Assignment

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Contents

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following files and execute the C program.

```
https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/1/exrand.c
```

```
https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/coeffs.h
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \le x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/1/cdf_plot.py
```

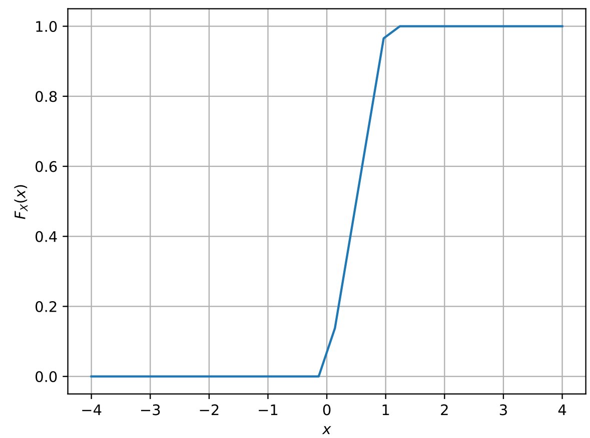

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: As U is uniform random variable distribution, (between 0 to 1); $CDF = F_U(x)$

$$F_U(x) = \int_0^x P_U(x)\,dx \tag{1.2}$$

As uniform distribution, $P_U(x_i) = t$ (t is constant)
So,

$$F_U(x) = \int_0^x t\,dx \tag{1.3}$$

We know that,

$$F_U(1) = 1 \tag{1.4}$$

$$\implies \int_0^1 t\,dx = 1 \tag{1.5}$$

$$\implies t = 1 \tag{1.6}$$

$$\implies F_U(x) = \int_0^x 1\,dx \tag{1.7}$$

$$\implies F_U(x) = x \tag{1.8}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.9}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.10}$$

Write a C program to find the mean and variance of U.

Solution: Code file can be found here:

```
https://github.com/PradeepMundlik/
    AI1110/blob/master/
    Assignment_Soln/codes/1/q1_4.
    c
```

Mean=0.500007
Variance=0.083301

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.11}$$

Solution: from question 1.3

$$F_U(x) = x \tag{1.12}$$

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dx \tag{1.13}$$

for k = 1,

$$E\left[U\right] = \int_{-\infty}^{\infty} x dx \tag{1.14}$$

$$= \int_{0}^{1} x dx \tag{1.15}$$

$$= \frac{1}{2} \tag{1.16}$$

for k = 2,

$$E\left[U^2\right] = \int_{-\infty}^{\infty} x^2 dx \tag{1.17}$$

$$= \int_{0}^{1} x^2 dx + 0 \tag{1.18}$$

$$= \frac{1}{3} \tag{1.19}$$

Mean = 0.5

$$var(x) = E[U^2] - (E\left[U\right])^2 \tag{1.20}$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.21}$$

$$= \frac{1}{3} - \frac{1}{4} \tag{1.22}$$

$$= \frac{1}{12} \tag{1.23}$$

$$var(x) = 0.083 \tag{1.24}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution:

```
https://github.com/PradeepMundlik/
    AI1110/blob/master/
    Assignment_Soln/codes/2/q2_1.
    c
```

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The CDF of X is plotted in Fig. 2.2

```
https://github.com/PradeepMundlik/
    AI1110/blob/master/
    Assignment_Soln/codes/2/
    cdf_plot.py
```

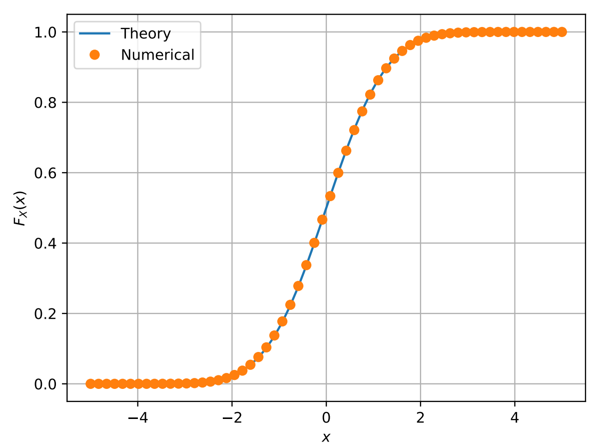

Fig. 2.2: The CDF of X

Properties of CDF:

- As x reaches 0 the CDF reaches 0.5
- As x approaches infinity the CDF approaches 1
- As x approaches minus infinity the CDF approaches 0

