Fourier Series

EE3900: Linear Systems and Signal Processing

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1. Periodic Function

Let

$$x(t) = A_0 \left|\sin\left(2\pi f_0 t\right)\right| \tag{1.1}$$

1.1 Plot $x(t)$
Solution: Download the following Python code that plots Fig. 1.1.
Run the code by executing

```
python 1.1.py
```

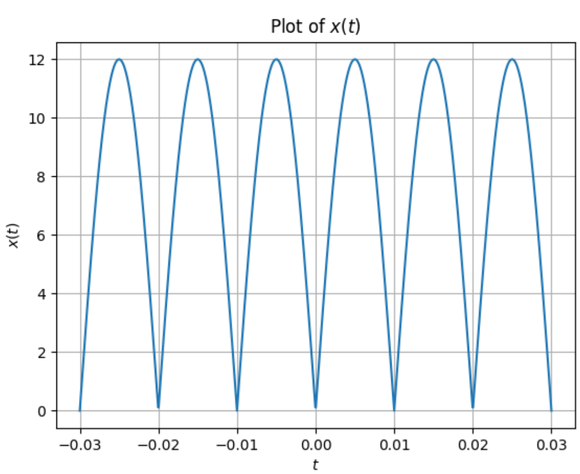

Fig. 1.1. Plot of $x(t)$

1.2 Show that $x(t)$ is periodic and find its period
Solution: Since $x(t)$ is the absolute value of a sinusoidal function, it is periodic, which is also evident from the plot

Consider $x(t + \frac{1}{2f_0})$

$$\begin{aligned} x\left(t + \frac{1}{2f_0}\right) &= A_0 \left|\sin\left(2\pi f_0 \left(t + \frac{1}{2f_0}\right)\right)\right| && (1.2)\\ &= A_0 \left|\sin\left(2\pi f_0 t + \pi f_0\right)\right| && (1.3)\\ &= A_0 \left|(-1)^{f_0} \sin\left(2\pi f_0 t\right)\right| && (1.4)\\ &= A_0 \left|\sin\left(2\pi f_0 t\right)\right| && (1.5)\\ &= x(t) && (1.6)\end{aligned}$$

Therefore, $x(t)$ is periodic with period $\frac{1}{2f_0}$

2. Fourier Series

Consider $A_0 = 12$ and $f_0 = 50$ for all numerical calculations

2.1 If

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.1}$$

show that

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.2}$$

Solution:

$$x(t)e^{-\jmath 2\pi n f_0 t} = \sum_{k=-\infty}^{\infty} c_k e^{-\jmath 2\pi (n-k) f_0 t} \tag{2.3}$$

$$\implies \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t)e^{-\jmath 2\pi n f_0 t}\, dt = \sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi (n-k) f_0 t}\, dt \tag{2.4}$$

But

$$\begin{aligned} \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi (n-k) f_0 t}\, dt &= \begin{cases} \frac{1}{f_0} & k = n \\ 0 & k \neq n \end{cases} && (2.5)\\ &= \frac{1}{f_0}\delta(n-k) && (2.6)\end{aligned}$$

$$\sum_{k=-\infty}^{\infty} c_k \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} e^{-\jmath 2\pi(n-k)f_0 t}\, dt = \sum_{k=-\infty}^{\infty} c_k \frac{1}{f_0}\delta(n-k) \tag{2.7}$$

$$= \frac{1}{f_0} c_n * \delta(n) \tag{2.8}$$

$$= \frac{1}{f_0} c_n \tag{2.9}$$

Therefore

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} x(t) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.10}$$

2.2 Find c_k for (1.1)

Solution:

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{\frac{1}{2f_0}} A_0 \left|\sin(2\pi f_0 t)\right| e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.11}$$

$$c_k = f_0 \int_{-\frac{1}{2f_0}}^{0} A_0 \left(-\sin(2\pi f_0 t)\right) e^{-\jmath 2\pi k f_0 t}\, dt + f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \left(\sin(2\pi f_0 t)\right) e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.12}$$

$$c_k = f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 u)\, e^{\jmath 2\pi k f_0 u}\, dt + f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t)\, e^{-\jmath 2\pi k f_0 t}\, dt \tag{2.13}$$

$$c_k = f_0 \int_{0}^{\frac{1}{2f_0}} A_0 \sin(2\pi f_0 t) \left(e^{\jmath 2\pi k f_0 t} + e^{-\jmath 2\pi k f_0 t}\right) dt \tag{2.14}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} 2\sin(2\pi f_0 t)\cos(2\pi k f_0 t)\, dt \tag{2.15}$$

$$= f_0 A_0 \int_{0}^{\frac{1}{2f_0}} \left\{\sin(2\pi(1+k)f_0 t) + \sin(2\pi(1-k)f_0 t)\right\} dt \tag{2.16}$$

$$= f_0 A_0 \left[-\frac{\cos(2\pi(1+k)f_0 t)}{2\pi(1+k)f_0} - \frac{\cos(2\pi(1-k)f_0 t)}{2\pi(1-k)f_0}\right]_0^{\frac{1}{2f_0}} \tag{2.17}$$

