ePG Pathshala

Subject: Information Technology
Paper : Numerical Methods
Module 6: Successive approximation method for finding roots of an equation f(x) = 0

1.1 Introduction

In our previous modules studied so far, we found that though closed methods (bracketing methods) namely Bisection Method and method of False Position guarantee convergence but are slow methods. On the other hand, non-bracketing methods are fast but do not guarantee convergence.

Today in this module, we shall explore one more method for finding roots of an equation $f(x)=0$, called Successive Approximation Method, or fixed point iteration method.

Let us first understand, what is meant by a fixed point of a function? The concept of fixed point would be applied to root finding problem and if properly applied after ensuring convergence, the method shows rapid convergence. So, what is a fixed point? As the name suggests, a fixed point of the function g is any real number, x_0 for which $g(x_0)=x_0$, that is, it is the argument where the value of function is same as the argument. In other words, argument remains fixed.

For example, let $g(x)=x^2$, now, to find fixed points, we need to solve $g(x)=x$

$\therefore x^2 = x$

$\therefore x(x-1)=0$

$\therefore x = 0 \text{ or } 1$

Thus, function $g(x)=x^2$, has two fixed points 0 and 1.

Consider another example, $g(x)=cx(1-x)$, where c is some nonzero constant, then, in order to find fixed points of $g(x)$, we solve for $x=g(x)$

$\therefore cx(1-x)=x$

$\therefore x=0$ or $c(1-x)=1$

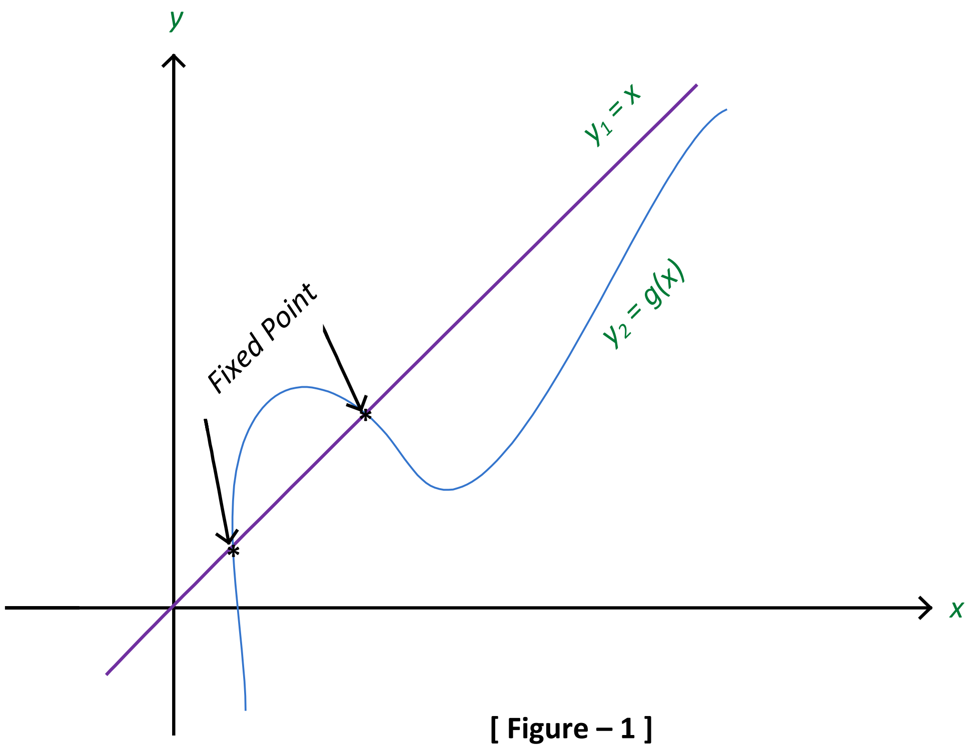
$c(1-x)=1$ gives, $1-x=\frac{1}{c}$

$$\therefore x = 1 - \frac{1}{c}$$

$$= \frac{(c-1)}{c}$$

So, the function $g(x) = cx(1-x)$ has two fixed points, namely 0 and (c-1)/c.

Fixed point can be found graphically also.



