

Random Numbers

AI1110

I Sai Pradeep
AI21BTECH11013

July 5, 2022

CONTENTS

1	Uniform Random Numbers	1
2	Central Limit Theorem	3
3	From Uniform to Other	5
4	Triangular Distribution	6

1. UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat

Solution: Download the C source code by executing the following commands

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/1.1/coeffs.h
```

Compile and run the C program by executing the following

```
cc exrand.c -lm
./a.out
```

- 1.2 Load the uni.dat file into Python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/Pradeep8802/
Random_numbers/tree/main/1.2/cdf_U.
py
```

Run the code by executing the below command

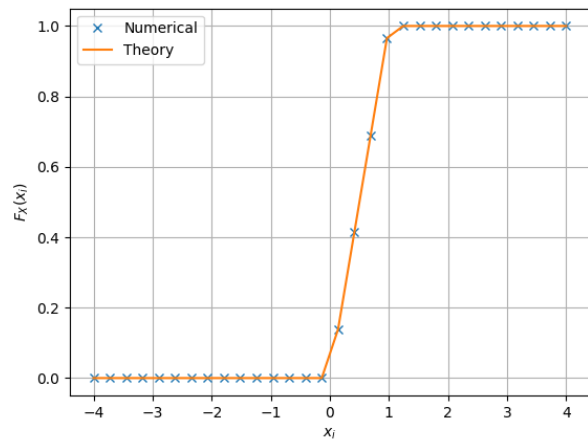


Fig. 1.2. The CDF of U

```
python3 cdf_U.py
```

- 1.3 Find a theoretical expression for $F_U(x)$

Solution: Here U is a uniform random variable in $[0, 1]$. The PDF of U is given as follows

$$p_U(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad (1.2)$$

The CDF of U is

$$F_U(x) = \Pr(U \leq x) \quad (1.3)$$

Since,

$$\Pr(U \leq x) = \int_{-\infty}^x p_U(x) \, dx \quad (1.4)$$

$$\therefore F_U(x) = \int_{-\infty}^x p_U(x) \, dx$$

If $x < 0$,

$$\int_{-\infty}^x p_U(x) dx = \int_{-\infty}^x 0 dx \quad (1.5)$$

$$= 0 \quad (1.6)$$

$$F_U(x) = 0 \quad (1.7)$$

If $0 \leq x \leq 1$,

$$\int_{-\infty}^x p_U(x) dx = \int_{-\infty}^0 0 dx + \int_0^x 1 dx \quad (1.8)$$

$$= 0 + x \quad (1.9)$$

$$= x \quad (1.10)$$

$$F_U(x) = x \quad (1.11)$$

If $x > 1$,

$$\begin{aligned} \int_{-\infty}^x p_U(x) dx \\ = \int_{-\infty}^0 0 dx + \int_0^1 1 dx + \int_1^x 0 dx \end{aligned} \quad (1.12)$$

$$\int_{-\infty}^x p_U(x) dx = 0 + 1 + 0 \quad (1.13)$$

$$= 1 \quad (1.14)$$

$$F_U(x) = 1 \quad (1.15)$$

Hence the CDF of U is

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (1.16)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.17)$$

and its variance as

$$\text{Var}[U] = E[(U - E[U])^2] \quad (1.18)$$

Write a C program to find the mean and variance of U

Solution: Download the following codes and execute the C program

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/1.4/
mean_variance.c
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/1.1/coeffs.h
```

Run the code by executing the below command

