

Assignment

Pradeep Mundlik

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1 UNIFORM RANDOM NUMBERS

Let U be a uniform random variable between 0 and 1.

- 1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/1/exrand.c

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/coeffs.h

- 1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \quad (1.1)$$

Solution: The following code plots Fig. 1.2

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/1/cdf_plot.py

- 1.3 Find a theoretical expression for $F_U(x)$.

Solution: As U is uniform random variable distribution, (between 0 to 1); $CDF = F_U(x)$

$$F_U(x) = \int_0^x P_U(x) dx \quad (1.2)$$

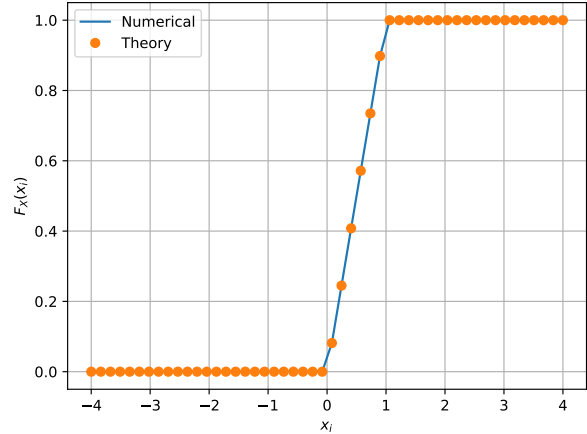


Fig. 1.2: The CDF of U

As uniform distribution, $P_U(x_i) = t$ (t is constant)
So,

$$F_U(x) = \int_0^x t dx \quad (1.3)$$

We know that,

$$F_U(1) = 1 \quad (1.4)$$

$$\Rightarrow \int_0^1 t dx = 1 \quad (1.5)$$

$$\Rightarrow t = 1 \quad (1.6)$$

$$\Rightarrow F_U(x) = \int_0^x 1 dx \quad (1.7)$$

$$\Rightarrow F_U(x) = x \quad (1.8)$$

- 1.4 The mean of U is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^N U_i \quad (1.9)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \quad (1.10)$$

Write a C program to find the mean and variance of U .

Solution: Code file can be found here:

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/1/q1_4.c

Mean=0.500007

Variance=0.083301

1.5 Verify your result theoretically given that

$$E[U^k] = \int_{-\infty}^{\infty} x^k dF_U(x) \quad (1.11)$$

Solution: from question 1.3

$$F_U(x) = x \quad (1.12)$$

$$E[U^k] = \int_{-\infty}^{\infty} x^k dx \quad (1.13)$$

for $k = 1$,

$$E[U] = \int_{-\infty}^{\infty} x dx \quad (1.14)$$

$$= \int_0^1 x dx \quad (1.15)$$

$$= \frac{1}{2} \quad (1.16)$$

for $k = 2$,

$$E[U^2] = \int_{-\infty}^{\infty} x^2 dx \quad (1.17)$$

$$= \int_0^1 x^2 dx + 0 \quad (1.18)$$

$$= \frac{1}{3} \quad (1.19)$$

Mean = 0.5

$$\text{var}(x) = E[U^2] - (E[U])^2 \quad (1.20)$$

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \quad (1.21)$$

$$= \frac{1}{3} - \frac{1}{4} \quad (1.22)$$

$$= \frac{1}{12} \quad (1.23)$$

$$\text{var}(x) = 0.083 \quad (1.24)$$

2 CENTRAL LIMIT THEOREM

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \quad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

Solution:

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/2/q2_1.c

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 2.2

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/2/cdf_plot.py

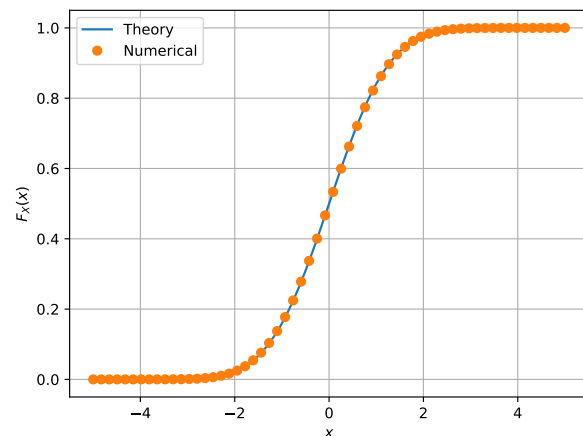


Fig. 2.2: The CDF of X

Properties of CDF:

- As x reaches 0 the CDF reaches 0.5
- As x approaches infinity the CDF approaches 1
- As x approaches minus infinity the CDF approaches 0

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \quad (2.2)$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 2.3 using the code below

```
https://github.com/PradeepMundlik/
AI1110/blob/master/
Assignment_Soln/codes/2/
pdf_plot.py
```

Properties of PDF of X :

