

Digital Signal Processing

EE3900

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1. DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

2. LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1 \mu C$. Then S is switched to position Q . After a long time, the charge on the capacitor is $q_2 \mu C$.

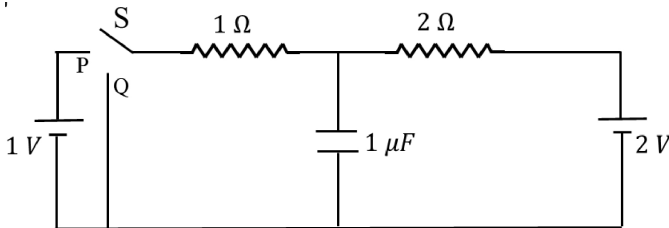


Fig. 2.1.

2. Draw the circuit using latex-tikz.

Solution: The following code yields Fig.2.2

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/blob/main/
cktsig/Tikz%20Circuits/2.2.tex
```

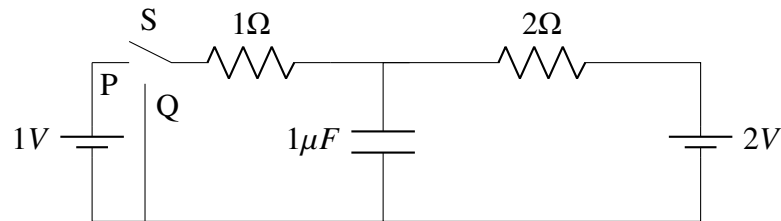


Fig. 2.2. Given Circuit

3. Find q_1 .

Solution: Before switching S to Q : At steady state, which is achieved when switch S is at P for a long time, the capacitor behaves as an open switch, hence the current through the capacitor is 0. Let i be the current flowing in the circuit at steady state. Applying KVL,

$$1 - i - 2i - 2 = 0 \quad (2.1)$$

$$3i = -1 \Rightarrow i = -\frac{1}{3} A \quad (2.2)$$

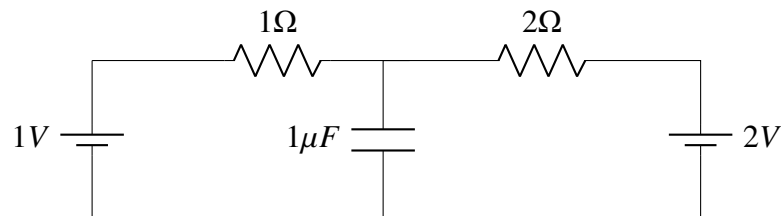


Fig. 2.3. Before switching S to Q

Potential Difference across the capacitor at steady state is

$$1 - \left(\frac{-1}{3} \right) = \frac{4}{3} V \quad (2.3)$$

$$q_1 = \frac{4}{3} \cdot 1 \quad (2.4)$$

$$= \frac{4}{3} \mu C \quad (2.5)$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution: We know that Laplace Transform for function $f(t)$ is given as $F(s)$,

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (2.6)$$

$$(2.7)$$

For $u(t)$, we have,

$$F(s) = \int_0^{\infty} u(t)e^{-st} dt \quad (2.8)$$

Using (1.1),

$$F(s) = \int_0^{\infty} u(t)e^{-st} dt \quad (2.9)$$

$$= \int_0^{\infty} e^{-st} dt \quad (2.10)$$

$$= -\left(0 - \frac{1}{s}\right) \quad (2.11)$$

$$= \frac{1}{s} \quad (2.12)$$

ROC is $Re(s) > 0$ since for $s > 0$, $e^{-st} < \infty$ for $t \rightarrow \infty$

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \quad (2.13)$$

and find the ROC.

Solution: From 2.6,

$$F(s) = \int_0^{\infty} u(t)e^{-at}e^{-st} dt \quad (2.14)$$

$$= \int_0^{\infty} u(t)e^{-(s+a)t} dt \quad (2.15)$$

$$= \int_0^{\infty} e^{-(s+a)t} dt \quad (2.16)$$

$$= -\left(0 - \frac{1}{s+a}\right) \quad (2.17)$$

$$= \frac{1}{s+a} \quad (2.18)$$

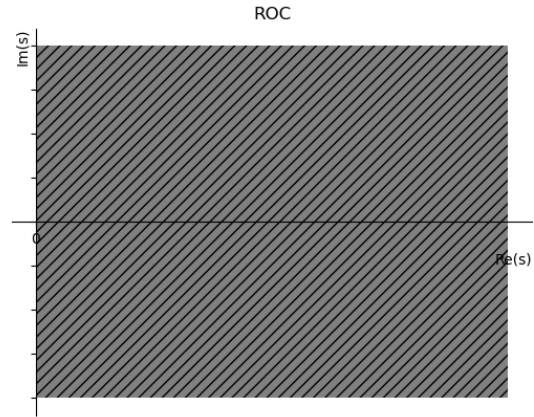


Fig. 2.4.

ROC is

$$Re(s) + a > 0 \Rightarrow Re(s) > -a \quad (2.19)$$

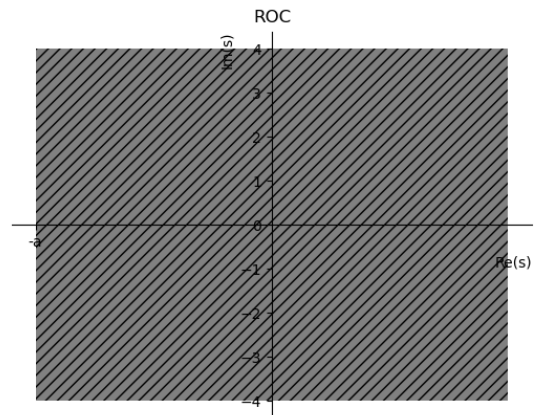


Fig. 2.5.

6. Now consider the following resistive circuit transformed from Fig. 2.1 where

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \quad (2.20)$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \quad (2.21)$$

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution:

$$R_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3} \Omega \quad (2.22)$$

$$V_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3} V \quad (2.23)$$



Fig. 2.6.

$$V_{C_0}(s) = V_S(s) \frac{C_0}{C_0 + R_{eff}} \quad (2.24)$$

$$= \left(\frac{4}{3s} \right) \left(\frac{\frac{1}{s}}{\frac{1}{s} + \frac{2}{3}} \right) \quad (2.25)$$

$$= \frac{3 + 4s}{3s \left(s + \frac{3}{2} \right)} \quad (2.26)$$

7. Find $v_{C_0}(t)$. Plot using python.

Solution: Running the following code gives the plot.

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/2.7.py
```

