LMU: Fakultät für Physik

Lecture:
When Machine Learning meets Complex Systems

PD Dr. Christoph Räth, Sebastian Baur, Daniel Köglmayr, Joel Steinegger

Exercise Sheet 1

(Please prepare your answers until Friday, October 31st)

Exercise 1.1: Consider the Lagrangian $L = T - U$ of the double pendulum with

$$T = \frac{1}{2} m_1 l_1^2 \dot{\phi}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\phi}_1^2 + l_2^2 \dot{\phi}_2^2 + 2 l_1 l_2 \dot{\phi}_1 \dot{\phi}_2 \cos(\phi_1 - \phi_2) \right]$$

and

$$U = -(m_1 + m_2) g l_1 \cos(\phi_1) - m_2 g l_2 \cos(\phi_2)$$

Derive the equations of motion for ϕ_1 and ϕ_2 and compare your result with that obtained for the (single) pendulum. What is similar in the equations? What is different? How can the equations of motion of the double pendulum become formally identical to the equation of motion of the pendulum? What does this mean physically?

Exercise 1.2: Derive the (quartic) equation for fixed points satisfying the condition $x_{n+2} = x_n$ for the logistic map and show that it has the roots given by $x = 0$, $x = 1 - 1/r$ and $x = \frac{r+1 \mp \sqrt{(r-3)(r+1)}}{2r}$.

Exercise 1.3: Consider the logistic map $x_{n+1} = r x_n (1 - x_n)$. Explain why it is sensible to restrict r and x to the intervals $r \in [0, 4]$ and $x \in [0, 1]$.

Exercise 1.4: (a) With $r = 2.8$ and $x_0 = 0.1$ iterate the logistic equation $x_{n+1} = r x_n (1 - x_n)$ many times. Show that they oscillate about and converge on the solution $x^* = 1 - 1/r$.
(b) Produce a bifurcation diagram by varying r. You can start with (almost) any initial values for $x_0, 0 < x_0 < 1$. Be sure to discard a sufficient number of initial iterates.
Have a closer look at the range $3.0 < r < 3.57$, where successive period doublings occur. How many period doublings can you realize? Use your plot to estimate the Feigenbaum number.
(c) Expand your plot in the vicinity of the period-3 window at about $r = 3.84$ and show that it also exhibits a sequence of period doublings prior to the onset of chaos.
(d) (optional) Show graphically the iterates of the logistic map using the cobweb-diagram.

Exercise 1.5: The Hénon Map is given by

$$
\begin{aligned}
x_{n+1} &= y_n + 1 - a x_n^2 \\
y_{n+1} &= b x_n
\end{aligned}
$$

where a and b are adjustable parameters.
(a) Show that the Hénon map is invertible, if $b \neq 0$ and find the inverse.
(b) Find all the fixed points of the Hénon map and show that they exist only if $a > a_0$, where a_0 is to be determined.
(c) Calculate the Jacobian matrix of the Hénon map and find its eigenvalues.
$a > a_1 = \frac{3}{4}(1-b)^2$.
(d) Explore numerically what happens in the Hénon map for other values of a while keeping $b = 0.3$. Demonstrate that period doubling can occur, leading to the onset of chaos at $a \approx 1.06$. Generate some nice looking plots of the chaotic attractor for the canonical parameter values of $a = 1.3$, $b = 0.3$. How far can you zoom into the attractor before it isn't interesting anymore?

Exercise 1.6: Find the equilibria and their eigenvalues for the damped anharmonic oscillator

$$
\frac{d^2x}{dt^2} + b\frac{dx}{dt} + x + x^2 = 0
$$

Exercise 1.7: Dynamics of Romance: Consider a love affair between Romeo and Juliet governed by the equations

$$
\begin{aligned}
\frac{dR}{dt} &= aR + bJ \\
\frac{dJ}{dt} &= cR + dJ
\end{aligned}
$$

where R is Romeo's love (or hate if $R < 0$) for Juliet and J is Juliet's love (or hate) for Romeo at time t. Discuss and interpret the dynamics and ultimate fate for the different combinations of romantic styles determined by a, b, c and d.