

## Evaluation of $v_e q_e$ at $y' = 0$

The question is the transformation of the last term in (B4),  $(v_e q_e)_{y'=0}$ , into (B5) in Huang and Nakamura (2016). Starting with the definitions

$$v_e(x, y, y', z, t) = v(x, y + y', z, t), \quad (1)$$

$$u_e(x, y, y', z, t) = u(x, y + y', z, t) - U_{\text{REF}}(y, z, t), \quad (2)$$

$$\theta_e(x, y, y', z, t) = \theta(x, y + y', z, t) - \Theta_{\text{REF}}(y, z, t), \quad (3)$$

$$q_e(x, y, y', z, t) = q(x, y + y', z, t) - Q_{\text{REF}}(y, z, t), \quad (4)$$

where  $y'$  is the displacement coordinate relative to  $y$ . Here PV is defined as

$$q = f(y) + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + f e^{z/H} \frac{\partial}{\partial z} \left( \frac{e^{-z/H} \theta}{\partial \tilde{\theta} / \partial z} \right), \quad (5)$$

and

$$Q_{\text{REF}} = f(y) - \frac{\partial U_{\text{REF}}}{\partial y} + f e^{z/H} \frac{\partial}{\partial z} \left( \frac{e^{-z/H} \Theta_{\text{REF}}}{\partial \tilde{\theta} / \partial z} \right). \quad (6)$$

Therefore

$$\begin{aligned} q_e &= \frac{\partial v_e}{\partial x} - \frac{\partial u_e}{\partial y} + f e^{z/H} \frac{\partial}{\partial z} \left( \frac{e^{-z/H} \theta_e}{\partial \tilde{\theta} / \partial z} \right) \\ &= \frac{\partial v_e}{\partial x} - \frac{\partial u_e}{\partial y'} + \frac{\partial U_{\text{REF}}}{\partial y} + f e^{z/H} \frac{\partial}{\partial z} \left( \frac{e^{-z/H} \theta_e}{\partial \tilde{\theta} / \partial z} \right). \end{aligned} \quad (7)$$

In the above we used  $\partial u_e / \partial y' = \partial u / \partial y = \partial u_e / \partial y + \partial U_{\text{REF}} / \partial y$ . Then multiplying by  $v_e$  and utilizing  $\partial u_e / \partial x + \partial v_e / \partial y' = 0$ , after arrangement,

$$v_e q_e = \frac{\partial}{\partial x} \left[ \frac{1}{2} \left( v_e^2 - u_e^2 - \frac{R}{H} \frac{e^{-\kappa z/H} \theta_e^2}{\partial \tilde{\theta} / \partial z} \right) \right] - \frac{\partial}{\partial y'} (u_e v_e) + f e^{z/H} \frac{\partial}{\partial z} \left( \frac{e^{-z/H} v_e \theta_e}{\partial \tilde{\theta} / \partial z} \right) + v_e \frac{\partial U_{\text{REF}}}{\partial y}. \quad (8)$$

So it seems that  $\partial(u_e v_e) / \partial y$  in (B5) is equivalent to  $\partial(u_e v_e) / \partial y' - v_e \partial U_{\text{REF}} / \partial y$ . Note that in the zonal average, Eq. (8) reduces to the usual Taylor identity

$$\overline{v'q'} = -\frac{\partial}{\partial y} (\overline{u'v'}) + f e^{z/H} \frac{\partial}{\partial z} \left( \frac{e^{-z/H} \overline{v'\theta'}}{\partial \tilde{\theta} / \partial z} \right). \quad (9)$$

In the subsequent papers we have used the spherical coordinate version of the budget (e.g. Huang and Nakamura 2017, Nakamura and Huang 2018, Valva and Nakamura 2020) – in those papers (and in Clare’s GitHub), the momentum flux convergence is calculated with respect to the displacement latitude  $\phi'$  but we might have neglected the correction term  $(v_e/a \cos \phi) \partial(U_{\text{REF}} \cos \phi) / \partial \phi$ .