Digital Signal Processing

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Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

```
wget https://raw.githubusercontent.com/
    gadepall/
EE1310/master/filter/codes/Sound_Noise.wav
```

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: We can get code from below link-

```
https://github.com/PradeepMundlik/
    EE3900/tree/master/Assignment1/
    codes/q2/Cancel−noise.py
```

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.

3 Difference Equation

3.1 Let

$$x(n) = \left\{ \underset{\uparrow}{1}, 2, 3, 4, 2, 1 \right\} \tag{3.1}$$

Sketch $x(n)$.
Solution:

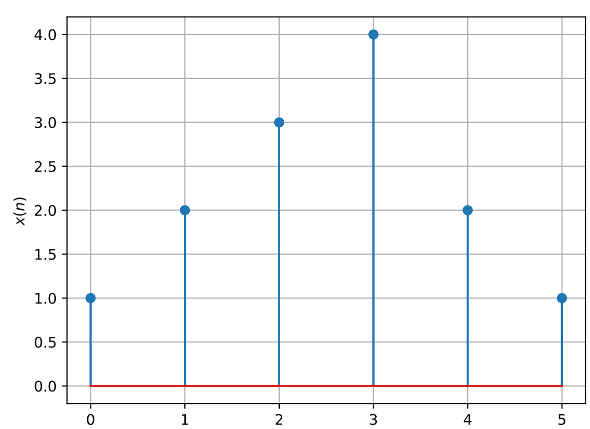

Fig. 3.1

```
https://github.com/PradeepMundlik/
    EE3900/blob/master/Assignment1/
    codes/q3/xn.py
```

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \quad (3.2)$$

Sketch $y(n)$.
Solution: The c code for yn:

```
https://github.com/PradeepMundlik/
    EE3900/blob/master/Assignment1/
    codes/q3/yn.c
```

Data file for $Y(n)$:

```
https://github.com/PradeepMundlik/
    EE3900/blob/master/Assignment1/
    codes/q3/yn.dat
```

The following code yields Fig. 3.2.

```
https://github.com/PradeepMundlik/
    EE3900/blob/master/Assignment1/
    codes/q3/yn.py
```

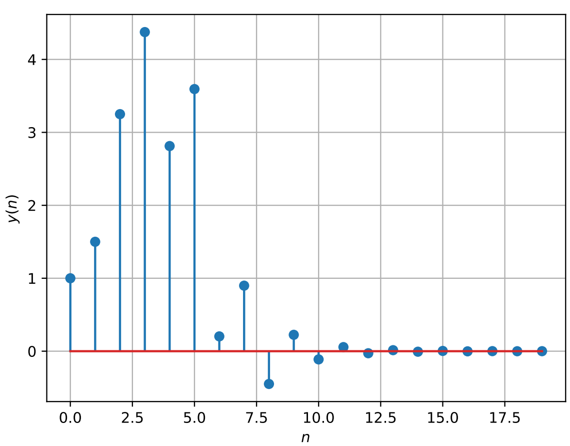

Fig. 3.2

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.1)$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \quad (4.2)$$

and find

$$\mathcal{Z}\{x(n-k)\} \quad (4.3)$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-1)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \quad (4.4)$$
$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.5)$$
$$= z^{-1}X(z) \quad (4.6)$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)z^{-n} \quad (4.7)$$
$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-k} \quad (4.8)$$
$$= z^{-k}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.9)$$
$$= z^{-k}X(z) \quad (4.10)$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1
Solution: from 3.1

$$x(n) = \{1, 2, 3, 4, 2, 1\} \quad (4.11)$$
$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \quad (4.12)$$
$$X(z) = 1z^{0} + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + 1z^{-5} \quad (4.13)$$
$$X(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + 1z^{-5} \quad (4.14)$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \quad (4.15)$$

from (3.2) assuming that the Z-transform is a linear operation.
Solution: Applying (4.7) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \quad (4.16)$$
$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \quad (4.17)$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4.18)$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.19}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.20}$$

Solution:

$$\mathcal{Z}\{\delta(n)\} = \sum_{n=-\infty}^{\infty} \delta(n) z^{-n} \tag{4.21}$$

$$= 1 \tag{4.22}$$

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.23}$$

and from (4.19),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.24}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.25}$$

using the fomula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.26}$$

Solution:

$$\mathcal{Z}\{a^n u(n)\} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} \tag{4.27}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.28}$$

$$= \sum_{n=0}^{\infty} \left(az^{-1}\right)^n \tag{4.29}$$

$$= \frac{1}{1 - az^{-1}}, \quad \left|az^{-1}\right| < 1 \tag{4.30}$$

using the fomula for the sum of an infinite geometric progression.

