INTRODUCTION TO LIMIT:-

The concept of limits plays an important role in calculus. Before defining the limit of a function near a point let us consider the following example

$$\text{Let } F(x) = \frac{x2-1}{x-1}$$

$$\text{Now } F(1) = \frac{1^2-1}{1-1} = \frac{0}{0} \text{ undefined}$$

But if we take x close to 1, we obtain different values for F(x) as follows

TABLE -1

X	0.91	0.93	0.99	0.9999	0.99999
F(X)	1.91	1.93	1.99	1.9999	1.99999

TABLE – 2

X	1.1	1.01	1.001	1.00001	1.000001
F(X)	2.1	2.01	2.001	2.00001	2.000001

In above we can see that when x gets closer to 1,F(x) gets closer to 2.however,in this case F(x) is not defined at x=1,but as x approaches to 1 F (x) approaches to 2.

This generates a new concept in setting the value of a function by approach method. The above value is called limiting value of a Function.

SOME DEFINATIONS ASSOCIATED WITH LIMIT:-

NEIGHBOURHOOD:-

For every a€R, the open interval (a-δ, a+ δ) is called a neighborhood of a where δ>0 is a very very small quantity.

Example (1.9,2.1) is a neighborhood of 2. (δ =0.1)

DELETED NEIGHBOURHOOD of 'a':-

(a- δ, a+ δ)- {a} is called deleted neighborhood of a.

Left neighborhood of a is given by (a- δ, a) .

Right neighborhood of a given by (a, a+ δ).

Example

(1.9.2.1)- {2} is a deleted neighborhood of 2.

(1.9, 2) is left neighborhood of 2.

(2, 2.1) is a right neighborhood of 2.

DEFINATION OF LIMIT:-

Given €>0, there exist δ>0 depending upon € only such that , |x-a|< δ => |f(x)-l|<€

Then $\lim_{x \to a} f(x)$ **= l**

EXPLANATION

If for every $\in$>0,we can able to find δ ,which depends upon $\in$ only such that x$\in$ (a- δ,a+ δ),=> f(x) $\in$ (l-$\in$,l+$\in$) . In other words when x gets closer to a then f(x) gets closer to l.

We read x→a as x tends to 'a' i.e. x is nearer to a but x≠a

$\lim_{x \to a} f(x)$→ limit x tends a f(x). 'l' is called limiting value of f(x)at x = a.

In 1st example $\lim_{x \to 1} \frac{x^2-1}{x-1}$= 2

i.e limiting value of f(x) at x=1 is 2.

Note:-

Functional value always gives the exact value of a function at a point where as limiting value gives an approximated value of function.

Functional value is either defined or undefined. Similarly limiting value is either exist or does not exist.

EXISTENCY OF LIMITING VALUE:-

In our first example if we observe table -1 then we see we approach 2 from left in that table. In table -2 we approach 2 from right.

So in table -1 x€ (2- δ,2)

And in table -2 x€ (2,2+δ)

These two approaches give rise to two definitions.

LEFT HAND LIMIT

When x approaches a from left then the value to which f(x) approaches is called left hand limit of f(x) at x=a written as L.H.L.= $\lim_{x \to a^-} f(x)$

X→a^- means x€$(a-\delta, a)$.

RIGHT HAND LIMIT: -

When x approaches a form right then the value to which f(x) approaches is called right hand limit.

Mathematically $\boxed{\text{R.H.L.} = \lim_{x \to a^+} f(x)}$

x→a^+ means x$\in$ (a,a+δ)

EXISTENCY OF LIMIT

If L.H.L = R.H.L i.e.

$$\boxed{\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = l}$$

Then the limit of the function exists and $\lim_{x\to a} f(x) = l$

Otherwise limit does not exist.

ALGEBRA OF LIMIT: -

IF $\lim_{x\to a} f(x) = l$ and $\lim_{x\to a} g(x) = m$

Then

i) $\lim_{x\to a}\{f(x) + g(x)\} = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = l + m$

ii)) $\lim_{x\to a}\{f(x) - g(x)\} = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = l - m$

iii) $\lim_{x\to a}\{f(x) . g(x)\} = \lim_{x\to a} f(x) . \lim_{x\to a} g(x) = l\, m$

iv)) $\lim_{x\to} \{\frac{f(x)}{g(x)}\} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \frac{l}{m}$ (provided m≠0)

v) $\lim_{x\to a} K = K$ (K is constant)

vi) $\lim_{x\to a} k\, f(x) = k \lim_{x\to a} f(x) = K\, l$

(Vii) $\lim_{x\to a} \log_b f(x) = \log_b \lim_{x\to a} f(x) = \log_b l$

Viii) $\lim_{x\to a} e^{f(x)} = e^{\lim_{x\to a} f(x)} = e^l$

(ix) $\lim_{x\to a} f(x)^n = \{\lim_{x\to a} f(x)\}^n = l^n$

(x) $\lim_{x\to a} f(x) = \lim_{y\to a} f(y) = l$

(xi) $\lim_{x\to a} |f(x)| = |\lim_{x\to a} f(x)| = |l|$

(xii) If $\lim_{x\to a} f(x) = \infty$, then $\lim_{x\to a} \frac{1}{f(x)} = 0$

EVALUATION OF LIMIT:-

When we evaluate limits it is not necessary to test the existency of limit always. So in this section we will discuss various methods of evaluating limits.

(1) EVALUATION OF ALGEBRAIC LIMITS :-

(2) Method -> (i) Direct substitution (ii) Factorisation Iii) Rationalisation

i) Direct Substitution :-

If f(x) is an algebraic function and f(a) is finite. Then $\lim_{x\to a} f(x)$ is equal to f(a) i.e. we can substitute x by a.

Let us consider following examples.

Example -1

Evaluate $\lim_{x\to 0}(x^2+2x+1)$

ANS →

$\lim_{x\to 0}(x^2+2x+1)$= 0^2+2x0+1 = 1

Example -2

Evaluate $\lim_{x\to -1} \frac{x-1}{x^2+2x-1}$

ANS→

$\lim_{x\to -1} \frac{x-1}{x^2+2x-1} = \frac{(-1)-1}{(-1)^2+2\times(-1)-1} = \frac{-2}{1-2-1} = \frac{-2}{-2} = 1$

Example – 3

$\lim_{x\to 1}\frac{\sqrt{x}}{\sqrt{x+2}}$ = ?

Ans :-

$\lim_{x\to 1}\frac{\sqrt{x}}{\sqrt{x+2}} = \frac{\sqrt{1}}{\sqrt{1+2}} = \frac{1}{\sqrt{3}}$

Example -4

Evaluate $\lim_{x\to 1}\frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = \frac{0}{0}$,Which cannot be determined.

NOTE: -

So here direct substitution method fails to find the limiting value. In this case we apply following method.

ii) FACTORISATION METHOD :-

If the given Function is a rational function $\frac{f(x)}{g(x)}$, and $\frac{f(a)}{g(a)}$ is in $\frac{0}{0}$ form then we apply factorisation method i.e we factorise f(x) and g (x) and cancel the common factor. After cancellation we again apply direct substitution, if result is a finite number otherwise we repeat the process .

This method is clearly explained in following example.

Example -4

Evaluate $\lim_{x\to 1}\frac{x^2-1}{x-1}$

Ans : - $\lim_{x\to 1}\frac{x^2-1}{x-1} = \lim_{x\to 1}\frac{(x+1)(x-1)}{(x-1)}$

$= \lim_{x\to 1}(x+1)$ {x→ 1 means x≠1 => (x-1) ≠ 0}

= 1+1 = 2 {after cancellation we can apply the direct substitution}

Example -5

Evaluate $\lim_{x\to-3} \frac{x^2+7x+12}{x^2+5x+6}$

ANS :-

$\lim_{x\to-3} \frac{x^2+7x+12}{x^2+5x+6}$ {by putting x=-3 we can easily check that the question is in $\frac{0}{0}$ form }

$= \lim_{x\to-3} \frac{x^2+4x+3x+12}{x^2+2x+3x+6}$

$= \lim_{x\to-3} \frac{x(x+4)+3(x+4)}{x(x+2)+3(x+2)}$

$= \lim_{x\to-3} \frac{(x+4)(x+3)}{(x+2)(x+3)}$ {x→ -3 then x+3 ≠ 0}

$= \lim_{x\to-3} \frac{(x+4)}{(x+2)} = \frac{-3+4}{-3+2} = \frac{1}{-1} =$ -1

Example – 6

Evaluate $\lim_{x\to4} \frac{x^3-3x^2-3x-4}{x^2-4x}$

ANS ->

$\lim_{x\to4} \frac{x^3-3x^2-3x-4}{x^2-4x}$ ($\frac{0}{0}$ form)

As x=4 gives $\frac{0}{0}$ form

⇨ X-4 is a factor of both polynomials.

x-4 $|x^3$-$3x^2-3x-4|$ x^2+x+1

$|x^3$ -$4x^2$ |

\- +

$|x^2-3x-4|$

$|x^2-4x$ |

\- +

| x - 4 |

| x - 4 |

0

Hence x^3 -$3x^2$-3x-4 = (x-4)(x^2+x+1)

Now $\lim_{x\to4} \frac{x^3-3x^2-3x-4}{x^2-4x} = \lim_{x\to4} \frac{(x-4)(x^2+x+1)}{x(x-4)} = \lim_{x\to4} \frac{x^2+x+1}{x} = \frac{4^2+4+1}{4} = \frac{21}{4}$.

iii) Rationalisation method :-

When either the numerator or the denominator contain some irrational functions and direct substitution gives $\frac{0}{0}$ form, then we apply rationalisation method. In this method we rationalize the irrational function to eliminate the $\frac{0}{0}$form.This can be better explained in following examples.

