5 DETERMINANTS & CRAMER'S RULE

: SYLLABUS :

Determinants, Cramer's Rule

1. Determinant of Second Order :

Determinant is a system of representing numbers in a square arrangement under a particular symbol.

$ad-bc$ can be represented in the from $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

This representation is called a determinant and $ad - bc$ is the value of the determinant.

$$\therefore \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

In this determinant there are two rows and two columns. Hence it is called a determinant of 2nd order. a b c and d are known as the elements of the determinant. The first element in the first row is the leading element of the determinant and the diagonal passing through it is the leading diagonal of the determinant.

We shall find the values of some determinants of 2nd order.

(i) $\begin{vmatrix} 3 & 4 \\ 1 & 5 \end{vmatrix} = (3 \times 5) - (1 \times 4) = 15 - 4 = 11$

(ii) $\begin{vmatrix} 7 & 8 \\ 3 & 2 \end{vmatrix} = (7 \times 2) - (8 \times 3) = -10$

(iii) $\begin{vmatrix} 5 & 20 \\ 3 & 12 \end{vmatrix} = (5 \times 12) - (20 \times 3) = 0$

(iv) $\begin{vmatrix} p & q \\ a & b \end{vmatrix} = pb - aq$

(v) $\begin{vmatrix} x+y & x \\ x & x-y \end{vmatrix} = (x + y)(x - y) - x^2$

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$$= x^2 - y^2 - x^2 = -y^2$$

(vi) $\begin{vmatrix} a+b & a-b \\ a-b & a+b \end{vmatrix} = (a+b)(a+b) - (a-b)(a-b)$

$$= (a+b)^2 - (a-b)^2$$
$$= a^2 + 2ab + b^2 - (a^2 - 2ab + b^2)$$
$$= a^2 + 2ab + b^2 - a^2 + 2ab - b^2$$
$$= 4ab$$

2. Determinant of Third Order :

Determinant of 3rd order consists of three rows and three columns with 9 elements.

e. g $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

In this determinant 9 elements are arranged in 3 rows and 3 columns. Similarly

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ is a determinant of 3rd order.

In this determinant a_1 is the leading element and the diagonal passing through it is the leading diagonal. The value of this determinant can be obtained by expanding it. Before we expand the determinant of 3rd order, it is necessary to know the meaning of minor of any element of a determinant.

Minor :

If the column and row passing through an element of a determinant are deleted, then the determinant obtained from the remaining elements is known as the minor of that particular element. It is obvious that minor of any element of a determinant of 3rd order will be a determinant of 2nd order.

In the above determinant of 3rd order.

minor of a_1 is $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

minor of b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$

minor of c_1 is $\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

Thus we, can have 9 minors for 9 elements of a 3×3 determinant. We can expand a given determinant with respect to any row or any column. Take the products of every element of a row or a column with its minor. Observe the order of that particular row or column. If the order is odd give the signs +, –, + respectively to the products and if the order is even give the signs –, +, – respectively to the products. Taking the summation of the products obtained in this way we can get the value of the determinant. e.g. Let us expand.

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ with respect to its first row.

The value of the determinant

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

(As the order of row is odd)

$$= a_1(b_2c_3 - c_2 b_3) - b_1(a_2 c_3 - a_3c_2) + c_1(a_2 b_3 - a_3 b_2)$$
$$= a_1b_2 c_3 - a_1 c_2 b_3 - b_1 a_2 c_3 + b_1 a_3 c_2 + c_1a_2b_3 - c_1a_3b_2$$
$$= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1b_3 c_2 - a_2 b_1c_3 - a_3b_2 c_1$$

Now let us expand the determinant with respect to second column.

