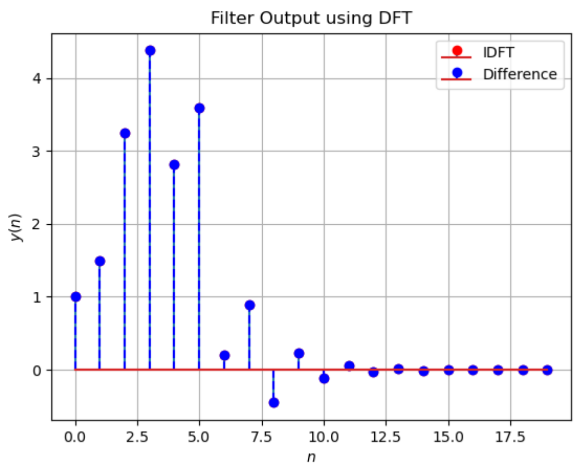

Fig. 6.3: $y(n)$

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Run the following code to plot

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/6.4.py
```

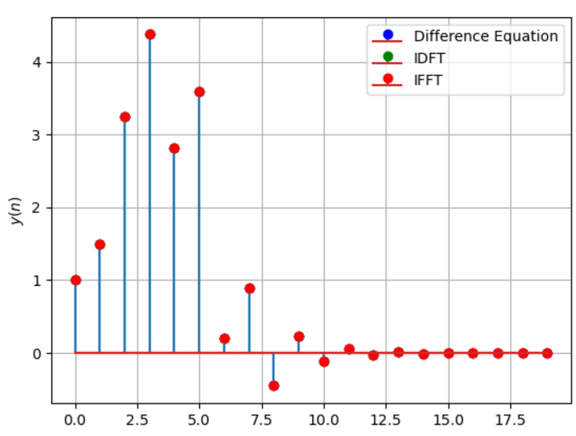

Fig. 6.4: $y(n)$

7 FFT

1. The DFT of $x(n)$ is given by
$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{7.1}$$
2. Let
$$W_N = e^{-j2\pi/N} \tag{7.2}$$
Then the N-point *DFT matrix* is defined as
$$\mathbf{F}_N = [W_N^{mn}] \tag{7.3}$$
where W_N^{mn} are the elements of $\mathbf{F}_N$.
3. Let
$$\mathbf{I}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.4}$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.5}$$
4. The 4 point *DFT diagonal matrix* is defined as
$$\mathbf{D}_4 = diag\begin{pmatrix} W_8^0 & W_8^1 & W_8^2 & W_8^3 \end{pmatrix} \tag{7.6}$$
5. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution: Given
$$W_N = e^{-j2\pi/N} \tag{7.8}$$
$$\implies W_N^2 = e^{2(-j2\pi/N)} \tag{7.9}$$
$$= e^{-j2\pi/(N/2)} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$
Therefore, hence proved that ,
$$W_N^2 = W_{N/2} \tag{7.12}$$
6. Show that
$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.13}$$
Solution: $\mathbf{I}_2$ is a 2×2 identity matrix, for any 2×2 A, we have
$$\mathbf{I}_2\mathbf{A} = \mathbf{A}\mathbf{I}_2 = \mathbf{A} \tag{7.14}$$
Consider
$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} \tag{7.15}$$

where

$$\mathbf{F}_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.16}$$

$$\mathbf{D}_2 = diag(W_4^0, W_4^1) = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \tag{7.17}$$

Since $W_4 = e^{-j2\pi/4}$
As,

$$\mathbf{P}_4 = \begin{pmatrix} \mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4 \end{pmatrix} \tag{7.18}$$

$$\implies \mathbf{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.19}$$

$$\mathbf{F}_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \tag{7.20}$$

$$\mathbf{D}_2\mathbf{F}_2 = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.21}$$

$$= \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \tag{7.22}$$

$$\begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix} \tag{7.23}$$

Using (7.19) and (7.23) in (7.13)

$$\begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix}\begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix}\mathbf{P}_4 = \begin{bmatrix} \mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2 \\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2 \end{bmatrix}\mathbf{P}_4 \tag{7.24}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -j & j \\ 1 & 1 & -1 & -1 \\ 1 & -1 & j & -j \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.25}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \tag{7.26}$$

The RHS of the equation (7.26) is $\mathbf{F}_4$, from (7.20).