```
cc mean_variance.c -lm
./a.out
```

The output of the code is

$$\mu = 0.500007 \quad (1.19)$$

$$\sigma^2 = 0.083301 \quad (1.20)$$

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.21)$$

Solution: The mean of U is given by

$$E[U] = \int_{-\infty}^{\infty} x dx \quad (1.22)$$

The CDF of U is

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \quad (1.23)$$

On differentiating the above CDF, we have

$$dF_U(x) = \begin{cases} d(0) & x < 0 \\ d(x) & 0 \leq x \leq 1 \\ d(1) & x > 1 \end{cases} \quad (1.24)$$

$$dF_U(x) = \begin{cases} 0 & x < 0 \\ dx & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases} \quad (1.25)$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \quad (1.26)$$

$$E[U^k] = \int_{-\infty}^{\infty} x^k d(x) \quad (1.27)$$

$$E[U] = \int_0^1 x dx = \frac{1}{2} \quad (1.28)$$

$$E[U^2] = \int_0^1 x^2 dx = \frac{1}{3} \quad (1.29)$$

$$\text{Var}(U) = E[U^2] - (E[U])^2 = \frac{1}{12} \quad (1.30)$$

Hence,

$$E[U] = \frac{1}{2} \approx 0.5 \quad (1.31)$$

$$\mu = 0.5 \quad (1.32)$$

$$\text{Var}(U) = \frac{1}{12} \approx 0.083333 \quad (1.33)$$

$$\sigma = 0.083333 \quad (1.34)$$

$$\Delta\mu \approx 0.000007 \quad (1.35)$$

$$\Delta\sigma \approx 0.000032 \quad (1.36)$$

2. CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following codes and execute the C program

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/2.1/2.1.c
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/2.1/coeffs
.h
```

Run the code by executing the below command

```
cc 2.1.c -lm
./a.out
```

2.2 Load gau.dat in Python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: Download the following code and execute the Python program

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/2.2/
cdf_gaussian.py
```

Run the code by executing the below command

```
python3 cdf_gaussian.py
```

Properties of CDF:

- (a) It is always continuous.
- (b) It always lies in $[0,1]$
- (c) Q function is defined as:

$$Q_X(x) = \Pr(x < X) \quad (2.2)$$

$$\therefore F_X(x) = 1 - Q_X(x) \quad (2.3)$$

- (d) CDF ($F_X(x)$) is non-decreasing function as it is the sum of non-negative probabilities.

2.3 Load gau.dat in Python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.4)$$

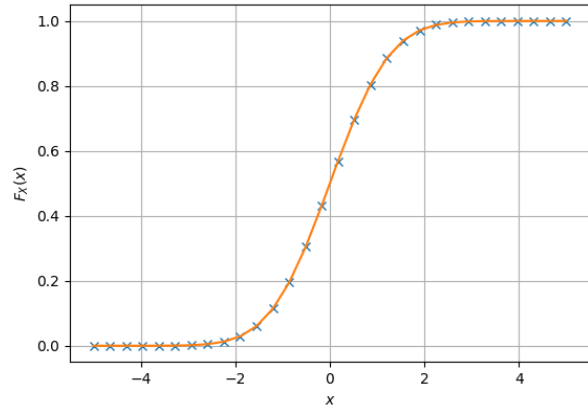


Fig. 2.5. The CDF of X

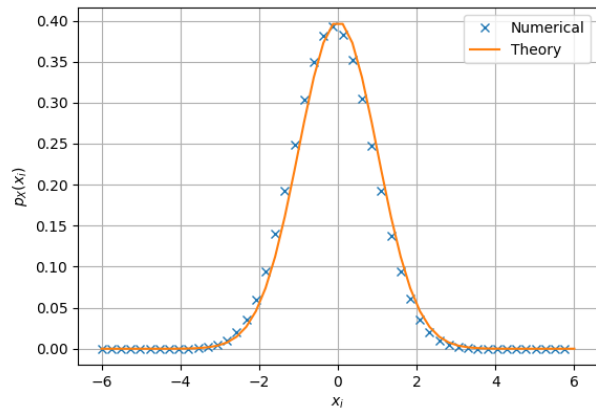


Fig. 2.5. The PDF of X

What properties does the PDF have?

Solution: Download the following code and execute the Python program

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/2.3/
pdf_gaussian.py
```

Run the code by executing the below command

```
python3 pdf_gaussian.py
```

Properties of PDF:

- (a) PDF($p_X(x)$) is bounded between $[0,1]$
- (b) PDF is a bell-shaped curve
- (c) PDF graph has symmetry around the mean of the distribution
- (d) If μ is the mean of the distribution, here $p_X(x)$ is symmetric with respect to 0,

since here $\mu = 0$ ($\mathcal{N}(0, 1)$)

