Sec 1.1 Basic Linear Algebra

1. Matrix * Vector Multiplication:

$A = (a_{ij})$ is $m\times n$ matrix, $\quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \in \mathbb{C}^{m\times n}$

m rows, n columns

$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{C}^{n\times 1}$

$A\vec{x} = ?$

- Familar Definition:

$$\begin{pmatrix} \boxed{a_{11} \ \cdots \ a_{1n}} \\ \vdots \\ \boxed{a_{m1} \ \cdots \ a_{mn}} \end{pmatrix} \begin{pmatrix} \boxed{\begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix}} \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + \cdots + a_{1n}x_n \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} \sum_{j=1}^n a_{1j}x_j \\ \vdots \\ \sum_{j=1}^n a_{mj}x_j \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} = \vec{b}$$

$$\Rightarrow \quad b_i = \sum_{j=1}^n a_{ij}x_j, \qquad i = 1, \cdots, m$$

We can consider A as a linear mapping. $\vec{x}$ is input, $\overset{\vec{b}}{\overset{\shortparallel}{A\vec{x}}}$ is output

The map $x \longmapsto Ax$ is linear: $\quad A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

$A(\alpha\vec{x}) = \alpha\cdot A\vec{x}$

- Another way to understand this:

$A = \left(\vec{a}_1 \middle| \vec{a}_2 \middle| \cdots \middle| \vec{a}_n \right)$, eg. $\vec{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix}$

$$A\vec{x} = \left[\vec{a}_1 \middle| \vec{a}_2 \middle| \cdots \middle| \vec{a}_n \right] \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \underbrace{x_1 \left[\vec{a}_1\right] + x_2\left[\vec{a}_2\right] + \cdots + x_n\left[\vec{a}_n\right]} = \left[\vec{b}\right]$$

$\vec{b}$ is a linear combination of columns of A

The coefficients are from $\vec{x}$

2. Matrix * Matrix Multiplication:

$A = (a_{ik})$ is $l\times m$, $\quad C = (c_{kj})$ is $m\times n \quad \Rightarrow \quad B = AC$ is $l\times n$

$B = (b_{ij})$, $\quad b_{ij} = (i^{th}$ row of $A)\cdot(j^{th}$ column of $C) = \sum_{k=1}^m a_{ik}c_{kj}$

Another way to understand this:

$$B = AC = A\left[\vec{c}_1 \middle| \vec{c}_2 \middle| \cdots \middle| \vec{c}_n\right] = \left[A\vec{c}_1 \middle| A\vec{c}_2 \middle| \cdots \middle| A\vec{c}_n\right] = \left[\vec{b}_1 \middle| \vec{b}_2 \middle| \cdots \middle| \vec{b}_n\right]$$

$$\Rightarrow \vec{b_j} = A\vec{c_j} = \left[\vec{a_1}\middle|\vec{a_2}\middle|\cdots\middle|\vec{a_m}\right]\begin{bmatrix} c_{1j} \\ \vdots \\ c_{mj} \end{bmatrix} = \vec{a_1}\cdot c_{1j} + \cdots + \vec{a_m}\cdot c_{mj} = \sum_{k=1}^{m} c_{kj}\,\vec{a_k}$$

The column $\vec{b_j}$ is a linear combination of the columns of A with coefficients from $\vec{c_j}$

eg. (Outer product)

$$\vec{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix},\ m\times 1, \qquad \vec{v} = [v_1, \cdots, v_n],\ 1\times n$$

$$\Rightarrow \vec{u}\vec{v} = \vec{u}\,[v_1, \cdots, v_n] = \left[v_1\vec{u}\middle|\cdots\middle|v_n\vec{u}\right] = \begin{bmatrix} v_1u_1 & \cdots & v_nu_1 \\ v_1u_2 & & v_nu_2 \\ \vdots & & \vdots \\ v_1u_m & \cdots & v_nu_m \end{bmatrix}$$

3. Range and Nullspace of A:

range (A) = the set of all vectors that can be expressed as $A\vec{x}$ for some $\vec{x}$
= all linear combinations of columns of A
= the space spanned by columns of A.

null (A) = the set of all vectors $\vec{x}$ that satisf $A\vec{x} = \vec{0}$
= coefficients $x_1, \cdots, x_n$ such that the linear combination $x_1\vec{a_1} + \cdots + x_n\vec{a_n} = \vec{0}$

4. Rank

dimension of a space = number of elements in a basis.

column rank of A = dimension of the column space of A
= dimesion of range(A)

row rank of A = dimension of the space spanned by rows of A

☆ rank of A = column rank of A = row rank of A

If $A \in \mathbb{C}^{m\times n}$ has the maximal possible rank, i.e. rank$(A) = \min(m, n)$ then A is of <u>full rank</u>.

5. Inverse matrix.

A is <u>nonsingular</u> or <u>invertible</u> $\Leftrightarrow$ A is a square matrix of full rank.
$\Leftrightarrow$ There is a unique matrix A^{-1} such that
$AA^{-1} = A^{-1}A = I \leftarrow$ identity matrix $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}$
$\Leftrightarrow$ $\det(A) \neq 0$

6. Matrix Inverse * vector multiplication:

$\vec{x} = A^{-1}\vec{b}$ = solution to $A\vec{x} = \vec{b}$

= the vector of coefficients when expanding $\vec{b}$ using the basis of columns of A.

$$\boxed{\begin{array}{l}\vec{b}: \text{ coefficients of expansion} \\ \text{of } \vec{b} \text{ using the basis} \\ \{\vec{e}_1, \cdots, \vec{e}_m\}\end{array}} \;\underset{\text{multiply by } A}{\overset{\text{multiply by } A^{-1}}{\rightleftarrows}}\; \boxed{\begin{array}{l}A^{-1}\vec{b}: \text{ coefficients of expansion} \\ \text{of } \vec{b} \text{ using the basis} \\ \{\vec{a}_1, \cdots, \vec{a}_m\}\end{array}}$$

Multiplication by A^{-1} is a <u>change of basis</u> operation!