Example -7

Evaluate $\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}$

ANS :-

$\lim_{x\to 0}\frac{x}{\sqrt{x+1}-1}$ { In order to rationalize $\sqrt{x+1}-1$ $we\ have\ to\ apply\ a^2-b^2\ formula$

$a^2-b^2 = (a+b)(a-b)\ so\ here\ a-b$ is $present, so\ we\ have\ to$

$multiply\ a+b$ i.e. $\sqrt{x+1}+1$}

$= \lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}$

$= \lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{\sqrt{x+1}^2-1^2}$

$= \lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{x+1-1} = \lim_{x\to 0}\frac{x(\sqrt{x+1}+1)}{x}$

$= \lim_{x\to 0}(\sqrt{x+1}+1) = \sqrt{0+1}+1 = 1+1 = 2$

Example – 8

Evaluate $\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$

Ans :-

$\lim_{x\to 0}\frac{\sqrt{1+x}-\sqrt{1-x}}{2x}$ ($\frac{0}{0}$form)

$= \lim_{x\to 0}\left(\frac{(\sqrt{1+x}-\sqrt{1-x})(\sqrt{1+x}+\sqrt{1-x})}{2x(\sqrt{1+x}+\sqrt{1-x})}\right)$

$= \lim_{x\to 0}\left(\frac{(\sqrt{1+x})^2-(\sqrt{1-x})^2}{2x(\sqrt{1+x}+\sqrt{1-x})}\right) = \lim_{x\to 0}\frac{(1+x)-(1-x)}{2x(\sqrt{1+x}+\sqrt{1-x})}$

$= \lim_{x\to 0}\frac{2x}{2x(\sqrt{1+x}+\sqrt{1-x})}$

$= \lim_{x\to 0}\frac{1}{\sqrt{1+x}+\sqrt{1-x}}$

$= \frac{1}{\sqrt{1+0}+\sqrt{1-0}} = \frac{1}{1+1}$

$= \frac{1}{2}$

(3) Evaluating limit when x-> ∞

In order to evaluate infinite limits we use some formulas and techniques.

Formulas (i) $\lim_{x\to\infty} x^n = \infty$, n>0

(ii) $\lim_{x\to\infty} \frac{1}{x^n} = 0$, n>0

When we evaluate functions in $\frac{f(x)}{g(x)}$ form , the we use the following technique

Divide both f(x) and g(x) by x^k where x^k is the highest order term in g(x).

It can be better understood by following examples.

Example – 9

Evaluate $\lim_{x\to\infty} \frac{3x^2+x-1}{2x^2-7x+5}$

ANS:- $\lim_{x\to\infty} \frac{3x^2+x-1}{2x^2-7x+5}$

{Dividing numerator and denominator by highest order term in denominator i.e. x^2}

$$= \lim_{x\to\infty} \frac{\frac{3x^2+x-1}{x^2}}{\frac{2x^2-7x+5}{x^2}}$$

$$= \lim_{x\to\infty} \frac{\frac{3x^2}{x^2}+\frac{x}{x^2}-\frac{1}{x^2}}{\frac{2x^2}{x^2}-\frac{7x}{x^2}+\frac{5}{x^2}} = \lim_{x\to\infty} \frac{3+\frac{1}{x}-\frac{1}{x^2}}{2-\frac{7}{x}+\frac{5}{x^2}}$$

$$= \frac{\lim_{x\to\infty} 3+\lim_{x\to\infty}\frac{1}{x}-\lim_{x\to\infty}\frac{1}{x^2}}{\lim_{x\to\infty} 2-\lim_{x\to\infty}\frac{7}{x}+\lim_{x\to\infty}\frac{5}{x^2}} \quad \text{(applying algebra of limits)}$$

$$= \frac{3+0-0}{2-0+0} = \frac{3}{2}$$

Example – 10

Evaluate $\lim_{x\to\infty} \frac{x^3+2x^2+3}{x^4-3x+1}$

ANS :-

$$\lim_{x\to\infty} \frac{x^3+2x^2+3}{x^4-3x+1}$$

{ Dividing numerator and denominator by highest order term x^4}

$$= \lim_{x\to\infty} \frac{\frac{x^3}{x^4}+\frac{2x^2}{x^4}+\frac{3}{x^4}}{\frac{x^4}{x^4}-\frac{3x}{x^4}+\frac{1}{x}}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}+\frac{2}{x^2}+\frac{3}{x^4}}{1-\frac{3}{x^3}+\frac{1}{x^4}} = \frac{0+0+0}{1-0+0} = \frac{0}{1} = 0$$

Example – 11

Evaluate $\lim_{x\to\infty} \frac{x^4+5x+2}{x^3+2}$

ANS :-

$$\lim_{x\to\infty} \frac{x^4+5x+2}{x^3+2}$$

{ Dividing numerator and denominator by highest order term of denominator i.e. x^3}

$$= \lim_{x\to\infty} \frac{\frac{x^4}{x^3}+\frac{5x}{x^3}+\frac{2}{x^3}}{\frac{x^3}{x^3}+\frac{2}{x^3}} = \lim_{x\to\infty} \frac{x+\frac{5}{x^2}+\frac{2}{x^3}}{1+\frac{2}{x^3}} = \frac{\lim_{x\to\infty} x+\lim_{x\to\infty} \frac{5}{x^2}+\lim_{x\to\infty} \frac{2}{x^3}}{\lim_{x\to\infty} 1+\lim_{x\to\infty} \frac{2}{x^3}}$$

$$= \frac{\lim_{x\to\infty} x+0+0}{1+0} = \lim_{x\to\infty} x = \infty$$

Example – 12

$$\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3}$$

ANS :- $\lim_{x\to\infty} \frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{4x+3} = \lim_{x\to\infty} \frac{\frac{\sqrt{3x^2-1}-\sqrt{2x^2-1}}{x}}{\frac{4x+3}{x}}$

{ Dividing numerator and denominator by highest order term in denominator i.e. x.}

$$= \lim_{x\to\infty} \frac{\sqrt{\frac{3x^2-1}{x^2}}-\sqrt{\frac{2x^2-1}{x^2}}}{\frac{4x}{x}+\frac{3}{x}} = \lim_{x\to\infty} \frac{\sqrt{\frac{3x^2}{x^2}-\frac{1}{x^2}}-\sqrt{\frac{2x^2}{x^2}-\frac{1}{x^2}}}{4+\frac{3}{x}}$$

$$= \lim_{x\to\infty} \frac{\sqrt{3-\frac{1}{x^2}}-\sqrt{2-\frac{1}{x^2}}}{4+\frac{3}{x}} = \frac{\left(\lim_{x\to\infty}(3-\frac{1}{x^2})\right)^{1/2}-\left(\lim_{x\to 0}(2-\frac{1}{x^2})\right)^{1/2}}{\lim_{x\to\infty} 4+\lim_{x\to\infty}\frac{3}{x}}$$

$$= \frac{(3-0)^{1/2}-(2-0)^{1/2}}{4+0}$$

$$= \frac{\sqrt{3}-\sqrt{2}}{4} \text{ (ans)}$$

Important note in ∞ limit evaluation:-

$$\lim_{x\to\infty} \frac{a_0+a_1x+------+a_mx^m}{b_0+b_1x+b_2x^2+------+b_nx^n} = \begin{cases} \frac{a_m}{b_n} & if\ m=n \\ 0 & if\ m<n \\ \infty & if\ m>n \end{cases}$$

Example-13

If $\lim_{x\to\infty} \left(\frac{x^2-1}{x+1} - ax - b\right) = 2, find\ the\ values\ of\ a\ and\ b.$

Solution -> Given $\lim_{x\to\infty} \left(\frac{x^2-1}{x+1} - ax - b\right) = 2$

⇨ $\lim_{x\to\infty}\left(\frac{x^2-1-ax^2-ax-bx-b}{x+1}\right) = 2$

⇨ $\lim_{x\to\infty} \frac{x^2(1-a)-x(a+b)-(b+1)}{x+1} = 2$

As result is finite non zero quantity

⇨ Degree of numerator polynomial = degree of denominator polynomial

⇨ $\boxed{\text{Degree of polynomial in numerator = 1}}$

{As x+1 has degree = 1}

⇨ 1-a = 0 => $\boxed{\text{a-=1}}$

Now putting a = 1 in above evaluation

$\lim_{x\to\infty} \frac{-x(1+b)-(b+1)}{x+1}$ = 2

⇨ $\frac{-(1+b)}{1} = 2$ {by important note}

⇨ $-1-b=2$ {$\lim_{x\to\infty} \frac{a_0+a_1x---+a_mx^m}{b_0+b_1x+---+b_nx^n} = \frac{a_m}{b_n}$ $where\ m=n$}

=> $\boxed{\text{b=-1-2 =-3}}$

Therefore a=1 and b=-3

(4) <u>Important Formulas in limit</u>

(1) $\lim_{x\to a} \frac{x^n-a^n}{x-a} = na^{n-1}$ where a>0 and n€R

(2) $\lim_{x\to 0} \frac{a^x-1}{x}$ = $\log_e a$

In particular $\lim_{x\to 0} \frac{e^x-1}{x}$ = 1

(3) $\lim_{x\to 0}(1+x)^{\frac{1}{x}}$ = e

(4) $\lim_{x\to\infty}(1+\frac{1}{x})^x$ = e

(5) $\lim_{x\to 0} \frac{\log_a(1+x)}{x} = \log_a e$

In particular $\lim_{x\to 0} \frac{\log_e(1+x)}{x} = \log_e e$ =1

(6) $\lim_{x\to 0} \cos x$ = 1

(7) $\lim_{x\to 0} \sin x$ =0

(8) $\lim_{x\to 0} \frac{\sin x}{x}$ = 1

(9) $\lim_{x\to 0} \frac{\tan x}{x}$ = 1

<u>SUBSTITUTION METHOD:-</u>

In order to apply known formula sometimes we apply substitution method. In this method x is replaced by another variable u,and then we apply formula on 'u'.

Let us consider the following example.

