Sec 2.3 LU Factorization

Goal: 1. Motivation: why LU factorization?

2. LU factorization and Gaussian Elimination (GE) without pivoting

3. $PA = LU$ factorization and GE with pivoting

1. Motivation:

If $A = LU$ (U: upper trangular, L: lower trangular), then $A\vec{x} = \vec{b} \iff L\underbrace{U\vec{x}}_{\vec{y}} = \vec{b}$, $L\vec{y} = \vec{b}$

step 1: Solve $L\vec{y} = \vec{b}$ for $\vec{y}$, forward substitution, $O(n^2)$ operations

step 2: Solve $U\vec{x} = \vec{y}$ for $\vec{x}$, backward substitution, $O(n^2)$ operations

Determine the factorization $A = LU$ requires $O(\frac{2}{3}n^3)$ operations. (only once)

But it can be used to solve $A\vec{x} = \vec{b}$ for different $\vec{b}$

2. $A = LU$ factorization and Gaussian Elimination without pivoting:

$A = A_1 \longrightarrow A_2$:

$(E_i - m_{i,1}E_1) \longrightarrow (E_i)$ where $m_{i,1} = \dfrac{a_{i1}^{(1)}}{a_{11}^{(1)}}$, $i = 2, \cdots, n$

$$A_2 = M_1 \cdot A_1 \quad \text{where } M_1 = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -m_{21} & 1 & \ddots & \vdots \\ \vdots & 0 & \ddots & 0 \\ -m_{n1} & 0 & \cdots\ 0 & 1 \end{pmatrix}$$

$$A\vec{x} = \vec{b} \iff \underbrace{M_1 \cdot A}_{A_2}\vec{x} = \underbrace{M_1 \cdot \vec{b}}_{\vec{b}_2}$$

$A_2\vec{x} = \vec{b}_2$

Similarly, $m_{i,2} = \dfrac{a_{i2}^{(2)}}{a_{22}^{(2)}}$, $i = 3, \cdots, n$,

$$M_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & -m_{32} & \ddots & \\ & \vdots & & \\ & -m_{n2} & & 1 \end{pmatrix}$$

$\underbrace{M_2 A_2}_{A_3}\vec{x} = \underbrace{M_2 \vec{b}_2}_{\vec{b}_3}$, where $A_3 = M_2 \cdot A_2 = M_2 \cdot M_1 \cdot A$

$\vec{b}_3 = M_2 \cdot \vec{b}_2 = M_2 \cdot M_1 \cdot \vec{b}$

In general,

$m_{i,k} = \dfrac{a_{i,k}^{(k)}}{a_{kk}^{(k)}}$, $i = k+1, \cdots, n$,

$$M_k = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & -m_{k+1,k} & \ddots & \\ & \vdots & & \\ & -m_{n,k} & & 1 \end{pmatrix}$$

$$A_{k+1}\vec{x} = \vec{b}_{k+1}, \quad \text{where } A_{k+1} = M_k \cdots M_1 A$$
$$\vec{b}_{k+1} = M_k \cdots M_1 \cdot \vec{b}$$

At the end, $A_n \cdot \vec{x} = \vec{b}_n$,

where $A_n = M_{(n-1)} \cdots M_1 \cdot A = \underbrace{\begin{pmatrix} a_{11}^{(1)} & a_{12}^{(1)} & \cdots & a_{1n}^{(1)} \\ 0 & a_{22}^{(2)} & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & a_{nn}^{(n)} \end{pmatrix}}_{U}$, upper triangular

$$\vec{b}_n = M_{n-1} \cdots M_1 \cdot \vec{b}$$

$$A_n = M_{n-1} \cdots M_1 \cdot A \Rightarrow \underbrace{[M_1]^{-1} \cdots [M_{n-1}]^{-1}}_{L} \cdot \underbrace{A_n}_{U} = A$$

<u>Claim</u>: $L = \begin{pmatrix} 1 & & & 0 \\ m_{21} & 1 & & \\ \vdots & & \ddots & \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{pmatrix}$, lower triangular

proof: $M_k : (E_i - m_{i,k}E_k) \to (E_i)$ for $i = k+1, \cdots, n$

$[M_k]^{-1} : (E_i + m_{i,k}E_k) \to (E_i)$

So $L_k := [M_k]^{-1} = \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & m_{k+1,k} & \ddots & \\ & \vdots & & \\ & m_{n,k} & & 1 \end{pmatrix} = I + \vec{v}^{(k)} e_k^T$

$$\vec{v}^{(k)} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ m_{k+1,k} \\ \vdots \\ m_{n,k} \end{pmatrix}, \quad e_k^T = (0, \cdots, 0, 1, 0 \cdots 0)$$

$$L = L_1 \cdots L_{n-1} = (I + \vec{v}^{(1)} e_1^T) \cdots (I + \cdots + \vec{v}^{(n-1)} e_{n-1}^T)$$
$$= I + \vec{v}^{(1)} e_1^T + \cdots + \vec{v}^{(n-1)} e_{n-1}^T = \begin{pmatrix} 1 & & & 0 \\ m_{21} & 1 & & \\ \vdots & & \ddots & \\ m_{n1} & \cdots & m_{n,n-1} & 1 \end{pmatrix}$$

$\vec{v}^{(i)} \underbrace{e_i^T \cdot \vec{v}^{(j)}}_{0 \text{ if } i \le j} e_j^T = 0$

lower triangular

Remark: • $A = LU \Rightarrow \det(A) = \underbrace{\det(L)}_{1} \cdot \det(U) = \det(U)$

• A square matrix A has an LU factorization if all leading principal minors are nonzero, i.e., $\det(A(1:k, 1:k)) \neq 0$, $k = 1, \cdots, n-1$.

