6 MATRICES

1. Meaning and Definition :

A matrix is an arrangment of numbers in rows and columns. If mn numbers are arranged in a rectangular array of m rows and n columns it is called a matrix of order $m \times n$. In writing a matrix it is usual to enclose this array by big brackets like [ ] or ( ) or { }. The matrix is denoted by capital letters like A, B, C etc. e.g.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots\cdots & a_{3n} \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ a_{m_1} & a_{m_2} & a_{m_3} & a_{mn} \end{bmatrix}$$

The individual quantities like $a_{11}, a_{12}, a_{23} \cdots\cdots a_{mn}$ are called the elements of the matrix. A matrix of m rows and n columns is said to be of order $m \times n$. It should be made clear at this stage that the matrix is simply an arrangement of numbers and it can not have a value. Matrix should only be looked upon as an operator. The use of matrix algebra is very common in higher Mathematics, Statistics, Economics and Business problems.

Consider the following matrix A

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots\cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots\cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} \cdots\cdots & a_{3n} \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots\cdots & \cdots \\ a_{m1} & a_{m2} & a_{m3} & a_{mn} \end{bmatrix}$$

In this matrix there are m rows and n columns. The matrix is therefore of order $m \times n$. It can be expressed as $A_{m \times n}$. All the mn numbers in the matrix are known as elements of the matrix. The element of ith row and jth column is denoted by a_{ij}. In particular,

a_{34} = element of 3rd row and 4th column.

a_{52} = element of 5th row and 2nd column.

a_{44} = element of 4th row and 4th column.

Consider the matrix.

$$A = \begin{bmatrix} 1 & -2 & 3 & 0 \\ 4 & 2 & 3 & 1 \\ 2 & 0 & -5 & -3 \end{bmatrix}$$

Here 12 elements are arranged in 3 rows and 4 columns.

Its order is 3×4

In the above matrix

$a_{11} = 1$, $a_{12} = -2$, $a_{13} = 3$, $a_{14} = 0$

$a_{21} = 4$, $a_{22} = 2$, $a_{23} = 3$, $a_{24} = 1$

$a_{31} = 2$, $a_{32} = 0$, $a_{33} = -5$, $a_{34} = -3$

Diagonal elements of a matrix and its principal diagonal

Consider the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

This is a matrix of order 3×3. The diagonal containing the elements a_{11}, a_{22}, a_{33} is called the principal diagonal of the matrix. a_{11}, a_{22}, a_{33} are diagonal elements.

In the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 5 \\ 2 & 0 & 3 & 4 \\ 5 & 6 & 7 & 2 \\ 4 & 1 & 4 & 3 \end{bmatrix}$$

1, 0, 7, 3 are diagonal elements.

Equal Matrices :

Two matrices are said to be equal if they satisfy the following conditions :

(1) The number of rows in both matrices should be equal and the number of columns should also be equal i.e. the order of both the matrices must be the same.

(2) The corresponding elements in both the matrices should be the same. i.e., two matrices are equal if they are equal in all respects.

Transpose of a matrix :

If we interchange rows and columns of a matrix A, the new matrix so obtained is known as the transpose of matrix A and it is denoted by A^T or A'.

e.g. If $A = \begin{bmatrix} 1 & 2 & 5 & 0 \\ 3 & 5 & 2 & 1 \\ 4 & 1 & 0 & 0 \end{bmatrix}$

$$A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 5 & 1 \\ 5 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

The order of A is 3×4 and that of A^T is 4×3.

i.e., If A is of order $(m \times n)$ then A^T is of order $(n \times m)$

2. Special Types of Matrices :

(i) **Row Matrix :** A matrix in which there is only one row and any number of columns is said to be a row matrix.

$A = [a_{11}, a_{12}, a_{13} \ldots\ldots a_{1n}]$ is a row matrix.

It is clear that order of this row matrix is $1 \times n$

e.g., [1 2 3] is a row matrix of order 1 x 3.

(ii) Column Matrix : A matrix in which there is only one column and any number of rows is said to be a column matrix.

e.g. $A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \cdots \\ \cdots \\ a_{m1} \end{bmatrix}$ is a column matrix of order $m \times 1$.

Similarly $\begin{bmatrix} 0 \\ 5 \\ 2 \\ 1 \end{bmatrix}$ is a column matrix of order 4×1

(iii) Zero Matrix or Null Matrix : If all the elements of matrix are zero, it is said to be a null matrix or a zero matrix.

e.g. $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a null matrix of order 2 x 4.

(**Note :** A Zero matrix can be a row matrix or a column matrix).

(iv) Square Matrix : A matrix in which number of rows and number of column are equal is said to be a square matrix.

e.g. $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ is a square matrix of order 3 x 3

Similarly $\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ is a square matrix of order 2 x 2

(v) Symmetric Matrix* : If the transpose of a square matrix gives the same matrix, it is known as a symmetric matrix i.e. for a symmetric matrix

a_{ij} = element of i^{th} row and j^{th} column

= element of j^{th} row and i^{th} column

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e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$ is a symmetric matrix of order 3×3

Here $a_{21} = a_{12} = 2$

$a_{13} = a_{31} = 3$

$a_{23} = a_{32} = 5$

$\therefore \quad A^T = A$

[Note : Every symmetric matrix must be a square matrix.]

(vi) **Skew Symmetric Matrix :*** If in a square matrix $a_{ij} = -a_{ji}$ i.e., the elements of i^{th} row and j^{th} column and j^{th} row and i^{th} column are equal in magnitude but opposite in sign, the matrix is known as a skew symmetric matrix.

Obviously each diagonal element of a skew symmetric matrix must be zero.

i.e. $(a_{11} = a_{22} = a_{33} = ... = 0)$

e.g. $A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}$ is a skew symmetric matrix of order 3×3.

Its diagonal elements are all zero. In a skew symmetric matrix $A^T = -A$.

