Digital Signal Processing
EE3900

I Sai Pradeep
AI21BTECH11013

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Contents

1. Definitions

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \tag{1.1}$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st}\, dt \tag{1.2}$$

2. Laplace Transform

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1\,\mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2\,\mu C$.

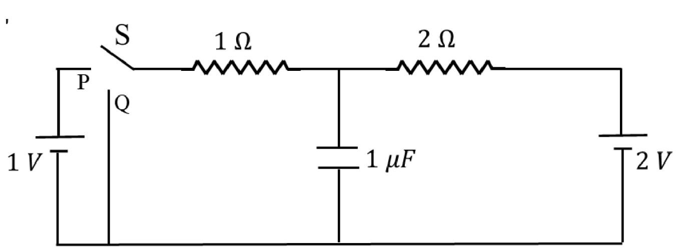

Fig. 2.1.

2. Draw the circuit using latex-tikz.
Solution: The following code yields Fig.2.2

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    cktsig/Tikz%20Circuits/2.2.tex
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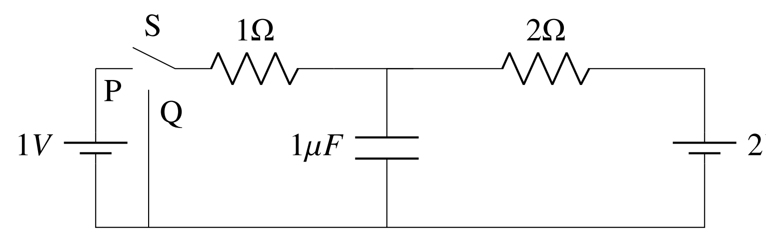

Fig. 2.2. Given Circuit

3. Find q_1.
Solution: Before switching S to Q: At steady state, which achieved when switch S is at P for long time capacoitor behaves as an open switch, hence current through capacitor is 0, Let i be the current flowing in the circuit at steady state. Applying KVL ,

$$1 - i - 2i - 2 = 0 \tag{2.1}$$

$$3i = -1 \Rightarrow i = \frac{-1}{3}A \tag{2.2}$$

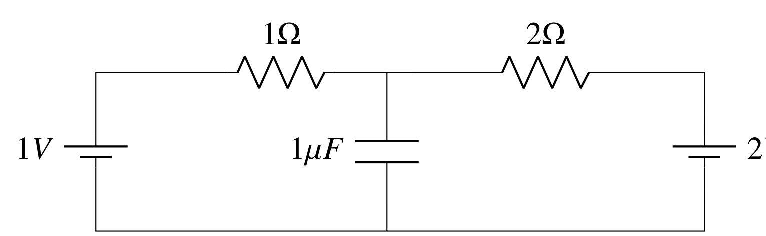

Fig. 2.3. Before switching S to Q

Potential Difference across the capacitor at steady state is

$$1 - \left(\frac{-1}{3}\right) = \frac{4}{3}V \tag{2.3}$$

$$q_1 = \frac{4}{3} \cdot 1 \tag{2.4}$$

$$= \frac{4}{3}\mu C \tag{2.5}$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.
Solution: We know that Laplace Transform fo function $f(t)$ is given as $F(s)$,