$$\therefore \mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix}\begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix}\mathbf{P}_4 \tag{7.27}$$

7. Show that

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix}\begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix}\mathbf{P}_N \tag{7.28}$$

Solution:
Consider the following properties:

$$W_N^{k(2n+1)} = W_N^k W_{N/2}^{kn} \tag{7.29}$$

$$W_N^{k+N/2} = e^{-j2\pi[k+N/2]/N} = e^{-j2\pi k/N}e^{-j\pi} \tag{7.30}$$

$$= -W_N^k \tag{7.31}$$

Consider $X[k]$,

$$X[k] = \sum_{n=0}^{N-1} W_N^{kn} x[n] \tag{7.32}$$

$$= \sum_{n=0}^{N/2-1} \left[x[2n] W_N^{k(2n)} + x[2n+1] W_N^{k(2n+1)}\right] \tag{7.33}$$

$k = 0, \cdots, N-1$
which is gathering odd and even terms seperately which allows us to write

$$X[k] = \sum_{n=0}^{N/2-1} x[2n] W_{N/2}^{kn} + W_N^k \sum_{n=0}^{N/2-1} x[2n+1] W_{N/2}^{kn} \tag{7.34}$$

$$= Y[k] + W_N^k Z[k] \quad k = 0, \cdots, N-1 \tag{7.35}$$

where $Y[k]$ and $Z[k]$ are DFTs of length N/2 of even numbered sequence $\{x[2n]\}$ and of the odd numbered sequence $\{x[2n+1]\}$, respectively. Here, we cannot proceed for $k \geq N/2$. So, for $k \geq N/2$:

$$X[k+N/2] = Y[k+N/2] + W_N^{k+N/2} Z[k+N/2] \tag{7.36}$$

Using the periodicity of $Y[k]$ and $Z[k]$ and (7.30),

$$X[k+N/2] = Y[k+N/2] - W_N^k Z[k] \tag{7.37}$$

Using (7.35) and (7.37), and linear transformation upon them

$$\mathbf{X}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{Y_{N/2}} \\ \mathbf{Z_{N/2}} \end{bmatrix} \tag{7.38}$$

where $\mathbf{I_{N/2}}$ is a unit matrix and $\mathbf{D_{N/2}}$ is a diagonal matrix with elements as $\left\{W_N^k, k = 0, \cdots, N/2 - 1\right\}$, both of dimension $N/2 \times N/2$ .Whereas
$\mathbf{Y_N/2}$ = DFT of even terms of $\mathbf{X_N} = \mathbf{F_{N/2}x_{even}}$
$\mathbf{Z_N/2}$ = DFT of odd terms of $\mathbf{X_N} = \mathbf{F_{N/2}x_{odd}}$
which gets us to

$$\begin{bmatrix} \mathbf{Y_{N/2}} \\ \mathbf{Z_{N/2}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{x} \tag{7.39}$$

which is permuting into combinations of even and odd terms of a sequence.As we have permuted it into odd and even parts, we have to reverse this process. So, a permutation matrix is multiplied.

$$\begin{bmatrix} \mathbf{Y_{N/2}} \\ \mathbf{Z_{N/2}} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P_N x} \tag{7.40}$$

Replacing in (7.38),

$$\mathbf{X}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P_N x} \tag{7.41}$$

As $\mathbf{X_N} = \mathbf{F_N x}$,

$$\mathbf{F_N x} = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P_N x} \tag{7.42}$$

Applying $\mathbf{x^{-1}}$,

$$\mathbf{F}_N = \begin{bmatrix} \mathbf{I}_{N/2} & \mathbf{D}_{N/2} \\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{N/2} & 0 \\ 0 & \mathbf{F}_{N/2} \end{bmatrix} \mathbf{P}_N \tag{7.43}$$

8. Find

$$\mathbf{P}_4 \mathbf{x} \tag{7.44}$$

