1) Given

$$A = \begin{pmatrix} 1 & 3 & -2 \\ -4 & 0 & -1 \\ 2 & -2 & 0 \end{pmatrix} \quad \text{and} \quad \vec{v} = \begin{pmatrix} -2 \\ 1+i \\ 1 \end{pmatrix}$$

a)

$$A\vec{v} = \begin{pmatrix} -2+3+3i-2 \\ 8+0-1 \\ -4-2-2i+0 \end{pmatrix} = \begin{pmatrix} -1+3i \\ 7 \\ -6-2i \end{pmatrix}$$

b)

$$\|\vec{v}\|_1 = |-2| + |1+i| + |1|$$
$$= 2 + \sqrt{2} + 1$$
$$= 3+\sqrt{2}$$

$$|1+i| = \sqrt{1^2+1^2} = \sqrt{2}$$

c)

$$\|\vec{v}\|_2 = \left(|-2|^2 + |1+i|^2 + |1|^2\right)^{1/2}$$
$$= \left(4 + (\sqrt{2})^2 + 1\right)^{1/2}$$
$$= (4+2+1)^{1/2}$$
$$= (7)^{1/2}$$
$$= \sqrt{7}$$

d)

$$\|\vec{v}\|_\infty = \max(2, \sqrt{2}, 1)$$
$$= 2$$

e)

$$\|A\|_1 = \max_{1\le j\le n} \sum_{i=1}^{m} |a_{ij}| \quad (\text{max Column sum})$$
$$= \max(1+4+2,\ 3+0+2,\ 2+1+0)$$
$$= \max(7, 5, 3)$$
$$= 7$$

f)

$$\|A\|_\infty = \max_{1\le i\le m} \sum_{j=1}^{n} |a_{ij}| \quad (\text{max row sum})$$
$$= \max(1+3+2,\ 4+0+1,\ 2+2+0)$$
$$= \max(6, 5, 4)$$
$$= 6$$

g)

$$\|A\|_F = \left(\sum_{i=1}^{m}\sum_{j=1}^{n} |a_{ij}|^2\right)^{1/2}$$
$$= (1+16+4+9+0+4+4+1+0)^{1/2}$$
$$= \sqrt{39}$$

2) Given

$$2x_1 - 6\alpha x_2 = 3$$
$$3\alpha x_1 - x_2 = \frac{3}{2}$$

$$3\alpha x_1 - x_2 = \frac{3}{2} \quad (E_1)$$
$$2x_1 - 6\alpha x_2 = 3 \quad (E_2)$$

$$m_2 = \frac{2}{3\alpha}$$

$$\boxed{E_2 - m_2 E_1 \to E_2} \Rightarrow \left[\begin{array}{cc|c} 3\alpha x_1 & -x_2 & \frac{3}{2} \\ 0 & \left(-6\alpha + \frac{2}{3\alpha}\right)x_2 & 3-1/\alpha \end{array}\right]$$

$$\Rightarrow 2x_1 - \frac{2}{3\alpha} \times 3\alpha x_1 - 6\alpha x_2 + \frac{2}{3\alpha} x_2 = 3 - \frac{2}{3\alpha}\cdot\frac{3}{2}$$

c)

$$\Rightarrow 0 + \left(-6\alpha + \frac{2}{3\alpha}\right) x_2 = 3 - \frac{1}{\alpha}$$

$$\left(\frac{-18\alpha^2+2}{3\alpha}\right) x_2 = \frac{3\alpha - 1}{\alpha}$$

$$(2-18\alpha^2)\, x_2 = 9\alpha - 3$$

$$x_2 = \left(\frac{9\alpha-3}{2-18\alpha^2}\right)$$

$$= \frac{3}{2}\,\frac{(3\alpha-1)}{(1^2-(3\alpha)^2)}$$

$$= -\frac{3}{2}\,\frac{(1-3\alpha)}{(1+3\alpha)(1-3\alpha)}$$

$$\boxed{x_2 = -\frac{3}{2}\left(\frac{1}{1+3\alpha}\right)}$$

backward Substitution

$$x_1 = \frac{3 + 6\alpha x_2}{2}$$

$$= \frac{3 + 6\alpha\left(-\frac{3}{2} \times \frac{1}{(1+3\alpha)}\right)}{2}$$

$$= \frac{3}{2}\left(1 - \frac{3\alpha}{(1+3\alpha)}\right)$$

$$= \frac{3}{2}\left(\frac{1+3\alpha-3\alpha}{1+3\alpha}\right)$$

$$\boxed{x_1 = \frac{3}{2(1+3\alpha)}}$$

For x_1 and x_2 System has unique solutions.

a) For the equation to have no solution

equation

$$-6\alpha + \frac{2}{3\alpha} = 0$$

$$-18\alpha^2 + 2 = 0$$

$$18\alpha^2 = 2$$
$$\alpha^2 = \frac{1}{9}$$
$$\alpha = \pm\sqrt{\frac{1}{9}}$$
$$\boxed{\alpha = \pm\frac{1}{3}}$$

