Optimization through School Geometry

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Contents

***Abstract*—This book provides an introduction to optimization based on the NCERT textbooks from Class 6-12. Links to sample Python codes are available in the text.**

Download python codes using

```
svn co https://github.com/gadepall/school/trunk/
    ncert/optimization/codes
```

1 Constrained Optimization

1. Express the problem of finding the distance of the point $\mathbf{P} = \begin{pmatrix} 3 \\ -5 \end{pmatrix}$ from the line

$$L: \quad \begin{pmatrix} 3 & -4 \end{pmatrix} \mathbf{x} = 26 \tag{1.1.1}$$

as an optimization problem.
Solution: The given problem can be expressed as

$$\min_{\mathbf{x}} g(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 \tag{1.1.2}$$

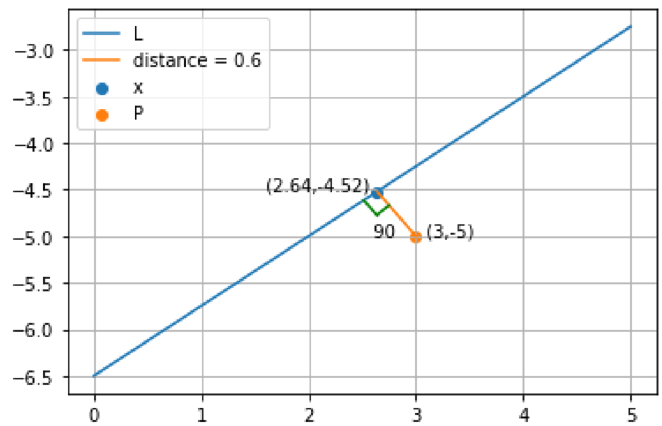
$$\text{s.t.} \quad \mathbf{n}^T \mathbf{x} = c \tag{1.1.3}$$

where

$$\mathbf{n} = \begin{pmatrix} 3 \\ -4 \end{pmatrix} \tag{1.1.4}$$

$$c = 26 \tag{1.1.5}$$

2. Explain Problem 1.1 through a plot and find a graphical solution.
Solution: The Problem 1.1 is solved by finding the closest point on the line L from the point P and subsequently finding distance between them.
Graphically this point is the point of intersection of a ray from point P and ⊥ to the line L.



3. Solve (1.1.2) using cvxpy.
Solution: The following code yields

```
codes/line_dist_cvx.py
```

$$\mathbf{x}_{\min} = \begin{pmatrix} 2.64 \\ -4.52 \end{pmatrix}, \tag{1.3.1}$$

$$g\left(\mathbf{x}_{\min}\right) = 0.6 \tag{1.3.2}$$

4. Convert (1.1.2) to an *unconstrained* optimization problem.
Solution: L in (1.1.1) can be expressed in terms of the direction vector $\mathbf{m}$ as

$$\mathbf{x} = \mathbf{A} + \lambda \mathbf{m}, \tag{1.4.1}$$

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where $\mathbf{A}$ is any point on the line and

$$\mathbf{m}^T\mathbf{n} = 0 \tag{1.4.2}$$

Substituting (1.4.1) in (1.1.2), an unconstrained optimization problem

$$\min_{\lambda} f(\lambda) = \|\mathbf{A} + \lambda\mathbf{m} - \mathbf{P}\|^2 \tag{1.4.3}$$

is obtained.

5. Solve (1.4.3).
Solution:

$$f(\lambda) = (\lambda\mathbf{m} + \mathbf{A} - \mathbf{P})^T(\lambda\mathbf{m} + \mathbf{A} - \mathbf{P}) \tag{1.5.1}$$
$$= \lambda^2\|\mathbf{m}\|^2 + 2\lambda\mathbf{m}^T(\mathbf{A} - \mathbf{P}) + \|\mathbf{A} - \mathbf{P}\|^2 \tag{1.5.2}$$

$$\because f^{(2)}\lambda = 2\|\mathbf{m}\|^2 > 0 \tag{1.5.3}$$

the minimum value of $f(\lambda)$ is obtained when

$$f^{(1)}(\lambda) = 2\lambda\|\mathbf{m}\|^2 + 2\mathbf{m}^T(\mathbf{A} - \mathbf{P}) = 0 \tag{1.5.4}$$

$$\implies \lambda_{\min} = -\frac{\mathbf{m}^T(\mathbf{A} - \mathbf{P})}{\|\mathbf{m}\|^2} \tag{1.5.5}$$

Choosing $\mathbf{A}$ such that

$$\mathbf{m}^T(\mathbf{A} - \mathbf{P}) = 0, \tag{1.5.6}$$

substituting in (1.5.5),

$$\lambda_{\min} = 0 \quad \text{and} \tag{1.5.7}$$
$$\mathbf{A} - \mathbf{P} = \mu\mathbf{n} \tag{1.5.8}$$

for some constant μ. (1.5.8) is a consequence of (1.4.2) and (1.5.6). Also, from (1.5.8),

$$\mathbf{n}^T(\mathbf{A} - \mathbf{P}) = \mu\|\mathbf{n}\|^2 \tag{1.5.9}$$
$$\implies \mu = \frac{\mathbf{n}^T\mathbf{A} - \mathbf{n}^T\mathbf{P}}{\|\mathbf{n}\|^2} = \frac{c - \mathbf{n}^T\mathbf{P}}{\|\mathbf{n}\|^2} \tag{1.5.10}$$

from (1.1.3). Substituting $\lambda_{\min} = 0$ in (1.4.3),

$$\min_{\lambda} f(\lambda) = \|\mathbf{A} - \mathbf{P}\|^2 = \mu^2\|\mathbf{n}\|^2 \tag{1.5.11}$$

upon substituting from (1.5.8). The distance between $\mathbf{P}$ and L is then obtained from (1.5.11) as

$$\|\mathbf{A} - \mathbf{P}\| = |\mu|\,\|\mathbf{n}\| \tag{1.5.12}$$
$$= \frac{\left|\mathbf{n}^T\mathbf{P} - c\right|}{\|\mathbf{n}\|} \tag{1.5.13}$$