Using (2.26),

$$\frac{3 + 4s}{3s \left(s + \frac{3}{2} \right)} = \frac{2}{3s} + \frac{2}{3 \left(\frac{3}{2} + s \right)} \quad (2.27)$$

Apply inverse Laplacian Transform,

$$V_{C_0}(s) \xleftrightarrow{\mathcal{L}^{-1}} V_{C_0}(t) \quad (2.28)$$

$$\mathcal{L}^{-1} [V_{C_0}(s)] = \mathcal{L}^{-1} \left[\frac{2}{3s} + \frac{2}{3 \left(\frac{3}{2} + s \right)} \right] \quad (2.29)$$

$$= \mathcal{L}^{-1} \left[\frac{2}{3s} \right] - \frac{2}{3} \mathcal{L}^{-1} \left[\frac{1}{\frac{3}{2} + s} \right] \quad (2.30)$$

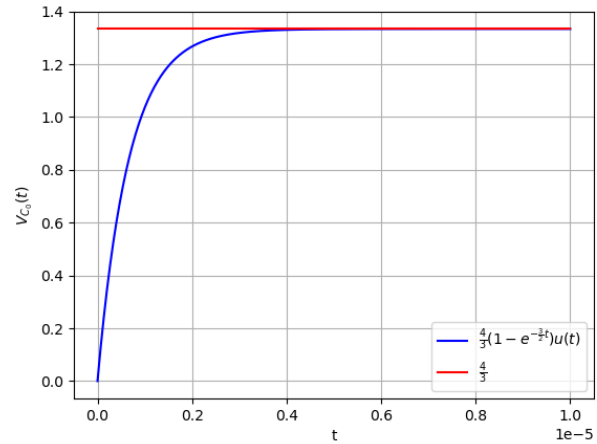
Since,

$$\mathcal{L}^{-1} \left[\frac{1}{s} \right] = u(t) \quad (2.31)$$

$$\mathcal{L}^{-1} \left[\frac{1}{s - a} \right] = e^{at} u(t) \quad (2.32)$$

Using the above equations,

$$V_{C_0}(t) = \frac{2}{3} \left(1 + e^{-\frac{3}{2}t} \right) u(t) \quad (2.33)$$