[Figure – 1]

If we plot, $y_1 = x$ and $y_2 = g(x)$ on the same set of coordinate axis, then at the point of intersection of both curves $(x, y_1) = (x, y_2)$

$$\Rightarrow (x, x) = (x, g(x))$$

$$\Rightarrow x = g(x)$$

So the point of intersection of $y = x$ and $y = g(x)$ gives the fixed point of $g(x)$. Examples considered x^2 and $cx(1-x)$ are found to have two fixed points. But it is not necessary for a function to have a fixed point.

$x^2 + 1$, e^x $\ln x$ serve as examples of functions having no fixed points.

1.2 Iterative Process

Fixed point iteration method is very simple. It consists of expressing the given equation $f(x)=0$ in the form of $x=g(x)$. Naturally, root of $f(x)=0$, satisfies $x=g(x)$, that is, it is the fixed point of $g(x)$.

This is how, root finding problem gets converted to fixed point finding problem. Iterative scheme at the face of it is very simple. Make an initial guess x_0. The sequence of approximations is generated by the formula:

$x_{n+1}=g(x_n),\ n=0,1,2,...$

The function g is called iteration function. First question, which arises in one's mind is: Is it always feasible to algebraically transform $f(x)=0$ to $x=g(x)$? Answer is yes. One of the ways is to write $f(x)=0$ is as $x=f(x)+x$, obtained by adding x on both the sides of $f(x)=0$. So, $g(x)=f(x)+x$. Really speaking, it is likely, that there would be many ways of transforming $f(x)=0$ into $x=g(x)$ and each of them could behave differently. Let us understand this with the same example that we considered in our previous modules.

The first equation, whose roots we would like to find is $x^3+x-1=0$

This equation can be expressed in the form $x=g(x)$ in different ways:

Transposing x^3-1 on the other side gives, $x=1-x^3$, whereas adding x on both sides in $f(x)=0$ gives $x=x^3+2x-1$. Next, transposing $x-1$ on the other side of equality sign gives $x^3=1-x$ and taking cube root, we obtain, $x=(1-x)^{\frac{1}{3}}$. Finally, write x^3+x-1 as $x(1+x^2)=1$ giving $x=\frac{1}{(1+x^2)}$. Let us apply our iterative scheme to each of g's and observe the convergence behavior. Results are displayed in the following table.

Table

X	g(X)=1- X^3
0.000000	1.000000
1.000000	0.000000
0.000000	1.000000
1.000000	0.000000
0.000000	1.000000
1.000000	0.000000
0.000000	1.000000
1.000000	0.000000
0.000000	1.000000
1.000000	0.000000

X	g(X)= X^3 +2 X – 1
1.000000	2.000000
2.000000	11.000000
11.000000	1352.000000
1352.000000	2471329024.000000

X	g(X)=$(1\text{-}X)^{1/3}$
0.000000	1.000000
1.000000	0.000000
0.000000	1.000000
1.000000	0.000000
0.000000	1.000000
1.000000	0.000000
0.000000	1.000000
1.000000	0.000000
0.000000	1.000000
1.000000	0.000000

We observe from above tables that sequences of approximations behave differently with different g's. For $g(x) = 1 - x^3$ and $g(x) = (1-x)^{\frac{1}{3}}$, values oscillate between 0 and 1 whereas for $g(x) = x^3 + 2x - 1$, values keep on increasing without bound. Unfortunately, in none of these choices of $g(x)$, method converges.

X	g(X)=1/(1+X^2)
0.000000	1.000000
1.000000	0.500000
0.500000	0.800000
0.800000	0.609756
0.609756	0.728968
0.728968	0.653000
0.653000	0.701061
0.701061	0.670472
0.670472	0.689878
0.689878	0.677538
0.677538	0.685374
0.685374	0.680394
0.680394	0.683557
0.683557	0.681547
0.681547	0.682824
0.682824	0.682013
0.682013	0.682528

X	g(X)=1/(1+X^2)
0.682528	0.682201
0.682201	0.682409
0.682409	0.682276
0.682276	0.682360
0.682360	0.682307
0.682307	0.682341
0.682341	0.682319
0.682319	0.682333
0.682333	0.682324
0.682324	0.682330
0.682330	0.682326
0.682326	0.682329
0.682329	0.682327
0.682327	0.682328
0.682328	0.682328
0.682328	0.682328

In case of $g(x) = \frac{1}{(1+x^2)}$, it is evident from the above table that the method converges to the desired root 0.682328 but convergence rate is very slow. This suggests that one needs to choose $g(x)$ in an appropriate way, so that the sequence of approximations converges and converges fast.