$$= \frac{f_0 A_0}{2\pi f_0}\left[\frac{1-(-1)^{1+k}}{1+k} + \frac{1-(-1)^{1-k}}{1-k}\right] \tag{2.18}$$

$$= \left(1+(-1)^k\right)\frac{A_0}{2\pi}\left[\frac{1}{1+k} + \frac{1}{1-k}\right] \tag{2.19}$$

$$= \left(1+(-1)^k\right)\frac{A_0}{\pi(1-k^2)} \tag{2.20}$$

Therefore

$$c_k = \begin{cases} \frac{2A_0}{\pi(1-k^2)} & k \text{ is even} \\ 0 & k \text{ is odd} \end{cases} \tag{2.21}$$

2.3 Verify (1.1) using Python

Solution: Download the following Python code that plots Fig. 2.3.

Run the code by executing

```
python 2.3.py
```

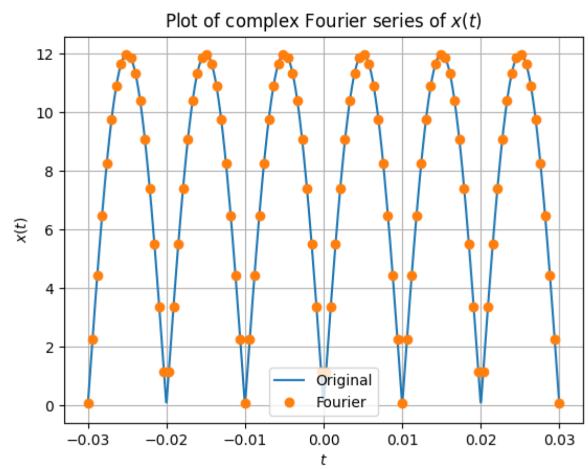

Fig. 2.3. Plot of $x(t)$ along with its complex Fourier series expansion

2.4 Show that

$$x(t) = \sum_{k=0}^{\infty}\left(a_k \cos(2\pi k f_0 t) + b_k \sin(2\pi k f_0 t)\right) \tag{2.22}$$

and obtain the formulae for a_k and b_k

Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{2.23}$$

$$= c_0 + \sum_{k=1}^{\infty} c_k e^{\jmath 2\pi k f_0 t} + c_{-k} e^{-\jmath 2\pi k f_0 t} \tag{2.24}$$

Thus

$$x(t) = c_0 + \sum_{k=1}^{\infty}(c_k + c_{-k})\cos(2\pi k f_0 t) + \sum_{k=1}^{\infty}\jmath\,(c_k - c_{-k})\sin(2\pi k f_0 t) \tag{2.25}$$

Therefore

$$a_k = \begin{cases} c_0 & k = 0 \\ c_k + c_{-k} & k > 0 \end{cases} \tag{2.26}$$

$$b_k = \jmath(c_k - c_{-k}) \quad k \geq 0 \tag{2.27}$$

2.5 Find a_k and b_k for (1.1)
Solution:

$$a_0 = c_0 = \frac{2A_0}{\pi} \tag{2.28}$$

For $k > 0$, if k is odd

$$a_k = 0 + 0 = 0 \tag{2.29}$$

and if k is even

$$a_k = \frac{2A_0}{\pi(1-k^2)} + \frac{2A_0}{\pi(1-k^2)} = \frac{4A_0}{\pi(1-k^2)} \tag{2.30}$$

For odd or even k, $c_k = c_{-k}$ always

$$b_k = 0 \quad \forall k \geq 0 \tag{2.31}$$

Therefore

$$a_k = \begin{cases} \frac{2A_0}{\pi} & k = 0 \\ \frac{4A_0}{\pi(1-k^2)} & k = 2m, m \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \tag{2.32}$$

$$b_k = 0 \qquad k \geq 0 \tag{2.33}$$

2.6 Verify (2.22) using Python
Solution: Download the following Python code that plots Fig. 2.6.
Run the code by executing

```
python 2.6.py
```