Fig. 2.7. Plot of $V_{C_0}(t)$

8. Verify your result using ngspice.

Solution:

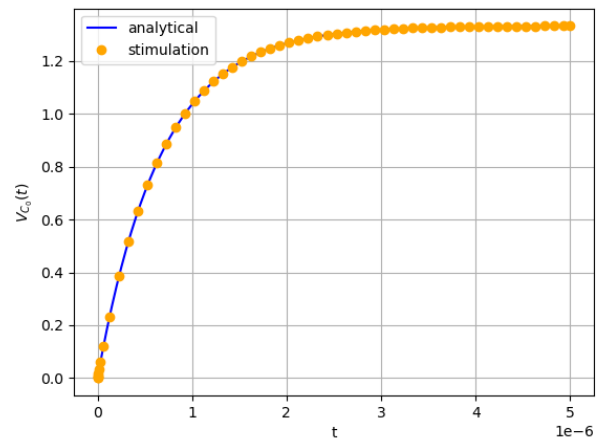


Fig. 2.8.

9. Obtain Fig. 2.7 using the equivalent differential equation

Solution: Results obtained can be verified by running the following code.

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/2.8.cir
```

And is plotted using the below code.

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/2.8.py
```

Using Kirchoff's junction law

$$\frac{v_c(t) - v_1(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (2.34)$$

where $q(t)$ is the charge on the capacitor

On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + (sQ(s) - q(0^-)) = 0 \quad (2.35)$$

But $q(0^-) = 0$ and

$$q(t) = C_0 v_c(t) \quad (2.36)$$

$$\Rightarrow Q(s) = C_0 V_c(s) \quad (2.37)$$

Thus

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sC_0 V_c(s) = 0 \quad (2.38)$$

$$\Rightarrow \frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - 0}{\frac{1}{sC_0}} = 0 \quad (2.39)$$

which is the same equation as the one we obtained from Fig. 2.7

3. INITIAL CONDITIONS

1. Find q_2 in Fig. 2.1.

Solution: At steady state capacitor behaves as an open switch. Hence $V_{C_0} = V_{1\Omega}$.

Let i be the current in the circuit. Using KVL,

$$2 - 2i - i = 0 \Rightarrow i = \frac{2}{3} \quad (3.1)$$

$$V_{1\Omega} = i \times 1 = \frac{2}{3} V \quad (3.2)$$

$$V_{C_0} = \frac{q_2}{C_0} = V_{1\Omega} = \frac{2}{3} \quad (3.3)$$

$$\Rightarrow q_2 = \frac{2}{3} \mu C \quad (3.4)$$

2. Draw the equivalent s -domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.

Solution:

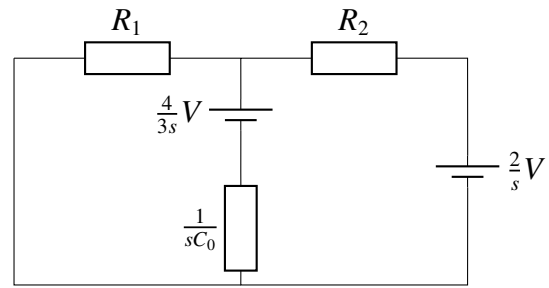


Fig. 3.1. After switching S to Q

3. $V_{C_0}(s) = ?$

Solution: Let voltage across capacitor be V . Using KCL at node in Fig. 3.1

$$\frac{V - 0}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0 \left(V - \frac{4}{3s} \right) = 0 \quad (3.5)$$

$$\Rightarrow V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \quad (3.6)$$

4. $v_{C_0}(t) = ?$ Plot using python.

Solution: Running the following code gives the plot.

