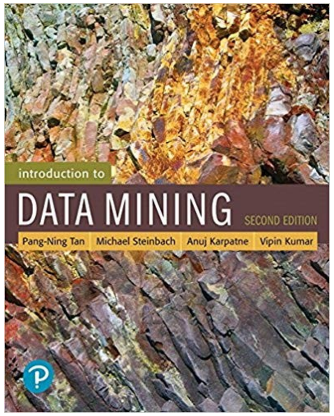

CIS 530—Advanced Data Mining

2- Linear Algebra and Matrix

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Courtesy of Fei-Fei Li: http://vision.stanford.edu/teaching/cs131_fall1617/

Outline

Vectors and Matrices

Basic operations

Special matrices

More Matrix Operations

Matrix inverse

Matrix rank

Singular Value Decomposition (SVD)

Use for image compression

Transformation Matrices

Homogeneous coordinates

Translation

Vector

- A column vector $\mathbf{v} \in \mathbb{R}^{n \times 1}$ where

$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- A row vector $\mathbf{v}^T \in \mathbb{R}^{1 \times n}$ where

$$\mathbf{v}^T = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

T denotes the transpose operation

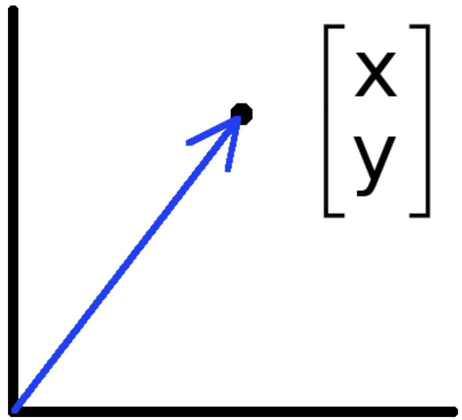
Vector

- We'll default to column vectors in this class

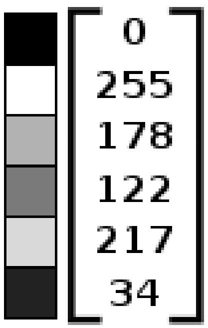
$$\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

- You'll want to keep track of the orientation of your vectors when programming in MATLAB
- In MATLAB means , indicating the transpose operation

Vectors have two main uses



- Vectors can represent an offset in 2D or 3D space
- Points are just vectors from the origin



- Data (pixels, gradients at an image keypoint, etc.) can also be treated as a vector
- Such vectors don't have a geometric interpretation, but calculations like "distance" can still have value

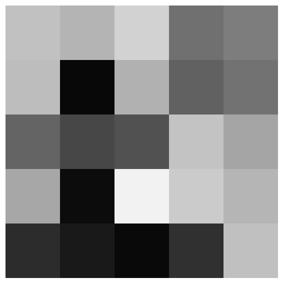
Matrix

- A matrix $\mathbf{A} \in \mathbb{R}^{m \times n}$ is an array of numbers with size by, i.e., rows and columns.

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & & & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

- If $m = n$, we say that $\mathbf{A}$ is square.

Images



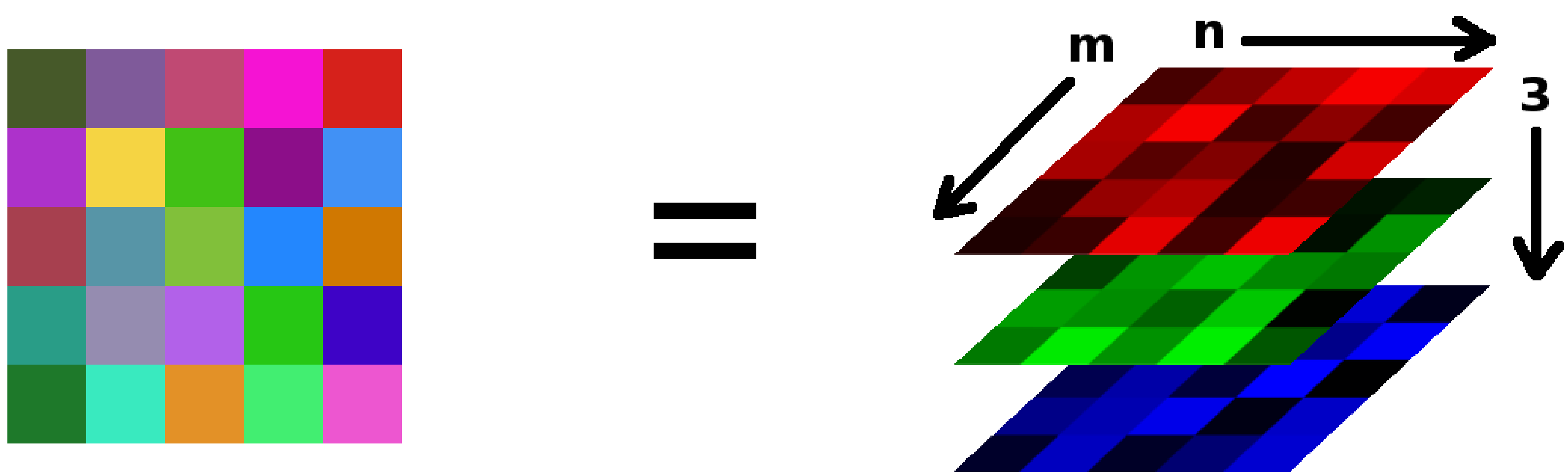
$$\begin{bmatrix} 193 & 180 & 210 & 112 & 125 \\ 189 & 8 & 177 & 97 & 114 \\ 100 & 71 & 81 & 195 & 165 \\ 167 & 12 & 242 & 203 & 181 \\ 44 & 25 & 9 & 48 & 192 \end{bmatrix}$$

MATLAB represents an image as a matrix of pixel brightnesses

Note that matrix coordinates are NOT Cartesian coordinates. The upper leg corner is [y,x] = (1,1)

Color Images

- Grayscale images have one number per pixel, and are stored as an m × n matrix.
- Color images have 3 numbers per pixel – red, green, and blue brightnesses (RGB)
- Stored as an m × n × 3 matrix



Basic Matrix Operations

- Addition
- Scaling
- Dot product
- Multiplication
- Transpose
- Inverse / pseudoinverse
- Determinant / trace

Matrix Operations

- Addition

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+1 & b+2 \\ c+3 & d+4 \end{bmatrix}$$

  – Can only add a matrix with matching dimensions, or a scalar.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + 7 = \begin{bmatrix} a+7 & b+7 \\ c+7 & d+7 \end{bmatrix}$$

- Scaling

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times 3 = \begin{bmatrix} 3a & 3b \\ 3c & 3d \end{bmatrix}$$