So $\alpha = \frac{1}{3}$ or $-\frac{1}{3}$

$\Rightarrow$ Substituting $\alpha = -\frac{1}{3}$

$$\Rightarrow -6\alpha + \frac{2}{3\alpha} = 3 - \frac{1}{\alpha}$$
$$\Rightarrow -6\left(-\frac{1}{3}\right) + \frac{2}{3(-1/3)} = 3 - \frac{1}{(-1/3)}$$
$$\Rightarrow 2 - 2 = 3 - (-3)$$
$$0 = 6 \qquad LHS \neq RHS$$

$\Rightarrow$ So for $\alpha = -\frac{1}{3}$, equation has no solution

b) $\Rightarrow$ Substituting $\alpha = +\frac{1}{3}$

$$\Rightarrow -6\alpha + \frac{2}{3\alpha} = 3 - \frac{1}{\alpha}$$
$$\Rightarrow -6\left(\frac{1}{3}\right) + \frac{2}{3}\left(\frac{1}{1/3}\right) = 3 - \frac{1}{(1/3)}$$
$$\Rightarrow -2 + 2 = 3 - 3$$
$$\Rightarrow 0 = 0 \qquad LHS = RHS$$

$\Rightarrow$ So with $\alpha = \frac{1}{3}$, equation has infinite solutions

2(i) $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad \tilde{b} = \begin{bmatrix} 1+\varepsilon \\ 1-\varepsilon \end{bmatrix}$

$$x - \tilde{x} = A^{-1}b - A^{-1}\tilde{b}$$

Finding A^{-1},

$$A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{1-(-1)}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$
$$A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}$$

So $A^{-1}b = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5-0.5 \\ 0.5+0.5 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$A^{-1}\tilde{b} = \begin{bmatrix} 0.5 & -0.5 \\ 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} 1+\varepsilon \\ 1-\varepsilon \end{bmatrix} = \begin{bmatrix} 0.5(1+\varepsilon) - 0.5(1-\varepsilon) \\ 0.5(1+\varepsilon) + 0.5(1-\varepsilon) \end{bmatrix}$$
$$= \begin{bmatrix} \cancel{0.5} + 0.5\varepsilon - \cancel{0.5} + 0.5\varepsilon \\ 0.5 + \cancel{0.5\varepsilon} + 0.5 - \cancel{0.5\varepsilon} \end{bmatrix} = \begin{bmatrix} \varepsilon \\ 1 \end{bmatrix}$$

So $x - \tilde{x} = A^{-1}b - A^{-1}\tilde{b}$

$$= \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} \varepsilon \\ 1 \end{bmatrix}$$
$$x - \tilde{x} = \begin{bmatrix} -\varepsilon \\ 0 \end{bmatrix}$$

ii) $A = \begin{bmatrix} -1+\varepsilon & 1 \\ -1 & 1 \end{bmatrix}$

So $A^{-1} = \frac{1}{\det(A)}\begin{bmatrix} 1 & -1 \\ 1 & -1+\varepsilon \end{bmatrix} = \frac{1}{(-1+\varepsilon)-(-1)}\begin{bmatrix} 1 & -1 \\ 1 & -1+\varepsilon \end{bmatrix}$

$$A^{-1} = \frac{1}{\varepsilon}\begin{bmatrix} 1 & -1 \\ 1 & -1+\varepsilon \end{bmatrix} = \begin{bmatrix} 1/\varepsilon & -1/\varepsilon \\ 1/\varepsilon & 1-1/\varepsilon \end{bmatrix}$$

finding $A^{-1}b = \begin{bmatrix} 1/\varepsilon - 1/\varepsilon \\ \cancel{1/\varepsilon} + 1 - \cancel{1/\varepsilon} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

finding $A^{-1}\tilde{b} = \begin{bmatrix} 1/\varepsilon & -1/\varepsilon \\ 1/\varepsilon & 1-1/\varepsilon \end{bmatrix}\begin{bmatrix} 1+\varepsilon \\ 1-\varepsilon \end{bmatrix} = \begin{bmatrix} 1/\varepsilon(1+\varepsilon) - 1/\varepsilon(1-\varepsilon) \\ 1/\varepsilon(1+\varepsilon) + (1-1/\varepsilon)(1-\varepsilon) \end{bmatrix}$

$$= \begin{bmatrix} 1 + \cancel{1/\varepsilon} - \cancel{1/\varepsilon} + 1 \\ \cancel{1/\varepsilon} + 1 + 1 - \varepsilon - \cancel{1/\varepsilon} + 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 3-\varepsilon \end{bmatrix}$$

$$x - \tilde{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -2 \\ 3-\varepsilon \end{bmatrix} = \begin{bmatrix} -2 \\ 1-3-\varepsilon \end{bmatrix} = \begin{bmatrix} -2 \\ -2+\varepsilon \end{bmatrix}$$