after substituting for μ from (1.5.10). Using the corresponding values from Problem (1.1) in (1.5.13),

$$\min_{\lambda} f(\lambda) = 0.6 \tag{1.5.14}$$

2 Convex Function

1. The following python script plots

$$f(\lambda) = a\lambda^2 + b\lambda + d \tag{2.1.1}$$

for

$$a = \|\mathbf{m}\|^2 > 0 \tag{2.1.2}$$
$$b = \mathbf{m}^T(\mathbf{A} - \mathbf{P}) \tag{2.1.3}$$
$$c = \|\mathbf{A} - \mathbf{P}\|^2 \tag{2.1.4}$$

where $\mathbf{A}$ is the intercept of the line L in (1.1.1) on the x-axis and the points

$$\mathbf{U} = \begin{pmatrix} \lambda_1 \\ f(\lambda_1) \end{pmatrix}, \mathbf{V} = \begin{pmatrix} \lambda_2 \\ f(\lambda_2) \end{pmatrix} \tag{2.1.5}$$
$$\mathbf{X} = \begin{pmatrix} t\lambda_1 + (1-t)\lambda_2 \\ f[t\lambda_1 + (1-t)\lambda_2] \end{pmatrix}, \tag{2.1.6}$$
$$\mathbf{Y} = \begin{pmatrix} t\lambda_1 + (1-t)\lambda_2 \\ tf(\lambda_1) + (1-t)f(\lambda_2) \end{pmatrix} \tag{2.1.7}$$

for

$$\lambda_1 = -3, \lambda_2 = 4, t = 0.3 \tag{2.1.8}$$

in Fig. 2.1. Geometrically, this means that any point $\mathbf{Y}$ between the points $\mathbf{U}, \mathbf{V}$ on the line UV is always above the point $\mathbf{X}$ on the curve $f(\lambda)$. Such a function f is defined to be *convex* function

```
codes/optimization/1.2.py
```

2. Show that

$$f[t\lambda_1 + (1-t)\lambda_2] \le tf(\lambda_1) + (1-t)f(\lambda_2) \tag{2.2.1}$$

for $0 < t < 1$. This is true for any convex function.
Solution: Consider a convex function f. By the definition of a convex function, a secant joining any two points on f should lie above the graph $y = f(x)$.
Consider the two points to be $(\lambda_1, f(\lambda_1))$ and $(\lambda_2, f(\lambda_2))$ and a value t such that $0 < t < 1$.
A point on the secant between those points will be

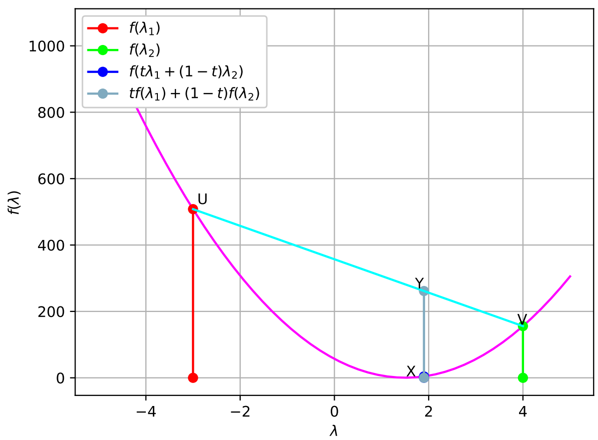

Fig. 2.1: $f(\lambda)$ versus λ
.

$(t\lambda_1 + (1-t)\lambda_2, tf(\lambda_1) + (1-t)f(\lambda_2))$
The point on the $y = f(x)$ at corresponding x coordinate will be
$(t\lambda_1 + (1-t)\lambda_2, f[t\lambda_1 + (1-t)\lambda_2])$
For the secant to lie above the graph, y-coordinate of point on secant should be above the y-coordinate of corresponding point on the graph.
i.e $f[t\lambda_1 + (1-t)\lambda_2] \leq f(\lambda_1) + (1-t)f(\lambda_2)$

3. Show that

$$(2.2.1) \implies f^{(2)}(\lambda) > 0 \tag{2.3.1}$$

Solution: Consider three points $(a, f(a))$, $(b, f(b))$ and $(c, f(C))$ such that $a < b < c$. (2.2.1) with $t = \frac{c-b}{c-a}$ and $\lambda_1 = a$, $\lambda_2 = c$ gives

$$f(b) \leq \left[\frac{c-b}{c-a}\right] f(a) + \left[\frac{b-a}{c-a}\right] f(c)$$

which solves to for any $a < b < c$,

$$\frac{f(b)-f(a)}{b-a} \leq \frac{f(c)-f(a)}{c-a} \tag{2.3.2}$$

Apply limits

$$\lim_{b-a\to 0} \frac{f(b)-f(a)}{b-a} \leq \lim_{c-b\to 0} \frac{f(c)-f(b)}{c-b}$$
$$\implies f^{(1)}(b) \leq f^{(1)}(c)$$
$$\implies f^{(1)}(c) - f^{(1)}(b) \geq 0$$

Because $c - b > 0$

$$\frac{f^{(1)}(c) - f^{(1)}(b)}{c-b} \geq 0$$

Apply limit

$$\lim_{c-b\to 0} \frac{f^{(1)}(c) - f^{(1)}(b)}{c-b} \geq 0$$
$$\implies f^{(2)}(c) \geq 0$$

As a,b,c are chosen arbitrarily with $a < b < c$ for any λ in domain of f,

$$f^{(2)}(\lambda) \geq 0 \tag{2.3.3}$$

4. Show that a convex function has a unique minimum.
Solution: Consider a convex function f having global minimum at x_1 and local minimum at x_2 $(x_1 \neq x_2)$
i.e $f(x_1) \leq f(x_2)$
From the definition of convexity, we have for $0 < \theta < 1$,
$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2)$
Since θ is positive

$$f(x_1) \leq f(x_2) \implies \theta f(x_1) \leq \theta f(x_2) \tag{2.4.1}$$

which justifies below condition

$$\theta f(x_1) + (1-\theta)f(x_2) \leq \theta f(x_2) + (1-\theta)f(x_2)$$
$$\leq f(x_2) \tag{2.4.2}$$

Replacing this condition to the definition of convexity
$f(\theta x_1 + (1-\theta)x_2) \leq f(x_2)$
If x_2 is a local minimum, the neighborhood must be defined such as $f(x) \geq f(x_2)$.
To satisfy both conditions it must be that
$f(x) = f(x_1) = f(x_2) \forall x_1 \leq x \leq x_2$
which shows that f has a unique minimum value.

