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On multi-parameter persistent homology

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1 Introduction.

Persistent homology is a topological tool for studying the global, non-linear, geometric features of data. In the last two decades it has been applied widely [12] and has been subject of a large body of theoretical work [2, 8]. Multi-dimensional persistence is considered to be even more useful than 1-dimensional persistence when studying real data (see for example the flying wing problem of [17]). However its study is inherently more difficult than that of the 1-dimensional case [6, 7]. The goal of these notes is that of introducing the reader to the theory relative to multi-dimensional persistent homology (part I) and study the distance between simple types of multi-dimensional persistent modules that we call indicator modules (part II). More precisely we will structure the document as follows.

We will start in section 2 by reviewing the basic concepts of 1-dimensional persistent homology including the basic definition, examples on how to obtain persistent homology modules from morse filtrations, definition of the barcode distance under tameness conditions and the stability theorem. Readers already initialized in 1-dimensional persistent homology can safely skip this section without resulting in an increased comprehension difficulty for the rest of the document.

As mentioned earlier part I will be dedicated to introducing the basic concepts related with multi-dimensional persistent homology. During this part we will be following steps similar to those performed in [18] with the main goals of making the reader familiar with the correspondence between multi-dimensional persistent homology and modules over posets (definition 3.1), introduce tame poset modules (definition 3.9) and prove that the category of tame poset modules is in fact abelian (proposition 6.9). An additional important result we will show in this part of the document is proposition 4.10 which generalizes [17, lemma 2.19] allowing us to describe all possible poset module homomorphisms (definition 4.7) between 2 indicator modules (definition 4.5). More precisely we will see that such homomorphisms are in fact nothing but multiplication by a scalar in some (the rest is multiplication by zero) of the connected components forming the intersection between the corresponding indicator sets. We find this result to be very useful for computing the interleaving distance between indicator modules.

Finally, in part II of the document we will take a closer look to the structure of poset modules and, more precisely, to distances between them. We will have two main goals during this part. The first one will be that of proving the often mentioned but rarely proven existence and uniqueness of remak decompositions for tame poset modules. The second goal will be that of defining interleaving distance (which generalizes the bottleneck distance), prove a generalized stability theorem and compute interleaving distances between indicator modules. Unluckily enough, due to the complexity of irreducible poset modules (see [6]) we cannot combine both these results in order to compute the interleaving distance between any two given poset modules. We can however give an upper bound (see lemma 9.65) and study more in depth the interleaving distance between indicator modules. For this particular type of poset modules we will in fact give an alternative definition of the interleaving distance in terms of set inclusion (see proposition 10.2 and corollary 10.4) as well as an upper bound (proposition 10.9) which will turn out to be exact in the 1-dimensional case (example 10.8). When the intersection of the indicator sets is connected we will additionally provide a lower bound for the interleaving distance (see proposition 10.16 and remark 10.18).

In order to allow a fast reading of this document almost every part, section and sub-section reported here starts with a summary of the results proven. When introducing new concepts these summaries usually include new notions of them. The reader not interested in an extensive read of the document can easily comprehend its contents just by reading these summaries. On the opposite side of the spectrum the reader interested in a more extensive reading can refer to the extended version of this document [19]. Besides containing mostly unchanged the results shown in both parts I and II this extended version contains an additional part spanning most of the document where a revision of the theory developed in [17] for building QR-codes is reported.

2 Review on 1-dimensional persistent homology.

In this brief section we will recall the most basic notions of 1-dimensional persistent homology. More precisely we will review the definitions of persistent homology, barcodes and bottleneck distance and will state without proof a particular case of the stability theorem that we attempt to generalize to the multi-dimensional persistent homology during these notes. The reader already familiar with these concepts can safely skip this section and without resulting in any added difficulty on the comprehension of the rest of the document.

Even though 1-dimensional persistent homology is usually studied as arising from either a point cloud or a morse filtration of a topological space it can be defined more in general from any sequence of chain complexes as follows.

Definition 2.1. Given a sequence of chain complexes $C = (C_*^i)_{i \in I}$ indexed by a totally ordered set I together with chain maps $x^{i,j} : C_*^i \to C_*^j$ with $j \geq i$ we define the **$(i,\, j)$-persistent homology** of C as the image of the induced morphism $x_*^{i,j} : H_*(C_*^i, k) \to H_*(C_*^j, k)$, for a fixed field k. We denote such an image by $\boldsymbol{H_*^{i \to j}(C)}$.

As mentioned earlier such sequence of chain complexes is usually studied as deriving either from a point cloud or from a morse filtration. This last case is the one we will be more interested in since, according to the nerve theorem (also known as Čech theorem) the Čech complex that can be obtained from a a point cloud has the same homotopy type as the union of all the spheres in that Čech complex and, therefore, studying the persistent homology derived from a family of Čech complexes based on a fixed point cloud is in fact equivalent to studying the persistent homology of a particular morse filtration of $\mathbb{R}^n$ as we will later see. For more information regarding persistent homology on point clouds refer to [11, 12]. The case of persistent homology arising from a morse filtration is better explained in the following example.

Example 2.2. Given a topological space X embedded in $\mathbb{R}^n$ (for example S^{n-1}) we can define a function $f : X \to \mathbb{R}$ (for example projection on the first coordinate) then, for every

$i \in \mathbb{R}$ we obtain the topological space $X_i = f^{-1}(\{x \leq i\})$ which leads to its simplicial chain complex C_*^i. Taking as chain maps $x^{i,j} : C_*^i \to C_*^j$ those induced by inclusion $X_i \hookrightarrow X_j$ we can derive the desired persistent homology.

Notice that, from what we have previously mentioned regarding Čech complexes and the nerve theorem, we can obtain a topological space with the same homology type as an ε-Čech complex centered on a point cloud $S \subseteq \mathbb{R}^n$ by defining $f : \mathbb{R}^n \to \mathbb{R}$ as the function sending any point $x \in \mathbb{R}^n$ to $f(x) = \min(\{\|x - y\| \mid y \in S\})$ and taking $f^{-1}(\{x \leq \varepsilon\})$. Thus persistent homology arising from point clouds can also be obtained from morse filtrations.

When studying persistent homology of a given sequence of chain complexes however we do not usually work with the raw persistent homology but rather with a more visual interpretation of it given by barcodes. In order to define barcodes notice that fixed an homology level l all persistent homologies $H_l^{i\to j}(C)$ of a sequence of chain complexes C (indexed by the ordered set I) can be interpreted as the map from degree i to degree j of the sequence of vector spaces $M = \{H_l(C_l^i, k)\}_{i\in I}$ (see example 3.3 for more detail). Under certain tameness conditions (for example those in def 3.9) we will have that the morphisms $H_l^{i\to j}(C)$ are mostly the identity morphism and $H_l^{i\to j}(C) \ncong H_l^{i-\varepsilon\to j+\varepsilon'}(C)$ for every $\varepsilon, \varepsilon' \in \mathbb{R}^+$ with $\varepsilon + \varepsilon > 0$ only for a finitely many cases. Under this conditions we can remove most of the indexes in I and think of I as $\mathbb{N}$ with $H_l^{n\to n+m}(C) = \mathrm{Id}$ for every positive m and every $n \geq n_0$ with fixed $n_0 \in \mathbb{N}$ and $H_l(C_l^n, k) = H_l(C_l^{n_0}, k)$ for any $n \geq n_0$. With this in mind we can think of M as a $k[x]$-module and from the structure theorem of PID's , since the only graded ideals of $k[x]$ are of the form $x^n \cdot k[x]$ we will have that

$$H_l(C,k) \cong \left(\bigoplus_{i\in I} x^{t_i} \cdot k[x]\right) \oplus \left(\bigoplus_{j\in J} x^{r_j} \cdot (k[x] / (x^{s_j} \cdot k[x]))\right),$$

with both I and J finite and $t_i, r_j, s_j \in \mathbb{R}^+$. With this in mind we can define barcodes as.

Definition 2.3. Given a sequence of chain complexes $C = (C_*^i)_{i\in I}$, a field k and $l \in \mathbb{N}\setminus\{0\}$ such that $H_l(C,k) \cong \left(\bigoplus_{i\in I} x^{t_i} \cdot k[x]\right) \oplus \left(\bigoplus_{j\in J} x^{r_j} \cdot (k[x] / (x^{s_j} \cdot k[x]))\right)$ with I and J finite and $t_i, r_j, s_j \in \mathbb{R}^+$ then we define the **barcodes** associated to the l-th persistent homology of the complexes in C as the set containing the possibly repeated sets

$$\boldsymbol{B_l(C)} = \{[t_i, \infty) \subseteq \mathbb{R} : i \in I\} \cup \{[r_j, r_j + s_j] \subseteq \mathbb{R} : j \in J\}.$$

Barcodes are in fact central in persistent homology theory since we can define a distance between barcodes which allows us to determine how different two persistent homologies are.

Definition 2.4. Given two intervals intervals $I, I' \subseteq \mathbb{R}$ then, denoting by $\{$ and $\}$ an either open or closed left and right endpoint respectively we define the function

$$\boldsymbol{d\left(I, I'\right)} = \begin{cases} 0 & \text{if } I = I' = \mathbb{R}, \\ \left|b - b'\right| & \text{if } I = \left(-\infty, b\right\} \text{ and } I' = \left(-\infty, b\right\}, \\ \left|a - a'\right| & \text{if } I = \left\{a, \infty\right) \text{ and } I' = \left\{a', \infty\right), \\ \infty & \text{if } \left|I\right| = \infty \text{ and } \left|I'\right| < \infty \text{ or viceversa}, \\ 0 & \text{if } I = I' = \emptyset, \\ \frac{b-a}{2} & \text{if } I = \emptyset \text{ and } I' = \left\{a, b\right\} \text{ or viceversa}, \\ \min\left(\max\left(\left|a - a'\right|, \left|b - b'\right|\right), \max\left(\frac{b-a}{2}, \frac{b'-a'}{2}\right)\right) & \text{if } I = \left\{a, b\right\} \text{ and } I' = \left\{a', b'\right\}, \end{cases}$$

Using definition 2.4 we can now define the bottleneck distance between two barcodes.

Definition 2.5. Given two barcodes $B = \left\{I_i\right\}_{i \in A}$ and $B' = \left\{I'_i\right\}_{i \in A'}$ we define the **bottleneck distance** between B and B' as

$$d_B\left(B, B'\right) = \inf_{f: B_{B'} \rightleftharpoons B'_B} \left(\sup_{I \in B_{B'}} \left(d\left(I,\ f\left(I\right)\right)\right)\right),$$

where, given sets A and C, then A_C denotes the set A plus $|C|$ copies of $\emptyset$ and f iterates over all bijective functions between $B_{B'}$ and B'_B.

Finally, one of the most important results arising from this definition is the stability theorem (see [2] for a more general statement and proof).

Theorem 2.6. *(stability) Given a topological space X and functions $f, g : X \to \mathbb{R}$ then denoting by B_f and B_g the barcodes corresponding to the level l persistent homology obtained from the morse filtrations derived from f and g then*

$$d_B\left(B_f,\ B_g\right) \leq \left|f - g\right|_\infty.$$

This theorem can be very useful when studying data clouds via morse filtrations as explained earlier since it can help to decide of two point clouds belong to the same distribution simply by computing the upper bound of the barcode distance given by $\left|f - g\right|_\infty$. Moreover this theorem tells us that slightly modifying the point clouds should not result in very different barcodes.

Part I
Modules over posets.

In this part of the document we will will introduce the basic concepts related with multi-dimensional persistent homology. The main goals will be those of making the reader familiar with the correspondence between multi-dimensional persistent homology and modules over posets (definition 3.1), introduce tame poset modules (definition 3.9) and prove that the category of tame poset modules is in fact abelian (proposition 6.9). This last result will ultimately be useful during part II for proving that tame poset modules accept unique remak decompositions. An additional important result we will show in this part of the document is proposition 4.10 which generalizes [17, lemma 2.19] allowing us to describe all possible poset module homomorphisms (definition 4.7) between 2 indicator modules (definition 4.5). This last result will also be of great importance during part II for computing the interleaving distance (9.57) between indicator modules.

We will start in section 3 by making explicit the mentioned correspondence between multi-parameter persistent homology and modules over a poset (poset modules) (definition 3.1). This correspondence will allow us to focus during the rest of this part on poset modules in order to study multi-parameter persistence. In this section we will also define tame poset module (definition 3.9) in which we will take a special interest since it has been empirically observed that data analysis tend to produce tame persistence modules [18, remark 2.14].

In section 4 we will introduce a very particular kind of poset modules called indicator modules (see definition 4.1). Besides serving as an example of tame poset modules these modules take great importance in works such as [18] due to the fact that they can be used in order to represent tame modules (see A.24). Moreover the introduction of this modules and, in particular, proposition 4.10 will be crucial for understanding approximately half of the results showed in part II.

We will continue in section 5 where we will prove in a constructive way that all tame poset modules can in fact be pulled back from pointwise finitely-dimensional modules over a finite poset. In fact we will see that this pull-back characterizes tame modules which allows us to effectively view tame modules as pointwise finitely-dimensional modules over a finite poset.

Finally, in section 6, we will use the results of section 5 in order to define the category of tame modules over a fixed poset (definition 6.8) and prove that such category is abelian (proposition 6.9). As mentioned earlier this result will be needed in part II in order to prove uniqueness and existence of a remak decomposition of tame poset modules. In fact for proving that last result it would suffice exactness of the category in the sense of [1]. However, since this stronger requirement is extensively used in [17] (on which we based the extended version of this work [19]) we will prove it.

We will conclude with section 7 where we will briefly review all shown results and point out how those could be used in order to describe tame poset modules via upset and downset resolutions (see appendix A).

3 Tame poset modules.

In this section we will define poset modules and tame poset modules.

Let us start by recalling the isomorphism presented in [14, theorem 2.6] between the category of r-parameter persistence modules and the category of $\mathbb{N}^r$-graded modules over $\mathbb{K}[x_1, \ldots, x_r]$. This is isomorphisms gives us a valid generalization of the 1-dimensional persistent modules, however through these notes we will work under an even more general definition that is not restricted to the poset $\mathbb{N}^r$ that is a poset module.

Definition 3.1. Given Q a partially ordered set (**poset**) and $\preceq$ its partial order. A **module over Q** (or **Q-module**) is a Q-graded vector space $M = \bigoplus_{q\in Q} M_q$ such that, for every pair of degrees $q, q' \in Q$ satisfying $q \preceq q'$ exists an homomorphism $M_q \to M_{q'}$ and those homomorphisms commute when composed. That is, given degrees $q, q', q" \in Q$ satisfying $q \preceq q' \preceq q"$ the homomorphism $M_q \to M_{q"}$ equals the composition $M_q \to M_{q'} \to M_{q"}$. We call these homomorphisms the **structure homomorphisms** of the Q-module. In the event that $q = q'$ we take structure homomorphism $M_q \to M_q$ to be the identity homomorphism.

Notation 3.2. Through all this notes we will denote by k the base field of all mentioned Q-modules. The studying of the relation between Q-modules with different base fields goes beyond the scope of this document and, therefore, won't be considered.

To see how this definition generalizes the usual definition of persistent homology an example is due.

Example 3.3. Take Q to be a totally ordered set, X and a function $f : X \to Q$. Then we can build $(X_q)_{q\in Q}$ a filtration of X indexed by Q (that is for every $q, q' \in Q$ satisfying $q \preceq q'$ then $X_q \subseteq X_{q'}$) by defining $X_q = f^{-1}(\{x \in Q : x \preceq q\})$. Denoting by H_i the i-th homology functor we can define the Q-module $M = \bigoplus_{q\in Q} M_q = \bigoplus_{q\in Q} H_i(X_q)$ having as structure homomorphisms the vector space homomorphisms $H_i(X_q) \to H_i(X_{q'})$ induced by the inclusion $X_q \subseteq X_{q'}$ for any $q, q' \in Q$ satisfying $q \preceq q'$. By functoriality of the homology functor we have that that the structure homomorphisms satisfy the desired commutativity properties and, therefore, we can conclude that M is in fact a Q-module.

It is important to keep in mind this example since it will be over poset modules of this form (with $Q \in \{\mathbb{R}^n, \mathbb{Z}^n\}$) that we will generalize the 1-dimensional stability theorem (see theorem 9.63).

Notice how, if the topological space X is not a degeneracy, then most of the structure homomorphisms of the poset module described in example 3.3 are in fact the identity homomorphism (for instance when the topological spaces X_q and $X_{q'}$ are homotopy equivalent). These redundant homomorphisms constitute a lot of unnecessary information for studying the nature of a Q-module. In order to remove all these redundant homomorphisms we partition the poset Q into multiple regions as explained in the following definition.

Definition 3.4. Given a poset Q and a Q-module M we say that a cover R of Q is a **constant subdivision** of Q subordinate to M if all the sets $I \in R$ are pairwise disjoint and for every set $I \in R$ there exists a single vector space M_I with isomorphisms $M_I \leftrightarrow M_i$ for every $i \in I$ such that, given I, $J \in R$ then any pair of elements $i \in I$ and $j \in J$ satisfying $i \preceq j$ induces the same composite homomorphism $M_I \to M_i \to M_j \to M_J$. We call the elements of R **constant regions**.

Example 3.5. With the notation of example 3.3, in the simple case where X is the n-th unit shell ($n > 1$) under the l^∞ norm, $f : X \to \mathbb{R}^{n+1}$ is the natural injection identity and the studied homology is the n-th then there is a trivial constant subdivision consisting in 2 regions. Namely those regions are:

1. The constant region formed by all those points with at least one coordinate strictly lower than -1 (which has n-th homology 0).

2. The constant region formed by all those points with all coordinates greater or equal than 1 but at least one coordinate strictly lower than 1 (which has n-th homology $k = \mathbb{R}$).

As we said definition 3.4 is introduced in order to avoid redundancies in the studied Q-modules. However this simplification is only useful if the set R of constant regions resulting from a constant subdivision shows a poset structure. When this happens we can perform a "change" of poset from Q to R as shown by the following remark.

Remark 3.6. Using the notation of definition 3.4, if the cover R has a poset structure compatible with the poset structure of Q then we can define $N = \bigoplus_{I \in R} M_I$ and we would have that N is in fact an R-module with structure homomorphisms given by the ones described in definition 3.4 and their compositions. In fact, if R had a poset structure, we could rephrase definition 3.4 by saying that a Q-module M has a constant subdivision if exists an R-module $N = \bigoplus N_q$ and a poset morphism (a set morphism preserving order) $\varphi : Q \to R$ such that

$$M \cong \boldsymbol{\varphi^* (N) = \bigoplus_{q \in Q} N_{\varphi(q)}}$$

where the identity is in fact the definition of pullback of φ. This observation is stated more precisely by theorem 5.21.

The cover R of definition 3.4 however does not always inherit a poset structure from the poset Q as noted in the following example.

Example 3.7. Take the poset $Q = \mathbb{R}^2$ with the usual poset structure and a Q-module M defined as

$$M_q = \begin{cases} k^2 & \text{if } q = 0 \\ k & \text{otherwise} \end{cases},$$

with structure homomorphism being projection into the first component from degree 0 to every degree above it, inclusion into the first component from any degree below the origin to the origin and identity for any other possible pair of degrees. Then we have a clear constant subdivision of M into 2 distinct constant regions. Namely the origin and its complement. However there is no clear way of ordering these two regions just from the ordering of $\mathbb{R}^2$. This example shows that the cover R of definition 3.6 does not always inherit a poset structure from the poset structure of Q.

We will later prove (proposition 5.14 and theorem 5.19) that constant subdivisions can be built in a particular way that allow the cover R to inherit a poset structure from Q. For that future purpose is however necessary to make now a final note on definition 3.4.

Lemma 3.8. *Any refinement of a constant subdivision is still a constant subdivision. In particular constant subdivisions are not unique.*

Proof. Here we will follow the same notation used in definition 3.4 and denote by R' the refinement of R. Given a constant region $I \in R$ with associated vector space M_I then we can associate to any constant region $I_j \in R'$ satisfying $I_j \subseteq I$ a copy M_{I_j} of the vector space M_I. The homomorphisms $M_{I_j} \to M_{I_{j'}}$ can then be taken to be the identity homomorphisms while the morphisms $M_{I_j} \to M_{J_k}$ with $I \neq J$ can be taken to be the same as the homomorphisms $M_I \to M_J$. $\square$

We conclude this section by using the concept of constant subdivision in order to define tame poset modules.

Definition 3.9. Given a poset Q and a Q-module M we say that:

1. A constant subdivision of Q is **finite** if it has finitely many regions.
2. M is ***Q*-finite**, or **pointwise finite dimensional** (**PFD**) if its components are vector fields with finite dimension.
3. M is **tame** if it is Q-finite and admits a finite constant subdivision subordinate to M.

These definitions are of considerable importance since it has been empirically observed that data analysis tend to produce tame persistence modules [18, remark 2.14]. In fact the main results of this document are only valid for tame modules.

4 Indicator modules and module homomorphisms.

This section is dedicated to the study of indicator modules and homomorphisms between them. The section is divided into two sub-sections. In sub-section 4.1 we will define upset and downset indicator modules and characterize all poset modules that are copies of k at every degree in a given set and 0 outside of it. Later in sub-section 4.2 we will characterize all possible homomorphisms between poset modules of this type (see proposition 4.10). This last characterization will be very important for the results shown in part II.

4.1 Indicator modules.

In section 3 we saw the definition of poset module, how it relates with multi-parameter persistent homology and introduced the concept of tame Q-module. It would now be interesting if we could define an extremely simple type of poset module that could be used as building blocks for all poset modules. Unfortunately, due to the existence of infinitely many types of irreducible poset modules (see [6]) this is not possible. What we can do however is define some special poset modules (indicator modules) that can be used to represent tame poset modules as the images of homomorphisms between those. In this sub-section we will introduce such types of poset modules, however the mentioned representations fall outside of the scope of these notes. For more information please refer to appendix A.

The idea of indicator modules is that of defining a poset module by making it equal to k for any degree inside a given set $S \subseteq Q$ and 0 outside and take as structure homomorphism between two degrees p and q the identity homomorphism whenever $p, q \in S$ and the zero homomorphism otherwise. However this cannot be done for any set S as can be seen from the following example.

Example 4.1. Take the poset $Q = \mathbb{R}^2$ and the sub-set $S = \{(x, x) \in \mathbb{R}^2 : x \in \mathbb{R}\}$. Define now $k[S]$ to be the Q-graded vector space which equals the 1-dimensional vector space k at every degree $q \in S$ and equals 0 on every other degree. Lets now try to give $k[S]$ a structure of Q-module. Take $x, x' \in \mathbb{R}$ with $x < x'$ and look at the structure homomorphism $k[S]_{(x,x)} \to k[S]_{(x',x')}$. By definition of Q-module we have that such structure homomorphism must be equal to the composition $k[S]_{(x,x)} \to k[S]_{(x,x')} \to k[S]_{(x',x')}$, however, by definition of $k[S]$ we have that $k[S]_{(x,x')} = 0$ which implies that the composition must be the 0 homomorphism. Thus the only Q-module structure that can be attributed to $k[S]$ is the trivial one with all non trivial structure homomorphisms being the 0-homomorphism.

This example shows us that we cannot take the set S to be any sub-set of Q. The exact description of the type of sets with which we can define this modules is given by proposition 4.4. Before proving it however we need some notation and lemma 4.3.

Definition 4.2. Given a poset Q and a sub-set $S \subseteq Q$ we say that S is a

downset if for every $q \in S$ and every $p \in Q$ satisfying $p \preceq q$ then $p \in S$.

upset if for every $q \in S$ and every $p \in Q$ satisfying $p \succeq q$ then $p \in S$.

Lemma 4.3. *Given a poset Q and a set $S \subseteq Q$ then S is the intersection of an upset and a downset if and only if for any degrees $p, q \in S$ satisfying $p \preceq q$ and any $x \in S$ satisfying $p \preceq x \preceq q$ then $x \in S$.*

Proof. Suppose that S is the result of intersecting an upset U with a downset D. then, given $p, q \in S$ and $x \in Q$ satisfying $p \preceq x \preceq q$ we would have that $x \in U$ since $p \in U$ and $x \succeq p$ and $x \in D$ since $q \in D$ and $x \preceq q$ and, therefore $x \in U \cap D = S$.

Suppose now that S satisfies the described property and take $S_D \subseteq Q$ to be the downset co-generated by S and $S_U \subseteq Q$ the upset generated by S. Since $S \subseteq S_U, S_D$ then we clearly have that $S \subseteq S_U \cap S_D$. On the other hand, for any $x \in S_U \cap S_D$ we have that since $x \in S_U$ then exists $p \in S$ satisfying $p \preceq x$ and, since $x \in S_D$, then exists $q \in S$ satisfying $x \preceq q$. By hypothesis we have that $x \in S$ and, therefore, $S_U \cap S_D \subseteq S$. Joining both inclusions we obtain that $S = S_U \cap S_D$ and therefore can be written as intersection of an upset with a downset just as we wanted to prove. $\square$

Proposition 4.4. *Given a poset Q and sub-set $S \subseteq Q$ then, if we define $k[S]$ as the Q-graded vector space having copies of k for every degree inside S and 0 for every degree outside S then $k[S]$ can be attributed with a Q-module structure such that the structure homomorphisms between two degrees in S is the identity if and only if S is the intersection of an upset and a downset.*

Proof. Suppose that S was not the result of intersecting an upset and a downset. Then, from lemma 4.3, we have $p, q \in S$ with $p \preceq q$ and $x \in Q \backslash S$ with $p \preceq x \preceq q$. If $k[S]$ had a Q-module structure then, by definition of Q-module, we would have that the structure homomorphism $k[S]_p \to k[S]_q$ would be equal to the composition $k[S]_p \to k[S]_x \to k[S]_q$. By construction of $k[S]$on the other hand we have that $k[S]_x = 0$ and, therefore, $k[S]_p \to k[S]_q$ must be the zero homomorphism. This contradicts the Q-module structure we want to attribute to $k[S]$ thus proving one implication.