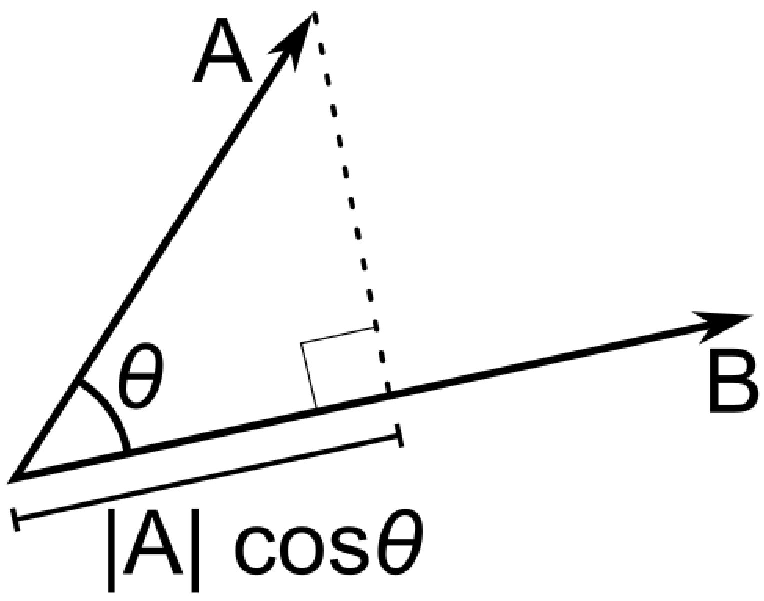
Matrix Operations

- Inner product (dot product) of vectors
  - Multiply corresponding entries of two vectors and add up the result
  - is also ||||cos(*the angle between x and y*)

$$\mathbf{x}^T\mathbf{y} = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \sum_{i=1}^{n} x_i y_i \qquad \text{(scalar)}$$

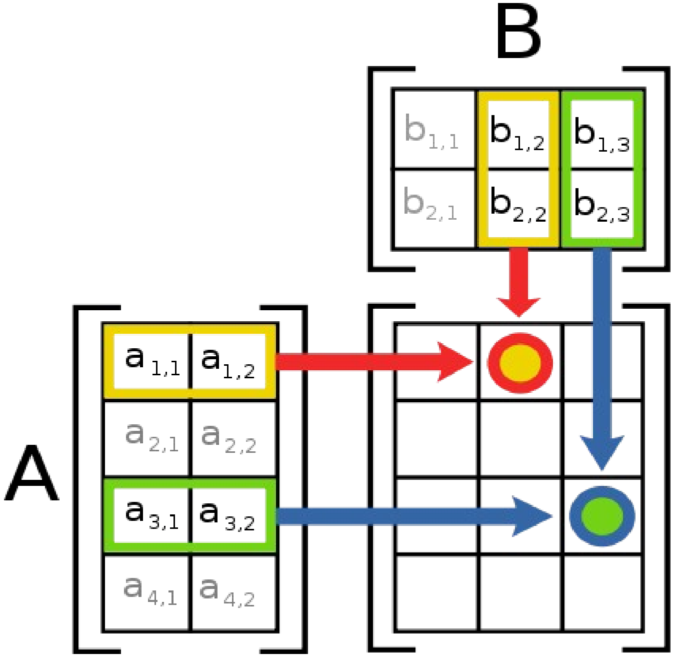
Matrix Operations

- Inner product (dot product) of **vectors**
  - If is a unit vector, then gives the length of which lies in the direction of



Matrix Operations

- Multiplication
- The product AB is:



- Each entry in the result is (that row of A) dot product with (that column of B)
- Many uses, which will be covered later

Matrix Operations

- Multiplication example:

A x B

$$\begin{bmatrix} 1 & \boxed{3} \\ 5 & \boxed{7} \end{bmatrix}$$

$$\begin{bmatrix} \boxed{0 \quad 2} \\ 4 \quad 6 \end{bmatrix} \quad \begin{bmatrix} 10 & \boxed{14} \\ 34 & 54 \end{bmatrix}$$

$$0 \cdot 3 + 2 \cdot 7 = \boxed{14}$$

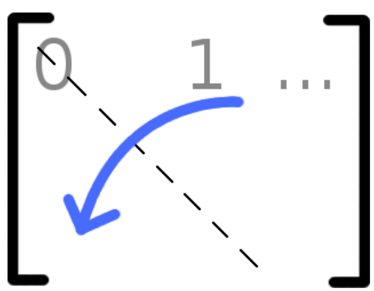
  - – Each entry of the matrix product is made by taking the dot product of the corresponding row in the leg matrix, with the corresponding column in the right one.

Matrix Operations

- Powers
  - By convention, we can refer to the matrix product AA as A^2, and AAA as A^3, etc.
  - Obviously only square matrices can be multiplied that way

Matrix Operations

- Transpose – flip matrix, so row 1 becomes column 1



$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \\ 4 & 5 \end{bmatrix}^T = \begin{bmatrix} 0 & 2 & 4 \\ 1 & 3 & 5 \end{bmatrix}$$

- A useful identity:

$$(ABC)^T = C^T B^T A^T$$

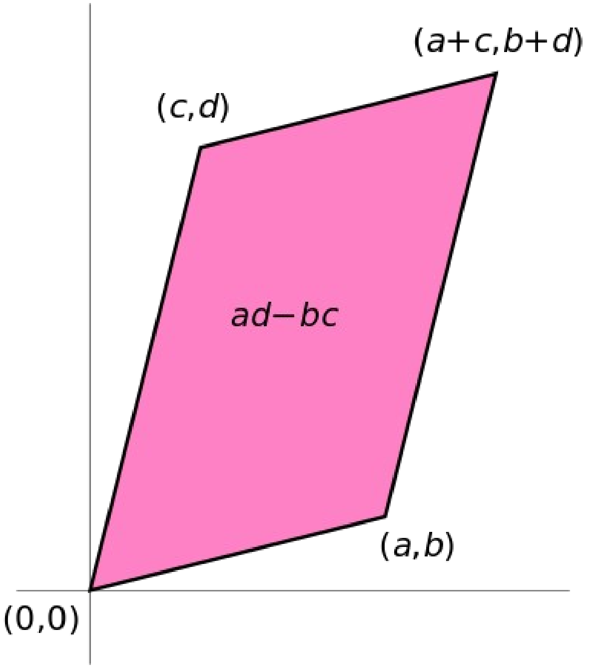
Matrix Operations

- Determinant
  - $\det(\mathbf{A})$ returns a scalar
  - Represents area (or volume) of the parallelogram described by the vectors in the rows of the matrix
  - For $\mathbf{A}, = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det(\mathbf{A}) = ad - bc$
  - Properties: $\det(\mathbf{AB}) = \det(\mathbf{BA})$

$$\det(\mathbf{A}^{-1}) = \frac{1}{\det(\mathbf{A})}$$

$$\det(\mathbf{A}^T) = \det(\mathbf{A})$$

$$\det(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A} \text{ is singular}$$



Matrix Operations

- Trace

$$\mathrm{tr}(\mathbf{A}) = \text{sum of diagonal elements}$$

$$\mathbf{tr}(\begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix}) = 1 + 7 = 8$$

  – Invariant to a lot of transformations, so it's used sometimes in proofs. (Rarely in this class though.)