Solution: From (7.19),

$$\mathbf{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.45}$$

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.46}$$

After proper zero padding of $\mathbf{P}_4$,

$$\mathbf{P}_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.47}$$

$$\mathbf{P}_4 \mathbf{x} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.48}$$

$$= \begin{pmatrix} 1 \\ 3 \\ 2 \\ 4 \\ 0 \\ 0 \end{pmatrix} \tag{7.49}$$

9. Show that

$$\mathbf{X} = \mathbf{F}_N \mathbf{x} \tag{7.50}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution: Given $\mathbf{x}, \mathbf{X}$ are the vector represen-

tations of $x(n), X(k)$ respectively.

$$\mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.51}$$

$$\mathbf{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \tag{7.52}$$

$$\mathbf{F}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{7.53}$$

As

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N} \tag{7.54}$$

Here, we have,

$$X(0) = \sum_{n=0}^{N-1} x(n) e^{0} \tag{7.55}$$

$$= \sum_{n=0}^{N-1} x(n) \tag{7.56}$$

$$X(1) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi n/N} \tag{7.57}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^n \tag{7.58}$$

$$\vdots \tag{7.59}$$

$$X(N-1) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi(N-1)n/N} \tag{7.60}$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{n\times(N-1)} \tag{7.61}$$

Representing the above equations in matrix form as matrix multiplication, we get,

$$\begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & \cdots & \\ \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{N-1} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{7.62}$$

$$\therefore \mathbf{X} = \mathbf{F}_N \mathbf{x} \tag{7.63}$$

10. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} + \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.64}$$

$$\begin{bmatrix} X(4) \\ X(5) \\ X(6) \\ X(7) \end{bmatrix} = \begin{bmatrix} X_1(0) \\ X_1(1) \\ X_1(2) \\ X_1(3) \end{bmatrix} - \begin{bmatrix} W_8^0 & 0 & 0 & 0 \\ 0 & W_8^1 & 0 & 0 \\ 0 & 0 & W_8^2 & 0 \\ 0 & 0 & 0 & W_8^3 \end{bmatrix} \begin{bmatrix} X_2(0) \\ X_2(1) \\ X_2(2) \\ X_2(3) \end{bmatrix} \tag{7.65}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix} X_1(0) \\ X_1(1) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.66}$$

$$\begin{bmatrix} X_1(2) \\ X_1(3) \end{bmatrix} = \begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} \tag{7.67}$$

$$\begin{bmatrix} X_2(0) \\ X_2(1) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} + \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.68}$$

$$\begin{bmatrix} X_2(2) \\ X_2(3) \end{bmatrix} = \begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} - \begin{bmatrix} W_4^0 & 0 \\ 0 & W_4^1 \end{bmatrix} \begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} \tag{7.69}$$