3 Gradient Descent

1. Find a numerical solution for (2.1.1)
Solution: A numerical solution for (2.1.1) is obtained as

$$\lambda_{n+1} = \lambda_n - \mu f'(\lambda_n) \tag{3.1.1}$$
$$= \lambda_n - \mu(2a\lambda_n + b) \tag{3.1.2}$$

where λ_0 is an inital guess and μ is a variable parameter. The choice of these parameters is very important since they decide how fast the algorithm converges.

2. Write a program to implement (3.1.2).
Solution: Download and execute

```
codes/optimization/gd.py
```

3. Find a closed form solution for λ_n in (3.1.2) using the one sided Z transform.
4. Find the condition for which (3.1.2) converges, i.e.

$$\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0 \tag{3.4.1}$$

4 Lagrange Multipliers

1. Find

$$\min_{\mathbf{x}} g(\mathbf{x}) = \|\mathbf{x} - \mathbf{P}\|^2 = r^2 \tag{4.1.1}$$

$$\text{s.t.} \quad h(\mathbf{x}) = \mathbf{n}^T\mathbf{x} - c = 0 \tag{4.1.2}$$

by plotting the circles $g(\mathbf{x})$ for different values of r along with the line $g(\mathbf{x})$.
Solution: The following code plots Fig. 4.1

```
codes/concirc.py
```

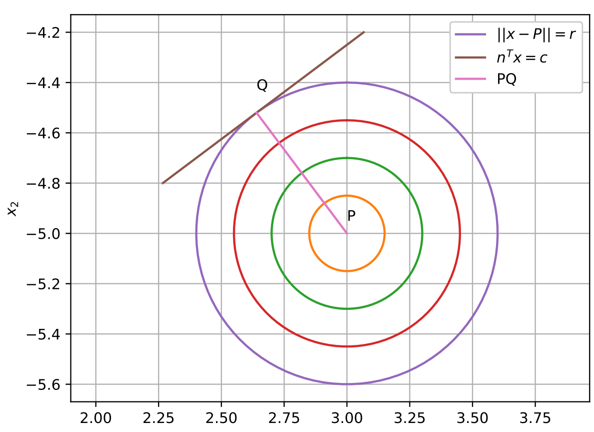

Fig. 4.1: Finding $\min_{\mathbf{x}} g(\mathbf{x})$

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2. By solving the quadratic equation obtained from (4.1.1), show that

$$\min_{\mathbf{x}} r = \frac{3}{5}, \mathbf{x}_{\min} = \mathbf{Q} = \begin{pmatrix} 2.64 \\ -4.52 \end{pmatrix} \tag{4.2.1}$$

In Fig. 4.1, it can be seen that $\mathbf{Q}$ is the point of contact of the line L with the circle of minimum radius $r = \frac{3}{5}$.

3. Show that

$$\nabla h(\mathbf{x}) = \begin{pmatrix} 3 \\ -4 \end{pmatrix} = \mathbf{n} \tag{4.3.1}$$

where

$$\nabla = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix} \tag{4.3.2}$$

4. Show that

$$\nabla g(\mathbf{x}) = 2\left\{\mathbf{x} - \begin{pmatrix} 3 \\ -5 \end{pmatrix}\right\} = 2\{\mathbf{x} - \mathbf{P}\} \tag{4.4.1}$$

5. From Fig. 4.1, show that

$$\nabla g(\mathbf{Q}) = \lambda \nabla h(\mathbf{Q}), \tag{4.5.1}$$

Solution: In Fig. 4.1, PQ is the normal to the line L, represented by $h(\mathbf{x})$. $\because$ the normal vector of L is in the same direction as PQ, for some constant k,

$$(\mathbf{Q} - \mathbf{P}) = k\mathbf{n} \tag{4.5.2}$$

which is the same as (4.5.1) after substituting from (4.3.1). and (4.4.1).

6. Use (4.5.1) and $\mathbf{h}(\mathbf{Q}) = 0$ from (4.1.2) to obtain $\mathbf{Q}$.
Solution: From the given equations, we obtain

$$(\mathbf{Q} - \mathbf{P}) - \lambda\mathbf{n} = 0 \tag{4.6.1}$$

$$\mathbf{n}^T\mathbf{Q} - c = 0 \tag{4.6.2}$$

which can be simplified to obtain

$$\begin{pmatrix} \mathbf{I} & -\mathbf{n} \\ \mathbf{n}^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{Q} \\ \lambda \end{pmatrix} = \begin{pmatrix} \mathbf{P} \\ c \end{pmatrix} \tag{4.6.3}$$

The following code computes the solution to (4.6.3)

```
codes/lagmul.py
```

7. Define

$$C(\mathbf{x}, \lambda) = g(\mathbf{x}) - \lambda h(\mathbf{x}) \tag{4.7.1}$$

and show that $\mathbf{Q}$ can also be obtained by solving the equations

$$\nabla C(\mathbf{x}, \lambda) = 0. \tag{4.7.2}$$

What is the sign of λ? C is known as the Lagrangian and the above technique is known as the Method of Lagrange Multipliers.