  – Properties:

$$\mathrm{tr}(\mathbf{AB}) = \mathrm{tr}(\mathbf{BA})$$

$$\mathrm{tr}(\mathbf{A} + \mathbf{B}) = \mathrm{tr}(\mathbf{A}) + \mathrm{tr}(\mathbf{B})$$

Special Matrices

- Identity matrix **`I`**
  - Square matrix, 1's along diagonal, 0's elsewhere
  - **`I`**[another matrix] = [that matrix]

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Diagonal matrix
  - Square matrix with numbers along diagonal, 0's elsewhere
  - A diagonal [another matrix] scales the rows of that matrix

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 2.5 \end{bmatrix}$$

Special Matrices

- Symmetric matrix

$$\mathbf{A}^T = \mathbf{A}$$

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 1 & 7 \\ 5 & 7 & 1 \end{bmatrix}$$

- Skew-symmetric matrix

$$\mathbf{A}^T = -\mathbf{A}$$

$$\begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & -7 \\ 5 & 7 & 0 \end{bmatrix}$$

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Inverse

- Given a matrix **A**, its inverse $\mathbf{A}^{-1}$ is a matrix such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
  - E.g. $$\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$
- Inverse does not always exist. If $\mathbf{A}^{-1}$ exists, **A** is *invertible* or *non--singular*. Otherwise, it's *singular*.
- Useful identities, for matrices that are invertible:

$$(\mathbf{A}^{-1})^{-1} = \mathbf{A}$$

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

$$\mathbf{A}^{-T} \triangleq (\mathbf{A}^{T})^{-1} = (\mathbf{A}^{-1})^{T}$$

Pseudoinverse

- Say you have the matrix equation AX=B, where A and B are known, and you want to solve for X
- You could use MATLAB to calculate the inverse and pre-multiply by it: $A^{-1}AX=A^{-1}B \rightarrow X=A^{-1}B$
- MATLAB command would be inv(A)*B
- But calculating the inverse for large matrices often brings problems with computer floating-point resolution (because it involves working with very small and very large numbers together).
- Or your matrix might not even have an inverse.

Pseudoinverse

Fortunately, there are workarounds to solve AX=B in these situations. And MATLAB can do them!

Instead of taking an inverse, directly ask MATLAB to solve for X in AX=B, by typing A\B, called “mldivide”

For complete instruction please check:

https://www.mathworks.com/help/matlab/ref/mldivide.html

Pseudoinverse

- MATLAB will try several appropriate numerical methods (including the pseudoinverse if the inverse doesn’t exist)
  - If there is no exact solution, it will return the closest one
  - If there are many solutions, it will return the smallest one

Pseudoinverse

- MATLAB example:

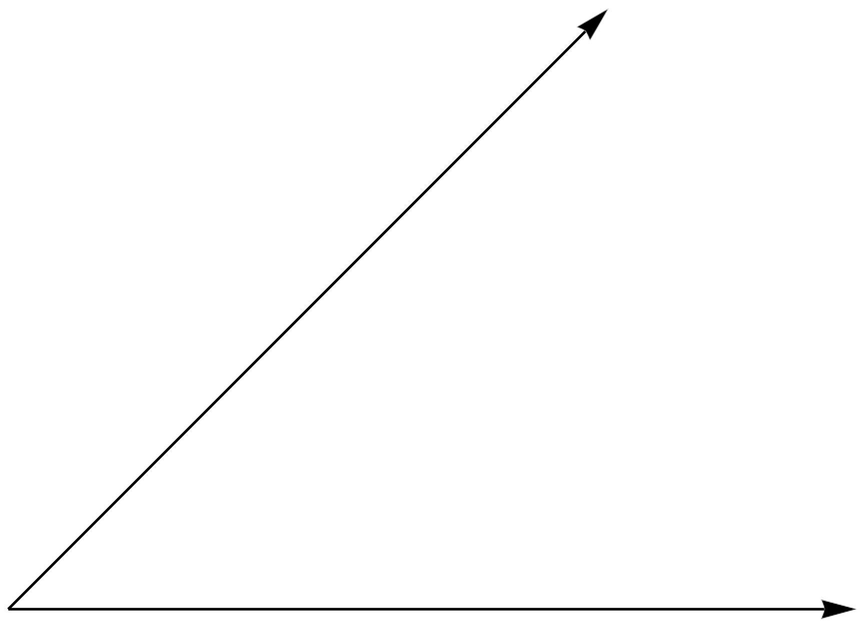
$$AX = B$$

$$A = \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

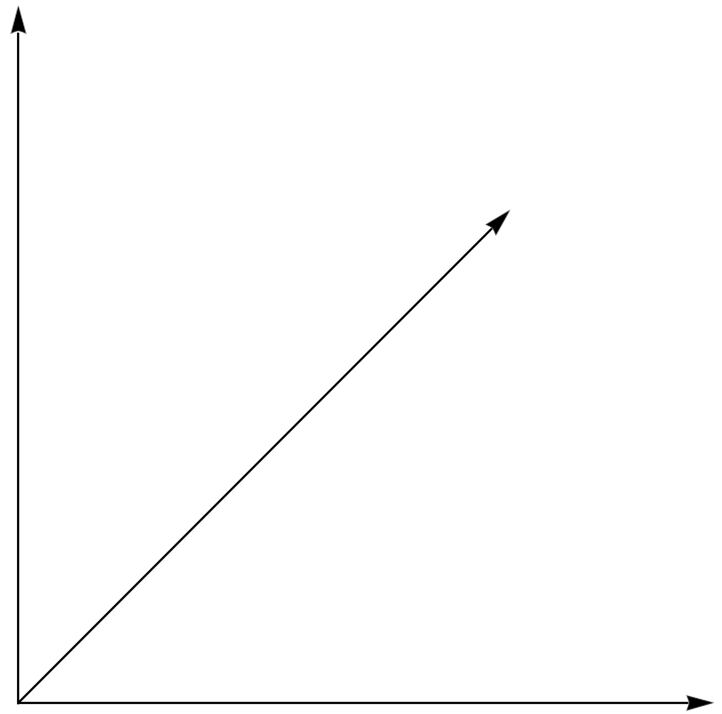
```
>> x = A\B
x =
     1.0000
    -0.5000
```

Linear Independence

Linearly independent set



Not linearly independent



Linear Independence

- Suppose we have a set of vectors
- If we can express as a linear combination of the other vectors then is linearly dependent on the other vectors

- If no vector is linearly dependent on the rest of the set, the set is linearly independent

Matrix Rank

- Column/row rank

$$\text{col-rank}(\mathbf{A}) = \text{ the maximum number of linearly independent column vectors of } \mathbf{A}$$
$$\text{row-rank}(\mathbf{A}) = \text{ the maximum number of linearly independent row vectors of } \mathbf{A}$$

- Column rank always equals row rank
- Matrix rank

$$\text{rank}(\mathbf{A}) \triangleq \text{col-rank}(\mathbf{A}) = \text{row-rank}(\mathbf{A})$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

What is the rank of the left matrix and why?

Matrix Rank

- For transformation matrices, the rank tells you the **dimensions** of the output
- E.g., if rank of **A** is 1, then the transformation

p'=Ap

maps points onto a line.

- Here's a matrix with rank 1:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ 2x + 2y \end{bmatrix} \leftarrow$$

All points get mapped to the line y=2x

Matrix Rank

If an m x m matrix is rank m, we say it's "full rank"

- Maps an *m x 1* vector uniquely to another *m x 1* vector
- An inverse matrix can be found

If rank < *m*, we say it's "singular"

At least one dimension is getting collapsed. No way to look at the result and tell what the input was

- Inverse does not exist

Inverse also doesn't exist for non-square matrices

Singular Value Decomposition

There are several computer algorithms that can “factorize” a matrix, representing it as the product of some other matrices

The most useful of these is the **Singular Value Decomposition (SVD)**.