2.4 Find the mean and variance of X by writing a C program

Solution: Download the following code and execute the C program

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/2.4/coeffs
.h
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/2.4/
mean_variance_gaussian.c
```

Run the code by executing the below command

```
cc mean_variance_gaussian.c -lm
./a.out
```

$$\mu = 0.000326 \quad (2.5)$$

$$\sigma = 1.000906 \quad (2.6)$$

$$\text{Mean} = 0.000326 \quad (2.7)$$

$$\text{Variance} = 1.000906 \quad (2.8)$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.9)$$

repeat the above exercise theoretically

Solution: As $p_X(x)$ is given, Hence, the mean of X is

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.10)$$

$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.11)$$

$$\begin{aligned} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^0 x \exp\left(-\frac{x^2}{2}\right) dx \\ &+ \frac{1}{\sqrt{2\pi}} \int_0^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \end{aligned} \quad (2.12)$$

Let $g(x) = x e^{-\frac{x^2}{2}}$

Consider $\int_{-\infty}^0 g(x) dx$

Substituting $z = -x$, we get

$$\int_{-\infty}^0 g(x) dx = \int_{\infty}^0 g(-x) d(-x) \quad (2.13)$$

$$= - \int_{\infty}^0 g(-x) d(x) \quad (2.14)$$

$$= \int_0^{\infty} g(-x) d(x) \quad (2.15)$$

$$g(-x) = -x e^{-\frac{(-x)^2}{2}} \quad (2.16)$$

$$g(-x) = -(x e^{-\frac{x^2}{2}}) \quad (2.17)$$

$$g(-x) = g(x) \quad (2.18)$$

$$\int_{-\infty}^0 g(x) dx = - \int_0^{\infty} g(x) d(x) \quad (2.19)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.20)$$

$$= \int_0^{\infty} g(x) d(x) - \int_0^{\infty} g(x) d(x) = 0 \quad (2.21)$$

\therefore Mean = 0

Let

$$h(x) = \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (2.22)$$

$$h(-x) = \frac{(-x)^2}{\sqrt{2\pi}} \exp\left(-\frac{(-x)^2}{2}\right) \quad (2.23)$$

$$= \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (2.24)$$

$$(2.25)$$

$\therefore h(-x) = h(x)$

$$\therefore \int_{-\infty}^0 \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx = \int_0^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.26)$$

$$E[X^2] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.27)$$

$$= 2 \int_0^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.28)$$

Using integration by parts,

$$E[X^2] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} (x) \cdot (x \exp\left(-\frac{x^2}{2}\right)) dx \quad (2.29)$$

$$= \sqrt{\frac{2}{\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right) dx \right) \Big|_0^\infty - \sqrt{\frac{2}{\pi}} \int_0^\infty 1 \cdot \left(\int x \exp\left(-\frac{x^2}{2}\right) dx \right) dx \quad (2.30)$$

Substituting $z = \frac{x^2}{2} \implies dz = x dx$

$$(2.31)$$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-z) dz \quad (2.32)$$

$$= -\exp(-z) \quad (2.33)$$

$$= -\exp\left(-\frac{x^2}{2}\right) \quad (2.34)$$

Substituting $x = \sqrt{2}t$

$$\int_0^\infty \exp(-t^2) \sqrt{2} dt = \sqrt{2} \int_0^\infty \exp(-t^2) dt \quad (2.35)$$

$$= \sqrt{\frac{\pi}{2}} \quad (2.36)$$

$$E[X^2] = 0 - \sqrt{\frac{2}{\pi}} \left(-\sqrt{\frac{\pi}{2}} \right) \quad (2.37)$$

$$= 1 \quad (2.38)$$

$$\therefore \text{Var}[X] = E[X^2] - (E[X])^2 \quad (2.39)$$

$$= 1 - 0 \quad (2.40)$$

$$= 1 \quad (2.41)$$

$$\text{Var}[X] = 1 \quad (2.42)$$

3. FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF

Solution: Download the following code and execute the C program

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/3.1/3.1.c
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/3.1/coeffs.h
```