(vii) **Unit Matrix or Identity Matrix :** A square matrix in which all diagonal elements are unity and all other elements are zero is known as a unit matrix or an identity matrix, and it is denoted by I.

e.g. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a unit matrix of order 3×3

Similarly $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2×2

[Note : Unit matrix holds special importance in the study of matrices.]

(viii) **Diagonal Matrix :** If all elements except diagonal elements of a square matrix are zero the matrix is said to be a diagonal matrix.

e.g. $A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is a diagonal matrix.

Define skew ... with illustration (Dec. 2015)

3. Simple Rules of Operations on Matrices :

3.1 Addition and Subtraction of Matrices :

The addition or subtraction of two or more matrices is possible only when they are of the same order.

Addition or subtraction of two or more matrices of the same order can be obtained by adding or subtracting the corresponding elements of these matrices. It is obvious that the order of the new matrix so obtained is also same as the order of given matrices.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 5 & 2 \end{bmatrix}$ is a matrix of order 2 x 3 and

$B = \begin{bmatrix} -1 & 2 & -1 \\ 3 & -5 & -4 \end{bmatrix}$ is also a matrix of order 2 x 3

$$\therefore \; C = A + B = \begin{bmatrix} 1+(-1) & 2+2 & 3+(-1) \\ 3+3 & 5+(-5) & 2+(-4) \end{bmatrix}$$

$$\therefore \quad C = \begin{bmatrix} 0 & 4 & 2 \\ 6 & 0 & -2 \end{bmatrix}$$

C is also of order 2 x 3

Similarly

$$D = A - B = \begin{bmatrix} 1-(-1) & 2-2 & 3-(-1) \\ 3-3 & 5-(-5) & 2-(-4) \end{bmatrix}$$

$= \begin{bmatrix} 2 & 0 & 4 \\ 0 & 10 & 6 \end{bmatrix}$ is of order 2 x 3

3.2 Scalar Product of a Matrix :

The scalar product of a matrix is obtained by multiplying each element of the matrix by that scalar.

e.g. $A = \begin{bmatrix} 1 & 0 & 3 \\ 4 & 5 & 7 \end{bmatrix}$ then

$2A = \begin{bmatrix} 2 & 0 & 6 \\ 8 & 10 & 14 \end{bmatrix}$ and

$$-3A = \begin{bmatrix} -3 & 0 & -9 \\ -12 & -15 & -21 \end{bmatrix}$$

Illustration 1 : Find the elements a_{11}, a_{23}, a_{13}, a_{34} and a_{44} from the following matrices. Give also the order of each matrix.

(i) $\begin{bmatrix} 2 & 3 & 4 & 2 \\ -8 & 7 & -5 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 2 & 9 & -3 \\ 1 & 0 & 5 & 6 \\ 6 & 0 & 7 & 9 \end{bmatrix}$

Ans. : (i) In the given matrix there are two rows and four columns.

$\therefore$ Its order is 2×4.

Here, $a_{11} = 2$, $a_{23} = -5$

$a_{13} = 4$ and the remaining elements are not possible.

(ii) Here the order of the given matrix is 4×4.

and $a_{11} = 0$ $a_{23} = 9$ $a_{13} = 1$ $a_{34} = 6$ $a_{44} = 9$.

Illustration 2 : The following matrices are of special type. Mention the type of each one of them.

(1) $\begin{bmatrix} 0 & 1 & 2 \\ 1 & -4 & -5 \\ 2 & -5 & 0 \end{bmatrix}$ (2) $\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$ (3) $[0\ 1\ 0\ 1]$

(4) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (5) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (6) $\begin{bmatrix} 0 & -3 & -2 \\ 3 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$

(7) $\begin{bmatrix} 3 & -4 \\ -3 & 4 \end{bmatrix}$ (8) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Ans. :

1) It is a symmetric matrix, because by interchanging rows and columns we get the same matrix. (i.e. $A = A^T$)

2) It is a column matrix, because there is only one column.

3) It is a row matrix because there is only one row.

4) It is a null matrix because all the elements are zero.

5) It is a unit matrix because the elements of principal diagonal are unity and all other elements are zero.

6) It is a skew symmetric matrix because $a_{ij} = -a_{ji}$ and the elements of principal diagonal are all zero.

7) It is a square matrix because the number of rows and columns are equal.

(8) It is a diagonal matrix because all the elements except diagonal elements are zero.

Illustration 3 : Find A + B, B – A and A – B from the following matrices :

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 5 & 7 & 6 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 0 & -2 \\ 7 & -6 & 0 \end{bmatrix}.$$

Ans. : Here,

Order of A = 2 x 3 and

Order of B = 2 x 3

∴ The addition and subtraction of A and B are possible.

$$A + B = \begin{bmatrix} 0+5 & 1+0 & 2-2 \\ 5+7 & 7-6 & 6+0 \end{bmatrix} = \begin{bmatrix} 5 & 1 & 0 \\ 12 & 1 & 6 \end{bmatrix}$$

$$B - A = \begin{bmatrix} 5-0 & 0-1 & -2-2 \\ 7-5 & -6-7 & 0-6 \end{bmatrix} = \begin{bmatrix} 5 & -1 & -4 \\ 2 & -13 & -6 \end{bmatrix}$$

$$A - B = \begin{bmatrix} 0-5 & 1-0 & 2-(-2) \\ 5-7 & 7-(-6) & 6-0 \end{bmatrix} = \begin{bmatrix} -5 & 1 & 4 \\ -2 & 13 & 6 \end{bmatrix}$$

Illustration 4 : Find A + 2B, A – B + C, A + B – 2C, A - C, A + 2C, B + D from the following matrices (if possible) :

$$A = \begin{bmatrix} 4 & 2 & 0 & 3 \\ 5 & -7 & -2 & 0 \\ 6 & 0 & -3 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 1 & -1 & -2 \\ 4 & -2 & 0 & -3 \\ 3 & 0 & 4 & 0 \\ 5 & 3 & 7 & 9 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 2 \\ -3 & 1 & 1 & 0 \\ 4 & 2 & 0 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Ans. :

Order of A = 3 x 4

Order of B = 4 x 4

Order of C = 3 x 4

Order of D = 3 x 4.

