Digital Signal Processing
EE3900

Fourier Series

I Sai Pradeep
AI21BTECH11013

October 13, 2022

Contents

***Abstract*—This manual provides a simple introduction to Fourier Series**

1. Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$.
Solution:

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    charger/codes/1.1.py
```

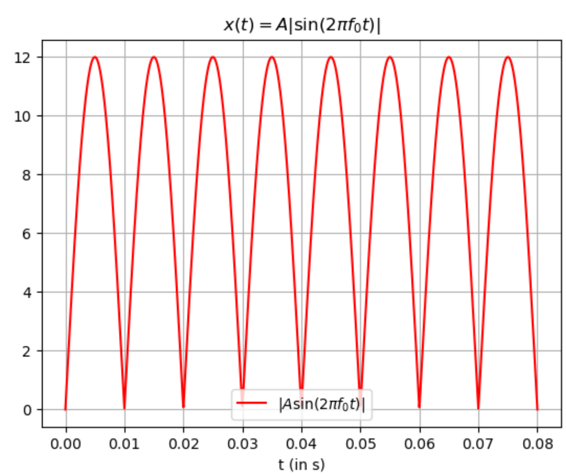

Fig. 1.1.

1.2 Show that $x(t)$ is periodic and find its period.
Solution: A signal $x(t)$ is said to be periodic with fundamental period T if

$$x(t + nT) = x(t)\forall n \in \mathbb{Z} \tag{1.2}$$

Let T be fundamental period of $x(t)$. Comparing (1.2) and (1.1), we get

$$A_0 \left|\sin\left(2\pi f_0 t\right)\right| = A_0 \left|\sin\left(2\pi f_0 (t + T)\right)\right| \tag{1.3}$$
$$\left|\sin\left(2\pi f_0 t\right)\right| = \left|\sin\left(2\pi f_0 (t + T)\right)\right| \tag{1.4}$$
$$\left|\sin\left(2\pi f_0 t\right)\right| = \left|\sin\left(2\pi f_0 t + 2\pi f_0 T\right)\right| \tag{1.5}$$

As $|sin\theta|$ is periodic with fundamental period $F = \pi$, Hence,

$$\left|\sin\left(t\right)\right| = \left|\sin\left(t + F\right)\right| \tag{1.6}$$

Hence,$2\pi f_0 T = \pi$, therefore, fundamental period(T) is

$$T = \frac{\pi}{2\pi f_0} = \frac{1}{2f_0} \tag{1.7}$$

2. Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations.

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.2}$$

Solution: From (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.3}$$

Mulitply $e^{-j2\pi l f_0 t}$ on both sides of (2.3), we get,

$$x(t)e^{-j2\pi l f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} e^{-j2\pi l f_0 t} \tag{2.4}$$

Integrating (2.4) w.r.t. t from $-T$ to T, and $T = \frac{1}{f_0}$, we get,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t}\,dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \sum_{k=-\infty}^{\infty} c_k e^{j2\pi(k-l)f_0 t}\,dt \tag{2.5}$$

$$= \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\,dt \tag{2.6}$$

Consider the following cases.
case-1:$k = l$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\,dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{0}\,dt \tag{2.7}$$

$$= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\,dt \tag{2.8}$$

case-2: $k \neq l$
Let $n = f_0(k-l)$, here $n \in \mathbb{Z}$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\,dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{2n\pi}\,dt \tag{2.9}$$

Here, $2n\pi T = 2f_0(k-l)T\pi$, and $2n\pi T = (k-l)\pi$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\,dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\,dt \tag{2.10}$$