• If in addition, $\det(A) \neq 0$, then the LU factorization is unique.

e.g. Find LU Factorization of $A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{pmatrix}$

$$\begin{pmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{pmatrix} \xrightarrow[(E_3 - \frac{1}{2}E_1) \to (E_3)]{(E_2 - \frac{1}{2}E_1) \to (E_2)} \begin{pmatrix} 2 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 1 & -1 \end{pmatrix} \xrightarrow{(E_3 - \frac{1}{3}E_2) \to E_3} \begin{pmatrix} 2 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & -\frac{7}{3} \end{pmatrix}$$

So $A = LU = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 4 & 4 \\ 0 & 3 & 4 \\ 0 & 0 & -\frac{7}{3} \end{pmatrix}$

<u>Algorithm</u>: (LU Factorization)

Inputs : $A = (a_{ij}) \in \mathbb{R}^{n \times n}$

Outputs : $L = (l_{ij})$, $U = (u_{ij})$.

step 1 : Set $U = A$, $L = I$

step 2 : For $k = 1, \cdots, n-1$

For $j = k+1, \cdots, n$

set $l_{jk} = a_{jk} / a_{kk}$

$a_{j,k:m} = a_{j,k:m} - l_{jk} \cdot a_{k,k:m}$

$U = \text{triu}(A)$ % taking the upper triangular part of A

2. $PA = LU$ factorization and Gaussian Elimination with partial pivoting

<u>Def</u> : <u>Permutation matrix</u> is a matrix obtained by rearranging the rows of I_n

e.g. $P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_3 \\ e_2 \end{pmatrix}$ is a permutation matrix.

suppose $A = (a_{ij})$. $PA = \begin{pmatrix} A(1,:) \\ A(3,:) \\ A(2,:) \end{pmatrix}$ $AP = (A(:,1), A(:,3), A(:,2))$

Remark: $PP^T = P^TP = I$, i.e., $\boxed{P^T = P^{-1}}$

- In general, Gaussian Elimination with row interchanges $\Rightarrow$

$$M_{n-1} P_{n-1} \cdots M_2 P_2 M_1 P_1 \cdot A = U.$$

$$\Rightarrow \underbrace{\tilde{M}_{n-1} \cdots \tilde{M}_2 \tilde{M}_1}_{L^{-1}} \cdot \underbrace{P_{n-1} \cdots P_2 \cdot P_1}_{P} \cdot A = U,$$ where $\tilde{M}_k$ has the same structure as M_k

Then $L^{-1} P A = U \Rightarrow PA = LU$,

upper triangular

permutation matrix

lower triangular

To solve $A\vec{x} = \vec{b}$: $\underbrace{PA}\,\vec{x} = \underbrace{P\vec{b}}_{\tilde{b}} \Leftrightarrow LU\vec{x} = \tilde{b} \Leftrightarrow$ solve $L\vec{y} = \tilde{b}$ for $\vec{y}$

solve $U\vec{x} = \vec{y}$ for $\vec{x}$

Remark: $PA = LU \Rightarrow A = P^{-1}LU = (P^T L)\cdot U$, $P^T L$ is not lower triangular unless $P = I_n$

eg. Determine a factorization of the form $A = (P^T L) U$ for $A = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix}$

$$A \xrightarrow{(E_1)\leftrightarrow(E_2)} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & 2 & 0 \\ 1 & 2 & 0 & 2 \end{pmatrix} \xrightarrow[(E_4 - E_1)\to(E_4)]{(E_3-(-1)E_1)\to(E_3)} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

$$\xrightarrow{E_2 \leftrightarrow E_4} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{(E_4-(-1)E_3)\to(E_4)} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix} = U$$

So $E_1 \leftrightarrow E_2$, $E_2 \leftrightarrow E_4 \Rightarrow P = \begin{pmatrix} e_2 \\ e_4 \\ e_3 \\ e_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$

$$PA = \begin{pmatrix} 1 & 1 & -1 & 2 \\ 1 & 2 & 0 & 2 \\ -1 & -1 & 2 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow[(E_3-(-1)E_1)\to(E_3)]{(E_2-E_1)\to(E_2)} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 1 \end{pmatrix} \xrightarrow{(E_4-(-1)E_3)\to E_3} \begin{pmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

So $L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \Rightarrow P^T L = \begin{pmatrix} e_4 \\ e_1 \\ e_3 \\ e_2 \end{pmatrix} L = \begin{pmatrix} L(4,:) \\ L(1,:) \\ L(3,:) \\ L(2,:) \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$