Thus the orders of A, C and D are equal while the order of B is different.

∴ A + 2B, A − B + C, A + B − 2C and B + D are not possible while.

$$A - C = \begin{bmatrix} 4-1 & 2-0 & 0-0 & 3-2 \\ 5+3 & -7-1 & -2-1 & 0-0 \\ 6-4 & 0-2 & -3-0 & -1-1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 & 0 & 1 \\ 8 & -8 & -3 & 0 \\ 2 & -2 & -3 & -2 \end{bmatrix}$$

$$A + 2C = \begin{bmatrix} 4+2 & 2+0 & 0+0 & 3+4 \\ 5-6 & -7+2 & -2+2 & 0+0 \\ 6+8 & 0+4 & -3+0 & -1+2 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 & 0 & 7 \\ -1 & -5 & 0 & 0 \\ 14 & 4 & -3 & 1 \end{bmatrix}$$

3 Multiplication of Two Matrices :

If $A = \begin{bmatrix} 2 & 5 & 7 \\ 1 & 2 & 3 \end{bmatrix}$ is a matrix of order 2×3

and $B = \begin{bmatrix} 3 & 2 & 1 \\ 4 & 7 & 9 \\ 5 & 2 & 4 \end{bmatrix}$ is a matrix of order 3×3

Then the multiplication AB can be obtained in the following way :

$$AB = \begin{bmatrix} \boxed{2 \quad 5 \quad 7} \\ \boxed{1 \quad 2 \quad 3} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 3 \\ 4 \\ 5 \end{matrix}} & \boxed{\begin{matrix} 2 \\ 7 \\ 2 \end{matrix}} & \boxed{\begin{matrix} 1 \\ 9 \\ 4 \end{matrix}} \end{bmatrix}$$

$$AB = \begin{bmatrix} 2\times3+5\times4+7\times5 & 2\times2+5\times7+7\times2 & 2\times1+5\times9+7\times4 \\ 1\times3+2\times4+3\times5 & 1\times2+2\times7+3\times2 & 1\times1+2\times9+3\times4 \end{bmatrix}$$

$$= \begin{bmatrix} 6+20+35 & 4+35+14 & 2+45+28 \\ 3+8+15 & 2+14+6 & 1+18+12 \end{bmatrix}$$

$$= \begin{bmatrix} 61 & 53 & 75 \\ 26 & 22 & 31 \end{bmatrix}$$

The order of matrix AB is 2×3.

If A is a matrix of order $m \times n$ and B is a matrix of order $n \times p$, then the product AB will be a matrix of order $m \times p$. Thus for the multiplication of two matrices A and B, the number of columns of matrix A and the number of rows of matrix B should be equal. For obtaining the first element of the first row of matrix AB, the elements of the first row are multiplied by the corresponding elements of the first column and they are added up. Similarly for obtaining the second element of first row, the elements of the first row of A are multiplied by the corresponding elements of the second column of B and they are added up. For obtaining the third element of the second row of AB, the elements of second row of A are multiplied by the corresponding elements of the third column of B and are added up. Similarly all the elements of the matrix AB can be obtained.

The multiplication of two matrices are explained in the following illustrations :

Illustration 5 :

If $A = \begin{bmatrix} 4 & 5 \\ 3 & 0 \\ 7 & 4 \end{bmatrix}$ **and**

$B = \begin{bmatrix} 5 & 6 & 7 \\ 1 & 1 & 0 \end{bmatrix}$ **find AB.**

Ans. : Here A is a matrix of order 3×2 and B is a matrix of order 2×3 i.e. the number of columns of A is equal to the number of rows of B. Here the product AB is possible and AB will be of order 3×3.

Now, $$AB = \begin{bmatrix} \boxed{4 \quad 5} \\ \boxed{3 \quad 0} \\ \boxed{7 \quad 4} \end{bmatrix} \begin{bmatrix} \boxed{\begin{matrix} 5 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 6 \\ 1 \end{matrix}} & \boxed{\begin{matrix} 7 \\ 0 \end{matrix}} \end{bmatrix}$$

$$AB = \begin{bmatrix} 4 \times 5 + 5 \times 1 & 4 \times 6 + 5 \times 1 & 4 \times 7 + 5 \times 0 \\ 3 \times 5 + 0 \times 1 & 3 \times 6 + 0 \times 1 & 3 \times 7 + 0 \times 0 \\ 7 \times 5 + 4 \times 1 & 7 \times 6 + 4 \times 1 & 7 \times 7 + 4 \times 0 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 29 & 28 \\ 15 & 18 & 21 \\ 39 & 46 & 49 \end{bmatrix}$$

Illustration 6 :

If $A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$

Find AB and BA. Is AB = BA ?

Ans. : Here A is 2 x 2 matrix and B is 2 x 2 matrix. Hence AB is possible.

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 10+0 & 2+9 \\ 5+0 & 1+12 \end{bmatrix}$$

$$= \begin{bmatrix} 10 & 11 \\ 5 & 13 \end{bmatrix}$$

B is 2 x 2 matrix and A is 2 x 2 matrix hence BA is also possible.

$$BA = \begin{bmatrix} 5 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 10+1 & 15+4 \\ 0+3 & 0+12 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 19 \\ 3 & 12 \end{bmatrix}$$

Thus $AB \neq BA$.

The matrix multiplication is not commutative.

