Random Numbers
AI1110

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Contents

1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat
Solution: Download the C source code by executing the following commands

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/1.1/exrand.c
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/1.1/coeffs.h
```

Compile and run the C program by executing the following

```
cc exrand.c −lm
./a.out
```

1.2 Load the uni.dat file into Python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
wget https://github.com/Pradeep8802/
    Random_numbers/tree/main/1.2/cdf_U.
    py
```

Run the code by executing the below command

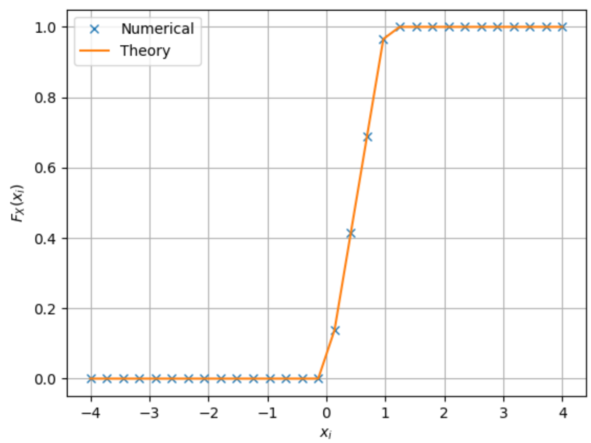

Fig. 1.2. The CDF of U

```
python3 cdf_U.py
```

1.3 Find a theoretical expression for $F_U(x)$
Solution: Here U is a uniform random variable in $[0, 1]$. The PDF of U is given as follows

$$p_U(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

The CDF of U is

$$F_U(x) = \Pr(U \leq x) \tag{1.3}$$

Since,

$$\Pr(U \leq x) = \int_{-\infty}^{x} p_U(x)\, \mathrm{d}x \tag{1.4}$$

$$\therefore F_U(x) = \int_{-\infty}^{x} p_U(x)\, \mathrm{d}x$$

If $x < 0$,

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x = \int_{-\infty}^{x} 0\, \mathrm{d}x \tag{1.5}$$
$$= 0 \tag{1.6}$$
$$F_U(x) = 0 \tag{1.7}$$

If $0 \leq x \leq 1$,

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x = \int_{-\infty}^{0} 0\, \mathrm{d}x + \int_{0}^{x} 1\, \mathrm{d}x \tag{1.8}$$
$$= 0 + x \tag{1.9}$$
$$= x \tag{1.10}$$
$$F_U(x) = x \tag{1.11}$$

If $x > 1$,

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x$$
$$= \int_{-\infty}^{0} 0\, \mathrm{d}x + \int_{0}^{1} 1\, \mathrm{d}x + \int_{1}^{x} 0\, \mathrm{d}x \tag{1.12}$$

$$\int_{-\infty}^{x} p_U(x)\, \mathrm{d}x = 0 + 1 + 0 \tag{1.13}$$
$$= 1 \tag{1.14}$$
$$F_U(x) = 1 \tag{1.15}$$

Hence the CDF of U is

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \tag{1.16}$$

1.4 The mean of U is defined as

$$E\left[U\right] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.17}$$

and its variance as

$$\mathrm{Var}\left[U\right] = E[(U - E[U])^2] \tag{1.18}$$

Write a C program to find the mean and variance of U

Solution: Download the following codes and execute the C program

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/1.4/
    mean_variance.c
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/1.1/coeffs.h
```

Run the code by executing the below command

