MDTS4411
Bayesian Data Analysis

Ideas of Objective and Subjective probabilities

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- How This Leads to Bayesian Data Analysis

Self Reading Material

After studying this Note, students should be able to:

Understand how the concept of probability evolved historically.

- Distinguish clearly between objective and subjective interpretations of probability.
- Recognize the philosophical foundations of frequentist and Bayesian statistics.
- Appreciate probability as a measure of uncertainty, not just randomness.
- Understand why Bayesian data analysis treats probability as a degree of belief.
- Prepare conceptually for formal Bayesian inference methods covered later in the course.

Jakob Bernoulli

Pierre-Simon Laplace

The Deterministic Worldview (17th–18th Century)

In the 17th and 18th centuries, most mathematicians and philosophers believed the world to be fundamentally deterministic.

According to this view, every event has a precise cause, and if all causes were known, the outcome could be predicted with certainty.

Jakob Bernoulli and Pierre-Simon Laplace viewed probability not as a property of the world, but as a measure of human ignorance.

Randomness was not considered real; it merely reflected incomplete knowledge.

Laplace: Probability quantifies what we do not know about a perfectly predictable system.

pov - A fair coin is said to have a probability of 0.5 of landing heads. This does not mean the coin behaves randomly. Rather, it reflects our inability to account for all physical factors involved in the toss.

Hence, Probability was framed as a mental construct, not physical randomness!

Philosophical Clarification of Objective vs Subjective (Late 18th Century)

Immanuel Kant (1724–1804) was a German philosopher and one of the most influential thinkers in the history of Western philosophy. He fundamentally reshaped how we understand knowledge, reality, and human reasoning.

Immanuel Kant did not develop a theory of probability, but his philosophical work clarified the meanings of the terms *objective* and *subjective*, which later became crucial for probability theory.

According to Kant:
•*Objective* refers to structures or rules that are universal and independent of individual minds.
•*Subjective* refers to personal sensations, beliefs, or mental states.

During the early 19th century, these meanings entered common scientific and philosophical usage. "Objective" came to mean facts about the external world, while "subjective" came to mean individual judgment or belief.

Summary: The objective vs subjective perspective is not originally statistical.
Terminology is grounded philosophically.

The 19th-Century Turning Point — Two Meanings of Probability Emerge

Between 1837 and 1842, several thinkers across Europe independently reached a striking conclusion: a single concept of probability was insufficient.

They argued that probability refers to two fundamentally different things.

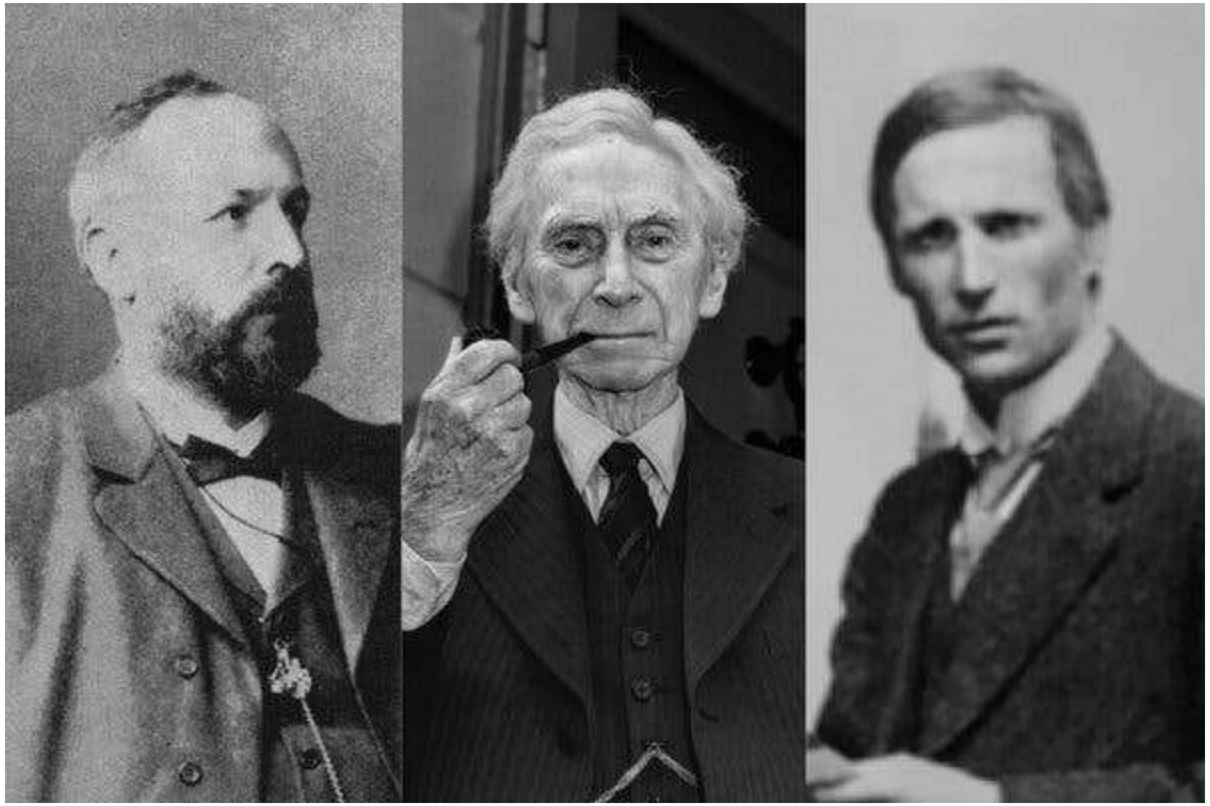
Objective Probability (Chance)