Run the code by executing the below command

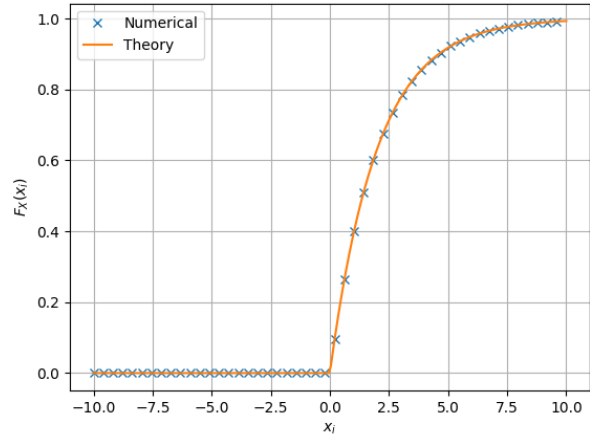


Fig. 3.5. The CDF of V

cc 3.1.c -lm
./a.out

Download the following code and execute the Python program

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/3.1/cdf_3.1.
py
```

Run the code by executing the below command

```
python3 cdf_3.1.py
```

3.6 Find a theoretical expression for $F_V(x)$

Solution: As $F_V(x)$ is the CDF of V,

$$F_V(x) = P(V \leq x) \quad (3.2)$$

$$= P(-2 \ln(1 - U) \leq x) \quad (3.3)$$

$$= -2 \ln(1 - U) \leq x \quad (3.4)$$

$$= \ln(1 - U) \geq -x/2 \quad (3.5)$$

Since $\ln(x)$ and $-x/2$ are monotonic, we can write

$$1 - U \geq e^{-x/2} \implies 1 - e^{-x/2} \leq U$$

Hence,

$$F_V(x) = P(-2 \log(1 - U) \leq x) = P(1 - e^{-\frac{x}{2}} \geq U) \quad (3.6)$$

Since,

$$p_U(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad (3.7)$$

$$P(U < x) = \int_0^x dx = x \quad (3.8)$$

$$\therefore P(U \leq 1 - e^{-\frac{x}{2}}) = 1 - e^{-\frac{x}{2}}, \forall x \geq 0$$

4. TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the following code and execute the C program

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/4.1/4.1.c
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/4.1/coeffs
.h
```

Run the code by executing the below command

```
cc 4.1.c -lm
./a.out
```

4.2 Find the CDF of T

Solution: Download the following code and execute the Python program

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/4.2/cdf_4
.2_num.py
```

Run the code by executing the below command

```
python3 cdf_4.2_num.py
./a.out
```

$$T = U_1 + U_2 \quad (4.2)$$

$$p_T(x) = p_{U_1+U_2}(x) \quad (4.3)$$

Consider the convolution of the two distributions,

$$p_T(x) = p_{U_1+U_2}(x) \quad (4.4)$$

$$= p_{U_1}(x) * p_{U_2}(x) d\tau \quad (4.5)$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau) p_{U_2}(x - \tau) d\tau \quad (4.6)$$

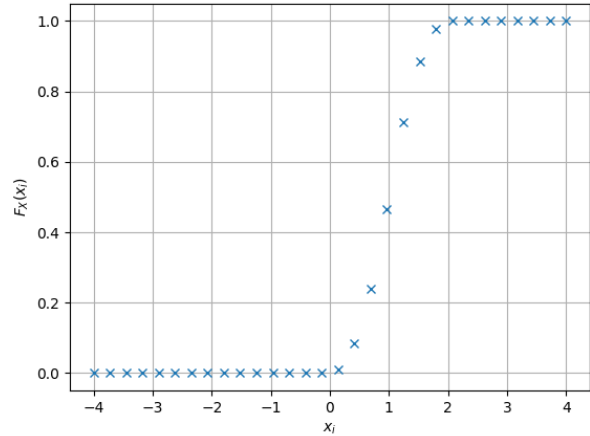


Fig. 4.6. The Numerical CDF of T

$$\text{For } \tau \in (-\infty, 0) \cup (1, \infty) \\ p_{U_1}(\tau) = 0 \quad (4.7)$$

Hence,

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau) p_{U_2}(x - \tau) d\tau \quad (4.8)$$

$$= \int_0^1 p_{U_2}(x - \tau) d\tau \quad (4.9)$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x 1 d\tau & 0 < x < 1 \\ \int_{x-1}^1 1 d\tau & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.10)$$

Hence

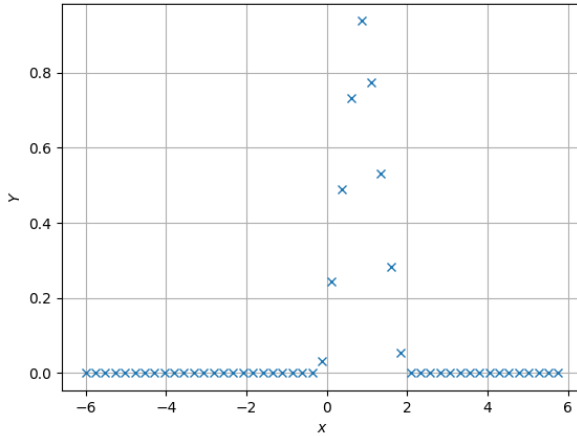
$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.11)$$

Expression for CDF can be obtained by Integrating $p_T(x)$ w.r.t. x

$$F_T(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \leq x < 2 \\ 1 & x > 2 \end{cases} \quad (4.12)$$

4.3 Find the PDF of T

Solution: Download the following code and execute the Python program

Fig. 4.6. The Numerical PDF of T

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/4.3/
pdf_4.3_num.py
```