8. Obtain $\mathbf{Q}$ using gradient descent.
Solution:

codes/gd_lagrange.py

5 Quadratic Programming

1. An apache helicopter of the enemy is flying along the curve given by
$$y = x^2 + 7 \tag{5.1.1}$$
A soldier, placed at
$$\mathbf{P} = \begin{pmatrix} 3 \\ 7 \end{pmatrix}. \tag{5.1.2}$$
wants to shoot the heicopter when it is nearest to him. Express this as an optimization problem.
Solution: The given problem can be expressed as
$$\min_{\mathbf{x}} \|\mathbf{x} - \mathbf{P}\|^2 \tag{5.1.3}$$
$$\text{s.t. } \mathbf{x}^T\mathbf{V}\mathbf{x} + \mathbf{u}^T\mathbf{x} + d = 0 \tag{5.1.4}$$
where
$$\mathbf{V} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{5.1.5}$$
$$\mathbf{u} = -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{5.1.6}$$
$$d = 7 \tag{5.1.7}$$
2. Show that the constraint in 5.1.3 is nonconvex.
3. Show that the following *relaxation* makes (5.1.3) a convex optimization problem.
$$\min_{\mathbf{x}} (\mathbf{x} - \mathbf{P})^T (\mathbf{x} - \mathbf{P}) \tag{5.3.1}$$
$$\text{s.t. } \mathbf{x}^T\mathbf{V}\mathbf{x} + \mathbf{u}^T\mathbf{x} \le 0 \tag{5.3.2}$$
4. Solve (5.3.1) using cvxpy.
Solution: The following code yields the minimum distance as 2.236 and the nearest point on the curve as
$$\mathbf{Q} = \begin{pmatrix} 1 \\ 8 \end{pmatrix} \tag{5.4.1}$$
codes/qp_cvx.py
5. Solve (5.3.1) using the method of Lagrange multipliers.
6. Graphically verify the solution to Problem 5.1.
Solution: The following code plots Fig. 5.6
codes/qp_parab.py
7. Solve (5.3.1) using gradient descent.

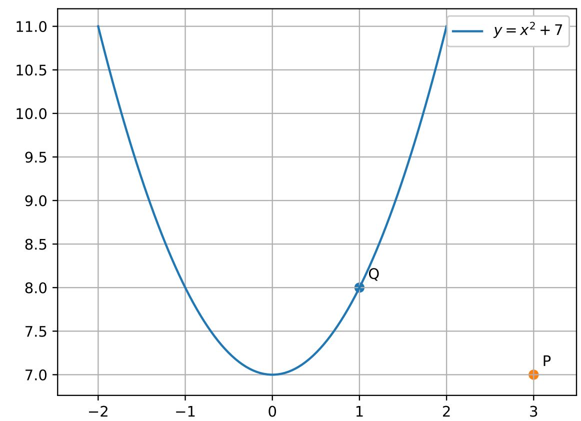

Fig. 5.6: **Q** is closest to **P**
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6 Semi Definite Programming

1. Express the problem of finding the point on the curve
$$x^2 = 2y \tag{6.1.1}$$
nearest to the point
$$\mathbf{P} = \begin{pmatrix} 0 \\ 5 \end{pmatrix}. \tag{6.1.2}$$
as an optimization problem.
Solution: The given problem can be expressed as
$$\min_{\mathbf{x}} \mathbf{x}^T\mathbf{Q}_0\mathbf{x} + \mathbf{q}_0^T\mathbf{x} + c_0 \tag{6.1.3}$$
$$\text{s.t. } \mathbf{x}^T\mathbf{Q}_1\mathbf{x} + \mathbf{q}_1^T\mathbf{x} + c_1 \le 0 \tag{6.1.4}$$
where
$$\mathbf{Q}_0 = \mathbf{I}, \mathbf{Q}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.1.5}$$
$$\mathbf{q}_0 = -2\mathbf{P}, \mathbf{q}_1 = -2\begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{6.1.6}$$
$$c_0 = \|\mathbf{P}\|^2, c_1 = 0 \tag{6.1.7}$$
2. Show that (6.1.3) is equivalent to
$$\begin{aligned} &\min_{\mathbf{x},\theta} \theta \\ \text{s.t. } &\begin{pmatrix} \mathbf{I} & \mathbf{M}_0\mathbf{x} \\ \mathbf{x}^T\mathbf{M}_0^T & -c_0 - q_0^T\mathbf{x} + \theta \end{pmatrix} \succeq 0 \\ &\begin{pmatrix} \mathbf{I} & \mathbf{M}_1\mathbf{x} \\ \mathbf{x}^T\mathbf{M}_1^T & -c_1 - q_1^T\mathbf{x} \end{pmatrix} \succeq 0 \end{aligned} \tag{6.2.1}$$

where

$$\mathbf{Q}_i = \mathbf{M}_i^T \mathbf{M}_i, i = 0, 1 \tag{6.2.2}$$

3. Solve (6.2.1) using cvxpy.
4. Graphically verify the solution to Problem 6.1.
5. Solve (6.1.3) using the method of Lagrange multipliers.

7 Linear Programming

1. Solve

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 4 & 1 \end{pmatrix} \mathbf{x} \tag{7.1.1}$$

$$s.t. \quad \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 50 \\ 90 \end{pmatrix} \tag{7.1.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{7.1.3}$$

using cvxpy.
Solution: The given problem can be expressed in general as

$$\max_{\mathbf{x}} \mathbf{c}^T \mathbf{x} \tag{7.1.4}$$

$$s.t. \quad \mathbf{A}\mathbf{x} \preceq \mathbf{b}, \tag{7.1.5}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{7.1.6}$$

where

$$\mathbf{c} = \begin{pmatrix} 4 \\ 1 \end{pmatrix} \tag{7.1.7}$$

$$\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 3 & 1 \end{pmatrix} \tag{7.1.8}$$

$$\mathbf{b} = \begin{pmatrix} 50 \\ 90 \end{pmatrix} \tag{7.1.9}$$

and can be solved using *cvxpy* through the following code

```
codes/lp_cvx.py
```

to obtain

$$\mathbf{x} = \begin{pmatrix} 30 \\ 0 \end{pmatrix}, Z = 120 \tag{7.1.10}$$

2. Graphically, show that the feasible region in Problem 7.1 result in the interior of a convex polygon and the optimal point is one of the vertices. **Solution:** The following code plots Fig. 7.2.

```
codes/lp_cvx.py
```

