Sec 2.4 Cholesky Factorization

Goal: 1. Symmetric Positive Definite (SPD) matrices

2. Cholesky Factorization

1. <u>Def</u>: $A \in \mathbb{R}^{n\times n}$ is <u>symmetric</u> $\iff$ $A = A^T$ i.e. $a_{ij} = a_{ji}$.

<u>Def</u>: $A \in \mathbb{R}^{n\times n}$ is <u>positive definite</u> if it is symmetric and $\vec{x}^T A \vec{x} > 0$ for $\forall\, \vec{x} \neq \vec{0}$

<u>Def</u>: Given $A \in \mathbb{R}^{n\times n}$, the <u>leading principal submatrix</u> of order k is

$$A_k = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} = A(1:k,\ 1:k). \quad \text{for some } 1 \le k \le n$$

<u>Theorem</u>: A symmetric matrix A is positive definite $\iff$ $\det(A_k) > 0$, $k = 1, \cdots, n$.

e.g. $A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$. $\det(A_1) = 2$, $\det(A_2) = 3$, $\det(A_3) = 8 - 2 - 2 = 4$

A symmetric, $\det(A_k) > 0$, $k = 1, 2, 3 \Rightarrow A$ is positive definite

<u>Theorem</u>: A is symmetric positive definite (SPD)

$\iff$ Gaussian Elimination without pivoting has all pivot elements positive.

Suppose A is SPD.

$$A = \left(\begin{array}{c|c} a_{11} & \vec{z}_1^{\,T} \\ \hline \vec{z}_1 & B_1 \end{array}\right) \xrightarrow{G.E.} M_1 A = \left(\begin{array}{c|c} a_{11} & z^T \\ \hline \vec{0} & \widetilde{B}_1 \end{array}\right) \Rightarrow M_1 A M_1^T = \left(\begin{array}{c|c} a_{11} & \vec{0} \\ \hline \vec{0} & \widetilde{B}_1 \end{array}\right)$$

A is SPD $\Rightarrow$ $M_1 A M_1^T$ is SPD $\Rightarrow$ $a_{11} > 0$, $\widetilde{B}_1$ is SPD

similarly, $M_2 M_1 A M_1^T M_2^T = \left(\begin{array}{c|cc} a_{11} & \multicolumn{2}{c}{\vec{0}} \\ \hline \vec{0} & a_{22}^{(2)} & \vec{0} \\ \cline{2-3} & \vec{0} & \widetilde{B}_2 \end{array}\right)$,

$$\Rightarrow \cdots \Rightarrow \underbrace{M_{n-1} \cdots M_1}_{L^{-1}} A \underbrace{M_1^T \cdots M_{n-1}^T}_{(L^T)^{-1}} = \underbrace{\begin{pmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn} \end{pmatrix}}_{D}, \text{ where } d_{ii} > 0$$

$\Rightarrow \quad A = LDL^T$.

D is diagonal, positive $\Rightarrow D = \tilde{D}^2 = \tilde{D}\cdot\tilde{D}^T$

$$A = LDL^T = L\tilde{D}\cdot\tilde{D}^T L^T = \underbrace{(L\tilde{D})}_{\tilde{L}}\cdot(L\tilde{D})^T = \tilde{L}\cdot\tilde{L}^T.$$

2. <u>Cholesky factorization</u> :

A is positive definite $\Leftrightarrow A = L\cdot L^T = R^T\cdot R$, where L is lower triangular, R is upper triangular

$\triangle \quad \nabla$

<u>Remark</u>: positive definite $A \Rightarrow$ no need for pivoting

Algorithm (Cholesky factorization)

Inputs: A, $n\times n$ matrix

Outputs: R

step 1: Set $R = \text{triu}(A)$, the upper triangular part of A

step 2: for $k = 1, \cdots, n$, do step 3 & 4

step 3: for $i = k+1, \cdots, n$

$m_{i,k} = R_{k,i}/R_{kk}$

$R_{i,i:n} = R_{i,i:n} - m_{i,k} * R_{k,i:n}$

step 4: $R_{k,k:n} = R_{k,k:n} / \sqrt{R_{k,k}}$

Operation Count :

$$\sum_{k=1}^{n}\sum_{i=k+1}^{n}\Big(\underbrace{1}_{\text{division}} + \underbrace{(n-i+1)}_{\text{multiplications}} + \underbrace{(n-i+1)}_{\text{subtractions}}\Big) + \sum_{k=1}^{n}\Big(\underbrace{1}_{\text{sqrt}} + \underbrace{(n-k+1)}_{\text{division}}\Big)$$

$$\sim \sum_{k=1}^{n}\sum_{i=k+1}^{n} 2\underbrace{(n-i+1)}_{j} \sim 2\sum_{k=1}^{n}\sum_{j=1}^{n-k} j = 2\sum_{k=1}^{n}\frac{(1+n-k)(n-k)}{2} \underset{j=n-k}{=} \sum_{j=1}^{n-1} j(j+1) \sim \sum_{j=1}^{n-1} j^2$$

$$= \frac{(n-1)n(2n-1)}{6} \sim \frac{1}{3}n^3 \text{ flops.}$$

Recall: Gauss Elimination $\sim \frac{2}{3}n^3$ flops.