Assignment

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Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
link of exrand.c
  and coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 0

```
link of cdf_ploy.
  py
```

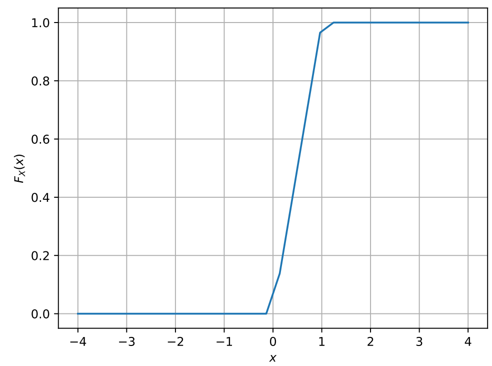

Fig. 0: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: As U is uniform random variable distribution, (between 0 to 1); $CDF = F_U(x)$

$$F_U(x) = \int_0^x P_U(x)\,dx \tag{1.2}$$

As uniform distribution, $P_U(x_i) = t$ (t is constant)
So,

$$F_U(x) = \int_0^x t\,dx \tag{1.3}$$

We know that,

$$F_U(1) = 1 \quad (1.4)$$

$$\implies \int_0^1 t\,dx = 1 \quad (1.5)$$

$$\implies t = 1 \quad (1.6)$$

$$\implies F_U(x) = \int_0^x 1\,dx \quad (1.7)$$

$$\implies F_U(x) = x \quad (1.8)$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \quad (1.9)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.10)$$

Write a C program to find the mean and variance of U.
Solution: Code file can be found here:

link of c file.

Mean=0.500007
Variance=0.083301

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.11)$$

Solution: from question 1.3

$$F_U(x) = x \quad (1.12)$$

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dx \quad (1.13)$$

for k = 1,

$$E[U] = \int_{-\infty}^{\infty} xdx \quad (1.14)$$

$$= \int_0^1 xdx \quad (1.15)$$

$$= \frac{1}{2} \quad (1.16)$$

for k = 2,

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2dx \quad (1.17)$$

$$= \int_0^1 x^2dx + 0 \quad (1.18)$$

$$= \frac{1}{3} \quad (1.19)$$

Mean = 0.5

$$var(x) = E[U^2] - (E[U])^2 \quad (1.20)$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.21)$$

$$= \frac{1}{3} - \frac{1}{4} \quad (1.22)$$

$$= \frac{1}{12} \quad (1.23)$$

$$var(x) = 0.083 \quad (1.24)$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

link of c file

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 0

link of cdf_plot.py file

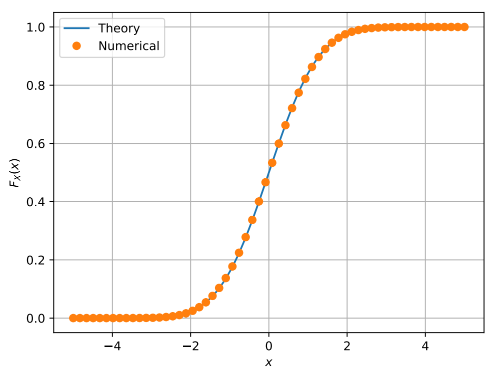

Fig. 0: The CDF of X

Properties of CDF:

- As x reaches 0 the CDF reaches 0.5
- As x approaches infinity the CDF approaches 1
- As x approaches minus infinity the CDF approaches 0

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 0 using the code below

link of pdf_plot.py

Proprties of PDF of X:

- Graph is simitrical about x=0
- Graph have its peak at x=0

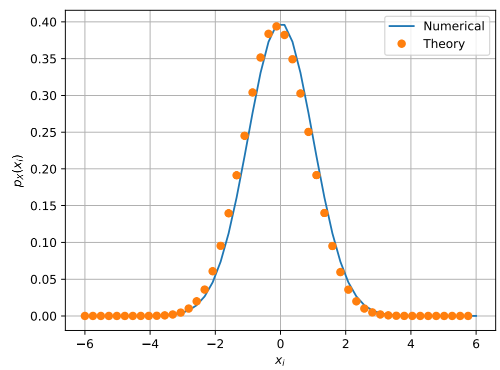

Fig. 0: The PDF of X

2.4 Find the mean and variance of X by writing a C program.
Solution:
Mean = 0.000294
Variance = 0.999560
c code file can be found at below link:

link of c file

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.

Solution:

$$E[x] = \int_{-\infty}^{\infty} x p_X(x)\, dx \tag{2.4}$$

$$E[x] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.5}$$

$$E[x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \tag{2.6}$$

$$\frac{x^2}{2} = t \tag{2.7}$$

$$xdx = dt \tag{2.8}$$

$$E[x] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \exp(-t)\, dt \tag{2.9}$$

$$E[x] = 0 \tag{2.10}$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x)\, dx \tag{2.11}$$

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) dx \tag{2.12}$$

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \left(x \exp\left(-\frac{x^2}{2}\right)\right) dx \tag{2.13}$$

$$\tag{2.14}$$

Integration by parts,

$$= xI\, dx - \int I\, dx \tag{2.15}$$

Where $I = \int x \exp\left(-\frac{x^2}{2}\right)$

$$\frac{x^2}{2} = t \tag{2.16}$$

$$I = \int \exp(-t)\, dt \tag{2.17}$$

$$I = -\exp(-t) \tag{2.18}$$

$$I = -\exp\left(-\frac{x^2}{2}\right) \tag{2.19}$$

$$\tag{2.20}$$

from (2.15),

$$= -x \exp\left(-\frac{x^2}{2}\right) - \int - \exp\left(-\frac{x^2}{2}\right) dx \tag{2.21}$$

$$= -x \exp\left(-\frac{x^2}{2}\right) + \int \exp\left(-\frac{x^2}{2}\right) dx \tag{2.22}$$

Substituting limits in (2.13),

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \tag{2.23}$$

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \tag{2.24}$$

$$E[x^2] = 1 \tag{2.25}$$

$$Var(x) = E[x^2] - (E[x])^2 \tag{2.26}$$

$$Var(x) = 1 - 0 \tag{2.27}$$

$$Var(x) = 1 \tag{2.28}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2 \ln (1 - U) \tag{3.1}$$

and plot its CDF.

