Chapter 3. QR factorization and least squares

Sec 3.1 Projectors

Topics: 1. projectors, orthogonal projectors

2. construct orthogonal projector ith an orthonormal basis.

3. Construct orthogonal projector with an arbitrary basis

1. Projectors:

- Def: A <u>projector</u> is a square matrix P that satisfies $P^2 = P$.

  eg. $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $P^2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P$. $P\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$

- Def: If P is a projector, $I-P$ is called the <u>complementary projector</u> to P.

  (It is easy to verify that $I-P$ is a projector: $(I-P)^2 = I - 2P + P^2 = I - 2P + P = I - P$)

  eg. $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $I - P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. $(I-P)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ x_2 \end{pmatrix}$

Theorem: $\text{range}(I-P) = \text{null}(P)$, $\text{null}(I-P) = \text{range}(P)$, $\text{range}(P) \cap \text{null}(P) = \{\vec{0}\}$

- Def: Suppose S_1 and S_2 are subspaces of $\mathbb{C}^m$. If $S_1 \cap S_2 = \{\vec{0}\}$ and $\underbrace{S_1 + S_2}_{\text{span of } S_1 \text{ and } S_2} = \mathbb{C}^m$,

  then S_1 and S_2 are said to be <u>complementary subspaces</u>.

Remark: A projector P separates $\mathbb{C}^m$ into S_1 and S_2 with $\text{range}(P) = S_1$ and $\text{null}(P) = S_2$.

We say that P is <u>the projector onto S_1 along S_2</u>.

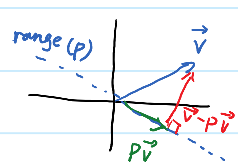
eg. $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

$$S_1 = \text{range}(P) = \left\{ \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \middle| \text{ any } x_1 \right\} = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

$$S_2 = \text{null}(P) = \left\{ \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \middle| \text{ any } x_2 \right\} = \text{span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$S_1 \cap S_2 = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}. \quad S_1 + S_2 = \mathbb{C}^2$$

- Def: An <u>orthogonal projector</u> is one that projects onto S_1 along S_2, where S_1 and S_2 are orthogonal.



<u>Theorem</u>: The projector P is orthogonal $\iff$ $P = P^*$

2. Construct orthogonal projector with an orthonormal basis:

Let $\{\vec{q}_1, \cdots, \vec{q}_m\}$ be an orthonormal basis of $\mathbb{C}^m$. Let $\hat{Q} = [\vec{q}_1, \cdots, \vec{q}_n] \in \mathbb{C}^{m\times n}$, $n \le m$

- Define $\boxed{P := \hat{Q} \cdot \hat{Q}^*}$. Then P is an orthogonal projector.

  (proof: $P^2 = \hat{Q}\hat{Q}^* \cdot \hat{Q}\hat{Q}^* = \hat{Q} \cdot I_n \cdot \hat{Q}^* = \hat{Q} \cdot \hat{Q}^* = P$, $P^* = (\hat{Q}\hat{Q}^*)^* = \hat{Q}\hat{Q}^* = P$)

$$P\vec{v} = \hat{Q}\hat{Q}^*\vec{v} = \underbrace{\begin{bmatrix} \vec{q}_1 & \cdots & \vec{q}_n \end{bmatrix}}_{m\times n} \underbrace{\begin{bmatrix} \vec{q}_1^{\,*} \\ \vdots \\ \vec{q}_n^{\,*} \end{bmatrix}}_{n\times n} \begin{bmatrix} \vec{v} \end{bmatrix} = \sum_{i=1}^{n} \vec{q}_i \cdot \vec{q}_i^{\,*}\vec{v} = \sum_{i=1}^{n} \vec{q}_i\,(\vec{q}_i^{\,*} \cdot \vec{v})$$

$\Rightarrow$ P projects onto the column space of $\hat{Q}$.

- $I - P = \boxed{I - \hat{Q}\hat{Q}^*}$ is also an orthogonal projector, because $(I-P)^* = I^* - P^* = I - P$

- special case: Given a unit vector $\vec{q}$,

  rank 1 orthogonal projector: $P_{\vec{q}} = \vec{q}\,\vec{q}^{\,*}$. It gives component in direction $\vec{q}$

  rank $m-1$ orthogonal projector: $P_{\perp\vec{q}} = I - \vec{q}\,\vec{q}^{\,*}$. It eliminates component in direction $\vec{q}$.

  For arbitrary nonzero vector $\vec{a}$, $P_{\vec{a}} = \dfrac{\vec{a}\,\vec{a}^{\,*}}{\vec{a}^{\,*}\vec{a}}$, orthogonal, rank 1

  $P_{\perp\vec{a}} = I - P_{\vec{a}} = I - \dfrac{\vec{a}\,\vec{a}^{\,*}}{\vec{a}^{\,*}\vec{a}}$, orthogonal, rank $m-1$

3. Construct orthogonal projection with an arbitrary basis.

Suppose $\vec{a}_1, \cdots, \vec{a}_n \in \mathbb{C}^m$ are linearly independent. Let $A = [\vec{a}_1, \cdots, \vec{a}_n] \in \mathbb{C}^{m\times n}$.

Construct an orthogonal projector $P: \mathbb{C}^m \to \text{range}(A)$:

P is orthogonal $\iff$ range$(P) \perp$ null(P) $\iff$ range$(A) \perp$ range$(I-P)$

For any $\vec{v} \in \mathbb{C}^m$, we need $\vec{a}_j \perp (I-P)\vec{v}$, i.e. $\vec{a}_j^{\,*}(v - \underbrace{P\vec{v}}_{A\vec{x} \text{ for some } \vec{x}}) = 0$, $j = 1, \cdots, n$.

$\Rightarrow A^*(\vec{v} - A\vec{x}) = 0 \Rightarrow A^*\vec{v} = A^*A\vec{x} \Rightarrow \vec{x} = (A^*A)^{-1}A^*\vec{v}$

$\Rightarrow P\vec{v} = A\vec{x} = A(A^*A)^{-1}A^*\vec{v}$. Therefore, $\boxed{P = A(A^*A)^{-1}A^*}$ is orthogonal.

Remark: 1. $P = A(A^*A)^{-1}A^*$ is a multidimensional generalization of $P_{\vec{a}} = \dfrac{\vec{a}\,\vec{a}^{\,*}}{\vec{a}^{\,*}\vec{a}}$.

2. When $A = \hat{Q}$ is orthonormal, $P = \hat{Q}\underbrace{(\hat{Q}^*\hat{Q})^{-1}}_{I_n}\hat{Q}^* = \hat{Q} \cdot \hat{Q}^*$