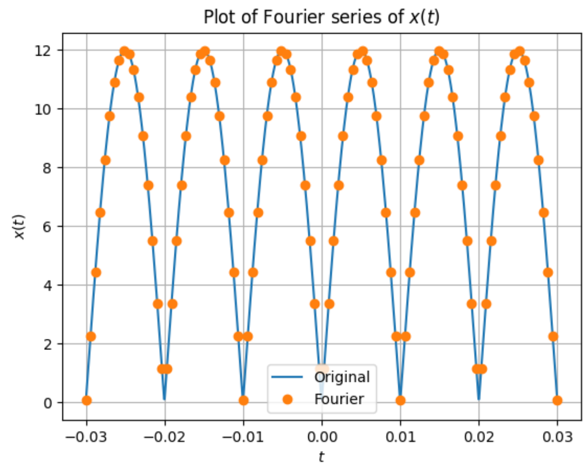

Fig. 2.6. Plot of $x(t)$ along with its Fourier series expansion

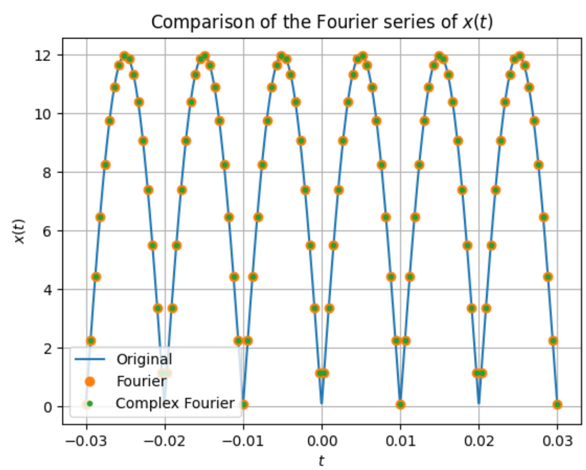

Fig. 2.6. Comparison of the Fourier series of $x(t)$

3. Fourier Transform

3.1

$$\delta(t) = 0 \qquad t \neq 0 \tag{3.1}$$

$$\int_{-\infty}^{\infty} \delta(t) \mathrm{d}t = 1 \tag{3.2}$$

3.2 The Fourier Transform of $g(t)$ is

$$G(f) = \int_{-\infty}^{\infty} g(t) e^{-\jmath 2\pi f t} \, \mathrm{d}t \tag{3.3}$$

3.3 Show that

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} G(f) e^{-\jmath 2\pi f t_0} \tag{3.4}$$

Solution:

$$g(t - t_0) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} g(t - t_0) e^{-\jmath 2\pi f t} \, \mathrm{d}t \tag{3.5}$$

$$= \int_{-\infty}^{\infty} g(u) e^{-\jmath 2\pi f (u + t_0)} \, \mathrm{d}u \tag{3.6}$$

$$= e^{-\jmath 2\pi f t_0} \int_{-\infty}^{\infty} g(u) e^{-\jmath 2\pi f u} \, \mathrm{d}u \tag{3.7}$$

$$= G(f) e^{-\jmath 2\pi f t_0} \tag{3.8}$$

3.4 Show that

$$G(t) \xleftrightarrow{\mathcal{F}} g(-f) \tag{3.9}$$

Solution:

$$G(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} G(t) e^{-\jmath 2\pi f t} \, \mathrm{d}t \tag{3.10}$$

But

$$g(t) = \int_{-\infty}^{\infty} G(f) e^{\jmath 2\pi f t} \, \mathrm{d}f \tag{3.11}$$

$$= \int_{-\infty}^{\infty} G(u) e^{\jmath 2\pi u t} \, \mathrm{d}u \tag{3.12}$$

$$\implies g(-f) = \int_{-\infty}^{\infty} G(u) e^{-\jmath 2\pi u f} \, \mathrm{d}u \tag{3.13}$$

$$= \mathcal{F}\{G(t)\} \tag{3.14}$$

3.5 Find the Fourier transform of $\delta(t)$
Solution:

$$\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} \, \mathrm{d}t \tag{3.15}$$

$$= e^{-j2\pi f t}\Big|_{t=0} \tag{3.16}$$

$$=1 \tag{3.17}$$

3.6 Find the Fourier transform of $e^{-\jmath 2\pi f_0 t}$
Solution:

$$\delta(t) \xleftrightarrow{\mathcal{F}} 1 \tag{3.18}$$

$$\implies \delta(t - f_0) \xleftrightarrow{\mathcal{F}} e^{-j2\pi f f_0} \tag{3.19}$$

$$\implies e^{-j2\pi t f_0} \xleftrightarrow{\mathcal{F}} \delta(-f - f_0) \tag{3.20}$$

$$\therefore e^{-j2\pi t f_0} \xleftrightarrow{\mathcal{F}} \delta(f + f_0) \tag{3.21}$$

3.7 Find the Fourier transform of $\cos(2\pi f_0 t)$
Solution:

$$\cos(2\pi f_0 t) = \frac{e^{\jmath 2\pi f_0 t} + e^{-\jmath 2\pi f_0 t}}{2} \tag{3.22}$$

$$\implies \cos(2\pi f_0 t) \xleftrightarrow{\mathcal{F}} \frac{\delta(f - f_0) + \delta(f + f_0)}{2} \tag{3.23}$$

3.8 Find the Fourier transform of $x(t)$ and plot it. Verify using Python
Solution:

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{\jmath 2\pi k f_0 t} \tag{3.24}$$

$$\mathcal{F}\{x(t)\} = \sum_{k=-\infty}^{\infty} c_k \mathcal{F}\left\{e^{\jmath 2\pi k f_0 t}\right\} \tag{3.25}$$

$$= \sum_{k=-\infty}^{\infty} c_k \delta(f - k f_0) \tag{3.26}$$

$$= \frac{2A_0}{\pi} \sum_{k \text{ is even}} \frac{\delta(f - k f_0)}{1 - k^2} \tag{3.27}$$