The value of the determinant

$$= -b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + b_2 \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix} - b_3 \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

(As the order of the column is even)

$$= -b_1(a_2 c_3 - a_3 c_2) + b_2(a_1c_3 - a_3 c_1) - b_3 (a_1 c_2 - a_2c_1)$$
$$= -a_2b_1c_3 + a_3 b_1 c_2 + a_1 b_2 c_3 - a_3 b_2 c_1 - a_1 b_3 c_2 + a_2 b_3 c_1$$
$$= a_1 b_2 c_3 + a_2 b_3 c_1 + a_3 b_1 c_2 - a_1 b_3 c_2 - a_2 b_1 c_3 - a_3 b_2 c_1$$

Thus the value of the determinant is the same, when it is expanded with respect to any row or any column.

We shall now obtain the values of some determinants.

Illustration 1 : Expand the following determinant :

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Ans. : Expanding the determinant with respect to first row, we get the value of the determinant.

$$D = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$= 1(45-48) - 2(36-42) + 3(32-35)$

$= -3 + 12 - 9$

$= 0$

Illustration 2 : Expand the determinant :

$$\begin{vmatrix} 4 & 7 & 8 \\ 3 & 2 & 6 \\ 1 & 5 & 0 \end{vmatrix}$$

Ans. :

$D = 4(2 \times 0 - 6 \times 5) - 7(3 \times 0 - 1 \times 6) + 8(3 \times 5 - 2 \times 1)$

$= 4(-30) - 7(-6) + 8(13)$

$= -120 + 42 + 104$

$= 26$

Illustration 3 : Find the value of :

$$\begin{vmatrix} 2 & 5 & -3 \\ 3 & -2 & 5 \\ -1 & 3 & 2 \end{vmatrix}$$

Ans. : Expanding with respect to first row .

$= 2(-4-15) - 5(6+5) - 3(9-2)$

$= 2(-19) - 5(11) - 3(7)$

$= -38 - 55 - 21$

$= -114$

Illustration 4 : Find the value of

$$\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix}$$

Ans. :

$$\begin{vmatrix} 1 & 1 & 1 \\ bc & ca & ab \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} \end{vmatrix}$$

$$= 1\{ca \times \frac{1}{c} - ab \times \frac{1}{b}\} - 1\{bc \times \frac{1}{c} - ab \times \frac{1}{a}\}$$

$$+ 1\{bc \times \frac{1}{b} - ca \times \frac{1}{a}\}$$

$= 1(a-a) - 1(b-b) + 1(c-c)$

$= 0 - 0 + 0$

$= 0$

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3. Rules of Determinant :

Let

$$D = \begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{vmatrix}$$

Expanding w.r.t. first row,

$$D = 4(6 - 6) - 5(3 - 3) + 6(2 - 2)$$
$$= 0.$$

We observe that in the above determinant the elements of two rows are identical.

Fot this we can give the following rule.

Rule I :

If the elements of any two rows or two columns of a determinant are identical, the value of the determinant is zero.

Now let us find the values of the following two determinants :

$$D_1 = \begin{vmatrix} 3 & 2 & 5 \\ 1 & 4 & 10 \\ 2 & 3 & 15 \end{vmatrix} \text{ and } D_2 = 5 \times \begin{vmatrix} 3 & 2 & 1 \\ 1 & 4 & 2 \\ 2 & 3 & 3 \end{vmatrix}$$

Expanding D_1

$$D_1 = 3(60 - 30) - 2(15 - 20) + 5(3 - 8)$$
$$= 90 + 10 - 25$$
$$= 75$$

Similarly

$$D_2 = 5\,[3(12 - 6) - 2(3 - 4) + 1(3 - 8)]$$
$$= 5\,[18 + 2 - 5]$$
$$= 5 \times 15$$
$$= 75$$

It is seen that in the first determinant from each element of third column 5 can be taken as the common factor. This factor is taken common in the second determinant, however the values of both determinants are the same. Thus, we have the following rule :

Rule II :

If there is a common factor in each element of any row or any column of a delerminant it can be taken as the common factor of the determinant.