1.3 Convergence of Successive Approximation

So, looking at the above discussion, we need to address two issues namely:

(i) Does the transformed equation $x = g(x)$ has a fixed point?

(ii) Out of different options for $g(x)$, could there be some indicator about appropriate choice of $g(x)$ to ensure convergence.

Answer to first question is found in the following theorem:

Theorem 1:

(i) If g is continuous on the closed interval $[a,b]$ with $g:[a,b] \to [a,b]$, then g has a fixed point $p \in [a,b]$.

(ii) Moreover, if g is differentiable on the open interval (a,b) and there exists a positive constant $k<1$ such that $|g'(x)| \le k < 1$ for all $x \in (a,b)$ then the fixed point is unique.

Proof of existence of a fixed point, which is part (i) is based on intermediate value theorem whereas proof of part (ii) is based on mean value theorem. Now, let us try to answer, second question about proper choice of $g(x)$. Also, whether initial choice x_0 can be chosen, that ensures the convergence of the sequence $x_n = g(x_{n-1})$, $\mathrm{n} = 1,2,\ldots$ to the root c of f.

Here is a proven theorem for the questions raised:

Theorem 2:

Let g be continuous on the closed interval $[a,b]$ with $g:[a,b] \to [a,b]$. If g is differentiable on the open interval (a,b) and there exists a positive constant $k<1$ such that $|g'(x)| \le k < 1$ for all $x \in (a,b)$ then

(i) The sequence of approximations $\{x_n\}$ generated by $x_n = g(x_{n-1})$ converges to fixed point $c \in [a,b]$

(ii) $|x_n - x_{n-1}| \le k^n \max\{x_0 - a, b - x_0\}$ and

(iii) $|x_n - c| \le \dfrac{k^n}{1-k}|x_1 - x_0|$

This condition of the theorem is sufficient for convergence, but is not necessary. There are instances, when the sufficient condition does not hold, but the method converges. But, more important, from the (ii) and (iii) of the theorem, it is evident that the smaller the k, more rapid would be the convergence as $k^n \to 0$ faster. With $k = \frac{1}{2}$, speed of convergence roughly would be same as that of Bisection Method.

1.4 Stopping Criterion

There is no difference in stopping criterion as compared to earlier methods. Any of the stopping criteria may be applied. So, one may stop if $|x_n - x_{n-1}| < \in - TOL$

1.5 Algorithm

Step 1	Input: $f(x)$ the given function , $g(x)$ x_0 the initial guess, $\in$: the error tolerance (X – TOL) N : the maximum number of iterations
Step 2	$k = 0$
Step 3	Compute $x_{k+1} = g(x_k)$
Step 4	If $|x_{k+1} - x_k| < \in$, Output: Estimate of the root is x_{k+1}, exit
Step 5	Else: $k = k+1$, if $k \le N$, go to step 3
Step 6	* Else Output: Does not converge in N iterations

$*$If $g(x)$ is selected, such that $|g'(x)| \le 1$ on the interval containing x_0, the method converges. If no such $g(x)$ is found, still the method may converge (not guaranteed) in that case upper limit on number of iterations is desirable.

1.6 Illustrations

Example 1:

Consider the equation $x^4 - x - 10 = 0$

Here $f(x) = x^4 - x - 10$

$f(0) = -10,\ f(1) = -10,\ f(2) = 4$

So the root lies between 1 and 2, as $f(x)$ changes its sign from –ve to +ve.

Therefore our choice for interval $[a,b]$ is $[1,2]$.

There are three obvious ways of transforming $f(x) = 0$ to $x = g(x)$

(i) $x = x^4 - 10\ :\ g(x) = x^4 - 10$

(ii) $x^4 = x + 10\ \therefore x = (x+10)^{1/4}:\ g(x) = (x+10)^{1/4}$

(iii) $x^4 - x = 10$,

$\therefore x(x^3 - 1) = 10$, which gives $x = \dfrac{10}{x^3 - 1}:\ g(x) = \dfrac{10}{x^3 - 1}$

First consider, $x = x^4 - 10$; $g'(x) = 4x^3 > 4$ on $(1,2)$

$\therefore g(x)$ does not satisfy the hypothesis of Theorem 2.