Example – 14:-

Evaluate $\lim_{x\to 0}\frac{\sin 2x}{x}$

ANS:- Let 2x=u => when x→0

U →0 (as u = 2x)

Now $\lim_{x\to 0}\frac{\sin 2x}{x} = \lim_{u\to 0}\frac{\sin u}{\frac{u}{2}} = 2\lim_{u\to 0}\frac{\sin u}{u}$

$= 2\times 1 = 2$

In general

Putting λx=u

$$\boxed{\lim_{x\to 0} f(\lambda x) = \lim_{u\to 0} f(u) = \lim_{x\to 0} f(x)}$$

Hence some of the formulas may be stated as fllows

1) $\lim_{x\to 0}\frac{a^x-1}{x} = \log_e a$

In particular $\lim_{x\to 0}\frac{e^x-1}{x} = 1$

2) $\lim_{x\to 0}(1+\lambda x)^{\frac{1}{x}} = e$

3) $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$

4) $\lim_{x\to 0}\frac{\log_a(1+x)}{x} = \log_a e$

In particular $\lim_{x\to 0}\frac{\log_e(1+x)}{x} = 1$

5) $\lim_{x\to 0}\frac{\sin\lambda x}{\lambda x} = 1$

6) $\lim_{x\to 0}\frac{\tan x}{x} = 1$

Some examples based on the formulas

(1) Evaluate $\lim_{x\to 0}\frac{\sin 3x}{\tan 5x}$

Ans :-

$\lim_{x\to 0}\frac{\sin 3x}{\tan 5x} = \lim_{x\to 0}\frac{\frac{3\sin 3x}{3x}}{\frac{5\tan 5x}{5x}}$

$= \frac{3}{5}\lim_{x\to 0}\frac{\left(\frac{\sin 3x}{3x}\right)}{\left(\frac{\tan 5x}{5x}\right)} = \frac{3}{5}\times\frac{1}{1} = \frac{3}{5}$

(2) Evaluate $\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (2014 S)

Ans :- $\lim_{x\to 0}\frac{1-\cos x}{x^2} = \lim_{x\to 0}\frac{2\sin^2\frac{x}{2}}{x^2}$

$= \lim_{x\to 0} 2.\frac{\sin\frac{x}{2}}{\frac{x}{2}.2}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}.2} = \frac{1}{2}\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{\sin\frac{x}{2}}{\frac{x}{2}} = \frac{1}{2}\times 1\times 1 = \frac{1}{2}$

(3) Evaluate $\lim_{x\to 0}\frac{e^{3x}-e^{x}}{x}$

Ans :- $\lim_{x\to 0}\frac{e^{3x}-e^{x}}{x} = \lim_{x\to 0}\frac{e^{3x}-1+1-e^{x}}{x}$

$= \lim_{x\to 0}\frac{(e^{3x}-1)}{x}$ - $\lim_{x\to 0}\frac{(e^{x}-1)}{x}$

$= \lim_{x\to 0} 3(\frac{e^{3x}-1}{3x}) - \lim_{x\to 0}\frac{e^{x}-1}{x}$ = 3-1 = 2

(4) Evaluate $\lim_{x\to \frac{\pi}{2}}(\sec x - \tan x)$

Ans :-

$\lim_{x\to \frac{\pi}{2}}(\sec x - \tan x) = \lim_{x\to \frac{\pi}{2}}(\frac{1}{\cos x} - \frac{\sin x}{\cos x})$

$= \lim_{x\to \frac{\pi}{2}}(\frac{1-\sin x}{\cos x}) = \lim_{x\to \frac{\pi}{2}}(\frac{(1-\sin x)(1+\sin x)}{\cos x(1+\sin x)})$

$= \lim_{x\to \frac{\pi}{2}}(\frac{1-sin^2x}{\cos x(1+\sin x)}) = \lim_{x\to \frac{\pi}{2}}\frac{cos^2x}{\cos x(1+\sin x)}$

$= \lim_{x\to \frac{\pi}{2}}(\frac{\cos x}{1+\sin x}) = \frac{\cos\frac{\pi}{2}}{1+sin\frac{\pi}{2}} = \frac{0}{1+1} = \frac{0}{2} = 0$

(5) Evaluate $\lim_{x\to 0}(\frac{\tan x-\sin x}{sin^3x})$

$\lim_{x\to 0}(\frac{\tan x-\sin x}{sin^3x}) = \lim_{x\to 0}(\frac{\frac{\sin x}{\cos x}-\sin x}{sin^3x})$

$= \lim_{x\to 0}(\frac{\sin x-\sin x\,cos x}{sin^3x\cos x}) = \lim_{x\to 0}(\frac{\sin x\,(1-\cos x)}{sin^3x\cos x}) = \lim_{x\to 0}(\frac{1-\cos x}{sin^2x\cos x})$

$= \lim_{x\to 0}(\frac{1-cosx}{(1-cos^2x)cosx}) = \lim_{x\to 0}(\frac{(1-cosx)}{(1-cosx)(1+cosx)cosx})$

$= \lim_{x\to 0}(\frac{1}{(1+\cos x)\cos x}) = \frac{1}{(1+cos0)cos0} = \frac{1}{(1+1).1} = \frac{1}{2}$

(6) Evaluate $\lim_{x\to 0}\frac{sin^{-1}x}{x}$.

Ans :-

$\lim_{x\to 0}\frac{sin^{-1}x}{x} = \lim_{u\to 0}\frac{u}{\sin u}$

{put $sin^{-1}x$ =u => x=sinu when x->0 u-> $sin^{-1}x$ -> 0 {as $sin^{-1}0$ = 0}}

$= \lim_{u\to 0}\frac{1}{\frac{\sin u}{u}} = \frac{1}{1} = 1$

(7) Evaluate $\lim_{x\to 0}\frac{tan^{-1}x}{x}$

Ans :-

$\lim_{x\to 0}\frac{tan^{-1}x}{x} = \lim_{u\to 0}\frac{u}{\tan u}$ {put $tan^{-1}x$ = u => x= tanu when x->0 u->0}

$= \lim_{u\to 0}\frac{1}{\frac{tanu}{u}} = {}^{1}/_{1} = 1$

(8) Evaluate $\lim_{x\to 2}\frac{x^3-8}{x^5-32}$

Ans :-

$$\lim_{x\to 2}\frac{x^3-8}{x^5-32} = \lim_{x\to 2}\frac{x^3-2^3}{x^5-2^5} \qquad \{ \text{ as } \lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\}$$

$$= \lim_{x\to 2}\frac{\frac{x^3-2^3}{x-2}}{\frac{x^5-2^5}{x-2}} = \frac{3\ \ 2^{3-1}}{5\ \ 2^{5-1}} = \frac{3\times 2^2}{5\times 2^4} = \frac{3}{5\times 2^2} = \frac{3}{20}\ .$$

(9) Evaluate $\lim_{x\to 0}\frac{(3+x)^3-27}{x}$

Ans :- $\lim_{x\to 0}\frac{(3+x)^3-27}{x}$ { put x+3 = u when x--> 0 then u-->3}

$= \lim_{u\to 0}\frac{u^3-3^3}{u-3}$ $\{\lim_{x\to a}\frac{x^n-a^n}{x-a} = na^{n-1}\}$

$= 3\ \ 3^{3-1} = 3\times 3^2 = 3\times 9 = 27$

(10) Evaluate $\lim_{x\to 0}\frac{tan^{-1}\,3x}{\sin 7x}$

Ans :- $\lim_{x\to 0}\frac{tan^{-1}\ 3x}{\sin 7x} = \lim_{x\to 0}\frac{\frac{tan^{-1}3x}{3x}.3}{\frac{sin7x}{7x}.7}$

$= \frac{3}{7}\lim_{x\to 0}\frac{(\frac{tan^{-1}3x}{3x})}{(\frac{sin7x}{7x})}$

$= \frac{3}{7}\times\frac{1}{1} = \frac{3}{7}$

(11) Evaluate $\lim_{x\to 1}\frac{\log_e 2x-1}{x-1}$

Ans :- $\lim_{x\to 1}\frac{\log_e 2x-1}{x-1}$

{ For applying log formula x→ 0 ,but here x→ 1, so we have to substitute a new variable u as, u = x-1}

$= \lim_{u\to 0}\frac{\log_e 2(u+1)-1}{u}$

$= \lim_{u\to 0}\frac{\log_e 2u+1}{u}$ when x-->1 then u=x-1 –>0}

$= \lim_{u-0}\frac{\log_e(1+2u)}{2u}.2 = 1\times 2 = 2$

(12) Evaluate $\lim_{x\to 0}\frac{4^x-5^x}{3^x-4^x}$.

Ans :- $\lim_{x\to 0}\frac{4^x-5^x}{3^x-4^x} = \lim_{x\to 0}\frac{\frac{4^x-1+1-5^x}{x}}{\frac{3^x-1+1-4^x}{x}}$

$= \lim_{x\to 0}\frac{\frac{(4^x-1)}{x}-\frac{(5^x-1)}{x}}{(\frac{3^x-1}{x})-(\frac{4^x-1}{x})} = \frac{\log_e 4-\log_e 5}{\log_e 3-\log_e 4}$

$= \frac{ln4-ln5}{ln3-ln4} = \frac{ln\frac{4}{5}}{ln\frac{3}{4}}$ $\{\log e\ is written as \ln\ \ i.e.\ natural logarithm\}$

(13) Evaluate $\lim_{x\to 0}(1+3x)^{\frac{1}{x}}$

Ans :-

$\lim_{x\to 0}(1+3x)^{\frac{1}{x}} \quad = \quad \lim_{x\to 0}\{(1+3x)^{\frac{1}{3x}}\}^3$

$= \{\lim_{x\to 0}(1+3x)^{\frac{1}{3x}}\}^3 \quad = e^{3}$

(14) Evaluate $\lim_{x\to 0}(1+\frac{2x}{3})^{\frac{1}{2x}}$

Ans:-

$\lim_{x\to 0}(1+\frac{2x}{3})^{\frac{1}{2x}} \quad = \quad \lim_{x\to 0}(1+\frac{2x}{3})^{\frac{1/3}{(\frac{2x}{3})}}$

$= \{\lim_{x\to 0}(1+\frac{2x}{3})^{\frac{1}{(\frac{2x}{3})}}\}^{1/3} = e^{1/3}$

Use of L.H.L and R.H.L to find limit of a function

L.H.L and R.H.L used to find limit of a function where the definition of a function changes. For example |x| at 0 or [x] at any integral point etc.