Let us expand the determinant.

$$\begin{vmatrix} 3 & 1 & 3 \\ 5 & 2 & 2 \\ 7 & 4 & 8 \end{vmatrix}$$

$= 3(16 - 8) - 1(40 - 14) + 3(20 - 14)$
$= 24 - 26 + 18$
$= 16$

Interchanging the second and third columns. We get the determinant

$$\begin{vmatrix} 3 & 3 & 1 \\ 5 & 2 & 2 \\ 7 & 8 & 4 \end{vmatrix}$$

Its value $= 3(8 - 16) - 3(20 - 14) + 1(40 - 14)$
$= - 24 - 18 + 26)$
$= - 16$

Thus only the sign of the value of the determinant is changed by interchanging two columns. The rule for this can be given as follows:

Rule III :

If two rows or columns of a determinant are interchanged the value of the determinant is changed only in sign.

Consider the determinant $\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{vmatrix}$

Its value $= 4(4 - 3) - 5(2 - 9) + 6(1 - 6)$
$= 4 + 35 - 30$
$= 9.$

Now multiplying each element of the second row by 2 and adding them to the corresponding elements of third row, we get the new determinant as

$$\begin{vmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 5 & 5 & 8 \end{vmatrix}$$

Its value $= 4(16 - 15) - 5(8 - 15) + 6(5 - 10)$
$= 4 + 35 - 30$
$= 9.$

Hence the value of the two determinants are the same. We can write the following rule from this.

Rule IV :

If all the elements of any row (or column) are multiplied by a constant number and added to the corresponding elements of any other

...(or column) the value of the determinant remains unchanged.

Now we shall show that

$$\begin{vmatrix} 3+2 & 2+3 & 1+3 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix} = \begin{vmatrix} 3 & 2 & 1 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 3 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix}$$

Here

$$\text{L.H.S.} = \begin{vmatrix} 3+2 & 2+3 & 1+3 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix} = \begin{vmatrix} 5 & 5 & 4 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix}$$

$$= 5(24 - 2) - 5(40 - 2) + 4(10 - 6)$$

$$= 110 - 190 + 16$$

$$= -64.$$

$$\text{R.H.S.} = \begin{vmatrix} 3 & 2 & 1 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix} + \begin{vmatrix} 2 & 3 & 3 \\ 5 & 3 & 1 \\ 2 & 2 & 8 \end{vmatrix}$$

$$= \{3(24 - 2) - 2(40 - 2) + 1(10 - 6)\} + \{2(24 - 2) - 3(40 - 2) + 3(10 - 6)\}$$

$$= \{66 - 76 + 4\} + \{44 - 114 + 12\}$$

$$= -6 - 58$$

$$= -64.$$

Here each element of the first row is expressed as the sum of two terms. We have shown that the determinant is expressible as the sum of two determinants we can write the following rule.

Rule V :

If each element of any row or column is the sum of two terms then the determinant can be shown as the sum of two determinants of the same order.

Now let us take the determinants

$$D_1 = \begin{vmatrix} 2 & 3 & 1 \\ 5 & 0 & 6 \\ 7 & 4 & 9 \end{vmatrix} \text{ and } D_2 = \begin{vmatrix} 2 & 5 & 7 \\ 3 & 0 & 4 \\ 1 & 6 & 9 \end{vmatrix}.$$

Expanding these two determinants we get,

$$D_1 = 2(0 - 24) - 3(45 - 42) + 1(20 - 0)$$

$$= -48 - 9 + 20$$

$$= -37$$

$$D_2 = 2(0 - 24) - 5(27 - 4) + 7(18 - 0)$$

$$= -48 - 115 + 126$$

$$= -37$$

In this case the second determinant is obtained by interchanging the rows and columns of the first determinant. Thus, we have the following rule.

Rule VI :

If the rows and columns of a determinant are interchanged, the value of the determinant remains the same.

4. Use of Determinant in Solving Simultaneous Equations (Cramer's Method) :

(A) For Two Liner Equations :

$a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are two linear equations of the first degree in x and y. $a_1, b_1, c_1, a_2, b_2, c_2$ are constants. These equations can be solved by comparing the co–efficients of x or y.