Objective probability refers to stable, long-run regularities observed in the world. It is based on repeated observations and is independent of who observes them.

Examples:

- Crime rates over many years
- Proportion of male and female births
- Long-run frequency of heads in repeated coin tosses

These probabilities are testable, measurable, and accessible to anyone who follows the same procedure.



Bruno de Finetti Bertrand Russell Frank P. Ramsey

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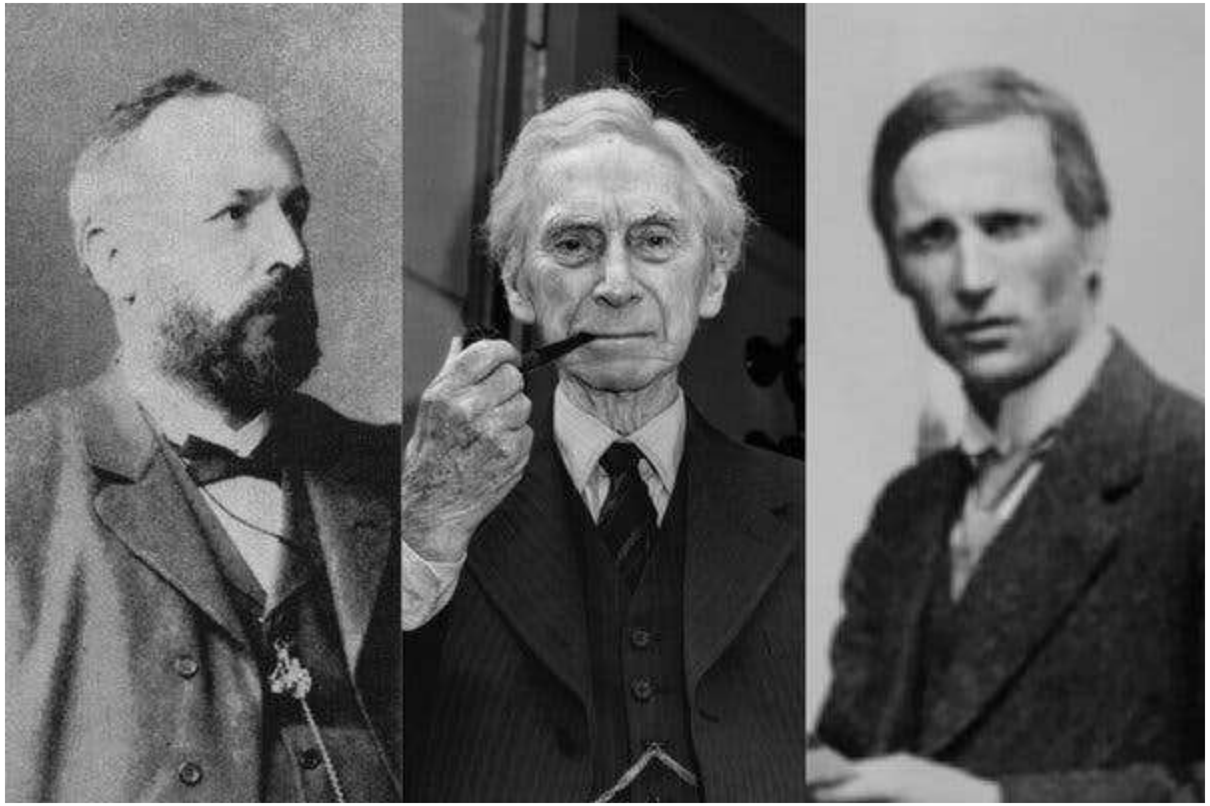
2. Subjective Probability (Belief)

Subjective probability refers to an individual's degree of belief about an uncertain event, given available information.