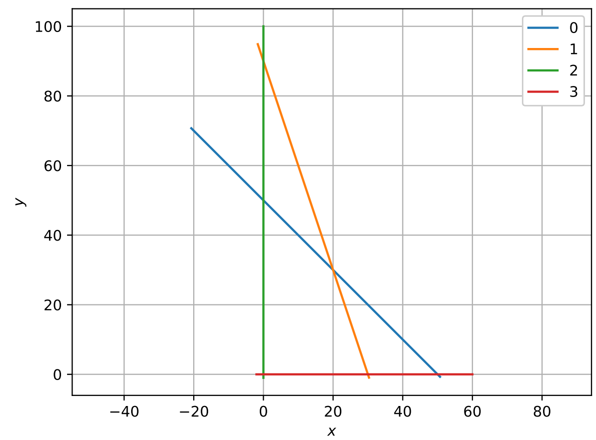

Fig. 7.2

3. Solve

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 3 & 9 \end{pmatrix} \mathbf{x} \tag{7.3.1}$$

$$s.t. \quad \begin{pmatrix} 1 & 3 \\ -1 & -1 \\ 1 & -1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 60 \\ -10 \\ 0 \end{pmatrix} \tag{7.3.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{7.3.3}$$

Solution: The following code

```
codes/lp_cvx_mult.py
```

is used to obtain

$$\mathbf{x} = \begin{pmatrix} 15 \\ 15 \end{pmatrix}, Z = 180 \tag{7.3.4}$$

4. Solve

$$\min_{\mathbf{x}} Z = \begin{pmatrix} -50 & 20 \end{pmatrix} \mathbf{x} \tag{7.4.1}$$

$$s.t. \quad \begin{pmatrix} -2 & 1 \\ -3 & -1 \\ 2 & -3 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 5 \\ -3 \\ 12 \end{pmatrix} \tag{7.4.2}$$

$$\mathbf{x} \succeq \mathbf{0} \tag{7.4.3}$$

Solution: The following code

```
codes/lp_cvx_nosol.py
```

shows that the given problem has no solution.
5. Verify all the above solutions using Lagrange multipliers.
6. Repeat the above exercise using the Simplex method.
7. **(Diet problem)**: A dietician wishes to mix two types of foods in such a way that vitamin con-

tents of the mixture contain atleast 8 units of vitamin A and 10 units of vitamin C. Food 'I' contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C. Food 'II' contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs Rs 50 per kg to purchase Food 'I' and Rs 70 per kg to purchase Food 'II'. Formulate this problem as a linear programming problem to minimise the cost of such a mixture.

Solution: Let the mixture contain x kg of food I and y kg of food II.

The given problem can be expressed as

Resources	Food		Requirement
	I	II	
Vitamin A	2	1	Atleast 8 Units
Vitamin C	1	2	Atleast 10 Units
Cost	50	70	

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 50 & 70 \end{pmatrix} \mathbf{x} \tag{7.7.1}$$

$$s.t. \quad \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x} \geq \begin{pmatrix} 8 \\ 10 \end{pmatrix} \tag{7.7.2}$$

$$\mathbf{x} \geq \mathbf{0} \tag{7.7.3}$$

The corner points of the feasible region are available in Table 7.7 and plotted in Fig. 7.7.

Corner Point	$Z = 50x + 70y$
(0,8)	560
(2,4)	380
(10,0)	500

TABLE 7.7

The smallest value of Z is 380 at the point (2,4). But the feasible region is unbounded therefore we draw the graph of the inequality

$$50x + 70y < 380 \tag{7.7.4}$$

to check whether the resulting open half has any point common with the feasible region but on checking it doesn't have any points in common. Thus the minimum value of Z is 380 attained at $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Hence optimal mixing strategy for the dietician would be to mix 2 Kg of Food I and 4 Kg of Food II. The following code provides the solution to (7.7.3).

```
codes/diet.py
```

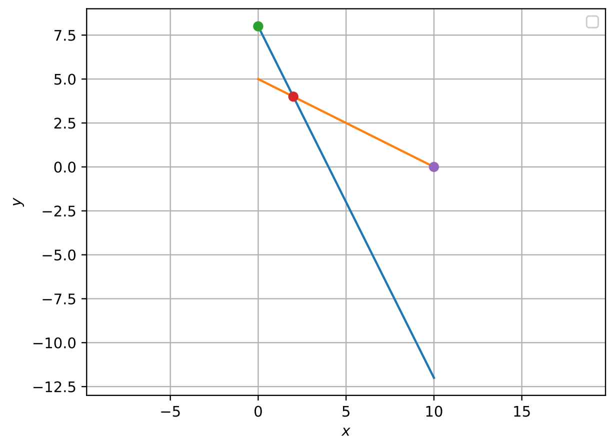

Fig. 7.7

8. **(Allocation problem)** A cooperative society of farmers has 50 hectare of land to grow two crops X and Y. The profit from crops X and Y per hectare are estimated as Rs 10,500 and Rs 9,000 respectively. To control weeds, a liquid herbicide has to be used for crops X and Y at rates of 20 litres and 10 litres per hectare. Further, no more than 800 litres of herbicide should be used in order to protect fish and wild life using a pond which collects drainage from this land. How much land should be allocated to each crop so as to maximise the total profit of the society?

   Solution: The given problem can be formulated as

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 10500 & 9000 \end{pmatrix} \mathbf{x} \tag{7.8.1}$$

$$s.t. \quad \begin{pmatrix} 20 & 10 \end{pmatrix} \mathbf{x} \leq 800 \tag{7.8.2}$$

$$\begin{pmatrix} 1 & 1 \end{pmatrix} \mathbf{x} = 50 \tag{7.8.3}$$

   Fig 7.8 shows the intersection of various lines and the optimal point as indicated.