Suppose now that S was the intersection of an upset and a downset. Then, once again from lemma 4.3, we would have that for every $p, q \in S$ satisfying $p \preceq q$ and any $x \in Q$ satisfying $p \preceq x \preceq q$ then $x \in S$. Thus we can take the composition $k[S]_p \to k[S]_x \to k[S]_q$ to be the identity homomorphism just like the structure morphism $k[S]_p \to k[S]_q$. Since this works for any p, x, q satisfying the specified properties this proves by definition of Q-module that the Q-module structure attributed to $k[S]$ is in fact well defined. $\square$

From the previous proposition we can finally give a definition of indicator module.

Definition 4.5. Given a poset Q an upset $U \subseteq Q$ and a downset $D \subseteq Q$ and denoting $S = U \cap D$ we call **indicator module** over S to the Q-module $\boldsymbol{k}[\boldsymbol{S}]$ defined in proposition 4.4. We call the set S an **indicator set**.

Of particular interests are the situations where either the upset or the downset in the previous definition are in fact equal to all of Q. Because of this before concluding with this sub-section we introduce the following definition.

Definition 4.6. Given a poset Q an upset $U \subseteq Q$ and a downset $D \subseteq Q$ we call

1. **indicator sub-module** over U or **upset module** over U the indicator module $k[U]$.
2. **indicator quotient module** over D or **downset module** over D the quotient module $k[D]$.

4.2 Connected module morphisms.

In this sub-section we perform two main actions. First, given a poset Q ans a sub-set $S \subseteq Q$ we define connected component of S (definition 4.9) and second we use this definition in order to characterize all homomorphisms between indicator modules (proposition 4.10) and use this as a motivation for defining connected homomorphism.

Let us start by defining homomorphism between poset modules.

Definition 4.7. Given a poset Q, two Q-modules M and N and a map $\varphi : M \to N$ we say that φ is a ***Q*-module homomorphism** (or simply homomorphism) between M and N if it is a degree preserving linear map (equivalently φ is a collection of vector space homomorphisms $\varphi_q : M_q \to N_q$) that commutes with the structure homomorphisms of M and N. We denote by $\mathbf{Hom}_{\boldsymbol{Q}}(M, N)$ the set of all Q-module morphisms between M and N.

The notion of connected Q-module homomorphism is slightly more complicated and before introducing it we must build some theory.

Definition 4.8. Given a poset Q we say that Q is **connected** if, for every pair of elements $p, q \in Q$, exists a finite sequence $(q_i, q_i')_{i=0,\dots,n} \subset Q$ such that

$$p = q_0 \preceq q_0' \succeq q_1 \preceq \cdots \succeq q_n \preceq q_n' = q$$

Such a sequence is called a **path** and n is its **length**.

Since paths can be combined in order to form longer paths it is straight forward to see that being connected by a path is in fact an equivalence relation. This observation leads to the following definitions.

Definition 4.9. Given a poset Q we call **connected component** of Q to any subset $S \subseteq Q$ formed by all those elements $q \in Q$ for which exists a path connecting them to any fixed element $q_0 \in S$. In other words a connected component of Q is an element of the quotient of Q with the equivalence relation of being connected by a path. The set of connected components of a poset Q is denoted by $\boldsymbol{\pi_0(Q)}$.

Using this definition of connected component we can now characterize all poset homomorphisms between two indicator modules. This characterization is given by the following proposition which generalizes [17, lemma 2.19].

Proposition 4.10. *Given indicator modules $k[S], k[S'] \subseteq k[Q]$ we have that*

$$Hom_Q(k[S], k[S']) = k^{|D|},$$

with

$$D = \{C \in \pi_0(S \cap S') : (S \backslash C) \cap C_D = \emptyset \text{ and } (S' \backslash C)_D \cap C = \emptyset\},$$

where S_D denotes the downset co-generated by a set $S \subseteq Q$. Moreover any homomorphism in $Hom_Q(k[S], k[S'])$ acts as scalar multiplication on every connected component of the set D.

Proof. Notice that, given any $\varphi \in \mathrm{Hom}_Q(k[S], k[S'])$ and any $q \notin S \cap S'$ then we must necessarily have $\varphi_q = 0$ since either $k[S]_q = 0$ or $k[S']_q = 0$. Thus if $S \cap S' = \emptyset$ then $\mathrm{Hom}_Q(k[S], k[S']) = 0$ which coincides with the desired result since there would be no connected components of $S \cap S'$. From now on we will therefore assume that $S \cap S' \neq \emptyset$.

Take any $q \in S \cap S'$ and pick the connected component $C \in \pi_0(S \cap S')$ such that $q \in C$. Take now any $q', q" \in C$ satisfying $q' \preceq q \preceq q"$. By definition of Q-module homomorphism we have that φ commutes with the structure homomorphisms $k[S]_{q'} \to k[S]_q \to k[S]_{q"}$ and $k[S']_{q'} \to k[S']_q \to k[S']_{q"}$. Since these homomorphisms are the identity morphisms by definition of indicator modules then we necessarily have that $\varphi_{q'} = \varphi_q = \varphi_{q"}$. Since this is valid for any q' below q and any $q"$ above q then we can proceed inductively in order to extend this result to all elements of the connected component C. Doing so we obtain that for every $q' \in C$ then $\varphi_q = \varphi_{q'} = \varphi_C$. In other words the actions of φ on every degree of the connected component C are all equal. Moreover, since for every $q \in S$ we have that $k[S]_q = k[S']_q = k$ by definition of indicator module then $\varphi_q : k[S]_q = k \to k = k[S]_{q'}$ can be identified with a multiplication by a scalar of k. If we now prove that this scalar must be zero for any connected component outside of D and can be non-zero for any connected component in D then we will have proven the desired result.

Suppose that $(S \backslash C) \cap C_D \neq \emptyset$, that is suppose that exists $q' \in S \backslash C$ such that $q' \preceq q$ for some $q \in C$. Then, by definition of Q-module homomorphism, we would have the commutative diagram

$$\begin{tikzcd} k[S]_{q'} \arrow[r] \arrow[d, "\varphi_{q'}"] & k[S]_q \arrow[d, "\varphi_q"] \\ k[S']_{q'} \arrow[r] & k[S']_q \end{tikzcd} .$$

Notice now that, since $q' \notin C$ and $q' \preceq q$ with $q \in C$ and $q' \in S$ then, by definition of connected component, we have that $q' \notin S'$. Thus $k[S']_{q'} = 0$ and, therefore, $k[S]_{q'} \xrightarrow{\varphi_{q'}} k[S']_{q'}$ is the zero homomorphism. This implies that $k[S]_{q'} \xrightarrow{\varphi_{q'}} k[S']_{q'} \to k[S']_q$ is the zero homomorphism and, therefore, by the previous commutative diagram, $k[S]_{q'} \to k[S]_q \xrightarrow{\varphi_q} k[S']_q$ is also the zero homomorphism. On the other hand, since $q, q' \in S$ then, by definition of indicator module, we have that $k[S]_{q'} \to k[S]_q$ is the identity homomorphism. We can thus conclude that $\varphi_q = \varphi_C$ is the zero homomorphism.

Analogously, if $(S' \backslash C)_D \cap C \neq \emptyset$ then we can take $q' \in S' \backslash C$ and $q \in C$ with $q \preceq q'$. Since $q' \notin C$ then, by construction we must have that $q' \notin S$ and, therefore $k[S]_q \to k[S]_{q'}$ is the zero homomorphism. Thus from commutativity with structure homomorphisms we obtain that $k[S]_q \xrightarrow{\varphi_q} k[S']_q \to k[S']_{q'}$ is also the zero homomorphism. Since $q, q' \in S'$ by construction then, by definition of indicator module, we must have that $k[S']_q \to k[S']_{q'}$ is the identity homomorphism and that $k[S']_q \cong k[S']_{q'} \cong k$. Thus we conclude that $\varphi_{q'} = \varphi_C$ is the zero homomorphism.

With this we have proven that φ must act as multiplication by 0 on every connected component of $S \cap S'$ outside of D. That is we have proven that $\mathrm{Hom}_Q(k[S], k[S']) \subseteq k^{|D|}$.

We now need to prove that for every $C \in D$ then φ_C can be taken to be multiplication by some non-zero scalar. For every $C \in D$ we can define $\varphi_C^\lambda \in \mathrm{Hom}_Q\left(k\left[S\right], k\left[S'\right]\right)$ by letting it be multiplication by any scalar element $\lambda \in k$ in the connected component $C \in D \subseteq \pi_0\left(S, S'\right)$ and the zero homomorphism everywhere else. This is in fact a Q-module homomorphism since for every $q \preceq q' \in Q$ we have the following 4 possible scenarios for all of which commutativity with the structure homomorphisms is satisfied:

1. $q, q' \notin C$ in this case we have that both $k\left[S\right]_q \to k\left[S\right]_{q'} \stackrel{\varphi_{q'}}{\to} k\left[S'\right]_{q'}$ and $k\left[S\right]_q \stackrel{\varphi_q}{\to} k\left[S'\right]_{q'} \to k\left[S'\right]_{q'}$ are the zero morphism since $\varphi_q = \varphi_{q'} = 0$ by construction.

2. $q \notin C$ and $q' \in C$ in this case we have that $k\left[S\right]_q \stackrel{\varphi_q}{\to} k\left[S'\right]_q \to k\left[S'\right]_{q'}$ is the zero morphism since $\varphi_q = 0$ by construction. Moreover, since $q \preceq q'$ then the only way that $q \notin C$ is that $q \notin S \cap S'$. We cannot have $q \in S$ since we would have $q \in S \backslash C$ and, therefore $\left(S \backslash C\right) \cap C_D \neq \emptyset$ contradicting definition of D. Thus we must conclude that $q \notin S$ which implies that $k\left[S\right]_q \to k\left[S\right]_{q'}$ is the zero homomorphism. This in turn implies that $k\left[S\right]_q \to k\left[S\right]_{q'} \stackrel{\varphi_{q'}}{\to} k\left[S'\right]_{q'}$ is also the zero homomorphism thus proving commutativity also in this case.

3. $q \in C$ and $q' \notin C$ in this case we have that $k\left[S\right]_q \to k\left[S\right]_{q'} \stackrel{\varphi_{q'}}{\to} k\left[S'\right]_{q'}$ is the zero morphism since $\varphi_{q'} = 0$ by construction. Moreover, since $q \preceq q'$ then the only way that $q' \notin C$ is that $q' \notin S \cap S'$. We cannot have $q' \in S'$ since then we would have $q \in S' \backslash C$ and, therefore $\left(S' \backslash C\right)_D \cap C$ contradicting definition of D. Thus we must conclude that $q \notin S'$ which implies that $k\left[S'\right]_q \to k\left[S'\right]_{q'}$ is the zero homomorphism. This in turn implies that $k\left[S\right]_q \stackrel{\varphi_q}{\to} k\left[S'\right]_q \to k\left[S'\right]_{q'}$ is also the zero homomorphism thus proving commutativity also in this case.

4. $q, q' \in C$ in this case $k\left[S\right]_q \to k\left[S\right]_{q'} \stackrel{\varphi_{q'}}{\to} k\left[S'\right]_{q'}$ is equal to $k\left[S\right]_q \stackrel{\varphi_q}{\to} k\left[S'\right]_{q'} \to k\left[S'\right]_{q'}$ since both compose the homomorphism consisting on multiplication by λ with the identity homomorphism (either on the left or on the right) and, therefore, both are just equal to multiplication by λ.

By adding together the Q-module morphisms of the form $\varphi_C^{\lambda_C}$ for every component $C \in D$ and modifying the values $\lambda_C \in k$ we obtain that $k^{|D|} \subseteq \mathrm{Hom}_Q\left(k\left[S\right], k\left[S'\right]\right)$. With this second inclusion the proof is now complete, that is we can conclude that$\mathrm{Hom}_Q\left(k\left[S\right], k\left[S'\right]\right) = k^{\pi_0\left(S \cap S'\right)}$. □

Remark 4.11. Notice that, in the case in which $S = U$ is an upset and $S' = D$ is a downset then, by definition of upset and downset the set D can be reduced to $D = \pi_0\left(S \cap S'\right)$.

The result given by proposition 4.10 motivates the following definition of connected Q-module homomorphism.

Definition 4.12. Given a poset Q and two non-zero general indicator modules $k\left[S\right]$ and $k\left[S'\right]$ we say that a Q-module homomorphism $\varphi : k\left[S\right] \to k\left[S'\right]$ is **connected** if there exists a scalar $\lambda \in k$ such that φ_q acts as multiplication by λ in the copy of k in degree q for all $q \in S \cap S'$.

Remark 4.13. According to proposition 4.10 and remark 4.11 then any homomorphism between indicator module $k[S]$ and $k[S']$ will be connected whenever $S \cap S'$ is connected.

5 Encodings of poset modules.

During this section we will explore the concept of encoding of a poset module which will allow to effectively translate some module over a given poset to a module over a different poset which is potentially easier to study.

We will start in sub-section 5.1 by giving the definition of encoding of a poset module and relate it with the concept of constant subdivision via remark 3.6.

We will then continue in sub-section 5.2 by describing a natural way of refining a constant subdivision in order to obtain a finer constant subdivision (see lemma 3.8) having a poset structure inherited from the original poset. This refinement will allow us, by remark 3.6, to always relate constant subdivisions of a poset module with encodings of such poset module.

We will finally conclude with sub-section 5.3 where we will relate the tame poset modules introduced in section 3 with encodings and, therefore, again by remark 3.6, with constant subdivisions.

5.1 Encodings of poset modules.

Remark 3.6 suggests a definition of constant subdivision of poset module different than the one given in definition 3.4. However this alternate definition is valid only in the case where the set formed by the constant regions of the constant subdivision inherits a poset structure from the poset it covers. However this is not always the case as showed by example 3.7. In order to cover for this we introduce the notion of encoding of a poset module.

Definition 5.1. Given a poset Q and a Q-module M an **encoding** of M by a poset P is a surjective poset morphism $\pi : Q \twoheadrightarrow P$ together with a P-module H such that $M \cong \pi^*(H) = \bigoplus_{q \in Q} H_{\pi(q)}$ where π^* denotes the pullback of H along π. Here the structure homomorphisms of $\pi^*(H)$ are obtained from those of H by setting $(\pi^*(H))_q \to (\pi^*(H))_{q'}$ equal to $H_{\pi(q)} \to H_{\pi(q')}$ for every $q \preceq q'$ in Q.

Notation 5.2. With the aim of simplifying notation we will often refer to the P-module H as the encoding of the Q-module M whenever either the poset P or the poset morphism $\pi : Q \to P$ can be omitted.

Just as with constant subdivision we also have notions of finiteness on encoding of poset modules which, in fact, are analogous to those in definition 3.9.

Definition 5.3. Given a poset Q, a Q-module M and an encoding of M represented by the poset P the P-module H and the poset morphism $\pi : Q \to P$ then we say that the encoding is **finite** if

1. The poset P is finite.

2. The vector space H_p has finite dimension for all $p \in P$.

Notice the close relation between definitions 3.9 and 5.3 which establish a first close relation between finitely encoded poset modules (modules admitting a finite encoding) and tame modules.

The following 3 examples may be useful for forming an intuition of encoding of a poset module.

Example 5.4. Given any poset Q, any field k and taking the Q-module $M = k[Q]$ then a finite encoding of M is given by the poset morphism $\pi : Q \to \{*\} = P$ together with the P-module $H = k$ where k is viewed as a 1-dimensional k-vector space.

Example 5.5. Given any poset Q, any upset $U \subseteq Q$, any field k and taking the Q-module $M = k[U]$ then a finite encoding of M is given by the poset $P = \{0, 1\}$ with $0 < 1$ together with the P-module $H = k[\{1\}]$ and the poset morphism $\pi : Q \to P$ defined as

$$\pi(q) = \begin{cases} 1 & \text{if } q \in U \\ 0 & \text{else} \end{cases}.$$

Example 5.6. Dually to the previous example we have that, if we replace the upset U with a downset $D \subseteq Q$ then a poset encoding is given by the same poset P but redefining $H = k[\{0\}]$ and taking the poset morphism $\pi : Q \to P$ defined as

$$\pi(q) = \begin{cases} 0 & \text{if } q \in D \\ 1 & \text{else} \end{cases}.$$

Example 5.7. Combining both previous examples we can take as Q-module any indicator module $M = k[S]$ such that S is neither an upset nor a downset. Then we can define the poset $P = \{a, b, b', c\}$ by giving it the order relations $a \preceq b \preceq c$ and $a \preceq b' \preceq c$. Let's now denote by S_U the upset generated by the set S and by S_D the downset co-generated by S. If exists $q \in Q$ such that $q \notin S_U, S_D$ then we can define the P-module $H = k[\{b\}]$ and the encoding of M will be given by the poset morphism $\pi : Q \to P$ defined as

$$\pi(x) = \begin{cases} a & \text{if } x \in S_D \backslash S \\ b & \text{if } x \in S \\ b' & \text{if } x \in Q \backslash (S_U \cup S_D) \\ c & \text{if } x \in S_U \backslash S \end{cases}.$$

On the other hand, if $Q \backslash (S_U \cup S_D) = \emptyset$ then we can take the poset $P' = \{a, b, c\} \subseteq P$ and define π as before in order to obtain the desired encoding.

Just as with constant subdivisions we said that they where *subordinate* to a given poset module the notion of poset encoding gives birth to a concept of subordination of a poset homomorphism with respect to a specific poset module.

Definition 5.8. Given posets P, Q and a Q-module M we say that a poset homomorphism $\pi : Q \to P$ is **subordinate** to M if exists a P-module H such that P, H and π constitute an encoding of M. That is π is subordinate to M if $M \cong \pi^* (H)$ for some P-module H.

Remark 5.9. Any poset homomorphism $\pi : Q \to P$ that is part of an encoding of a Q-module M is in fact subordinate to M by definition and, conversely, any poset homomorphism $\pi : Q \to P$ leads to a poset encoding of M also by definition.

5.2 Uptight poset.

Remark 3.6 and definition 5.1 suggest a close relation between constant subdivisions subordinate to a given poset module and and encoding of such poset module. In fact, in remark 3.6, we notice that a poset module admits a constant subdivision with poset structure if and only if it admits an encoding to that poset. In this section we will see how, given a constant subdivision subordinate to a certain poset module M we can obtain another constant subdivision subordinate to M (theorem 5.19) that has a poset structure (definition 5.16) and therefore, according to remark 3.6 leads to an encoding of M. This will allow us to prove in the next section that any poset module admitting a finite constant subdivision subordinate to it also admits an encoding into a finite poset.

In order to construct the mentioned constant subdivision with poset structure we first need to introduce the concept of uptight region on which we will base our construction.

Definition 5.10. Given a poset Q and a family Υ of upsets of Q then, for every element $q \in Q$ we define the sub-family $\Upsilon_q \subseteq \Upsilon$ as the sub-family containing all elements of Υ containing q. That is for every $q \in Q$ we define

$$\Upsilon_q = \{U \in \Upsilon \,:\, q \in U\}\,.$$

Moreover we say that two elements $p, q \in Q$ belong to the same **uptight region** if $\Upsilon_p = \Upsilon_q$.

Remark 5.11. Belonging to the same uptight region is trivially an equivalence relation. Thus we can take quotient of a poset Q by this equivalent relation.

At first glance it may seem that uptight regions could be arbitrarily complex sets. However the following lemma proves in a constructive way that uptight regions can be obtained by intersecting an upset with a downset. This will serve as an example of uptight region and will be needed in future proofs.

Lemma 5.12. *Given a poset Q and any family Υ of upsets of Q then all the uptight regions induced by Υ on Q can be expressed as the intersection of exactly one upset (not necessarily in Υ) with exactly one downset.*

Proof. By definition we have that the uptight region R containing an element $q \in Q$ is given by

$$R = \left(\bigcap_{U \in \Upsilon_q} U \right) \cap \left(\bigcap_{U \in \Upsilon \setminus \Upsilon_q} U^C \right),$$

where the super-index C denotes the complementary. Since the intersection of any family of upsets is an upset, the complementary of an upset is a downset and the arbitrary intersection of downsets is a downset we obtain the desired result. □

Remark 5.13. By the characterization of indicator modules (proposition 4.4) lemma 5.12 allows us to take indicator modules on uptight regions.

Our interest on uptight regions is motivated by the following proposition which allows us to dote the set of all uptight regions with a poset structure inherited from the original poset Q.

Proposition 5.14. *Given a poset Q and any family Υ of upsets in Q then, if we define the relation $\preceq$ between any two uptight regions $A, B \subseteq Q$ deriving from Υ by saying that $A \preceq B$ whenever $a \preceq b$ for some $a \in A$ and some $b \in B$ then we will have that $\preceq$ is a reflexive and acyclic (that is implies the identity of all those elements) relation between uptight regions. That is $A \preceq A$ for every uptight regions (reflexivity) and any sequence of uptight regions of the form $A_1 \preceq \cdots \preceq A_n$ with $A_1 = A_n$ implies identity of all the uptight regions in between.*

Proof. Since $a \preceq a$ for any $a \in Q$ then we clearly have that $A \preceq A$ for any uptight region A. This proves that the relation $\preceq$ between uptight regions is in fact reflexive.

Let's now prove the acyclic part. Given an element a belonging to an uptight region A then, for every $b \in Q$ such that $a \preceq b$ we will have that $\Upsilon_b \supseteq \Upsilon_a$ since the elements in Υ_a are all upsets and, therefore, all contain b or, equivalently, are included in Υ_b. Therefore the existence of a cycle $A \preceq A_1 \preceq \cdots \preceq A_n \preceq A$ of uptight regions would imply a sequence of families of upsets of the form $\Upsilon_a \subseteq \Upsilon_{a_1} \subseteq \cdots \subseteq \Upsilon_{a_n} = \Upsilon_a$. Since inclusion is acyclic this implies that $\Upsilon_a = \Upsilon_{a_1} = \cdots = \Upsilon_{a_n}$ and therefore, by definition of uptight region, we can conclude that $A = A_1 = \cdots = A_n$ thus proving that the relation $\preceq$ is acyclic. □

Despite being reflexive and acyclic, the previously defined relation is not a relation of order since it fails to be transitive. In fact we can see how this relation fails transitivity with the following simple example

Example 5.15. Define the poset $Q = \{a, b, b', c\}$ by setting the relations $a \preceq b$ and $b' \preceq c$. If we define the upset family $\Upsilon = \{\{a, b, b', c\}, \{b, b', c\}, \{c\}\}$ then, following the constructive method given by proof of lemma 5.12, we obtain that all uptight regions are either $A = \{a\}$, $B = \{b, b'\}$ or $C = \{c\}$. The relation $\preceq$ described in proposition 5.14 will then give us that $A \preceq B$ and $B \preceq C$ h since $a \preceq b$ and $b' \preceq c$ respectively. However, since $a \not\preceq c$ we will lack the relation $A \preceq C$ thus failing transitivity.

However, acyclicity and reflexiveness of the relation provided by proposition 5.14 allows us to solve this problem by taking the transitive closure of this relation as specified in the following definition.

Definition 5.16. Given a poset Q and and a family Υ of upsets of Q we define the **uptight poset** of Q associated to Υ (and denote it by $\boldsymbol{P_\Upsilon}$) as the set of all uptight regions induced by Υ on Q together with the relation $\preceq$ that sets $A \preceq B$ whenever exists a sequence $A = C_0, C_1, \ldots, C_n = B$ of uptight regions and a sequence $a = c_0, c'_1, c_1, c'_2, c_2, \ldots, c'_n = b$ of elements of Q such that $c_i, c'_i \in C_i$ and $c_i \preceq c'_{i+1}$. In other words the relation $\preceq$ of the uptight poset is the transitive closure P_Υ of the directed acyclic graph of uptight regions given by the relation defined in proposition 5.14.

Remark 5.17. The relation defined in 5.16 dotes the set P_Υ with a poset structure since it is trivially reflexive, its transitive by construction and antysimmetry follows from the acyclic property of the relation described in proposition 5.14 on which this order relation is based.

We are now able to obtain a poset from any family of upsets, however, in addition to that we need post the regions of the uptight region to constitute a refinement of some constant subdivision (definition 3.4). In order to make the shift from upsets to constant regions we introduce the following concept of constant upset.

Definition 5.18. Given a poset Q and a Q-module M then we define a **constant upset** of Q subordinated to M as either an upset A_U generated by a constant region A or the complement $(A_D)^C$ of a downset A_D co-generated by such a constant region.

If we now take a constant subdivision of a Q-module, use all constant regions of that subdivision in order to obtain constant upsets and then build an uptight poset from this new family of upsets we will obtain a new constant subdivision refining the first as proved by the following theorem.

Theorem 5.19. *Given a poset Q, a Q-module M and a constant subdivision R of Q subordinate to M then, denoting by Υ the family of all constant upsets that can be obtained from the constant regions in R, we obtain that the partition of Q given by the elements of the uptight poset P_Υ also form a constant subdivision of Q subordinate to M.*

Proof. Since there is no guarantee that the resulting constant subdivision will be a refinement of R we cannot make use of lemma 3.8. Thus we are forced to prove the theorem from the definition of constant subdivision.

Let us recall that, by definition of constant subdivision, what we need to prove is that for any uptight region $A \in P_\Upsilon$ exists a vector space M_A such that for every $a \in A$ we have an isomorphism $M_A \leftrightarrow M_a$ while for any other uptight region $B \in P_\Upsilon$ and every $b \in B$ such that $a \preceq b$ the morphism $M_A \to M_a \to M_b \to M_B$ is well defined, that is it does not depend on the choice of neither a nor b.

In order to prove this we will start by finding the vector spaces M_A. Given an uptight region $A \in P_\Upsilon$ there are two possibilities, either A intersects a single constant region of the constant subdivision R or it does not. In the first scenario, denoting by P the intersected constant region, we can simply take $M_A = M_I$ where we are using the same notation as in definition 3.4.