Solution: Code for plot can be found at below link:

link of file

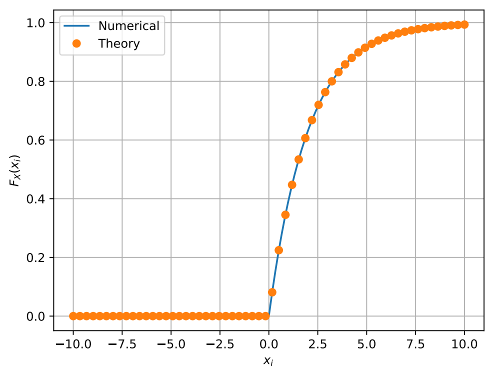

Fig. 0: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = P(V < x) \tag{3.2}$$
$$F_V(x) = P(-2\ln(1-U) < x) \tag{3.3}$$
$$F_V(x) = P\left((1-U) > \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$
$$F_V(x) = P\left(U < 1-\exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$
$$P(U < x) = x \tag{3.6}$$
$$P\left(U < 1-\exp\left(-\frac{x}{2}\right)\right) = 1-\exp\left(-\frac{x}{2}\right) \tag{3.7}$$
$$F_V(x) = 1-\exp\left(-\frac{x}{2}\right) \tag{3.8}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: T.dat and code files can be found here:

link of file

4.2 Find the CDF of T.
Solution:
CDF of T,

$$F_T(t) = P(T < t) \tag{4.2}$$
$$F_T(t) = P(U_1 + U_2 < t) \tag{4.3}$$
$$0 \le U_1 \le 1 \tag{4.4}$$
$$0 \le U_2 \le 1 \tag{4.5}$$
$$0 \le U_1 + U_2 \le 2 \tag{4.6}$$
$$\forall t > 2, P(U_1 + U_2 < t) = 1 \tag{4.7}$$
$$\forall t < 0, P(U_1 + U_2 < t) = 0 \tag{4.8}$$

for $0 \le t \le 1$, from fig 0 Code for plot

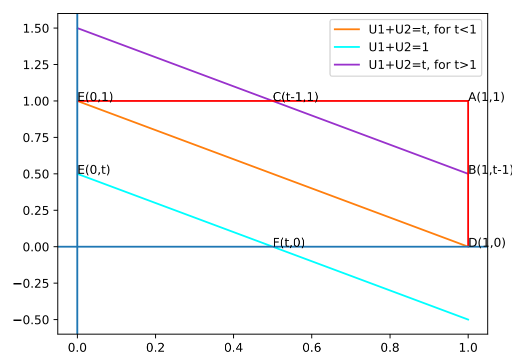

Fig. 0: q4.2

can be found at below link:

link for plot code

$$P(U_1 + U_2 < t) = \frac{\Delta EOF}{\Delta AEOD} \tag{4.9}$$
$$= \frac{t^2}{2} \tag{4.10}$$
$$\tag{4.11}$$

for $1 \leq t \leq 2$,

$$P(U_1 + U_2 < t) = \frac{\Delta ABC}{\Delta AEOD} \tag{4.12}$$

$$= 1 - \frac{(2-t)^2}{2} \tag{4.13}$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t < 1 \\ 1 - \frac{(2-t)^2}{2} & 1 \leq t < 2 \\ 1 & t \geq 2 \end{cases} \tag{4.14}$$

4.3 Find the PDF of T.
Solution:

$$P_T(t) = \frac{d(F_T(t))}{dt} \tag{4.15}$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} \tag{4.16}$$

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution:

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \therefore\leq t < 1 \\ 1 - \frac{(2-t)^2}{2} & 1 \leq t < 2 \\ 1 & t \geq 2 \end{cases} \tag{4.17}$$

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} \tag{4.18}$$

4.5 Verify your results through a plot.
Solution: Code files for plots can be found at:

link for pdf and cdf plots

PDF of T: fig 0
CDF of T: fig 0

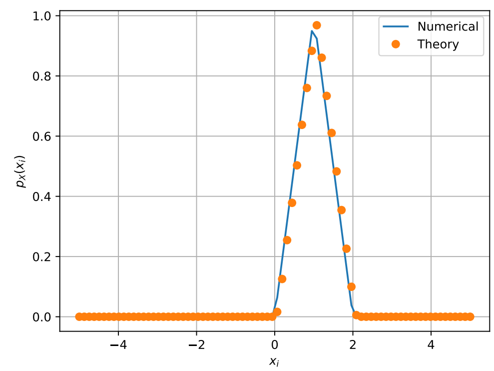

Fig. 0: PDF of T

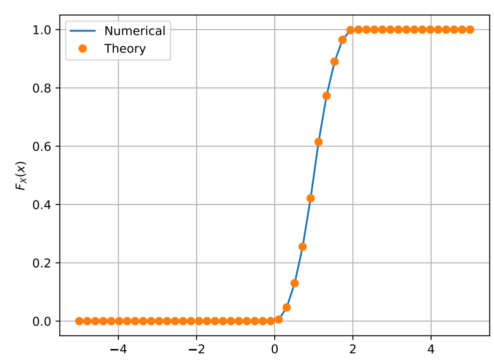

Fig. 0: CDF of T