- Graph is simitrical about $x=0$
- Graph have its peak at $x=0$

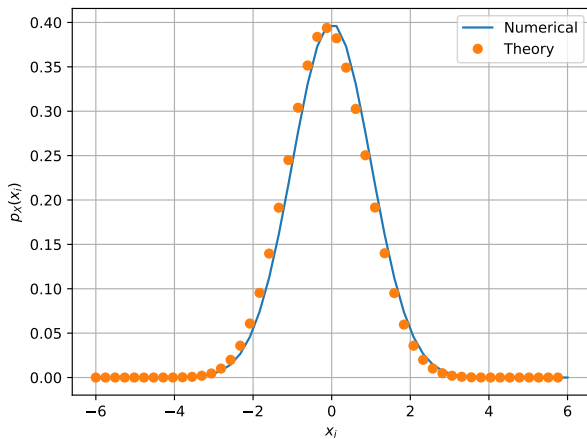


Fig. 2.3: The PDF of X

2.4 Find the mean and variance of X by writing a C program.

Solution:

Mean = 0.000294

Variance = 0.999560

c code file can be found at below link:

```
https://github.com/PradeepMundlik/
AI1110/blob/master/
Assignment_Soln/codes/2/q2_4.
c
```

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically. **Solution:**

$$E[x] = \int_{-\infty}^{\infty} x p_X(x) dx \quad (2.4)$$

$$E[x] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \quad (2.5)$$

$$E[x] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \exp\left(-\frac{x^2}{2}\right) dx \quad (2.6)$$

$$\frac{x^2}{2} = t \quad (2.7)$$

$$x dx = dt \quad (2.8)$$

$$E[x] = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} \exp(-t) dt \quad (2.9)$$

$$E[x] = 0 \quad (2.10)$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \quad (2.11)$$

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \exp\left(-\frac{x^2}{2}\right) dx \quad (2.12)$$

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \left(x \exp\left(-\frac{x^2}{2}\right) \right) dx \quad (2.13)$$

$$(2.14)$$

Integration by parts,

$$= x I dx - \int I dx \quad (2.15)$$

Where $I = \int x \exp\left(-\frac{x^2}{2}\right)$

$$\frac{x^2}{2} = t \quad (2.16)$$

$$I = \int \exp(-t) dt \quad (2.17)$$

$$I = -\exp(-t) \quad (2.18)$$

$$I = -\exp\left(-\frac{x^2}{2}\right) \quad (2.19)$$

$$(2.20)$$

from (2.15),

$$= -x \exp\left(-\frac{x^2}{2}\right) - \int -\exp\left(-\frac{x^2}{2}\right) dx \quad (2.21)$$

$$= -x \exp\left(-\frac{x^2}{2}\right) + \int \exp\left(-\frac{x^2}{2}\right) dx \quad (2.22)$$

$$\left(x \exp\left(-\frac{x^2}{2}\right) \right)_{-\infty}^{\infty} = 0 \quad (2.23)$$

Substituting limits in (2.13),

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{2}\right) \quad (2.24)$$

$$E[x^2] = \frac{1}{\sqrt{2\pi}} \times \sqrt{2\pi} \quad (2.25)$$

$$E[x^2] = 1 \quad (2.26)$$

$$\text{Var}(x) = E[x^2] - (E[x])^2 \quad (2.27)$$

$$\text{Var}(x) = 1 - 0 \quad (2.28)$$

$$\text{Var}(x) = 1 \quad (2.29)$$

3 FROM UNIFORM TO OTHER

3.1 Generate samples of

$$V = -2 \ln(1 - U) \quad (3.1)$$

and plot its CDF.

Solution: Code for plot can be found at below link:

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/3/q3_1.py

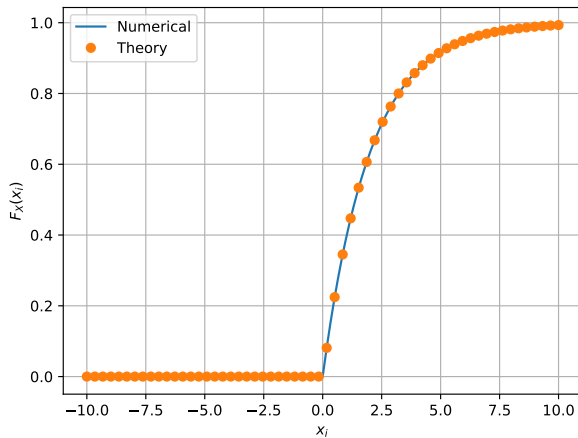


Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.

Solution:

$$F_V(x) = P(V < x) \quad (3.2)$$

$$F_V(x) = P(-2 \ln(1 - U) < x) \quad (3.3)$$

$$F_V(x) = P\left((1 - U) > \exp\left(-\frac{x}{2}\right)\right) \quad (3.4)$$

$$F_V(x) = P\left(U < 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$P(U < x) = x \quad (3.6)$$

$$P\left(U < 1 - \exp\left(-\frac{x}{2}\right)\right) = 1 - \exp\left(-\frac{x}{2}\right) \quad (3.7)$$

$$F_V(x) = 1 - \exp\left(-\frac{x}{2}\right) \quad (3.8)$$

4 TRIANGULAR DISTRIBUTION

4.1 Generate

$$T = U_1 + U_2 \quad (4.1)$$

Solution: T.dat and code files can be found here:

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/4/T.py

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/4/T.dat

4.2 Find the CDF of T.

Solution:

CDF of T,

$$F_T(t) = P(T < t) \quad (4.2)$$

$$F_T(t) = P(U_1 + U_2 < t) \quad (4.3)$$

$$0 \leq U_1 \leq 1 \quad (4.4)$$

$$0 \leq U_2 \leq 1 \quad (4.5)$$

$$0 \leq U_1 + U_2 \leq 2 \quad (4.6)$$

$$\forall t > 2, P(U_1 + U_2 < t) = 1 \quad (4.7)$$

$$\forall t < 0, P(U_1 + U_2 < t) = 0 \quad (4.8)$$

for $0 \leq t \leq 1$, from fig 4.2 Code for plot can be found at below link:

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/4/q4_2.py

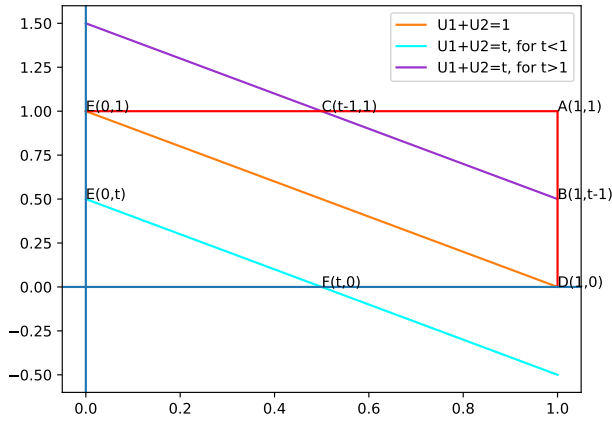


Fig. 4.2: q4.2

$$P(U_1 + U_2 < t) = \frac{\Delta EOF}{\Delta AEO D} \quad (4.9)$$

$$= \frac{t^2}{2} \quad (4.10)$$

$$(4.11)$$

for $1 \leq t \leq 2$,

$$P(U_1 + U_2 < t) = \frac{\Delta ABC}{\Delta AEO D} \quad (4.12)$$

$$= 1 - \frac{(2-t)^2}{2} \quad (4.13)$$

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t < 1 \\ 1 - \frac{(2-t)^2}{2} & 1 \leq t < 2 \\ 1 & t \geq 2 \end{cases} \quad (4.14)$$

4.3 Find the PDF of T .

Solution:

$$P_T(t) = \frac{d(F_T(t))}{dt} \quad (4.15)$$

$$\therefore P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} \quad (4.16)$$

4.4 Find the theoretical expressions for the PDF and CDF of T .

Solution:

$$F_T(t) = \begin{cases} 0 & t < 0 \\ \frac{t^2}{2} & 0 \leq t < 1 \\ 1 - \frac{(2-t)^2}{2} & 1 \leq t < 2 \\ 1 & t \geq 2 \end{cases} \quad (4.17)$$

$$P_T(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t < 1 \\ 2-t & 1 \leq t < 2 \\ 0 & t \geq 2 \end{cases} \quad (4.18)$$

4.5 Verify your results through a plot.

Solution: Code files for plots can be found at:

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/4/T_pdf.py

https://github.com/PradeepMundlik/AI1110/blob/master/Assignment_Soln/codes/4/T_cdf.py

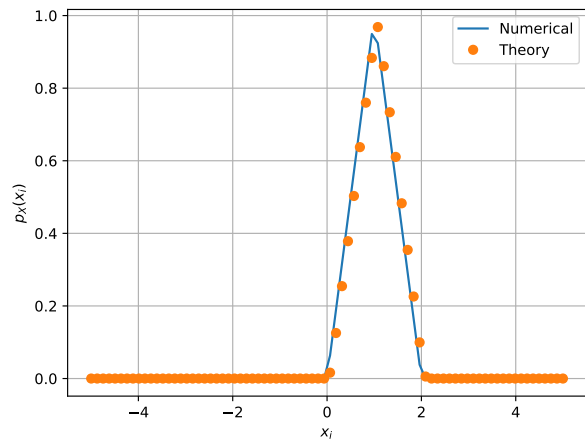


Fig. 4.5: PDF of T

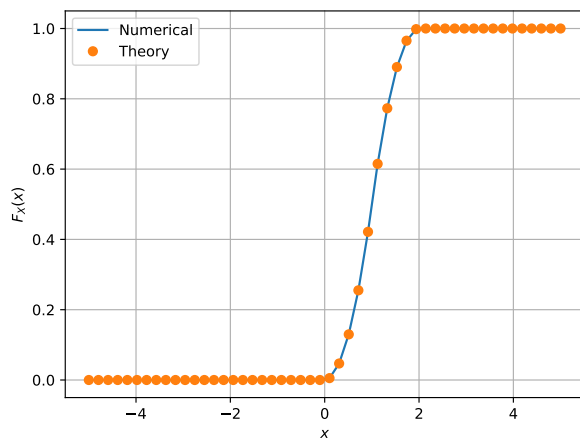


Fig. 4.5: CDF of T