$$F(s) = \int_0^\infty f(t)e^{-st}\,dt \tag{2.6}$$

$$\tag{2.7}$$

For $u(t)$, we have,

$$F(s) = \int_0^\infty u(t)e^{-st}\,dt \tag{2.8}$$

Using (1.1),

$$F(s) = \int_0^\infty u(t)e^{-st}\,dt \tag{2.9}$$

$$= \int_0^\infty e^{-st}\,dt \tag{2.10}$$

$$= -\left(0 - \frac{1}{s}\right) \tag{2.11}$$

$$= \frac{1}{s} \tag{2.12}$$

ROC is $Re(s) > 0$ since for $s > 0$, $e^{-st} < \infty$ for $t \to \infty$

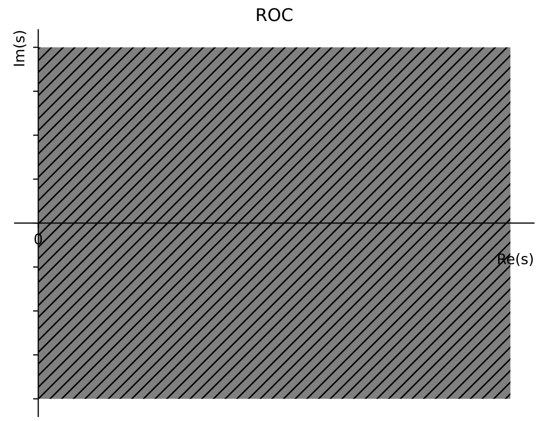

Fig. 2.4.

5. Show that

$$e^{-at}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s+a}, \quad a > 0 \tag{2.13}$$

and find the ROC.
Solution: From 2.6,

$$F(s) = \int_0^\infty u(t)e^{-at}e^{-st}\,dt \tag{2.14}$$

$$= \int_0^\infty u(t)e^{-(s+a)t}\,dt \tag{2.15}$$

$$= \int_0^\infty e^{-(s+a)t}\,dt \tag{2.16}$$

$$= -\left(0 - \frac{1}{s+a}\right) \tag{2.17}$$

$$= \frac{1}{s+a} \tag{2.18}$$

ROC is

$$Re(s) + a > 0 \Rightarrow Re(s) > -a \tag{2.19}$$

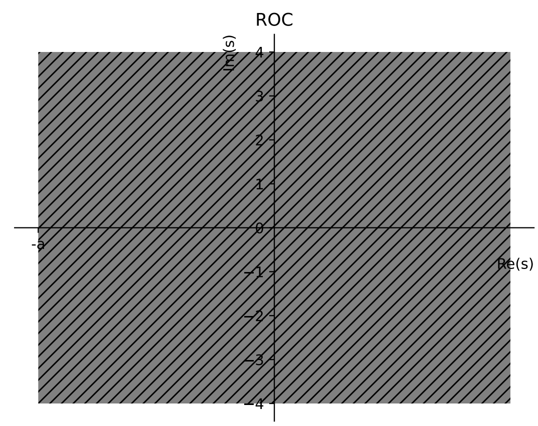

Fig. 2.5.

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_1(s) \tag{2.20}$$

$$2u(t) \overset{\mathcal{L}}{\longleftrightarrow} V_2(s) \tag{2.21}$$

Find the voltage across the capacitor $V_{C_0}(s)$.
Solution:

$$R_{eff} = \frac{\frac{2}{2} + \frac{1}{1}}{1 + \frac{1}{2}} = \frac{4}{3}\Omega \tag{2.22}$$

$$V_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}V \tag{2.23}$$

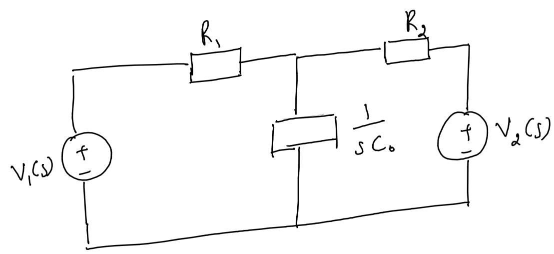

Fig. 2.6.

$$V_{C_0}(s) = V_S(s)\frac{C_0}{C_0 + R_{eff}} \tag{2.24}$$
$$= \left(\frac{4}{3s}\right)\left(\frac{\frac{1}{s}}{\frac{1}{s}+\frac{4}{3}}\right) \tag{2.25}$$
$$= \frac{4}{s\,(4s+3)} \tag{2.26}$$