For comparing the co–efficients of y, multiply the first equations by b_2 and the second by b_1.

$$\therefore \quad a_1b_2x + b_1b_2y + c_1b_2 = 0$$
$$a_2b_1x + b_1b_2y + b_1c_2 = 0$$
$$- \qquad - \qquad -$$

$$(a_1b_2 - a_2b_1)x + c_1b_2 - b_1c_2 = 0$$

$$\therefore \quad x = \frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}$$

$$\text{Similarly} \quad y = \frac{c_1a_2 - a_1c_2}{a_1b_2 - a_2b_1}$$

$$\therefore \quad -y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}$$

$$\text{i.e.} \quad \frac{x}{b_1c_2 - b_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\text{and} \quad \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad \frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}$$

$$\therefore \quad \frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

Thus, we can get the values of x and y. This method of solving simultaneous equations is known as **Cramer's method.**

(B) Cramer's method for solving three simultaneous linear equations

If we have three simultaneous linear equations

$a_1x + b_1y + c_1z + d_1 = 0$

$a_2x + b_2y + c_2z + d_2 = 0$

$a_3x + b_3y + c_3z + d_3 = 0$

We can obtain the solutions as $\dfrac{x}{D_x} = \dfrac{-y}{D_y} = \dfrac{z}{D_z} = \dfrac{-1}{D}$

Where

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad D_x = \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

$$D_y = \begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix} \qquad D_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Then, $x = \dfrac{-D_x}{D}$, $y = \dfrac{D_y}{D}$, $z = \dfrac{-D_z}{D}$

It should be noted that the equation can be solved only when $D \neq 0$.

Illustration 5 : Solve the equations :

$$3x + 7y + 4 = 0$$
$$4x + y - 3 = 0$$

Ans. : Here $a_1 = 3$, $b_1 = 7$, $c_1 = 4$, $a_2 = 4$, $b_2 = 1$, $c_2 = -3$.

$$\therefore \frac{x}{\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}$$

$$\therefore \frac{x}{\begin{vmatrix} 7 & 4 \\ 1 & -3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 3 & 4 \\ 4 & -3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 3 & 7 \\ 4 & 1 \end{vmatrix}}$$

$$\frac{x}{-25} = \frac{-y}{-25} = \frac{1}{-25}$$

$$x = \frac{-25}{-25} \text{ and } y = \frac{25}{-25}$$

$$x = 1 \text{ and } y = -1$$

Illustration 6 : Solve the equations :

$$2x - 9 = 5y$$
$$x - y = 3$$

Ans. : We shall arrange the equations in proper form.

$$2x - 5y - 9 = 0$$

$\therefore$ $$x - y - 3 = 0$$

$\therefore$ $$\frac{x}{\begin{vmatrix} -5 & -9 \\ -1 & -3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 2 & -9 \\ 1 & -3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 2 & -5 \\ 1 & -1 \end{vmatrix}}$$

$\therefore$ $$\frac{x}{15-9} = \frac{-y}{-6+9} = \frac{1}{-2+5}$$

$\therefore$ $$\frac{x}{6} = \frac{-y}{3} = \frac{1}{3}$$

$\therefore$ $$x = \frac{6}{3}; -y = \frac{3}{3}$$

$\therefore$ $$x = 2; y = -1$$

Illustration 7 : Solve the equations :

$$y = x + 7$$
$$y = 2x + 15$$

Ans. : Rearranging the equations, we get,

$$-x + y - 7 = 0$$
$$-2x + y - 15 = 0$$

$\therefore$ $$\frac{x}{\begin{vmatrix} 1 & -7 \\ 1 & -15 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} -1 & -7 \\ -2 & -15 \end{vmatrix}} = \frac{1}{\begin{vmatrix} -1 & 1 \\ -2 & 1 \end{vmatrix}}$$