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/3.4.py
```

From (3.6),

$$V_{C_0}(s) = \frac{4}{3} \left(\frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(\frac{1}{s} - \frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (3.7)$$

Taking an inverse Laplace Transform,

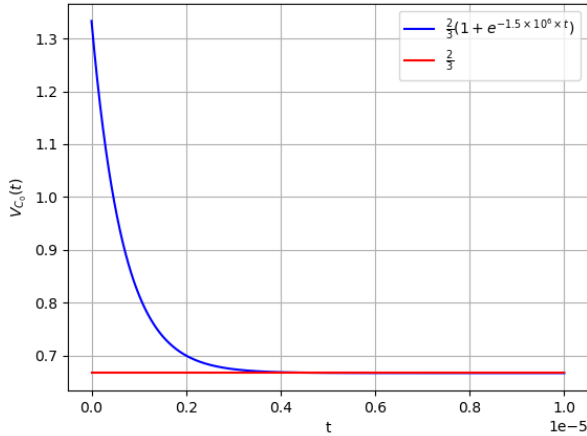
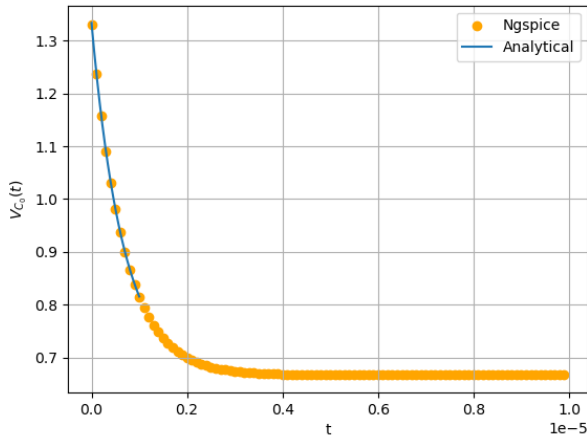
$$v_{C_0}(t) = \frac{4}{3} e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} u(t) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} \right) u(t) \quad (3.8)$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3} \left(1 + e^{-(1.5 \times 10^6)t} \right) u(t) \quad (3.9)$$

5. Verify your result using ngspice.

Solution: Results obtained can be verified by running the following code.

Fig. 3.2. Plot of $V_{C_0}(t)$ Fig. 3.3. ngspice plot of $V_{C_0}(t)$

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/3.5.cir
```

Runningn the below code plots the figure 3.3, and verifies our result.

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/3.5.py
```

6. Find $v_{C_0}(0^-)$, $v_{C_0}(0^+)$ and $v_{C_0}(\infty)$.

Solution: From the initial conditions,

$$v_{C_0}(0^-) = \frac{q_1}{C} = \frac{4}{3}V \quad (3.10)$$

Using (3.9),

$$v_{C_0}(0^+) = \lim_{t \rightarrow 0^+} v_{C_0}(t) = \frac{4}{3}V \quad (3.11)$$

$$v_{C_0}(\infty) = \lim_{t \rightarrow \infty} v_{C_0}(t) = \frac{2}{3}V \quad (3.12)$$

7. Obtain Fig. 3.2 using the equivalent differential equation

Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (3.13)$$

where $q(t)$ is the charge on the capacitor. On taking the Laplace transform on both sides of the equation (3.13), we get,

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sQ(s) - q(0^-) = 0 \quad (3.14)$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \quad (3.15)$$

$$\Rightarrow Q(s) = C_0 V_c(s) \quad (3.16)$$

Thus

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sC_0 V_c(s) - \frac{4}{3}C_0 \right) = 0 \quad (3.17)$$

$$\Rightarrow \frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \quad (3.18)$$

which is the same equation as the one we obtained from Fig. 3.2

4. BILINEAR TRANSFORM

- 4.1. In Fig. 2.1, consider the case when S is switched to Q right in the beginning. Formulate the differential equation

Solution: Considering KCL on the circuit 4.1, we get the differential equation as

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (4.1)$$

$$\Rightarrow \frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0 \frac{dv_c}{dt} = 0 \quad (4.2)$$

Here we have $q(0) = 0$, since initially the capacitor is uncharged.

- 4.2. Find $H(s)$ considering the output voltage at the capacitor

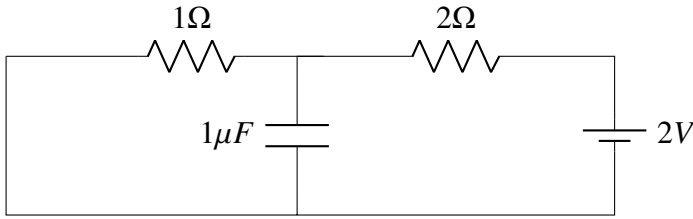


Fig. 4.1. Switch S connected to Q initially

Solution: Applying laplace transform to the equation (4.1), we get

$$\frac{V_c(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \mathcal{L}\left(\frac{dq}{dt}\right) = 0 \quad (4.3)$$

$$\frac{V_c(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sQ(s) - q(0) = 0 \quad (4.4)$$

$$\frac{V_c(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sQ(s) - 0 = 0 \quad (4.5)$$

$$\Rightarrow V_c(s) \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + sC_0 V_c(s) = \frac{V_2(s)}{R_2} \quad (4.6)$$

$$\Rightarrow \frac{V_c(s)}{V_2(s)} = \frac{\frac{1}{R_2}}{\frac{1}{R_1} + \frac{1}{R_2} + sC_0} \quad (4.7)$$

Here, $Q(s)$ is the laplace transform of q , $V_c(s)$ is laplace transform of $v_c(t)$. Hence, the transform function($H(s)$) is

$$H(s) = \frac{V_c(s)}{V_2(s)} \quad (4.8)$$

$$= \frac{\frac{1}{R_2 C_0}}{s + \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0}} \quad (4.9)$$

Substituting values of $R_1 = 1\Omega$, $R_2 = 2\Omega$ and $C_0 = 1\mu F$, we get,

$$H(s) = \frac{0.5}{s \cdot 10^{-6} + 1.5} \quad (4.10)$$

$$\Rightarrow H(s) = \frac{5 \times 10^5}{s + 1.5 \times 10^6} \quad (4.11)$$

4.3. Plot $H(s)$. What kind of filter is it?

Solution: THE below python code plots the Figure 4.2

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/4.3.py
```