Illustration 7 :

If $A = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & -2 \\ 1 & 0 & -1 \end{bmatrix}$

$B = \begin{bmatrix} 1 & 0 \\ -5 & 2 \\ 4 & 7 \end{bmatrix}$ **find AB. Is BA possible ?**

Ans. : As A is 3 x 3 matrix and B is 3 x 2 matrix, AB is possible.

$$AB = \begin{bmatrix} -2 & 1 & 0 \\ 3 & 4 & -2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -5 & 2 \\ 4 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} -2-5+0 & 0+2+0 \\ 3-20-8 & 0+8-14 \\ 1+0-4 & 0+0-7 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & 2 \\ -25 & -6 \\ -3 & -7 \end{bmatrix}$$

Now, B is 3 x 2 matrix and A is 3 x 3 matrix, hence BA is not possible. Here number of columns of B is not equal, to number of rows of A.

Illustration 8 :

If $A = \begin{bmatrix} 4 & 5 & 7 \\ -2 & 3 & 0 \end{bmatrix}$ **and**

$B = \begin{bmatrix} 0 & 1 & 4 & 7 \\ -2 & 1 & 0 & 5 \\ 3 & 0 & 1 & 4 \end{bmatrix}$**, find AB.**

Ans. : Here A is 2 x 3 matrix and B is 3 x 4 matrix, hence AB is possible.

$$AB = \begin{bmatrix} 4 & 5 & 7 \\ -2 & 3 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 4 & 7 \\ -2 & 1 & 0 & 5 \\ 3 & 0 & 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0-10+21 & 4+5+0 & 16+0+7 & 28+25+28 \\ 0-6+0 & -2+3+0 & -8+0+0 & -14+15+0 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 9 & 23 & 81 \\ -6 & 1 & -8 & 1 \end{bmatrix}$$

Illustration 9 :

If $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ **find the value of** $A^2 - A + I$

Ans. : Here A is a square matrix, hence $A^2 = A \times A$ is possible.

$$A^2 = A \times A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2+4 & 2+4+4 & 2+4+2 \\ 1+2+4 & 2+4+4 & 2+4+2 \\ 2+2+2 & 4+4+2 & 4+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 10 & 8 \\ 7 & 10 & 8 \\ 6 & 10 & 9 \end{bmatrix}$$

$$\text{Now, } A^2 - A + I = \begin{bmatrix} 7 & 10 & 8 \\ 7 & 10 & 8 \\ 6 & 10 & 9 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 8 & 6 \\ 6 & 9 & 6 \\ 4 & 8 & 9 \end{bmatrix}$$

Illustration 10 : Find the value of : **(March, April, 2007, 2009)**

$$[1\ 2\ 3] \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 9 \\ 9 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

Ans. : The order of the first matrix is 1×3 and that of the second matrix is 3×3, hence their product is a matrix of order 1×3. If this product matrix is multiplied by the third matrix of order 3×1, we get a matrix of order 1×1.

$$[1\ 2\ 3] \begin{bmatrix} 2 & 3 & 0 \\ 0 & 4 & 9 \\ 9 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$$= [2+0+27 \quad 3+8+3 \quad 0+18+0] \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$$= [29 \quad 14 \quad 18] \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix}$$

$$= [58 + 70 + 18]$$

$$= [146]$$

Illustration 11 : If $A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$ **find matrix B such that**

$$A + 2B = A^2$$

Ans. : $A^2 = A \times A$

$$= \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 16+2 & 4+3 \\ 8+6 & 2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix}$$

Now $A + 2B = A^2$

$\therefore \quad 2B = A^2 - A$

$$\therefore \quad 2B = \begin{bmatrix} 18 & 7 \\ 14 & 11 \end{bmatrix} - \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 6 \\ 12 & 8 \end{bmatrix}$$

$$\therefore \quad B = \frac{1}{2} \begin{bmatrix} 14 & 6 \\ 12 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 3 \\ 6 & 4 \end{bmatrix}$$

Illustration 12 :

If $A = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$ **Prove that** $A^2 = I$

Ans. : $A^2 = A \times A$

$$A^2 = \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 4 & 3 \\ 1 & -3 & -3 \\ -1 & 4 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0+4-3 & 0-12+12 & 0-12+12 \\ 0-3+3 & 4+9-12 & 3+9-12 \\ 0+4-4 & -4-12+16 & -3-12+16 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I.$$

4. Laws of Matrix Algebra :

4.1 Properties of Addition of Matrices :

(1) If A and B be any two matrices of the same order $m \times n$, then $A + B = B + A$ (Commutative law)

(2) If A, B, C be any three matrices of the same order $m \times n$, then $A + (B + C) = (A + B) + C$ (Associative law for addition)

(3) If K be a scalar and A, B be two matrices of the same order $m \times n$ then,

$$K(A + B) = KA + KB$$

(4) If A be a $m \times n$ matrix and 0 be a null matrix of the same order, then

(i) $A + 0 = 0 + A = A$

(ii) $A + (-A) = (-A) + A = 0$

(5) If A, B, C be any three matrices of the same order $m \times n$, then $A + C = B + C$ gives $A = B$.

4.2 Properties of Matrix Multiplication

(1) If A be a square matrix of order $n \times n$ and I be a unit matrix of the same order then. $AI = IA = A$

(2) If A is a $m \times n$ matrix and 0 is a $n \times m$ matrix, then $A\,0 = 0\,A = 0$

(3) If A, B, C be three matrixes of order $m \times n$, $n \times p$, $p \times q$ respectively, then $A(BC) = (AB)C$ (Associative Law for multiplication)

(4) If A, B, C be three matrices of order $m \times n$, $n \times p$, $n \times p$ respectively, then

$A(B + C) = A \cdot B + A \cdot C$ [Distributive Law]

(5) If A, B, C are three matrices such that $AB = AC$ then, in general $B \neq C$

(6) If $AB = O$ where A, B are two matrices, then in general $A \neq O$ or, $B \neq O$, or $A \neq O$ and $B \neq O$.