$\therefore$ $$\frac{x}{-15+7} = \frac{-y}{15-14} = \frac{1}{-1+2}$$

$\therefore$ $$\frac{x}{-8} = \frac{-y}{1} = \frac{1}{1}$$

$$x = -8; y = -1$$

Illustration 8 : Solve the equations :

$$\frac{x}{3} + \frac{y}{4} = 1$$

$$\frac{2x}{9} - \frac{y}{2} = 6$$

Ans. : Multiplying the 1st equation by 12 and 2nd equation by 18 and **rearranging the equations, we get.**

$4x + 3y - 12 = 0$

$4x - 9y - 108 = 0$

$$\frac{x}{\begin{vmatrix} 3 & -12 \\ -9 & -108 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 4 & -12 \\ 4 & -108 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 4 & 3 \\ 4 & -9 \end{vmatrix}}$$

$$\frac{x}{-324 - 108} = \frac{-y}{-432 + 48} = \frac{1}{-36 - 12}$$

$$\frac{x}{-432} = \frac{-y}{-384} = \frac{1}{-48}$$

$$x = \frac{-432}{-48}; \ y = \frac{384}{-48}$$

$$x = 9, \quad y = -8$$

Illustration 9 : Solve the equations :

$$4x + 10y = 2xy$$

$$5x + 16y = 3xy$$

Ans. : Dividing both the equations by xy, we get

$$\frac{4}{y} + \frac{10}{x} = 2, \ \& \ \frac{5}{y} + \frac{16}{x} = 3$$

Putting $\frac{1}{x} = a$ and $\frac{1}{y} = b$, we get

$10a + 4b - 2 = 0$

$16a + 5b - 3 = 0$

Now, by Cramer's method,

$$\frac{a}{\begin{vmatrix} 4 & -2 \\ 5 & -3 \end{vmatrix}} = \frac{-b}{\begin{vmatrix} 10 & -2 \\ 16 & -3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} 10 & 4 \\ 16 & 5 \end{vmatrix}}$$

$$\frac{a}{-12 + 10} = \frac{-b}{-30 + 32} = \frac{1}{50 - 64}$$

$$\frac{a}{-2} = \frac{-b}{2} = \frac{1}{-14}$$

$$a = \frac{2}{14}, \ b = \frac{2}{14}$$

$$a = \frac{1}{7}, \ b = \frac{1}{7}$$

$\therefore \quad \frac{1}{x} = \frac{1}{7}, \frac{1}{y} = \frac{1}{7}$

$\therefore \quad x = 7$ & $y = 7$

Illustration10 : Solve the following equations, using Cramer's ru[le] (April, 20[..])

$$x + 2y + 3z = 14$$
$$2x + y + z = 7$$
$$5x + 2y + z = 12$$

Ans : The equations can be expressed as

$$x + 2y + 3z - 14 = 0$$
$$2x + y + z - 7 = 0$$
$$5x + 2y + z - 12 = 0$$

$$\therefore \quad \frac{x}{D_x} = \frac{-y}{D_y} = \frac{z}{D_z} = \frac{-1}{D}$$

Where,

$$D = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 5 & 2 & 1 \end{vmatrix} = 1(1-2) - 2(2-5) + 3(4-5)$$

$$= -1 + 6 - 3$$
$$= 2$$

$$D_x = \begin{vmatrix} 2 & 3 & -14 \\ 1 & 1 & -7 \\ 2 & 1 & -12 \end{vmatrix} = 2(-12+7) - 3(-12+14) - 14(1-$$

$$= -10 - 6 + 14$$
$$= -2$$

$$D_y = \begin{vmatrix} 1 & 3 & -14 \\ 2 & 1 & -7 \\ 5 & 1 & -12 \end{vmatrix} = 1(-12+7) - 3(-24+35) - 14(2-$$

$$= -5 - 33 + 42$$
$$= 4$$

$$D_z = \begin{vmatrix} 1 & 2 & -14 \\ 2 & 1 & -7 \\ 5 & 2 & -12 \end{vmatrix} = 1(-12+14) - 2(-24+35) - 14(4-$$

$$= 2 - 22 + 14$$
$$= -6$$

$$\therefore \quad \frac{x}{-2} = \frac{-y}{4} = \frac{z}{-6} = \frac{-1}{2}$$

$\therefore \quad x = 1, y = 2, z = 3$

Illustration 11 : Solve the equations by Cramer's rule :

$$-\frac{4}{y}-\frac{2}{z}=1\,;\ \frac{1}{x}+\frac{2}{y}+\frac{1}{z}=2\,;\ \frac{2}{x}+\frac{5}{y}-\frac{2}{z}=3$$

