Sec 1.3 Norms

Goal: 1. Vector norms

2. Matrix norms

1. Vector norms.

- A <u>norm</u> is a function $\|\cdot\| : \mathbb{C}^m \longrightarrow \mathbb{R}$ satisfying
  - (1) $\|\vec{x}\| \geq 0$, and $\|\vec{x}\| = 0$ only if $\vec{x} = \vec{0}$
  - (2) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$ (triangle inequality)
  - (3) $\|\alpha \vec{x}\| = |\alpha| \cdot \|\vec{x}\|$, $\forall \alpha \in \mathbb{C}$

  eg. The Euclidean length $\|\vec{x}\|_2 = \sqrt{x^* x}$ is a norm.

- The p-norms: $\|\vec{x}\|_p = \left(\sum_{i=1}^{m} |x_i|^p \right)^{1/p}$, $1 \leq p < \infty$

  special cases: 1-norm: $\|\vec{x}\|_1 = \sum_{i=1}^{m} |x_i|$

  2-norm: $\|\vec{x}\|_2 = \left(\sum_{i=1}^{m} |x_i|^2 \right)^{1/2} = \sqrt{\vec{x}^* \vec{x}}$

  infinity-norm: $\|\vec{x}\|_\infty = \max_{1 \leq i \leq m} |x_i|$

  eg. $\vec{x} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$. $\Longrightarrow$ $\|x\|_1 = 1 + 2 + 3 = 6$

  $\|x\|_2 = (1 + 4 + 9)^{1/2} = \sqrt{14}$

  $\|x\|_\infty = \max\{1, 2, 3\} = 3$

- The weighted norm:

  Given any norm $\|\cdot\|$ and a diagonal matrix $W = \begin{pmatrix} w_1 & & & \\ & w_2 & & \\ & & \ddots & \\ & & & w_m \end{pmatrix}$ with all $w_i \neq 0$, a weighted norm can be defined as

  weights

  $$\|\vec{x}\|_W := \|W\vec{x}\| = \left\| \begin{pmatrix} w_1 x_1 \\ \vdots \\ w_m x_m \end{pmatrix} \right\|$$

2. Matrix norm

- <u>Def</u>: A <u>matrix norm</u> $\|\cdot\| : \mathbb{C}^{m \times n} \longrightarrow \mathbb{R}$ must sastisfy:

(1) $\|A\| \geq 0$, and $\|A\| = 0$ only if $A = 0$

(2) $\|A+B\| \leq \|A\| + \|B\|$ (triangle inequality)

(3) $\|\alpha A\| = |\alpha| \cdot \|A\|$

Example: $\|A\|_1 = \max_{1 \leq j \leq n} \|A(:,j)\|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}|$ (max. column sum)

$\|A\|_\infty = \max_{1 \leq i \leq m} \|A(i,:)\|_1 = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}|$ (max. row sum)

$\|A\|_F = \left(\sum_{i=1}^{m} \sum_{j=1}^{n} |a_{ij}|^2 \right)^{1/2}$ (The Frobenius norm)

In general, $\|AB\| \leq \|A\| \cdot \|B\|$

- For a matrix $A \in \mathbb{C}^{m \times n}$, the <u>induced matrix norm</u> is

$$\|A\|_{(m,n)} = \sup_{\substack{\vec{x} \in \mathbb{C}^n \\ \vec{x} \neq 0}} \frac{\|A\vec{x}\|_{(m)}}{\|\vec{x}\|_{(n)}} = \sup_{\substack{\vec{x} \in \mathbb{C}^n \\ \|\vec{x}\| = 1}} \|A\vec{x}\|_{(m)}$$

where $\|\cdot\|_{(m)}$ and $\|\cdot\|_{(n)}$ are given vector norms

$\|A\|_{(m,n)}$ is the maximum factor by which A can "stretch" a vector.

	MATLAB Syntax
$\|\vec{x}\|_p$	norm (x, p)
$\|\vec{x}\|_\infty$	norm (x, inf)
$\|\vec{x}\|_2$	norm (x)

	MATLAB Syntax
$\|A\|_p$	norm (A, p)
$\|A\|_\infty$	norm (A, inf)
$\|A\|_F$	norm (A, 'fro')