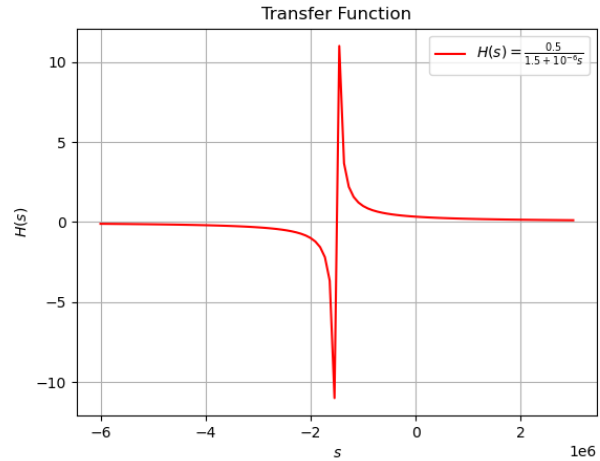


Fig. 4.2. Plot of $H(s)$

Considering the frequency-domain transfer function ($H(s = e^{j\omega})$), from (4.11), we get

$$H(s = j\omega) = \frac{5 \times 10^5}{j\omega + 1.5 \times 10^6} \quad (4.12)$$

$$\Rightarrow |H(s = j\omega)| = \frac{5 \times 10^5}{\sqrt{\omega^2 + 2.25 \times 10^{12}}} \quad (4.13)$$

Clearly from (4.12), as ω increases, $H(s = j\omega)$ decreases (inverse proportionality). When high frequency signals (large values of ω) pass through this transfer function ($H(s = j\omega)$), they become negligible, which results in removing high frequency signals and allowing only low frequency signal to pass. Hence, this is a low-pass filter.

4.4. Using trapezoidal rule for integration, formulate the difference equation by considering

$$y(n) = y(t)|_{t=n} \quad (4.14)$$

Solution: In the equation (4.1), we have

$$\frac{dq}{dt} = C_0 \frac{dv_c}{dt} \quad (4.15)$$

$$v_2(t) = 2u(n) \quad (4.16)$$

Hence,

$$\frac{v_c(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + C_0 \frac{dv_c}{dt} = 0 \quad (4.17)$$

$$\Rightarrow C_0 \frac{dv_c}{dt} = \frac{2u(t) - v_c(t)}{R_2} - \frac{v_c(t)}{R_1} \quad (4.18)$$

$$\Rightarrow \frac{dv_c}{dt} = \frac{2u(t) - v_c(t)}{R_2 C_0} - \frac{v_c(t)}{R_1 C_0} \quad (4.19)$$

$$\Rightarrow v_c(t)|_{t=n}^{n+1} = \int_n^{n+1} \left(\frac{2u(t) - v_c(t)}{R_2 C_0} - \frac{v_c(t)}{R_1 C_0} \right) dt \quad (4.20)$$

From trapezoidal rule of integration

$$\int_a^b f(t) dt \approx \frac{b-a}{2} (f(a) + f(b)) \quad (4.21)$$

Apply (4.21), to the RHS of the equation (4.20), we get,

$$\begin{aligned} \int_n^{n+1} \frac{2u(t) - v_c(t)}{R_2 C_0} - \frac{v_c(t)}{R_1 C_0} dt &= \frac{1}{R_2 C_0} (u(n) + u(n+1)) \\ &\quad - \frac{1}{2} (y(n+1) + y(n)) \left(\frac{1}{R_1 C_0} + \frac{1}{R_2 C_0} \right) \end{aligned} \quad (4.22)$$