Also the same concept is used when we come across following terms.

$$\boxed{\begin{array}{l} \lim_{x\to 0^+}\frac{1}{x} = \infty \\ \lim_{x\to 0^-}\frac{1}{x} = -\infty \\ \lim_{x\to 0^+} e^{\frac{1}{x}} = \infty \\ \lim_{x\to 0^-} e^{\frac{1}{x}} = 0 \end{array}}$$

Examples :-

(1) Evaluate $\lim_{x\to 0}\frac{|x|}{x}$

Ans :- L.H.L = $\lim_{x\to 0-}\frac{|x|}{x}$ {x → 0^- => x€(-δ,0)i.e. x<0 => |x| = -x}

$= \quad \lim_{x\to 0^-}\frac{-x}{x}$

$= \quad \lim_{x\to 0^-}(-1) \quad = -1$

R.H.L = $\lim_{x\to 0^+}\frac{|x|}{x} \quad = \lim_{x\to 0^+}\frac{x}{x}$ {x→0^+ => x € (0,δ) i.e.x>0 => |x| = x}

$= \quad \lim_{x\to 0^+} 1 \quad = 1$

From above

L.H.L ≠ R.H.L =>$\lim_{x\to 0}\frac{|x|}{x}$ does not exist

(2) Evaluate $\lim_{x\to -1}\frac{x+1}{|x+1|}$

Ans :- $\lim_{x\to -1}\frac{x+1}{|x+1|}$ $= \lim_{u-0}\frac{u}{|u|}$ {Let x+1=u . when x→-1 then u→0}

L.H.L. $=\lim_{u\to 0^-}\frac{u}{|u|}$ $=\lim_{u\to 0^-}\frac{u}{-u}$ {u→0^- => u<0 => IuI = -u }

$=\lim_{u\to 0^-}(-1)$ = - 1

R.H.L $=\lim_{u\to 0^+}\frac{u}{|u|}$ $=\lim_{u\to 0^+}\frac{u}{u}$ $=\lim_{u\to 0^+}1$ = 1 {u→0^+ => u>0 => IuI =u }

Hence $\lim_{u\to 0}\frac{u}{|u|}$ does not exist

Therefore $\lim_{x\to 1}\frac{x+1}{|x+1|}$ does not exist.

(3) Find $\lim_{x\to 0^+}\{[x]+10\}$

Ans $\lim_{x\to 0^+}\{[x]+10\}$

$=\lim_{x\to 0^+}(0+10)$ {As x-->0^+ => x € (0,δ) i.e 0 <x <1 } => [x]=0 }

$=\lim_{x\to 0^+}10 = 10$

(4) Find $\lim_{x\to 3.7}[x]$

Ans :-

$\lim_{x\to 3.7}[x]$ =[3.7] = 3

(5) Find $\lim_{x\to -1}[x]$

Ans :-

[x] changes its definition at each integral point. So, we have to go through L.H.L and R.H.L.

L.H.L. $=\lim_{x\to -1^-}[x]$ $=\lim_{x\to -1^-}(-2)$ = -2

{As x→ -1^- => x€(-1-δ,-1) i.e.-2<x<-1 =>[x]=-2}

R.H.L. $=\lim_{x\to -1^+}[x] = \lim_{x\to -1^+}-1$ = -1 {as x-->-1^+ => -1<x<0 =>[x]=-1}

As from above L.H.L ≠ R.H.L.

⇨ $\lim_{x\to -1}[x]$ does not exist.

(6) Evaluate $\lim_{x\to\frac{4}{3}}[3x-1]$

Ans :-

$\lim_{x\to\frac{4}{3}}[3x-1]$ $=\lim_{u\to 3}[u]$ {Put 3x-1 = u => when x →$\frac{4}{3}$ u→ 3 X $\frac{4}{3}-1$ i.e.u→3}

Now L.H.L = $\lim_{u\to 3^-}[u]$ $=\lim_{u\to 3^-}2$ = 2 {As $u\to 3^-$ => $2<u<3$ => $[u]=2$}

R.H.L = $\lim_{u\to 3^+}[u]$ $=\lim_{u\to 3^+}3$ =3 {As $u\to 3^+$ => $3<u<4$ => $[u]=3$}

Hence L.H.L ≠ R.H.L =>$\lim_{u\to 3}[u]$ does not exist.

Therefore $\lim_{x\to\frac{4}{3}}[3x-1]$ does not exist.

(7) Evaluate $\lim_{x\to 2} f(x)$ where

$$f(x) = \begin{cases} -x & x<1 \\ x+1 & x\geq 1 \end{cases}$$

Ans :- As f(x) does not changes its definition at '2' so,

$\lim_{x\to 2} f(x) = \lim_{x\to 2}(x+1) = 2+1 = 3$

{As x-->2 => x ∈ (2-δ,2+δ) => x> 1 => f(x)= x+1 }

(8) Evaluate $\lim_{x\to 1} f(x)$ and $\lim_{x\to 2} f(x)$ if $f(x) = \begin{cases} x^2 & x<1 \\ 2x+1 & 1\leq x\leq 2 \\ 5 & x>2 \end{cases}$

Ans :- As function changes its definition at x=1 and 2, so we have to go through L.H.L and R.H.L. step.

$\lim_{x\to 1} f(x)$

L.H.L = $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^2 = 1^2 = 1$

R.H.L = $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+}(2x+1) = 2 \times 1 + 1 = 3$

{ when x-->1^- => x<1 so we use f(x) = x^2 }

{when x-->1^+ => x > 1 i.e. 1 < x < 2 = > f(x) = 2x+1} From above L.H.L ≠R.H.L

⇨ $\lim_{x\to 1} f(x)$ *does not exist*

$\lim_{x\to 2} f(x)$

L.H.L = $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}(2x+1)$ = 2 x 2 + 1 = 5

{when x-->2^- => x ε (2-δ,2) i.e. 1<x<2 => f(x)=2x+1}

R.H.L = $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 5$ = 5

{x-->2^+ => x>2 => f(x) = 5 from definition}

As L.H.L = R.H.L

Therefore $\boxed{\lim_{x\to 2} f(x) = 5}$

(9) Evaluate $\lim_{x\to 0} \frac{1}{x}$

Ans :- L.H.L. = $\lim_{x\to 0^-} \frac{1}{x} = -\infty$

R.H.L. = $\lim_{x\to 0^+} \frac{1}{x} = \infty$

L.H.L ≠ R.H.L.

=>$\lim_{x\to 0^-} \frac{1}{x}$ does not exist.

Note

X-->a $\nearrow$ X-->a$^+$

X-->a $\searrow$ X-->a$^-$

So when we use direct substitution method either for x-->a$^+$ or x-->a$^-$ in both case we have to replace x by a.

Sandwich theorem or squeezing theorem

If $\lim_{x\to a} f(x)$ = $\lim_{x\to a} g(x)$ = l and a function h (x) is such that f(x)≤ $h(x) \leq g(x)$ for all xε(a-$\delta, a+\delta$) ,then $\boxed{\lim_{x\to a} h(x) = l}$

Example

Find $\lim_{x\to 0} x \sin\frac{1}{x}$

Solution: - We know $|\sin\frac{1}{x}| \leq 1$

⇨ $|x \sin\frac{1}{x}| \leq |x|$

Again $|x \sin\frac{1}{x}| \geq 0$

Hence $0 \leq |x \ \sin\frac{1}{x}| \leq |x|$

Now $\lim_{x\to 0} 0 = 0$

And $\lim_{x\to 0} |x| = 0$

{ s $\lim_{x\to 0^-} |x| = \lim_{x\to 0^-} (-x)$ =- 0 = 0 and $\lim_{x\to 0^+} |x| = \lim_{x\to 0^+} x = 0$ }

Hence by sandwich theorm

$$\lim_{x\to 0} |x \ \sin\frac{1}{x}| = 0$$

When x--> 0 , x $\sin\frac{1}{x}$ = (+)ve. So $|x \ \sin\frac{1}{x}| = \ x\sin\frac{1}{x}$

{when x-->0^- then x∈(-δ, 0), x = (-)ve, $\sin\frac{1}{x}$ = -ve => x $\sin \frac{1}{x}$ = +ve}

{When x-->0^+then x ∈ $(0, \delta)$, x = +ve, $\sin\frac{1}{x}$ = +ve => x $\sin\frac{1}{x}$ = +ve}

Hence, $\boxed{\lim_{x\to 0} x \sin\frac{1}{x} = 0}$

ILLUSTRATIVE EXAMPLES

1. Evaluate $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$ $a, b \neq 0$ (2015-S) (2019-w)

Ans.

$$\lim_{x\to 0}\frac{\sin ax}{\sin bx} = \lim_{x\to 0}\frac{\frac{\sin ax}{ax}.a}{\frac{\sin bx}{b}.b}$$

$$= \frac{a}{b}\lim_{x\to 0}\frac{(\frac{\sin ax}{ax})}{(\frac{\sin bx}{bx})}$$

$$= \frac{a}{b} \text{ X } \frac{1}{1} \quad = \frac{a}{b}$$

2. Evaluate $\lim_{x\to 0}\frac{x - x\cos 2x}{\sin^3 2x}$ (2015-S)

Ans.

$$\lim_{x\to 0}\frac{x-x\cos 2x}{\sin^3 2x} = \lim_{x\to 0}\frac{x(1-\cos 2x)}{\sin^3 2x}$$

$$= \lim_{x\to 0}\frac{x\,2\sin^2 x}{\sin^3 2x} = 2\lim_{x\to 0}\frac{\frac{x\sin^2 x}{x^3}}{\frac{\sin^3 2x}{(2x)^3}.2^3}$$

$$=2\lim_{x\to 0}\frac{\frac{\sin^2 x}{x^2}}{\frac{\sin^3 2x}{(2x)^3}.8}$$

$$= \frac{2}{8}\lim_{x\to 0}\frac{(\frac{\sin x}{x})^2}{(\frac{\sin 2x}{2x})^3}$$

$$= \frac{1}{4} \text{ X } \frac{1^2}{1^3}$$

$$= \frac{1}{4} \text{ (Ans)}$$

3. Evaluate $\lim_{x\to 0}\frac{\cos mx - \cos nx}{x^2}$ (2017-s old)

Ans.