Examples:

- Probability that it will rain tomorrow
- A doctor's belief that a patient has a disease
- Confidence in an exam answer

These probabilities can differ between individuals and change when new information arrives.



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Refinements by Poisson, Bolzano, Ramsey, and De Finetti

Poisson (1837): Poisson distinguished between chance and probability.

He argued that *chance* refers to objective regularities in nature, while *probability* represents our belief about those chances, especially in legal and social contexts.
Example: A coin's "chance" of heads might be 0.51 due to tiny flaws (objective). But if you don't know, your "probability" is 0.5 (subjective ignorance).

Siméon Denis Poisson

Bolzano (1837): Bolzano argued that probabilities apply to propositions, not events. A probability statement is objective if it corresponds to true facts, and subjective if it reflects mistaken or incomplete knowledge. Example: "It will rain" is objective if facts back it; subjective if your belief is wrong.

Bernard Bolzano

Ramsey and De Finetti (20th century): They formalized subjective probability as a coherent system of beliefs, governed by consistency rules.
De Finetti famously stated: “Probability does not exist.”

By this he meant that probability is not a physical property of the world, but a logical representation of uncertainty in the mind of an observer.

Why Does This Matter to us Today?

Today we use both concepts every day!

• Objective → Weather apps, sports stats, science experiments

• Subjective → Doctors guessing how a medicine will work, or we betting on a game

From Probability to Statistical Inference

These two interpretations of probability gave rise to two major schools of statistics:

Frequentist Statistics

Treats probability as an objective property of repeated experiments.

Bayesian Statistics

Treats probability as a degree of belief, updated using data.

Both approaches aim to reason under uncertainty, but they differ in how probability itself is understood.

Certainty, Uncertainty, and Probability: The Bayesian Way of Thinking

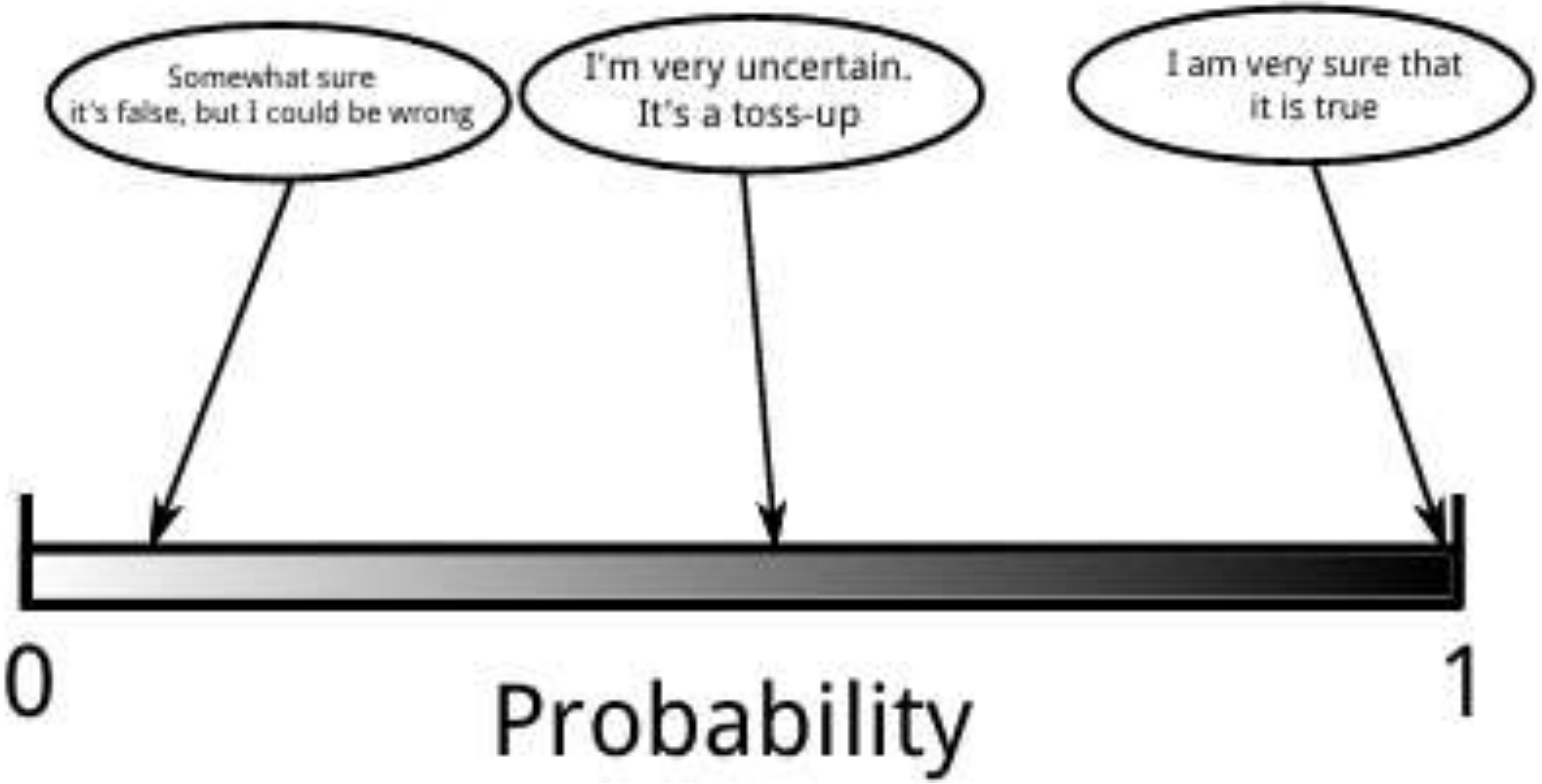
Think about everyday situations: sometimes you know something for sure, and sometimes you’re just guessing with confidence.