For $g(x) = (x+10)^{\frac{1}{4}}$: $g'(x) = \dfrac{1}{4}(x+10)^{-\frac{3}{4}}$, for $1 < x < 2$

Now $x + 10 > 11$ for $x > 1$

$\therefore (x+10)^{\frac{3}{4}} > 11^{\frac{3}{4}}$, which gives $\dfrac{1}{4(x+10)^{\frac{3}{4}}} < \dfrac{1}{4*(11)^{\frac{3}{4}}} = \dfrac{1}{4*(1331)^{\frac{1}{4}}} < 1$

For $g(x)$: $g(x) = 10(x^3 - 1)^{-1}$

$$g'(x) = 10(-1)(x^3 - 1)^{-2} * 3x^2$$

$$= -10\frac{3x^2}{(x^3 - 1)^2}$$

This expression for $g'(x)$ does not guarantee $|g'(x)| \le 1$

Now, let us generate the sequence of approximations for all above three cases.

By Theorem 2, convergence is guaranteed for $g(x) = (x+10)^{\frac{1}{4}}$ only.

$f(x) = x^4 - x - 10 \quad , \quad x_0 \in (1,2)$

X	g(X)=10/(X^3-1)
1.500000	4.210526
4.210526	0.135784
0.135784	-10.025098
-10.025098	-0.009915
-0.009915	-9.999990
-9.999990	-0.009990
-0.009990	1.000000
1.000000	-0.009990
-0.009990	-9.999990
-9.999990	-0.009990

X	g(X)=X^4 – 10
1.500000	-4.937500
-4.937500	584.331055
584.331055	1352.000000
1352.000000	16583170048.000000

From above tables it is clear that for $g(x) = \dfrac{10}{x^3 - 1}$, values oscillate and the method does not converge, whereas for $g(x) = x^4 - 10$, the sequence of approximations rapidly increases without bound. Now let us generate the sequence of approximations for the function $g(x) = (x+10)^{1/4}$

X	g(X)= (X+10)$^{1/4}$
1.500000	1.841512
1.841512	1.855034
1.855034	1.855563
1.855563	1.855584
1.855584	1.855585
1.855585	1.855585

X	g(X)= (X+10)$^{1/4}$
2.000000	1.861210
1.861210	1.855805
1.855805	1.855593
1.855593	1.855585
1.855585	1.855585

X	g(X)= (X+10)$^{1/4}$
1.000000	1.821160
1.821160	1.854236
1.854236	1.855532
1.855532	1.855582
1.855582	1.855584
1.855584	1.855585

X	g(X)= (X+10)$^{1/4}$
4.000000	1.934336
1.934336	1.858658
1.858658	1.855705
1.855705	1.855589
1.855589	1.855585
1.855585	1.855585

As expected the sequence of approximations are converting to the root 1.855585 quite fast for different initial guesses.

Example 2: $3x - \cos x - 1 = 0$

$$f(0) = -2 < 0$$

$$f\left(\frac{\pi}{2}\right) = \frac{3\pi}{2} - 1 > 0$$

So the root lies between 0 and 1.

Rewriting the equation as $x = \dfrac{1+\cos x}{3} = g(x)$ gives, $g'(x) = -\dfrac{1}{3}\sin x$

