Formulation and Simulation of an Initial-Boundary Value Problem for an Elastic String

Praful Sigdel

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1 Introduction

To formulate and simulate an initial-boundary value problem for the motion of an elastic string with specific conditions, we consider the following approach.

2 Assumptions

- The string is inextensible and has uniform mass per unit length.
- The string is fastened at both ends, which are fixed.
- The initial position of the string is given by the function $f(x)$.
- The string is released from rest.
- Air resistance acts on the string, with a force proportional to the square of the velocity at each point.
- The string has internal damping.

3 Governing Equation

The motion of the elastic string can be described by the following partial differential equation (PDE):

$$\frac{\partial^2 u}{\partial t^2} + 2\zeta\omega_0 \frac{\partial u}{\partial t} + \omega_0^2 u = c^2 \frac{\partial^2 u}{\partial x^2} - k \left| \frac{\partial u}{\partial t} \right| \frac{\partial u}{\partial t}$$

where:

- $u(x, t)$ is the transverse displacement of the string at position x and time t.

- ζ is the damping ratio due to internal damping.
- ω_0 is the natural frequency of the string.
- c is the speed of wave propagation in the string.
- k is the air resistance coefficient.

4 Initial Conditions

At $t = 0$:

$$u(x,0) = f(x) \quad \text{(given initial position)}$$

$$\frac{\partial u}{\partial t}(x,0) = 0 \quad \text{(string is released from rest)}$$

5 Boundary Conditions

For $0 \leq x \leq L$:

$$u(0,t) = 0 \quad \text{(left end is fixed)}$$

$$u(L,t) = 0 \quad \text{(right end is fixed)}$$

6 Simulation

To simulate the motion of the elastic string, we can discretize the PDE using numerical methods such as the finite difference method or the finite element method. The resulting system of equations can be solved numerically using appropriate software or programming languages.

The simulation will provide the transverse displacement $u(x,t)$ of the string at different positions x and time instants t, considering the initial position $f(x)$, damping ratio ζ, natural frequency ω_0, wave propagation speed c, and air resistance coefficient k.

The simulation results can be visualized using plots or animations to better understand the motion of the elastic string under the given conditions.