   The following code provides the solution to (7.8.3) at $\begin{pmatrix} 30 \\ 20 \end{pmatrix}$.

```
codes/allocation.py
```

9. **(Manufacturing problem)** A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated for atleast 5 hours a day. She produces only two items M

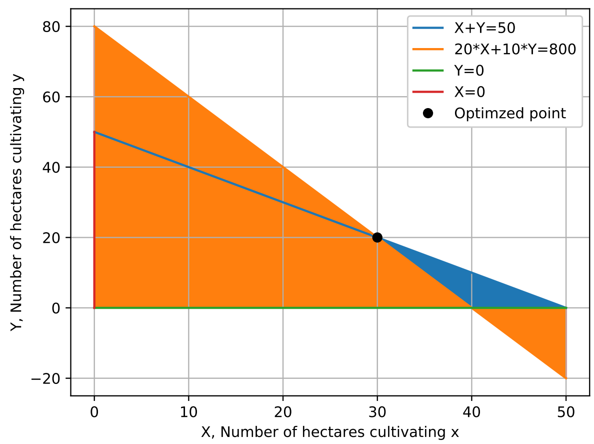

Fig. 7.8: Feasible region for allocation Problem

Fig. 7.8

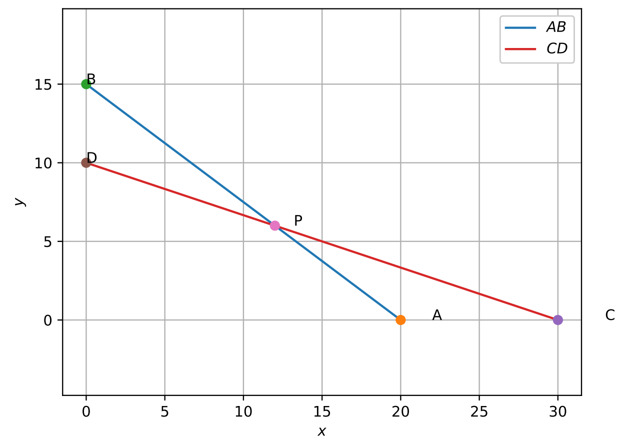

Fig. 7.9: Feasible region for manufacturing Problem

Fig. 7.9

and N each requiring the use of all the three machines. The number of hours required for producing 1 unit of each of M and N on the three machines are given in the following table:

Number of hours required on machines			
Items	**I**	**II**	**III**
M	1	2	1
N	2	1	1.25

She makes a profit of Rs 600 and Rs 400 on items M and N respectively. How many of each item should she produce so as to maximise her profit assuming that she can sell all the items that she produced? What will be the maximum profit?

Solution: The given problem can be formulated as

$$\max_{\mathbf{x}} Z = \begin{pmatrix} 80000 & 12000 \end{pmatrix} \mathbf{x} \tag{7.9.1}$$

$$s.t. \quad \begin{pmatrix} 3 & 4 \\ 1 & 3 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 60 \\ 30 \end{pmatrix} \tag{7.9.2}$$

Fig 7.9 shows the intersection of various lines and the optimal point as indicated.
The following code provides the solution to (7.9.2) at $\begin{pmatrix} 12 \\ 6 \end{pmatrix}$.

```
codes/Manufacturing.py
```

10. **(Transportation problem)** There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below where A,B,C are cost in ruppes:

From/To	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum. What will be the minimum transportation cost?

Solution: The given problem can be formulated as

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 10 & -70 \end{pmatrix} \mathbf{x} \tag{7.10.1}$$

$$s.t. \quad \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \mathbf{x} \preceq \begin{pmatrix} 8 \\ -4 \end{pmatrix} \tag{7.10.2}$$

$$\mathbf{x} \preceq \begin{pmatrix} 5 \\ 5 \end{pmatrix} \tag{7.10.3}$$

Fig 7.10 shows the intersection of various lines and the optimal point indicated as OPT PT.
The following code provides the solution to (7.10.3) at $\begin{pmatrix} 0 \\ 5 \end{pmatrix}$.

```
codes/Transportation.py
```

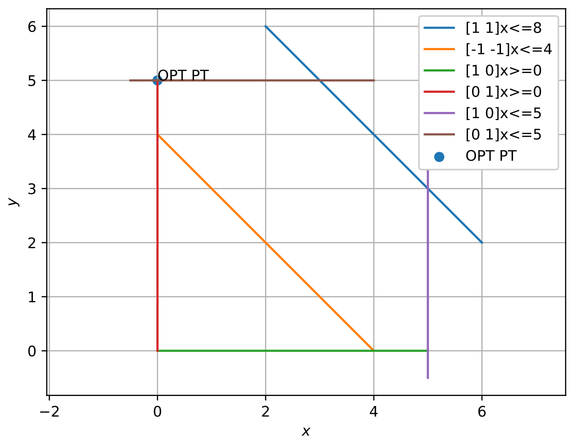

Fig. 7.10: Feasible region for Transportation Problem

Fig. 7.10

8 Exercises

1. Solve

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 3 & 2 \end{pmatrix} \mathbf{x} \tag{8.0.1.1}$$

$$s.t. \quad \begin{pmatrix} -1 & -1 \\ 3 & 5 \end{pmatrix} \mathbf{x} \le \begin{pmatrix} -8 \\ 15 \end{pmatrix} \tag{8.0.1.2}$$

$$\mathbf{x} \ge \mathbf{0} \tag{8.0.1.3}$$

2. Solve

$$\min_{\mathbf{x}} Z = \begin{pmatrix} 200 & 500 \end{pmatrix} \mathbf{x} \tag{8.0.2.1}$$

$$s.t. \quad \begin{pmatrix} -1 & -2 \\ 3 & 4 \end{pmatrix} \mathbf{x} \le \begin{pmatrix} -10 \\ 24 \end{pmatrix} \tag{8.0.2.2}$$