$$\lim_{x\to 0}\frac{\cos mx-\cos nx}{x^2} = \lim_{x\to 0}\frac{2\sin(\frac{mx+nx}{2})\sin\frac{nx-mx}{2}}{x^2} \quad \{\cos C-\cos D= 2\sin\frac{C+D}{2}\sin\frac{D-C}{2}\}$$

$$=\lim_{x\to 0} 2\,\frac{\sin(\frac{n+m}{2})x}{x}\,\frac{\sin(\frac{n-m}{2})x}{x}$$

$$= \lim_{x\to 0} 2\,(\frac{m+n}{2})\,\frac{\sin(\frac{m+n}{2})x}{(\frac{m+n}{2})x}\,(\frac{n-m}{2})\,\frac{\sin(\frac{n-m}{2})x}{(\frac{n-m}{2})x}$$

$$= 2\,(\frac{m+n}{2})\,(\frac{n-m}{2})\lim_{x\to 0}\frac{\sin(\frac{m+n}{2})x}{(\frac{m+n}{2})x}\,\frac{\sin(\frac{n-m}{2})x}{(\frac{n-m}{2})x}$$

$= \quad 2\,\frac{(m+n)}{(n\;2m)}\,\frac{\quad}{2}$x1x1

$= \quad \frac{(m+n)(n-m)}{2}$

$= \quad \frac{n^2-m^2}{2}$

4. Evaluate $\lim_{x\to\frac{\pi}{2}}(\frac{\pi}{2}-x)\tan x$

Ans.

$\lim_{x\to\frac{\pi}{2}}(\frac{\pi}{2}-x)\tan x = \lim_{u\to 0} u\tan(\frac{\pi}{2}-u)$ $\quad$ {put $\frac{\pi}{2}-x = u \; when \; x\to\frac{\pi}{2} \quad u\to 0$}

$= \lim_{u\to 0} u \; \cot u$

$= \lim_{u\to 0}\frac{u}{\tan u}$

$= \lim_{u\to 0}\frac{u/u}{\tan u/u}$

$= \lim_{u\to 0}\frac{1}{(\frac{\tan u}{u})}$

$= \frac{1}{1} = 1$

5. Evaluate $\lim_{x\to 1}\frac{x^2-2x+1}{x^2-x}$ (2017 S)

Ans.

$\lim_{x\to 1}\frac{x^2-2x+1}{x^2-x} = \lim_{x-1}\frac{(x-1)^2}{x(x-1)}$

$= \lim_{x-1}\frac{x-1}{x}$

$= \frac{1-1}{1} = \frac{0}{1} = 0$

6. Evaluate $\lim_{x\to}\frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$ (a>b) (2017 W)

Ans.

$\lim_{x\to}\frac{\sqrt{x-b}-\sqrt{a-b}}{x^2-a^2}$ $\quad$ $(\frac{0}{0}\,form)$

$= \lim_{x\to a}\frac{(\sqrt{x-b}-\sqrt{a-b})(\sqrt{x-b}+\sqrt{a-b})}{(x^2-a^2)(\sqrt{x-b}+\sqrt{a-b})}$

$= \lim_{x\to a}\frac{(x-b)-(a-b)}{(x-a)(x+a)(\sqrt{x-b}+\sqrt{a-b})} \quad = \lim_{x\to a}\frac{x-b-a+}{(x-a)(x+a)(\sqrt{x-b}+\sqrt{a-b})}$

$= \lim_{x\to a}\frac{(x-a)}{(x-a)(x+a)(\sqrt{x-b}+\sqrt{a+b})} = \lim_{x\to a}\frac{1}{(x+a)(\sqrt{x-b}+\sqrt{a-b})} = \frac{1}{(a+a)(\sqrt{a-b}+\sqrt{a-b})}$

$= \frac{1}{2a\;2\sqrt{a-b}} = \frac{1}{4a\sqrt{a-}}$

7. Evaluate $\lim_{x\to\frac{1}{2}} \frac{|2x-1|}{2x-1}$

Ans. $\lim_{x\to\frac{1}{2}} \frac{|2x-1|}{2x-1}$

{Put 2x-1 = u => when x -->$\frac{1}{2}$, u --> 2X $\frac{1}{2}$ - 1 = 0 }

= $\lim_{u\to0} \frac{|u|}{u}$

L.H.L = $\lim_{u\to0^-} \frac{|u|}{u}$ = $\lim_{u\to0^-} \frac{-u}{u}$ = $\lim_{u\to0^-} (-1)$ = -1

R.H.L = $\lim_{u\to0^+} \frac{|u|}{u}$ = $\lim_{u\to0^+} \frac{u}{u}$ = $\lim_{u\to0^+}$ 1 = 1

As L.H.L G R.H.L, so $\lim_{u\to0} \frac{|u|}{u}$ does not exist.

=>$\lim_{x\to\frac{1}{2}} \frac{|2x-1|}{2x-1}$ does not exist.

8. Evaluate $\lim_{x\to\infty} \frac{x}{[x]}$

Ans:- From definition of [x] we know that ,

x – 1 < [x] ≤ x

⇨ $\frac{x}{x-1} > \frac{x}{[x]} \geq \frac{x}{x}$

⇨ $1 \leq \frac{x}{[x]} < \frac{x}{x-1}$

Now, $\lim_{x\to\infty}$ 1 = 1

$\lim_{x\to\infty} \frac{x}{x-1}$ = $\lim_{x\to\infty} \frac{\frac{x}{x}}{\frac{x}{x}-\frac{1}{x}}$ = $\lim_{x\to\infty} \frac{1}{1-\frac{1}{x}}$ = $\frac{1}{1-0}$ = 1

Hence by sandwich theorem $\lim_{x\to\infty} \frac{x}{[x]}$ = 1

9. Evaluate$\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$

Ans.

$$\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$$

= $\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3}$

= $\frac{1}{6} \lim_{n\to\infty} \frac{n}{n} \left(\frac{n+1}{n}\right)\left(\frac{2n+1}{n}\right)$

= $\frac{1}{6} \lim_{n\to\infty} 1(1+\frac{1}{n})(2+\frac{1}{n})$

= $\frac{1}{6}$ X 1(1+0)(2+0) = $\frac{2}{6}$ = $\frac{1}{3}$

10. Evaluate $\lim_{x\to\alpha}\frac{x\sin\alpha-\alpha\sin x}{x-\alpha}$

Ans.

$\lim_{x\to\alpha}\frac{x\sin\alpha-\alpha\sin x}{x-\alpha}$ Put x-a = u, when x-->a , then u-->0

$= \lim_{u\to0}\frac{(a+u)\sin\alpha-\alpha\sin(u+a)}{u}$

$= \lim_{u\to0}\frac{a\sin\alpha+u\sin a-\alpha\sin u\cos a-a\cos u\sin a}{u}$

$= \lim_{u\to0}\frac{a\sin\alpha-\alpha\cos u\sin a+u\sin a-a\sin u\cos a}{u}$

$= \lim_{u\to0}\left\{\frac{a\sin a(1-\cos u)}{u}+\sin a-a\cos a\frac{\sin u}{u}\right\}$

$= \lim_{u\to0}\left\{\frac{a\sin a\,2\sin^2\frac{u}{2}}{u}+\sin a-a\cos a\frac{\sin u}{u}\right\}$

$= \lim_{u\to0}\left\{2a\sin a\frac{\sin\frac{u}{2}}{\frac{u}{2}\cdot2}\cdot\sin\frac{u}{2}+\sin a-a\cos a\frac{\sin u}{u}\right\}$

$= \lim_{u\to0}\left\{a\sin a\frac{\sin\frac{u}{2}}{\frac{u}{2}}\cdot\sin\frac{u}{2}+\sin a-a\cos a\frac{\sin u}{u}\right\}$

$= a\sin a\cdot1\cdot0+\sin a-a\cos a\cdot1 = \sin a-a\cos a$

11. Evaluate $\lim_{x\to5}\frac{\log_e x-\log_e 5}{x-5}$

Ans. $\lim_{x\to5}\frac{\log_e x-\log_e 5}{x-5}$ (Put u = x – 5, when x-->5 then u --> 0)

$= \lim_{u\to0}\frac{\log_e(u+5)-\log_e 5}{u}$

$= \lim_{u\to0}\frac{\log_e\left(\frac{u+5}{5}\right)}{u}$

$= \lim_{u\to0}\frac{\log_e\left(\frac{u}{5}+1\right)}{u}$

$= \lim_{u\to0}\frac{\log_e\left(\frac{u}{5}+1\right)}{\frac{u}{5}\cdot5}$

$= \frac{1}{5}\lim_{u\to0}\frac{\log_e\left(1+\frac{u}{5}\right)}{\frac{u}{5}}$

$= \frac{1}{5}\cdot1 = \frac{1}{5}$

12. Evaluate $\lim_{x\to0}\frac{\sqrt{1+x}-1}{\log_e(1+x)}$

Ans. $\lim_{x\to0}\frac{\sqrt{1+x}-1}{\log_e(1+x)}$

$= \lim_{x\to0}\frac{(\sqrt{1+x}-1)(\sqrt{1+x}+1)}{\log_e(1+x)(\sqrt{1+x}+1)}$

$= \lim_{x\to0}\frac{1+x-1}{\log_e(1+x)(\sqrt{1+x}+1)}$

$= \lim_{x\to 0} \frac{1}{\frac{\log_e(1+x)}{x}.(\sqrt{1+x}+1)}$

$= \frac{1}{1K\sqrt{1+0}+1} \quad = \frac{1}{\sqrt{1}+1} \quad = \frac{1}{2}$

13. Evaluate $\lim_{x\to 2} \frac{\log_7(2x-3)}{(x-2)}$

Ans. $\lim_{x\to 2} \frac{\log_7(2x-3)}{(x-2)}$ { Put x-2 = u when x-->2 then u-->0 }

