2.2 Pivoting Strategies

Goal: 1. Motivating example
2. Partial Pivoting
3. Complete Pivoting

1. eg. 1. $\begin{cases} 0.003000x_1 + 59.14x_2 = 59.17 & (E_1) \\ 5.291x_1 - 6.130x_2 = 46.78 & (E_2) \end{cases}$

exact solution: $x_1 = 10.00$, $x_2 = 1.000$

Now we apply Gaussian Elimination using four-digit arithmetic with rounding.

$$m_2 = \frac{5.291}{0.003000} = 1763.6\bar{6} \approx 1764$$

$$(E_2 - m_2E_1) \to (E_2): \left(\begin{aligned} &(-6.130 - \underbrace{1764 * 59.14}_{104322.76})x_2 = 46.78 - \underbrace{1764 * 59.17}_{104375.88} \\ &(-6.130 - 104300)x_2 \approx 46.78 - 104400 \end{aligned} \right)$$

$$104300\, x_2 \approx 104400$$

$$x_2 \approx 1.001$$

$$x_1 = (59.17 - 59.14x_2)/0.003000 = (59.17 - 59.14 * 1.001)/0.003000$$

$$= (59.17 - 59.19914)/0.003000 \approx (59.17 - 59.20)/0.003000$$

$$= -0.03000/0.003000 = -10$$

But in fact, $x_1 = 10$. Round-off error dominated the calculation!

Reason: ① $m_i = \dfrac{a_{ik}^{(k)}}{a_{kk}^{(k)}}$, if $a_{kk}^{(k)} < a_{ik}^{(k)}$, then $m_i > 1$, round-off error is magnified.

② $x_i = \dfrac{a_{i,n+1}^{(n)} - \Sigma a_{ij}^{(n)} x_j}{a_{ii}^{(n)}} = \dfrac{a_{i,n+1}^{(i)} - \sum_{j=i+1}^{n} a_{ij}^{(i)} x_j}{a_{ii}^{(i)}}$

If $a_{ii}^{(i)}$ is small, then error in the numerator will be magnified.

2. Partial Pivoting:

Before eliminating the k^{th} column below the diagonal, find $P \geq k$ such that

$$|a_{pk}^{(k)}| = \max_{k \leq i \leq n} |a_{ik}^{(k)}| \quad \text{and perform } (E_k) \leftrightarrow (E_p).$$

9. Using partial pivoting and 4-digit arithmetic with rounding to solve

$$\begin{cases} 0.003000 x_1 + 59.14 x_2 = 59.17 \\ 5.291 x_1 - 6.130 x_2 = 46.78 \end{cases}$$

partial pivoting $\Longrightarrow$ $\begin{cases} 5.291 x_1 - 6.130 x_2 = 46.78 & (E_1) \\ 0.003000 x_1 + 59.14 x_2 = 59.17 & (E_2) \end{cases}$

$$m_2 = \frac{0.003000}{5.291} \approx 0.0005670$$

$$(E_2 - m_2 E_1) \to (E_2): \quad \underbrace{(59.14 + \underbrace{0.0005670 * 6.130}_{\approx 0.003476})}_{59.143476 \approx 59.14} x_2 = \underbrace{59.17 - \underbrace{0.0005670 * 46.78}_{\approx 0.02652}}_{59.1438 \approx 59.14}$$

$$59.14 x_2 \approx 59.14$$

$$x_2 \approx 1.000$$

$$x_1 = (46.78 + 6.130 x_2)/5.291 = (46.78 + 6.130 * 1.000)/5.291$$

$$= (46.78 + 6.130)/5.291 = 52.91/5.291 = 10.00$$

Algorithm (Gaussian Elimination with partial Pivoting)

elimination

step 1: For $k = 1, \cdots, n-1$, do step 2 & 3

pivoting { step 2: Find $P \ge k$ such that $|a_{pk}^{(k)}| = \max_{k \le i \le n} |a_{ik}^{(k)}|$ and perform $(E_k) \longleftrightarrow (E_p)$.

eliminating kth column { step 3: For $i = k+1, \cdots, n$, do step 4

step 4: set $m_i = a_{ik}/a_{kk}$ and perform $A(i, k+1:n+1) = A(i, k+1:n+1) - m_i * A(k, k+1:n+1)$

backward substitution { step 5: set $x_n = a_{n,n+1}/a_{nn}$

step 6: For $i = n-1, \cdots, 1$, set $x_i = (a_{i,n+1} - \sum_{j=i+1}^{n} a_{ij} x_j)/a_{ii}$

step 7: Output $x_1, \cdots, x_n$

\# flops in Gaussian Elimination without pivoting $\approx \frac{2}{3} n^3$

\# of comparisons due to partial pivoting:

$$\sum_{k=1}^{n-1} (n-k) = \sum_{i=1}^{n-1} i = \frac{n(n-1)}{2} = \frac{n^2}{2} + L.O.T. \approx \frac{n^2}{2}.$$

$\Bigg\} \approx \frac{2}{3} n^3$

3. Complete pivoting:

Find $p, q \geq k$ such that $|a_{p,q}^{(k)}| = \max\limits_{k \leq i,j \leq n} |a_{i,j}|$ and then

① interchange the k^{th} row and the p^{th} row

② interchange the kth column and the q^{th} colum

So that $a_{pq}^{(k)}$ is the pivot.

# of comparisons introduced by complete pivoting:

$$\sum_{k=1}^{n} (k^2-1) = \sum_{k=1}^{n} k^2 - n = \frac{n(n+1)(2n+1)}{6} - n \approx \frac{n^3}{3}, \qquad \text{expensive!}$$