4.3 Properties of the Transpose of a Matrix

If A^T & B^T be the transposes of two matrices A and B - then

(1) $(A^T)^T = A$

(2) $(A + B)^T = A^T + B^T$

(3) $(AB)^T = B^T \ A^T$

5. Determinant :

Another type of arrangement of numbers is called determinant. In a matrix the number of rows and number of columns are not necessarily equal, but in a determinant number of rows and number of columns should be equal. Thus a determinant is a square arrangement of numbers. A matrix cannot have a value, while the value of a determinants can be found out. The elements of a determinant are shown within two vertical lines.

Thus we have seen two types of arrangements of numbers, i.e. Determinant and Matrix. We shall now understand the difference between the two :

Determinant	Matrix
(1) In determinant the number of rows and columns are equal.	(1) In matrix the number of rows and columns are not necessarily equal.
(2) In determinant the elements are shown between two vertical lines like | |.	(2) In matrix the elements are shown in brackets like ( ), { } or [ ].
(3) A determinant has a value.	(3) A matrix can not have a value. It is merely an arrangement.

Determinant of a Square Matrix :

If $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

is called a determinant of matrix A and it is denoted by $|A|$.

Illustration 13 : Find the values of the determinant of the following matrices :

(i) $A = \begin{bmatrix} 2 & 3 \\ 7 & 18 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 5 & 6 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$

(iii) $B = \begin{bmatrix} -2 & 3 \\ -4 & 5 \end{bmatrix}$ (iv) $B = \begin{bmatrix} 5 & -1 & 1 \\ -2 & 3 & 4 \\ 1 & 1 & 7 \end{bmatrix}$

Ans. : (i) $|A| = \begin{vmatrix} 2 & 3 \\ 7 & 18 \end{vmatrix} = 36 - 21 = 15$

(ii) $|A| = \begin{vmatrix} 5 & 6 & 1 \\ 0 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix}$

$= 5(4-3) - 6(0-3) + 1(0-2)$

$= 5 + 18 - 2$

$= 21$

(iii) $|B| = \begin{vmatrix} -2 & 3 \\ -4 & 5 \end{vmatrix} = -10 + 12 = 2$

(iv) $|B| = \begin{vmatrix} 5 & -1 & 1 \\ -2 & 3 & 4 \\ 1 & 1 & 7 \end{vmatrix}$

$= 5(21-4) + 1(-14-4) + 1(-2-3)$

$= 5(17) + 1(-18) + 1(-5)$

$= 85 - 18 - 5$

$= 62$

Illustration 14 : Find the transpose of the following matrices :

$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix}$; $B = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 1 & 1 & 7 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 2 & 5 \\ 3 & 4 & 1 \end{bmatrix}$

Ans. :

(i) $A^T = \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix}$

(ii) $B^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 3 & 0 & 7 \end{bmatrix}$; $C^T = \begin{bmatrix} 1 & -1 & 3 \\ 2 & 2 & 4 \\ 3 & 5 & 1 \end{bmatrix}$

Adjoint of a Square Matrix :

$A = \begin{bmatrix} 2 & 3 & 4 \\ 1 & 0 & 6 \\ 5 & 7 & 8 \end{bmatrix}$ is a 3 x 3 square matrix. The determinant of this matrix is

$$|A| = \begin{vmatrix} 2 & 3 & 4 \\ 1 & 0 & 6 \\ 5 & 7 & 8 \end{vmatrix}$$

The minor of any element of a matrix can be obtained by eliminating the row and column in which that element lies. If this minor is given proper sign it can be called co-factor of that element. The signs of minors are taken alternatively positive and negative as given below, to obtain co-factors :

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Thus for the above matrix

minor of 2 $= \begin{vmatrix} 0 & 6 \\ 7 & 8 \end{vmatrix}$ co-factor of 2 $= +\begin{vmatrix} 0 & 6 \\ 7 & 8 \end{vmatrix}$

minor of 3 $= \begin{vmatrix} 1 & 6 \\ 5 & 8 \end{vmatrix}$ co-factor of 3 $= -\begin{vmatrix} 1 & 6 \\ 5 & 8 \end{vmatrix}$

minor of 4 $= \begin{vmatrix} 1 & 0 \\ 5 & 7 \end{vmatrix}$ co-factor of 4 $= +\begin{vmatrix} 1 & 0 \\ 5 & 7 \end{vmatrix}$

minor of 1 $= \begin{vmatrix} 3 & 4 \\ 7 & 8 \end{vmatrix}$ co-factor of 1 $= -\begin{vmatrix} 3 & 4 \\ 7 & 8 \end{vmatrix}$

minor of 0 $= \begin{vmatrix} 2 & 4 \\ 5 & 8 \end{vmatrix}$ co-factor of 0 $= +\begin{vmatrix} 2 & 4 \\ 5 & 8 \end{vmatrix}$

Thus co-factor of each element is found out by giving proper sign to its minor.

co-factor of 6 $= -\begin{vmatrix} 2 & 3 \\ 5 & 7 \end{vmatrix}$

co-factor of 5 $= +\begin{vmatrix} 3 & 4 \\ 0 & 6 \end{vmatrix}$

co-factor of 7 $= -\begin{vmatrix} 2 & 4 \\ 1 & 6 \end{vmatrix}$

cofactor of 8 = $+\begin{vmatrix} 2 & 3 \\ 1 & 0 \end{vmatrix}$

Adjoint of a square matrix is the transpose of the matrix of the co-factors of a given matrix. In order to obtain adjoint of a matrix first of all in place of each element write its cofactor and then take the transpose of the matrix obtained. The adjoint of a square matrix A is denoted by adj. A. In a 2 x 2 matrix the minors are given signs as $\begin{vmatrix} + & - \\ - & + \end{vmatrix}$ to obtain cofactors.

