# 3.2 QR Factorization

Topics: 1. Reduced QR factorization
2. Full QR factorization
3. Gram-Schmidt Orthogonalization
4. Solving $A\vec{x} = \vec{b}$ by QR factorization

## 1. Reduced QR factorization:

Suppose $A \in \mathbb{C}^{m\times n}$ $(m \geq n)$, $A = \left[\vec{a}_1 \middle| \cdots \middle| \vec{a}_n\right]$

successive spaces spanned by columns of $A$: $\langle \vec{a}_1 \rangle \subseteq \langle \vec{a}_1, \vec{a}_2 \rangle \subseteq \cdots \subseteq \langle \vec{a}_1, \cdots, \vec{a}_n \rangle$.

We want to find orthonormal vectors $\vec{q}_1, \cdots, \vec{q}_n$ such that

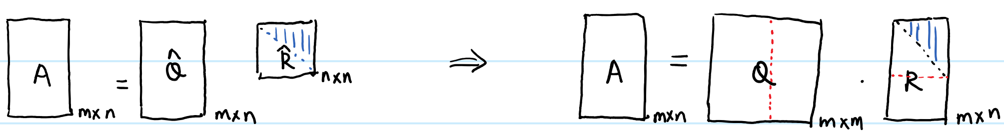
$$\langle \vec{q}_1, \cdots, \vec{q}_j \rangle = \langle \vec{a}_1, \cdots, \vec{a}_j \rangle, \quad j = 1, \cdots, n$$

i.e.
$$\underbrace{\begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix}}_{\hat{Q}} \cdot \underbrace{\begin{bmatrix} r_{11} & \cdots & r_{1n} \\ & \ddots & \vdots \\ & & r_{nn} \end{bmatrix}}_{\hat{R}}$$

$A = \hat{Q}\cdot\hat{R}$, where $\hat{Q} \in \mathbb{C}^{m\times n}$ has orthonormal columns, $\hat{R} \in \mathbb{C}^{n\times n}$ is upper triangular } reduced QR factorization of A

## 2. Full QR factorization:

If we add $(m-n)$ orthonormal columns to $\hat{Q}$ and $(m-n)$ zero rows to $\hat{R}$, then




(reduced QR factorization) (full QR factorization)

$A = QR$, where $Q \in \mathbb{C}^{m\times m}$ unitary, $R \in \mathbb{C}^{m\times n}$ upper triangular } full QR factorization

Remark: For $j = n+1, \cdots, m$, $q_j \perp \text{range}(A)$

## 3. Gram-Schmidt Orthogonalization:

Goal: Find unit vector $\vec{q}_j \in \langle \vec{a}_1, \cdots, \vec{a}_j \rangle$ that is orthogonal to $\langle \vec{q}_1, \cdots, \vec{q}_{j-1} \rangle$.

Let $\vec{v}_j = \vec{a}_j - \underbrace{(\vec{q}_1^{\,*} \vec{a}_j)}_{r_{1j}} \vec{q}_1 - \underbrace{(\vec{q}_2^{\,*} \vec{a}_j)}_{r_{2j}} \vec{q}_2 - \cdots \underbrace{(\vec{q}_{j-1}^{\,*} \vec{a}_j)}_{r_{j-1,j}} \vec{q}_{j-1}$.

Then $(\vec{q}_k, \vec{v}_j) = 0$ for $k = 1, \cdots, j-1$, i.e. $\vec{v}_j \perp \langle \vec{q}_1, \cdots, \hat{q}_{j-1} \rangle$ (verify).

Let $\vec{q}_j = \vec{v}_j / \|\vec{v}_j\|$, then $\{\vec{q}_1, \cdots, \vec{q}_j\}$ orthonormal and $\langle \vec{q}_1, \cdots, \vec{q}_j \rangle = \langle \vec{a}_1, \cdots, \vec{a}_j \rangle$.

So the process is:

$$\vec{q}_1 = \vec{a}_1 / r_{11}$$
$$\vec{q}_2 = (\vec{a}_2 - r_{12}\vec{q}_1) / r_{22}$$
$$\vec{q}_3 = (\vec{a}_3 - r_{13}\vec{q}_1 - r_{23}\vec{q}_2) / r_{33}$$
$$\vdots$$
$$\vec{q}_n = (\vec{a}_n - r_{1n}\vec{q}_1 - \cdots - r_{n-1,n}\vec{q}_{n-1}) / r_{nn} = \left(\vec{a}_n - \sum_{i=1}^{n-1} r_{in}\vec{q}_i\right) / r_{nn}$$

where $r_{ij} = (\vec{q}_i^{\,*} a_j)$ for $i = 1, \cdots, j-1$,

$$r_{jj} = \left\| \vec{a}_j - \sum_{i=1}^{j-1} r_{ij}\vec{q}_i \right\|_2 .$$

Algorithm (Classical Gram-Schmidt Orthogonalization (unstable)):

for $j = 1 : n$
- $v_j = a_j$
- for $i = 1 : (j-1)$
  - $r_{ij} = q_i^* a_j$
  - $v_j = v_j - r_{ij} q_i$
- $r_{jj} = \|v_j\|_2$
- $q_j = v_j / r_{jj}$

Theorem (Existence and Uniqueness of QR factorization):

Every $A \in \mathbb{C}^{m \times n}$ $(m \geq n)$ has a reduced QR factorization and a full QR factorization.

If $A$ has full rank, then it has a <u>unique</u> reduced QR factorization with $r_{jj} > 0$.

4. Solving $A\vec{x} = \vec{b}$ by QR factorization:

$$A\vec{x} = \vec{b} \Rightarrow QR\vec{x} = \vec{b} \Rightarrow R\vec{x} = Q^*\vec{b}.$$ upper triangular system

step 1: Compute QR factorization: $A = QR$

step 2: Compute $\vec{y} = Q^*\vec{b}$

step 3: Use backward substitution to solve $R\vec{x} = \vec{y}$

Remark: Gaussian Elimination (or LU) is generally used in practice, requiring only half # of flops.