$$P_8 \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \\ x(4) \\ x(5) \\ x(6) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \\ x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} \tag{7.70}$$

$$P_4 \begin{bmatrix} x(0) \\ x(2) \\ x(4) \\ x(6) \end{bmatrix} = \begin{bmatrix} x(0) \\ x(4) \\ x(2) \\ x(6) \end{bmatrix} \tag{7.71}$$

$$P_4 \begin{bmatrix} x(1) \\ x(3) \\ x(5) \\ x(7) \end{bmatrix} = \begin{bmatrix} x(1) \\ x(5) \\ x(3) \\ x(7) \end{bmatrix} \tag{7.72}$$

Therefore,

$$\begin{bmatrix} X_3(0) \\ X_3(1) \end{bmatrix} = F_2 \begin{bmatrix} x(0) \\ x(4) \end{bmatrix} \tag{7.73}$$

$$\begin{bmatrix} X_4(0) \\ X_4(1) \end{bmatrix} = F_2 \begin{bmatrix} x(2) \\ x(6) \end{bmatrix} \tag{7.74}$$

$$\begin{bmatrix} X_5(0) \\ X_5(1) \end{bmatrix} = F_2 \begin{bmatrix} x(1) \\ x(5) \end{bmatrix} \tag{7.75}$$

$$\begin{bmatrix} X_6(0) \\ X_6(1) \end{bmatrix} = F_2 \begin{bmatrix} x(3) \\ x(7) \end{bmatrix} \tag{7.76}$$

11. For

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.77}$$

compte the DFT using (7.50)
Solution:

$$\mathbf{F}_6 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3} \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3} \\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3} \end{bmatrix} \tag{7.78}$$

Using (7.77),

$$\mathbf{X} = \mathbf{F}_6\mathbf{x} \tag{7.79}$$

$$\mathbf{X} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-j\pi/3} & e^{-j2\pi/3} & e^{-j\pi} & e^{-j4\pi/3} & e^{-j5\pi/3} \\ 1 & e^{-j2\pi/3} & e^{-j4\pi/3} & e^{-j2\pi} & e^{-j8\pi/3} & e^{-j10\pi/3} \\ 1 & e^{-j\pi} & e^{-j2\pi} & e^{-j3\pi} & e^{-j4\pi} & e^{-j5\pi} \\ 1 & e^{-j4\pi/3} & e^{-j8\pi/3} & e^{-j4\pi} & e^{-j16\pi/3} & e^{-j20\pi/3} \\ 1 & e^{-j5\pi/3} & e^{-j10\pi/3} & e^{-j5\pi} & e^{-j20\pi/3} & e^{-j25\pi/3} \end{bmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \tag{7.80}$$

$$\implies X = \begin{bmatrix} 13 \\ -4 - \sqrt{3}j \\ 1 \\ -1 \\ 1 \\ -4 + \sqrt{3}j \end{bmatrix} \tag{7.81}$$

12. Repeat the above exercise using the FFT after zero padding **x**.
Solution: After padding, **x** becomes an 8×1 matrix, as shown below

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{7.82}$$

Using (7.43),

$$\mathbf{F}_8 = \begin{bmatrix} \mathbf{I}_4 & \mathbf{D}_4 \\ \mathbf{I}_4 & -\mathbf{D}_4 \end{bmatrix} \begin{bmatrix} \mathbf{F}_4 & 0 \\ 0 & \mathbf{F}_4 \end{bmatrix} \mathbf{P}_8 \tag{7.83}$$

$$\mathbf{F}_4 = \begin{bmatrix} \mathbf{I}_2 & \mathbf{D}_2 \\ \mathbf{I}_2 & -\mathbf{D}_2 \end{bmatrix} \begin{bmatrix} \mathbf{F}_2 & 0 \\ 0 & \mathbf{F}_2 \end{bmatrix} \mathbf{P}_4 \tag{7.84}$$

$$\mathbf{F}_2 = \begin{bmatrix} \mathbf{I}_1 & \mathbf{D}_1 \\ \mathbf{I}_1 & -\mathbf{D}_1 \end{bmatrix} \begin{bmatrix} \mathbf{F}_1 & 0 \\ 0 & \mathbf{F}_1 \end{bmatrix} \mathbf{P}_2 \tag{7.85}$$