Ans. : Put $\frac{1}{x}=X,\ \frac{1}{y}=Y,\ \frac{1}{z}=Z$

The equations can be expressed as

$$3X-4Y-2Z-1=0$$
$$X+2Y+Z-2=0$$
$$2X+5Y-2Z-3=0$$

$$\therefore\ \frac{X}{D_X}=\frac{-Y}{D_Y}=\frac{Z}{D_Z}=\frac{-1}{D}$$

Where,

$$D=\begin{vmatrix}3 & -4 & -2\\ 1 & 2 & 1\\ 2 & 5 & -2\end{vmatrix}=3(-4-5)+4(-2-2)-2(5-4)$$
$$=-45$$

$$D_X=\begin{vmatrix}-4 & -2 & -1\\ 2 & 1 & -2\\ 5 & -2 & -3\end{vmatrix}=-4(-3-4)+2(-6+10)-1(-4-5)$$
$$=45$$

$$D_Y=\begin{vmatrix}3 & -2 & -1\\ 1 & 1 & -2\\ 2 & -2 & -3\end{vmatrix}=3(-3-4)+2(-3+4)-1(-2-2)$$
$$=-15$$

$$D_Z=\begin{vmatrix}3 & -4 & -1\\ 1 & 2 & -2\\ 2 & 5 & -3\end{vmatrix}=3(-6+10)+4(-3+4)-1(5-4)$$
$$=15$$

$$\text{Now, }\frac{X}{45}=\frac{-Y}{-15}=\frac{Z}{15}=\frac{-1}{-45}$$

$$\therefore\ X=1,\ Y=\frac{1}{3},\ Z=\frac{1}{3}$$

$$\frac{1}{x}=1,\ \frac{1}{y}=\frac{1}{3},\ \frac{1}{z}=\frac{1}{3}$$

$$x=1,\ y=3,\ z=3$$

Illustration : 12 Evaluate $\begin{vmatrix} 6 & 3 & 9 \\ 1 & 0 & 2 \\ 40 & 50 & 20 \end{vmatrix}$

Ans. : Taking 3 from the first row and 10 from the third row common factors we get,

$$D = 3 \times 10 \begin{vmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 4 & 5 & 2 \end{vmatrix}$$

$$= 30[2(0-10) - 1(2-8) + 3(5-0)]$$
$$= 30[-20 + 6 + 15]$$
$$= 30(1)$$
$$= 30$$

Illustration 13 : Evaluate $\begin{vmatrix} 2 & 5 & 7 \\ 6 & 15 & 21 \\ 1300 & 1248 & 1871 \end{vmatrix}$

Ans. :

Taking 3 as common factor from second row we get

$$= 3\begin{vmatrix} 2 & 5 & 7 \\ 2 & 5 & 7 \\ 1300 & 1248 & 1871 \end{vmatrix}$$

$$= 3 \times 0 \; [\because \text{Two rows are identical.}]$$
$$= 0$$

Illustration 14 : Evaluate $\begin{vmatrix} x+y & z & 1 \\ y+z & x & 1 \\ z+x & y & 1 \end{vmatrix}$

Ans. : $D = \begin{vmatrix} x+y & z & 1 \\ y+z & x & 1 \\ z+x & y & 1 \end{vmatrix}$

Adding the elements of second column to the corresponding elements of first column we get,

$$D = \begin{vmatrix} x+y+z & z & 1 \\ x+y+z & x & 1 \\ x+y+z & y & 1 \end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix} 1 & z & 1 \\ 1 & x & 1 \\ 1 & y & 1 \end{vmatrix}$$