Represent any matrix **A** as a product of three matrices: $\mathbf{U\Sigma V^T}$

MATLAB command: [U,S,V]=svd(A)

Singular Value Decomposition

$$\mathbf{U\Sigma V^T = A}$$

- Where **U** and **V** are rotation matrices, and **Σ** is a scaling matrix. For example:

$$\begin{array}{ccccccc} U & & \Sigma & & V^T & & A \\ \begin{bmatrix} -.40 & .916 \\ .916 & .40 \end{bmatrix} & \times & \begin{bmatrix} 5.39 & 0 \\ 0 & 3.154 \end{bmatrix} & \times & \begin{bmatrix} -.05 & .999 \\ .999 & .05 \end{bmatrix} & = & \begin{bmatrix} 3 & -2 \\ 1 & 5 \end{bmatrix} \end{array}$$

Singular Value Decomposition

U and **V** are always rotation matrices.

- Geometric rotation may not be an applicable concept, depending on the matrix. So, we call them "unitary" matrices – each column is a unit vector.

Σ is a diagonal matrix

- The number of nonzero entries = rank of **A**
- The algorithm always sorts the entries high to low

$$\overset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \overset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

SVD Applications

- We've discussed SVD in terms of geometric transformation matrices
- But SVD of an image matrix can also be very useful
- To understand this, we'll look at a less geometric interpretation of what SVD is doing
- An **outer-product view**

SVD Applications

$$\begin{array}{cccc} U & \Sigma & V^T & A \\ \begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix} \times & \begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix} \times & \begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix} = & \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \end{array}$$

- Look at how the multiplication works out, left to right:
- Column 1 of **U** gets scaled by the first value from **Σ**.

$$\begin{array}{ccc} U\Sigma & V^T & A_{partial} \\ \begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix} \times & \begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix} & \begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix} \end{array}$$

- The resulting vector gets scaled by row 1 of $\mathbf{V^T}$ to produce a contribuLon to the columns of **A**

SVD Applications

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \qquad \overset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

$$+ \quad \overset{U\Sigma}{\begin{bmatrix} -3.67 & -.71 & 0 \\ -8.8 & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \qquad \overset{A_{partial}}{\begin{bmatrix} -.6 & -.1 & .4 \\ .2 & 0 & -.2 \end{bmatrix}}$$

$$= \qquad \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

- Each product of (column i of **U**)·(value i from **Σ**)·(row i of **V**^{T}) produces a component of the final **A**.

SVD Applications

$$\overset{U\Sigma}{\begin{bmatrix} \boxed{-3.67} & -.71 & 0 \\ \boxed{-8.8} & .30 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \overset{A_{partial}}{\begin{bmatrix} 1.6 & 2.1 & 2.6 \\ 3.8 & 5.0 & 6.2 \end{bmatrix}}$$

$$\overset{U\Sigma}{\begin{bmatrix} -3.67 & \boxed{-.71} & 0 \\ -8.8 & \boxed{.30} & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} \quad \overset{A_{partial}}{\begin{bmatrix} -.6 & -.1 & .4 \\ .2 & 0 & -.2 \end{bmatrix}}$$

$$\overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

- We can call those first few columns of **U** the
- *Principal Components* of the data
- They show the major patterns that can be added to produce the columns of the original matrix
- The rows of **V**^{T} show how the *principal components*
- are mixed to produce the columns of the matrix

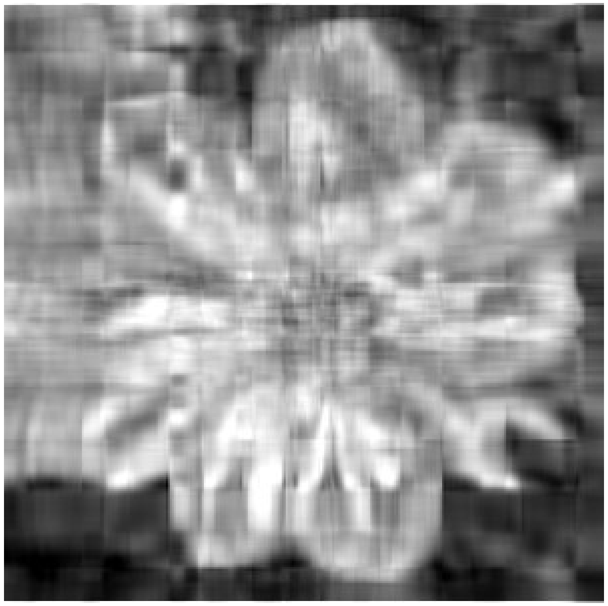
SVD Applications

$$\overset{U}{\begin{bmatrix} -.39 & -.92 \\ -.92 & .39 \end{bmatrix}} \times \overset{\Sigma}{\begin{bmatrix} 9.51 & 0 & 0 \\ 0 & .77 & 0 \end{bmatrix}} \times \overset{V^T}{\begin{bmatrix} -.42 & -.57 & -.70 \\ .81 & .11 & -.58 \\ .41 & -.82 & .41 \end{bmatrix}} = \overset{A}{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}}$$

We can look at **Σ** to see that the first column has a large effect

while the second column has a much smaller effect in this example

SVD Applications



For this image, using **only the first 10** of 300 principal components produces a recognizable reconstrucLon

So, SVD can be used for image compression

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- Homogeneous coordinates
- Translation

Transformation

- Matrices can be used to transform vectors in useful ways, through multiplication: x'= Ax
- Simplest is scaling:

$$\begin{bmatrix} s_x & 0 \\ 0 & s_y \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} s_x x \\ s_y y \end{bmatrix}$$

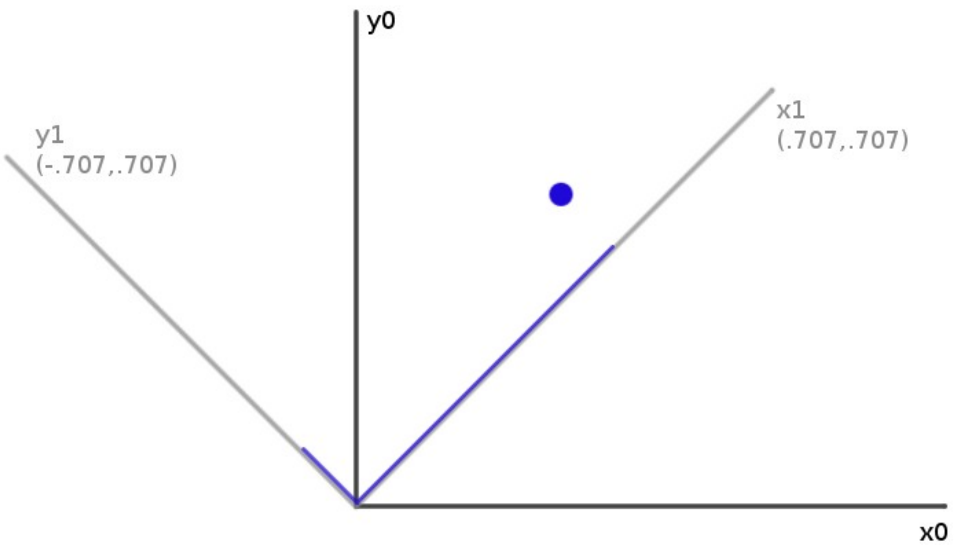
- (Verify to yourself that the matrix multiplication works out this way)

Rotation

- How can you convert a vector represented in frame “0” to a rotated coordinate frame “1”?
- Remember what a vector is:

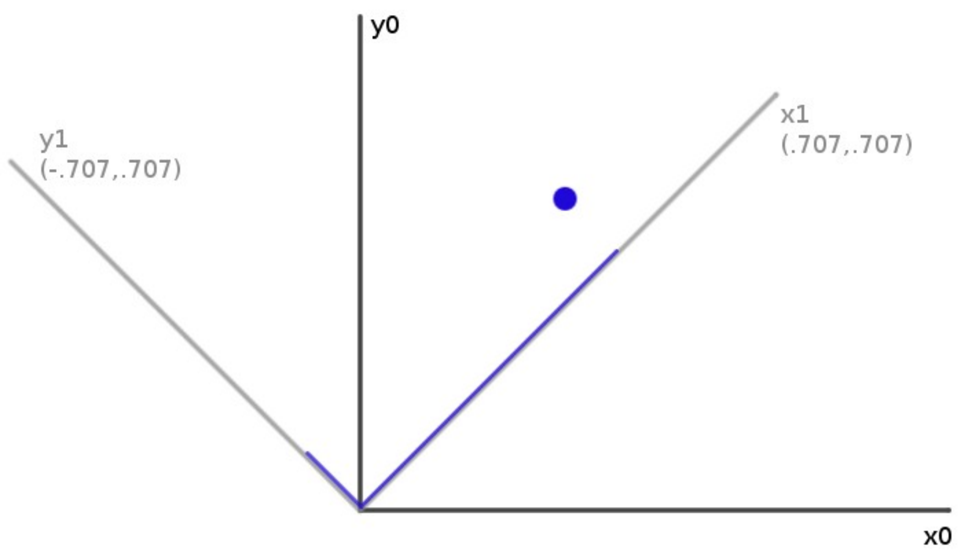
[component in direction of the frame’s x axis, and

component in direction of y axis]



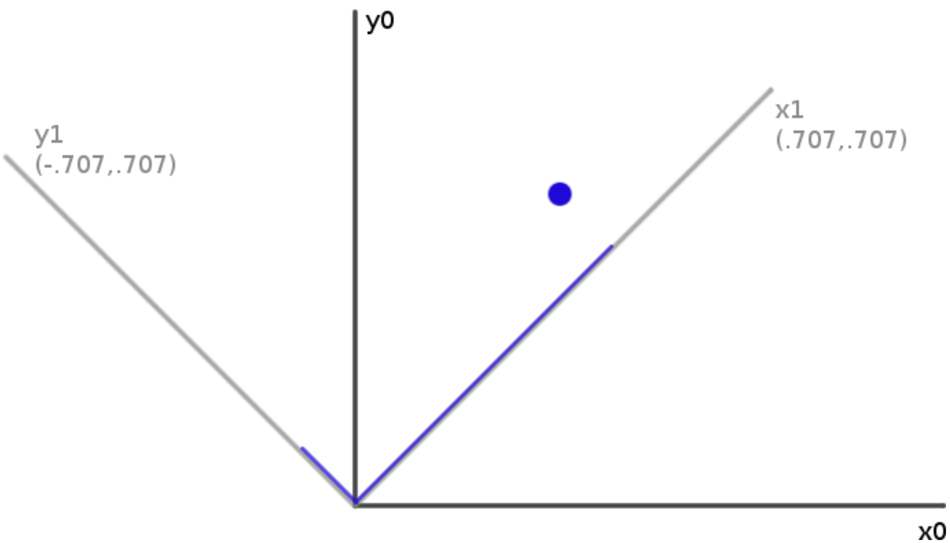
Rotation

- So to rotate it we must produce this vector: [component in direction of **new** x axis, component in direction of **new** y axis]
- We can do this easily with ***dot products***!
- New x coordinate is [original vector] **dot** [the new x axis]
- New y coordinate is [original vector] **dot** [the new y axis]



Rotation

- Insight: this is what happens in a matrix*vector multiplication
  - Result x coordinate is:
    [original vector] **dot** [matrix row 1]
  - So matrix multiplication can rotate a vector p:



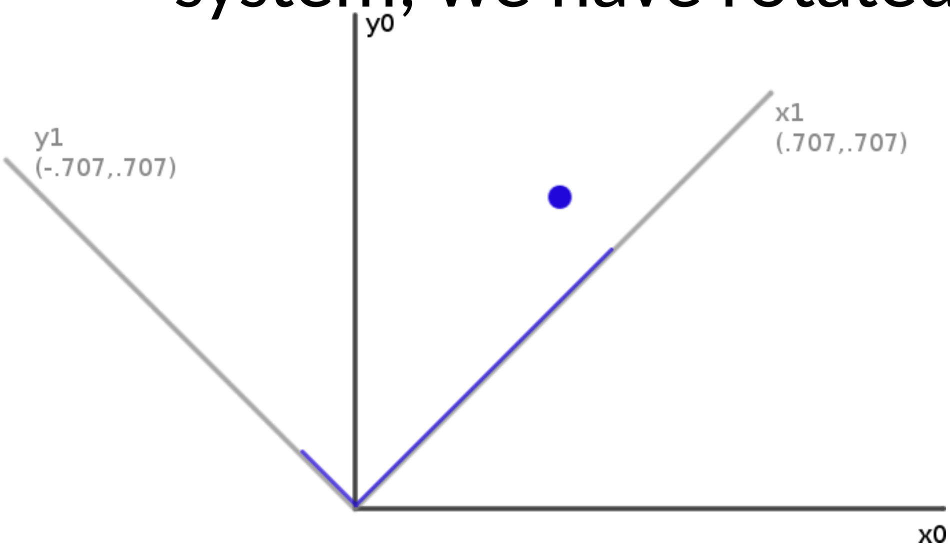
R x p =
rotated p'

$$\begin{array}{c} p \\ \begin{bmatrix} px \\ py \end{bmatrix} \end{array}$$

$$\begin{array}{c} R \\ \begin{bmatrix} .707 & .707 \\ -.707 & .707 \end{bmatrix} \end{array} \quad \begin{bmatrix} px' \\ py' \end{bmatrix}$$

Rotation

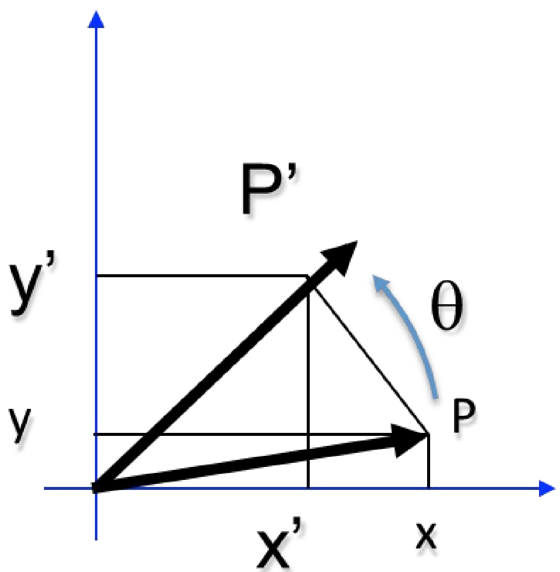
- Suppose we express a point in the new coordinate system which is rotated left
- If we plot the result in the **original** coordinate system, we have rotated the point right



  – Thus, rotation matrices can be used to rotate vectors. We'll usually think of them in that sense---- as operators to rotate vectors

2D Rotation Matrix Formula

Counter-clockwise rotation by an angle θ



$x' = \cos\theta\, x - \sin\theta$

y

$y' = \cos\theta\, y + \sin$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P' = R\ P}$$

Transformation Matrices

Multiple transformation matrices can be used to transform a point: $p' = R_2 R_1 S p$

The effect of this is to apply their transformations one ager the other, from **right to left**.