$$\tag{7.86}$$

As $\mathbf{F_1} = \begin{bmatrix} W_1^0 \end{bmatrix}$, we have

$$\mathbf{F_1} = [1] \tag{7.87}$$

Value of $\mathbf{F_2}$ is,

$$\mathbf{F_2} = \begin{bmatrix} \mathbf{F}_1 & \mathbf{D_1F_1} \\ \mathbf{F}_1 & -\mathbf{D_1F_1} \end{bmatrix} \mathbf{P_2} \tag{7.88}$$

$$= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.89}$$

value of $\mathbf{F_4}$ is,

$$\mathbf{D}_2 = diag(W_4^0, W_4^1) \tag{7.90}$$

$$= diag(1, -j) \tag{7.91}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix} \tag{7.92}$$

$$\mathbf{D_2F_2} = \begin{bmatrix} 1 & 0 \\ 0 & -j \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{7.93}$$

$$= \begin{bmatrix} 1 & 1 \\ -j & j \end{bmatrix} \tag{7.94}$$

$$\mathbf{F_4} = \begin{bmatrix} \mathbf{F}_2 & \mathbf{D_2F_2} \\ \mathbf{F}_2 & -\mathbf{D_2F_2} \end{bmatrix}\mathbf{P_4} \tag{7.95}$$

$$\mathbf{F_4} = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -j & j \\ 1 & 0 & -1 & -1 \\ 0 & 1 & j & -j \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{7.96}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -j & 1 & j \\ 1 & -1 & 0 & j \\ 0 & j & 1 & -j \end{bmatrix} \tag{7.97}$$

value of $\mathbf{F_8}$ is,

$$\mathbf{D_4} = diag\left(1, W_8, W_8^2, W_8^3\right) \tag{7.98}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix} \tag{7.99}$$

$$\mathbf{D_4F_4} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1-j}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & -j & 1 & j \\ 1 & -1 & 0 & j \\ 0 & j & 1 & -j \end{bmatrix} \tag{7.100}$$

$$= \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ -1 & 1 & 0 & -j \\ 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \end{bmatrix} \tag{7.101}$$

$F_8 = \mathbf{ABP_8}$ where

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{-1-j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 1 & 0 & \frac{1-j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{1+j}{\sqrt{2}} & \frac{-1+j}{\sqrt{2}} & \frac{-1-j}{\sqrt{2}} \\ 0 & 0 & 1 & 0 & 1 & -1 & 0 & j \\ 0 & 0 & 0 & 1 & 0 & \frac{-1+j}{\sqrt{2}} & \frac{1+j}{\sqrt{2}} & \frac{1-j}{\sqrt{2}} \end{bmatrix} \tag{7.102}$$

$$\mathbf{B} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -j & 1 & j & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & j & 0 & 0 & 0 & 0 \\ 0 & j & 1 & -j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & j & -1 & j \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & -j \\ 0 & 0 & 0 & 0 & 0 & -j & -1 & j \end{bmatrix} \tag{7.103}$$

$$\mathbf{F_8} = \tag{7.104}$$

And $\mathbf{P_8}$ is a permutation matrix.

$$\mathbf{X} = \begin{bmatrix} 13 \\ -4-8j \\ j \\ 2-2j \\ -1 \\ 2+2j \\ -j \\ -4+8j \end{bmatrix} \tag{7.105}$$

13. Write a C program to compute the 8-point FFT.
Solution:

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/7_13.c
```