Suppose now that A does not intersect a single constant region. In this scenario then the uptight region A would intersect at least two constant regions $I, J \in R$. Notice now that, using the same notation of definition 5.18, then we have that $I \subset I_U \cap I_D$ for every $I \in R$. Therefore, for every element $i \in A \cap I$ we have that $i \in U_I$ and $i \in D_I$. Since A is an uptight region then, by definition of uptight region, we have that this must be true for all elements of A. In particular, for every $j \in A \cap J$ we will have that $j \in I_U$ and $j \in I_D$. In fact, since $j \in A$ we have that $j \in A \cap I_U$ and $j \in A \cap I_D$. By definition of the constant upsets I_U and I_D this implies that exist $i \in A \cap I \subseteq A \cap I_U$ and $i' \in A \cap I \subseteq A \cap I_D$ such that $i \preceq j \preceq i'$. Switching I and J and following the same reasoning we obtain $j' \in A \cap J \subseteq A \cap J_D$ such that $i' \preceq j'$. In summary we have constructed a chain of the form $i \preceq j \preceq i' \preceq j'$ with $i, i' \in A \cap I$ and $j, j' \in A \cap J$. By definition of constant subdivision this chain induces the homomorphisms

$$M_I \to M_i \to M_j \to M_{i'} \to M_I,$$
$$M_J \to M_j \to M_{i'} \to M_{j'} \to M_J,$$

where $M_I \to M_i$, $M_{i'} \to M_I$, $M_J \to M_j$ and $M_{j'} \to M_J$ are the isomorphisms given by definition of constant region and the homomorphisms in between are the structure homomorphisms of M given by the order relation in Q. Moreover, always by definition of constant region, we have that the composites $M_i \to M_j \to M_{i'}$ and $M_j \to M_{i'} \to M_{j'}$ are also isomorphisms and, therefore, the composition $M_I \to M_i \to M_j \to M_J$ must also be an isomorphism as is the composition $M_J \to M_j \to M_{i'} \to M_I$. Thus, if A intersects more than one constant region we can just take $M_A = M_I$ for any one of the intersected constant regions I since for all such regions the resulting vector spaces M_I are isomorphic. We can thus suppose without loss of generality that all constant regions I intersecting A have the same associated vector space M_I and we can simply take $M_A = M_I$ with the corresponding homomorphisms $M_A \to M_a$ for any $a \in A$. This completes half of the proof.

We are now just left with proving that for any uptight posets $A, B \in P_\Upsilon$ such that $A \preceq B$ then the induced morphism $M_A \to M_a \to M_b \to M_B$ does not depend on the choice of elements $a \in A$ and $b \in B$. This however follows from the fact that the vector spaces M_A and M_B where in fact chosen from the vector spaces M_I associated to the constant subdivision and, therefore the statement follows from definition of constant subdivision together with the observation that for any constant region I intersecting A the associated vector space M_I is equal to M_A and the same is valid replacing A with B. □

This theorem tells us that any constant subdivision subordinate to a given poset module can be modified to obtain another constant subdivision subordinate to the same poset module, which, in addition, has a poset structure. As we mentioned this allows us to obtain an encoding of a poset module from any constant subdivision subordinate to such module. However the previous proof gives us more than that. In fact we have that the method of construction of such new constant subdivisions preserves finiteness as proven by the following lemma.

Lemma 5.20. *Given a poset Q and, a Q-module M and a finite constant subdivision R of Q subordinate to M then the constant subdivision obtained from the constant upsets deriving from R as detailed in theorem 5.19 is also finite.*

Proof. Being R finite the set Υ of constant upsets is also finite since it has $2\,|R|$ elements. On the other hand, by the characterization of uptight regions given in lemma 5.12, we have that there can be at most $2^{|\Upsilon|}$ uptight regions associated to Υ. We can thus conclude that the resulting constant subdivision has at most $2^{2|R|}$ elements and, therefore, is finite whenever $|R|$ is finite. $\square$

5.3 Encodings and tame modules.

In this final brief sub-section we will finally prove the already mentioned relation existing between tame poset modules and poset modules admitting finite encodings. More precisely we will prove that a poset module is tame if and only if it admits a finite encoding.

This desired result is in fact but a corollary of the following theorem.

Theorem 5.21. *Given a poset Q and a Q-finite Q-module M then we have that M admits a finite encoding if and only if there exists a finite constant subdivision of Q subordinate to M. More precisely, given a finite constant subdivision subordinate to M then the uptight poset obtained from that constant subdivision yields to a finite encoding of M.*

Proof. The "only if" direction of the first statement comes directly from definition of constant subdivision as noted in remark 3.6. In fact we have that a finite encoding $(\pi : Q \to P,\, H)$ with P a poset and H a P-module leads to a finite constant subdivision having as regions the fibers of π, as vector spaces $M_{\pi^{-1}(p)}$ the vector spaces H_p and with morphisms given by the P-module structure of H.

For the "if" direction we are actually going to prove the second statement which implies it. From theorem 5.19 and lemma 5.20 we have that, given a finite constant subdivision R of Q subordinate to M then, taking constants upsets and uptight regions from R we obtain a finite constant subdivision subordinate to M which, according to proposition 5.14 and definition 5.16 has in fact a poset structure. Denote this poset by P and the vector spaces associated to it by H_I where I are the mentioned uptight regions. Since M is Q-finite and $H_I \cong M_i$ for every $i \in I$ by definition of constant subdivision then, defining $\pi : Q \to P$ as $\pi(i) = I$ (which is a poset morphism by definition of the poset structure of P) and doting $H = \{H_I\}_{I \in P}$ with a P-module structure by using the homomorphisms between H_I and H_J whenever $I \preceq J$ (which are given by definition of constant subdivision) we obtain in fact the desired finite encoding of M. $\square$

Corollary 5.22. *Given a poset Q and a Q-module M then M is a tame Q-module if and only if it admits a finite encoding.*

Proof. If M is tame then, by definition, it is Q-finite and admits a finite constant subdivision which, by theorem 5.21 implies that it admits a finite encoding.

On the other hand, if M is finitely encoded then, by definition of finite encoding we have that the encoding of H of M is pointwise finite and, therefore, M is Q-finite. Thus, again by theorem 5.21, we have that M admits a finite constant subdivision and, therefore, since it is also Q-finite, it is a tame module by definition. $\square$

Because of this last result from now on we will arbitrarily use the terms tame module and finitely encoded module to refer to the same concept.

6 The category of tame modules.

Up until now we have introduced the concepts of tame poset module (section 3) and of encoding of poset module (section 5). Moreover we have also proven (see corollary 5.22) that a poset module is tame if and only if it admits a finite encoding. In this section we will properly define the category of tame modules and prove that it is in fact an abelian category. The idea for proving this will be that of inheriting the zero object and construction of products and coproducts from the abelian category of graded vector spaces of which the category of tame modules is just a sub-category. In order to prove preadditiveness and existence of kernels and cokernels we will then need to define the concept of tame homomorphism (see definition 6.3). This concept of tame homomorphism will be defined so that given two tame modules and a tame homomorphism between them then both the modules and the morphism can be pulled back from poset modules and a poset module homomorphism having a finite poset. Then preadditiveness and existence of kernels and cokernels will follow from abelianess of graded vector spaces indexed by a (finite) poset.

Abelianess of the category of tame modules will not only be of great utility in part II for proving existence and uniqueness of remak decompositions (for which in fact we only need exactness of the category) but is also used in other works such as [17] (on which we based the extended version of this work [19]).

Let's start by recalling the definition of abelian category.

Definition 6.1. A category is called **abelian** if it satisfies all the following properties:

1. It has a **zero object** (0). That is an object that is both final and initial meaning that for every object O in the category exists exactly one morphism from O to 0 and one morphism from 0 to O.

2. It is **preadditive**. That is all sets of homomorphisms are abelian groups and the composition of morphisms is bilinear in the sense that it distributes over morphisms addition in the specified groups.

3. It has all binary biproducts (**products** and **coproducts**).

4. It has all **kernels** and **cokernels**.

5. All monomorphisms and epimorphisms are **normal**. That is all monomorphisms are kernels of some morphism and all morphisms are cokernels of some morphism.

Let's now try to define the category of tame modules over a fixed posed so that it turns out to be abelian. Fixed poset Q a first idea for defining the category of tame Q-modules would be that of defining such category as the full sub-category of the category of Q-modules having as objects only the tame Q-modules. However, as shown by the following example, this approach does not guarantee the existence of kernels and co-kernels which would imply that the category is not abelian.

Example 6.2. Let Q be the poset given by $\mathbb{Z}^2$ with the standard order and define the upset $U = \{(x, y) \in \mathbb{Z}^2 \,:\, x + y \geq 1\}$, the downset $D = \{(x, y) \in \mathbb{Z}^2 \,:\, x + y \leq 1\}$ and the indicator set $L = U \cap D$ (see definition 4.5). Using these indicator sets we can build the Q-modules $M = k\,[U] \oplus k\,[U]$ and $N = k\,[L]$. These Q-modules have a finite encoding shown in examples 5.5, 5.6 and 5.7. We can therefore conclude by corollary 5.22 that M and N are tame. Take now the poset homomorphism $\varphi : M \to N$ that for every degree $q \notin L$ sends M_q to zero and for every $q = (x, y) \in L$ sends the element $(v, w) \in k^2 = M_q$ to $v \cdot x - w \cdot y \in k$. Let us denote by K the kernel of this homomorphism and by R a constant subdivision of Q subordinate to K. We will now prove that for any two different degrees $p, q \in L$ and a constant region $C \in R$ such that $p \in C$ we have that $q \notin C$. Suppose that existed $C \in R$ such that $p, q \in C$. Then, since Q is upper directed, we could take $r \in Q$ satisfying $r \gneq p, q$ and we would have that r belongs to a different constant region since $K_r \cong k^2 \ncong k \cong K_p, K_q$. Then, by definition of constant subdivision, the images of the structure homomorphism $\varphi_{p,r} : K_p \to K_r$ and $\varphi_{q,r} : K_q \to K_r$ should be equal. However, denoting $p = (x, 1 - x)$ and $q = (x', 1 - x')$ we have by construction that

$$\begin{aligned}\operatorname{Im}(\varphi_{p,r}) &= \left\{v\,(x,\, 1 - x) \in k^2 \,:\, v \in k\right\},\\ \operatorname{Im}(\varphi_{q,r}) &= \left\{v\,(x',\, 1 - x') \in k^2 \,:\, v \in k\right\}.\end{aligned}$$

Since $x \neq x'$ by construction this leads us to a contradiction. Thus we can conclude that all elements in L necessarily belong to different constant regions of R, Since $|L| = \infty$ this implies that K cannot have a finite constant subdivision which, by definition, means that it is not tame.

In order to fix these cases we must restrict poset modules homomorphism to a special kind of homomorphisms called tame homomorphism.

Definition 6.3. Given a poset Q and tame Q-modules M and N we say that a homomorphism $\varphi : M \to N$ is **tame homomorphism** if exists a finite constant subdivision R of Q subordinate to both M and N and such that for each constant region $I \in R$ the composite morphism $M_I \to M_i \xrightarrow{\varphi_i} N_i \to N_I$ does not depend on the choice of the degree $i \in I$. We denote by Mor $\boldsymbol{(M, N)}$ the set of tame homomorphisms from M to N.

Remark 6.4. Since given two constant subdivisions we can obtain a new finer constant subdivision by intersecting the elements of the previous constant subdivisions we can conclude that composing tame homomorphisms leads to a tame homomorphism.

Example 6.5. From proposition 4.10 and example 5.7 we know that any homomorphism between indicator modules $k\,[S]$ and $k\,[S']$ is in fact a tame homomorphism with associated constant subdivision

$$R = \pi_0\,(S \cap S') \cup \{Q \backslash\,(S \cup S')\,, S \backslash\,(S \cap S')\,, S' \backslash\,(S \cap S')\}\,.$$

Notation 6.6. In order to simplify notation from now on, given a poset Q and tame Q-modules M and N, we will use the term *homomorphism* to refer to homomorphism from between M and N (tame or not) and the term *morphism* to refer to a tame homomorphism between M and N.

Using now tame homomorphism instead of just homomorphism we obtain that both kernels and cokernels of such functions are also tame poset modules. More precisely we have the following lemma.

Lemma 6.7. *Given a poset Q, any two tame Q-modules M and N and a tame homomorphism $\varphi : M \to N$ then both* $\ker(\varphi)$ *and coker* (φ) *are tame Q-modules.*

Proof. $\ker(\varphi)$ and coker (φ) are sub-modules of M and N respectively which makes them Q-finite (since M and N are Q-finite). On the other hand the same finite constant subdivision subordinate to both M and N, which exists by hypothesis, is also subordinate to $\ker(\varphi)$ and coker (φ) since φ is by definition identical for all degrees within any fixed constant region of this subdivision. Finally, for every constant region I we can associate to $\ker(\varphi)$ and coker (φ) the corresponding sub-vector spaces M_I and N_I associated to M and N respectively. We can therefore conclude that both $\ker(\varphi)$ and coker (φ) have also finite constant subdivisions and, therefore, are tame Q-modules just as we wanted to prove. □

We can now finally define the category of tame modules as follows.

Definition 6.8. Given a poset Q the **category of tame Q-modules** is the subcategory of the category of Q-modules whose objects are the tame Q-modules and whose morphisms are the tame homomorphisms between them.

We conclude this section by proving that the previously defined category is in fact abelian.

Proposition 6.9. *Given a poset Q the category of tame Q-modules is an abelian category.*

Proof. Since composition of tame morphisms is tame by remark 6.4 we have that the category is well defined.

By construction the category of tame Q-modules is in fact a subcategory of the abelian category of Q-modules. Moreover the zero object (the trivial zero module) of this larger category is also tame and since the zero homomorphism is also trivially tame when acting between tame modules then we can conclude that the category of tame modules has a zero object.

Since the larger category of all Q-modules is known to be abelian then it is also preadditive. Therefore, in order to prove preadditiveness of the category of tame Q-modules we only need to prove that we can add tame homomorphisms to obtain another tame homomorphism. Given Q-modules M and N and morphisms $\varphi, \varphi' \in \mathrm{Mor}(M, N)$then, by definition 6.3, if we intersect all constant regions of the constant subdivisions associated to φ and φ', we obtain the constant subdivision associated to the tame homomorphism $\varphi + \varphi'$ and, by definition of both φ and φ' (which are tame) and construction of these constant regions then for any such constant region I the composite morphism $M_I \to M_i \xrightarrow{\varphi_i+\varphi_i'} N_i \to N_I$ does not depend on the choice of $i \in I$ which proves that $\varphi + \varphi'$ is indeed tame.

On the other hand, given Q-modules M and N we know from theorem 5.21 that they are tame if and only they have a finite encodings $(\pi : Q \to P, H)$ and $(\pi' : Q \to P', H')$ respectively. Since the direct sums and product of M and N have the finite encodings $\pi \oplus \pi' : Q \to P \oplus P'$, $H \oplus H'$ and $\pi \times \pi' : Q \to P \times P'$, $H \times H'$ respectively then we can deduce that $M \oplus N$ and $M \times N$ are also tame Q-modules. This proves the existence of products and coproducts.

The existence of kernels and cokernels in this category is proven by lemma 6.7.

Finally since, in the category of Q-modules, every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel then the existence of kernels and cokernels also implies normality of monomorphisms and epimorphisms.

We can thus conclude that the category of tame modules is in fact abelian. □

7 Conclusions.

During this part of the document we have seen how the study of multi-dimensional persistent homology coincides in fact with the study of poset modules (graded vector spaces indexed by a poset together with structure homomorphisms 3.1). We have then seen how, in many cases, poset modules can be simplified by "joining" all indexes with isomorphic vector spaces thus leading to a constant subdivision of the poset (definition 3.4). Then in section 5.1 we have seen how such constant subdivisions can in fact be modified in order to obtain a poset which, in practice, allows us to view the original poset module as a module over this new potentially simpler poset. Finally we have used this last fact in order to define the category of tame modules and prove that such category is abelian. This last result not only will allow us, in part II to prove existence and uniqueness of remak decompositions for tame poset modules but is also used in [18] in order to prove existence of finite upset and downset resolutions for tame poset modules. Even though these results fall outside of the scope of these current notes they are included as part of the extended version of this document [19] where we follow an approach different from [18] in order to reach the same results. In order to simplify the work of the interested reader a slightly reduced version of the mentioned results with the approach followed in [19] are reported in appendix A. Abelianess of the category of tame modules is also used in [17] (and again reported in the extended version of this document [19]) for proving that, what he denominates *socle functor,* preserves tameness (see [17, 19] for more information).

The other important result we obtained during this part of the document is proposition 4.10 which allows us to classify all poset module homomorphisms between two indicator modules (see definition 4.5). This result is proved in a weaker form in [17, lemma 2.19], however we were not able to find the stronger version proven in this document. Since the interleaving distance (see definition 9.57), which generalizes the bottleneck distance to the multi-dimensional case, is based on existence of "special" homomorphism between two poset modules, this classification will in fact be crucial in part II for computing distances between indicator modules.

Part II

Decomposition and interleaving distance.

In this second part of the document we will take a closer look to the structure of poset modules and, more precisely, to distances between them. We will have two main goals. The first goal will be that of proving the often mentioned but rarely proven existence and uniqueness of remak decompositions for tame poset modules. The second goal will be that of defining interleaving distance (which generalizes the bottleneck distance), prove a generalized stability theorem and compute interleaving distances between indicator modules. Unluckily enough, due to the complexity of irreducible poset modules (see [6]) we cannot combine both these results in order to compute the interleaving distance between any two given poset modules. We can however give an upper bound (see lemma 9.65) and study more in depth the interleaving distance between indicator modules. For this particular type of poset modules we will in fact give an alternative definition of the interleaving distance in terms of set inclusion (see proposition 10.2 and corollary 10.4) as well as an upper bound (proposition 10.9) which will turn out to be exact in the 1-dimensional case (example 10.8). When the intersection of the indicator sets is connected we will additionally provide a lower bound for the interleaving distance (see proposition 10.16 and remark 10.18).

In order to achieve the above described goals this part of the document will be structured as follows.

We will start in section 8 by proving existence and uniqueness of remak decomposition for tame poset modules. This result is often mentioned in many works (see for example [10, 5]) and is proven in [5] without uniqueness, however we were unable to find an explicit proof of such result other than the fact that it follows from [1].

We will then proceed in section 9 by shifting from general poset modules to modules over real or discrete polyhedral groups (see examples 9.9 and 9.10). Such groups take great importance in the theory developed in [17] which is reported in the extension of this work (see [19, part II]). Moreover, the introduction of real and discrete polyhedral groups will allow us to define interleaving distance which generalizes the bottleneck distance used in the 1-dimensional case. The definition of interleaving distance that we will be giving by following the same categorical point of view introduced in [2] will then allow us to easily obtain a stability theorem for this distance.

Finally, in section 10, we will study more in depth the interleaving distance between indicator modules by re-defining it in terms of inclusions of sets and giving both upper and lower bounds for it. Notice how the indicator modules studied in this section are in fact more general than the interval modules used in [9].

We will conclude with section 11 where we will briefly review all shown results.

8 Decomposition of tame poset modules.

In this section we will prove existence and uniqueness of remak decompositions of tame poset modules dominating a given constant subdivision. The idea of decomposing poset modules into irreducible parts is however not new. In fact in [5] Bakke and Crawley-Boevey prove the existence of such a decomposition and in [10] Dey and Xin propose an algorithm for finding such decomposition. However the first does not prove uniqueness of the decomposition while the second only mentions that such unique decomposition follows from [1]. We where in fact unable to find an explicit proof of existence and uniqueness of such decomposition as we report it below.

To proof is highly based in the results shown by Atiyah in [1]. In this article Atiyah proves a generalization of the Krull-Schmidt theorem that works in more general categories with a much weaker conditions (bi-chain condition definition 8.5). Because of this, before proving the desired result in corollary 8.8 we will first briefly expose without proving the results shown in [1].

Let us start by recalling the definition of an exact category.

Definition 8.1. Given a category C we cay that C is **exact** if it satisfies the following conditions:

1. It contains a zero object.
2. The set of homomorphisms $\mathrm{Hom}\,(A, B)$ between any pair of objects $A, B \in \mathrm{obj}\,(C)$ constitutes an abelian group.

Remark 8.2. By definition of exact category and abelian category (see definition 6.1) we have that any abelian category is in fact exact. In fact, since the set of morphisms between tame modules forms a group we know have that any full sub-category of an abelian category is exact whenever it contains the zero object.

In order to translate the previous definition to our case of tame poset modules we have the following lemma.

Lemma 8.3. *Given a tame poset module M with finite constant subdivision D and defining by C the category having as objects all tame sub-modules of M to which D is subordinate and as homomorphisms the morphisms between them then C is exact.*

Proof. The zero object of the category of tame modules trivially dominates any constant subdivision and, therefore, belongs to C. Moreover, by definition, we have that C is a full sub-category of the category of all tame poset modules. Since this last category is abelian by proposition 6.9 we can thus conclude from remark 8.2 that C is in fact an exact category. $\square$

The Krull-Schmidt theorem proved in [1] replaces the ascending and descending chain conditions by the following notions of bi-chain and bi-chain condition.

Definition 8.4. Given a category C a **bi-chain** of C is defined as a sequence of triples $B = \{(M_n, f_n, g_n)\}_{n\in\mathbb{N}\cup\{0\}}$ such that:

1. $M_n \in \mathrm{obj}\,(C)$ for every $n \in \mathbb{N} \cup \{0\}$.
2. $f_n : M_n \to M_{n-1}$ is a monomorphism for every $n \geq 1$.
3. $g_n : M_{n-1} \to M_n$ is an epimorphism for every $n \geq 1$.

We say that B **terminates** if there exists $n_0 \in \mathbb{N}$ such that, for every $n \geq n_0$ both f_n and g_n are isomorphisms.

Definition 8.5. Given a category C we say that C **satisfies the bi-chain condition** if every bi-chain of C terminates.

An example of category satisfying the bi-chain condition is given once again by the same category of described in lemma 8.3 as proven by the following lemma.

Lemma 8.6. *Given a tame poset module M with finite constant subdivision D and defining by C the category having as objects all tame sub-modules of M to which D is subordinate and as homomorphisms the morphisms between them then C satisfies the bi-chain condition.*

Proof. By construction we have that all elements in the category C are point-wise finite dimensional and, moreover, all elements in such category are subordinate to the same finite constant subdivision. Because of these finiteness conditions we can take n to be the maximum among the dimensions of the vector spaces M_p for any degree p. Given any $N \in \mathrm{obj}\,(C)$ denote now by $\dim(N)$ the sum of the dimensions of the vectors spaces associated to N for each constant region. That is, using the same notation as in 3.4 and denoting by k the the field below each vector space we define

$$\dim(N) = \sum_{I\in D} \dim_k(N_I)\,.$$

From what we said earlier we have that, for every, $N \in \mathrm{obj}\,(C)$ then $\dim(N) \leq n \cdot |D| < \infty$. Take now a bi-chain $B = \{(M_n, f_n, g_n)\}_{n\in\mathbb{N}}$ of C. Since every g_n is an epimorphism then, using again the same notation as in 3.4, we must necessarily have that $\dim_k((M_n)_I) \geq \dim_k((M_{n+1})_I)$ for every $n \in \mathbb{N}$ and every $I \in D$. In particular we have that $\dim(M_n) \geq \dim(M_{n+1})$ with identity happening if and only if g_n is an isomorphism. For the same reasons we have that, for every $n \in \mathbb{N}$ then $\dim(M_n) = \dim(M_{n+1})$ if and only if f_n is an isomorphism. Thus we have that $\dim(M_n) \geq \dim(M_{n+1})$ for every $n \in \mathbb{N}$ with identity happening if and only if both f_n and g_n are isomorphisms. Since $\dim(M_1) \leq n \cdot |D| < \infty$ and as we explained earlier and $\dim(M_n) \geq 0$ by definition then there cannot exist infinitely many values of $n \in \mathbb{N}$ such that either f_n or g_n are not isomorphisms. Thus we can conclude that the bi-chain B terminates. Since this argument is valid for every bi-chain of C we can conclude that C satisfies the bi-chain condition just as we wanted to prove. $\square$

Using this concept of bi-chain Atiyah proves in [1, theorem 1] the following extension of the Krull-Schmidt theorem which we state without proof.

Theorem 8.7. *Given C an exact category satisfying the bi-chain condition then C also satisfies the Krull-Schmidt theorem. More precisely, for every non-zero object $M \in obj(C)$ then M admits a decomposition as a direct sum of irreducible objects (i.e that cannot be written as a non-trivial direct sum of other objects) and this decomposition is unique up to isomorphism.*

Using theorem 8.7 and lemmas 8.3 and 8.6 we can now prove the main result of this section.

Corollary 8.8. *Given a non-zero tame poset module M and a decomposition D subordinate to M then M can be uniquely decomposed as a direct sum of irreducible poset modules dominating the decomposition D.*

Proof. Applying theorem 8.7, to the category described in lemmas 8.3 and 8.6 we obtain that M has a unique decomposition as a direct sum of tame modules that are irreducible among those dominating the constant subdivision D subordinate to M. $\square$

9 Interleaving distance.

In this section we will define a more general version of the interleaving distance which was first defined in [8] for the 1-dimensional case and then generalized to modules over $\mathbb{R}^n$ (with the usual order) in [15]. We will do so by proceeding similarly to how Bebenik and Scott proceed in [2] to define a categorical interleaving distance in the 1-dimensional case. Besides giving this definition of categorical interleaving distance for the multi-dimensional case our goal will be that of proving a multi-dimensional stability theorem similar to the ones proven in [2, 15] but now working for modules over real or discrete polyhedral groups (see examples 9.9 and 9.10). In order to achieve this goal the section will be structured as follows.

We will start in sub-section 9.1 by introducing the concept of polyhedral group and, more precisely, of real and discrete polyhedral groups (see examples 9.9 and 9.10). Real and discrete polyhedral groups can be thought of a generalization of $\mathbb{R}^n$ and $\mathbb{Z}^n$ that will replace general posets in the definition of interleaving distance (definition 9.57). Such groups play an important role in [17] for defining QR-codes (a functional generalization of the 1-dimensional barcodes to the multi-dimensional case). For more information on QR-codes the interested reader can refer to either [17] or to our revision of that work in [19, part II].

We will continue in sub-section 9.2 where we will generalize poset modules (definition 3.1) by adopting a categorical point of view. With this generalization we will be able to relate poset modules with persistence diagrams (see [5]). This relation will in fact tell us that, in the 1-dimensional case, when the posets are either $\mathbb{R}$ or $\mathbb{Z}$, the interleaving distance defined in this section coincides with the one given in [2].