$$\therefore |g'(x)| = \frac{1}{3}|\sin x| \le \frac{1}{3} < 1, \forall x$$

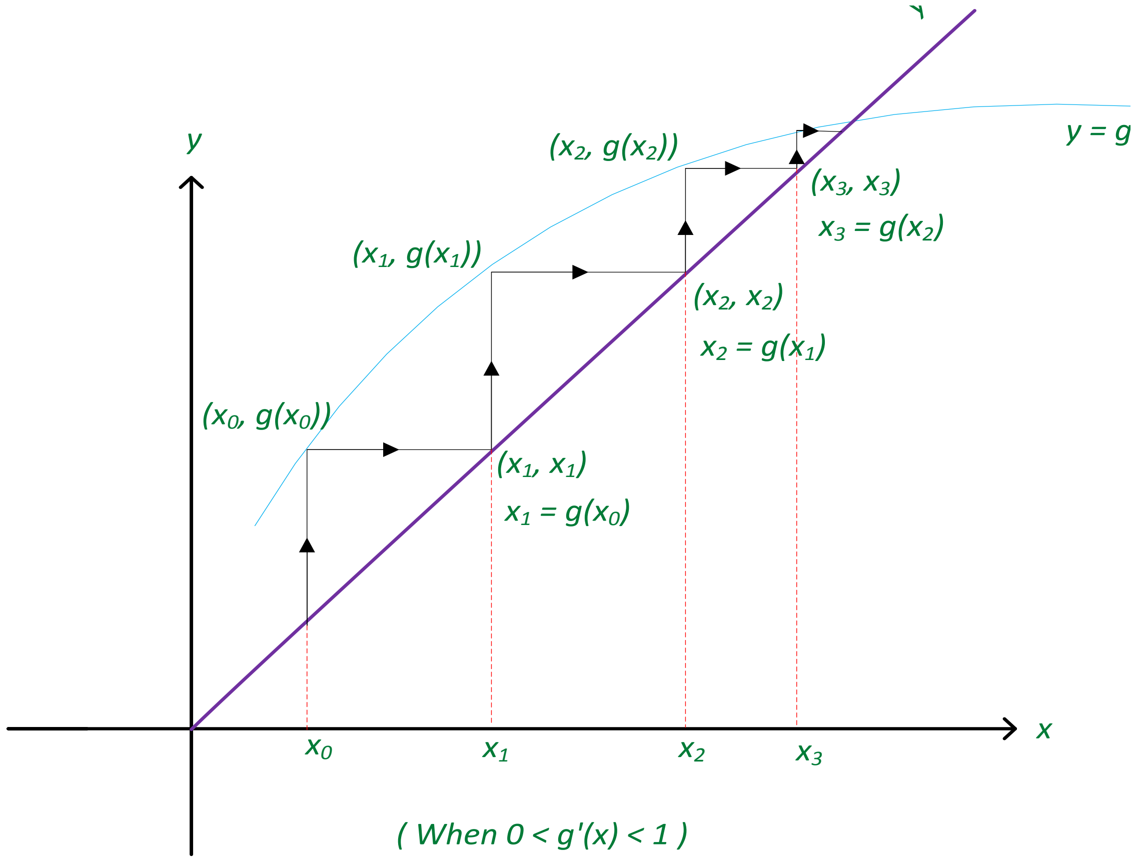
Take $x_0 = 0$ and generate the sequence of approximations:

X	**g(X)=(1+cos X)/3**
0.000000	0.666667
0.666667	0.595296
0.595296	0.609328
0.609328	0.606678
0.606678	0.607182
0.607182	0.607086
0.607086	0.607105
0.607105	0.607101
0.607101	0.607102
0.607102	0.607102

As expected, method converges to the root 0.607102 correct to six decimal places quite fast

