Original basis $\vec{v}_1, \vec{v}_2 \dots \vec{v}_n$ ⟶ orthogonal basis $\vec{u}_1, \vec{u}_2 \dots \vec{u}_n$

1) $\vec{u}_1 = \vec{v}_1$

2) $\vec{u}_2 = \vec{v}_2 - \frac{(\vec{v}_2 \cdot \vec{u}_1)}{|\vec{u}_1|^2} \vec{u}_1$

3) $\vec{u}_3 = \vec{v}_3 - \frac{\vec{v}_3 \cdot \vec{u}_1}{|\vec{u}_1|^2} \vec{u}_1 - \frac{(\vec{v}_3 \cdot \vec{u}_2)}{|\vec{u}_2|^2} \vec{u}_2$

Summary

$$\vec{u}_k = \vec{v}_k - \sum_{i=1}^{k-1} \frac{(\vec{v}_k \cdot \vec{u}_i)}{|u_i|^2} \vec{u}_i$$

Q) $\mathbb{R}^3$ $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

Find orthogonal & orthonormal vectors

$$u_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$u_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1} \vec{u}_1$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \frac{(1+1)}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$u_3 = \vec{v_3} - \frac{v_3 u_1}{u_1 u_1} u_1 - \frac{v_3 u_2}{u_2 u_2} u_2$$

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{(1 + (-1) + 2)}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \frac{(1/3 + 2/3 + 2/3)}{(1/9 + 4/9 + 1/9)} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$u_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2/3 \\ -2/3 \\ 2/3 \end{bmatrix} - \frac{5/3}{6/9} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 5/3 \\ 8/3 \end{bmatrix} - \frac{5}{2} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 \\ 5/3 \\ 4/3 \end{bmatrix} - \begin{bmatrix} 5/6 \\ 5/3 \\ 5/6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2-5}{6} \\ 0 \\ \frac{8-5}{6} \end{bmatrix} \quad \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

Orthonormal basis

$$\hat{u}_1 = \frac{\vec{u}_1}{\|u_1\|_2} = \frac{\vec{u}_1}{\sqrt{\vec{u}_1 \cdot \vec{u}_1}} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

$$\hat{u}_2 = \frac{\vec{u}_2}{\|\vec{u}_2\|_2} = \frac{1}{\sqrt{2/3}} \begin{bmatrix} 1/3 \\ 2/3 \\ 1/3 \end{bmatrix} = \begin{bmatrix} \frac{1 \times \sqrt{3}}{\sqrt{3}\sqrt{2}} \\ \frac{2 \times \sqrt{3}}{3 \times \sqrt{2}} \\ \frac{1 \times \sqrt{3}}{3\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$$