In sub-section 9.3 we will will complete our shift to upper directed real or discrete polyhedral groups by defining the unit positive vector (definition 9.31) and polyhedral distance (definition 9.35). This presumably novel distance generalizes the usual l_∞ distance in $\mathbb{R}^n$ and $\mathbb{Z}^n$ and is the one we will be using in our stability theorem (theorem 9.63).

In sub-section 9.3 we will use the unit positive vector in order to define both the ε translate of a diagram over a real or discrete polyhedral group and the ε-translation diagram homomorphism between a diagram and its ε-translate. These concepts generalize the corresponding concepts used in [2, 15] in order to define the categorical interleaving distance.

In sub-section 9.5 we will follow the same steps performed by Bubenik and Scott in [2, section 3] in order to define the interleaving distance between two persistence diagrams and, by extension, between two poset modules. Because of the results shown in both [2, theorem 4.16] and [5, theorem 3.4] the interleaving distance thus defined will coincide in fact, under some tameness conditions, with the well known bottleneck distance on the 1-dimensional case.

We will conclude in sub-section 9.6 where we will prove the stability theorem for modules over real or discrete polyhedral groups (see theorem 9.63). Besides the upper bound given by the stability theorem we will also give an additional upper bound that follow directly from the decomposition given in section 8.

9.1 Polyhedral groups.

In this sub-section we will shift our attention from the general posets we have been working with up until now to posets showing an additional group structure that will help us being more precise for the purpose of defining directions inside the poset.

Definition 9.1. Given an abelian group Q we say that Q is **partially ordered** if exists a sub-monoid (a subset of Q closed under addition and containing the identity) having trivial unit group (i.e. the only element of the sub-monoid with inverse inside the sub-monoid is the identity). We call this sub-monoid the **positive cone** of the partially ordered Q and denote it by $\boldsymbol{Q_+}$. Q_+ induces a partial order on Q by setting $q \preceq q'$ for two elements $q, q' \in Q$ whenever $q' - q \in Q_+$.

Remark 9.2. The order induced by Q_+ is well defined since it is:

reflexive: Since $0 \in Q_+$ then $q \preceq q + 0 = q$ for every $q \in Q_+$

transitive: Given $q \preceq q' \preceq q"$ we have by definition that $q' = q + v$ and $q" = q' + v'$ with $v, v' \in Q_+$ and, since Q_+ is closed under addition then $v + v' \in Q_+$ which implies that $q \preceq q + (v + v') = q"$.

anti-symmetric: If $q \preceq q'$ and $q' \preceq q$ then, by definition $q' - q = v \in Q_+$ and $q - q' = -v \in Q_+$. Since Q_+ has trivial unit group this implies that $v = -v = 0$ and, therefore $q = q'$.

Notation 9.3. Because of the introduction of partially ordered groups from now on, unless otherwise specified, we will use the symbol Q to refer to partially ordered abelian groups In what follows, unless otherwise specified we will use the symbol Q to denote an abelian partially ordered group and Q_+ to denote its positive cone.

Notation 9.4. Moreover, since we will be working exclusively with abelian groups we will usually avoid specifying that the group is abelian and implicitly assume it.

The notion of partially ordered group is quite known and probably known to the reader and has many simple examples such as $\mathbb{R}^I$ or $\mathbb{Z}^I$ for any cardinal I finite or not. Much less known however is the concept of partially ordered polyhedral group with which we will be working in this notes. Such concept is based on the notion of face of a polyhedral group.

Definition 9.5. Given a partially ordered group Q with positive cone Q_+ we say that a sub-monoid $\tau \subseteq Q_+$ is a **face** of Q_+ if $Q_+ \backslash \tau$ is an ideal of Q_+. That is if for every $q \in Q_+ \backslash \tau$ and every $r \in Q_+$ then $q + r \notin \tau$.

Remark 9.6. The finite intersection of faces is still a face. That is given faces τ and σ of a partially ordered group Q then $\sigma \cap \tau$ is still a sub-monoid of Q_+ (since intersection of sub-monoids is a sub-monoid) and for every $q \in Q_+ \backslash (\sigma \cap \tau)$ then either $q \in Q_+ \backslash \sigma$ or $q \in Q_+ \backslash \tau$. In either case if existed $r \in Q_+$ such that $q + r \in \sigma \cap \tau \subseteq \sigma, \tau$ then either σ or τ would not be faces by definition contradicting hypothesis. This proves that $\sigma \cap \tau$ is indeed a face.

Remark 9.7. Notice how any face τ of Q_+can be written as the intersection between Q_+ and the downset co-generated by τ. In fact, by definition of face, any sub-monoid of Q_+ satisfying this property is in fat a face.

We can now finally define polyhedral group.

Definition 9.8. Given a partially ordered group Q we say that Q is **polyhedral** if it has finitely many faces.

Intuitively speaking the faces of a partially ordered abelian polyhedral group Q are the boundaries of the positive cone Q_+. In order to make this concept clearer the following examples of polyhedral groups can be useful.

Example 9.9. For any $n \in \mathbb{N} \backslash \{0\}$ take $Q = \mathbb{R}^n$ and define the positive cone Q_+ as the intersection of sets of the form $S_a = \{(x_1, \dots, x_n) \in \mathbb{R}^n : a_1 x_1 + a_n x_n \geq 0\}$ for some $a = (a_1, \dots, a_n) \in \mathbb{R}^n \backslash \{0\}$. It is important to choose the sets S_a so that Q_+ has in fact trivial unit group for example taking $n = 3$ and only the set $S_{(0,0,1)} = Q_+$ the resulting positive cone contains both $(x, y, 0)$ and $-(x, y, 0)$ for any $x, y \in \mathbb{R}^+$ and, therefore, it does not have trivial unit group. Any partially ordered abelian polyhedral group obtained by applying the described method is known as a **real polyhedral group**. Taking as a particular example $Q_+ = (\mathbb{R}^+)^n$ then the faces of Q_+ would be given by the origin, the positive semi-axes and linear combinations of any set of positive semi-axes. That is the faces are sets of the form

$$\tau = \left\{ (x_1, \dots, x_n) \in \left(\mathbb{R}^+\right)^n \mid x_{i_1} = 0 \wedge \cdots \wedge x_{i_r} = 0 \right\},$$

with $\{i_1, \ldots, i_r\} \subseteq \{1, \ldots, n\}$. More in general we have that the positive cone of a real polyhedral group is of the form $Q_+ = \{x \in Q \,|\, l_1(x) \geq 0 \wedge \cdots \wedge l_r(x) \geq 0\}$ for some linear functions l_i such that the unit of Q_+ are trivial and the faces of Q_+ are of the form

$$\tau = \{x \in Q_+ \,|\, l_i(x) \,\forall i \in I\},$$

with $I \subseteq \{1, \ldots, r\}$. Moreover we have that two faces $\tau, \tau' \subseteq Q_+$ with τ defined as before and $\tau' = \{x \in Q_+ \,|\, l_i(x) \,\forall i \in I'\}$ with $I' \subseteq \{1, \ldots, r\}$ satisfy $\tau \subseteq \tau'$ if and only if $I' \subseteq I$.

Example 9.10. Any finitely generated partially ordered free abelian group having as positive any finitely generated sub-monoid (for example $\mathbb{Z}^n$ with the usual order) is also a type of partially ordered abelian polyhedral group known as **discrete polyhedral group**. Notice how all discrete polyhedral groups can in fact be interpreted as sub-groups of $\mathbb{Z}^n$ with a particular positive cone. As with real polyhedral groups taking the discrete polyhedral group $Q = \mathbb{Z}^n$ with positive cone $Q_+ = (\mathbb{Z}^+)^n$ then the faces of Q will be sets of the form

$$\tau = \left\{(x_1, \ldots, x_n) \in \left(\mathbb{Z}^+\right)^n \,:\, x_{i_1} = 0 \wedge \cdots \wedge x_{i_r} = 0\right\},$$

with $(i_1, \ldots, i_r) \subseteq \{1, \ldots, n\}$. The same observations made in example 9.9 and relative to the general case are valid also in the discrete case.

It would now be nice, and necessary for what follows, that, given a face $\sigma \subseteq Q_+$ then existed a face $\sigma^T \subseteq Q_+$ such that for every $q \in Q_+$ exist $x \in \sigma$ and $y \in \sigma'$ such that $q = x + y$. This however is not possible for every polyhedral group. As a counter-example take $Q = \mathbb{Z}^2$ with positive cone $Q_+ = \left\{(x, y) \in (\mathbb{Z}^+)^2 \,|\, \pi x \geq y\right\}$. Then Q has only two faces namely the face $\tau_0 = \{0\}$ and $\tau_x = \{(x, 0) \in \mathbb{Z}^2 \,|\, x \geq 0\}$. It is therefore impossible to write the element $(1, 1) \in Q_+$ as a sum of faces of Q_+ thus and therefore Q is a valid contra-example.
In order to solve this problem we need to introduce another concept known as closed polyhedral group which is based on the concept of ray.

Definition 9.11. Given a partially ordered abelian group Q we say that a face $\rho \subseteq Q_+$ is a **ray** if it is totally ordered as a sub-set of Q.

Remark 9.12. Notice that, by the characterization of faces in either real or discrete polyhedral groups given in examples 9.9 and 9.10, the rays in both real and discrete polyhedral groups are semi-lines. In other words they are 1-dimensional and contain a single linearly independent element.

One would think that, from definition of ray we would automatically have that for every face τ and every ray ρ then either $\rho \cap \tau = \{0\}$ or $\rho \subseteq \tau$. However this is not the case as shown by the following example.

Example 9.13. Take $Q = \mathbb{Z}^2$ with $Q_+ = \{(x, y) \in \mathbb{Z}^2 \,|\, x \geq 1\} \cup \{(0, y) \in \mathbb{Z}^2 \,|\, y \geq 0\}$ in this situation we have that Q_+ is in fact totally ordered and is therefore a ray but so is $\rho_x = \{(0, y) \in \mathbb{Z}^2 \,|\, y \geq 0\}$ and we do not have $Q_+ \subseteq \rho_x$.

In order to obtain this we also need to introduce closed

Definition 9.14. Given a partially ordered abelian polyhedral group Q we say that Q is closed if:

1. Every bounded subset of any ray has maximum.
2. For any ray ρ and any face τ either $\rho \cap \tau = \{0\}$ or $\rho \subseteq \tau$.
3. For every face $\tau \subseteq Q_+$, the complement $Q_+ \backslash \tau$ is generated as an upset by $\rho \backslash \{0\}$ for all rays $\rho \nsubseteq \tau$.

Examples of closed partially ordered abelian groups are given by real polyhedral groups (see example 9.9) and discrete polyhedral groups (see example 9.10) as proven by [18, example 5.9] and [16, Lemma 7.12] respectively (with the missing conditions of items 1 and 2 which can be proven for these cases but that we will assume as extra condition of definitions of real and discrete polyhedral groups).

Notation 9.15. In order to simplify notation from now on, unless otherwise specified, we will use the term *polyhedral group* to refer to partially ordered abelian closed polyhedral groups.

With this new definition example 9.13 can no longer be applied and we have in fact the following result regarding rays. With this in mind we can now prove the following lemma.

Lemma 9.16. *Given a closed polyhedral group Q and a face $\tau \subseteq Q_+$ then, denoting by $\rho_1, \ldots, \rho_n$ all the distinct rays of Q_+ such that $\rho_i \nsubseteq \tau$, we have that any $q \in Q_+$ can be written in a unique way as $q = x + \sum_{i=1}^{r} y_{j_r}$ with $x \in \tau$, $\{j_1, \ldots, j_r\} \subseteq \{1, \ldots, n\}$ and $y_i \in \rho_i \backslash \{0\}$.*

Proof. Take $q \in Q_+$. If $q \in \tau$ then we are done. Otherwise, from definition 9.14 we have that exists $i = 1, \ldots, n$ and $q' \in Q_+$ such that $q = y_i + q'$ for some $y_i \in \rho_i \backslash \{0\}$. Taking y_i to be the greatest element in ρ_i such that $y_i \preceq \rho_i$ (we can take such maximum by definition of closed) we can assure that there is no $y_i' \in \rho_i$ such that $y_i' \preceq q'$. Thus, repeating the process at most n times we will reach the desired result. Uniqueness follows from the fact that, since all rays are different and none of them is included in τ then the intersection between any two of them is a sub-set of the units set and since Q_+ has no units besides the value zero uniqueness follows. □

If we now prove that we can add faces together in order to obtain yet another face then the property we were looking for will follow. This adding of faces is given by the next lemma.

Lemma 9.17. *Given a closed polyhedral group Q and a faces $\tau, \tau' \subseteq Q_+$ then the set $\tau + \tau' = \{x + y \,|\, x \in \tau \wedge y \in \tau'\}$ is also a face.*

Proof. Since Q_+ is multiplicatively closed then we clearly have that $\tau + \tau' \subseteq Q_+$. On the other hand, since the sum of sub-monoids is also a sub-monoid then $\tau + \tau'$ is also a sub-monoid of Q_+. Thus we only need to prove that for every $q \in Q_+ \backslash (\tau + \tau')$ and every $r \in Q_+$

then $q+r \notin Q_+ \setminus (\tau+\tau')$. Suppose not, then exists $q \in Q_+ \setminus (\tau+\tau')$, $r \in Q_+$, and $s \in \tau+\tau'$ such that $q+r=s$. Use now lemma 9.16 in order to write $q=q'+x_q$, $r=r'+x_r$ and $s=s'+x_s$ with $x_q, x_r, x_s \in \tau$ and $q', r', s' \in Q_+$. By uniqueness of lemma 9.16 we have that $x_s = x_q + x_r$. Thus the previous identity can be re-written as $q'+r'=s'$. Notice that, since, by definition of $\tau+\tau'$ we can write $s=x+y$ with $x \in \tau$ and $y \in \tau'$ then, using lemma 9.16 to write $y=x_y+y'$ with $x_y \in \tau$ and we will have that $x+x_y=x_s$ which implies that $y'=s'$. Moreover, since $y' \preceq y$ then we necessarily have $y' \in \tau'$. Finally we cannot have $q' \in \tau'$ since otherwise, since $x_q \in \tau$ we would have that $q = x_q + q' \in \tau+\tau'$. With this in mind the identity $q'+r'=s'=y'$ contradicts the hypothesis that τ' is a face. We can thus conclude that $\tau+\tau'$ is a face. □

Using both previous lemmas we obtain the desired result in the form of the following corollary.

Corollary 9.18. *Given a closed polyhedral group Q and any face $\sigma \subseteq Q_+$ then there exists a face $\sigma^T \subseteq Q_+$ such that any $q \in Q_+$ can be written uniquely as a sum of an element of σ and an element of σ^T. Equivalently $\sigma \cap \sigma^T = \{0\}$ and $\sigma + \sigma^T = Q_+$*

Proof. Denoting by $\rho_1, \dots, \rho_n$ all rays of Q_+ such that $\rho_i \nsubseteq \sigma$ then we can define $\sigma^T = \rho_1 + \cdots + \rho_n$ which is a face by lemma 9.17 and result follows from lemma 9.16. □

Remark 9.19. Notice that, by construction of σ^T we have that $\left(\sigma^T\right)^T = \sigma$.

Remark 9.20. Since σ^T is obtained by adding together all rays ρ' such that $\rho' \nsubseteq \sigma$ then, since $\left(\sigma^T\right)^T = \sigma$ by remark we have that σ is obtained by adding together all rays ρ such that $\rho \nsubseteq \sigma^T$. Since for every $\rho \subseteq \sigma$ we have that $\rho \nsubseteq \sigma^T$ because $\sigma \cap \sigma^T = \{0\}$ and since for every $\rho \nsubseteq \sigma$ we have that $\rho \subseteq \sigma^T$ by construction of σ^T we can thus conclude that σ can be written as the sum of the rays included in σ.

Corollary 9.21. *Given a closed polyhedral group Q then any face $\sigma \subseteq Q_+$ can be written as the sum of the rays ρ_i that satisfy $\rho_i \subseteq \sigma$. Moreover every element in σ can be written uniquely as a sum of elements in each face.*

Proof. Take the face σ^T afforded by corollary 9.18. Then remark 9.19 tells us that $\left(\sigma^T\right)^T = \sigma$. Since for any ray ρ and construction of σ^T we have that $\rho \nsubseteq \sigma^T \Leftrightarrow \rho \subseteq \sigma$ then proof of corollary 9.18 gives us the first half for the statement. The second half on the other hand follows from the same arguments used for uniqueness in lemma 9.16. □

9.2 Diagrams indexed by posets.

In order to define the interleaving distance we first need to generalize the concept of poset module by looking at them from a categorical point of view.

Definition 9.22. Given a poset P and a category C we call **diagram in $\boldsymbol{C}$ indexed by $\boldsymbol{P}$** to any collection $F = \{F_p\}_{p \in P}$ of, not necessarily distinct, objects of D indexed by the poset P together with structure homomorphisms $F_p \to F_q \in \hom(C)$ for every $p, q \in P$

satisfying $p \preceq q$. The structure homomorphisms must additionally satisfy that, given any $p, q, r \in P$ such that $p \preceq q \preceq r$ then the homomorphism composition $F_p \to F_q \to F_r$ equals the homomorphism $F_p \to F_r$. In the event that $p = q$ we assume the convention that the structure homomorphism $F_p \to F_q$ is the identity homomorphism. We denote by $\boldsymbol{C^P}$ the set of diagrams in C indexed by P. If $C = \text{Vect}_k$ for some field k then we call any element in Vect_k^P a **persistence module**.

Remark 9.23. By definition of poset modules (see definition 3.1) and the just showed definition of persistence modules we have that persistence modules are just a collection of all homogeneous components of a poset module and vice-versa a poset module is just a persistence module together with an addition operation that, once restricted to every homogeneous component, equals that of the homogeneous component. In other words persistence modules and poset modules can be thought of as the same thing. Because of this we will from now on use the term poset module in order to refer to a persistence module and vice-versa.

Remark 9.24. Given categories C and D, a functor $\Psi : C \to D$, a poset P and a diagram $M \in C^P$ then Ψ induces a diagram $\Psi(M) \in D^P$ defined at every degree $p \in P$ by $(\Psi(M))_p = \Psi(M_p)$ and with structure homomorphisms obtained by taking the image of the structure homomorphisms in M by the functor Ψ.

Before proceeding an example of diagram indexed by a poset is due. The following example will in fact play an important role on the statement and proof of the stability theorem 9.63. We define such type of diagram as follows.

Definition 9.25. Given a poset P, a topological space X and a function $f : X \to P$ we define the **diagram in $\boldsymbol{X}$ indexed by $\boldsymbol{P}$ through $\boldsymbol{f}$** as the diagram $\boldsymbol{X_P^f}$ in the category category of topological spaces and indexed by the posed P whose objects are defined by $\left(X_P^f\right)_p = f^{-1}(\{q \in X \,:\, q \preceq p\})$ and whose structure homomorphisms are given by inclusion. We denote by $\boldsymbol{X^P}$ the set of all diagrams in X indexed by P through any function $f : X \to P$.

Remark 9.26. Since any homology functor sends a topological space to a vector space then, by remark 9.24, applying the homology functor (of any homology level) to any diagram X_P^f we obtain a module over the poset P (see remark 9.23 and example 3.3).

Remark 9.27. In the particular case where $P = \mathbb{R}$ both definitions 9.22 and 9.25 coincide with the subjects of study of [2]. Therefore, in this cases, we will be able to use the results shown in [2].

As with poset modules we can also define homomorphism between diagrams indexed by a poset.

Definition 9.28. Given a category C, a poset P and diagrams $F, G \in C^P$ we call **diagram homomorphism** from F to G to any collection of homomorphisms $\varphi = \{\varphi_p : F_p \to G_p\}_{p \in P}$ such that, for any $p, q \in P$ satisfying $p \preceq q$ then the homomorphism composition $F_p \xrightarrow{\varphi_p} G_p \to G_q$ equals the homomorphism composition $F_p \to F_q \xrightarrow{\varphi_q} G_q$.

Remark 9.29. In the particular case where F and G are poset modules (see remark 9.23) a diagram homomorphism between F and G is, by definition, a poset module homomorphisms.

Remark 9.30. Just as with remark 9.24, given categories C and D, a functor $\Psi : C \to D$, a poset P, diagrams $F, G \in C^P$ and a diagram homomorphism $\varphi = \{\varphi_p\}_{p \in P} : F \to G$ then the functor Ψ induces a natural diagram homomorphism $\Psi(\varphi)$ between the diagrams $\Psi(F), \Psi(G) \in D^P$ (see remark 9.24). More precisely the diagram homomorphism $\Psi(\varphi)$ is obtained by applying the functor Ψ to φ_p for every $p \in P$. Moreover, by construction and definition of functor we have that the following diagram commutes

$$\begin{tikzcd} F \arrow[r, "\varphi"] \arrow[d, "\Psi"] & G \arrow[d, "\Psi"] \\ \Psi(F) \arrow[r, "\Psi(\varphi)"] & \Psi(G) \end{tikzcd} .$$

9.3 The polyhedral distance.

In this sub-section we will introduce the concept of unit positive vector of a real or discrete polyhedral group, shift our attention to upper connected real or discrete polyhedral groups and define both the polyhedral distance and the ε-translate of a diagram indexed by an upper connected real or discrete polyhedral group.

Let us start by defining the unit positive vector.

Definition 9.31. Given a real or discrete polyhedral group $Q \in \{\mathbb{R}^n, \mathbb{Z}^n\}$ and denoting by $\{\rho_i\}_{i=1,\ldots,r}$ the set of all its rays (see definition 9.11) then, remark 9.12 allows us to take for each one of these rays a generator $1_{\rho_i} \in \rho_i$ which, in the discrete case is unique and, in the real case, can be taken to have l_2 norm equal to 1. With this notation we define the **unit positive vector of Q** as the element of the positive cone Q_+ given by

$$\boldsymbol{u_1} = \sum_{i=1}^{r} 1_{\rho_i}.$$

Given any element $\varepsilon \in \mathbb{R}, \mathbb{Z}$ depending on the discrete or real case then we denote $\boldsymbol{u_\varepsilon} = \varepsilon \cdot u_1$ where u_1 is seen as an element of the $\mathbb{R}$-vector space $\mathbb{R}^n$ (real case) or the $\mathbb{Z}$-module $\mathbb{Z}^n$ (discrete case).

Remark 9.32. For every $x, y \in \mathbb{R}, \mathbb{Z}$ we have that $u_x + u_y = u_{x+y}$ by construction.

Remark 9.33. By construction of the unit positive vector and applying lemma 9.16 with $\sigma = \{0\}$ then for any element $v \in Q_+$ exists M big enough so that $v \preceq u_M$. In particular, if Q is upper directed (that is given two elements $a, b \in Q$ exists $c \in Q_+$ such that $a, b \preceq c$) then for any $a, b \in Q$ exists M big enough so that $b \preceq a + u_M$ and $a \preceq b + u_M$.

Intuitively speaking the set $\{u_\varepsilon\}_{\varepsilon \geq 0}$ can be though of as the central axis of the positive cone, that is the set of points equidistant to all axes favoring none. An example that better illustrates this intuition is the following.

Example 9.34. In the special cases where $Q \in \{\mathbb{R}^n, \mathbb{Z}^n\}$ has the standard order then, since the rays are the positive semi-axes, we have that u_ε is just the vector having ε in all its coordinates. That is the same as the $\overrightarrow{\varepsilon}$ vector defined in [15]. In the even more particular case where $n = 1$ then we have that $u_\varepsilon = \varepsilon$.

Thanks to remark 9.33 we can now use the unit positive vector in order to define polyhedral distance.

Definition 9.35. Given an upper connected real or discrete polyhedral group $Q \in \{\mathbb{R}^n, \mathbb{Z}^n\}$, two elements $a, b \in Q$ and denoting by $\boldsymbol{Q_B}$ either $\mathbb{R}$ or $\mathbb{Z}$ depending on if Q is real or discrete respectively, then we define the **polyhedral distance** between a and b as the infimum value of M such that $a \preceq b + u_M$ and $b \preceq a + U_M$. That is

$$\boldsymbol{d_{Q_+}}(a, b) = \inf\left(\{M \in Q_B \,:\, (a \preceq b + u_M) \wedge (b \preceq a + U_M) \wedge (M \geq 0)\}\right).$$

Remark 9.36. Given any sequence $S = \{M_n\}_{n \in \mathbb{N}} \subseteq Q_B$ with $M_n \geq 0$ satisfying $a \preceq b + u_{M_n}$ and $b \preceq a + U_{M_n}$ for every $n \in \mathbb{N}$ and such that S converges to $M \in Q_B$ then we will have that $a \preceq b + u_M$ and $b \preceq a + U_M$ since Q_+ is closed. In other words the set

$$D = \{M \in Q_B \,:\, (a \preceq b + u_M) \wedge (b \preceq a + U_M) \wedge (M \geq 0)\}$$

is closed and, therefore, we can write

$$\boldsymbol{d_{Q_+}}(a, b) = \min(D).$$

Because of definitions 9.31 and 9.35 from now on we will stir our focus to upper directed real or discrete polyhedral groups. More precisely we will adopt the following notation.

Notation 9.37. Unless otherwise specified from now until the end of the document we will use the term "**discrete polyhedral group**" in order to refer to an upper directed discrete polyhedral group of the form $\mathbb{Z}^n$. Conversely we will use the term "**real polyhedral group**" in order to refer to an upper directed real polyhedral group (that is of the form $\mathbb{R}^n$ by definition). Moreover if Q is a real (conversely discrete) polyhedral group we will use the symbol $\boldsymbol{Q_B}$ in order to indicate $\mathbb{R}$ (conversely $\mathbb{Z}$).

It may not be immediately clear from definition that the function d_{Q_+} described in definition 9.35 is in fact a distance. Therefore we dedicate the following proposition to prove this fact.