In the example above, the result is $(R_2 (R_1 (S p)))$

The result is exactly the same if we multiply the matrices first, to form a single transformation matrix:

$p' = (R_2 R_1 S) p$

Homogeneous System

- In general, a matrix multiplication lets us linearly combine components of a vector

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

  – This is sufficient for scale, rotate, skew transformations.

  – But notice, we can't add a constant!

Homogeneous System

- The (somewhat hacky) solution? Stick a "1" at the end of every vector:

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

- Now we can rotate, scale, and skew like before, AND translate (note how the multiplication works out, above)
- This is called "homogeneous coordinates"

Homogeneous System

- In homogeneous coordinates, the multiplication works out so the rightmost column of the matrix is a vector that gets added.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} ax + by + c \\ dx + ey + f \\ 1 \end{bmatrix}$$

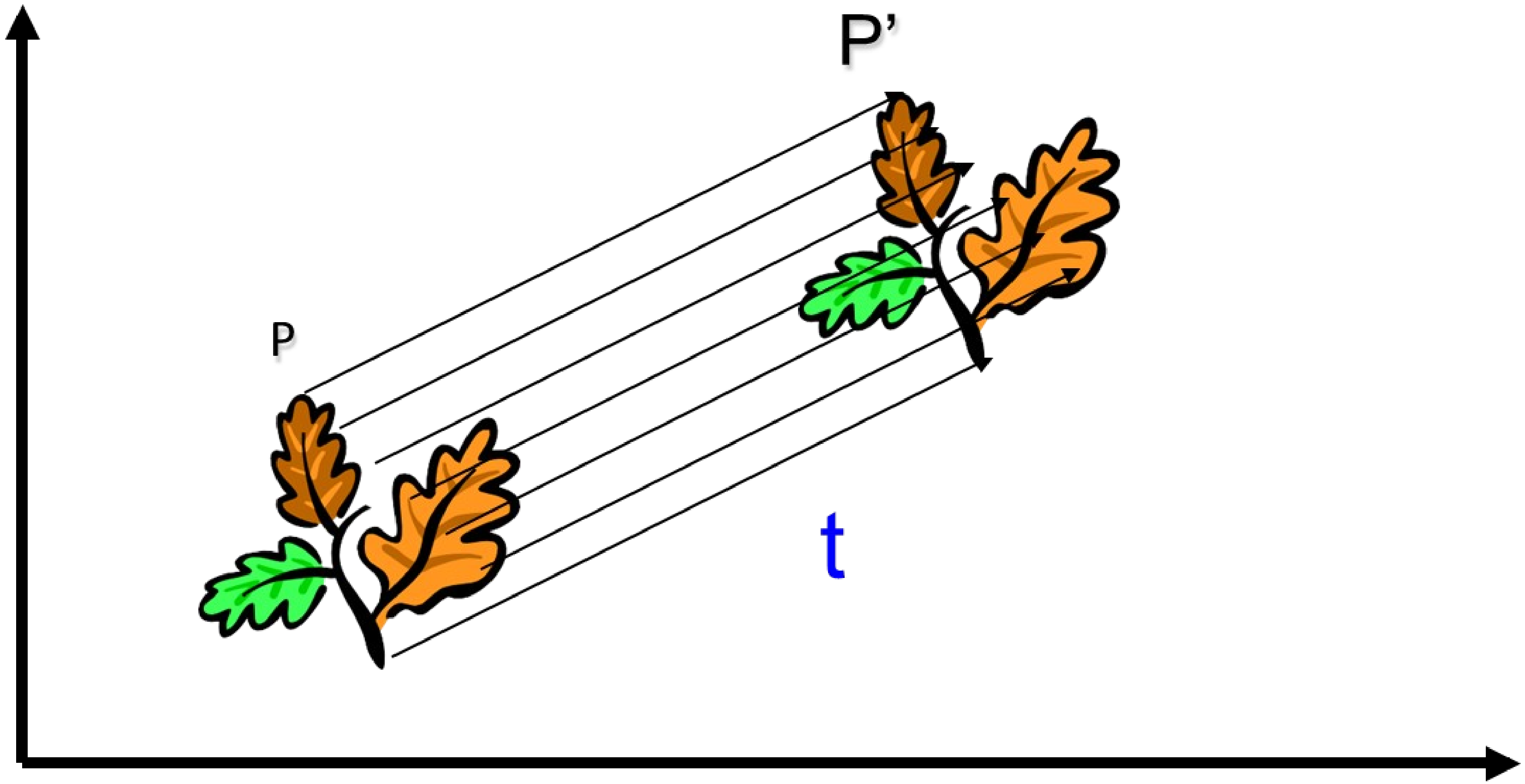
- Generally, a homogeneous transformation matrix will have a bottom row of [0 0 1], so that the result has a "1" at the bottom too.

Homogeneous System

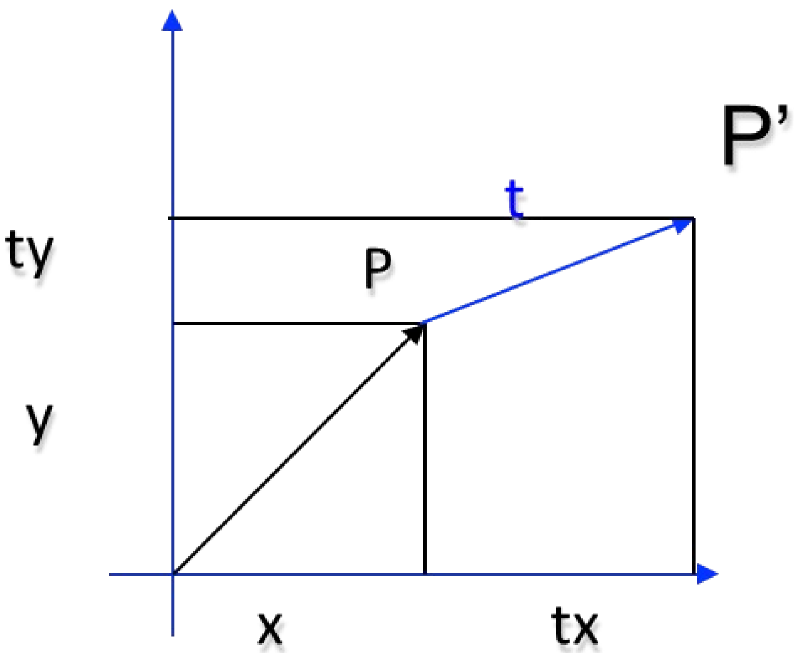
- One more thing we might want: to divide the result by something
  - For example, we may want to divide by a coordinate, to make things scale down as they get farther away in a camera image
  - Matrix multiplication can't actually divide
  - So, **by convention**, in homogeneous coordinates, we'll divide the result by its last coordinate after doing a matrix multiplication

$$\begin{bmatrix} x \\ y \\ 7 \end{bmatrix} \Rightarrow \begin{bmatrix} x/7 \\ y/7 \\ 1 \end{bmatrix}$$