The method of obtaining adjoint of a matrix is shown in the following illustration :

Illustration 15 : Find the adjoint of the following matrix :

$$A = \begin{bmatrix} 2 & 5 \\ 1 & 4 \end{bmatrix}$$

Ans. : The matrix of the co–factors is,

$$\begin{bmatrix} 4 & -1 \\ -5 & 2 \end{bmatrix}$$

Taking transpose,

$$adj \cdot A = \begin{bmatrix} 4 & -5 \\ -1 & 2 \end{bmatrix}$$

Illustration 16 : Find adj. A

$$A = \begin{bmatrix} 7 & 8 \\ 2 & 10 \end{bmatrix}$$

Ans. : The matrix of the co–factors is,

$$A = \begin{bmatrix} 10 & -2 \\ -8 & 7 \end{bmatrix}$$

Taking transpose,

$$adj.\ A = \begin{bmatrix} 10 & -8 \\ -2 & 7 \end{bmatrix}$$

Illustration 17 : Find adj. of A.

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 5 \\ 3 & 4 & 2 \end{bmatrix}$$

Ans. : The matrix of the co–factors is

$$\begin{bmatrix} \begin{vmatrix} 2 & 5 \\ 4 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 4 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 5 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} \end{bmatrix}.$$

$$= \begin{bmatrix} -16 & 13 & -2 \\ -2 & 1 & 1 \\ 13 & -9 & 1 \end{bmatrix}$$

Taking transpose,

$$adj \cdot \mathrm{A} = \begin{bmatrix} -16 & -2 & 13 \\ 13 & 1 & -9 \\ -2 & 1 & 1 \end{bmatrix}$$

Illustration 18 : Find adj. A and obtain the value of A x (adj. A

$$\mathrm{A} \quad \begin{bmatrix} 1 & 0 & 7 \\ 2 & 2 & 5 \\ 0 & 3 & 6 \end{bmatrix}$$

Ans. : The matrix of the co–factors is

$$\begin{bmatrix} \begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 0 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} \\ -\begin{vmatrix} 0 & 7 \\ 3 & 6 \end{vmatrix} & \begin{vmatrix} 1 & 7 \\ 0 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} \\ \begin{vmatrix} 0 & 7 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 7 \\ 2 & 5 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 2 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -12 & 6 \\ 21 & 6 & -3 \\ -14 & 9 & 2 \end{bmatrix}$$

$$adj \cdot \mathrm{A} = \begin{bmatrix} -3 & 21 & -14 \\ -12 & 6 & 9 \\ 6 & -3 & 2 \end{bmatrix}$$

Now, A x (adj•A)

$$= \begin{bmatrix} 1 & 0 & 7 \\ 2 & 2 & 5 \\ 0 & 3 & 6 \end{bmatrix} \begin{bmatrix} -3 & 21 & -14 \\ -12 & 6 & 9 \\ 6 & -3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} -3-0+42 & 21+0-21 & -14+0+14 \\ -6-24+30 & 42+12-15 & -28+18+10 \\ 0-36+36 & 0+18-18 & 0+27+12 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 0 & 0 \\ 0 & 39 & 0 \\ 0 & 0 & 39 \end{bmatrix}$$

6. Inverse of a Matrix :*

If A is a square matrix and if there exists another square matrix B of the same order such that AB = BA = I then B is called inverse of matrix A and it is denoted by A^{-1}. The necessary condition for a matrix to have an inverse is that its determinant should not be equal to zero. A square matrix whose determinant is not equal to zero, is called a non singular matrix. Thus a non singular square matrix can have an inverse.

To obtain inverse of a Matrix :

If A is a non singular square matrix then its inverse can be obtained in the following way :

$$A^{-1} = \frac{\text{adj}\,A}{|A|}$$

Thus for finding inverse of a given matrix, we should first obtain the value of its determinant. If the value is not equal to zero the inverse is possible. We shall now find inverse of some matrices.

Illustration 19 : Find inverse of the following matrices :

(i) $A = \begin{bmatrix} 2 & 3 \\ 4 & 10 \end{bmatrix}$ **(ii)** $A = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$ (Dec. 2014)

Ans. : (i) Here $|A| = \begin{vmatrix} 2 & 3 \\ 4 & 10 \end{vmatrix} = 20 - 12 = 8 \neq 0$

Hence the matrix is non sigular. We shall first of all find adj. A. The matrix of the co-factors is

$$\begin{bmatrix} 10 & -4 \\ -3 & 2 \end{bmatrix}$$

Taking transpose

$$adj \cdot A = \begin{bmatrix} 10 & -3 \\ -4 & 2 \end{bmatrix}$$

Now, $$A^{-1} = \frac{adj \cdot A}{|A|}$$

$$= \frac{1}{8}\begin{bmatrix} 10 & -3 \\ -4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 5/4 & -3/8 \\ -1/2 & 1/4 \end{bmatrix}$$

(ii) $$A = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

Here, $$|A| = \begin{vmatrix} 5 & 1 \\ 4 & 2 \end{vmatrix} = 10 - 4 = 6 \neq 0$$

Thus the matrix is non singular.

We shall now obtain adj. A. The matrix of the co-factors is

$$\begin{bmatrix} 2 & -4 \\ -1 & 5 \end{bmatrix}$$

$$adj \cdot A = \begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

Now, $$A^{-1} = \frac{adj \cdot A}{|A|}$$

$$= \frac{1}{6}\begin{bmatrix} 2 & -1 \\ -4 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & -1/6 \\ -2/3 & 5/6 \end{bmatrix}$$

Illustration 20 : Find A^{-1} and verify that $AA^{-1} = I$.

$$A = \begin{bmatrix} 6 & 3 \\ 4 & 5 \end{bmatrix}$$

Ans. : Here $|A| = \begin{vmatrix} 6 & 3 \\ 4 & 5 \end{vmatrix} = 30 - 12 = 18 \neq 0$

Thus that matrix is non singular.