7. Find $v_{C_0}(t)$. Plot using python.
Solution: Using (2.26),

$$\frac{4}{s\,(4s+3)} = \frac{4}{3s} - \frac{4}{3(\frac{3}{2}+s)} \tag{2.27}$$

Apply inverse Laplacian Transform,

$$V_{C_0}(s) \overset{\mathcal{L}^{-\infty}}{\longleftrightarrow} V_{C_0}(t) \tag{2.28}$$
$$\mathcal{L}^{-1}\left[V_{C_0}(s)\right] = \mathcal{L}^{-1}\left[\frac{4}{3s} - \frac{4}{3(\frac{3}{2}+s)}\right] \tag{2.29}$$
$$= \mathcal{L}^{-1}\left[\frac{4}{3s}\right] - \frac{4}{3}\mathcal{L}^{-1}\left[\frac{1}{\frac{3}{2}+s}\right] \tag{2.30}$$

Since,

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = u(t) \tag{2.31}$$
$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}u(t) \tag{2.32}$$

Using the above equations,

$$V_{C_0}(t) = \frac{4}{3}\left(1 - e^{\frac{-3}{2}t}\right)u(t) \tag{2.33}$$

8. Verify your result using ngspice.
Solution:

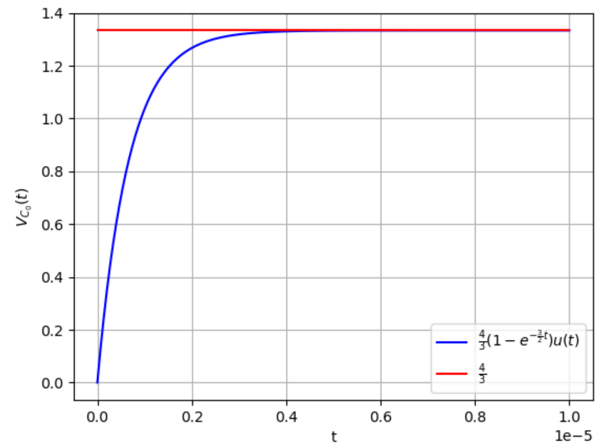

Fig. 2.7. Plot of $V_{C_0}(t)$

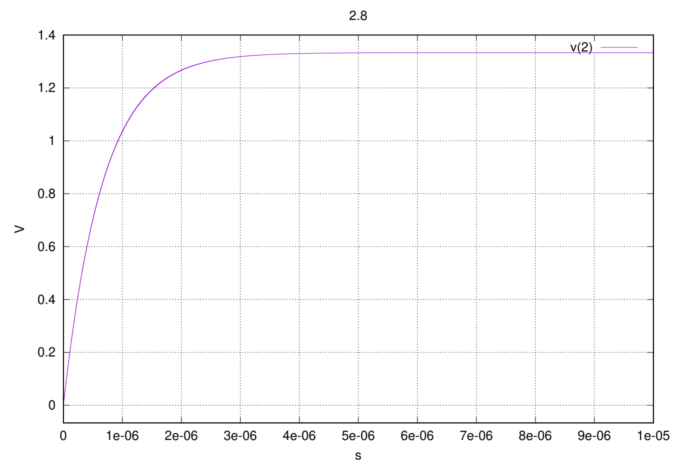

Fig. 2.8.

3. Initial Conditions

1. Find q_2 in Fig. 2.1.
Solution: At steady state capacitor behaves as an open switch. Hence $V_{C_0} = V_{1\Omega}$.
Let i be the current in the circuit. Using KVL,

$$2 - 2i - i = 0 \implies i = \frac{2}{3} \tag{3.1}$$
$$V_{1\Omega} = i \times 1 = \frac{2}{3}V \tag{3.2}$$
$$V_{C_0} = \frac{q_2}{C_0} = V_{1\Omega} = \frac{2}{3} \tag{3.3}$$
$$\implies q_2 = \frac{2}{3}\mu C \tag{3.4}$$

2. Draw the equivalent s-domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-

tikz.
Solution:

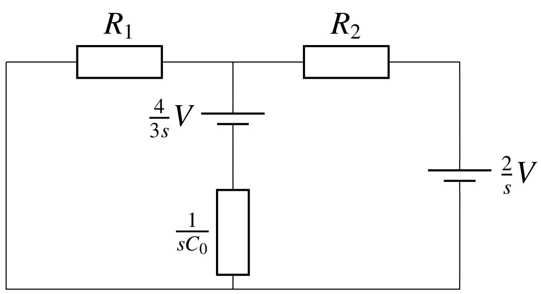

Fig. 3.1. After switching S to Q

3. $V_{C_0}(s) = ?$
Solution: Let voltage across capacitor be V. Using KCL at node in Fig. 3.1

$$\frac{V-0}{R_1} + \frac{V-\frac{2}{s}}{R_2} + sC_0\left(V - \frac{4}{3s}\right) = 0 \tag{3.5}$$

$$\implies V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \tag{3.6}$$

4. $v_{C_0}(t) = ?$ Plot using python.
Solution: From (3.6),

$$V_{C_0}(s) = \frac{4}{3}\left(\frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(\frac{1}{s} - \frac{1}{\frac{1}{C_0}\left(\frac{1}{R_1}+\frac{1}{R_2}\right)+s}\right) \tag{3.7}$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3}e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}u(t) + \frac{2}{R_2\left(\frac{1}{R_1}+\frac{1}{R_2}\right)}\left(1 - e^{-\left(\frac{1}{R_1}+\frac{1}{R_2}\right)\frac{t}{C_0}}\right)u(t) \tag{3.8}$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\left(1.5\times10^6\right)t}\right)u(t) \tag{3.9}$$

5. Verify your result using ngspice.
Solution: The figure 3.3 verifies our result.
6. Find $v_{C_0}(0-), v_{C_0}(0+)$ and $v_{C_0}(\infty)$.
Solution: From the initial conditions,

$$v_{C_0}(0-) = \frac{q_1}{C} = \frac{4}{3}V \tag{3.10}$$

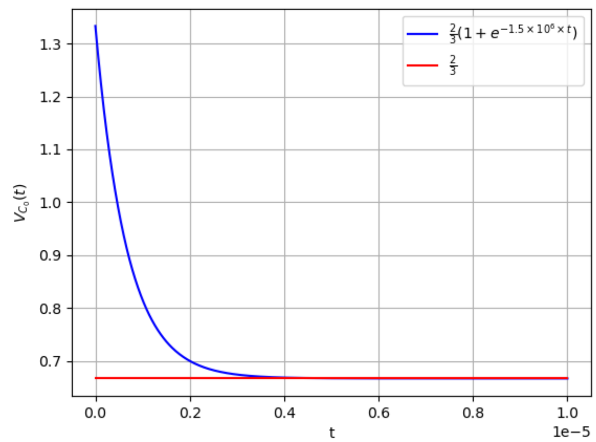

Fig. 3.2. Plot of $V_{C_0}(t)$

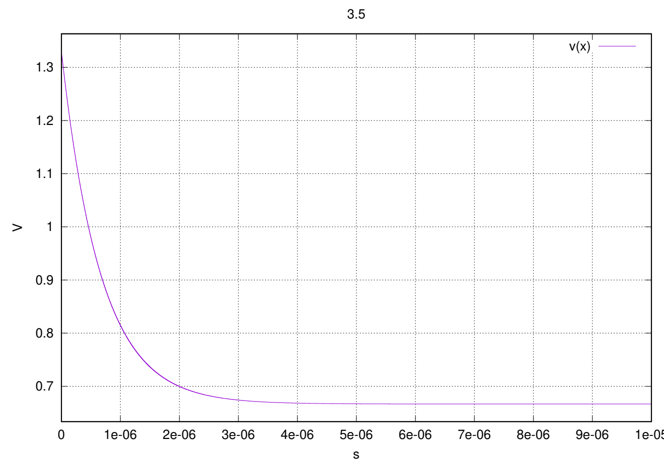

Fig. 3.3. ngspice plot of $V_{C_0}(t)$

Using (3.9),

$$v_{C_0}(0+) = \lim_{t\to0+} v_{C_0}(t) = \frac{4}{3}V \tag{3.11}$$

$$v_{C_0}(\infty) = \lim_{t\to\infty} v_{C_0}(t) = \frac{2}{3}V \tag{3.12}$$