Proposition 9.38. *The function d_{Q_+} given in definition 9.35 is indeed a distance.*

Proof. By remark 9.33 we know that the function d_{Q_+} is **well defined**.
The **symmetry** property follows directly from the definition which is symmetric.
Using the same notation of definition 9.35 then, given any $a \in Q$, we have that $a = a + u_0$ and, in particular, $a \preceq a + u_0$ which implies that $d_{Q_+}(a, a) \leq 0$ and, therefore, that $\boldsymbol{d_{Q_+}(a, a) = 0}$ since the image of d_{Q_+} is not null by construction. On the other hand if $d_{Q_+}(a, b) = 0$ then, by definition, there exists a sequence of elements $\{u_{M_n}\}_{n \in \mathbb{N}} \subseteq Q_+$ such that $\lim_{n \to \infty} u_{M_n} = 0$

and $a \preceq b + u_{M_n}$ and $b \preceq a + U_{M_n}$ for every $n \in \mathbb{N}$. Taking limits, we can conclude that $b \preceq a$ and $a \preceq b$ and, therefore, that $a = b$. Joining this result with the previous one we have that $\boldsymbol{d_{Q_+}(a,b) = 0 \Leftrightarrow a = b}$.

We are now only left with proving the **triangular inequality**. Given $a, b, c \in Q$ then, by remark 9.32 we have that

$$\begin{aligned} a + u_{d_{Q_+}(a,b)+d_{Q_+}(b,c)} &= a + u_{d_{Q_+}(a,b)} + u_{d_{Q_+}(b,c)}, \\ &\succeq b + u_{d_{Q_+}(b,c)} \succeq c, \end{aligned}$$

and, conversely, $a \preceq c + u_{d_{Q_+}(a,b)+d_{Q_+}(b,c)}$ which proves that $d_{Q_+}(a,b) + d_{Q_+}(b,c) \geq d_{Q_+}(a,c)$ as the triangular inequality states. □

The definition of polyhedral distance may at first seem a somewhat unfamiliar. The following lemma can help shaking this unfamiliarity by showing that the polyhedral distance is in fact nothing but a generalization of the familiar l_∞ distance.

Lemma 9.39. *In the special case where the real or discrete polyhedral group Q has the standard order then the polyhedral distance equals the l_∞ distance.*

Proof. Take any $a, b \in Q$. By definition of l_∞ distance we know that $d = l_\infty(a,b) = \|a - b\|_\infty$ is the maximum absolute element-wise difference between the coordinates in a and the coordinates in b. Thus, by example 9.34, we have that $b \preceq a + u_d$ and that $a \preceq b + u_d$. By definition of polyhedral distance this implies that $\boldsymbol{d_{Q_+}(a,b) \leq l_\infty(a,b)}$.

On the other hand, again by definition of l_∞, we know that, for any $d' < d$ exists at least one pair of coordinates from a and b such that their difference is greater than d'. Lets assume without loss of generality that b is greater in this coordinate. Then, by example 9.34 and since Q works with the standard order, we have that $a + u_{d'} \not\preceq b$. Thus, by definition of polyhedral distance, we must necessarily have $\boldsymbol{d_{Q_+}(a,b) \geq l_\infty(a,b)}$.

Combining both results we obtain that $d_{Q_+}(a,b) = l_\infty(a,b)$ and since a, b can be any pair of elements in Q this proves that $d_{Q_+} = l_\infty$ just as we wanted to prove. □

9.4 ε-translate and ε-translation.

In this brief sub-section we will use the notion of unit positive vector (definition 9.31) in order to define both the ε-translate of a diagram indexed by either a real or discrete polyhedral group and the ε-translation homomorphism between a diagram and its ε-translate. These concepts will be needed in sub-section 9.5 in order to greatly simplify the notation needed for the definition of the interleaving distance.

Let us start by defining ε-translate of a diagram which, intuitively speaking, is nothing but a copy of the diagram shifted by $-u_\varepsilon$.

Definition 9.40. Given a real or discrete polyhedral group Q, a category C, a diagram $M \in C^Q$ and any element $\varepsilon \in Q_B$ we define the **$\boldsymbol{\varepsilon}$-translate** of M as the diagram $\boldsymbol{M^\varepsilon} \in C^Q$ satisfying $(M^\varepsilon)_q = M_{q+\varepsilon}$ for every $q \in Q$ and having as structure homomorphisms the same ones that M has.

Example 9.41. Intuitively speaking the ε-translation of a diagram M is just a copy of such diagram shifted in the direction $u_{-\varepsilon}$. For example take the poset module $M_{[0,1]} \in \mathrm{Vect}_k^{\mathbb{R}}$ which is 0 for every degree except for the degrees in the interval $[0, 1]$ where it equals k and whose poset homomorphisms are the identity whenever possible. Let us in fact denote by M_I the poset modules satisfying such characteristics for some interval I. Then we have that M^ε is the poset module which is 0 at every degree except for the degrees in the interval $[-\varepsilon, 1-\varepsilon]$ where it equals k.

Remark 9.42. It follows immediately from definition that for any $\varepsilon, \varepsilon' \in Q_B$ we have we have that $(M^\varepsilon)^{\varepsilon'} = M^{\varepsilon+\varepsilon'} = \left(M^{\varepsilon'}\right)^{\varepsilon}$.

Remark 9.43. The ε-translate preserves diagram homomorphisms. More precisely, given a category C, a real or discrete polyhedral group Q, any two diagrams $M, N \in C^Q$ and a diagram homomorphism $f : M \to N$ then, for any $\varepsilon \in Q_B$ we have that f induces a diagram homomorphism $f^\varepsilon : M^\varepsilon \to N^\varepsilon$ defined at degree $q + \varepsilon$ as the homomorphism f at degree q.

Remark 9.44. With the same notation as in the definition, given another category D and a functor $\Psi : C \to D$ then, by definition of the diagram induced by Ψ (remark 9.24) we have that $\Psi(M^\varepsilon) = \Psi(M)^\varepsilon$.

There exists a natural homomorphism from any diagram to its ε-translate which is given by the structure homomorphisms of that diagram. More precisely we have the following definition.

Definition 9.45. Given a real or discrete polyhedral group Q, a category C, a diagram $M \in C^Q$ and a non negative element $\varepsilon \in Q_B$ we define the **$\boldsymbol{\varepsilon}$-translation homomorphism** as the natural diagram homomorphism $\boldsymbol{T_M^\varepsilon} : M \to M^\varepsilon$ given by the structure homomorphisms of M and having as structure homomorphisms the ones induced by the structure homomorphisms in M.

Example 9.46. It is important to notice that $T_M^\varepsilon(M) \neq M^\varepsilon$. In fact, by definition, this identity is true if and only if $M_q \to M_{q+\varepsilon}$ is an epimorphism for every $q \in Q$. For example, following the notation of example 9.41 we can take the poset module $M_{[0,1]} \in \mathrm{Vect}_k^{\mathbb{R}}$ and, choosing any $\varepsilon > 0$ we will have that $M_{[0,1]}^\varepsilon = M_{[-\varepsilon,1-\varepsilon]}$ while

$$T_{M_{[0,1]}}^\varepsilon\left(M_{[0,1]}\right) = M_{[0,1-\varepsilon]} \subsetneq M_{[-\varepsilon,1-\varepsilon]}.$$

Remark 9.47. Since structure homomorphisms commute then, for any $\varepsilon, \varepsilon' \geq 0$ we have we have that $T_{M^{\varepsilon'}}^\varepsilon \circ T_M^{\varepsilon'} = T_M^{\varepsilon+\varepsilon'} = T_{M^\varepsilon}^{\varepsilon'} \circ T_M^\varepsilon$.

Remark 9.48. By construction of T_M^ε and definition of diagram homomorphism (definition 9.28) we have that for any diagram homomorphism $f : M \to N$ and any $\varepsilon \geq 0$ the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "T_M^\varepsilon"] & N \arrow[d, "T_N^\varepsilon"] \\ M^\varepsilon \arrow[r, "f^\varepsilon"] & N^\varepsilon \end{tikzcd}$$

Remark 9.49. With the same notation as in definition 9.45, given another category D and a functor $\Psi : C \to D$ then, by definition of the diagram and diagram homomorphism induced by Ψ (remarks 9.24 and 9.30) we have that $\Psi\left(T_M^\varepsilon\right) = T_{\Psi(M)}^\varepsilon$.

9.5 Interleaving extended pseudo distance.

In this sub-section we will follow the same steps performed by Bebenik and Scott in [2, section 3] in order to define interleaving distance (see [8, 15, 2]) between diagrams indexed by real or discrete polyhedral groups. The interleaving distance is in fact an extended pseudo-metric as we prove in proposition 9.56.

Let us start by defining ε-interleaving.

Definition 9.50. Given a category C, a real or discrete polyhedral group Q, two diagrams $M, N \in C^Q$ and a non negative value $\varepsilon \in Q_B$ we say that M and N are **ε-interleaved** if exist diagram homomorphisms $F : M \to N^\varepsilon$ and $G : N \to M^\varepsilon$ (see definition 9.40) such that $G^\varepsilon \circ F = T_M^{2\varepsilon}$ and $F^\varepsilon \circ G = T_N^{2\varepsilon}$ (see remark 9.43 and definition 9.45). We refer to the pair of homomorphisms F, G as an **ε-interleaving** of M and N.

Remark 9.51. The identities that an ε-interleaving of M and N given by $F : M \to N^\varepsilon$ and $G : N \to M^\varepsilon$ must satisfy can be replaced by the more visual yet equivalent commutativeness of the following diagrams

$$\begin{tikzcd} M \arrow[rr, "T_M^{2\varepsilon}"] \arrow[dr, "F"'] & & M^{2\varepsilon} \\ & N^\varepsilon \arrow[ur, "G^\varepsilon"'] & \end{tikzcd}, \quad \begin{tikzcd} N \arrow[rr, "T_N^{2\varepsilon}"] \arrow[dr, "G"'] & & N^{2\varepsilon} \\ & M^\varepsilon \arrow[ur, "F^\varepsilon"'] & \end{tikzcd}$$

Remark 9.52. By example 9.34 and definition 9.40 we have that, in the 1-dimensional case where $Q \in \{\mathbb{R}, \mathbb{Z}\}$ the given definition of ε-interleaving coincides with the 1-dimensional definition of ε-interleaving given in [2, definition 3.1]. Moreover, from the same example we deduce that, in the special case where $Q = \mathbb{R}^n$ with the usual order then ε-interleaving coincides with the definition of ε-interleaving given in [15]. However our definition presents the advantage of being valid not only for the standard order but in fact with any positive cone of $\mathbb{R}^n$ as long as it has finitely many faces.

The following lemma, besides being crucial for proving the stability theorem in sub-section 9.6, gives us a very simple example of ε-interleaving which may be very useful for visualizing the just defined ε-interleaving.

Lemma 9.53. *Given a topological space X, a real or discrete polyhedral group Q, a non negative value $\varepsilon \in Q_B$ and functions $f, g : X \to Q$ such that* $\sup_{x \in X}\left(d_{Q_+}\left(f(x), g(x)\right)\right) \leq \varepsilon$ *then the diagrams X_Q^f and X_Q^g are ε-interleaved.*

Proof. Since $\sup_{x\in X}\left(d_{Q_+}\left(f\left(x\right),g\left(x\right)\right)\right)\leq\varepsilon$ then, by definition of polyhedral distance and remark 9.36 we have that $g\left(x\right)\preceq f\left(x\right)+u_\varepsilon$ and $f\left(x\right)\preceq g\left(x\right)+u_\varepsilon$ for every $x\in X$. From those inequalities we obtain respectively the inclusions

$$\left(X_Q^f\right)_q=f^{-1}\left(\left\{p\in Q\,:\,p\preceq q\right\}\right)\subseteq g^{-1}\left(\left\{p\in Q\,:\,p\preceq q+u_\varepsilon\right\}\right)=\left(X_Q^g\right)_{q+\varepsilon}=\left(\left(X_Q^g\right)^\varepsilon\right)_q,$$

and

$$\left(X_Q^g\right)_q=g^{-1}\left(\left\{p\in Q\,:\,p\preceq q\right\}\right)\subseteq f^{-1}\left(\left\{p\in Q\,:\,p\preceq q+u_\varepsilon\right\}\right)=\left(X_Q^f\right)_{q+\varepsilon}=\left(\left(X_Q^f\right)^\varepsilon\right)_q.$$

Thus we can now take desired ε-interleaving $F:X_Q^f\to\left(X_Q^g\right)^\varepsilon$ and $G:X_Q^g\to\left(X_Q^f\right)^\varepsilon$ to be the diagram homomorphisms deriving from these inclusions. Since, by definition 9.25 the structure homomorphisms of both X_Q^f and X_Q^g are given by inclusion and since the diagram homomorphism $T_M^{2\varepsilon}$ is obtained from the structure homomorphisms then we can conclude that $G^\varepsilon\circ F=T^{2\varepsilon}_{X_Q^f}$ and $F^\varepsilon\circ G=T^{2\varepsilon}_{X_Q^g}$ thus proving that F and G form the desired ε-interleaving. $\square$

Notation 9.54. In order to simplify the notation involved in the statement of the previous lemma, given functions $f,g:X\to Q$ we will denote

$$\|\boldsymbol{f}-\boldsymbol{g}\|_{\boldsymbol{Q}_+}=\sup_{x\in X}\left(d_{Q_+}\left(f\left(x\right),g\left(x\right)\right)\right).$$

If we now manage to prove that taking the infimum ε for which two diagrams are ε-interleaved is an extended pseudometric (proposition 9.56) then we will be able to define interleaving distance as taking this infimum (definition 9.57). Then, by lemma 9.53, for any topological space X and any functions $f,g:X\to Q$ satisfying $\|f-g\|_{Q_+}=\varepsilon$ we will have that the interleaving distance between X_Q^f and X_Q^g is at most ε Finally, by functoriality of the homology functor, we will have that the ε-interleaving of the diagrams X_Q^f and X_Q^g will induce an ε-interleaving of the persistence modules $H\left(X_Q^f\right)$ and $H\left(X_Q^g\right)$ (lemma 9.62). Since this implies by definition that the interleaving distance between $H\left(X_Q^f\right)$ and $H\left(X_Q^g\right)$ is lower or equal than ε then we will have proven the desired stability theorem (theorem 9.63). Moreover the, by remark 9.52 the interleaving distance thus defined will coincide with the one given in [2, definition 3.2] and, therefore, by [2, theorem 4.16] it will coincide under tameness conditions with the usual barcode distance when restricted to the 1-dimensional case.

The remaining of this section will be dedicated to defining the interleaving distance while the remaining steps of the reasoning detailed above will be made in more detail in sub-section 9.6.

In order to define the interleaving distance and, more precisely, for proving that it is in fact an extended pseudo-distance we will need the following lemma.

Lemma 9.55. *Given a real or discrete polyhedral group Q, a category C and diagrams $M,N\in C^Q$ that are ε interleaved for some positive $\varepsilon\in Q_B$ then M and N are also ε' interleaved for any $\varepsilon'\geq\varepsilon$.*

Proof. Since M and N are ε-interleaved then we can take diagram homomorphisms $F : M \to N^{\varepsilon}$ and $G : N \to M^{\varepsilon}$ that form an ε-interleaving of M and N. Since $\varepsilon' - \varepsilon \geq 0$ by hypothesis then, using remarks 9.48 and 9.51 we obtain the following commutative diagrams

$$\begin{tikzcd}
M \arrow[rr, "T_M^{2\varepsilon}"] \arrow[dr, "F"'] & & M^{2\varepsilon} \arrow[rr, "T_{M^{2\varepsilon}}^{\varepsilon'-\varepsilon}"] & & M^{\varepsilon+\varepsilon'} \arrow[r, "T_{M^{\varepsilon+\varepsilon'}}^{\varepsilon'-\varepsilon}"] & M^{2\varepsilon'} \ , \\
& N^{\varepsilon} \arrow[ur, "G^{\varepsilon}"] \arrow[rr, "T_{N^{\varepsilon}}^{\varepsilon'+\varepsilon}"] & & N^{\varepsilon'} \arrow[ur, "G^{\varepsilon'}"] & &
\end{tikzcd}$$

$$\begin{tikzcd}
N \arrow[rr, "T_N^{2\varepsilon}"] \arrow[dr, "G"'] & & N^{2\varepsilon} \arrow[rr, "T_{N^{2\varepsilon}}^{\varepsilon'-\varepsilon}"] & & N^{\varepsilon+\varepsilon'} \arrow[r, "T_{N^{\varepsilon+\varepsilon'}}^{\varepsilon'-\varepsilon}"] & N^{2\varepsilon'} \ . \\
& M^{\varepsilon} \arrow[ur, "F^{\varepsilon}"] \arrow[rr, "T_{M^{\varepsilon}}^{\varepsilon'+\varepsilon}"] & & M^{\varepsilon'} \arrow[ur, "F^{\varepsilon'}"] & &
\end{tikzcd}$$

If we now define the diagram homomorphisms $F' = T_{N^{\varepsilon}}^{\varepsilon'-\varepsilon} \circ F : M \to N^{\varepsilon'}$ and $G' = T_{M^{\varepsilon}}^{\varepsilon'-\varepsilon} \circ G : N \to M^{\varepsilon'}$ then, by remark 9.47 we can rewrite the previous commutative diagrams as

$$\begin{tikzcd}
M \arrow[rr, "T_M^{2\varepsilon'}"] \arrow[dr, "F'"'] & & M^{2\varepsilon'} \ , \ N \arrow[rr, "T_N^{2\varepsilon'}"] \arrow[dr, "G"'] & & N^{2\varepsilon'} \ . \\
& N^{\varepsilon'} \arrow[ur, "G'^{\varepsilon'}"] & & M^{\varepsilon'} \arrow[ur, "F'^{\varepsilon'}"] &
\end{tikzcd}$$

By remark 9.51 commutativeness of these these diagrams proves that the diagram homomorphisms F' and G' form an ε'-interleaving of M and N. Equivalently M and N are ε'-interleaved just as we wanted to prove. □

Using this lemma we can now prove that taking infimum of the values ε for which two diagrams are ε-interleaved is in fact an extended pseudometric.

Proposition 9.56. *Given a real or discrete polyhedral group Q and a category C the function $d_I : C^Q \times C^Q \to Q_B$ defined as*

$$d_I(M,N) = \begin{cases} \inf\left(\{\varepsilon \in Q_B \,|\, M \text{ and } N \text{ are } \varepsilon\text{-interleaved}\}\right) & \text{if the set is not empty} \\ \infty & \text{else} \end{cases},$$

for any $M, N \in C^Q$ is an extended pseudo-metric in C^Q.

Proof. Since for every $M \in C^Q$ we have that the identity homomorphism $\mathrm{Id} : M \to M$ constitutes a 0-interleaving of M with itself then we have that $\boldsymbol{d_I(M,M) = 0}$.

On the other hand the **symmetry** property of the function d_I follows immediately from the symmetric definition of ε-interleaving (definition 9.50).

Thus we are only left with proving the **triangular inequality** in order to show that d_I is, at least, a pseudometric. Take any diagrams $M, N, L \in C^Q$ and denote $a = d_I(M,N)$ and

$b = d_I(N, L)$. Then, by definition of infimum, ε-interleaving and lemma 9.55 we have that for every $\varepsilon > 0$ exists an $(a+\varepsilon)$-interleaving $F_{M,N} : M \to N^{a+\varepsilon}$, $F_{N,M} : N \to M^{a+\varepsilon}$ and a $(b+\varepsilon)$-interleaving $F_{N,L} : N \to L^{b+\varepsilon}$, $F_{L,N} : L \to N^{a+\varepsilon}$. From these interleavings we obtain the following diagrams

$$\begin{tikzcd}
M \arrow[rr, "T_M^{2(a+\varepsilon)}"] \arrow[dr, "F_{M,N}"'] & & M^{2(a+\varepsilon)} \arrow[rr, "T_{M^{2(a+\varepsilon)}}^{2(b+\varepsilon)}"] & & M^{2(a+b+2\varepsilon)} \\
& N^{a+\varepsilon} \arrow[ur, "F_{N,M}^{a+\varepsilon}"'] \arrow[rr, "T_{N^{a+\varepsilon}}^{2(b+\varepsilon)}"] \arrow[dr, "F_{N,L}^{a+\varepsilon}"'] & & N^{a+\varepsilon+2(b+\varepsilon)} \arrow[ur, "F_{N,M}^{a+\varepsilon+2(b+\varepsilon)}"'] & \\
& & L^{a+b+2\varepsilon} \arrow[ur, "F_{L,N}^{a+b+2\varepsilon}"'] & &
\end{tikzcd},$$

$$\begin{tikzcd}
L \arrow[rr, "T_L^{2(b+\varepsilon)}"] \arrow[dr, "F_{L,N}"'] & & L^{2(b+\varepsilon)} \arrow[rr, "T_{L^{2(b+\varepsilon)}}^{2(a+\varepsilon)}"] & & L^{2(a+b+2\varepsilon)} \\
& N^{b+\varepsilon} \arrow[ur, "F_{N,L}^{b+\varepsilon}"'] \arrow[rr, "T_{N^{b+\varepsilon}}^{2(a+\varepsilon)}"] \arrow[dr, "F_{N,M}^{b+\varepsilon}"'] & & N^{b+\varepsilon+2(a+\varepsilon)} \arrow[ur, "F_{N,L}^{b+\varepsilon+2(a+\varepsilon)}"'] & \\
& & M^{a+b+2\varepsilon} \arrow[ur, "F_{M,N}^{a+b+2\varepsilon}"'] & &
\end{tikzcd}.$$

By remarks 9.48 and 9.51 and definition of ε-translate (definition 9.40) we know that each component of these diagram commutes and, therefore, the whole diagram commutes. Defining $F_{M,L} = F_{N,L}^{a+\varepsilon} \circ F_{M,N} : M \to L^{a+b+2\varepsilon}$ and $F_{L,M} = F_{N,M}^{b+\varepsilon} \circ F_{L,N} : L \to M^{a+b+2\varepsilon}$ then we can rewrite the previous commutative diagrams as

$$\begin{tikzcd}
M \arrow[rr, "T_M^{2(a+b+2\varepsilon)}"] \arrow[dr, "F_{M,L}"'] & & M^{2(a+b+2\varepsilon)} \\
& L^{a+b+2\varepsilon} \arrow[ur, "F_{L,M}^{a+b+2\varepsilon}"'] &
\end{tikzcd}, \quad
\begin{tikzcd}
L \arrow[rr, "T_L^{2(a+b+2\varepsilon)}"] \arrow[dr, "F_{L,M}"'] & & L^{2(a+b+2\varepsilon)} \\
& M^{a+b+2\varepsilon} \arrow[ur, "F_{M,L}^{a+b+2\varepsilon}"'] &
\end{tikzcd}.$$

By remark 9.51 commutativeness of these these diagrams proves that the diagram homomorphisms $F_{M,L}$ and $F_{L,N}$ constitute in fact an $(a+b+2\varepsilon)$-interleaving of M and L. Since we can take ε to be as small as we want to then we can deduce that $d_I(M, L) \leq a+b$. In other words we have that $d_I(M, L) \leq d_I(M, N) + d_I(N, L)$ thus proving the triangular inequality. Since we have proven that d_I satisfies $d_I(M, M) = 0$, symmetry and the triangular inequality and since we have defined it so that it takes only non-negative values or infinity we can conclude that d_I is in fact an extended pseudo-metric. $\square$

Using proposition 9.56 we can now finally define interleaving distance.

Definition 9.57. Given a real or discrete polyhedral group Q, a category C we call **interleaving distance** the extended pseudo-metric $\boldsymbol{d_I} : C^Q \times C^Q \to Q_B$ defined in proposition 9.56.

Remark 9.58. By remark 9.52 we have that the definition of interleaving distance we have just provided does in fact coincide in the 1-dimensional case with the interleaving distance given in [2, definition 3.2].

Before concluding this sub-section we deem necessary to show a few examples in order to help consolidate the notion of interleaving distance. Among these example 9.59 shows us that the interleaving distance is in fact extended since it can take the ∞ while example 9.60 shows us that it cannot be more than a pseudo-metric since it can take the zero value with different modules. Example 9.61 will be studied in more detail in section 10.

Example 9.59. Take $C = \mathrm{Vect}_k$ for some field k and $Q = \mathbb{R}$ with the usual order. Then we can define $M = 0$ and $N = \bigoplus_{q \in Q} k$ with structure homomorphisms given by the identity. For any $\varepsilon \geq 0$ we have that, by construction of M the only possible homomorphisms $F : M \to N^{\varepsilon}$ and $G : N \to M^{\varepsilon}$ are the 0 homomorphism. Thus $H = F^{\varepsilon} \circ G$ must also be the 0 homomorphism and, therefore, cannot be equal to $T_N^{2\varepsilon}$ which is the identity homomorphism by construction. Thus there is no possible ε-interleaving between M and N for any $\varepsilon \geq 0$ and, therefore $d_I(M, N) = \infty$ by definition. This example shows that the interleaving distance is extended since it can in fact assume infinite values.

Example 9.60. Take C, Q and M as in example 9.59 and define N as $N_0 = k$ and $N_q = 0$ for every $q \in Q \setminus \{0\}$. Notice that, by construction, all structure homomorphisms besides the trivial ones are in fact equal to the 0 homomorphism for both M and N. Thus, for every $\varepsilon > 0$, the pair of zero diagram homomorphisms constitutes an ε-interleaving between M and N. Taking infimum between all ε we deduce that $d(M, N) = 0$. This example shows us that the interleaving distance is not a metric since we can have diagrams $M \neq N$ such that $d(M, N) = 0$.

Example 9.61. As a last example take $C = \mathrm{Vect}_k$ for some field k and $Q = \mathbb{R}^2$ with the usual order. Define now the poset module M_1 as the module that has value 0 for every degree except for the square $[0, 1] \times [0, 1]$ where it has copies of k and take the structure homomorphisms of M to be the identity homomorphism whenever possible. Define now $M_{0.5}$ in a similar way but replacing the square $[0, 1] \times [0, 1]$ with the square $[0, 0.5] \times [0, 0.5]$. With this definition we have that, for any $\varepsilon > 0.5$, then $T_{M_1}^{2\varepsilon} = T_{M_{0.5}}^{2\varepsilon} = 0$ since, by construction, all structure homomorphisms between degrees that are apart for strictly more than 1 in any of the two modules is the zero homomorphism. Thus, taking as ε-interleaving the pair of 0 homomorphisms and taking infimum we obtain that $d_I(M_1, M_{0.5}) \leq 0.5$.