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
https://github.com/PradeepMundlik/
    AI1110/blob/master/
    Assignment_Soln/codes/2/
    pdf_plot.py
```

Proprties of PDF of X:

- Graph is simitrical about x=0
- Graph have its peak at x=0

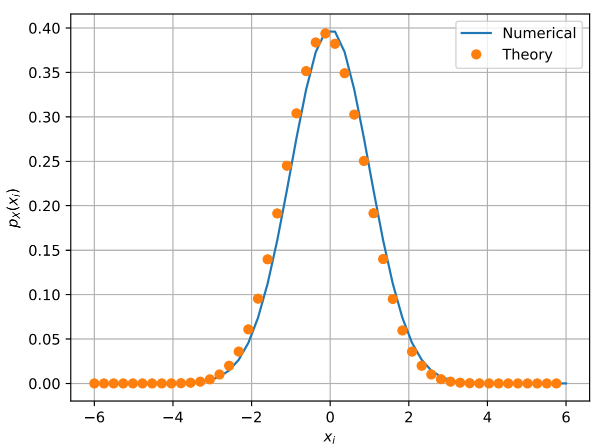

Fig. 2.3: The PDF of X

2.4 Find the mean and variance of X by writing a C program.
Solution:
Mean = 0.000294
Variance = 0.999560
c code file can be found at below link:

```
https://github.com/PradeepMundlik/
    AI1110/blob/master/
    Assignment_Soln/codes/2/q2_4.
    c
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically. **Solution:**

$$E[x] = \int_{-\infty}^{\infty} x p_X(x)\,dx \tag{2.4}$$
$$E[x] = \int_{-\infty}^{\infty} x\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)dx \tag{2.5}$$
$$E[x] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\exp\left(-\frac{x^2}{2}\right)dx \tag{2.6}$$
$$\frac{x^2}{2} = t \tag{2.7}$$
$$xdx = dt \tag{2.8}$$
$$E[x] = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{\infty} \exp(-t)\,dt \tag{2.9}$$
$$E[x] = 0 \tag{2.10}$$
$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x)\,dx \tag{2.11}$$
$$E[x^2] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2\exp\left(-\frac{x^2}{2}\right)dx \tag{2.12}$$
$$E[x^2] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x\left(x\exp\left(-\frac{x^2}{2}\right)\right)dx \tag{2.13}$$
$$\tag{2.14}$$

Integration by parts,

$$= xI\,dx - \int I\,dx \tag{2.15}$$

Where $I = \int x\exp\left(-\frac{x^2}{2}\right)$

$$\frac{x^2}{2} = t \tag{2.16}$$
$$I = \int \exp(-t)\,dt \tag{2.17}$$
$$I = -\exp(-t) \tag{2.18}$$
$$I = -\exp\left(-\frac{x^2}{2}\right) \tag{2.19}$$
$$\tag{2.20}$$

from (2.15),

$$= -x\exp\left(-\frac{x^2}{2}\right) - \int -\exp\left(-\frac{x^2}{2}\right)dx \tag{2.21}$$
$$= -x\exp\left(-\frac{x^2}{2}\right) + \int \exp\left(-\frac{x^2}{2}\right)dx \tag{2.22}$$

Substituting limits in (2.13),

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \tag{2.23}$$

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \tag{2.24}$$

$$E[x^2] = 1 \tag{2.25}$$

$$Var(x) = E[x^2] - (E[x])^2 \tag{2.26}$$

$$Var(x) = 1 - 0 \tag{2.27}$$

$$Var(x) = 1 \tag{2.28}$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1 - U) \tag{3.1}$$

and plot its CDF.
Solution: Code for plot can be found at below link:

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/3/q3_1.py

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/3/q3_1.dat

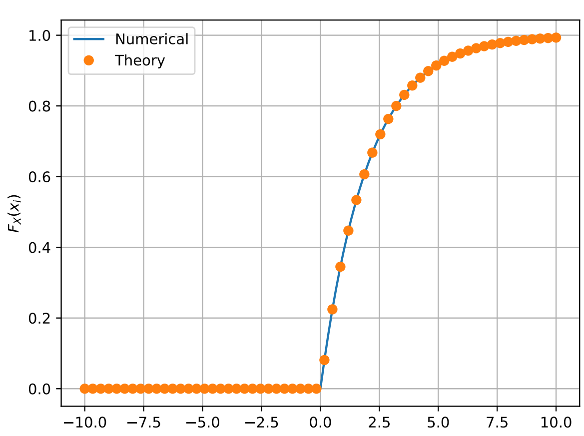

Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = P(V < x) \tag{3.2}$$

$$F_V(x) = P(-2\ln(1 - U) < x) \tag{3.3}$$

$$F_V(x) = P\left((1 - U) > \exp\left(-\frac{x}{2}\right)\right) \tag{3.4}$$

$$F_V(x) = P\left(U < 1 - \exp\left(-\frac{x}{2}\right)\right) \tag{3.5}$$

$$P(U < x) = x \tag{3.6}$$

$$P\left(U < 1 - \exp\left(-\frac{x}{2}\right)\right) = 1 - \exp\left(-\frac{x}{2}\right) \tag{3.7}$$

$$F_V(x) = 1 - \exp\left(-\frac{x}{2}\right) \tag{3.8}$$

4 Triangular Distribution

4.1 Generate

$$T = U_1 + U_2 \tag{4.1}$$

Solution: T.dat and code files can be found here:

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/4/T.py

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/4/T.dat

4.2 Find the CDF of T.
Solution:
CDF of T,

$$F_T(t) = P(T < t) \tag{4.2}$$

$$F_T(t) = P(U_1 + U_2 < t) \tag{4.3}$$

$$0 \le U_1 \le 1 \tag{4.4}$$

$$0 \le U_2 \le 1 \tag{4.5}$$

$$0 \le U_1 + U_2 \le 2 \tag{4.6}$$

$$\forall t > 2, P(U_1 + U_2 < t) = 1 \tag{4.7}$$

$$\forall t < 0, P(U_1 + U_2 < t) = 0 \tag{4.8}$$

for $0 \le t \le 1$, from fig 4.2 Code for plot can be found at below link:

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/4/q4_2.py

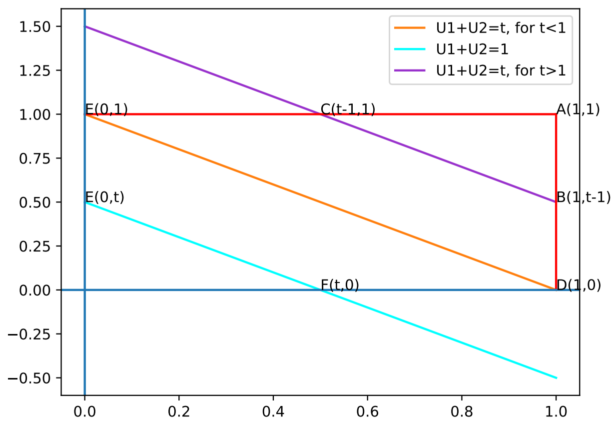

Fig. 4.2: q4.2

$$P(U_1 + U_2 < t) = \frac{\Delta EOF}{\Delta AEOD} \tag{4.9}$$
$$= \frac{t^2}{2} \tag{4.10}$$
$$\tag{4.11}$$

for $1 \leq t \leq 2$,

$$P(U_1 + U_2 < t) = \frac{\Delta ABC}{\Delta AEOD} \tag{4.12}$$
$$= 1 - \frac{(2-t)^2}{2} \tag{4.13}$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t < 1 \\ 1 - \frac{(2-t)^2}{2} & 1 \leq t < 2 \\ 1 & t \geq 2 \end{cases} \tag{4.14}$$

4.3 Find the PDF of T.
Solution:

$$P_T(t) = \frac{d(F_T(t))}{dt} \tag{4.15}$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2 - t & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} \tag{4.16}$$

4.4 Find the theoretical expressions for the PDF and CDF of T.
Solution:

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \therefore\leq t < 1 \\ 1 - \frac{(2-t)^2}{2} & 1 \leq t < 2 \\ 1 & t \geq 2 \end{cases} \tag{4.17}$$

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2 - t & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} \tag{4.18}$$

4.5 Verify your results through a plot.
Solution: Code files for plots can be found at:

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/4/T_pdf.py

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/4/T_cdf.py

PDF of T: fig 4.5
CDF of T: fig 4.5

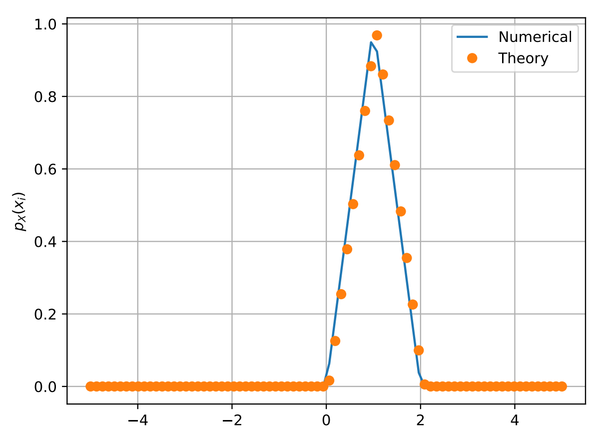

Fig. 4.5: PDF of T

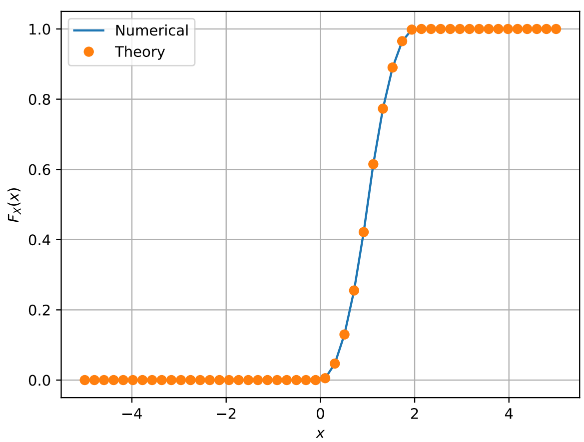

Fig. 4.5: CDF of T