```
cc mean_variance.c −lm
./a.out
```

The output of the code is

$$\mu = 0.500007 \tag{1.19}$$
$$\sigma^2 = 0.083301 \tag{1.20}$$

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k \mathrm{d}F_U(x) \tag{1.21}$$

Solution: The mean of U is given by

$$E\left[U\right] = \int_{-\infty}^{\infty} x\, \mathrm{d}x \tag{1.22}$$

The CDF of U is

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \tag{1.23}$$

On differentiating the above CDF, we have

$$\mathrm{d}F_U(x) = \begin{cases} \mathrm{d}(0) & x < 0 \\ \mathrm{d}(x) & 0 \leq x \leq 1 \\ \mathrm{d}(1) & x > 1 \end{cases} \tag{1.24}$$

$$\mathrm{d}F_U(x) = \begin{cases} 0 & x < 0 \\ \mathrm{d}x & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases} \tag{1.25}$$

$$E\left[U\right] = \int_{-\infty}^{\infty} x\, \mathrm{d}F_U(x) \tag{1.26}$$

$$E[U^k] = \int_{-\infty}^{\infty} x^k \mathrm{d}(x) \tag{1.27}$$

$$E[U] = \int_{0}^{1} x\mathrm{d}(x) = \frac{1}{2} \tag{1.28}$$

$$E[U^2] = \int_{0}^{1} x^2 \mathrm{d}x = \frac{1}{3} \tag{1.29}$$

$$Var(U) = E[U^2] - (E[U])^2 = \frac{1}{12} \tag{1.30}$$

Hence,

$$E[U] = \frac{1}{3} \approx 0.5 \tag{1.31}$$
$$\mu = 0.5 \tag{1.32}$$
$$Var(U) = \frac{1}{12} \approx 0.083333 \tag{1.33}$$
$$\sigma = 0.083333 \tag{1.34}$$
$$\Delta\mu \approx 0.000007 \tag{1.35}$$
$$\Delta\sigma \approx 0.000032 \tag{1.36}$$

2. Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution: Download the following codes and execute the C program

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/2.1/2.1.c
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/2.1/coeffs
    .h
```

Run the code by executing the below command

```
cc 2.1.c −lm
./a.out
```

2.2 Load gau.dat in Python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: Download the following code and execute the Python program

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/2.2/
    cdf_gaussian.py
```

Run the code by executing the below command

```
python3 cdf_gaussian.py
```

Properties of CDF:

(a) It is always continuous.
(b) It always lies in [0,1]
(c) Q function is defined as:

$$Q_X(x) = \Pr(x < X) \tag{2.2}$$

$$\therefore F_X(x) = 1 - Q_X(x) \tag{2.3}$$

(d) CDF ($F_X(x)$) is non-decreasing function as it is the sum of non-negative probabilities.

2.3 Load gau.dat in Python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.4}$$

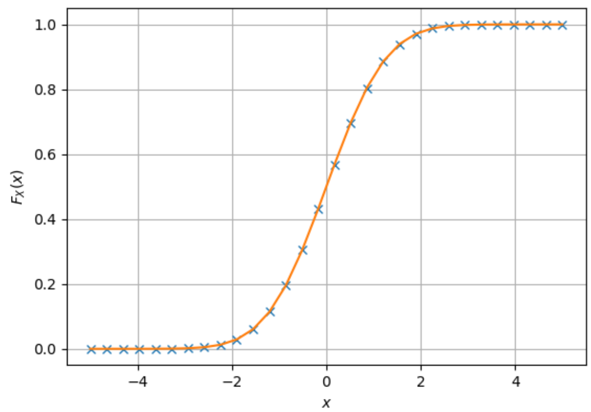

Fig. 2.5. The CDF of X

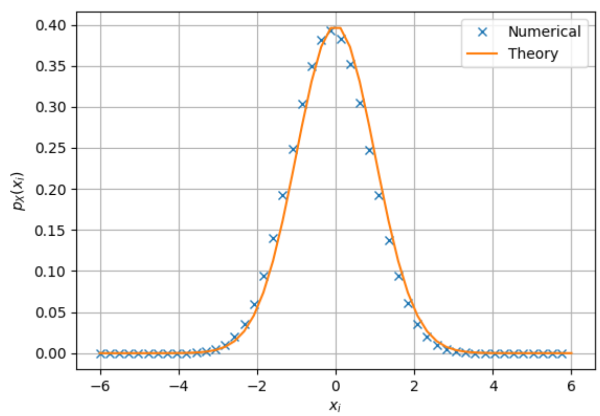

Fig. 2.5. The PDF of X

What properties does the PDF have?

Solution: Download the following code and execute the Python program

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/2.3/
    pdf_gaussian.py
```

Run the code by executing the below command

```
python3 pdf_gaussian.py
```

Properties of PDF:

(a) PDF($p_X(x)$) is bounded between [0,1]
(b) PDF is a bell-shaped curve
(c) PDF graph has symmetry around the mean of the distribution
(d) If μ is the mean of the distribution, here $p_X(x)$ is symmetric with respect to 0,

since here $\mu = 0$ ($\mathcal{N}(0,1)$)

2.4 Find the mean and variance of X by writing a C program

Solution: Download the following code and execute the C program

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/2.4/coeffs
    .h
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/2.4/
    mean_variance_gaussian.c
```

Run the code by executing the below command

