Sec 1.2 Orthogonal Vectors and Matrices

Goal: 1. Transpose & adjoint

2. inner product

3. orthogonal vectors

4. unitary matrices

1. Transpose & Adjoint

real matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix}$. (transpose of A)

Recall: for $z = a + bi \in \mathbb{C}$, $\bar{z} = a - bi \leftarrow$ complex conjugate of z

Complex matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} \Rightarrow A^* = \begin{bmatrix} \bar{a}_{11} & \bar{a}_{21} & \bar{a}_{31} \\ \bar{a}_{12} & \bar{a}_{22} & \bar{a}_{32} \end{bmatrix}$ adjoint of A (conjugate transpose)

For real matrix A, $A^* = A^T$.

Def: • If real $A = A^T$, then A is <u>symmetric</u>.

• If $A = A^*$, then A is <u>hermitian</u>

2. Inner Product.

• Inner product of two column vectors $\vec{x}, \vec{y} \in \mathbb{C}^m$: $\vec{x}^* \vec{y} := \sum_{i=1}^{m} \bar{x}_i y_i$

$(\bar{x}_1, \cdots, \bar{x}_m) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$

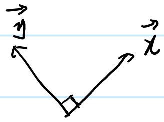
• Euclidian length of $\vec{x}$: $\|\vec{x}\| := \sqrt{\vec{x}^* \vec{x}} = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2}$

• Angle α between $\vec{x}$ and $\vec{y}$: $\cos\alpha = \dfrac{\vec{x}^* \vec{y}}{\|x\| \cdot \|y\|}$

	MATLAB Syntax	Comment
transpose A^T	A.'	transpose only
adjoint A^*	A'	conjugate transpose
inner product $\vec{x}^* \vec{y}$	x' * y	assume column vectors
Length $\|\vec{x}\|$	length(x), or norm(x)	

3. Orthogonal Vectors:

- The vectors $\vec{x}, \vec{y}$ are <u>orthognal</u> if $\vec{x}^* \vec{y} = 0$.

(if $\vec{x}, \vec{y} \in \mathbb{R}^m$, $\vec{x}^T \vec{y} = 0 \iff$  , e.g. $\vec{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{y} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\vec{x}^T y = 0$)

- Two sets of vectors X and Y are <u>orthogonal</u> if every $\vec{x} \in X$ is orthogonal to every $\vec{y} \in Y$

- A set of nonzero vectors S is <u>orthogonal</u> if its vectors are pairwise orthogonal, i.e., for any $\vec{x}, \vec{y} \in S$, $\vec{x} \neq \vec{y} \Rightarrow \vec{x}^* \vec{y} = 0$, and <u>orthonormal</u> if, in addition, every $\vec{x} \in S$ has $\|\vec{x}\| = 1$.

4. Unitary Matrices

- A square matrix $Q \in \mathbb{C}^{m \times m}$ is <u>unitary</u> (orthonormal in real case) if

$$Q^* = Q^{-1} \quad \text{i.e.} \quad Q^* Q = Q Q^* = I$$

$$Q^* Q = \begin{bmatrix} \vec{q}_1^* \\ \hline \vec{q}_2^* \\ \hline \vdots \\ \hline \vec{q}_m^* \end{bmatrix} \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \cdots & \vec{q}_m \end{bmatrix} = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

$$\Rightarrow q_i^* q_j = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

so the columns of a unitary matrix Q form an orthnormal basis of $\mathbb{C}^m$

- $(Q\vec{x})^* (Q\vec{y}) = \vec{x}^* Q^* Q \vec{y} = \vec{x}^* \cdot \vec{y}$

$\Rightarrow$ Inner product is preserved under multiplication by unitary Q.

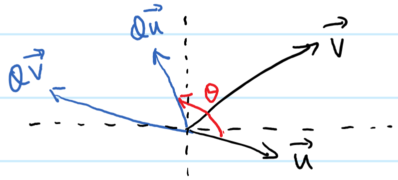
$\Rightarrow$ Lengths of vectors and angles between vectors are also presered. (ex.)

In the real case,

$$\det(QQ^T) = \det(I) \Rightarrow (\det(Q))^2 = 1 \Rightarrow \det(Q) = \pm 1.$$

Multiplication by an orthonormal matrix Q corresponds to a rigid rotation (if $\det Q = 1$) or reflection (if $\det Q = -1$)

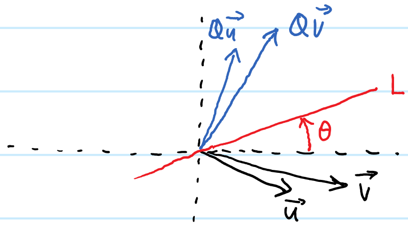
• rotation about the origin by an angle θ :



$$Q = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

eg. $\theta = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. rotate counterclockwise by 90°

• reflection about a line L through the origin which makes an angle θ with the x-axis



$$Q = \begin{pmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}$$

eg. $\theta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. reflection about the line $y = x$