$$= \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \cos(2n\pi) + j\sin(2n\pi)\,dt \tag{2.11}$$

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\,dt = -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.12}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.13}$$

$$= -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.14}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.15}$$

$$= -\sin(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.16}$$

$$+ j\cos(2n\pi t)\Big|_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} \tag{2.17}$$

$$\tag{2.18}$$

$$= -\sin((k-l)\pi) + \sin(-(k-l)\pi) \tag{2.19}$$

$$+ j\cos((k-l)\pi) - j\cos(-(k-l)\pi) \tag{2.20}$$

$$\tag{2.21}$$

Since $k - l \in \mathbb{Z}$,$\sin((k-l)\pi) = 0$ and $\sin(-(k-l)\pi) = 0$, similarly, as $\cos(\theta) = \cos(-\theta)$, we get $\cos((k-l)\pi) - \cos(-(k-l)\pi) = 0$
From (2.18),

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2n\pi}\,dt = \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\,dt \tag{2.22}$$

$$= 0 + j0 = 0 \tag{2.23}$$

Hence, we have,

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\,dt = \begin{cases} 0 & k \neq l \\ \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\,dt & k = l \end{cases} \tag{2.24}$$

From (2.5),

$$\int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t}\,dt = \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{j2\pi(k-l)f_0 t}\,dt \tag{2.25}$$

$$= c_k \times \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} 1\,dt \tag{2.26}$$

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t}\,dt \tag{2.27}$$

$$\therefore c_k = \frac{2}{T} \int_{-\frac{1}{T}}^{\frac{1}{T}} x(t)e^{-j2\pi k f_0 t}\,dt \tag{2.28}$$

2.2 Find c_k for (1.1)
Solution: We know that,

$$c_k = 2f_0 \int_0^{\frac{1}{2f_0}} x(t)e^{-j2\pi k f_0 t}\,dt \tag{2.29}$$

when $t \in \left(0, \frac{1}{2f_0}\right)$, $x(t) = A_0 \sin(2\pi f_0 t)$

$$c_k = 2f_0 \int_0^{\frac{1}{2f_0}} A_0 \left(\frac{e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}}{2j}\right) e^{-j2\pi k f_0 t}\,dt \tag{2.30}$$

$$= A_0 f_0 \int_0^{\frac{1}{2f_0}} \left(\frac{e^{j2\pi(1-k)f_0 t} - e^{j2\pi(-1-k)f_0 t}}{j}\right) dt \tag{2.31}$$

$$= A_0 f_0 \Bigg(\frac{e^{j2\pi(1-k)f_0 t}}{-2\pi(1-k)f_0}\Bigg|_0^{\frac{1}{2f_0}} \tag{2.32}$$

$$- \frac{e^{j2\pi(-1-k)f_0 t}}{-2\pi(-1-k)f_0}\Bigg|_0^{\frac{1}{2f_0}} \Bigg) \tag{2.33}$$

$$= A_0 \left[\frac{e^{j\pi(1-k)} - 1}{2\pi(k-1)} - \frac{e^{-j\pi(1+k)} - 1}{2\pi(k+1)}\right] \tag{2.34}$$

Hence,

$$c_k = \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k = even \\ 0 & k = odd \end{cases} \tag{2.35}$$

2.3 Verify (1.1) using python.
Solution:

```
wget https://github.com/Pradeep8802/EE3900
    -Digital-Signal-Processing/blob/main/
    charger/codes/2.3.py
        python3 2.3.py
```

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.36}$$

and obtain the formulae for a_k and b_k.
Solution: Using (2.1),

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t} \tag{2.37}$$

As,

$$e^{j2\pi k f_0 t} = \cos(2\pi k f_0 t) + j\sin(2\pi k f_0 t) \tag{2.38}$$

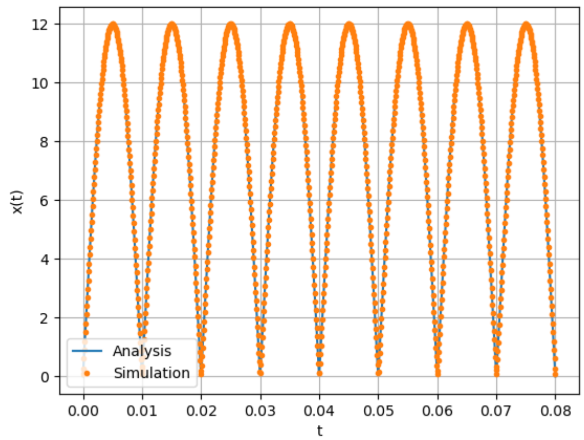

Fig. 2.3.

From (2.1), we have,

$$x(t) = \sum_{k=-\infty}^{\infty} c_k \left[\cos(2\pi k f_0 t) + j\sin(2\pi k f_0 t)\right] \tag{2.39}$$

$$= \sum_{k=-\infty}^{\infty} c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t) \tag{2.40}$$

$$= \sum_{k=-\infty}^{-1} \left[c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t)\right] \tag{2.41}$$

$$+ c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t)\right] \tag{2.42}$$

$$= \sum_{k=1}^{\infty} \left[c_{-k} \cos(2\pi k f_0 t) - jc_{-k} \sin(2\pi k f_0 t)\right] \tag{2.43}$$

$$+ c_0 + \sum_{k=1}^{\infty} \left[c_k \cos(2\pi k f_0 t) + jc_k \sin(2\pi k f_0 t)\right] \tag{2.44}$$

$$= c_0 + \sum_{k=1}^{\infty} \Bigg((c_k + c_{-k}) \cos(2\pi k f_0 t) \tag{2.45}$$

$$+ j(c_k - c_{-k}) \sin(2\pi k f_0 t) \Bigg) \tag{2.46}$$

Substituting $a_k = c_k + c_{-k}$ and $b_k = j(c_k - c_{-k})$,we get,

$$x(t) = c_0 + \sum_{k=1}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.47}$$

$$= \sum_{k=0}^{\infty} (a_k \cos 2\pi k f_0 t + b_k \sin 2\pi k f_0 t) \tag{2.48}$$

$$\therefore a_k = \begin{cases} c_k + c_{-k} & k \neq 0 \\ c_0 & k = 0 \end{cases} \tag{2.49}$$

$$b_k = j(c_k - c_{-k}) \tag{2.50}$$

Using (2.2),

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-j2\pi k f_0 t}\, dt \tag{2.51}$$

$$c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{j2\pi k f_0 t}\, dt \tag{2.52}$$

$$a_k = c_k + c_{-k} = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \left[e^{-j2\pi k f_0 t} + e^{j2\pi k f_0 t} \right] dt \tag{2.53}$$

$$= 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \cos(2\pi k f_0 t)\, dt \tag{2.54}$$

Similarly, for b_k, we get,

$$b_k = -j \left\{ 2f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) \sin\{2\pi k f_0 t\}\, dt \right\} \tag{2.55}$$

2.5 Find a_k and b_k for (1.1)
Solution: Using (2.49) and (2.50) with (2.35),

$$a_k = c_k + c_{-k} = \begin{cases} \frac{4A_0}{\pi(1-k^2)} & k = even \\ \frac{2A_0}{\pi} & k = 0 \\ 0 & k = odd \end{cases} \tag{2.56}$$

$$b_k = j(c_k - c_{-k}) = 0 \tag{2.57}$$

2.6 Verify (2.36) using python.
Solution:

```
wget https://github.com/Pradeep8802/EE3900
    −Digital−Signal−Processing/blob/main/
    charger/codes/2.6.py
        python3 2.3.py
```

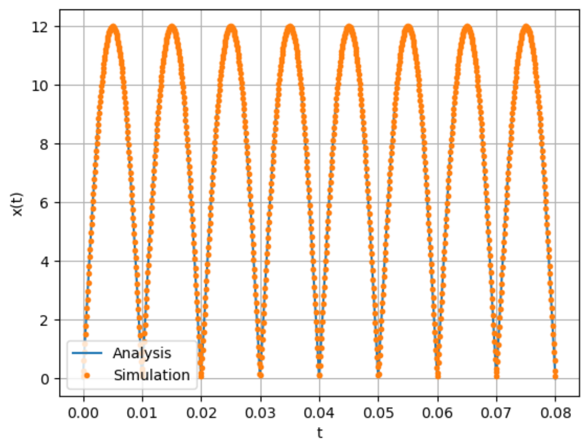

Fig. 2.6.