```
cc mean_variance_gaussian.c -lm
./a.out
```

$$\mu = 0.000326 \tag{2.5}$$
$$\sigma = 1.000906 \tag{2.6}$$
$$\text{Mean} = 0.000326 \tag{2.7}$$
$$\text{Variance} = 1.000906 \tag{2.8}$$

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.9}$$

repeat the above exercise theoretically

Solution: As $p_X(x)$ is given, Hence, the mean of X is

$$E[X] = \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.10}$$
$$E[X] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.11}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.12}$$

Let $g(x) = xe^{-\frac{-x^2}{2}}$

Consider $\int_{-\infty}^{0} g(x)\mathrm{d}x$

Substituting z=-x, we get

$$\int_{-\infty}^{0} g(x)\mathrm{d}x = \int_{\infty}^{0} g(-x)\mathrm{d}(-x) \tag{2.13}$$
$$= -\int_{\infty}^{0} g(-x)\mathrm{d}(x) \tag{2.14}$$
$$= \int_{0}^{\infty} g(-x)\mathrm{d}(x) \tag{2.15}$$
$$g(-x) = -xe^{-\frac{-(-x)^2}{2}} \tag{2.16}$$
$$g(-x) = -(xe^{-\frac{-(x)^2}{2}}) \tag{2.17}$$
$$g(-x) = g(x) \tag{2.18}$$
$$\int_{-\infty}^{0} g(x)\mathrm{d}x = -\int_{0}^{\infty} g(x)\mathrm{d}(x) \tag{2.19}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.20}$$
$$= \int_{0}^{\infty} g(x)\mathrm{d}(x) - \int_{0}^{\infty} g(x)\mathrm{d}(x) = 0 \tag{2.21}$$

$\therefore$ Mean $= 0$

Let

$$h(x) = \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.22}$$
$$h(-x) = \frac{(-x)^2}{\sqrt{2\pi}} \exp\left(-\frac{(-x)^2}{2}\right) \tag{2.23}$$
$$= \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.24}$$
$$\tag{2.25}$$

$\therefore h(-x) = h(x)$

$$\therefore \int_{-\infty}^{0} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x = \int_{0}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.26}$$

$$E\left[X^2\right] = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.27}$$
$$= 2\int_{0}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \tag{2.28}$$

Using integration by parts,

$$E\left[X^2\right] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} (x) \cdot (x \exp\left(-\frac{x^2}{2}\right)) dx \tag{2.29}$$

$$= \sqrt{\frac{2}{\pi}}\left(x\int x\exp\left(-\frac{x^2}{2}\right)dx\right)\Big|_0^\infty$$
$$-\sqrt{\frac{2}{\pi}}\int_0^\infty 1\cdot(\int x\exp\left(-\frac{x^2}{2}\right))dx \quad (2.30)$$

$$\text{Substituting } z = \frac{x^2}{2} \implies dz = xdx \quad (2.31)$$

$$\int x\exp\left(-\frac{x^2}{2}\right)dx = \int \exp(-z)dz \quad (2.32)$$
$$= -\exp(-z) \quad (2.33)$$
$$= -\exp\left(-\frac{x^2}{2}\right) \quad (2.34)$$

Substituting $x = \sqrt{2}t$

$$\int_0^\infty \exp(-t^2)\sqrt{2}dt = \sqrt{2}\int_0^\infty \exp(-t^2)dt \quad (2.35)$$
$$= \sqrt{\frac{\pi}{2}} \quad (2.36)$$
$$E[X^2] = 0 - \sqrt{\frac{2}{\pi}}\left(-\sqrt{\frac{\pi}{2}}\right) \quad (2.37)$$
$$= 1 \quad (2.38)$$
$$\therefore Var[X] = E\left[X^2\right] - (E[X])^2 \quad (2.39)$$
$$= 1 - 0 \quad (2.40)$$
$$= 1 \quad (2.41)$$
$$Var[X] = 1 \quad (2.42)$$

3. From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \quad (3.1)$$

and plot its CDF

Solution: Download the following code and execute the C program

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/3.1/3.1.c
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/3.1/coeffs.h
```