Graphical Representation of the Method

Case (i) 0 <g'(x) < 1

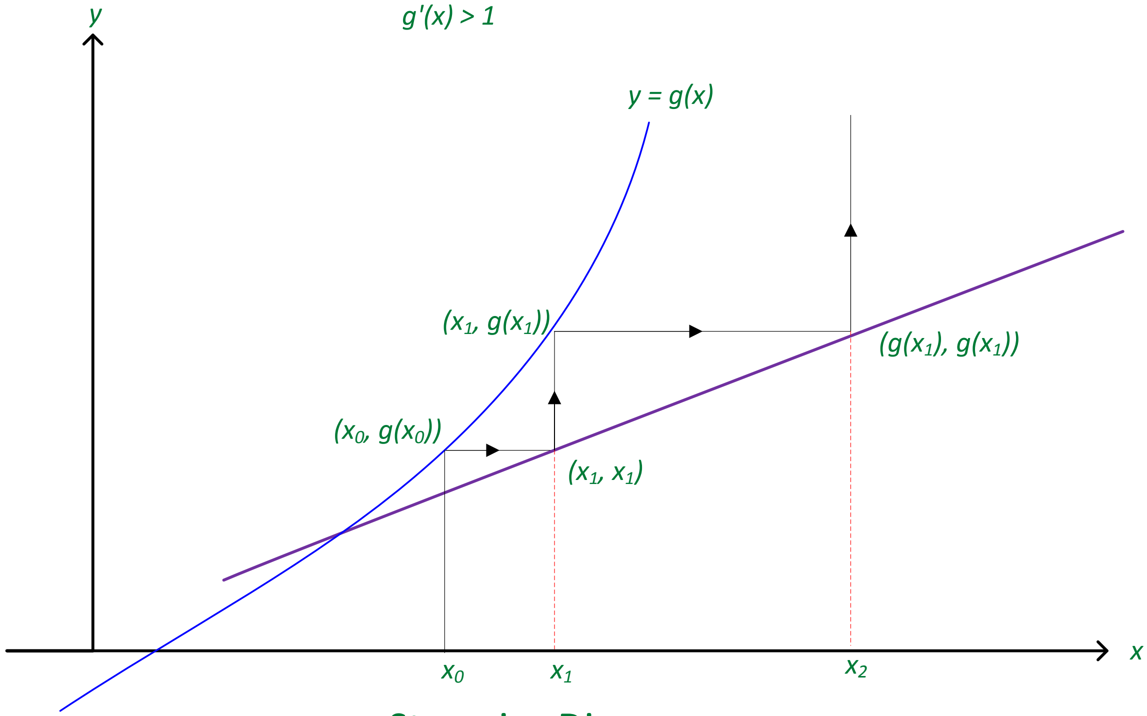


Stepwise Convergence

[Figure – 2]

Here initial guess is x_0. The point on the curve corresponding to $x = x_0$ is $(x_0, g(x_0))$. It is on the vertical line $x = x_0$ at intersection point with $y = g(x)$. If we move horizontally, the horizontal line meets $y = x$, at $(g(x_0), g(x_0))$ (y coordinate does not change, x coordinate value is same as y value). Thus x_1 is obtained. Again to get x_2, move vertically up to the curve and horizontally to $y = x$. As expected, sequence of approximations is converging and this convergence is called stepwise convergence as movement looks like steps. Sequence of approximations in the table would steadily reach towards the root.

Case (ii)



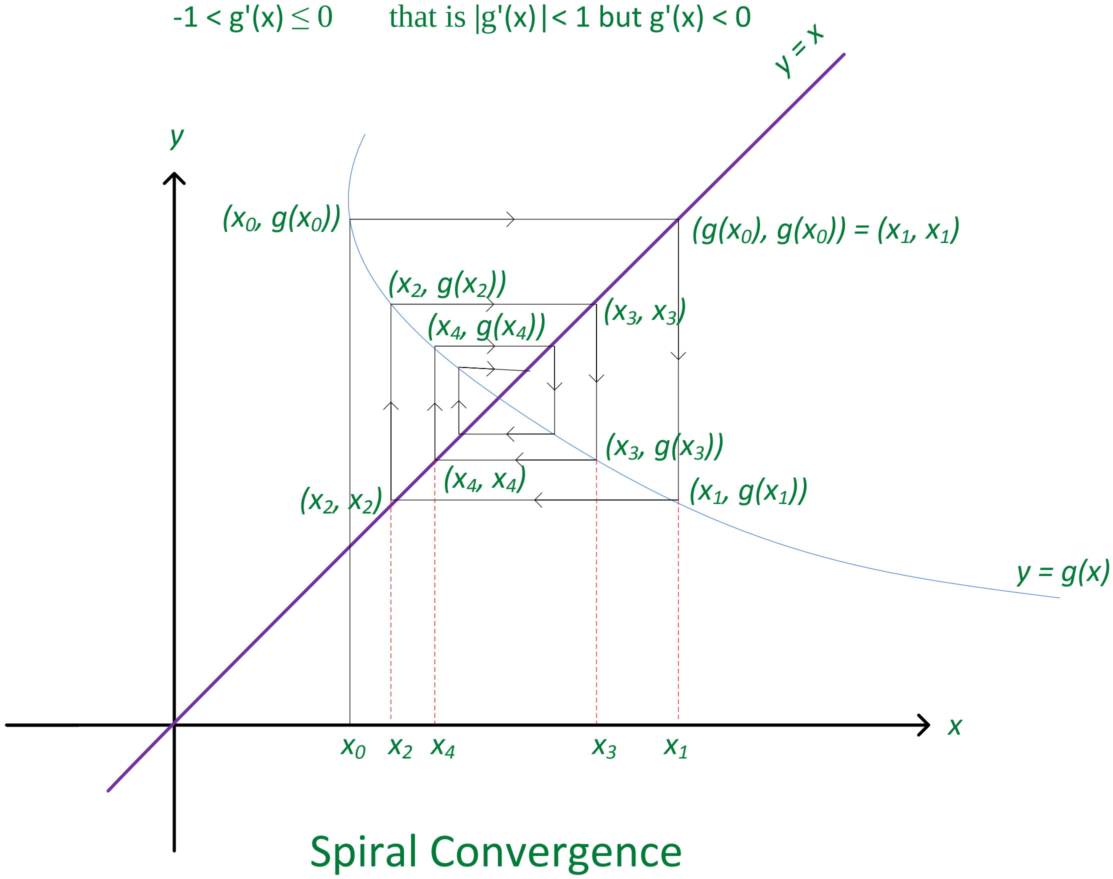
Stepwise Divergence

[Figure – 3]

This is an example of stepwise divergence. No matter, how close to the root, the initial guess is chosen, sequences of approximations move away from the root, steadily. Movement in the graph gives step like appearance and that is why it is called stepwise divergence.