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.31}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.

Solution:

$$H\left(e^{\jmath\omega}\right) = \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} \tag{4.32}$$

$$\implies \left|H\left(e^{\jmath\omega}\right)\right| = \frac{\left|1 + \cos 2\omega - \jmath \sin 2\omega\right|}{\left|1 + \frac{1}{2}\cos\omega - \frac{1}{2}\sin\omega\right|} \tag{4.33}$$

$$= \sqrt{\frac{(1 + \cos 2\omega)^2 + (\sin 2\omega)^2}{(1 + \frac{1}{2}\cos\omega)^2 + (\frac{1}{2}\sin\omega)^2}} \tag{4.34}$$

$$= \sqrt{\frac{2 + 2\cos 2\omega}{\frac{5}{4} + \cos\omega}} \tag{4.35}$$

$$= \sqrt{\frac{2(2\cos^2\omega)4}{5 + 4\cos\omega}} \tag{4.36}$$

$$\left|H\left(e^{\jmath\omega}\right)\right| = \frac{4\left|\cos\omega\right|}{\sqrt{5 + 4\cos\omega}} \tag{4.37}$$

So,

$$\frac{4\left|\cos(\omega + 2\pi)\right|}{\sqrt{5 + 4\cos(\omega + 2\pi)}} = \frac{4\left|\cos\omega\right|}{\sqrt{5 + 4\cos\omega}} \tag{4.38}$$

It is clear that $\left|H\left(e^{\jmath\omega}\right)\right|$ is periodic with period 2π.

The following code plots Fig. 4.6.

```
https://github.com/PradeepMundlik/
    EE3900/blob/master/Assignment1/
    codes/q4/dtft.py
```

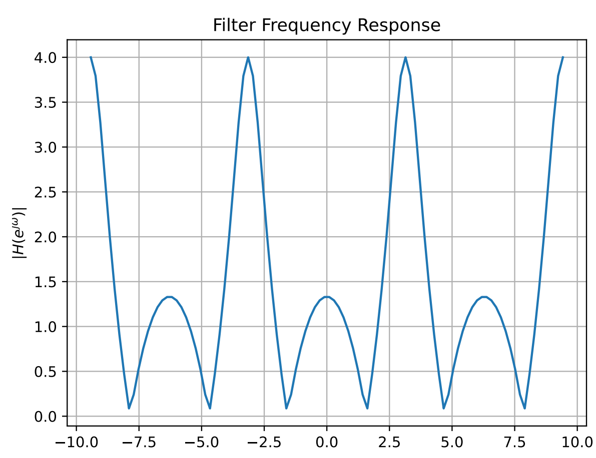

Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

4.7 Express $h(n)$ in terms of $H\left(e^{\jmath\omega}\right)$.

Solution:

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{\jmath\omega}\right) e^{j\omega n}\, d\omega \tag{4.39}$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}} e^{j\omega n}\, d\omega \tag{4.40}$$

5 Impulse Response

5.1 Using long division, find

$$h(n), n < 5 \tag{5.1}$$

for $H(z)$ in (4.17)
Solution:

$$H(z) = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.2}$$
$$1 + z^{-2} = \left(1 + \frac{1}{2}z^{-1}\right) * \left(2z^{-1} - 4\right) + 5 \tag{5.3}$$
$$H(z) = \frac{\left(1 + \frac{1}{2}z^{-1}\right) * \left(2z^{-1} - 4\right) + 5}{1 + \frac{1}{2}z^{-1}} \tag{5.4}$$
$$= 2z^{-1} - 4 + \frac{5}{1 + \frac{1}{2}z^{-1}} \tag{5.5}$$