In a quiz competition you’re asked a question you vaguely remember, you might say: “I’m about 70% sure the answer is C.”

This is not a fact about the world, it’s a fact about your mind.

Someone else might be 100% sure because they studied that exact topic last night.

Another person might say 25% because they have no clue and are simply choosing among four options.



Probability can be used to describe degrees of certainty, or how plausible some statement is. 0 and 1 are the two extremes of the scale and correspond to complete certainty. However, probabilities are not static quantities. When you get more information, your probabilities can change.

Objective vs Subjective Probability: *"The double sense of probability"*

Objective Probability:

- A measure of the possibility of things independently of the knowledge we have of them.

- **Classical definition**
- **Frequency definition**

- P(E): Same for all

Subjective Probability:

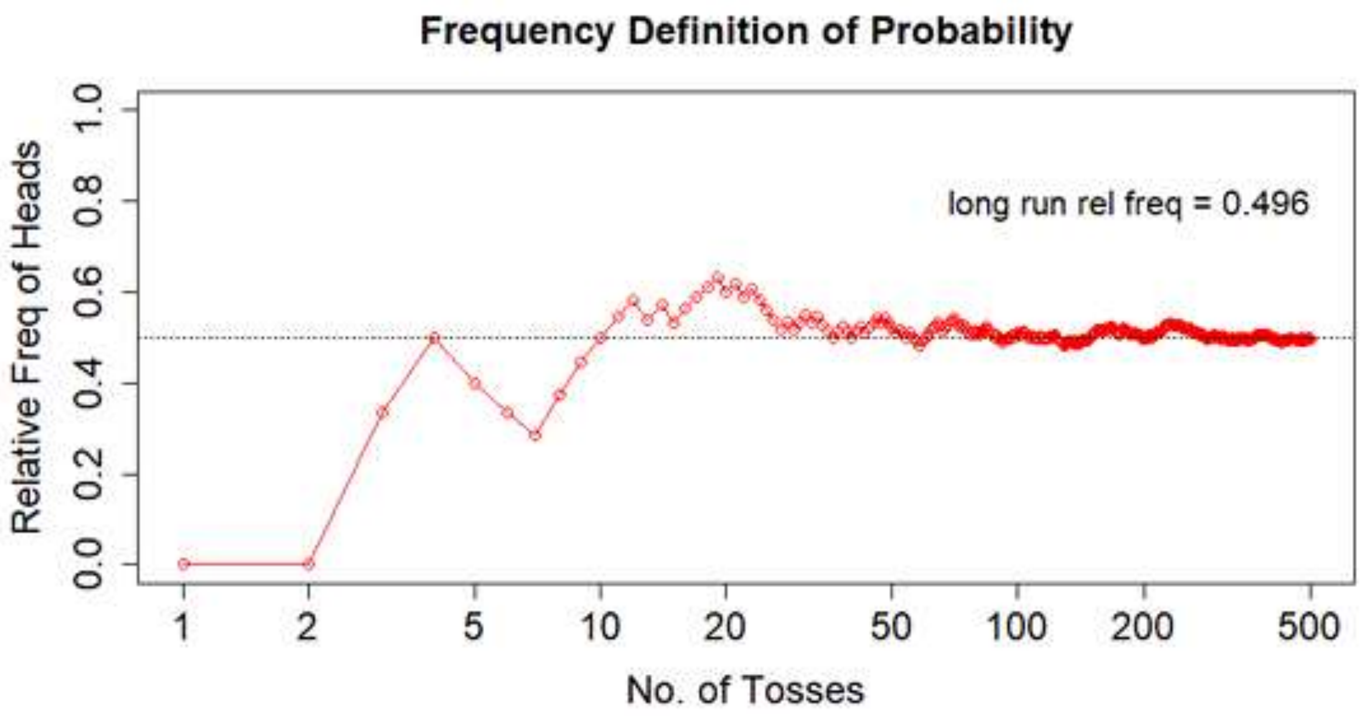
- A certain measure of our knowledge
- Varies from person to person
- based on individual knowledge, understanding, and experience of the likelihood of an event.
- P(E) is not an intrinsic characteristic of the event but depends on the *state of information* available to whoever evaluates.