$= \lim_{u\to 0} \frac{\log_7\{2(u+2)-3\}}{u} = \lim_{u\to 0} \frac{\log_7(2u+4-3)}{u}$

$= \lim_{u\to 0} \frac{2\log_7(1+2u)}{2u}$

$= 2\lim_{u\to 0} \frac{\log_7(1+2u)}{2u} = 2 . \log_7 e$

14. Find the value of a for which $\lim_{x\to 1} \frac{5^x-5}{(x-1)\log_e a} = 5$

Ans. Given $\lim_{x\to 1} \frac{5^x-5}{(x-1)\log_e a} = 5$ ------------------------ (1)

Now, $\lim_{x\to 1} \frac{5^x-5}{(x-1)\log_e a}$

$= \lim_{u\to 0} \frac{5^{u+1}-5}{u \ \log_e a}$ (Put x-1 = u, when x--> 1, then u-->0)

$= \lim_{u\to 0} \frac{5^u.5-5}{u \ \log_e a}$

$= \frac{5}{\log_e a} \lim_{u\to 0} \frac{5^u-1}{u}$

$= \frac{5}{\log_e a} \log_e 5$ -------------------------- (2)

From (1) and (2) we have,

$\frac{5}{\log_e a} \log_e 5 = 5$

$=> \log_e 5 = \log_e a$

=> a = 5

15. Evaluate $\lim_{x\to 1} \frac{x^2-4x+3}{x^2-6x+5}$

Ans. $\lim_{x\to 1} \frac{x^2-3x-x+3}{x^2-5x-x+5}$ ($\frac{0}{0}$ form)

$= \lim_{x\to 1} \frac{x(x-3)-1(x-3)}{x(x-5)-1(x-5)}$

$= \lim_{x\to 1} \frac{(x-3)(x-1)}{(x-5)(x-1)}$

$= \lim_{x\to 1} \frac{x-3}{x-5}$

$= \frac{1-3}{1-5} = \frac{-2}{-4} = \frac{1}{2}.$

16. Evaluate $\lim_{x\to 0}\frac{3^x+3^{-x}-2}{x^2}$

Ans.

$$\lim_{x\to 0}\frac{3^x+3^{-x}-2}{x^2} \quad = \lim_{x\to 0}\frac{3^x+\frac{1}{3^x}-2}{x^2}$$

$$= \lim_{x\to 0}\frac{3^{2x}+1-2\ 3^x}{3^x x^2} \ = \ \lim_{x\to 0}\frac{(3^x)^2-2.3^x.1+1^2}{3^x x^2}$$

$$= \lim_{x\to 0}\frac{(3^x-1)^2}{3^x x^2} \ = \ \lim_{x\to 0}\frac{1}{3^x}\left(\frac{3^x-1}{x}\right)^2$$

$$= \frac{1}{3^{\circ}}(\log_e 3)^2$$

$$= (\ln 3)^2$$

17. Evaluate $\lim_{x\to\infty}\{\sqrt{x^2+1}-\sqrt{x^2-1}\}x$

Ans.

$$\lim_{x\to\infty}\{\sqrt{x^2+1}-\sqrt{x^2-1}\}x$$

$$= \lim_{x\to\infty}\frac{\{(\sqrt{x^2+1}-\sqrt{x^2-1})(\sqrt{x^2+1}+\sqrt{x^2-1})\}x}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$= \lim_{x\to\infty}\frac{\{x^2+1-x^2+1\}x}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$= \lim_{x\to\infty}\frac{2x}{\sqrt{x^2+1}+\sqrt{x^2-1}}$$

$$= \lim_{x\to\infty}\frac{\frac{2x}{x}}{\frac{\sqrt{x^2+1}}{x}+\frac{\sqrt{x^2-1}}{x}}$$

$$= \lim_{x\to\infty}\frac{2}{\sqrt{\frac{x^2}{x^2}+\frac{1}{x^2}}+\sqrt{\frac{x^2}{x^2}-\frac{1}{x^2}}}$$

$$= \lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{1}{x^2}}+\sqrt{1-\frac{1}{x^2}}}$$

$$= \frac{2}{\sqrt{1+0}+\sqrt{1-0}}$$

$$= \frac{2}{2} = 1$$

18. Evaluate $\lim_{x\to 0}\frac{e^{\tan x}-1}{x}$

Ans.

$$\lim_{x\to 0}\frac{e^{\tan x}-1}{x}$$ {Put u = tanx when x-->0 then u-->0}

$$= \lim_{u\to 0}\frac{e^{u}-1}{\tan^{-1}u}$$

$$= \lim_{u\to 0}\frac{\frac{e^{u}-1}{u}}{\frac{\tan^{-1}u}{u}} = \frac{1}{1} = 1$$

19. Evaluate $\lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2+5x+7}$

Ans.

$$\lim_{x\to\infty}\frac{3x^3-4x^2+6x-1}{2x^3+x^2+5x+7} = \lim_{x\to\infty}\frac{\frac{3x^3}{x^3}-\frac{4x^2}{x^3}+\frac{6x}{x^3}-\frac{1}{x^3}}{\frac{2x^3}{x^3}+\frac{x^2}{x^3}+\frac{5x}{x^3}+\frac{7}{x^3}}$$

$$= \lim_{x\to\infty}\frac{3-\frac{4}{x}+\frac{6}{x^2}-\frac{1}{x^3}}{2+\frac{1}{x^2}+\frac{5}{x^2}+\frac{7}{x^3}} = \frac{3-0+0-0}{2+0+0+0} = \frac{3}{2}$$

20. Evaluate $\lim_{x\to 2}\frac{\frac{1}{x^2}-\frac{1}{4}}{x-2}$

Ans.

$$\lim_{x\to 2}\frac{\frac{1}{x^2}-\frac{1}{4}}{x-2} = \lim_{x\to 2}\frac{\frac{4-x^2}{4x^2}}{x-2}$$

$$= -\frac{1}{4}\lim_{x\to 2}\frac{x^2-4}{x^2(x-2)}$$

$$= -\frac{1}{4}\lim_{x\to 2}\frac{(x-2)(x+2)}{x^2(x-2)}$$

$$= -\frac{1}{4}\left(\frac{2+2}{2^2}\right) = -\frac{1}{4} \quad \text{(Ans)}$$

Exercise

1. Evaluate the following limits(2 marks)

(i) $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$ (2016-S) (2018-S)

(ii) $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$ (a;b G 0) (2015-S)

(iii) $\lim_{x\to 0}\frac{\sin 3x}{\sin 2x}$ (2019-w)

(iv) $\lim_{x\to 1}\frac{x^2-\sqrt{x}}{\sqrt{x}-1}$

(v) $\lim_{x\to 0}\frac{1-cos}{x^2}$ (2014-S)

(vi) $\lim_{x\to \pi}\frac{\sin x}{\pi-x}$ (2016-S)

(vii) $\lim_{x\to 0} ln(1+bx)^{\frac{1}{x}}$ (2016-S)

(viii) $\lim_{x\to 0}\frac{tan5}{tan7x}$ (2017-w)

2. Evaluate the following limits (5 marks)

(i) $\lim_{x\to 1} \frac{2^{x-1}-1}{\sqrt{x}-1}$

(ii) $\lim_{x\to 5} \frac{\sqrt{x-1}-2}{x-5}$

(iii) $\lim_{x\to 1} \frac{x^{\frac{1}{m}}-1}{x^{\frac{1}{n}}-1}$

(iv) $\lim_{n\to\infty} \frac{1^3+2^3+3^3+\cdots+n^3}{n^4}$

(v) $\lim_{x\to\infty} x^2\{\sqrt{x^4+a^2}-\sqrt{x^4-a^2}\}$

(vi) $\lim_{x\to 3} \frac{x^2-4x+3}{x^2-2x-3}$

(vii) $\lim_{x\to -5} \frac{2x^2+9x-5}{x+5}$

(viii) $\lim_{x\to 1} (\frac{2}{1-x^2}+\frac{1}{x-1})$

(ix) $\lim_{x\to 2} \frac{x^3-6x^2+11x-6}{x^2-6x+}$

(x) $\lim_{x\to 2} \frac{8\ \log_e(x-1)}{x^2-3x+2}$

(xi) $\lim_{x\to 0} \frac{tanx-sinx}{x^3}$ (2018-S)

(xii) $\lim_{x\to 0} \frac{\sqrt[3]{1+x}-\sqrt[3]{1-x}}{x}$

(xiii) $\lim_{x\to 0} \frac{a^x-b^x}{c^x-d^x}$ (2016-S)

(xiv) $\lim_{x\to 0} \frac{cos2x-cos3x}{x^2}$ (2017-S)

(xv) $\lim_{x\to 0} \frac{\sqrt{3-2x}-\sqrt{3}}{x}$

(xvi) $\lim_{x\to 0} \frac{\sqrt{1+x}-\sqrt{1-x}}{\sin^{-1} x}$ (2017-w)

3. Find the value of a on following cases.(5 marks)

(i) $\lim_{x\to\alpha} \frac{tana(x-\alpha)}{x-\alpha} = \frac{1}{2}$

(ii) $\lim_{x\to 0} \frac{e^{ax}-e^x}{x} = 2$

(iii) $\lim_{x\to 2} \frac{\log_e(2x-3)}{a(x-2)} = 1$

Answers

1. i) $\frac{1}{2}$ ii) $\frac{a}{b}$ iii) $\frac{3}{2}$ iv) 3 v) $\frac{1}{2}$ vi) 1 vii) b viii) 5/7

2. i) 2 ln2 ii) ¼ iii)n/m iv)1/4 v) a^2 vi) ½ vii) -11 viii) 1/2

ix) ½ x) 1 xi) $\frac{1}{2}$ xii) $\frac{2}{3}$ xiii) $\frac{\ln\frac{a}{b}}{\ln\frac{c}{d}}$ xiv) 5/2 xv) $-1/\sqrt{3}$ xvi) 1

3. (i) ½ ii) 3 iii) 2

Continuity and Discontinuity of Function

In the figure we observe that the 1st graph of a function in Fig-1 can be drawn on a paper without raising pencil i.e. 1st graph is continuously moving where as Fig -2 represents a graph , which cannot be drawn without raising the pencil. Because there are gaps or breaks. So, it is discontinuous.