Now,

$$\frac{5}{1 + \frac{1}{2}z^{-1}} = 5\left(1 - \frac{z^{-1}}{2} + \frac{z^{-2}}{4} - \frac{z^{-3}}{8} + \dots\right) \tag{5.6}$$
$$= 5 - \frac{5}{2}z^{-1} + \frac{5}{4}z^{-2} - \frac{5}{8}z^{-3} + \dots \tag{5.7}$$
$$= \sum_{n=0}^{\infty} 5\left(\frac{-z^{-1}}{2}\right)^n \tag{5.8}$$

$$H(z) = 2z^{-1} - 4 + \frac{5}{1 + \frac{1}{2}z^{-1}} \tag{5.9}$$
$$= 2z^{-1} - 4 + \sum_{n=0}^{\infty} 5\left(\frac{-z^{-1}}{2}\right)^n \tag{5.10}$$

As $n < 5$,

$$H(z) = 2z^{-1} - 4 + \sum_{n=0}^{4} 5\left(\frac{-z^{-1}}{2}\right)^n \tag{5.11}$$
$$H(z) = 1 - \frac{1}{2}z^{-1} + \frac{5}{4}z^{-2} - \frac{5}{8}z^{-3} + \frac{5}{16}z^{-4} \tag{5.12}$$
$$\implies h(n) = \left(1, \frac{-1}{2}, \frac{5}{4}, \frac{-5}{8}, \frac{5}{16}\right) \tag{5.13}$$

for general n,

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} & n \geq 2 \end{cases} \tag{5.14}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.15}$$

and there is a one to one relationship between $h(n)$ and $H(z)$. $h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.17),

$$H(z) = \frac{1}{1 + \frac{1}{2}z^{-1}} + \frac{z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{5.16}$$
$$\implies h(n) = \left(-\frac{1}{2}\right)^n u(n) + \left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.17}$$

using (4.26) and (4.7).

5.3 Sketch $h(n)$. Is it bounded? Justify theoratically.
Solution:

$$|u(n)| \leq 1 \tag{5.18}$$
$$\left|\left(-\frac{1}{2}\right)^n\right| \leq 1 \tag{5.19}$$
$$\implies \left|\left(-\frac{1}{2}\right)^n u(n)\right| \leq 1 \tag{5.20}$$

Similarly,

$$\left|\left(-\frac{1}{2}\right)^{n-2} u(n-2)\right| \leq 1 \tag{5.21}$$
$$\implies h(n) \leq 2 \tag{5.22}$$

Hence, $h(n)$ is bounded. The following code plots Fig. 5.3.

```
https://github.com/PradeepMundlik/
    EE3900/blob/master/Assignment1/
    codes/q5/hn.py
```

Fron fig.5.3 it is clear that, $h(n)$ converges to 0 and it is bounded as well.

5.4 Convergent? Justify using the ratio test.

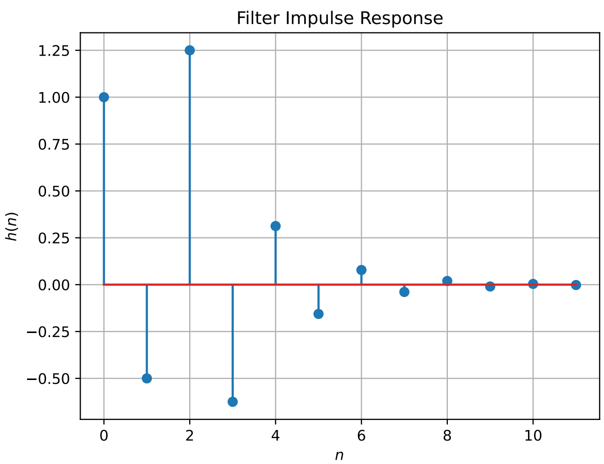

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

Solution: for $n > 2$,

$$h(n) = \left(-\frac{1}{2}\right)^n + \left(-\frac{1}{2}\right)^{n-2} \quad (5.23)$$

$$h(n) = 5\left(-\frac{1}{2}\right)^n \quad (5.24)$$

$$\left|\frac{h(n+1)}{h(n)}\right| = \frac{1}{2} < 1 \quad (5.25)$$

Hence, $h(n)$ is convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \quad (5.26)$$

Is the system defined by (3.2) stable for the impulse response in (5.15)?

Solution:

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad (5.27)$$

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n + \sum_{n=2}^{\infty} \left(-\frac{1}{2}\right)^{n-2} \quad (5.28)$$

These are both sums of infinite geometric progressions with first terms 1 and common ratios $-\frac{1}{2}$

$$\sum_{n=-\infty}^{\infty} h(n) = \frac{1}{1-\left(-\frac{1}{2}\right)} + \frac{1}{1-\left(-\frac{1}{2}\right)} \quad (5.29)$$

$$= \frac{4}{3} < \infty \quad (5.30)$$

Therefore, the system is stable.