Case (iii)

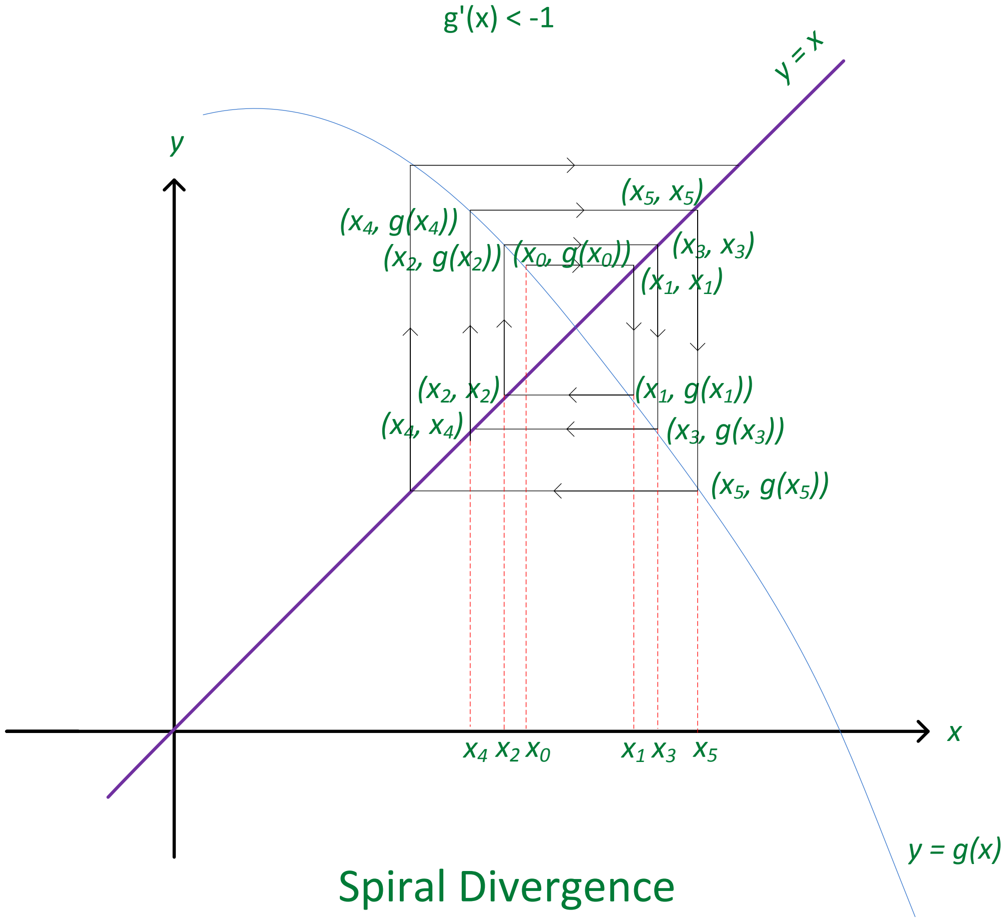
$-1 < g'(x) \leq 0$ that is $|g'(x)| < 1$ but $g'(x) < 0$



[Figure – 4]

Here sequences of approximations converge to the root by oscillating about the root. Graphically, movement looks like spirals. Therefore it is called spiral convergence.

Case (iv)



[Figure – 5]

Sequence of approximations diverges by oscillating about either side of root. Therefore, it is called spiral divergence.

1.7 Advantages and Disadvantages

Advantages:

(1) It is very simple, just evaluation of $g(x)$ per iteration, so quite easy to implement.

(2) If $g(x)$ is chosen wisely, it guarantees convergence.

(3) It requires only one initial guess.

Disadvantages:

(1) Sometimes, it may require good skill to transform $f(x)=0$ to the appropriate form $x=g(x)$, so that $|g'(x)|<1$ in the interval containing the root to ensure convergence.

(2) Its convergence is of linear or super linear depends upon $g(x)$.