Run the code by executing the below command

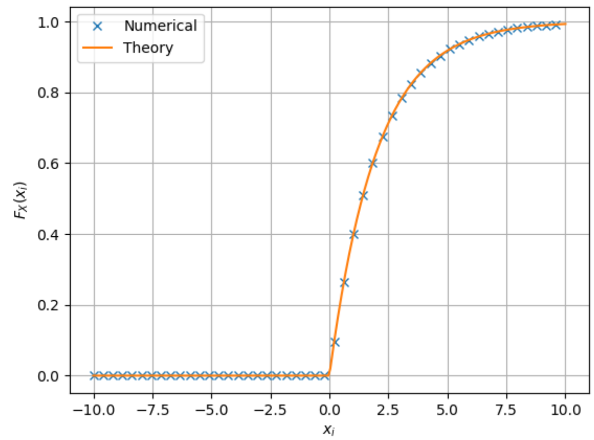

Fig. 3.5. The CDF of V

```
cc 3.1.c -lm
./a.out
```

Download the following code and execute the Python program

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/3.1/cdf_3.1.
    py
```

Run the code by executing the below command

```
python3 cdf_3.1.py
```

3.6 Find a theoretical expression for $F_V(x)$
Solution: As $F_V(x)$ is the CDF of V,

$$F_V(x) = P(V \le x) \quad (3.2)$$
$$= P(-2ln(1-U) \le x) \quad (3.3)$$
$$= -2ln(1-U) \le x \quad (3.4)$$
$$= ln(1-U) \ge -x/2 \quad (3.5)$$

Since ln(x) and -x/2 are monotonic, we can write

$$1 - U \ge e^{-x/2} \implies 1 - e^{-x/2} \le U$$

Hence,

$$F_V(x) = P(-2log(1-U) \le x) \quad = P(1 - e^{\frac{-x}{2}} \ge U) \quad (3.6)$$

Since,

$$p_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \quad (3.7)$$

$$P(U < x) = \int_0^x \mathrm{d}x = x \quad (3.8)$$

$$\therefore P(U \leq 1 - e^{\frac{-x}{2}}) = 1 - e^{\frac{-x}{2}}, \forall x \geq 0$$

4. Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: Download the following code and execute the C program

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/4.1/4.1.c
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/4.1/coeffs
    .h
```

Run the code by executing the below command

```
cc 4.1.c -lm
./a.out
```

4.2 Find the CDF of T

Solution: Download the following code and execute the Python program

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/4.2/cdf_4
    .2_num.py
```

Run the code by executing the below command

```
python3 cdf_4.2_num.py
./a.out
```

$$T = U1 + U2 \quad (4.2)$$

$$p_T(x) = p_{U_1+U_2}(x) \quad (4.3)$$

Consider the convolution of the two distributions,

$$p_T(x) = p_{U_1+U_2}(x) \quad (4.4)$$

$$= p_{U_1}(x) * p_{U_2}(x)\mathrm{d}\tau \quad (4.5)$$

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau)p_{U_2}(x-\tau)\mathrm{d}\tau \quad (4.6)$$

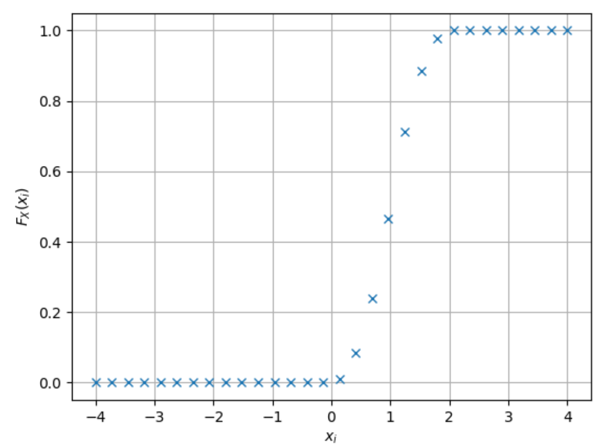

Fig. 4.6. The Numerical CDF of T

$$\text{For} \quad \tau \in (-\infty, 0) \cup (1, \infty)$$

$$p_{U_1}(\tau) = 0 \quad (4.7)$$

Hence,

$$p_T(x) = \int_{-\infty}^{\infty} p_{U_1}(\tau)p_{U_2}(x-\tau)\mathrm{d}\tau \quad (4.8)$$

$$= \int_0^1 p_{U_2}(x-\tau)\mathrm{d}\tau \quad (4.9)$$

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x 1d\tau & 0 < x < 1 \\ \int_{x-1}^1 1d\tau & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.10)$$

Hence

$$p_T(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 2 - x & 1 \leq x < 2 \\ 0 & x > 2 \end{cases} \quad (4.11)$$

Expression for CDF can be obtained by Integrating $p_T(x)$ w.r.t. x

$$F_T(x) = \begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 < x < 1 \\ -\frac{x^2}{2} + 2x - 1 & 1 \leq x < 2 \\ 1 & x > 2 \end{cases} \quad (4.12)$$

4.3 Find the PDF of T

Solution: Download the following code and execute the Python program

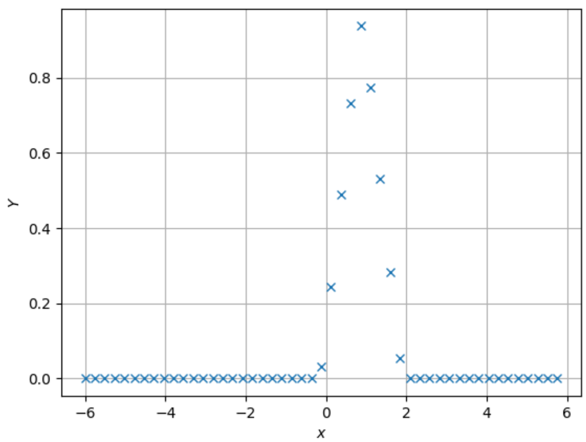

Fig. 4.6. The Numerical PDF of T

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/4.3/
    pdf_4.3_num.py
```