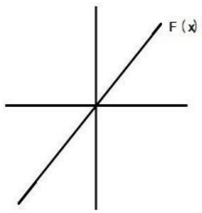

Fig-1

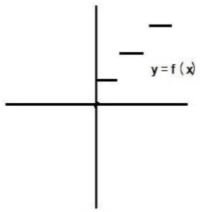

Fig-2

The feature of the graph of a function displays an important property of the function called continuity of a function.

<u>Continuity of a Function at a point</u>

Definition – A function f(x) is said to be continuous at x = a, if it satisfies the following conditions

(i) $\lim_{x \to a} f(x)$ exists.

(ii) f(a) is defined i.e. finite

(iii) $\lim_{x \to a} f(x) = f(a)$

If one or more of the above condition fail, the function f(x) is said to be discontinuous at x= a.

<u>Continuous Function</u>

A function is said to be continuous if it is continuous at each point of its domain.

Working procedure for testing continuity at a point x = a

<u>1st step</u> – First find $\lim_{x \to a} f(x)$ by using concepts from previous chapter.

If $\lim_{x \to a} f(x)$ does not exit then, f(x) is discontinuous at x = a.

If $\lim_{x \to a} f(x) = l$, then go to 2nd step.

2nd step – Find f(a) from the given data

If f(a) is undefined then f(x) is not continuous at x =a.

If f(a) has finite value then go to 3rd step.

3rd step – Compare $\lim_{x\to a} f(x)$ and f(a)

If $\lim_{x\to a} f(x)$ = f(a), then f(x) is continuous at x= a, otherwise f(x) is discontinuous at x = a.

Examples

Q1. Examine the continuity of the function f(x) at x = 3.

$$f(x) = \begin{cases} \frac{x^2-9}{x-3} & x \neq 3 \\ 6 & x = 3 \end{cases}$$

Ans:-

$$\lim_{x\to 3} f(x) = \lim_{x\to 3} \frac{x^2-9}{x-3} = \lim_{x\to 3} \frac{(x+3)(x-3)}{(x-3)}$$

$$= \lim_{x\to 3}(x+3) = 3+3 = 6 \qquad \{\text{As } x \to 3,\ x \neq 3 \Rightarrow x-3 \neq 0\}$$

From given data f(3) = 6

Now from above $\lim_{x\to 3} f(x)$ = f (3)

Therefore, f(x) is continuous at x = 3.

Q2. Test continuity of f(x) at '0' where,

$$f(x) = \begin{cases} (1+3x)^{\frac{1}{x}} & x \neq 0 \\ e^3 & x = 0 \end{cases}$$

Ans:- $\lim_{x\to 0} f(x) = \lim_{x\to 0}(1+3x)^{\frac{1}{x}}$

$$= \lim_{x\to 0} (1+3x)^{\frac{1}{3x}\cdot 3}$$

$$= \lim_{x\to 0} \{(1+3x)^{\frac{1}{3x}}\}^3$$

$$= \{\lim_{x\to 0}(1+3x)^{\frac{1}{3x}}\}^3$$

$$= e^3$$

{As $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$

$\lim_{x\to 0}(1+\lambda x)^{\frac{1}{x}}$ = e

In particular $\lim_{x\to 0}(1+3x)^{\frac{1}{3x}}$ = e

and we know, $\lim_{x\to a}\{f(x)\}^n = \{\lim_{x\to a} f(x)\}^n$

} From given data f(0) = e^3

Hence, $\lim_{x\to 0} f(x)$ = f(0)

Therefore, f(x) is continuous at x = 0.

Q3. Test continuity of f(x) at x = 0

$$f(x) = \begin{cases} \frac{|x|}{x} & x \text{ G } 0 \\ 0 & x = 0 \end{cases}$$

Ans. $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{|x|}{x}$

As |x| is present and x→0 , so we have to evaluate the above limit by L.H.L and R.H.L method

L.H.L = $\lim_{x\to 0^-} \frac{|x|}{x}$ {x → 0^- => x <0 }

$= \lim_{x\to 0^-} \frac{-x}{x}$

$= \lim_{x\to 0^-} (-1)$ = (-1)

R.H.L = $\lim_{x\to 0^+} \frac{|x|}{x}$ {x →0^+ => x> 0 }

$= \lim_{x\to 0^+} \frac{x}{x}$ = $\lim_{x\to 0^+} (1)$ = 1

Hence, L.H.L ≠ R.H.L

Therefore, f(x) does not exist.

Hence f(x) is not continuous at x = 0.

Q4. Test continuity of $\frac{x^2-4}{x-2}$ at x = 2.

Ans. Here, f(2) = $\frac{2^2-4}{2-2}$ = $\frac{0}{0}$ undefined.

Hence, f(x) is not continuous at x = 2.

Q5. Test continuity of f(x) at '0'.

$$f(x) = \begin{cases} \frac{\sin 3x}{\tan 5x} & x \text{ G } 0 \\ \frac{5}{3} & x = 0 \end{cases}$$

Ans. $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 3x}{\tan 5x} = \lim_{x\to 0} \frac{\frac{\sin 3x}{x}}{\frac{\tan 5x}{x}}$

$= \lim_{x\to 0} \frac{\frac{\sin 3x}{3x}.3}{\frac{\tan 5x}{5x}.5} = \frac{3}{5} \lim_{x\to 0} \{(\frac{\sin 3x}{3x})/(\frac{\tan 5x}{5x})\}$

$= \frac{3}{5} \quad (\frac{1}{1}) \quad = \frac{3}{5}$

Given that, f(0) = $\frac{5}{3}$

Thus, $\lim_{x\to 0} f(x)$ ≠ f(0)

Hence f(x) is not continuous at x = 0.

Q6. Test continuity of f(x) at x = $\frac{1}{2}$

$$f(x) = \begin{cases} 1-x & x \le 1/2 \\ x & x > 1/2 \end{cases}$$

Ans. First understand the function properly

When $x < \frac{1}{2}$, $f(x) = 1 - x$

$x > \frac{1}{2}$, $f(x) = x$

When $x = \frac{1}{2}$, $f(x) = 1 - x = 1 - \frac{1}{2} = \frac{1}{2}$

Now let us find the $\lim_{x\to 1/2} f(x)$

L.H.L $= \lim_{x\to \frac{1^-}{2}} f(x) = \lim_{x\to \frac{1^-}{2}} (1-x)$ {As $x \to \frac{1^-}{2}$ i.e. $x < \frac{1}{2}$, so $f(x) = 1 - x$ }

$= 1 - \frac{1}{2} = \frac{1}{2}$

R.H.L $= \lim_{x\to \frac{1^+}{2}} f(x)$ {As $x \to \frac{1^+}{2}$ i.e. $x > \frac{1}{2}$. So, f(x) = x from definition of f(x)}

$= \lim_{x\to \frac{1^+}{2}} x = \frac{1}{2}$

Now from above L.H.L = R.H.L

⇨ $\lim_{x\to \frac{1}{2}} f(x) = \frac{1}{2}$ ----------- (1)

From definition $f(\frac{1}{2}) = \frac{1}{2}$ ----------- (2)

From (1) and (2)

$\lim_{x\to \frac{1}{2}} f(x) = f(\frac{1}{2})$

Hence, f(x) is continuous at x = $\frac{1}{2}$.

Q7. Test continuity of f(x) at x = 0, 1

$$f(x) = \begin{cases} 2x+1 & \text{if } x \le 0 \\ x & \text{if } 0 < x \le 1 \\ 2x-1 & \text{if } x > 1 \end{cases}$$

Ans. Here given that

$f(x) = 2x+1$ for $x<0$ ---------- (1)

When x = 0, f(x) = f(0) = 2x+1 = 2X0+1 = 1 ---------- (2)

When 0<x<1, f(x) = x ----------- (3)

When x = 1 , f(x) = f(1)= x = 1 ----------- (4)

When x>1 , f(x) = 2x-1 ---------- (5)

Continuity test at x = 0

L.H.L = $\lim_{x\to 0^-} f(x)$ {x→0^-=>x<0 => f(x) = 2x+1 from (1)}

= $\lim_{x\to 0^-}(2x+1)$

= (2X0)+1 = 1

R.H.L = $\lim_{x\to 0^+} f(x)$ {x→0^+=> x>0 => 0<x<1 => f(x) = x}

= $\lim_{x\to 0^+} x$ = 0

As L.H.L ≠R.H.L

⇨ $\lim_{x\to 0} f(x)$ does not exist

Hence, f(x) is not continuous at x = 0.

Continuity test at x = 1

L.H.L = $\lim_{x\to 1^-} f(x)$ = $\lim_{x\to 1^-} x$ {x →1^- => x<1 i.e. 0<x<1 => f(x) = x from (3)}

= $\lim_{x\to 1^-} x$ = 1

R.H.L = $\lim_{x\to 1^+} f(x)$ = $\lim_{x\to 1^+} 2x-1$ {x →1^+ => x>1 , f(x) = 2x-1 from (5)}

= 2 X 1 − 1 = 1

As L.H.L = R.H.L

$\lim_{x\to 1} f(x)$ = 1

From given data f(1) = 1 {from equation (4)}

Hence, $\lim_{x\to 1} f(x)$ = f(1)

Therefore, f(x) is continuous at x = 1

Q8. Examine continuity of f(x) = [3x + 11] at x = -$\frac{11}{3}$ (2016-S)

Ans.

$\lim_{x\to -\frac{11}{3}} f(x) = \lim_{x\to -\frac{11}{3}}[3x+11]$ {Let u = 3x+11 when x→ - $\frac{11}{3}$, u = 3 X →$3\times -\frac{11}{3}$ +11 =0}

= $\lim_{u\to 0}[u]$ ------------- (1)

Now, $\lim_{u\to 0^-}[u] = \lim_{u\to 0^-} -1$ = -1 {As u→0^- => -1<u<0 => [u] = -1

And $\lim_{u\to 0^+}[u] = \lim_{u\to 0^+} 0$ = 0 {As u→0^+=> 0<u<1 => [u] = 0 }

As, L.H.L ≠ R.H.L

⇨ $\lim_{u\to 0}[u]$ does not exist => $\lim_{x\to -\frac{11}{3}} f(x)$ does not exist.