FFT matrix multiplication code :

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/FFTmul.py
```

Time Complexity comparision between FFT-IFFT and convolution:

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/time.c
```

The figure 7.13 is ploted using below code:

```
wget https://github.com/Pradeep8802/EE3900
    -Digital-Signal-Processing/blob/main/
    Assignment1/codes/time.py
```

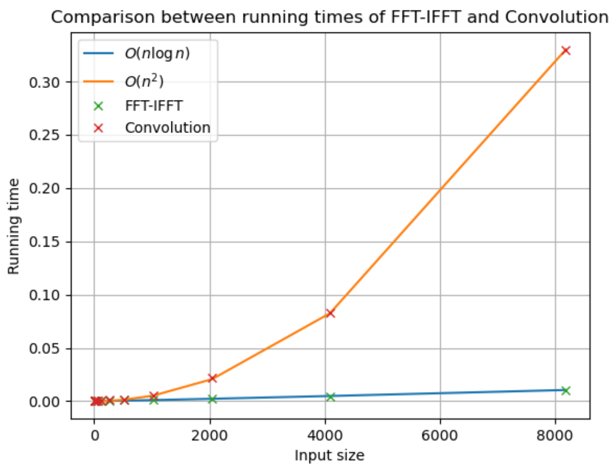

Fig. 7.13: Time Complexity

8 Exercises

Answer the following questions by looking at the python code in Problem 2.3

8.1 The command

```
output_signal = signal.lfilter(b, a,
    input_signal)
```

in Problem 2.3 is executed through the following difference equation

$$\sum_{m=0}^{M} a\left(m\right) y\left(n-m\right) = \sum_{k=0}^{N} b\left(k\right) x\left(n-k\right) \quad (8.1)$$

where the input signal is $x(n)$ and the output signal is $y(n)$ with initial values all 0. Replace **signal.filtfilt** with your own routine and verify.
Solution: On taking the Z-transform on both sides of the difference equation

$$\sum_{m=0}^{M} a\left(m\right) z^{-m} Y(z) = \sum_{k=0}^{N} b\left(k\right) z^{-k} X(z) \quad (8.2)$$

$$\implies H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b\left(k\right) z^{-k}}{\sum_{m=0}^{M} a\left(m\right) z^{-m}} \quad (8.3)$$

For obtaining the discrete Fourier transform, put $z = \jmath\frac{2\pi i}{I}$ where I is the length of the input signal and $i = 0, 1, \ldots, I-1$

Download the following Python code that does the above

```
wget https://github.com/Pradeep8802/EE3900
    -Digital-Signal-Processing/blob/main/
    Assignment1/codes/8.1.py
```

8.2 Repeat all the exercises in the previous sections for the above a and b
Solution: The polynomial coefficients obtained are

$$\mathbf{a} = \begin{pmatrix} 1.000 \\ -2.519 \\ 2.561 \\ -1.206 \\ 0.220 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} 0.003 \\ 0.014 \\ 0.021 \\ 0.014 \\ 0.003 \end{pmatrix} \quad (8.4)$$

The difference equation is then given by

$$\mathbf{a}^\top \mathbf{y} = \mathbf{b}^\top \mathbf{x} \quad (8.5)$$

where

$$\mathbf{y} = \begin{pmatrix} y(n) \\ y(n-1) \\ y(n-2) \\ y(n-3) \\ y(n-4) \end{pmatrix} \qquad \mathbf{x} = \begin{pmatrix} x(n) \\ x(n-1) \\ x(n-2) \\ x(n-3) \\ x(n-4) \end{pmatrix} \quad (8.6)$$

We have

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{N} b\left(k\right) z^{-k}}{\sum_{m=0}^{M} a\left(m\right) z^{-m}} \quad (8.7)$$

By using partial fraction decomposition, we can write this as

$$H(z) = \sum_i \frac{r(i)}{1 - p(i)z^{-1}} + \sum_j k(j)z^{-j} \quad (8.8)$$

On taking the inverse Z-transform on both sides by using (4.22)

$$H(z) \overset{\mathcal{Z}}{\rightleftharpoons} h(n) \quad (8.9)$$

$$\frac{1}{1-p(i)z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} (p(i))^n \, u(n) \quad (8.10)$$

$$z^{-j} \overset{\mathcal{Z}}{\rightleftharpoons} \delta(n-j) \quad (8.11)$$

Thus

$$h(n) = \sum_i r(i) \left(p(i)\right)^n u(n) + \sum_j k(j)\delta(n-j) \quad (8.12)$$