On the other hand, since diagram homomorphisms commute with structure homomorphisms then, for any diagram M any diagram homomorphism $F : M_1 \to M$ is uniquely determined by the image of the identity element at degree $(0, 0)$ and the same is true for any homomorphism $G : M_{0.5} \to M$. Because of this observation we will identify a diagram homomorphism from either M_1 or $M_{0.5}$ with the image of the identity element at $(0, 0)$. Notice now that, If we take $0 < \varepsilon < 0.5$ then the diagram homomorphism $T_{M_1}^{2\varepsilon}$ will be identified with the value $1 \in k$. Thus, in order for an ε-interleaving to exist between M_1 and $M_{0.5}$ the existence of a non-zero diagram homomorphism between $M_{0.5}$ and M_1^{ε} is needed. However such a diagram homomorphism does not exist. Suppose in fact that we could identify any $G : M_{0.5} \to M_1^{\varepsilon}$

with a value $a \in k \setminus \{0\}$. Since, by construction, we have that M_1^ε has copies of k for any degree in the square $[-\varepsilon, 1-\varepsilon] \times [-\varepsilon, 1-\varepsilon]$ then the image of the identity at $(0,0)$ by the composition $(M_{0.5})_{(0,0)} \xrightarrow{G_{(0,0)}} (M_1^\varepsilon)_{(0,0)} \to (M_1^\varepsilon)_{(1-\varepsilon,1-\varepsilon)}$ will still be a. On the other hand we have that the image of such identity by the composition $(M_{0.5})_{(0,0)} \to (M_{0.5})_{(1-\varepsilon,1-\varepsilon)} \xrightarrow{G_{(1-\varepsilon,1-\varepsilon)}} (M_1^\varepsilon)_{(1-\varepsilon,1-\varepsilon)}$ is 0 since the structure homomorphism $(M_{0.5})_{(0,0)} \to (M_{0.5})_{(1-\varepsilon,1-\varepsilon)}$ is 0 by construction of $M_{0.5}$ and ε. Since diagram homomorphisms commute with structure homomorphisms by definition these two results contradict one another and we must therefore conclude that any diagram homomorphism $G : M_{0.5} \to M_1^\varepsilon$ for $0 < \varepsilon < 0.5$ is 0. For what we said earlier we can thus conclude that there is no ε-interleaving between M_1 and $M_{0.5}$ with $\varepsilon < 0.5$ and, therefore, $d_I(M_1, M_{0.5}) \geq 0.5$. We can thus conclude that $d_I(M_1, M_{0.5}) = 0.5$.

This example can be made more general but for that purpose we will wait until section 10 where we will explore it in more detail.

9.6 Stability theorem and bounds.

In this last sub-section we will prove that then use this distance together with the polyhedral distance defined in the previous sub-section in order to prove the stability theorem 9.63. While doing this we will motivate with references to [2] why the interleaving distance and theorem 9.63 are in fact generalizations of the barcode distance and the 1-dimensional stability theorem. We will then conclude by using the decomposition of tame modules (see corollary 8.8) in order to give an upper bound of the interleaving distance. This upper bound will in fact have a closer resemblance with the bottleneck distance defined in the 1-dimensional case.

In order to prove the stability theorem we first need to prove that interleaving distance decreases when we apply an homology functor of any level. We will in fact prove the following more general lemma.

Lemma 9.62. *Given a real or discrete polyhedral group Q, a non negative value $\varepsilon \in Q_B$, categories C, D, a functor $\Psi : C \to D$ and diagrams $M, N \in C^Q$, if M and N are ε-interleaved then the induced diagrams $\Psi(M), \Psi(N) \in D^Q$ (see remark 9.24) are also ε-interleaved. In particular we have that $d_I(\Psi(M), \Psi(N)) \leq d_I(M, N)$.*

Proof. The second part of the statement follows immediately from the first, therefore we only need to prove the first part. Since M and N are ε-interleaved then, by definition, exist $F : M \to N^\varepsilon$ and $G : N \to M^\varepsilon$ such that $T_M^{2\varepsilon} = G^\varepsilon \circ F$ and $T_N^{2\varepsilon} = F^\varepsilon \circ G$. Then, by remarks

9.30, 9.44 and 9.49 we have the following commutative diagrams

$$\begin{tikzcd}
M \arrow[rrrr, "T_M^{2\varepsilon}"] \arrow[d, "\Psi"] \arrow[ddrr, "F"] & & & & M^{2\varepsilon} \arrow[d, "\Psi"] \\
\Psi\left(M\right) \arrow[rrrr, "T^{2\varepsilon}_{\Psi(M)}"] \arrow[dddrr, "\Psi(F)"'] & & & & \Psi\left(M\right)^{2\varepsilon} \\
& & N^{\varepsilon} \arrow[uurr, "G^{\varepsilon}"] \arrow[d, "\Psi"] & & \\
& & \Psi\left(N\right)^{\varepsilon} \arrow[uurr, "\Psi(G)^{\varepsilon}"] & &
\end{tikzcd}
,
\begin{tikzcd}
N \arrow[rrrr, "T_N^{2\varepsilon}"] \arrow[d, "\Psi"] \arrow[ddrr, "G"] & & & & N^{2\varepsilon} \arrow[d, "\Psi"] \\
\Psi\left(N\right) \arrow[rrrr, "T^{2\varepsilon}_{\Psi(N)}"] \arrow[dddrr, "\Psi(G)"'] & & & & \Psi\left(N\right)^{2\varepsilon} \\
& & M^{\varepsilon} \arrow[uurr, "F^{\varepsilon}"] \arrow[d, "\Psi"] & & \\
& & \Psi\left(M\right)^{\varepsilon} \arrow[uurr, "\Psi(F)^{\varepsilon}"] & &
\end{tikzcd}$$

By remark 9.51 the lower triangular section of these diagrams tells us that $\Psi\left(F\right)$ and $\Psi\left(G\right)$ constitute a ε-interleaving of $\Psi\left(M\right)$ and $\Psi\left(N\right)$ thus proving that those diagrams are ε-interleaved just as we wanted to prove. $\square$

We can now finally express more formally the reasoning made after proof of lemma 9.53 by stating and proving the following multi-dimensional stability theorem.

Theorem 9.63. *(Stability) Given a real or discrete polyhedral group Q, a topological space X and functions $f, g : X \to Q$ then we have that $d_I\left(H\left(X_Q^g\right), H\left(X_Q^f\right)\right) \leq \|f-g\|_{Q_+}$ where H denotes the homology functor at any level.*

Proof. By lemma 9.53 and definition 9.57 we know that $d_I\left(X_Q^g, X_Q^f\right) \leq \|f-g\|_{Q_+}$. On the other hand lemma 9.62 tells us that $d_I\left(H\left(X_Q^g\right), H\left(X_Q^f\right)\right) \leq d_I\left(X_Q^g, X_Q^f\right)$. The result follows from combining both inequalities. $\square$

Remark 9.64. Combining remarks 9.39 and 9.58 with [2, theorem 4.16] we can conclude that theorem 9.63 is in fact a generalization of the 1-dimensional stability theorem.

We conclude this section by showing how existence and uniqueness of decomposition of tame poset modules (see corollary 8.8) can be used in order to find upper bounds of the interleaving distance.

Lemma 9.65. *Given a real or discrete polyhedral group Q and tame Q-modules M and N that can be decomposed as direct sum of irreducible components as $M = \bigoplus_{i\in I} M_i$ and $N = \bigoplus_{i\in J} N_j$ then the interleaving distance between M and N satisfies*

$$d_I\left(M, N\right) \leq \inf_{f:\overline{IJ}\leftrightarrow\overline{IJ}} \left(\sup_{i\in\overline{IJ}} \left(d_I\left(M_i, N_{f(i)}\right)\right)\right) = \boldsymbol{d_B}\left(\left\{M_i\right\}_{i\in I}, \left\{N_j\right\}_{j\in J}\right),$$

where $\overline{IJ}$ is the disjoint union of I and J, f iterates over all automorphisms of $\overline{IJ}$ and we use the convention $M_i = 0$ if $i \notin I$ and $N_j = 0$ if $j \notin J$.

Proof. Given an automorphism $f : \overline{IJ} \leftrightarrow \overline{IJ}$ we just need to prove that existence of an ε-interleaving between M_i and $N_{f(i)}$ for every $i \in \overline{IJ}$ implies existence of an ε-interleaving between M and N. However such an ε-interleaving arises from the identities $M = \bigoplus_{i\in\overline{IJ}} M_i$ and $N = \bigoplus_{j\in\overline{IJ}} N_j$ taking direct sum the ε-interleavings between M_i and $N_{f(i)}$ for every $i \in \overline{IJ}$. $\square$

As we can see the upper bound given in the previous lemma is much more similar to the bottleneck distance of the one dimensional case (see definition 2.5). We would like for this inequality to be in fact an equality thus allowing us to compute the interleaving distance just from the interleaving distance of its components. However this is not possible even in the simple case where both M and N are decomposable as direct sum of simple indicator modules over $\mathbb{R}^2$ as shown by the following example provided by Bjerkevik in [3, example 5.1].

Example 9.66. Take $Q = \mathbb{R}^2$ with the usual order and define the Q-modules M and N as $M = \bigoplus_{i=1}^{3} I_i$ and $N = J$ with

$$I_1 = k\left[(-3,1)\times(-1,3)\right], \qquad I_2 = k\left[(-1,3)\times(-3,1)\right],$$
$$I_3 = k\left[(-1,1)\times(-1,1)\right], \qquad J = k\left[(-2,2)\times(-2,2)\right],$$

then we have a 1-interleaving between M and N obtained by defining the functions $f : M \to N^1$ and $g : N \to M^1$ at each component as multiplication by one except for the restriction $g_3 : J \to I_3$ which is given as multiplication by -1. That is $f_i : I_i \to J^1$ is the identity at every possible degree and 0 everywhere else for every $i = 1, 2, 3$ and the same can be said about $g_i : J \to I_i^1$ for $i = 1, 2$ while $g_3 : J \to I_3^1$ is multiplication by -1 at every possible degree and 0 everywhere else. Using proposition 4.10 it is easily checked that we have that $\mathrm{Hom}\left(I_i, I_j^2\right) = \{0\}$ for every $i \neq j$. Thus we have that the homomorphism $g^1 \circ f : M \to M^2$, when restricted to a component I_i with $i = 1, \ldots, 3$, is in fact equal to the composition $I_i \to J \to I_i$. By definition of these homomorphisms and since $\mathrm{Hom}\left(I_3, I_3^2\right) = \{0\}$ we can therefore conclude that $T_M^2 = g^1 \circ f$.

On the other hand, taking $(a, b) \in (-2, 2) \times (-2, 2)$ and any homogeneous element $x \in N_{(a,b)}$ we have that

$$T_N^2(x) = \begin{cases} 0 & \text{if } a \geq 0 \text{ or } b \geq 0 \\ x & \text{else} \end{cases}, \qquad f_1^1 \circ g_1(x) = T_N^2(x),$$
$$f_2^1 \circ g_2(x) = T_N^2(x), \qquad f_3^1 \circ g_3(x) = -T_N^2(x),$$

where we are looking at the right hand elements as homogeneous elements in $N^2_{(a,b)}$. Adding together all these results we can conclude that $T_N^2 = f^1 \circ g$. Joining both results we have that f and g form a 1-interleaving between M and N thus proving that $d_I(M, N) \leq 1$.

On the other hand it is trivially seen that $d_I(I_i, 0) \geq 2$ for $i = 1, 2$ and therefore, using the same notation as in lemma 9.65, we have that $d_B(\{I_1, I_2, I_3\}, \{J\}) \geq 2$ thus proving that the identity does not occur even in simple cases for multi-dimensional persistence.

Even though the previous example tells us that $d_B \not\leq d_I$ we still have some upper bounds of the $d_B \leq a \cdot d_I$ as proven in [4, 3]. More precisely Bjerkevik proves in [3] that, for modules over $\mathbb{R}^n$ that can be decomposed as a direct sum of indicator modules of the form $k\left[\{x \in \mathbb{R}^n \mid a < x < b\}\right]$ with $a, b \in \mathbb{R}^n$, we have $d_B \leq (2n-1)\, d_I$. Moreover he finds a counter example proving that such bound is in fact optimal.

10 Distance between indicator modules, examples and bounds.

In this last section we will study more in detail the interleaving distance between indicator modules (see definition 4.5). Even though a similar work was already performed by Dey and Cheng in [9] the results shown in this section were obtained independently from the ones in [9]. We will have three main goals in this section and for each of them we will dedicate a separate sub-section.

In sub-section 10.1 we will recall study interleaving distances between indicator modules and we will use proposition 4.10 in order to prove two distinct formulas (proposition 10.2 and corollary 10.4) that can be used in order to compute them.

In sub-section 10.2 we will introduce some new notation and use it in order to give an upper bound of the interleaving distance computed in sub-section 10.1 (see proposition 10.9). This upper bound will have the advantage to be simpler to compute than the formulas given by proposition 10.2 and corollary 10.4 and it closely resembles the formula of the bottleneck distance between 2 intervals (see example 10.8).

Finally in sub-section 10.3 we will introduce a new, more restrictive, type of indicator module (see definition 10.11) and compute lower bounds for the interleaving distance between two such indicator modules (see proposition 10.16).

10.1 Distance between indicator modules.

In order to be able to use lemma 9.65 for computing upper bounds of the interleaving distance it would be nice for all irreducible modules over real or discrete polyhedral groups to be indicator modules (definition 4.5). However, as proven by Buchet and Escolar in [6] this is not only false but there are actually infinitely many families of irreducible modules and such families of modules can appear from real data analysis and are therefore not product of degeneracies.

Example 10.1. A simple example of such irreducible poset module can be obtained by taking the $\mathbb{Z}^2$-module M defined as

$$M_x = \begin{cases} k & \text{if } x \in \{(1,0)\,,(0,1)\,,(2,1)\}\,, \\ k^2 & \text{if } x = (1,1)\,, \\ 0 & \text{else,} \end{cases}$$

this $\mathbb{Z}^2$-module is provided with the structure homomorphisms

$$\underset{a}{M_{(1,0)}} \to \underset{(a,0)}{M_{(1,1)}}, \underset{b}{M_{(0,1)}} \to \underset{(0,b)}{M_{(1,1)}}, \underset{(a,b)}{M_{(1,1)}} \to \underset{a+b}{M_{(2,1)}}.$$

It is easily seen that this $\mathbb{Z}^2$-module is irreducible and yet it is not an indicator module.

In order to make our goal achievable in this sub-section we will not consider all the possible families of irreducible poset modules but we will instead restrict ourselves to indicator modules obtained from connected sub-posets (see definition 4.8) which are clearly irreducible. In order to simplify notation we will call such indicator modules simply **connected indicator modules**. Such connected indicator modules coincide in fact with the interleaving modules used in [9].

Let us start by viewing how the interleaving distance between connected indicator modules can be computed in terms of the underlying sets.

Proposition 10.2. *Given a real or discrete polyhedral group Q and indicator modules $k[S]$ and $k[S']$ with $S, S' \subseteq Q$ then $k[S]$ and $k[S']$ are ε-interleaved for some $\varepsilon \geq 0$ if and only if $S \cap S^{2\varepsilon} \subseteq \mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon})$ and $S' \cap S'^{2\varepsilon} \subseteq \mathcal{F}(S', S^{\varepsilon}) \cap \mathcal{F}(S^{\varepsilon}, S'^{2\varepsilon})$ where for any $D, D' \subseteq Q$ we define $\boldsymbol{D^{\varepsilon}} = \{x - u_{\varepsilon} : x \in D\} \subseteq Q$ and*

$$\boldsymbol{\mathcal{F}}(D, D') = \bigcup_{C \in \mathcal{G}(D,D')} C,$$
$$\mathcal{G}(D, D') = \{C \in \pi_0(D \cap D') : (S \backslash C) \cap C_D = \emptyset \text{ and } (S' \backslash C)_D \cap C = \emptyset\},$$

where we are using the same notation as in proposition 4.10. In fact, when we have the above inclusions we have in fact an identity.

Proof. By definition of indicator module and ε-translation (definition 9.45) we know that, given and homogeneous element $x \in k[S]_q$ with $q \in S$ then

$$T^{\varepsilon}_{k[S]}(x) = \begin{cases} 0 & \text{if } (x+\varepsilon) \notin S, \\ x & \text{if } (x+\varepsilon) \in S, \end{cases}$$

where the value x in the right hand side equation is seen as an homogeneous element in $k[S]^{\varepsilon}_q$. Moreover, by definition of S^{ε} and of ε-translate (definition 9.40) we know that $k[S^{\varepsilon}] = k[S]^{\varepsilon}$. Thus we can conclude that $T^{2\varepsilon}_{k[S]}(k[S]) = k[S \cap S^{2\varepsilon}] \subseteq k[S]^{2\varepsilon}$ and, analogously, $T^{2\varepsilon}_{k[S']}(k[S']) = k[S' \cap S'^{2\varepsilon}] \subseteq k[S']^{2\varepsilon}$ where we are using that S and S' are both intersections of upsets and downsets.

On the other hand we know from the proof of proposition 4.10 that any homomorphism $F : S \to S'^{\varepsilon}$ acts as multiplication by some scalar λ_C on every connected component $C \in \mathcal{G}(S, S'^{\varepsilon})$. The analogous result is true for any homomorphism $G : S' \to S^{\varepsilon}$. Lets now identify any homomorphism between two indicator modules with this set of scalars λ_C. We now have that for any choice of homomorphisms F and G the composition $G^{\varepsilon} \circ F$ will be equal to multiplication by $\lambda_C \cdot \lambda_{C'}$ for every degree in the set $C \cap C'$ with $C \in \mathcal{G}(S, S'^{\varepsilon})$ and $C' \in \mathcal{G}(S'^{\varepsilon}, S^{\varepsilon})$ and will be 0 for every other degree. This implies that the set of degrees that are not mapped to 0 by the homomorphism composition $G^{\varepsilon} \circ F$ is included in $\mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon})$ and in fact is exactly that if all λ_C and $\lambda_{C'}$ are non-zero. Since $T^{2\varepsilon}_{k[S]}(k[S]) = k[S \cap S^{2\varepsilon}]$ for every $\varepsilon \geq 0$ then the previous assertion tells us that existence of a ε-interleaving between $k[S]$ and $k[S']$ implies the inclusion $S \cap S^{2\varepsilon} \subseteq \mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon})$. Analogously, existence of a ε-interleaving between these two modules also implies the inclusion $S' \cap S'^{2\varepsilon} \subseteq \mathcal{F}(S', S^{\varepsilon}) \cap \mathcal{F}(S^{\varepsilon}, S'^{2\varepsilon})$.

If we now prove that $D \cap D^{2\varepsilon} \supseteq \mathcal{F}(D, D'^{\varepsilon}) \cap \mathcal{F}(D'^{\varepsilon}, D^{2\varepsilon})$ for any sets $D, D' \subseteq Q$ then, whenever $S \cap S^{2\varepsilon} \subseteq \mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon})$, we will have that these inclusions are in fact identities. Since $T^{2\varepsilon}_{k[S]}$ is in fact multiplication by 1 for degrees in $S \cap S^{2\varepsilon}$ as proved earlier then, taking all λ_C and $\lambda_{C'}$ values equal to 1 we will have that $G^{\varepsilon} \circ F = T^{2\varepsilon}_{k[S]}$. Moreover, if we also have that $S' \cap S'^{2\varepsilon} \subseteq \mathcal{F}(S', S^{\varepsilon}) \cap \mathcal{F}(S^{\varepsilon}, S'^{2\varepsilon})$, then the same homomorphisms F and G will lead to the identity $F^{\varepsilon} \circ G = T^{2\varepsilon}_{k[S']}$ thus proving the existence of a ε-interleaving.

Thus we are only left with proving that $D \cap D^{2\varepsilon} \supseteq \mathcal{F}(D, D'^{\varepsilon}) \cap \mathcal{F}(D'^{\varepsilon}, D^{2\varepsilon})$ for any sets $D, D' \subseteq Q$. It is immediate from the definitions of $\mathcal{G}$ and $\mathcal{F}$ that for any sets $D, D' \in Q$ we have $\mathcal{F}(D, D') \subseteq D \cap D'$. Thus we have the inclusion

$$\begin{aligned} \mathcal{F}(D, D'^{\varepsilon}) \cap \mathcal{F}\left(D'^{\varepsilon}, D^{2\varepsilon}\right) &\subseteq (D \cap D'^{\varepsilon}) \cap \left(D'^{\varepsilon} \cap D^{2\varepsilon}\right), \\ &= D \cap D^{2\varepsilon} \cap D'^{\varepsilon} \subseteq D \cap D^{2\varepsilon}. \end{aligned}$$

□

Remark 10.3. Notice that the second part of the proof of proposition 10.2 tells us that the inclusion $S \cap S^{2\varepsilon} \subseteq \mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon})$ leads in fact to an equality. In fact, since $\mathcal{F}(D, D'^{\varepsilon}) \cap \mathcal{F}(D'^{\varepsilon}, D^{2\varepsilon}) \subseteq D \cap D^{2\varepsilon} \cap D'^{\varepsilon} \subseteq D \cap D^{2\varepsilon}$ then existence of a ε interleaving implies the inclusions $S \cap S^{2\varepsilon} \subseteq S'^{\varepsilon}$ and $S' \cap S'^{2\varepsilon} \subseteq S^{\varepsilon}$.

From remark 10.3 and proposition 10.2 we can derive the following corollary.

Corollary 10.4. *Given a real or discrete polyhedral group Q and indicator modules $k[S]$ and $k[S']$ with $S, S' \subseteq Q$ then $k[S]$ and $k[S']$ are ε-interleaved for some $\varepsilon \geq 0$ if and only if $S \cap S^{2\varepsilon} \subseteq S'^{\varepsilon}$, $S \cap S^{2\varepsilon} \cap S'^{\varepsilon} \subseteq \mathcal{F}(S, S'^{\varepsilon}), \mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon})$ and analogously inverting S and S'.*

Proof. By proposition 10.2 we must only prove that the given inclusions are in fact equivalent to the inclusions $S \cap S^{2\varepsilon} \subseteq \mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon})$ and $S' \cap S'^{2\varepsilon} \subseteq \mathcal{F}(S', S^{\varepsilon}) \cap \mathcal{F}(S^{\varepsilon}, S'^{2\varepsilon})$. In fact, by remark 10.3 we know that these inclusion imply in fact the identity and, therefore, we must only prove that these identities are in fact equivalent to the given inclusions. Since we have the inclusion chain

$$\begin{aligned} \mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}\left(S'^{\varepsilon}, S^{2\varepsilon}\right) &\subseteq \mathcal{F}(S, S'^{\varepsilon}) \cap \left(S'^{\varepsilon} \cap S^{2\varepsilon}\right), \quad \text{analogously} \subseteq (S \cap S'^{\varepsilon}) \cap \mathcal{F}\left(S'^{\varepsilon}, S^{2\varepsilon}\right), \\ &\subseteq (S \cap S'^{\varepsilon}) \cap \left(S'^{\varepsilon} \cap S^{2\varepsilon}\right), \\ &= S \cap S^{2\varepsilon} \cap S'^{\varepsilon}, \\ &\subseteq S \cap S^{2\varepsilon}, \end{aligned}$$

then the identity $S \cap S^{2\varepsilon} = \mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon})$ tells us that all inclusions in the previous chain are in fact identities and, therefore identities are in fact equivalent to the identity $S \cap S^{2\varepsilon} = \mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon})$. Repeating the same process but inverting S and S' we obtain the desired solution. □

Due to the extremely verbose definition of $\mathcal{F}$ proposition 10.2 and corollary 10.4 fail to be immediately intuitive. However corollary 10.4 gives us an immediate lower bound for the interleaving distance.

Example 10.5. Take Q a real or discrete polyhedral or discrete polyhedral group, $U \subseteq Q$ and upset and $D \subsetneq Q$ a downset. If D contained any upset then, since Q is upper directed (see notation 9.37), given any $x \in Q$ we would have $y \in D$ such that $x \preceq y$. Thus, since D is a downset, we would have that $x \in D$ and, therefore, $D = Q$ thus contradicting the hypothesis. Therefore D cannot contain any upset. Since for every $\varepsilon \in Q_B$ we have that $U^{2\varepsilon} \cap U$ is an upset and $D^{\varepsilon} \subsetneq Q$ is a downset then we can conclude that there is no $\varepsilon \geq 0$ such that $U \cap U^{2\varepsilon} \subseteq D^{\varepsilon}$. From corollary 10.4 this implies that there is no ε-interleaving between $k[U]$ and $k[D]$ and, therefore $d_I(k[U], k[D]) = \infty$. Notice that if we take any other upset $U' \subseteq Q$ and define $S = U' \cap D$ then, since $S^{\varepsilon} \subseteq D^{\varepsilon}$, we can conclude from the previous result that there is no $\varepsilon \geq 0$ such that $U \cap U^{2\varepsilon} \subseteq S^{\varepsilon}$. Thus we can conclude that $d_I(k[U], k[S]) = \infty$ for any upset U and any set S obtained from intersecting an upset $U' \subseteq Q$ and a strict downset $D \subsetneq Q$. Notice also that upper directedness of a real or discrete polyhedral group Q implies lower directedness of Q since, given any $x, y \in Q$ and $a \in Q$ satisfying $x, y \preceq a$ we can define $b = x - (a - y)$ and we will have that $b + (a - x) = y$ and, therefore $b \preceq x, y$. Thus the previous arguments work replacing upsets and downsets and we can conclude that $d_I(k[D], k[S]) = \infty$ for any downset D and any set S obtained from intersecting a downset $D' \subseteq Q$ and a strict upset $U \subsetneq Q$.

10.2 Upper bound.

Moreover the previous proposition also leads us to an upper bound of the interleaving distance. Before giving it however we need some notation.

Definition 10.6. Given a real or discrete polyhedral group Q and a sub-set $S \subseteq Q$ we denote by $\boldsymbol{S_U}$ the upset generated by S and by $\boldsymbol{S_D}$ the downset co-generated by S.