Hence, f(x) is not continuous at x = 0.

Q9. Determine the value of K for which f(x) is continuous at x = 1.

$$f(x) = \begin{cases} \frac{x^2-3x+2}{x-1} & x \text{ G } 1 \\ K & x = 1 \end{cases}$$

Ans.

Given function is continuous at x = 1.

=> $\lim_{x\to 1} f(x)$ = f(1)

=> $\lim_{x\to 1} f(x)$ = K -------------- (1)

Now, let us find $\lim_{x\to 1} f(x)$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2-3x+2}{x-1}$ $(\frac{0}{0} form)$

$= \lim_{x\to 1} \frac{x^2-2x-x+2}{x-1}$

$= \lim_{x\to 1} \frac{x(x-2)-1(x-2)}{x-1}$

$= \lim_{x\to 1} \frac{(x-2)(x-1)}{(x-1)}$ {As x → 1, x ≠ 1, x -1 ≠ 0}

$= \lim_{x\to 1}(x-2)$ = 1 – 2 = -1 ------------ (2)

Hence, From (1) and (2) we have K = -1 . (Ans)

Q10. If f(x) = $\begin{cases} ax^2+b & \text{if } x<1 \\ 1 & \text{if } x=1 \\ 2ax-b & \text{if } x>1 \end{cases}$

is continuous at x = 1, then find a and b.

Ans.

Given that f(x) is continuous at x = 1

⇨ $\lim_{x\to 1} f(x)$ = f(1)

⇨ $\lim_{x\to 1} f(x)$ = 1 ------------- (1) {As f(1) = 1 given}

From (1) as $\lim_{x\to 1} f(x)$ exists

=> $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = \lim_{x\to 1} f(x)$ ----------------------(2)

From (1) and (2) we have,

$\lim_{x\to 1^-} f(x)$ = 1

=> $\lim_{x\to 1^-}(ax^2 + b)$ = 1 {As x→1^-=> x< 1 => f(x) = ax^2 +b from def^n of f(x)}

$\Rightarrow\ a \times 1^2 + b = 1$

$\Rightarrow\ a + b = 1$ ---------------- (3)

Again from (1) and (2)

$\lim_{x\to 1^+} f(x) = 1$ $\{x \to 1^+ \Rightarrow x>1\ ,\ \Rightarrow f(x) = 2ax - b\}$

$\Rightarrow \lim_{x\to 1^+}(2ax - b) = 1$

$\Rightarrow\ (2 \times a \times 1) - b = 1$

$\Rightarrow\ 2a - b = 1$ ---------------(4)

Eq^n (3) $a + b = 1$

Eq^n (4) $2a - b = 1$

$3a = 2$

$\Rightarrow\ a = \frac{2}{3}$

From (3) $a + b = 1$

$\Rightarrow\ b = 1 - a = 1 - \frac{2}{3} = \frac{1}{3}$

Hence, $a = \frac{2}{3}$ and $b = \frac{1}{3}$

Q11. Find the value of 'a' such that

$$f(x) = \begin{cases} \frac{\sin ax}{\sin x} & x \neq 0 \\ \frac{1}{a} & x = 0 \end{cases}$$

is continuous at x = 0

Ans. f(x) is continuous at x = 0

$\Rightarrow \lim_{x\to 0} f(x) = f(0)$

$\Rightarrow \lim_{x\to 0} \frac{\sin ax}{\sin x} = \frac{1}{a}$

$\Rightarrow \lim_{x\to 0} \frac{a\left(\frac{\sin a}{a}\right)}{\left(\frac{\sin x}{x}\right)} = \frac{1}{a}$

$\Rightarrow a\ \frac{1}{1} = \frac{1}{a}$

$\Rightarrow a^2 = 1$

$\Rightarrow\ a = \pm 1$ (Ans)

Q12. Examine the continuity of the function

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \text{ G } 0 \\ 0 & x = 0 \end{cases} \quad \text{at } x = 0.$$

Ans.

Let us evaluate $\lim_{x\to 0} x^2 \sin\frac{1}{x}$.

We know that $-1 \le \sin\frac{1}{x} \le 1$

$\Rightarrow (-1)x^2 \le x^2 \sin\frac{1}{x} \le x^2 \cdot 1$

$\Rightarrow -x^2 \le x^2 \sin\frac{1}{x} \le x^2$

Now, $\lim_{x\to 0}(-x^2) = -0^2 = 0$

$\lim_{x\to 0} x^2 = 0^2 = 0$

Hence, by sandwich theorem

$\lim_{x\to 0} x^2 \sin\frac{1}{x} = 0$

Given $f(0) = 0$

Hence $\lim_{x\to 0} f(x) = f(0)$

Therefore, f(x) is continuous at x = 0.

Q13. Test continuity of f(x) at x = 0

$$f(x) = \begin{cases} \frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1} & x \text{ G } 0 \\ 0 & x = 0 \end{cases}$$

Ans:-Evaluation of $\lim_{x\to 0} f(x)$ is not possible directly.

L.H.L $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}}+1}$

{when $x \to 0^-$ then $\frac{1}{x} \to -\infty \Rightarrow e^{\frac{1}{x}} \to 0$ }

$= \frac{0-1}{0+1} = -1$

R.H.L. $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{e^{\frac{1}{x}} - \frac{1}{e^{\frac{1}{x}}}}{e^{\frac{1}{x}} + 1}$

$\{$when x→0^+ then $\frac{1}{x}$→∞ =>$e^{\frac{1}{x}}$→∞ =>$\frac{1}{e^{\frac{1}{x}}}$→0 $\}$

$$= \lim_{x\to 0^+} \frac{\frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}} - \frac{1}{e^{\frac{1}{x}}}}{\frac{e^{\frac{1}{x}}}{e^{\frac{1}{x}}} + \frac{1}{e^{\frac{1}{x}}}} = \lim_{x\to 0^+} \frac{1 - \frac{1}{e^{\frac{1}{x}}}}{1 + \frac{1}{e^{\frac{1}{x}}}}$$

$= \frac{1-0}{1+0} = 1$

From above L.H.L ≠ R.H.L

=>$\lim_{x\to 0} f(x)$ does not exist.

Therefore, f(x) is not continuous at x = 0.

Q14. Discuss the continuity of the function

$$f(x) = \begin{cases} x - \frac{|x|}{x} & x \text{ G } 0 \\ 2 & x = 0 \end{cases} \qquad \text{at } x=0$$

Ans: -

L.H.L $= \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} x - \frac{|x|}{x}$

$= \lim_{x\to 0^-}\{ x - \frac{(-x)}{x} \}$ $\{$x-->0^- => x<0 => |x| = -x$\}$

$= \lim_{x\to 0^-}\{ x - (-1)\} = \lim_{x\to 0^-}\{ x + 1\}$

$= 0 + 1 = 1$

R.H.L. $= \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x - \frac{|x|}{x}$

$= \lim_{x\to 0^+}\{ x - \frac{x}{x} \}$ $\{$x-->0^+ => x >0 => |x| = x$\}$

$= \lim_{x\to 0^-}\{ x - 1)\}$ = 0 - 1 = -1

So, L.H.L ≠ R.H.L =>$\lim_{x\to 0} f(x)$ does not exist.

Therefore, f(x) is not continuous at x = 0.

Exercise

Q1. Find the value of the constant K, so that the function given below is continuous at x = 0.

$$f(x) = \begin{cases} \frac{1-\cos 2x}{2x^2} & x \neq 0 \\ K & x = 0 \end{cases}$$ (5 marks)

Q2. Test the continuity of f(x) at x = 1, where

$$f(x) = \begin{cases} x^2 + 1 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 3x - 1 & \text{if } x > 1 \end{cases}$$ (5 marks)

Q3. Show that the function f(x) given by

$$f(x) = \begin{cases} \frac{\sin x}{x} + \cos x & x \neq 0 \\ 2 & x = 0 \end{cases}$$ is continuous at x = 0. (5 marks)

Q4. Test continuity of f(x) at x = 1

$$f(x) = \begin{cases} \frac{x^7-1}{x-1} & x \neq 1 \\ 7 & x = 1 \end{cases}$$ (5 marks)

Q5. Test continuity of f(x) at x = 0

$$f(x) = \begin{cases} (1+2x)^{\frac{1}{x}} & \text{if } x \neq 0 \\ e^2 & \text{if } x = 0 \end{cases}$$ (2017-W) (5 marks)

Q6. Test continuity of f(x) at x = 2

$$f(x) = \begin{cases} \frac{|x-2|}{x-2} & x \neq 2 \\ 1 & x = 2 \end{cases}$$ (10 marks)

Q7. Find the value of K for which f(x) is continuous at x = 0.

$$f(x) = \begin{cases} \frac{8^x - 4^x - 2^x + 1}{x^2} & x \neq 0 \\ K & x = 0 \end{cases}$$ (2016-S) (10 marks)

Q8. Test the continuity of the function f(x) at x = 0.

$$f(x) = \begin{cases} \frac{\sin 3x}{\tan^{-1} 7x} & x \neq 0 \\ 3/7 & x = 0 \end{cases}$$ (5 marks)

Q9. Test continuity of the function f(x) at x = 1

$$f(x) = \begin{cases} \frac{x^2-4x+3}{x-1} & x \text{ G } 1 \\ 2 & x = 1 \end{cases}$$ (5 marks)

Q10. Examine the continuity of the function of f(x) at x=0.

$$f(x) = \begin{cases} 2x+1 & \text{if} x < 0 \\ 0 & \text{if} x = 0 \\ x^2+1 & \text{if} x > 0 \end{cases}$$ (2014-S) (5 marks)

Answers

1) K = 1,

Q no. 2, 4, 5, 8 are continuous .

6, 9, 10 are discontinuous

7.2(ln2)^2