Run the code by executing the below command

```
python3 pdf_4.3_num.py
./a.out
```

The PDF of T is given by

$$p_T(t) = \frac{d(F_T(t))}{dt} \tag{4.13}$$

$$\therefore p_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 1 < t < 2 \\ 0 & t \geq 2 \end{cases} \tag{4.14}$$

4.4 Find the theoretical expressions for the PDF and CDF of T
Solution:

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 2-t & 0 < t \leq 2 \\ 0 & t > 2 \end{cases} \tag{4.15}$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t \leq 1 \\ 2t - \dfrac{t^2}{2} - 1 & 1 < t \leq 2 \\ 1 & t > 2 \end{cases} \tag{4.16}$$

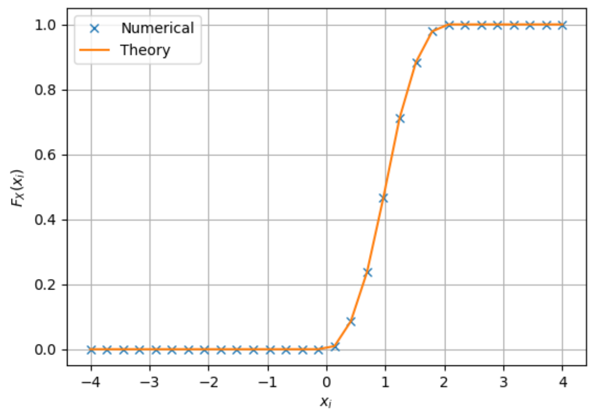

Fig. 4.6. The CDF of T

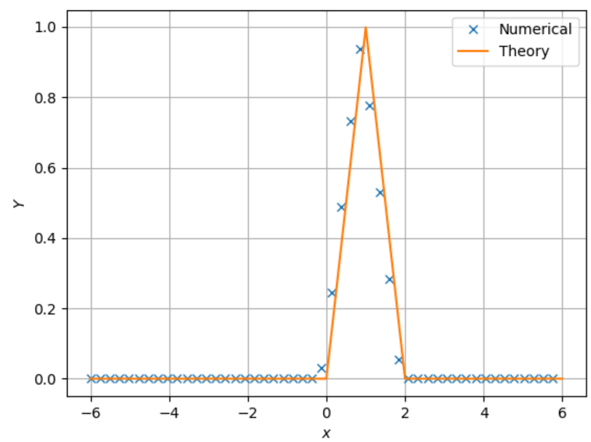

Fig. 4.6. The PDF of T

4.5 Verify your results through a plot
Solution: Download the following code and execute the Python programs

```
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/4.4/cdf_4
    .2.py
wget https://github.com/Pradeep8802/
    Random_numbers/blob/main/4.4/
    pdf_4.3.py
```

Run the codes by executing the below commands for cdf and pdf respectively

```
python3 cdf_4.2.py
python3 pdf_4.3.py
```