$= (x + y + z)(0)$ ..... [∵ Two columns are identical]

$= 0$

Illustration 15 : Evaluate $\begin{vmatrix} x+1 & 2 & 3 \\ 1 & x+2 & 3 \\ 1 & 2 & x+3 \end{vmatrix}$

Ans. :

$$D = \begin{vmatrix} x+1 & 2 & 3 \\ 1 & x+2 & 3 \\ 1 & 2 & x+3 \end{vmatrix}$$

Adding the elements of second and third columns to the corresponding elements of first column we get,

$$D = \begin{vmatrix} x+6 & 2 & 3 \\ x+6 & x+2 & 3 \\ x+6 & 3 & x+3 \end{vmatrix} = (x+6)\begin{vmatrix} 1 & 2 & 3 \\ 1 & x+2 & 3 \\ 1 & 2 & x+3 \end{vmatrix}$$

Now subtracting the elements of first row from the corresponding elements of second and third rows we get,

$$D = (x + 6)\begin{vmatrix} 1 & 2 & 3 \\ 0 & x & 0 \\ 0 & 0 & x \end{vmatrix}$$

Expanding with respect to first column we get

$$D = (x + 6)\,[1\,(x^2 - 0)]$$
$$= x^2 (x + 6)$$

Illustration 16 : Evaluate $\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$

Ans. : $D = \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix}$

Expanding with respect to first row we get

$$D = x(x^2 - yz) - y(xz - y^2) + z(z^2 - xy)$$
$$= x^3 - xyz - xyz + y^3 + z^3 - xyz$$
$$= x^3 + y^3 + z^3 - 3xyz.$$

Illustration 17 : Evaluate $\begin{vmatrix} 3x+11 & 3x+10 & 3x+8 \\ 3x+10 & 3x+9 & 3x+7 \\ 1965 & 1964 & 1962 \end{vmatrix}$

Ans. :

Here, $D = \begin{vmatrix} 3x+11 & 3x+10 & 3x+8 \\ 3x+10 & 3x+9 & 3x+7 \\ 1965 & 1964 & 1962 \end{vmatrix}$

Subtracting second column from first column and third column from second column we get.

$$D = \begin{vmatrix} 1 & 2 & 3x+8 \\ 1 & 2 & 3x+7 \\ 1 & 2 & 1962 \end{vmatrix}$$

$$= 2 \times \begin{vmatrix} 1 & 1 & 3x+8 \\ 1 & 1 & 3x+7 \\ 1 & 1 & 1962 \end{vmatrix}$$

$= 2(0)$.....($\because$ Two columns are identical)

$= 0$

Illustration 18 : Prove that *(March, April 2009)*

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

Ans. :

$$\text{L.H.S.} = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Subtracting second row from first row, and third row from second row we get.

$$= \begin{vmatrix} 0 & a-b & a^2-b^2 \\ 0 & b-c & b^2-c^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 1 & a+b \\ 0 & 1 & b+c \\ 1 & c & c^2 \end{vmatrix}$$

$= (a-b)(b-c)\{1(b+c-a-b)\}$.....

[Expanding with respect to first column]

$= (a-b)(b-c)(c-a)$

$=$ R.H.S.

Illustration 19: Solve the equation :

$$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$$

Ans. : $$\begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix} = 0$$

Subtracting second row from the first and third row from secon[d]
we get,

$$\begin{vmatrix} x-1 & 1-x & 0 \\ 0 & x-1 & 1-x \\ 1 & 1 & x \end{vmatrix} = 0$$

$$\therefore \quad (x-1)(x-1)\begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 1 & 1 & x \end{vmatrix} = 0$$

$\therefore \quad (x-1)^2 \{1(x+1) + 1(0+1) + 0\} = 0$

$\therefore \quad (x-1)^2 \{x+1+1\} = 0$

$\therefore \quad (x-1)^2 (x+2) = 0$

$\therefore \quad x - 1 = 0$ or $x + 2 = 0$

$\therefore \quad x = 1$ or $x = -2$