Considering $y(t) = v_c(t)$, from (4.20), we get,

$$\begin{aligned} y(n+1) - y(n) &= \frac{1}{R_2 C_0} (u(n) + u(n+1)) \\ &\quad - \frac{1}{2} (y(n+1) + y(n)) \left(\frac{1}{R_1 C_0} + \frac{1}{R_2 C_0} \right) \end{aligned} \quad (4.24)$$

Thus, the difference equation is

$$\begin{aligned} \Rightarrow y(n+1) &\left(1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \\ &= y(n) \left(1 - \frac{1}{2R_1 C_0} - \frac{1}{2R_2 C_0} \right) \\ &\quad + \frac{1}{R_2 C_0} (u(n) + u(n+1)) \end{aligned} \quad (4.25)$$

4.5. Find $H(z)$

Solution: Let $\mathcal{Z}\{y(n)\} = Y(z)$

On taking \mathcal{Z} -transform on both sides of the difference equation, we get,

$$\begin{aligned} zY(z) &\left(1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \\ &= Y(z) \left(1 - \frac{1}{2R_1 C_0} - \frac{1}{2R_2 C_0} \right) \\ &\quad + \frac{1}{R_2 C_0} \left(\frac{1}{1-z^{-1}} + \frac{z}{1-z^{-1}} \right) \end{aligned} \quad (4.26)$$

$$\begin{aligned} Y(z) &\left(z + \frac{z}{2R_1 C_0} + \frac{z}{2R_2 C_0} - 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) \\ &= \frac{1}{R_2 C_0} \frac{1+z}{1-z^{-1}} \end{aligned} \quad (4.27)$$

Here, since $v_2(t) = 2Vt \geq 0$

Initial voltage is given as,

$$\Rightarrow x(n) = 2u(n) \quad (4.28)$$

$$\Rightarrow X(z) = \frac{2}{1-z^{-1}} \quad |z| > 1 \quad (4.29)$$

Thus, the transfer function in z -domain is

$$\begin{aligned} H(z) &= \frac{Y(z)}{X(z)} \quad (4.30) \\ &= \frac{\frac{1+z}{2R_2 C_0}}{z + \frac{z}{2R_1 C_0} + \frac{z}{2R_2 C_0} - 1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0}} \quad (4.31) \\ &= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} - z^{-1} + \frac{z^{-1}}{2R_1 C_0} + \frac{z^{-1}}{2R_2 C_0}} \quad (4.32) \end{aligned}$$

Substituting the values of R_1, R_2 and C_0 , we get,

$$H(z) = \frac{2.5 \times 10^5 (1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.33)$$

Where, ROC of $H(z)$ is,

$$|z| > 1 \cap |z| > \left| \frac{7.5 \times 10^5 - 1}{7.5 \times 10^5 + 1} \right| \quad (4.34)$$

$$\Rightarrow |z| > 1 \quad (4.35)$$

4.6. How can you obtain $H(z)$ from $H(s)$?

Solution: The Z -transform can be obtained from the Laplace transform by the substitution

$$s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}} \quad (4.36)$$

where T is the sampling time period, used in the trapezoidal rule. Here, its value is 1. This

known as the bilinear transform.

From (4.20), we have,

$$H(z) = \frac{\frac{1}{R_2 C_0}}{2 \frac{1-z^{-1}}{1+z^{-1}} + \frac{1}{R_1 C_0} + \frac{1}{R_2 C_0}} \quad (4.37)$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 - z^{-1} + \left(\frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} \right) (1 + z^{-1})} \quad (4.38)$$

$$= \frac{\frac{1+z^{-1}}{2R_2 C_0}}{1 + \frac{1}{2R_1 C_0} + \frac{1}{2R_2 C_0} - z^{-1} + \frac{z^{-1}}{2R_1 C_0} + \frac{z^{-1}}{2R_2 C_0}} \quad (4.39)$$

$$= \frac{2.5 \times 10^5 (1 + z^{-1})}{7.5 \times 10^5 + 1 + (7.5 \times 10^5 - 1)z^{-1}} \quad (4.40)$$

Here, this result obtained in the equation (4.40) is same as the result we obtained in (4.33)