Run the code by executing the below command

```
python3 pdf_4.3_num.py
./a.out
```

The PDF of T is given by

$$p_T(t) = \frac{d(F_T(t))}{dt} \quad (4.13)$$

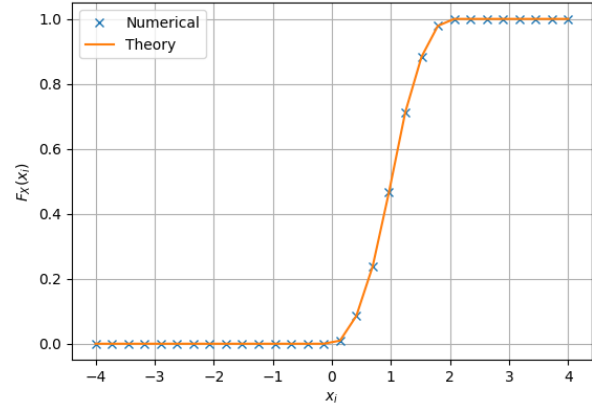
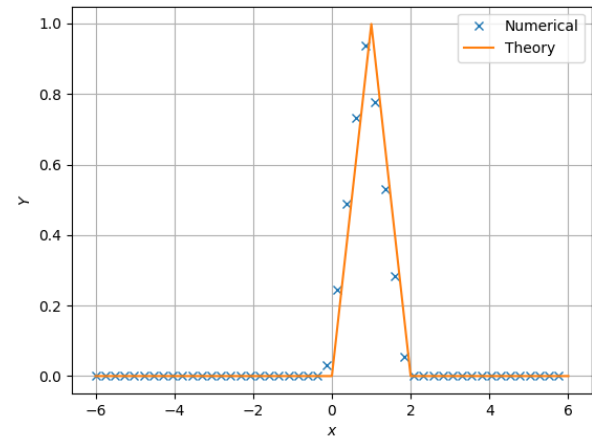
$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \quad (4.14)$$

4.4 Find the theoretical expressions for the PDF and CDF of T

Solution:

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 0 < t \leq 2 \\ 0 & t > 2 \end{cases} \quad (4.15)$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \frac{t^2}{2} - 1 & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \quad (4.16)$$

Fig. 4.6. The CDF of T Fig. 4.6. The PDF of T

4.5 Verify your results through a plot

Solution: Download the following code and execute the Python programs

```
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/4.4/cdf_4
.2.py
wget https://github.com/Pradeep8802/
Random_numbers/blob/main/4.4/
pdf_4.3.py
```

Run the codes by executing the below commands for cdf and pdf respectively

```
python3 cdf_4.2.py
python3 pdf_4.3.py
```