Definition 10.7. Given a real polyhedral group Q and indicator modules $S, S' \subseteq Q$ we define the following functions.

$$\begin{aligned}
\boldsymbol{d_{sup}(S, S')} &= \inf\left(\left\{\varepsilon \,:\, \varepsilon \geq 0,\, (S'_D)^{\varepsilon} \cap (S \cap S'^{\varepsilon})_U \subseteq S_D \text{ and } (S_D)^{\varepsilon} \cap (S' \cap S^{\varepsilon})_U \subseteq S'_D\right\}\right), \\
\boldsymbol{d_{sub}(S, S')} &= \inf\left(\left\{\varepsilon \,:\, \varepsilon \geq 0,\, S_U \cap (S \cap S'^{\varepsilon})_D \subseteq (S'_U)^{\varepsilon} \text{ and } S'_U \cap (S' \cap S^{\varepsilon})_D \subseteq (S_U)^{\varepsilon}\right\}\right), \\
\boldsymbol{d_{comb}(S, S')} &= \inf\left(\left\{\varepsilon \,:\, \varepsilon \geq 0,\, S \cap S^{2\varepsilon} \subseteq S'^{\varepsilon} \text{ and } S' \cap S'^{2\varepsilon} \subseteq S^{\varepsilon}\right\}\right), \\
\operatorname{len}(\boldsymbol{S}) &= \inf\left(\left\{\varepsilon \,:\, \varepsilon \geq 0,\, (S_D)^{\varepsilon} \cap S_U = \emptyset\right\}\right),
\end{aligned}$$

In order to gain an intuitive notion of the previous definition an example for the 1-dimensional case is due.

Example 10.8. Take the 1-dimensional real polyhedral group $Q = \mathbb{R}$ with the standard order and the intervals $S = (a, b)$, $S' = (a', b')$ with $a < b$ and $a' < b'$ then we have

$$\begin{aligned} S_D &= (-\infty, b)\,, & S_U &= (a, \infty)\,, \\ S'_D &= (-\infty, b')\,, & S_U &= (a', \infty)\,. \end{aligned}$$

On the other hand we have that

$$(S \cap S'^{\varepsilon})_U = \begin{cases} (\max(a, a' - \varepsilon), \infty) & \text{if } S \cap S'^{\varepsilon} \neq \emptyset \\ \emptyset & \text{else} \end{cases},$$

and, therefore,

$$(S'_D)^{\varepsilon} \cap (S \cap S'^{\varepsilon})_U = \begin{cases} (\max(a, a' - \varepsilon), b' - \varepsilon) & \text{if } S \cap S'^{\varepsilon} \neq \emptyset \\ \emptyset & \text{else} \end{cases}.$$

Since $S \cap S'^{\varepsilon} = (\max(a, a' - \varepsilon), \min(b, b' - \varepsilon))$ then $b > b' - \varepsilon$ implies that $(\max(a, a' - \varepsilon), b' - \varepsilon) \Leftrightarrow S \cap S'^{\varepsilon} = \emptyset$. Thus we can conclude that

$$(S'_D)^{\varepsilon} \cap (S \cap S'^{\varepsilon})_U \subseteq S_D \Leftrightarrow \qquad b' - \varepsilon \leq b,$$

and, analogously,

$$\begin{aligned} (S_D)^{\varepsilon} \cap (S' \cap S^{\varepsilon})_U \subseteq S'_D \Leftrightarrow & \qquad b - \varepsilon \leq b', \\ S_U \cap (S \cap S'^{\varepsilon})_D \subseteq (S'_U)^{\varepsilon} \Leftrightarrow & \qquad a \geq a' - \varepsilon, \\ S'_U \cap (S' \cap S^{\varepsilon})_D \subseteq (S_U)^{\varepsilon} \Leftrightarrow & \qquad a' \geq a - \varepsilon, \end{aligned}$$

and, therefore

$$\begin{aligned} d_{sub}(S, S') &= |a - a'|\,, \\ d_{sup}(S, S') &= |b - b'|\,. \end{aligned}$$

For computing $\operatorname{len}(S)$ we have the equivalence

$$(S_D)^{\varepsilon} \cap S_U = \emptyset \Leftrightarrow b \leq a - \varepsilon,$$

which leads us to the identity

$$\operatorname{len}(S) = b - a.$$

Finally, for the d_{comb} we have that the equivalences

$$\begin{aligned} S \cap S^{2\varepsilon} \subseteq S'^{\varepsilon} &\Leftrightarrow (a, b - 2\varepsilon) \subseteq (a' - \varepsilon, b' - \varepsilon)\,, \\ &\Leftrightarrow (a \geq (a' - \varepsilon) \wedge b \leq (b' - \varepsilon)) \vee (b - a) < 2\varepsilon. \end{aligned}$$

Subtracting the inequalities $b \leq (b' - \varepsilon)$ and $a \geq (a' - \varepsilon)$ we obtain that $b - a \leq b' - a'$. Therefore if we also have $b' - a' \leq 2\varepsilon$ we will have that $b - a \leq 2\varepsilon$. Thus we can deduce that satisfying the conditions $S \cap S^{2\varepsilon} \subseteq S'^{\varepsilon}$ and $S' \cap S'^{2\varepsilon} \subseteq S^{\varepsilon}$ simultaneously is equivalent to satisfying either $|a - a'| \leq \varepsilon$ and $|b - b'| \leq \varepsilon$ or $\frac{b-a}{2}, \frac{b'-a'}{2} \leq \varepsilon$. In other words we have that, in the 1-dimensional case

$$
\begin{aligned}
d_{comb}(S, S') &= \min\left(\max\left(d_{sub}(S, S'), d_{sup}(S, S')\right), \frac{1}{2}\max\left(\operatorname{len}(S), \operatorname{len}(S')\right)\right), \\
&= \min\left(\max\left(|a - a'|, |b - b'|\right), \max\left(\frac{b-a}{2}, \frac{b'-a'}{2}\right)\right),
\end{aligned}
$$

Notice how this formula coincides in the 1-dimensional case with the values used for computing the bottleneck distance (see definition 2.5) between 2 finite, non empty intervals.

With this notation we can now give the following upper bound of interleaving distance.

Proposition 10.9. *Given a real or discrete polyhedral group Q and indicator modules $k[S]$ and $k[S']$ with $S, S' \subseteq Q$ then*

$$d_I(k[S], k[S']) \leq \min\left(\max\left(d_{sub}(S, S'), d_{sup}(S, S'), d_{comb}(S, S')\right), \frac{1}{2}\max\left(len(S), len(S')\right)\right).$$

Proof. Suppose that exists $\varepsilon \geq 0$ such that $(S_D)^{2\varepsilon} \cap S_U = (S'_D)^{2\varepsilon} \cap S'_U = \emptyset$, then we would have that

$$
\begin{aligned}
S \cap S^{2\varepsilon} &= (S_U \cap S_D) \cap \left((S_U)^{2\varepsilon} \cap (S_D)^{2\varepsilon}\right), \\
&= \left(S_D \cap (S_U)^{2\varepsilon}\right) \cap \left((S_D)^{2\varepsilon} \cap S_U\right), \\
&= \left(S_D \cap (S_U)^{2\varepsilon}\right) \cap \emptyset = \emptyset.
\end{aligned}
$$

and, analogously $S' \cap S'^{2\varepsilon} = \emptyset$. Therefore, by proposition 10.2, an ε-interleaving between $k[S]$ and $k[S']$ would exist. We can thus conclude that

$$d_I(k[S], k[S']) \leq \frac{1}{2}\max\left(\operatorname{len}(S), \operatorname{len}(S')\right).$$

Suppose now that exists $\varepsilon \geq 0$ such that $S_U \cap (S \cap S'^{\varepsilon})_D \subseteq (S'_U)^{\varepsilon}$. Take now C any connected component of $S \cap S'^{\varepsilon}$ and choose any $q' \in S \backslash C$ and any $q \in C$. Since C is a connected component then, by definition, $q' \preceq q$ would imply that $q' \notin S'^{\varepsilon} \cap S$ and, therefore, $q' \notin S'^{\varepsilon}$ since $q' \in S$ by construction. On the other hand, since $(S'_D)^{\varepsilon}$ is a downset, $q' \preceq q$ and $q \in S'^{\varepsilon} = (S'_D)^{\varepsilon} \cap (S'_U)^{\varepsilon}$ then we have that $q' \in (S'_D)^{\varepsilon}$. Moreover, since $q' \in S \subseteq S_U$ and $q' \in (S \cap S'^{\varepsilon})_D$ by construction then $q' \in S_U \cap (S \cap S'^{\varepsilon})_D \subseteq (S'_U)^{\varepsilon}$. From this we can deduce that

$$q' \in (S \cap S'^{\varepsilon})_D \cap (S'_U)^{\varepsilon} \subseteq (S'_D)^{\varepsilon} \cap (S'_U)^{\varepsilon} = S'^{\varepsilon},$$

thus reaching contradiction. We can thus conclude that there are no degrees $q' \in S \backslash C$ and $q \in C$ satisfying $q' \preceq q$.

Assuming now that $S_D \supseteq (S'_D)^{\varepsilon} \cap (S \cap S'^{\varepsilon})_U$ then an analogous reasoning leads us to proving that there are no degrees $q' \in S'^{\varepsilon} \backslash C$ and $q \in C$ satisfying $q \preceq q'$. Thus we can conclude that for any ε satisfying $S_U \cap (S \cap S'^{\varepsilon})_D \subseteq (S'_U)^{\varepsilon}$ and $S_D \supseteq (S'_D)^{\varepsilon} \cap (S \cap S'^{\varepsilon})_U$ we have $\mathcal{F}(S, S'^{\varepsilon}) = S \cap S'^{\varepsilon}$.

Repeating the same reasoning now switching S and S' we obtain that $\mathcal{F}(S', S^{\varepsilon}) = S' \cap S^{\varepsilon}$. Combining both results with some shifting we obtain the identities

$$\begin{aligned} S^{\varepsilon} \cap S^{2\varepsilon} \cap S'^{\varepsilon} =& \mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}\left(S'^{\varepsilon}, S^{2\varepsilon}\right), \\ S'^{\varepsilon} \cap S'^{2\varepsilon} \cap S^{\varepsilon} =& \mathcal{F}(S', S^{\varepsilon}) \cap \mathcal{F}\left(S^{\varepsilon}, S'^{2\varepsilon}\right). \end{aligned}$$

Thus, from proposition 10.2, we just need to prove the following inclusions

$$S^{\varepsilon} \cap S^{2\varepsilon} \subseteq S^{\varepsilon} \cap S^{2\varepsilon} \cap S'^{\varepsilon}, \qquad\qquad S^{\varepsilon} \cap S^{2\varepsilon} \subseteq S^{\varepsilon} \cap S^{2\varepsilon} \cap S'^{\varepsilon}.$$

These inclusions however are equivalent to the inclusions

$$S^{\varepsilon} \cap S^{2\varepsilon} \subseteq S'^{\varepsilon}, \qquad\qquad S^{\varepsilon} \cap S^{2\varepsilon} \subseteq S'^{\varepsilon}.$$

By definition these inclusions are satisfied if and only if $\varepsilon \geq d_{comb}(S, S')$. Thus we can conclude that

$$d_I(k[S], k[S']) \leq \max\left(d_{sub}(S, S'), d_{sup}(S, S'), d_{comb}(S, S')\right).$$

Joining both results we reach the desired inequality. □

Remark 10.10. Notice the close similarity between this formula and the known bottleneck distance for the 1-dimensional case where this upper bound is in fact exact.

10.3 lower bound.

We would now like to give a lower bound of the interleaving distance. However we will provide this lower bound solely for a sub-family of all connected indicator modules. We will need for this family of indicator modules to satisfy the property that, whenever the degrees of any two modules of the family the result is a connected set. This will allow us to avoid using the function $\mathcal{F}$ in proposition 10.2 and corollary 10.4 by some simpler function that will give us the desired identity. With this goal in mind we define strongly connected sub-set of a real polyhedral group.

Definition 10.11. Given a real polyhedral group Q and a sub-set $S \subseteq Q$ we say that S is **strongly connected** if it is open (in the usual topology) and convex.

Remark 10.12. Since intersections of open sets are open and intersections of convex sets are convex then we can conclude that intersections of strongly connected sets are strongly connected.

Remark 10.13. By lemma 4.3 we have that strongly connected sets can be written as the intersection of an upset and a downset. Namely $S = S_U \cap S_D$.

Strongly connected sets are in fact connected as proven by the following lemma.

Lemma 10.14. *Given a real polyhedral group Q and a strongly connected set $S \subseteq Q$ then S is connected.*

Proof. Take any pair of elements $x, y \in S$. Since S is convex then for any $t \in [0,1]$ we have that $tx + (1-t)\,y \in S$. On the other hand we have that the set $l = \{tx + (1-t)\,y \,:\, t \in [0,1]\}$ is compact in the usual topology for being closed and bounded. Moreover, since S is open then its complementary S^C is closed. Thus, taking d to be the l_2 distance between l and S^C, we have that $d > 0$ since $l \cap S^C = \emptyset$ by convexity of S. Since Q is upper directed by notation 9.37 we can now take $a \in Q$ satisfying $x, y \preceq a$ and choose $n \in \mathbb{N}$ big enough so that $\max\left(\|a-x\|_2, \|a-y\|_2\right) < n \cdot d$. Since both $a - x$ and $a - y$ belong to Q_+ by construction then we will also have that $\frac{a-x}{n}, \frac{a-y}{n} \in Q_+$. We can now define the sequence $a_0, b_0, \ldots, b_{n-1}, a_n$ by setting $a_0 = x$, $b_i = a_i + \frac{a-x}{n}$ and $a_{i+1} = b_i - \frac{a-y}{n}$. By construction we will have that $a_i, b_i \in S$ for every i, that $a_i, a_{i+1} \preceq b_i$ and that $a_n = y$. In other words we have found a connected path between x and y thus proving that S is connected. $\square$

With this definition of strongly connected set we can now simplify the function $\mathcal{F}$ defined in proposition 10.2.

Proposition 10.15. *Given a real polyhedral group Q and strongly connected sets $S, S' \subseteq Q$ then we can re-write the function $\mathcal{F}$ defined in proposition 10.2 as*

$$\mathcal{F}(S, S') = \begin{cases} S \cap S' & \textit{if } S'_D \cap (S \cap S')_U \subseteq S \textit{ and } S_U \cap (S \cap S')_D \subseteq S' \\ \emptyset & \textit{else} \end{cases}.$$

Proof. Since, by remark 10.12 $S \cap S'$ has a single connected component (namely $S \cap S'$) then we have that

$$\mathcal{G}(S, S') = \begin{cases} \{S \cap S'\} & \text{if } (S \backslash S') \cap (S \cap S')_D = \emptyset \text{ and } (S' \backslash S)_D \cap (S \cap S') = \emptyset \\ \emptyset & \text{else} \end{cases}.$$

Since the identity $(S' \backslash S)_D \cap (S \cap S') = \emptyset$ is in fact equivalent to the identity $S' \backslash S \cap (S \cap S')_U = \emptyset$ and we have the equivalences

$$\begin{aligned} (S \backslash S') \cap (S \cap S')_D = \emptyset \Leftrightarrow \qquad & S \cap (S \cap S')_D \subseteq S', \\ (S' \backslash S) \cap (S \cap S')_U = \emptyset \Leftrightarrow \qquad & S' \cap (S \cap S')_U \subseteq S, \end{aligned}$$

then we can re-write

$$\mathcal{F}(S, S') = \begin{cases} S \cap S' & \text{if } S' \cap (S \cap S')_U \subseteq S \text{ and } S \cap (S \cap S')_D \subseteq S' \\ \emptyset & \text{else} \end{cases}.$$

The result then follows from the identities below

$$
\begin{aligned}
S' \cap (S \cap S')_U &= (S'_U \cap S'_D) \cap (S \cap S')_U = S'_D \cap (S'_U \cap (S \cap S')_U) = S'_D \cap (S \cap S')_U \,, \\
S \cap (S \cap S')_D &= (S_U \cap S_D) \cap (S \cap S')_D = S_U \cap (S_D \cap (S \cap S')_D) = S_U \cap (S \cap S')_D \,.
\end{aligned}
$$

□

We can now give a lower bound of the interleaving distance for connected indicator modules very similar to the upper bound given by proposition 10.9.

Proposition 10.16. *Given a real polyhedral group Q and indicator modules $k[S]$ and $k[S']$ with $S, S' \subseteq Q$ strongly connected then*

$$
d_I(k[S], k[S']) \geq \min\left(\max\left(d_{sub}(S, S'), d_{sup}(S, S')\right), \frac{1}{2}\max\left(len(S), len(S')\right)\right).
$$

Proof. Since proposition 10.9 already gives us one of the inequalities then, by proposition 10.2 we just need to prove that any $\varepsilon \geq 0$ satisfying the inclusions $S \cap S^{2\varepsilon} \subseteq \mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon})$ and $S' \cap S'^{2\varepsilon} \subseteq \mathcal{F}(S', S^{\varepsilon}) \cap \mathcal{F}(S^{\varepsilon}, S'^{2\varepsilon})$ is greater or equal than the right hand side of the identity. In other words we need to prove that, whenever those inclusions are satisfied then we have either both of the following

$$
(S_D)^{2\varepsilon} \cap S_U = \emptyset, \qquad\qquad (S'_D)^{2\varepsilon} \cap S'_U = \emptyset,
$$

or all the following

$$
\begin{aligned}
(S'_D)^{\varepsilon} \cap (S \cap S'^{\varepsilon})_U \subseteq S_D, &\quad (S_D)^{\varepsilon} \cap (S' \cap S^{\varepsilon})_U \subseteq S'_D, \\
S_U \cap (S \cap S'^{\varepsilon})_D \subseteq (S'_U)^{\varepsilon}, &\quad S'_U \cap (S' \cap S^{\varepsilon})_D \subseteq (S_U)^{\varepsilon}.
\end{aligned}
$$

Let's assume without loss of generality that $(S_D)^{2\varepsilon} \cap S_U \neq \emptyset$ and $S \cap S^{2\varepsilon} \subseteq \mathcal{F}(S, S'^{\varepsilon}) \cap \mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon})$. Then we have that

$$
\begin{aligned}
S \cap S^{2\varepsilon} &= (S_U \cap S_D) \cap \left((S_U)^{2\varepsilon} \cap (S_D)^{2\varepsilon}\right), \\
&= \left(S_U \cap (S_U)^{2\varepsilon}\right) \cap \left(S_D \cap (S_D)^{2\varepsilon}\right), \\
&= S_U \cap (S_D)^{2\varepsilon} \neq \emptyset.
\end{aligned}
$$

Thus we must have $\mathcal{F}(S, S'^{\varepsilon}) \neq \emptyset$ and $\mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon}) \neq \emptyset$. Since $\mathcal{F}(S'^{\varepsilon}, S^{2\varepsilon}) = \mathcal{F}(S', S^{\varepsilon})^{\varepsilon}$ trivially we also have that $\mathcal{F}(S', S^{\varepsilon}) \neq \emptyset$. By proposition 10.15 we obtain from $\mathcal{F}(S, S'^{\varepsilon}) \neq \emptyset$ the inclusions

$$
(S'_D)^{\varepsilon} \cap (S \cap S'^{\varepsilon})_U \subseteq S, \qquad\qquad S_U \cap (S \cap S'^{\varepsilon})_D \subseteq S'^{\varepsilon},
$$

and from $\mathcal{F}(S, S'^{\varepsilon}) \neq \emptyset$ the inclusions

$$
(S_D)^{\varepsilon} \cap (S' \cap S^{\varepsilon})_U \subseteq S', \qquad\qquad S'_U \cap (S' \cap S^{\varepsilon})_D \subseteq S^{\varepsilon},
$$

thus proving the desired result. □

Corollary 10.17. *Given a real polyhedral group Q and indicator modules $k[S]$ and $k[S']$ with $S, S' \subseteq Q$ strongly connected such that*

$$\max\left(d_{sub}\left(S, S'\right), d_{sup}\left(S, S'\right)\right) \geq d_{comb}\left(S, S'\right),$$

then the inequality of proposition 10.16 is in fact an identity.

Proof. This is just an immediate consequence of propositions 10.16 and 10.9. □

We conclude with an example of how propositions 10.16 and 10.9 can be used in order to compute distances between indicator modules.

Remark 10.18. Proof of both proposition 10.16 and corollary 10.17 only require the sets S and S' to be such that the intersections between their translations are connected. We could in fact replace the statements of these results replacing the strongly connected condition with this weaker condition and the result would still be valid.

Example 10.19. Let Q be a real polyhedral group and get degrees $a, a', b, b' \in Q$ such that $a \preceq b$ and $a' \preceq b'$. Using these degrees define the sub-sets $S, S' \subseteq Q$ as

$$S = \left\{x \in Q \,:\, a \precneqq x \precneqq b\right\}, S' = \left\{x \in Q \,:\, a' \precneqq x \precneqq b'\right\}.$$

The sets S and S' are clearly open in the usual topology, result from intersections of upsets and downsets and are connected. Thus we can apply propositions 10.16 and 10.9 to the indicator modules $k[S]$ and $k[S]$.

Given now two degrees $p, q \in Q$ define the following functions

$$\underline{\boldsymbol{d}}_{\boldsymbol{Q}_+}(p, q) = \inf\left\{\varepsilon \geq 0 \,:\, p + u_\varepsilon \not\succneqq q\right\},$$
$$\overline{\boldsymbol{d}}_{\boldsymbol{Q}_+}(p, q) = \inf\left\{\varepsilon \geq 0 \,:\, p - u_\varepsilon \precneqq q\right\}.$$

To understand intuitively this definitions notice that, if $Q = \mathbb{R}^n$ with the usual order then, denoting by p_i and q_i the i-th coordinate of p and q respectively we have that.

$$\underline{d}_{Q_+}(p, q) = \min\left\{\max\left(0, p_i - q_i\right)\right\}_{i=1,\dots,n},$$
$$\overline{d}_{Q_+}(p, q) = \max\left\{\max\left(0, p_i - q_i\right)\right\}_{i=1,\dots,n},$$

moreover, when $p = q + u_\varepsilon$ for some $\varepsilon \in Q_B$ then we have that

$$\max\left(\underline{d}_{Q_+}(p, q), \underline{d}_{Q_+}(q, p)\right) = \max\left(\overline{d}_{Q_+}(p, q), \overline{d}_{Q_+}(q, p)\right) = d_{Q_+}(p, q)$$

where d_{Q_+} is as in definition 9.35 and the second identity is always true.

Lets now see how the functions len, d_{sub}, d_{sup} and d_{comb} can be written in terms of $\underline{d}_{Q_+}$ and $\overline{d}_{Q_+}$.

By construction, we have that

$$\operatorname{len}(S) = \underline{d}_{Q_+}(a, b), \qquad \operatorname{len}(S') = \underline{d}_{Q_+}(a', b').$$

Notice now that the inclusion $\left(S'_D\right)^{\varepsilon} \cap \left(S \cap S'^{\varepsilon}\right)_U \subseteq S_D$ can be re-written as

$$\{x \in Q \,:\, a' - \varepsilon, a \precneqq x \precneqq b' - \varepsilon\} \subseteq \{x \in Q \,:\, x \precneqq b\},$$

and equivalently inverting S and S'. Thus we have that

$$d_{sup}\left(S, S'\right) = \max\left(\min\left(\overline{d}_{Q_+}\left(b', b\right), \underline{d}_{Q_+}\left(a, b'\right)\right), \min\left(\overline{d}_{Q_+}\left(b, b'\right), \underline{d}_{Q_+}\left(a', b\right)\right)\right).$$

Proceeding analogously we find that

$$d_{sub}\left(S, S'\right) = \max\left(\min\left(\overline{d}_{Q_+}\left(a', a\right), \underline{d}_{Q_+}\left(a, b'\right)\right), \min\left(\overline{d}_{Q_+}\left(a, a'\right), \underline{d}_{Q_+}\left(a', b\right)\right)\right),$$

$$d_{comb}\left(S, S'\right) = \max\left(\min\left(\frac{1}{2}\underline{d}_{Q_+}\left(a, b\right), \max\left(\overline{d}_{Q_+}\left(a', a\right), \overline{d}_{Q_+}\left(b, b'\right)\right)\right),\right.$$
$$\left.\min\left(\frac{1}{2}\underline{d}_{Q_+}\left(a', b'\right), \max\left(\overline{d}_{Q_+}\left(a, a'\right), \overline{d}_{Q_+}\left(b', b\right)\right)\right)\right).$$

Finally, in the particular case where a, b, a' and b' are aligned in the direction u_1 it is easily proven from what we have checked that the upper and lower bounds given in propositions 10.9 and 10.16 respectively coincide and, from what we have already mentioned we have that

$$d_I\left(k\left[S\right], k\left[S'\right]\right) = \min\left(\max\left(d_{Q_+}\left(a, a'\right), d_{Q_+}\left(b, b'\right)\right), \frac{1}{2}\max\left(\underline{d}_{Q_+}\left(a, b\right), \underline{d}_{Q_+}\left(a', b'\right)\right)\right),$$

just as would happens in the 1-dimensional case.

11 Conclusions.

In this part of the document we have proven that non-zero tame poset modules admit a unique decomposition as direct sum of irreducible poset modules. However, unlike with the 1-dimensional case, we have also noticed that the irreducible components of poset modules do not necessarily have to look like the intervals of barcodes. That is the irreducible components do not necessarily have to be connected indicator modules (see example 10.1).

We have then proceeded with generalizing the categorical interleaving distance described in [2] for the one dimensional case to the case of modules over real and discrete polyhedral groups. The resulting distance between two poset modules M and N consisted on retrieving the minimum value of ε such that exist poset homomorphisms $F : M \to N^{\varepsilon}$ and $G : N \to M^{\varepsilon}$ such that $F^{\varepsilon} \circ G = T_N^{2\varepsilon}$ and $G^{\varepsilon} \circ F = T_M^{2\varepsilon}$ where L^{ε} is the ε-translate of a poset module of poset homomorphism (see definition 9.40 and remark 9.43 respectively) and $T_L^{2\varepsilon}$ is the 2ε-translation homomorphism (see definition 9.45). The given definition coincidentally resulted to be equal, in the cases of the poset being either $\mathbb{R}^n$ or $\mathbb{Z}^n$, to the interleaving distance given in [15] from which we can deduce that, in this special case, the given distance is in fact maximal among the distances satisfying the stability theorem (see [15, theorem 5.5]).