We shall now obtain adj. A.

The matrix of the co-factors is

$$\begin{bmatrix} 5 & -4 \\ -3 & 6 \end{bmatrix}$$

Taking transpose,

$$adj \cdot A = \begin{bmatrix} 5 & -3 \\ -4 & 6 \end{bmatrix}$$

$$A^{-1} = \frac{adj \cdot A}{|A|}$$

$$= \frac{1}{18}\begin{bmatrix} 5 & -3 \\ -4 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{18} & \frac{-1}{6} \\ \frac{-2}{9} & \frac{1}{3} \end{bmatrix}$$

$$\text{Now, } AA^{-1} = \begin{bmatrix} 6 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} \frac{5}{18} & -\frac{1}{6} \\ -\frac{2}{9} & \frac{1}{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{5}{3} - \frac{6}{9} & -1+1 \\ \frac{10}{9} - \frac{10}{9} & -\frac{2}{3} + \frac{5}{3} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= I$$

Illustration 21 : Find inverse of the following matrix :

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{bmatrix}$$ *(March/April, 2007, 2009)*

Ans. : Here,

$$|A| = \begin{vmatrix} 2 & 3 & 1 \\ 0 & 5 & 6 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(10-6) - 3(0-6) + 1(0-5)$$

$$= 8 + 18 - 5$$

$$= 21 \ (\neq 0)$$

Thus ...

The matrix of the co–factors is

$$\begin{bmatrix} \begin{vmatrix} 5 & 6 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 6 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 0 & 5 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 3 & 1 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 0 & 6 \end{vmatrix} & \begin{vmatrix} 2 & 3 \\ 0 & 5 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 6 & -5 \\ -5 & 3 & 1 \\ 13 & -12 & 10 \end{bmatrix}$$

Taking transpose,

$$adj \cdot A = \begin{bmatrix} 4 & -5 & 13 \\ 6 & 3 & -12 \\ -5 & 1 & 10 \end{bmatrix}$$

Now, $A^{-1} = \dfrac{adj \cdot A}{|A|}$

$$= \frac{1}{21}\begin{bmatrix} 4 & -5 & 13 \\ 6 & 3 & -12 \\ -5 & 1 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{4}{21} & -\frac{5}{21} & \frac{13}{21} \\ \frac{6}{21} & \frac{1}{7} & -\frac{4}{7} \\ -\frac{5}{21} & \frac{1}{21} & \frac{10}{21} \end{bmatrix}$$

Illustration 22 : Find the inverse of the following matrix and verify that $AA^{-1} = I$:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}$$

Ans. : Here,

$$|A| = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{vmatrix}$$

$$= 2(0+1) - 1(2+1) - 1\,(1-0)$$
$$= 2 - 3 - 1$$
$$= -\,2\ (\neq 0)$$

Thus the matrix is non-singular.

The matrix of the co–factors is

$$\begin{bmatrix} \begin{vmatrix} 0 & -1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \\ -\begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 2 & -1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \\ \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & -1 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 1 & 0 \end{vmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 1 \\ -3 & 5 & -1 \\ -1 & 1 & -1 \end{bmatrix}$$

Taking transpose,

$$adj \cdot A = \begin{bmatrix} 1 & -3 & -1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{adj \cdot A}{|A|}$$

$$= -\frac{1}{2}\begin{bmatrix} 1 & -3 & -1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

Now, $AA^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \times -\frac{1}{2}\begin{bmatrix} 1 & -3 & -1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}$

$$= -\frac{1}{2}\begin{bmatrix} 2 & 1 & -1 \\ 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 1 & -3 & -1 \\ -3 & 5 & 1 \\ 1 & -1 & -1 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} 2-3-1 & -6+5+1 & -2+1+1 \\ 1-0-1 & -3+0+1 & -1+0+1 \\ 1-3+2 & -3+5-2 & -1+1-2 \end{bmatrix}$$

$$= -\frac{1}{2}\begin{bmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$= I$$

Illustration 23 : Show that $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ **Satisfies the equation** $A^3 - 6A^2 + 9A - 4I = 0$

Hence deduce A^{-1}.

Ans. : $A^2 = A \times A$

$$= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4+1+1 & -2-2-1 & 2+1+2 \\ -2-2-1 & 1+4+1 & -1-2-2 \\ 2+1+2 & -1-2-2 & 1+1+4 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$$

$$A^3 = A^2 \times A$$

$$= \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} \times \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 12+5+5 & -6-10-5 & 6+5+10 \\ -10-6-5 & 5+12+5 & -5-6-10 \\ 10+5+6 & -5-10-6 & 5+5+12 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Now, $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - \begin{bmatrix} 36 & -30 & 30 \\ -30 & 36 & -30 \\ 30 & -30 & 36 \end{bmatrix} + \begin{bmatrix} 18 & -9 & 9 \\ -9 & 18 & -9 \\ 9 & -9 & 18 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

$\therefore A^3 - 6A^2 + 9A - 4I = O$

To obtain A^{-1}, we multiply this equation by A^{-1}

$\therefore \quad A^{-1} A^3 - 6A^{-1}A^2 + 9A^{-1}A - 4A^{-1}I = A^{-1} 0$

$\therefore \quad A^{-1} AA^2 - 6A^{-1}AA + 9I - 4A^{-1} = 0$

$\therefore \quad IA^2 - 6IA + 9I = 4A^{-1}$

$\therefore \quad A^2 - 6A + 9I = 4A^{-1}$

$$\therefore \quad 4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - \begin{bmatrix} 12 & -6 & 6 \\ -6 & 12 & -6 \\ 6 & -6 & 12 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$4A^{-1} = \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Properties of the Inverse of a Matrix :

If A & B be two square matrices and A^{-1}, B^{-1} are the inverses of A & B respectively then,

(1)

(2) $A A^{-1} = A^{-1} A = I$

(3) $(A^{-1})^{-1} = A$

(4) $(A^T)^{-1} = (A^{-1})^T$

$(AB)^{-1} = B^{-1} A^{-1}$

Illustration 24 : A firm has three offices, one each in Ahmedabad, Surat and Rajkot. In Ahmedabad office there is 1 office superintendent, 2 head clerks, 6 clerks and 4 peons. In Surat office there are 2 head clerks, 5 clerks and 3 peons. In Rajkot office there is 1 office superintendent, 1 head clerk, 6 clerks, and 3 peons. The average monthly salary of office superintendent, head clerk, clerk and peon are respectively Rs. 12,000, Rs. 10,000, Rs. 6000 and Rs 3000. Using matrix multiplication find total monthly salary bill of each office.