5.6 Verify result using python code.

Solution:

https://github.com/PradeepMundlik/EE3900/blob/master/Assignment1/codes/q5/q5.6.py

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \quad (5.31)$$

This is the definition of $h(n)$.

Solution:

$$h(0) = 1 \quad (5.32)$$

Now, for $n = 1$,

$$h(1) + \frac{1}{2}h(0) = \delta(1) + \delta(-1) = 0 \quad (5.33)$$

$$\implies h(1) = -\frac{1}{2}h(0) = -\frac{1}{2} \quad (5.34)$$

For $n = 2$,

$$h(2) + \frac{1}{2}h(1) = \delta(2) + \delta(0) = 1 \quad (5.35)$$

$$\implies h(2) = 1 - \frac{1}{2}h(1) = \frac{5}{4} \quad (5.36)$$

For $n > 2$, the right hand side of the equation is always zero. Thus,

$$h(n) = -\frac{1}{2}h(n-1) \qquad n > 2 \quad (5.37)$$

$$h(3) = \frac{5}{4}\left(-\frac{1}{2}\right) \quad (5.38)$$

$$h(4) = \frac{5}{4}\left(-\frac{1}{2}\right)^2 \quad (5.39)$$

$$\vdots \quad (5.40)$$

$$h(n) = \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} \quad (5.41)$$

Therefore,

$$h(n) = \begin{cases} 1 & n = 0 \\ -\frac{1}{2} & n = 1 \\ \frac{5}{4}\left(-\frac{1}{2}\right)^{n-2} & n \geq 2 \end{cases} \tag{5.42}$$

Thus, it is bounded and convergent to 0

$$\lim_{n\to\infty} h(n) = 0 \tag{5.43}$$

The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
https://github.com/PradeepMundlik/
    EE3900/blob/master/Assignment1/
    codes/q5/hndef.py
```

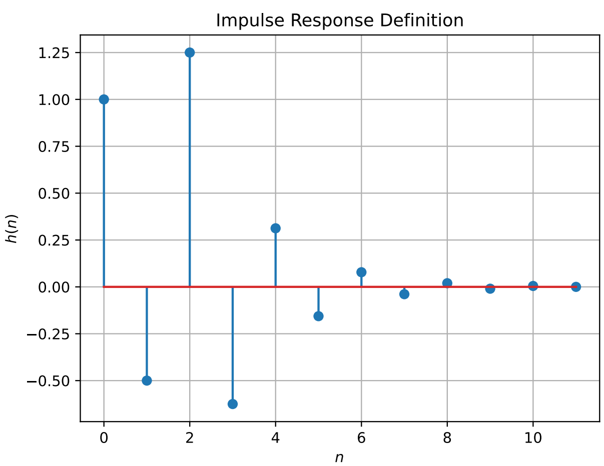

Fig. 5.7: $h(n)$ from the definition

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.44}$$

Comment. The operation in (5.44) is known as *convolution*.

Solution:

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.45}$$

$$= \sum_{k=0}^{5} x(k)h(n-k) \tag{5.46}$$

The following code plots Fig. 5.8. Note that this is the same as $y(n)$ in Fig. 3.2.