Download the following Python code

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    Assignment1/codes/8.2.py
```

Run the code by executing The above code outputs the values of $r(i), p(i), k(i)$

$$h(n) = \Re\left((0.24 - 0.71\jmath)(0.56 + 0.14\jmath)^n\right) u(n) + \Re\left((0.24 + 0.71\jmath)(0.56 - 0.14\jmath)^n\right) u(n) + 0.016\delta(n) \quad (8.13)$$

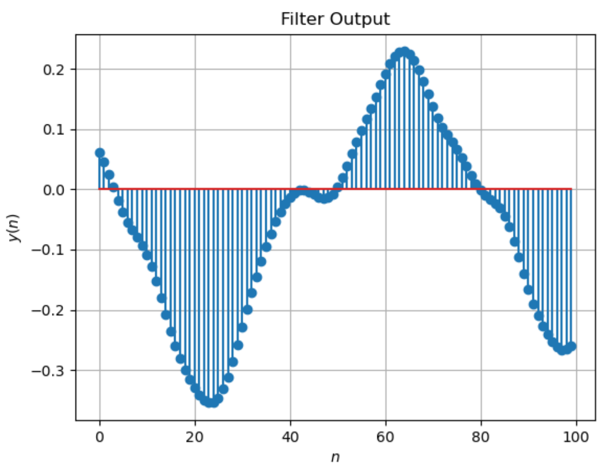

Fig. 8.2: Plot of $y(n)$

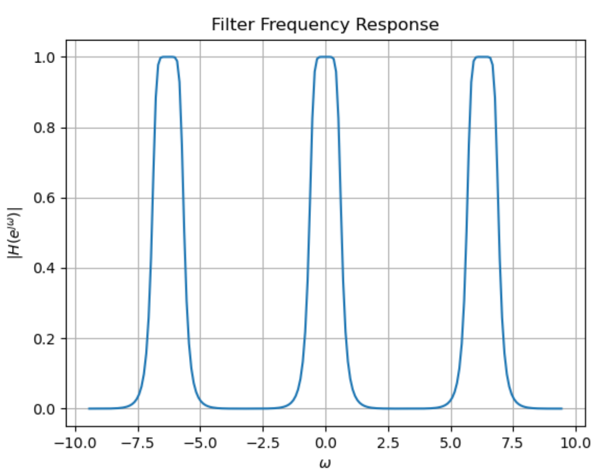

Fig. 8.2: Plot of $\left|H(e^{j\omega})\right|$

8.3 What is the sampling frequency of the input signal?
Solution: Sampling frequency(fs)=44.1kHZ.

8.4 What is type, order and cutoff-frequency of the above butterworth filter

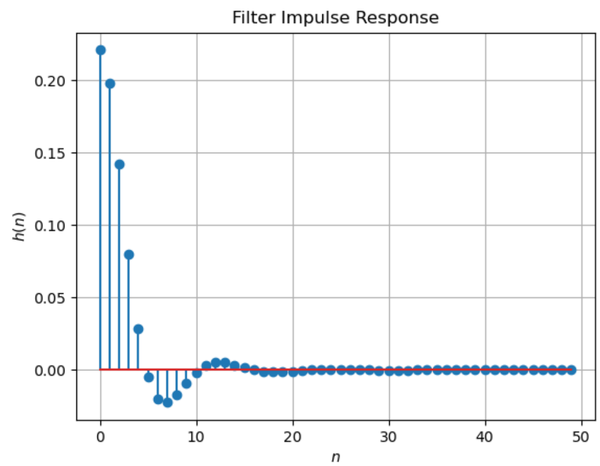

Fig. 8.2: Plot of $h(n)$

Solution: The given butterworth filter is low pass with order=4 and cutoff-frequency=4kHz.

8.5 Modifying the code with different input parameters and to get the best possible output.
Solution: Order: 10 Cutoff frequency: 3000 Hz