Ans. : The staff of the office in three different cities can be represented as following matrix :

$$\begin{array}{l} \\ \text{Ahmedabad} \\ \text{Surat} \\ \text{Rajkot} \end{array} \begin{array}{c} \begin{array}{cccc} \text{OS} & \text{HC} & \text{C} & \text{P} \end{array} \\ \begin{bmatrix} 1 & 2 & 6 & 4 \\ 0 & 2 & 5 & 3 \\ 1 & 1 & 6 & 3 \end{bmatrix} \end{array}$$

their monthly salary can be represented as a matrix.

$$\begin{array}{l} \text{OS} \\ \text{HC} \\ \text{C} \\ \text{P} \end{array} \begin{bmatrix} 12{,}000 \\ 10{,}000 \\ 6{,}000 \\ 3{,}000 \end{bmatrix}$$

The salary bill of the three offices can be obtained by multiplication of two matrices, as shown below :

$$\begin{array}{l} \text{Ahmedabad} \\ \text{Surat} \\ \text{Rajkot} \end{array} \begin{bmatrix} 1 & 2 & 6 & 4 \\ 0 & 2 & 5 & 3 \\ 1 & 1 & 6 & 3 \end{bmatrix} \times \begin{bmatrix} 12000 \\ 10000 \\ 6000 \\ 3000 \end{bmatrix}$$

$$= \begin{bmatrix} 12000 + 20000 + 36000 + 12000 \\ 0 + 20000 + 30000 + 9000 \\ 12000 + 10000 + 36000 + 9000 \end{bmatrix}$$

$$= \begin{bmatrix} 80000 \\ 59000 \\ 67000 \end{bmatrix}$$

∴ Salary bills of Ahmedabad, Surat and Rajkot offices are respectively Rs. 80,000, Rs. 59,000 and R

Illustration 25 : A factory produces two types of items A and B. For the production of these two items two machines are used. For producing one unit of item A first machine is used for 2 hours and second machine is used for 5 hours, and producing one unit of item B first machine is used for 3 hours and second machine is used for 1 hour. If the total time available on these two machines are respectively 85 hours and 115 hours, find the number of units of A and B that should be produced.

Ans. :

Suppose x units of A and y units of B are produced.

$\therefore \ 2x + 3y = 85$

$5x + y = 115$

Use Cramer's method and get the solution $x = 20,\ y = 15$.

Illustration 26 : If $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix}$ $B^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ then find $(AB)^{-1}$.

Ans. : We know that $(AB)^{-1} = B^{-1}, A^{-1}$

$$\therefore (AB)^{-1} = B^{-1}\ A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 3 & 0 & 4 \\ 1 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0+0+1 & 0+0+1 & 0+0+3 \\ 1+6+3 & -1+0+3 & 2+8+9 \\ 1+9+4 & -1+0+4 & 2+12+12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 3 \\ 10 & 2 & 19 \\ 14 & 3 & 26 \end{bmatrix}$$

Illustration 27 : Find the inverse of the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$

Ans. : Here $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ is the matrix of the co-efficients of the variables x, y, z in the given equations.

$$\therefore \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 36 \end{bmatrix}$$ and

Let, us find first inverse of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} = A$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 1(18-12) - 1(9-3) + 1(4-2)$$

$$= 6 - 6 + 2$$

$$= 2 \neq 0$$

Matrix of the co-factors

$$= \begin{bmatrix} 18-12 & -(9-3) & 4-2 \\ -(9-4) & 9-1 & -(4-1) \\ 3-2 & -(3-1) & 2-1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\therefore \text{adj } A = \begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{bmatrix}$$

$$\text{Now, } A^{-1} = \frac{1}{2}\begin{bmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & +1 \end{bmatrix}$$

Illustration 28 : If $A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ $B = \begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix}$ Find a and b such that AB = BA. Also compute 3A + 4B.

Ans. : $AB = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 3a+6 & 3b+10 \\ 4a+3 & 4b+5 \end{bmatrix}$$

Now, $BA = \begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$

$[3a + 4b$ ...

$$= \begin{bmatrix} 3a+4b & 2a+b \\ 29 & 11 \end{bmatrix}$$

We have $AB = BA$

$$\therefore \begin{bmatrix} 3a+6 & 3b+10 \\ 4a+3 & 4b+5 \end{bmatrix} = \begin{bmatrix} 3a+4b & 2a+b \\ 29 & 11 \end{bmatrix}$$

$\therefore$ $3a + 6 = 3a + 4b$ $\qquad$ $3b + 10 = 2a + b$

$4a + 3 = 29 \Rightarrow a = 6.5$ $\qquad$ $4b + 5 = 11 \Rightarrow b = 1.5$

$a = 6.5,\ b = 1.5$

$$\text{Now, } 3A + 4B = 3\begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} + 4\begin{bmatrix} 6.5 & 1.5 \\ 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 6 \\ 12 & 3 \end{bmatrix} + \begin{bmatrix} 26 & 6 \\ 12 & 20 \end{bmatrix}$$

$$= \begin{bmatrix} 35 & 12 \\ 24 & 23 \end{bmatrix}$$