LMU: Fakultät für Physik

Lecture:
When Machine Learning meets Complex Systems

PD Dr. Christoph Räth, Sebastian Baur, Daniel Köglmayr, Joel Steinegger

Exercise Sheet 2

(Please prepare your answers until Wednesday, November 12th)

Exercise 2.1: Recreate Lorenz's discovery of the sensitive dependence on initial conditions. The Lorenz equations are

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$

with the canonical parameter values $\sigma = 10, \rho = 28$ and $\beta = 8/3$.
(a) Start with an arbitrary initial condition at starting time $t = 0$ reasonably close to the attractor and solve the Lorenz equation numerically using the 4th order Runge Kutta method with the parameters given above. Make a graph in the zy-plane and demonstrate that different initial conditions are drawn to the same attractor. How long do you have to run the simulation until you can be reasonably sure that any trajectory has reached the attractor?
(b) Chose initial conditions x_0, y_0, z_0 on the attractor and create a graph of x vs t. Then rerun the simulation using the same initial conditions displaced by a small amount of, say, $\delta = 1 \times 10^{-8}$. Show that $x(t)$ for the two cases initially coincides, but then the small error eventually grows to the size of the attractor. Does anything change qualitatively if you vary the size of δ? What happens for $\delta = 1 \times 10^{-20}$?
(c) Choosing $\delta = 1 \times 10^{-8}$ again, plot the difference between the two trajectories as a function of t. What is a good way to visualize the exponential growth of the error?
(d) Also simulate the Halvorsen, Rucklidge and Thomas systems. If you have written your code of part (a) in a modular way, this should not be much additional work. Create some pretty plots of each attractor. Note that you might need to change the size of your integration time step to get sensible results. The systems are given by

$$\frac{dx}{dt} = -ax - 4y - 4z - y^2 \qquad \frac{dx}{dt} = -\kappa x + \lambda y - yz \qquad \frac{dx}{dt} = -bx + \sin y$$
$$\frac{dy}{dt} = -ay - 4z - 4x - z^2 \qquad \frac{dy}{dt} = x \qquad \frac{dy}{dt} = -by + \sin z$$
$$\frac{dz}{dt} = -az - 4x - 4y - x^2 \qquad \frac{dz}{dt} = -z + y^2 \qquad \frac{dz}{dt} = -bz + \sin x$$

system	parameter values
Halvorsen	$a = 1.27$
Rucklidge	$\kappa = 2.0, \lambda = 6.7$
Thomas	$b = 0.18$

Exercise 2.2: Consider the Rössler system

$$\frac{dx}{dt} = -y - z$$
$$\frac{dy}{dt} = x + ay$$
$$\frac{dz}{dt} = b + z(x - c)$$

(a) Start with an arbitrary initial condition at $t = 0$ reasonably close to the attractor and solve the equations numerically using 4th order Runge Kutta method with $a = b = 0.2$ and different values of c. A sensible choice may be a time step size on the order of 10^{-2} time units. Discard the transitory parts of the trajectory not on the attractor. Plot the trajectories in the xy plane for $c = 2.5, c = 3.5, c = 4$ and $c = 5$.
(b) The continuous Rössler system can be converted to a one-dimensional map by collecting the local maxima $x^{\max}$ of $x(t)$. For $c = 5$, plot the $x_{i+1}^{\max}$ vs. the $x_i^{\max}$. You should see a one-dimensional curve quite similar to the logistic map. Hint: For the purpose of finding the local maxima of x the Rössler system is quite well behaved, so you don't have to over-engineer your solution.
(c) Obtain the bifurcation diagram of this one-dimensional map in the range $2 < c < 6$.

Exercise 2.3: The Wiener-Khinchin theorem.
(a) Show that

$$FT\left[A_n\right] = \left|\widetilde{x}_k\right|^2$$

where A_n is the cyclically defined autocorrelation

$$A_n = \frac{1}{N} \sum_{n'} x_{n'} x_{n'+n}, \qquad x_n = x_{n-N} \text{ for } n > N$$

Use the following formulas

$$FT\left[x_n\right] = \frac{1}{N}\sum_n x_n e^{-2\pi ikn/N} \equiv \widetilde{x}_k \qquad \widetilde{x}_{-k} = \widetilde{x}_k^*$$

$$FT^{-1}\left[\widetilde{x}_k\right] = \sum_k \widetilde{x}_k e^{2\pi ikn/N} \qquad \delta_{nm} = \frac{1}{N}\sum_k e^{2\pi ik(n-m)/N}$$

(b) Show that Gaussian white noise given by

$$P(X) = \frac{1}{\sqrt{2\pi}}e^{-X^2/2}$$

has the property $\int_{-\infty}^{\infty} P(X)dX = 1$. Show further that the mean is zero, the variance is one and the full width at half maximum is $2\sqrt{2\ln(2)}$.

(c) Generate a time series of Gaussian white noise and numerically calculate the autocorrelation function and the power spectrum. Calculate the phases ϕ for all Fourier modes and plot them over frequency. Determine the probability distribution of ϕ. Comment your results. (d) Generate each of the time series below and numerically calculate the autocorrelation function and the power spectrum. Check the validity of the Wiener-Khinchin theorem. (1.) Logistic map with r = 3.5. (2.) Logistic map with r = 4. (3.) Linear superpositions of (1.), (2.) and Gaussian random noise from part (a).

Exercise 3.4: The Lyapunov exponent of the Logistic Map.
(a) Numerically calculate the Lyapunov exponent λ for the logistic map for $r = 3$, $r = 3.2$ and $r = 3.6$. Do the values of λ correspond to what you would expect from the bifurcation diagram? What happens for $r = 2$?
(b) By varying r, numerically estimate λ as a function of r and compare it to the bifurcation diagram. Does it match up with what you expected? What importance does the sign of λ have?