```
https://github.com/PradeepMundlik/
    EE3900/blob/master/Assignment1/
    codes/q5/ynconv.py
```

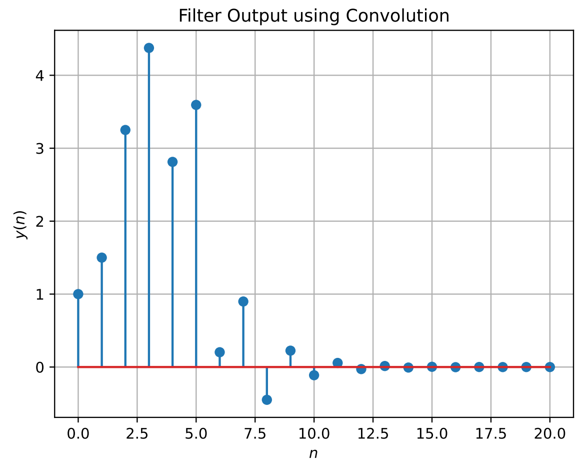

Fig. 5.8: $y(n)$ from the definition of convolution

5.9 Express the above convolution using a Toeplitz matrix.

Solution:

$$\vec{x} = \begin{pmatrix} 1 & 2 & 3 & 4 & 2 & 1 \end{pmatrix}^{\top} \tag{5.47}$$

$$\vec{h} = \begin{pmatrix} h_0 & h_1 & \cdots & h_{N-1} \end{pmatrix}^{\top} \tag{5.48}$$

$$\vec{y} = \vec{x} \circledast \vec{h} \tag{5.49}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{N+5} \end{pmatrix} = \begin{pmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_{N-6} \\ 0 & h_{N-1} & h_{N-2} & \cdots & h_{N-5} \\ 0 & 0 & h_{N-1} & \cdots & h_{N-4} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{N-1} \end{pmatrix} \begin{pmatrix} 1.0 \\ 2.0 \\ 3.0 \\ 4.0 \\ 2.0 \\ 1.0 \end{pmatrix} \tag{5.50}$$

5.10 Show that

$$y(n) = \sum_{n=-\infty}^{\infty} x(n-k)h(k) \tag{5.51}$$

Solution:

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.52}$$

Substitute $k = n - i$

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{n-i=-\infty}^{\infty} x(n-i)h(n-(n-i)) \tag{5.53}$$

$$= \sum_{i=\infty}^{-\infty} x(n-i)h(i) \tag{5.54}$$

$$= \sum_{i=-\infty}^{\infty} x(n-i)h(i) \tag{5.55}$$

since the order of limits does not matter for a summation. Thus,

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.56}$$

$$\implies x(n) * h(n) = h(n) * x(n) \tag{5.57}$$

Therefore, convolution is commutative.

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-\jmath 2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.
Solution: The following code plots Fig.6.1

https://github.com/PradeepMundlik/EE3900/blob/master/Assignment1/codes/q6/6.1.py

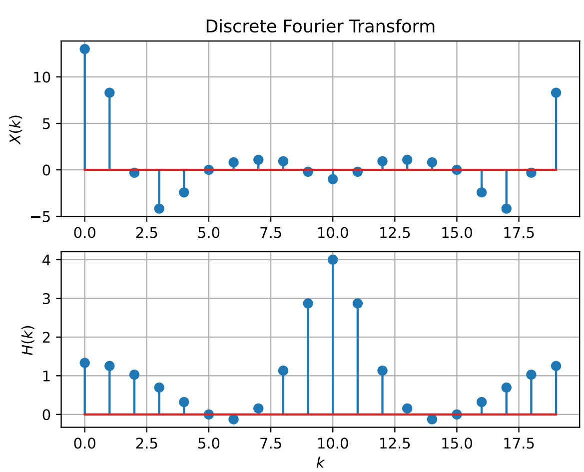

Fig. 6.1: Discret Fourier Transform

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: The following code plots Fig.6.2

https://github.com/PradeepMundlik/EE3900/blob/master/Assignment1/codes/q6/6.2.py

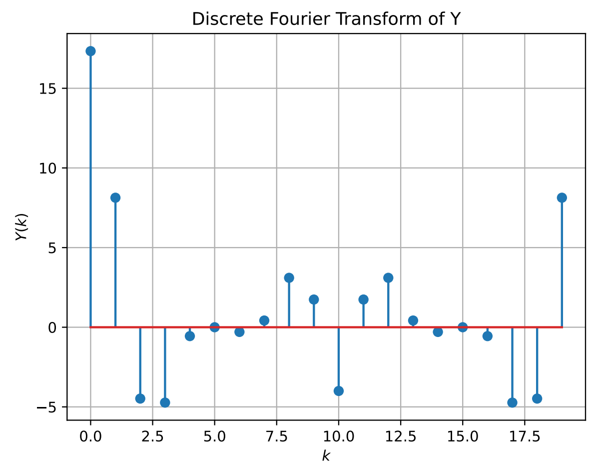

Fig. 6.2: Discret Fourier Transform of $Y(k)$

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\cdot e^{\jmath 2\pi kn/N}, \quad n = 0, 1, \dots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 6.3. Note that this is the same as $y(n)$ in Fig. 3.2.