In fact, following the steps in [2], we proved that the defined interleaving distance satisfies a stability theorem (see theorem 9.63) that is somewhat stronger than the one in [15] since it does not require any condition on the functions from which the poset modules are retrieved.

We have then seen in example 9.66 retrieved from [3, example 5.1] that, even though computing the interleaving distance between the irreducible components of two modules can lead us to an upper bound of the interleaving distance of the modules themselves such upper bound needs not to be exact even for the simplest cases.

Finally we have concluded by giving multiple formulas for computing interleaving distance between indicator modules (see proposition 10.2 and corollary 10.4), upper bounds of such distance (see proposition 10.9) and a lower bound in the special case where the intersection of the corresponding indicator sets resulted in a connected set (see proposition 10.16 and remark 10.18). We deem these found identities and bounds to be much easier to work with than the raw definition of interleaving distance and we think that further work on this direction could help gain a better comprehension on how QR-codes (see [17] or the extension of this work [19]) are modified under small perturbations of the input data.

A Existence of fringe presentations.

In this appendix we will see how poset modules can be described via the use of simple upset or downset modules (see definition 4.6), more precisely we will introduce the concepts of indicator resolutions and presentations (see definitions A.4 and A.1 respectively) which are but specific types of flat or injective resolutions. The main goal of this appendix will be that of proving in theorem A.24 that a poset module is tame if and only if admits finite indicator resolutions and presentations (see definitions A.6 and A.3 respectively).

To achieve this goal we divide this appendix in two sub-sections. In sub-section A.1 we will rigorously define finite and non finite indicator resolutions and presentations while sub-section A.2 will be dedicated to proving the mentioned theorem A.24.

A.1 Modules from upsets and downsets.

Let us start by defining indicator presentation.

Definition A.1. Given a poset Q and a Q-module M we call

- **upset presentation** of M to any expression of M as the co-kernel of an homomorphism $F_1 \overset{\varphi}{\to} F_0$ where both F_0 and F_1 can be expressed as direct sums of upset modules $F_i = \bigoplus_{j_i} k\left[U_i^{j_i}\right]$ and every component $k\left[U_1^{j_1}\right] \to k\left[U_0^{j_0}\right]$ is a connected homomorphism (see definition 4.12).

- **downset co-presentation** of M to any expression of M as the kernel of an homomorphism $E^0 \overset{\varphi}{\to} E^1$ where both E^0 and E^1 can be expressed as direct sums of downset modules $E^i = \bigoplus_{j_i} k\left[D_i^{j_i}\right]$ and every component $k\left[D_0^{j_0}\right] \to k\left[D_1^{j_1}\right]$ is a connected homomorphism.

We will use the term **indicator presentation** to refer to either an upset presentation or a downset co-presentation.

Remark A.2. From the above definition follows that upset presentations and downset co-presentations can be thought as short exact sequences respectively of the form

$$\bigoplus_{j_1} k\left[U_1^{j_1}\right] \xrightarrow{\varphi} \bigoplus_{j_0} k\left[U_0^{j_0}\right] \twoheadrightarrow M,$$

and

$$M \hookrightarrow \bigoplus_{j_0} k\left[D_0^{j_0}\right] \xrightarrow{\varphi} \bigoplus_{j_1} k\left[D_1^{j_1}\right],$$

where the components of the homomorphisms φ are connected.

Associated to indicator presentations we have the following notion of finiteness.

Definition A.3. Given a poset Q and a Q-module M, then following notation of definition A.1, we say that an indicator presentation of M

- is **finite** if the number of summands of the poset modules F_i or E^i (depending on the upset or downset case respectively) is finite.
- **dominates a constant subdivision** (or **encoding**) of M if the subdivision (or encoding) is subordinate to each indicator module $k\left[U_i^{j_i}\right]$ or $k\left[D_i^{j_i}\right]$ (depending on the upset or downset case respectively).

The concepts of upset presentations and downset co-presentations can be extended to longer sequences that in fact constitute resolutions of the module .

Definition A.4. Given a poset Q and a Q-module M we call

- **upset resolution** of M to any complex F_* of Q-modules, each of which can be expressed as a direct sum of upset modules (i.e. $F_i = \bigoplus_{j_i} k\left[U_i^{j_i}\right]$), with differential $F_i \overset{\varphi_i}{\to} F_{i-1}$ such that every component $k\left[U_i^{j_i}\right] \to k\left[U_{i-1}^{j_{i-1}}\right]$ of φ_i is a connected homomorphism. Moreover F_* must satisfy that $H_0\left(F_*\right) \cong M$ while all other homologies are zero.
- **downset resolution** of M to any complex E^* of Q-modules, each of which can be expressed as a direct sum of downset modules (i.e. $E^i = \bigoplus_{j_i} k\left[D_i^{j_i}\right]$), with differential $D_i \overset{\varphi_i}{\to} F_{i+1}$ such that every component $k\left[D_i^{j_i}\right] \to k\left[D_{i+1}^{j_{i+1}}\right]$ of φ_i is a connected homomorphism. Moreover E^* must satisfy that $H^0\left(E^*\right) \cong M$ while all other co-homologies are zero.

We will use the term **indicator resolutions** to refer to either an upset or downset resolution.

Remark A.5. As with indicator presentations upset and downset resolutions can thought of as exact sequences respectively of the form

$$\cdots \to \bigoplus_{j_1} k\left[U_1^{j_1}\right] \overset{\varphi_1}{\to} \bigoplus_{j_0} k\left[U_0^{j_0}\right] \to M \to 0,$$

and

$$0 \to M \to \bigoplus_{i_0} k\left[D_0^{i_0}\right] \overset{\varphi_0}{\to} \bigoplus_{i_1} k\left[D_1^{i_1}\right] \to \cdots,$$

where the components of the homomorphisms φ_i are connected.

As with indicator presentations, indicator resolutions also have associated notions of finiteness. Unlike usual such notion is not based on finiteness of the exact sequence length but rather on finiteness of each direct sum in the sequence's elements.

Definition A.6. Given a poset Q and a Q-module M with an indicator resolution I, then following notation of definition A.1, we say that I

- is **finite** if the number of summands of the poset modules F_i or E^i (depending on the upset or downset case respectively) is finite.
- **dominates a constant subdivision** (or **encoding**) of M if the subdivision (or encoding) is subordinate to each indicator module $k\left[U_i^{j_i}\right]$ or $k\left[D_i^{j_i}\right]$ (depending on the upset or downset case respectively).

A.2 The syzygy theorem.

In this sub-section we will prove that every tame poset module admits both a finite upset resolution and a finite downset resolution. In fact we will prove in theorem A.24 that a poset module is tame if and only admits such finite indicator resolutions. In order to prove theorem A.24 we will divide this sub-section into 3 different sub-sub-sections each of them showing results needed for the proof.

First in sub-sub-section A.2.1 we will prove that existence of finite indicator presentations (and therefore also of finite indicator resolutions) of a module imply tameness of the module.

In sub-sub-section A.2.2 we will introduce the concept of Matlis duality in order to prove the inverse implication.

Finally, in sub-sub-section A.2.3 we will put together both results in order to state and prove theorem A.24.

A.2.1 Finite indicator presentation or resolution implies tameness.

First of all we need to prove that all modules appearing in the definition of finite indicator presentations and resolutions (besides the module M) are tame. More precisely we need the following lemma.

Lemma A.7. *Given a poset Q and finite set of indicator modules $\{k[S_i]\}_{i=1,\dots,n}$ of Q then the Q-module $M = \bigoplus_{i=1}^{n} k[S_i]$ is a tame module.*

Proof. The case $n = 1$ is covered by examples 5.5, 5.5 and 5.7 which give us a finite encoding of M and, therefore, by corollary 5.22, tell us that M is tame.

For the more general case $n \geq 1$ the results follows from tameness of each $k[S_i]$ and abelianess of the category of tame modules (see proposition 6.9) which tells us that direct sum of tame modules is also a tame module. □

Not only the modules forming a finite indicator presentation or resolution are tame but so are the component-wise connected homomorphism between such modules. more precisely we have the following lemma.

Lemma A.8. *Given a poset Q, two Q-modules $M = \bigoplus_{i=1}^{n} k\left[S_i^M\right]$ and $N = \bigoplus_{j=1}^{m} k\left[S_j^N\right]$ and a poset homomorphism $\varphi : M \to N$ such that φ is component-wise connected, as in definitions A.1 and A.6, then φ is a tame Q-module homomorphism (see definition 6.3).*

Proof. Let Υ denote the family of all the upsets that are either generated by a set in the family $\Phi = \left\{S_i^M\right\}_{i=1,\ldots,m} \cup \left\{S_j^N\right\}_{j=1,\ldots,n}$ or are the complementary of a downset co-generated by a set in such family (see theorem 5.19). Use now the family Υ in order to generate an uptight poset P_Υ (see theorem 5.19). Notice that P_Υ is finite since the family Υ is finite (see proof of lemma 5.20).

Notice now that, for indicator set $S \in \Phi$ and any degree $q \in S$ we have that q belongs to both the upset S_U generated by S and the downset S_D co-generated by S. Thus, by the characterization of the uptight regions in P_Υ (see lemma 5.12) we can conclude that the uptight region $R \in P_\Upsilon$ satisfying $q \in R$ also satisfies $R \subseteq S_U \cap S_D$. By characterization of indicator sets (see second half of proof of 4.3) we also have that $S = S_U \cap S_D$ and, therefore $R \subseteq S$. This implies that for every $R \in P_\Upsilon$ and every $S \in \Phi$ either $R \subseteq S$ or $R \cap S = \emptyset$.

For every $R \in P_\Upsilon$ we can now define the vector spaces H_R^M and H_R^N as

$$H_R^M := \bigoplus_{i=1}^{m} \bigoplus_{R \subseteq S_i^M} k,$$

$$H_R^N := \bigoplus_{i=1}^{n} \bigoplus_{R \subseteq S_i^N} k.$$

By construction we now have that, for every $R \in P_\Upsilon$ and every $q \in R$ then $H_R^M \cong M_q$ and $H_R^N \cong N_q$ with isomorphism given by the identity.

Since the homomorphism $\varphi : M \to N$ is, by hypothesis, component-wise connected then, by definition of connected homomorphism (see definition 4.12), given any degree $q \in Q$ the restriction φ_q of φ to degree q depends only on the sets $S \in \Phi$ such that $q \in S$. Thus, since for every $R \in P_\Upsilon$ all elements $q \in R$ belong by construction exactly to the same sets $S \in \Phi$ then we can conclude that the homomorphism $\varphi : M \to N$ induces an homomorphism

$$\tilde{\varphi} : H^M = \bigoplus_{R \in P_\Upsilon} H_R^M \to H^N = \bigoplus_{R \in P_\Upsilon} H_R^N$$

given by the compositions $H_R^M \to M_q \to N_q \to H_R^N$ for any $q \in R$ and every $R \in P_\Upsilon$.

Since the uptight poset P_Υ is, by construction a constant subdivision of Q subordinate to both M and N what we have just proven tells us in fact that φ is a tame poset homomorphism thus concluding the proof. $\square$

We can now prove the main result of this sub-sub-section.

Proposition A.9. *Given a poset Q and a Q-module M if M admits either any finite indicator presentation or a finite indicator resolution then M is tame.*

Proof. Since existence of a finite upset (conversely downset) resolution implies existence of a finite upset (conversely downset) presentation we just need to prove the result for finite upset (conversely downset) presentations.

If M admits a finite upset presentation $F_1 \xrightarrow{\varphi} F_0 \twoheadrightarrow M$ (conversely a finite downset co-presentation $M \hookrightarrow E_0 \xrightarrow{\psi} E_1$) then, by definition, we have that M is the co-kernel of φ (conversely the kernel of ψ). On the other hand, from finiteness of the presentation and lemmas A.7 and A.8 we have that F_1, F_0 and φ (conversely E_0, E_1 and ψ) are all tame. Thus, since the category of tame modules is abelian (see proposition 6.9) we can conclude that M is also tame thus proving the result. □

A.2.2 Tameness implies finite indicator presentations and resolutions.

In this sub-sub-section we will prove the result inverse to the one given by proposition A.9. That is we will prove any tame module admits an indicator presentation or resolution of any kind (upset or downset).

In order to prove this Miller (see [18, proposition 6.7] and [17, proposition 4.7]) bases its prove on the existence of minimal injective hulls and resolutions for finitely generated $\mathbb{Z}^n$-modules as proven by Goto and Watanabe (see [13]). He then uses the finite encoding of tame modules given by corollary 5.22 to think of the encoding poset as a sub-poset of $\mathbb{Z}^n$ and uses this to prove that the encoding module has a finite injective hull. He continues by proving that finite injective hulls can in fact be written as direct sums of downset modules to obtain a downset hull of the encoded module. Finally he uses the encoding function in order to pull back the found downset hull to a downset hull of the original tame module (see lemma A.10). With this trick he is able to build both downset co-presentations and downset resolutions of a given tame module. Then he uses Matlis duality (see definition A.17) in order to switch from downset co-presentations and downset resolution to upset presentations and upset resolutions.

In these notes we will however opt for a different, simpler approach. First of all we will prove in lemma A.10 that we can pullback both upset and downset modules from encodings. Then we will this in lemma A.11 to prove that for any tame module M exists a finite direct sum $E = \bigoplus_i k[U_i]$ of upset modules and a surjective poset morphism $\varphi : E \twoheadrightarrow M$. Then we will use this result in order to obtain both a finite upset presentation and a finite upset resolution of M. Finally we will use Matlis duality (see definition A.17) similarly to [17, 18] but in the opposite direction in order to prove that downset co-presentations and downset resolutions of tame modules also exist. We find this alternative prove to be much simpler than the one proposed in [17, 18], however this simplification comes at the cost of loosing minimality of the indicator presentations and resolutions. Since minimality is not needed in any of the successive proofs of [17, 18] we deem this loss to be acceptable.

Lemma A.10. *Given two posets P, Q and an upset (dually downset) module $k[S]$ of Q then $k[S]$ is constant on every fiber of a poset morphism $\pi : Q \to P$ if and only if S is the pullback of π of an upset (dually downset) P-module.*

Proof. If S is the pullback of π of an upset (conversely downset) in P then $k[S]$ is constant on every fiber of π by definition.

For the other implication notice that the upset (dually a downset) S must be the union of fibers of P by hypothesis. Then denoting by S_P the minimal upset (dually downset) of P containing $\pi(S)$ we have that $\pi^{-1}(S_P) = S$. This implies that $k[S] = \pi^*(k[S_P])$ which proves the remaining implication. $\square$

Lemma A.11. *Given a poset Q and a tame Q-module M then exists a tame surjective morphism $\varphi : E = \bigoplus_{i=1}^{n} k[U_i] \twoheadrightarrow M$ for some finite family of upsets $\{U_i\}_{i=1,\dots,n}$.*

Proof. Since M is tame then, by corollary 5.22 we know that exists a finite encoding of M given by a finite poset P, a P-finite P-module H and a poset morphism $\pi : Q \twoheadrightarrow P$. The idea is to prove the result for P and H and then use lemma A.10 in order to pullback the desired surjection to the module M.

Since H is P-finite then for every $p \in P$ we can take $B_p = \left\{e_1^p, \dots, e_{n_p}^p\right\}$ to be a finite basis of H_p. Now, for every basis element e_i^p we can define the upset $V_i^p \subseteq P$ as the upset generated by the degree p. Since P is finite and $|B_p|$ is finite for every $p \in P$, then we have that there are finitely many such upsets V_i^p. We can now define the P-module E' to be the finite direct sum of the upset modules obtained from such upsets

$$E' := \bigoplus_{p \in P} \bigoplus_{i=1,\dots,n_p} k\left[V_i^p\right].$$

If we now denote by 1_i^p the identity element on $k\left[V_i^p\right]_p$ we can define a poset homomorphism $\varphi' : E' \twoheadrightarrow H$ by setting $\varphi'(1_i^p) = e_i^p$, extending by linearity on each degree p and setting $\varphi'\left(\psi_{p,p'}^{k\left[V_i^p\right]}(1_i^p)\right) := \psi_{p,p'}^{H}(e_i^p)$ where $\psi_{p,p'}^{N}$ denotes the structure morphism $N_p \to N_{p'}$ of the P-module N. Since by construction the basis elements e_i^p generate H then we can conclude that φ' is in fact surjective.

Applying lemma A.10 we can now pull-back every upset module $k\left[V_i^p\right]$ of P to an upset module $k\left[U_i^p\right]$ of Q. Using these new upset modules we can define the Q-module

$$E := \bigoplus_{p \in P} \bigoplus_{i=1,\dots,n_p} k\left[U_i^p\right].$$

Finally we obtain the surjective poset homomorphism $\varphi : E \twoheadrightarrow M$ defined by the composition $E_q \leftrightarrow E_p \to H_p \leftrightarrow M_p$ for some $p = \pi(q)$. Notice that, by definition of φ we have that φ is in fact a tame poset morphism and not just a poset homomorphism. $\square$

Remark A.12. Notice how the proof of the previous lemma gives us something more that just the statement of it. In fact, by construction of E and $\varphi : E \twoheadrightarrow M$ we have that E dominates the constant subdivision of Q subordinate to P given by the pre-images of the poset morphism $\pi : Q \twoheadrightarrow P$.

The tame surjection given by the previous lemma can lead us to an upset presentation and an upset resolution as can be deduced from the following lemma.

Lemma A.13. *Given a poset Q, a finite family of upsets $\{U_i\}_{i=1,\dots,n}$, a tame Q-module M and a tame morphism $\varphi : E = \bigoplus_{i=1,\dots,n} k[U_i] \to M$ then exists another finite family of upsets $\{U_i'\}_{i=1,\dots,n'}$, a tame Q-module $E' = \bigoplus_{i=1,\dots,n} k[U_i']$ and a tame morphism $\varphi' : E' \to E$ such that $Im(\varphi') = Ker(\varphi)$ and φ' is component-wise connected as in definitions A.1 and A.4. Moreover if both E and M admit finite encodings in the same finite poset P and by the same poset morphism $\pi : Q \twoheadrightarrow P$ them E' also admits a finite encoding in the poset P and by the poset morphism π.*

Proof. By lemma A.7 we know that E is tame and since the category of tame modules is abelian (see proposition 6.9) then we have that $\mathrm{Ker}(\varphi)$ is also a tame module. By lemma A.11 we can therefore deduce the existence of the Q-module E' and the tame morphism φ'.

Component-wise connectedness of the morphism φ' follows directly from construction of the morphism φ' in lemma A.11 which is completely determined by the image of the identity element in the generator degree of each upset.

The second part of the statement can be obtained with the same reasoning performed in lemma A.11. That is restricting to the finite case and then pulling back the desired Q-module E' which, by remark A.12 admits a finite encoding in the poset P and by the same poset morphism π. □

Proposition A.14. *Given a tame poset module M then M admits a finite upset presentation and a finite upset resolution. Moreover if M dominates a given encoding the obtained presentation and resolution can be taken so that they dominate that same encoding.*

Proof. The result follows immediately from combining lemmas A.11 and A.14. □

With proposition A.14 we have obtained half of the results we wanted to obtain in this sub-sub-section. Now we only need to prove the same result for downset co-presentations and downset resolutions in order to complete this sub-sub-section's goal. In order to do this we first need to introduce matlis duality. In order to define matlis dual of poset module however we first need to define the inverse of a poset..

Definition A.15. Given a poset P we call the **inverse poset** of P to the poset

$$\boldsymbol{-P} = \{-p : p \in P\},$$

with the order $-p \preceq -q \Leftrightarrow q \preceq p$. For every $p \in P$ we denote by $\boldsymbol{-p}$ the image of p by the natural set isomorphism $P \to -P$.

Remark A.16. For any type of poset P we have that$-(-P) \cong P$. Because of this we will write $-(-P) = P$.

We can now define the matlis dual of a poset module.

Definition A.17. Given a poset Q and a Q-module M we define the **matlis dual** of M as the $-Q$-module $M^\vee$ whose vector space at any given degree $q \in Q$ is defined as

$$\left(M^\vee\right)_{-q} = \mathrm{Hom}_k\left(M_q,\, k\right),$$

where a basis element $e_i \in M_q$ is sent to the homomorphism that sends all basis elements to 0 except for that one that is sent to 1. In this $-Q$-module the structure homomorphisms $\psi_{-q,-p}\left(M^\vee\right)_{-q} \to \left(M^\vee\right)_{-p}$ are inherited from the structure homomorphisms $\psi_{p,q} : M_p \to M_q$ for any $p, q \in Q$ with $p \preceq q$ by the pullback. That is for every $\varphi \in \left(M^\vee\right)_{-q}$ and every $x \in M_p$ we have

$$\psi_{-q,-p}\left(\varphi\right)\left(x\right) = \varphi\left(\psi_{p,q}\left(x\right)\right).$$

Matlis duality is important to us because of the three following 3 interesting properties.

Remark A.18. Given a poset Q and a Q-module M since we have the canonical isomorphisms

$$\begin{aligned} Q &\cong -\left(-Q\right), \\ M_q &\cong \mathrm{Hom}_k\left(\mathrm{Hom}_k\left(M_q,\, k\right),\, k\right), \end{aligned}$$

for any finite-dimensional vector space M_q and the structure homomorphisms commute with applying matlis duality by definition then we can conclude that for any Q-finite Q-module M we have the Q-module isomorphism $M \cong \left(M^\vee\right)^\vee$.

Remark A.19. Since taking dual is a contravariant exact functor in the category of finite vector spaces and matlis duality commutes with structure morphisms by definition then matlis duality is a contravariant exact functor from the category of Q-finite Q-modules (conversely tame Q-modules) to the category of Q-finite $-Q$-modules or (conversely tame $-Q$-modules) .

Remark A.20. Matlis duality sends downset modules to upset modules and vice-versa as follows immediately from definition. Moreover, since

$$\mathrm{Hom}_k\left(V \oplus W,\, k\right) \cong \mathrm{Hom}_k\left(V,\, k\right) \oplus \mathrm{Hom}_k\left(W,\, k\right)$$

then we can conclude that matlis duality sends direct sums of upset modules to direct sums of downset modules.

Using matlis duality and, in particular, remarks A.18, A.19, A.20 we can now prove the results analogue to lemmas A.11 and A.13 but for downset co-presentations and resolutions.

Lemma A.21. *Given a poset Q and a tame Q-module M then exists a tame injective morphism $\varphi : M \hookrightarrow F = \bigoplus_{i=1}^n k\left[D_i\right]$ for some finite set of downsets $\left\{D_i\right\}_{i=1,\ldots,n}$.*

Proof. Applying lemma A.11 to the matlis dual $M^\vee$ we obtain a tame surjective homomorphism of the form

$$\varphi^\vee : F^\vee = \bigoplus_i k\left[U_i\right] \twoheadrightarrow M^\vee$$

for some upsets $U_i \subseteq -Q$. Now apply again matlis duality to $M^\vee$, $F^\vee$ and $\varphi^\vee$ we obtain that $(F^\vee)^\vee \cong F = \bigoplus_i k[D_i]$ for downsets D_i matlis dual to the upsets U_i (as follows from remark A.20), that $(M^\vee)^\vee \cong M$ (as noted in remark A.18) and the injective tame morphism $(\varphi^\vee)^\vee : (M^\vee)^\vee \hookrightarrow (F^\vee)^\vee$ (as follows from contravariant exactness stated in remark A.19). Combining these isomorphisms, we obtain the desired injective tame morphism $\varphi : M \hookrightarrow F = \bigoplus_i k[D_i]$. ☐

Lemma A.22. *Given a poset Q, a finite family of downsets $\{D_i\}_{i=1,\dots,n}$, a tame Q-module M and a tame morphism $\varphi : M \to F = \bigoplus_{i=1,\dots,n} k[F_i] \to M$ then exists another finite family of downsets $\{D'_i\}_{i=1,\dots,n'}$, a tame Q-module $F' = \bigoplus_{i=1,\dots,n} k[D'_i]$ and a tame morphism $\varphi' : F \to F'$ such that $Im(\varphi) = Ker(\varphi')$ and φ' is component-wise connected as in definitions A.1 and A.4. Moreover if both F and M admit finite encodings in the same finite poset P and by the same poset morphism $\pi : Q \twoheadrightarrow P$ them F' also admits a finite encoding in the poset P and by the morphism π.*

Proof. Similarly to lemma A.21 this lemma follows from applying lemma A.13 to the matlis dual of both M and F and then applying matlis duality again in order to obtain Q-modules isomorphic to the original ones. Component-wise connectedness and the second part of the statement also follow from matlis duality and lemma A.13. ☐

Continuing with the analogy we can now use lemmas lemmas A.21 and A.22 in order to prove a result analogue to proposition A.14.

Proposition A.23. *Given a tame poset module M then M admits a finite downset co-presentation and a finite downset resolution. Moreover if M dominates a given encoding the obtained co-presentation and resolution can be taken so that they dominate that same encoding.*

Proof. The result follows immediately from combining lemmas A.21 and A.22 ☐

A.2.3 Tameness equivalent to finite indicator presentations and resolutions.

In this brief final sub-sub-section we will put together the results shown in sub-sub-sections A.2.1 and A.2.2 in order to state and prove the main result we wanted to prove in this appendix a syzygy theorem for tame poset modules.

Theorem A.24. *(syzygy) Given a poset Q then, for any Q-finite Q-module M tameness of M is equivalent to any of the following equivalent conditions:*

1. *There is a finite constant subdivision of Q subordinate to M.*
2. *There is a finite poset encoding subordinate to M.*
3. *M admits a finite upset presentation.*
4. *M admits a finite downset co-presentation.*

5. *M admits a finite upset resolution.*

6. *M admits a finite downset resolution.*

Proof. Since M is Q-finite then, by definition of tameness (see def 3.9), tameness of M is in fact equivalent to item 1.

On the other hand item 2 is equivalent to tameness of M by corollary 5.22.

Items 5 and 6 imply item 3 and 4 respectively by simply cutting on the second morphism. On the other hand items 3 and 4 imply tameness by proposition A.9. Finally tameness implies items 3 and 5 by proposition A.14 and items 4 and 6 by proposition A.23. $\square$

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