$$\mathbf{x} \ge \mathbf{0} \tag{8.0.2.3}$$

3. Maximise Z=3x+4y
subject to the constraints : x+y≤4, x≥0, y≥ 0.

4. Minimise Z=-3x+4y
subject to x+2y≤8, 3x+2y≤12, x≥0, y≥0.

5. Maximise Z=5x+3y subject to 3x+5y≤15, 5x+2y≤10, x≥0, y≥0.

6. Minimise Z=3x+5y such that x+3y≥3, x+y≥2, x,y≥0.

7. Maximise Z=3x+2y subject to x+2y≤10, 3x+y≤15, x,y≥0.

8. Minimise Z=x+2y subject to 2x+y≥3, x+2y≥6, x,y≥0.
Show that the minimum of Z occurs at more than two points.

9. Minimise and Maximise Z=5x+10y subject to x+2y≤120, x+y≥60, x-2y≥0, x,y≥0.

10. Minimise and Maximise Z=x+2y subject to x+2y≥100, 2x-y≤0, 2x+y≤200; x,y≥0.

11. Maximise Z=-x+2y, subject to the constraints: x≥3, x+y≥5, x+2y≥6, y≥0.

12. Maximise Z=x+y, subject to x-y≤-1,-x+y≤0, x,y≥0.

13. Reshma wishes to mix two types of food P and Q in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food P costs Rs 60/kg and Food Q costs Rs 80/kg. Food P contains 3 units/kg of Vitamin A and 5 units/kg of Vitamin B while food Q contains 4 units/kg of Vitamin A and 2 units/kg of vitamin B. Determine the minimum cost of the mixture.

14. One kind of cake requires 200g of flour and 25g of fat, and another kind of cake requires 100g of flour and 50g of fat. Find the maximum number of cakes which can be made from 5kg of flour and 1 kg of fat assuming that there is no shortage of the other ingredients used in making the cakes.

15. A factory makes tennis rackets and cricket bats. A tennis racket takes 1.5 hours of machine time and 3 hours of craftman's time in its making while a cricket bat takes 3 hour of machine time and 1 hour of craftman's time. In a day, the factory has the availability of not more than 42 hours of machine time and 24 hours of craftsman's time.
(i) What number of rackets and bats must be made if the factory is to work at full capacity?
(ii)If the profit on a racket and on a bat is Rs 20 and Rs 10 respectively, find the maximum profit of the factory when it works at full capacity.

16. A manufacturer produces nuts and bolts. It takes 1 hour of work on machine A and 3 hours on machine B to produce a package of nuts. It takes 3 hours on machine A and 1 hour on machine B to produce a package of bolts. He earns a profit of Rs17.50 per package on nuts and Rs 7.00 per package on bolts. How many packages of each should be produced each day so as to maximise his profit, if he operates his machines for at the most 12 hours a day?

17. A factory manufactures two types of screws, A and B. Each type of screw requires the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of Rs 7 and screws B at a profit of Rs 10. Assuming that he can sell all the screws he manufactures, how many packages of each type should the factory owner produce in a day in order to maximise his profit? Determine the maximum profit.

18. A cottage industry manufactures pedestal lamps and wooden shades, each requiring the use of a grinding/cutting machine and a sprayer. It takes 2 hours on grinding/cutting machine and 3 hours on the sprayer to manufacture a pedestal lamp. It takes 1 hour on the grinding/cutting machine and 2 hours on the sprayer to manufacture a shade. On any day, the sprayer is available for at the most 20 hours and the grinding/cutting machine for at the most 12 hours. The profit from the sale of a lamp is Rs 5 and that from a shade is Rs 3. Assuming that the manufacturer can sell all the lamps and shades that he produces, how should he schedule his daily production in order to maximise his profit?

19. A company manufactures two types of novelty souvenirs made of plywood. Souvenirs of type A require 5 minutes each for cutting and 10 minutes each for assembling. Souvenirs of type B require 8 minutes each for cutting and 8 minutes each for assembling. There are 3 hours 20 minutes available for cutting and 4 hours for assembling. The profit is Rs 5 each for type A and Rs 6 each for type B souvenirs. How many souvenirs of each type should the company manufacture in order to maximise the profit?

20. A merchant plans to sell two types of personal computers – a desktop model and a portable model that will cost Rs 25000 and Rs 40000 respectively. He estimates that the total monthly demand of computers will not exceed 250 units. Determine the number of units of each type of computers which the merchant should stock to get maximum profit if he does not want to invest more than Rs 70 lakhs and if his profit on the desktop model is Rs 4500 and on portable model is Rs 5000.

21. A diet is to contain at least 80 units of vitamin A and 100 units of minerals. Two foodsF_1 and F_2 are available. Food F_1 costs Rs 4 per unit food and F_2 costs Rs 6 per unit. One unit of food F_1 contains 3 units of vitamin A and 4 units of minerals. One unit of food F_2 contains 6 units of vitamin A and 3 units of minerals. Formulate this as a linear programming problem. Find the minimum cost for diet that consists of mixture of these two foods and also meets the minimal nutritional requirements.

22. There are two types of fertilisers F_1 and F_2.F_1 consists of 10% nitrogen and 6% phosphoric acid and F_2 consists of 5% nitrogen and 10% phosphoric acid. After testing the soil conditions, a farmer finds that she needs atleast 14 kg of nitrogen and 14 kg of phosphoric acid for her crop. If F_1 costs Rs 6/kg and F_2 costs Rs 5/kg, determine how much of each type of fertiliser should be used so that nutrient requirements are met at a minimum cost. What is the minimum cost?

23. The corner points of the feasible region determined by the following system of linear

inequalities: $2x+y\leq10$, $x+3y\leq15$, $x,y\geq0$ are (0,0), (5,0),(3,4) and (0,5).Let $Z=px+qy$, where $p,q>0$.Condition on p and q so that the maximum of Z occurs at both (3,4) and (0,5) is
(A) $p = q$
(B) $p = 2q$
(C) $p = 3q$
(D) $q = 3p$

24. Refer to Example 9. How many packets of each food should be used to maximise the amount of vitamin A in the diet? What is the maximum amount of vitamin A in the diet?

25. A farmer mixes two brands P and Q of cattle feed. Brand P, costing Rs 250 per bag, contains 3 units of nutritional element A, 2.5 units of element B and 2 units of element C. Brand Q costing Rs 200 per bag contains 1.5 units of nutritional element A, 11.25 units of element B, and 3 units of element C. The minimum requirements of nutrients A, B and C are 18 units, 45 units and 24 units respectively. Determine the number of bags of each brand which should be mixed in order to produce a mixture having a minimum cost per bag? What is the minimum cost of the mixture per bag?