2D Translation



Using Homogeneous Coordinates



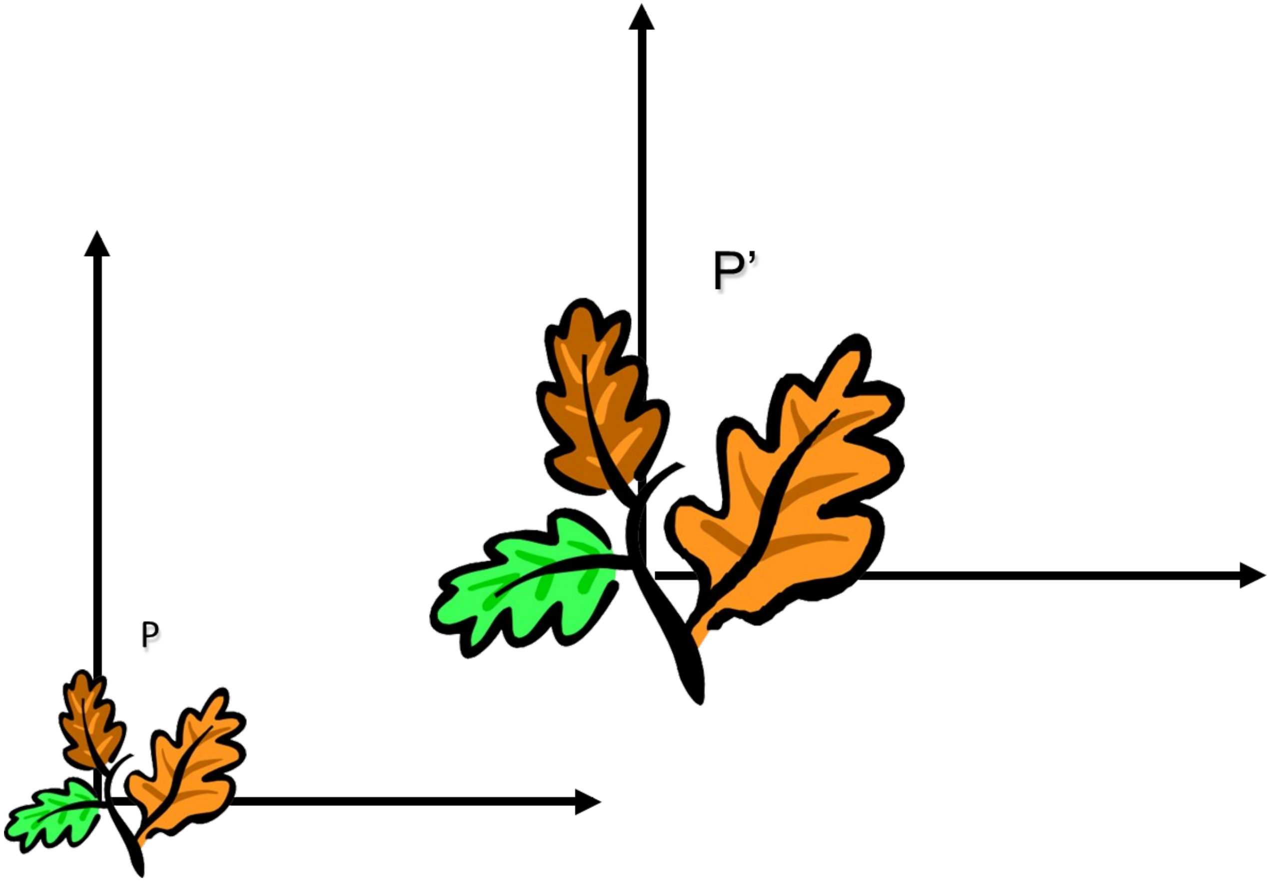
$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{t} = (t_x, t_y) \rightarrow (t_x, t_y, 1)$$

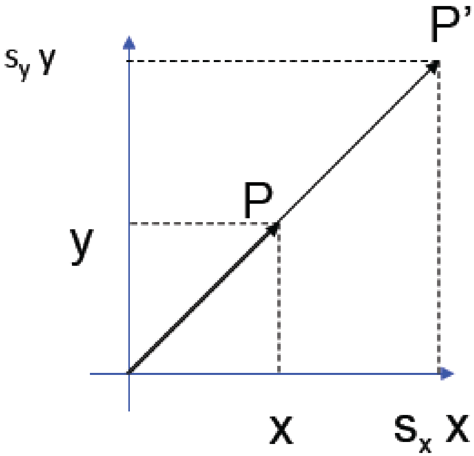
$$\mathbf{P}' \rightarrow \begin{bmatrix} x + t_x \\ y + t_y \\ 1 \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} & & \text{t} \end{array} \\ \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \end{array} \cdot \begin{array}{c} \text{P} \\ \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \end{array}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{t} \\ 0 & 1 \end{bmatrix} \cdot \mathbf{P} = \mathbf{T} \cdot \mathbf{P}$$

Scaling



Scaling Equation



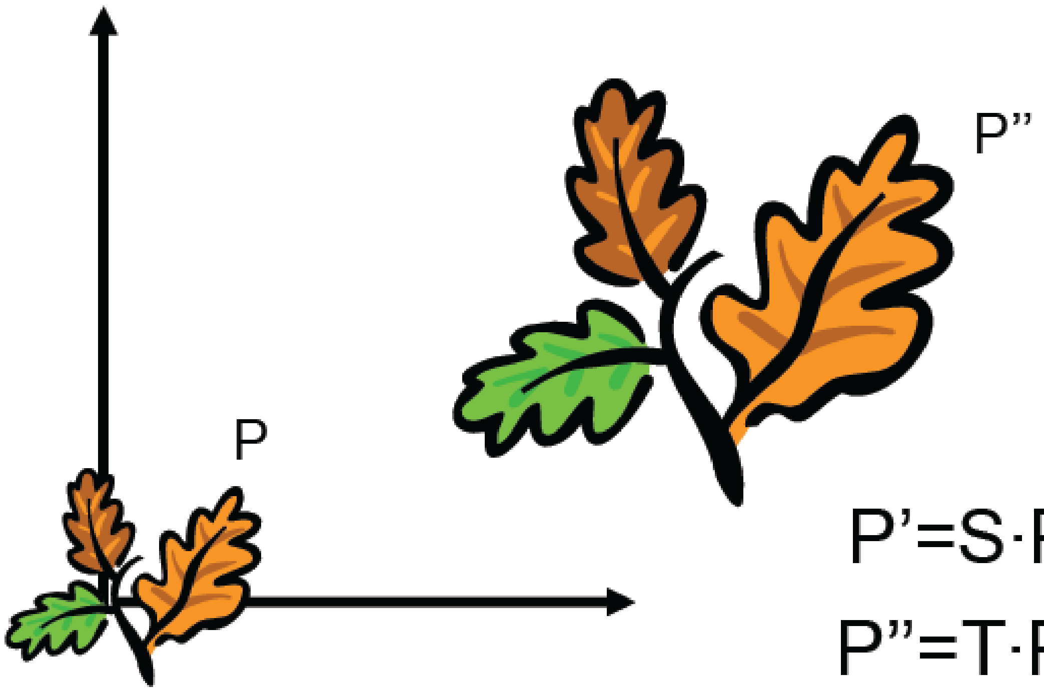
$$\mathbf{P} = (\mathrm{x}, \mathrm{y}) \rightarrow \mathbf{P'} = (\mathrm{s_x x}, \mathrm{s_y y})$$

$$\mathbf{P} = (x, y) \rightarrow (x, y, 1)$$

$$\mathbf{P'} = (s_x x, s_y y) \rightarrow (s_x x, s_y y, 1)$$

$$\mathbf{P'} \rightarrow \begin{bmatrix} s_x x \\ s_y y \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{S}} \cdot \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{S'} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix} \cdot \mathbf{P} = \mathbf{S} \cdot \mathbf{P}$$