6.3 code

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Run the code to generate the Fig. 6.4

code xk

6.5 Wherever possible, express all the above equations as matrix equations.
Solution:

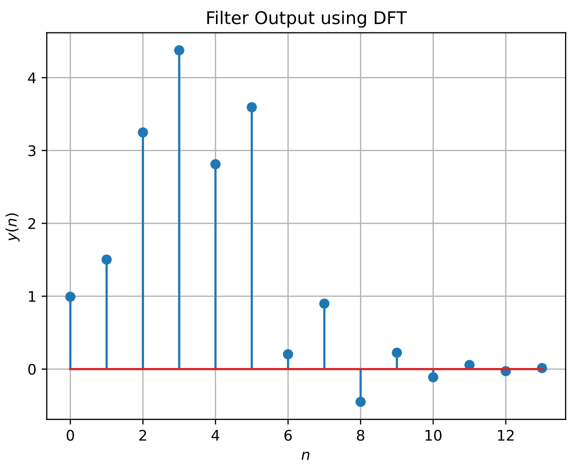

Fig. 6.3: $y(n)$ from the DFT

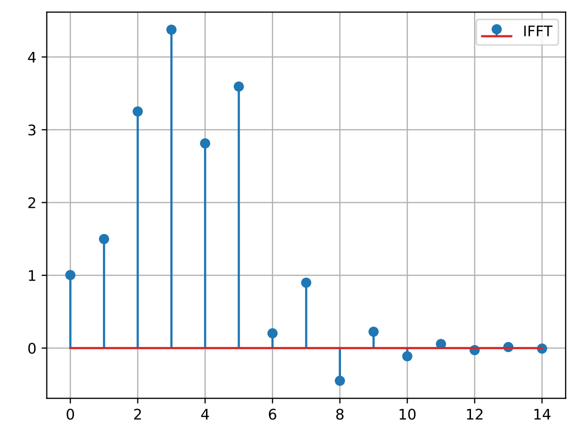

Fig. 6.4: IFFT $X(k)$

$$\vec{x} = \begin{pmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{pmatrix}^\top \tag{6.4}$$

$$\vec{h} = \begin{pmatrix} x_0 & x_1 & \cdots & x_{N-1} \end{pmatrix}^\top \tag{6.5}$$

$$\vec{y} = \vec{x} \circledast \vec{h} \tag{6.6}$$

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{2N-1} \end{pmatrix} = \begin{pmatrix} h_0 & 0 & 0 & \cdots & 0 \\ h_1 & h_0 & 0 & \cdots & 0 \\ h_2 & h_1 & h_0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ h_{N-1} & h_{N-2} & h_{N-3} & \cdots & h_0 \\ 0 & h_{N-1} & h_{N-2} & \cdots & h_1 \\ 0 & 0 & h_{N-1} & \cdots & h_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & h_{N-1} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \tag{6.7}$$

The convolution can be written using a Toeplitz matrix.
Consider the DFT matrix

$$\vec{W} = \begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \omega^3 & \cdots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \omega^6 & \cdots & \omega^{2(N-1)} \\ 1 & \omega^3 & \omega^6 & \omega^9 & \cdots & \omega^{3(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & \omega^{3(N-1)} & \cdots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{6.8}$$

where $\omega = e^{-\jmath 2\pi/N}$ is the N^{th} root of unity
Then the discrete Fourier transforms of $\vec{x}$ and $\vec{h}$ are given by

$$\vec{X} = \vec{W}\vec{x} \tag{6.9}$$

$$\vec{H} = \vec{W}\vec{h} \tag{6.10}$$

$\vec{Y}$ is then given by

$$\vec{Y} = \vec{X} \circ \vec{H} \tag{6.11}$$

where $\circ$ denotes the Hadamard product (element-wise multiplication)
But $\vec{Y}$ is the discrete Fourier transform of the filter output $\vec{y}$

$$\vec{Y} = \vec{W}\vec{y} \tag{6.12}$$

Thus,

$$\vec{W}\vec{y} = \vec{X} \circ \vec{H} \tag{6.13}$$

$$\implies \vec{y} = \vec{W}^{-1}\left(\vec{X} \circ \vec{H}\right) \tag{6.14}$$

$$= \vec{W}^{-1}\left(\vec{W}\vec{x} \circ \vec{W}\vec{h}\right) \tag{6.15}$$

This is the inverse discrete Fourier transform of $\vec{Y}$