Frequency Notion of Probability

Frequentists interpret probability as the limiting relative frequency of the occurrence of an event as the number of trials goes to infinity.

Example: The probability of a head for a fair coin is 0.5 means the relative frequency of heads should be approximate 0.5 if we flip the coin many times.

This definition is intuitively reasonable, but makes several technical assumptions about the existence of the limit and the fact that the ratio converges to a single value for all possible sequences of experimental outcomes.



Consider the following questions:

1. Suppose Real Madrid plays FC Barcelona in soccer tomorrow: what is the probability of Real Madrid winning?

2. What is the probability of rain tomorrow?

3. What is the probability that Real Madrid wins, if it rains tomorrow?

4. What is the probability that a specified rocket launch will fail?

5. Probability of having a specific disease.

Coherence or Consistency Condition for Subjective Probability

Subjective probabilities must obey certain "coherence" (consistency) conditions in order to be workable.

For example, if you believe that the probability that chances of raining tomorrow is 70%, then to be consistent you cannot believe that the chances of not raining tomorrow is 70%.

It is easy to fall into subjective probabilities that are not coherent.

The subjective perspective of probability fits well with Bayesian statistics, which are an alternative to the more common frequentist statistical methods.

Summary

- Classical and frequentist definitions give useful rules for evaluating probability. **<u>They do not define the concept</u>**.

- **The axiomatic approach doesn't define what the probability is and how to evaluate it**:

  it just says probability is just any real number which satisfies the axioms. It is easy to demonstrate that the probabilities evaluated using the classical and the frequentist approaches do in fact satisfy the axioms.

- The subjective approach to probability, together with the coherence requirement, defines what probability is and provides the rules which its evaluation must obey; these rules turn out to be the same as the axioms.

- Axiomatic approach is a unifying perspective. The coherence conditions needed for subjective probability can be proved to hold for the classical and empirical definitions. The axiomatic perspective codifies these coherence conditions, so can be used with any of the above three perspectives.

To ensure coherence, a subjective probability function P(·) must satisfy the following conditions:

1. Non-negativity - P(A)≥0 for every event
2. Normalization - P(Ω)=1
3. Finite additivity **-** For any finite collection of mutually exclusive events $A_1, A_2, \dots, A_n$

$$P(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$$

These conditions ensure that subjective probabilities are internally consistent and represent rational degrees of belief. Importantly, coherence requires finite additivity, but does not, by itself, require countable additivity.

Subjective Probability and Bayes' Theorem as Belief Updating

In the subjective interpretation, probability quantifies an agent's degree of belief given available information.
When new data are observed, rationality demands that these beliefs be updated consistently. Bayes' theorem provides the mathematical rule for this updating process.

The updated belief (posterior) is given by: $P(\theta|D) = \frac{P(D|\theta)\, P(\theta)}{\mathrm{P(D)}}$

In this framework: The prior $P(\theta)$ represents subjective belief before observing data. The likelihood $P(D|\theta)$ encodes how the data relate to the unknown quantity. The posterior $P(\theta|D)$ represents updated belief after incorporating the evidence. Bayesian inference therefore treats learning from data as a coherent process of belief revision, governed by the axioms of subjective probability and implemented through Bayes' theorem.

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