26. A dietician wishes to mix together two kinds of food X and Y in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg food is given below:

Food	**Vitamin A**	**Vitamin B**	**VitaminC**
X	1	2	3
Y	2	2	1

One kg of food X costs Rs 16 and one kg of food Y costs Rs 20. Find the least cost of the mixture which will produce the required diet?

27. A manufacturer makes two types of toys A and B. Three machines are needed for this purpose and the time (in minutes) required for each toy on the machines is given below:

	Machines		
Types of toys	**I**	**II**	**III**
A	12	18	6
B	6	0	9

Each machine is available for a maximum of 6 hours per day. If the profit on each toy of type A is Rs 7.50 and that on each toy of type B is Rs 5, show that 15 toys of type A and 30 of type B should be manufactured in a day to get maximum profit.

28. An aeroplane can carry a maximum of 200 passengers. A profit of Rs 1000 is made on each executive class ticket and a profit of Rs 600 is made on each economy class ticket. The airline reserves at least 20 seats for executive class. However, at least 4 times as many passengers prefer to travel by economy class than by the executive class. Determine how many tickets of each type must be sold in order to maximise the profit for the airline. What is the maximum profit?

29. Two godowns A and B have grain capacity of 100 quintals and 50 quintals respectively. They supply to 3 ration shops, D, E and F whose requirements are 60, 50 and 40 quintals respectively. The cost of transportation per quintal from the godowns to the shops are given in the following table:

Transportation cost per qunital (in Rs)		
From/To	**A**	**B**
D	6	4
E	3	2
F	2.50	3

How should the supplies be transported in order that the transportation cost is minimum? What is the minimum cost?

30. An oil company has two depots A and B with capacities of 7000 L and 4000 L respectively. The company is to supply oil to three petrol pumps, D, E and F whose requirements are 4500L, 3000L and 3500L respectively. The distances (in km) between the depots and the petrol pumps is given in the following table:

Distance in (km.)		
From/To	**A**	**B**
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost of 10 litres of oil is Re 1 per km, how should the delivery be scheduled in order that the transportation cost is minimum? What is the minimum cost?

31. A fruit grower can use two types of fertilizer in his garden, brand P and brand Q. The amounts (in kg) of nitrogen, phosphoric acid, potash, and chlorine in a bag of each brand are given in the table. Tests indicate that the garden needs at least 240 kg of phosphoric acid, at least 270 kg of potash and at most 310 kg of chlorine. If the grower wants to minimise the amount of nitrogen added to the garden, how many bags of each brand should be used? What is the minimum amount of nitrogen added in the garden?

kg per bag		
	Brand P	**Brand Q**
Nitrogen	3	3.5
Phospheric acid	1	2
Potash	3	1.5
Chlorine	1.5	2

32. Refer to Question 29. If the grower wants to maximise the amount of nitrogen added to the garden, how many bags of each brand should be added? What is the maximum amount of nitrogen added?

33. A toy company manufactures two types of dolls, A and B. Market research and available resources have indicated that the combined production level should not exceed 1200 dolls per week and the demand for dolls of type B is at most half of that for dolls of type A. Further, the production level of dolls of type A can exceed three times the production of dolls of other type by at most 600 units. If the company makes profit of Rs 12 and Rs 16 per doll respectively on dolls A and B, how many of each should be produced weekly in order to maximise the profit?
34. Find the shortest distance of the point $\begin{pmatrix} 0 \\ c \end{pmatrix}$ from the parabola $y = x^2$, where $\frac{1}{2} \le c \le 5$.
35. Find the maximum area of an isosceles triangle inscribed in the ellipse

$$\mathbf{x}^T \begin{pmatrix} a^2 & 0 \\ 0 & b^2 \end{pmatrix} \mathbf{x} = a^2 b^2 \tag{8.0.35.1}$$

with its vertex at one end of the major axis.
36. Find the maximum and minimum values, if any, of the following functions given by
    a) $f(x) = (2x-1)^2 + 3$
    b) $f(x) = 9x^2 + 12x + 2$
    c) $f(x) = -(x-1)^2 + 10$
    d) $f(x) = x^2$.
37. Find the absolute maximum and absoute minimum value of the following functions in the given intervals
    a) $f(x) = 4x - \frac{1}{2}x^2, x \in \left(-2, \frac{9}{2}\right)$
    b) $f(x) = (x-1)^2 + 3, x \in (-3, 1)$
38. Find the maximum profit that a company can make, if the profit function is given by

$$p(x) = 41 - 72x - 18x^2 \tag{8.0.38.1}$$

39. Find two positive numbers whose sum is 15 and the sum of whose squares is minimum.
40. Find two numbers whose sum is 24 and whose product is as large as possible.
41. Find two positive numbers whose sum is 16 and the sum of whose cubes is minimum.
42. The sum of the perimeter of a circle and square is k, where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.
43. A window is in the form of a rectangle surmounted by a semicircular opening. The total perimeter of the window is 10 m. Find the dimensions of the window to admit maximum light through the whole opening.
44. **(Manufacturing problem)** A manufacturing company makes two models A and B of a product. Each piece of Model A requires 9 labour hours for fabricating and 1 labour hour for finishing. Each piece of Model B requires 12 labour hours for fabricating and 3 labour hours for finishing. For fabricating and finishing, the maximum labour hours available are 180 and 30 respectively. The company makes a profit of Rs 8000 on each piece of model A and Rs 12000 on each piece of Model B. How many pieces of Model A and Model B should be manufactured per week to realise a maximum profit? What is the maximum profit per week?

45. **(Diet problem)** A dietician has to develop a special diet using two foods P and Q. Each

packet (containing 30 g) of food P contains 12 units of calcium, 4 units of iron, 6 units of cholesterol and 6 units of vitamin A. Each packet of the same quantity of food Q contains 3 units of calcium, 20 units of iron, 4 units of cholesterol and 3 units of vitamin A. The diet requires atleast 240 units of calcium, atleast 460 units of iron and at most 300 units of cholesterol. How many packets of each food should be used to minimise the amount of vitamin A in the diet? What is the minimum amount of vitamin A?