Scaling & Translation



$$P' = S \cdot P$$

$$P'' = T \cdot P'$$

$$P'' = T \cdot P' = T \cdot (S \cdot P) = T \cdot S \cdot P = A \cdot P$$

Scaling & Translation

$$\mathbf{P''} = \mathbf{T} \cdot \mathbf{S} \cdot \mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \underbrace{\begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}}_{\text{A}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix} = \begin{bmatrix} S & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

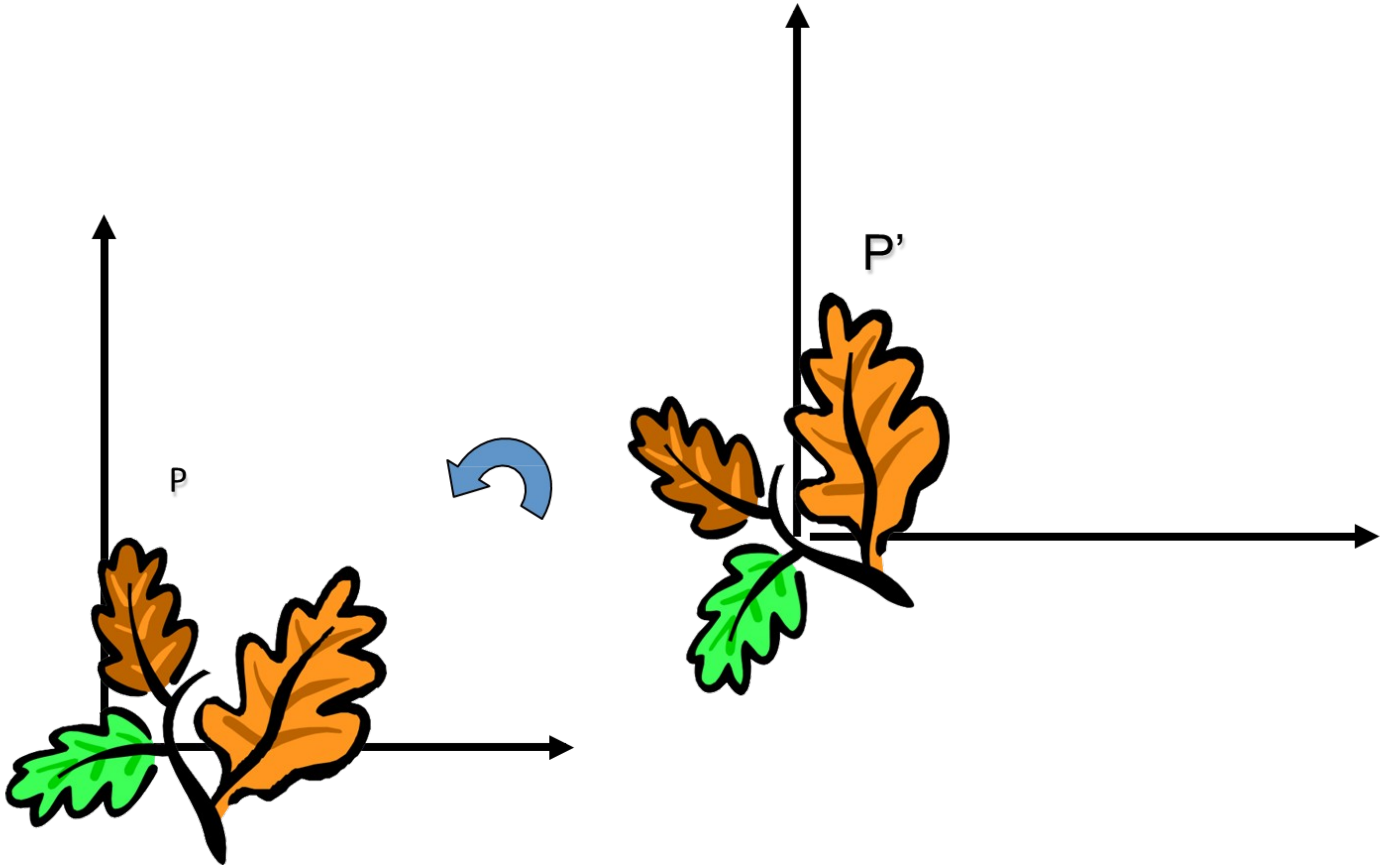
Order Matters

$$\mathbf{P'''} = \mathbf{T}\cdot\mathbf{S}\cdot\mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x & 0 & t_x \\ 0 & s_y & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + t_x \\ s_y y + t_y \\ 1 \end{bmatrix}$$

$$\mathbf{P'''} = \mathbf{S}\cdot\mathbf{T}\cdot\mathbf{P} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

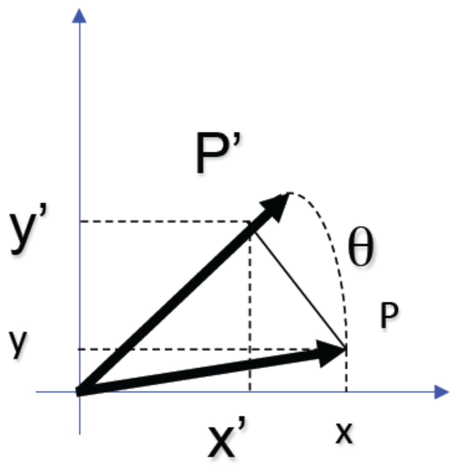
$$= \begin{bmatrix} s_x & 0 & s_x t_x \\ 0 & s_y & s_y t_y \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} s_x x + s_x t_x \\ s_y y + s_y t_y \\ 1 \end{bmatrix}$$

Rotation



Rotation Equation

Counter-clockwise rotation by an angle θ



$$x' = \cos\theta\ x - \sin\theta\ y$$

$$y' = \cos\theta\ y + \sin\theta\ x$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\mathbf{P'} = \mathbf{R}\ \mathbf{P}$$

Rotation Matrix Properties

- Transpose of a rotation matrix produces a rotation in the opposite direction

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

- The rows of a rotation matrix are always mutually perpendicular (a.k.a. orthogonal) unit vectors
  - (and so are its columns)

Properties

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

A 2D rotation matrix is 2x2

Note: R belongs to the category of *normal* matrices and satisfies many interesting properties:

$$\mathbf{R} \cdot \mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}} \cdot \mathbf{R} = \mathbf{I}$$

$$\det(\mathbf{R}) = 1$$

Scaling + Translation + Rotation

P'= (T R S) P

$$\mathbf{P}' = \mathbf{T}\cdot\mathrm{R}\cdot\mathbf{S}\cdot\mathbf{P} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & t_x \\ \sin\theta & \cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} =$$

$$= \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} R\,S & t \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

This is the form of the general-purpose transformation matrix